- 观察 data X
  缺失 data Z
  完全 data $Y = (X, Z)$
  参数 θ.

似然：
- $L(\theta|X) = P(X|\theta) \rightarrow$ 直接求解困难.
  $L(\theta|Y) = P(Y|\theta) \rightarrow$ 简单，容易计算. $\Big]$ 利用
  $P(Z|Y, \theta) \rightarrow$ 简单

计算 已知观察 data 下，θ 下的后验期望.

在 $\theta^{(t)}$ 下，在完全数据下 θ 的期望 完全似然 缺失 data.

E-step.
- $Q(\theta|\theta^{(t)}) = \int \log L(\theta|Y) \, P(Z|X, \theta^{(t)})$
  $= E[\int \log L(\theta|Y) | X, \theta^{(t)}]$.

M-step
- $\theta^{(t+1)} = \arg\max\limits_{\theta} Q(\theta|\theta^{(t)})$.

极大似然
$\downarrow$
Q 函数.

- $\log P(X|\theta) = \int \log P(X, Z|\theta) \, dz$
  $= \log \int \dfrac{P(X, Z|\theta)}{g(Z)} \, g(Z) \, dz \qquad$ 令 $g(z) = P(Z|X, \theta^{(t)})$.
  $\Rightarrow \int \log \dfrac{P(X, Z|\theta)}{P(Z|X, \theta^{(t)})} \cdot P(Z|X, \theta^{(t)}) \, dz$
  $= \underbrace{\int \log P(X, Z|\theta) \, P(Z|X, \theta^{(t)}) \, dz}_{Q(\theta, \theta^{(t)})} - \underbrace{\int \log P(Z|X, \theta^{(t)}) \, dz}_{\text{与 } \theta \text{ 无关 const.}}$

- $Y_1, Y_2 \overset{iid}{\sim} Exp(\theta) = \theta\, exp\{-y\theta\}$

Y_2 为缺失的数据.

$\qquad Y_1 = 5 \qquad Y_2 = ?$

观察data. $\quad P(y_1|\theta) = \theta\, exp\{-y_1\theta\}$

$\qquad log\, P(y_1|\theta) = log\,\theta - y_1\theta.$

最大似然估计.

$\qquad\qquad \dfrac{1}{\theta} - y_1 = 0 \qquad \theta = \dfrac{1}{y_1} \quad = 0.2 \leftarrow$

完整data. $\quad P(y_1, y_2|\theta) = \theta^2\, exp\{-(y_1+y_2)\theta\}$

$\qquad log\, P(y_1, y_2|\theta) = 2 log\,\theta - (y_1+y_2)\theta$

EM.

$\qquad\qquad P(y_2|y_1,\theta^{(t)}) = P(y_2|\theta^{(t)}) = \theta^{(t)} exp\{-y_2\theta^{(t)}\}$

$\qquad Q(\theta|\theta^{(t)}) = E[log\, P(y_1, y_2|\theta)\,|\,y_1, \theta^{(t)}]$

$\qquad\qquad = 2 log\,\theta - y_1\theta - \theta \cdot E[y_2]$

E-step. $\qquad \theta = \dfrac{2\theta^{(t)}}{5\theta^{(t)}+1}$

最终 $\qquad \theta^* = \dfrac{2\theta^*}{5\theta^*+1} \qquad \Rightarrow \theta^* = 0.2$

两种帮法
同解率
但 EM
解决了缺
失去数据

- $\vec{y} = (y_1, y_2, y_3, y_4)^T$ 多项式分布.

$$\frac{1}{2} + \frac{1}{4}\theta, \quad \frac{1}{4}(1-\theta), \quad \frac{1}{4}(1-\theta), \quad \frac{1}{4}\theta$$

$$\vec{y} = (125, 18, 20, 34)^T, \quad n = 197$$

最大似然法

$$P(\vec{y} \mid \theta) = \frac{n!}{y_1! \, y_2! \, y_3! \, y_4!} \left(\frac{1}{2} + \frac{1}{4}\theta\right)^{y_1} \left[\frac{1}{4}(1-\theta)\right]^{y_2} \left[\frac{1}{4}(1-\theta)\right]^{y_3} \left(\frac{1}{4}\theta\right)^{y_4}.$$

$$\log P(\vec{y} \mid \theta) = \qquad y_1 \log\left(\frac{2+\theta}{4}\right) + y_2 \log \frac{1}{4}(1-\theta) + y_3 \log(1-\theta)\cdot\frac{1}{4} + y_4 \log \frac{1}{4}\theta$$

$$\frac{y_1}{2+\theta} - \frac{y_2+y_3}{1-\theta} + \frac{y_4}{\theta} = 0$$

$W = (W_1, W_2)^T.$ $\qquad W = N(\mu, \Sigma)$ $\mu = \binom{\mu_1}{\mu_2}$ $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

$P(W|\theta) = \frac{1}{(2\pi)|\Sigma|^{\frac{1}{2}}} exp\{-\frac{1}{2}(W-\mu)^T \Sigma^{-1}(W-\mu)\}$

$\begin{matrix} W_1 \\ \vdots \\ W_m \end{matrix}$ ⬅ 都有 $\qquad W_1 \cdots W_m$

⬅ 部分值缺失. $\qquad W_{2j}$ ($j = m, m+1, \cdots m+m_1$) 第1个缺失,

⬅ 部分值缺失. $\qquad W_{1j}$ ($j = m_1, \cdots, N$) 第2个缺失.

通过后验公布而期望补自数据.

$\vec{y} = (y_1, \cdots, y_m)$

$\log L(\theta|y) = -n\log(2\pi) - \frac{1}{2}n\log|\Sigma| - \frac{1}{2}\sum_{i=1}^{n}(\vec{W_j} - \mu)^T \Sigma^{-1}(\vec{W_j} - \mu)$

$|\Sigma| = \begin{vmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{vmatrix} = \Sigma_{11}\Sigma_{22} - \Sigma_{21}\Sigma_{12}$

$T_i = \sum_{j=1}^{n} W_{ij}$ ($i = 1, 2, \cdots$)