2.4. 二元分布 $P(X=x)=\mu^x(1-\mu)^{1-x}$ $P(x=1)=\mu$ $P(x=0)=1-\mu$

$$E(X)=\int x P(x)\,dx = 1\cdot P(x=1)+0\cdot P(x=0)=\mu$$

$$Var(x)=E[(x-Ex)^2]=\int(x-\mu)^2P(x)\,dx$$
$$=\int[(1-\mu)^2\mu+\mu^2(1-\mu)]\,dx$$
$$=\mu-\mu^2$$
$$=\mu(1-\mu)$$

二项分布 $P(Y=m)=\binom{N}{m}\mu^m(1-\mu)^{N-m}$

$$Y=X_1+X_2+\cdots+X_N,\ X_i\sim B(x|\mu),\ i=1,2,\cdots,N$$

$$EY=E(X_1+X_2+\cdots+X_N)=EX_1+EX_2+\cdots+EX_N$$
$$=N\mu$$

$$Var(Y)=E[(Y-EY)^2]=E(Y^2)-(EY)^2$$

$$E(Y^2)=\int Y^2P(Y)\,dy$$
$$=\int m^2\binom{N}{m}\mu^m(1-\mu)^{N-m}\,dm$$
$$=\int[m(m-1)+m]\frac{N!}{m!(N-m)!}\mu^m(1-\mu)^{N-m}\,dm$$
$$=\int\frac{N!}{(m-2)!(N-m)!}\mu^m(1-\mu)^{N-m}\,dm+m\int\frac{N!}{m!(N-m)!}\mu^m(1-\mu)^{N-m}\,dm$$
$$=N(N-1)\mu^2\int\frac{(N-2)!}{(m-2)![(N-2)-(m-2)]!}\mu^{m-2}(1-\mu)^{[(N-2)-(m-2)]}\,dm$$
$$+\int mP(Y=m)\,dm$$
$$=N(N-1)\mu^2+N\mu$$
$$=N^2\mu^2-N\mu^2+N\mu$$

$$Var(Y)=E(Y^2)-(EY)^2$$
$$=N^2\mu^2-N\mu^2+N\mu-N^2\mu^2$$
$$=N\mu(1-\mu)$$

2.7 $\{x_n\}_{n=1}^{N}$, i.i.d, $1\le x_n\le 6$

记6个面的概率分别为μ_i，$1\le i\le 6$，$\sum_{n=1}^{6}\mu_i=1$

$\{x_n\}$符合二元分布，概率函数为

$$P(x_n|\mu_i)=\mu_i^{x_n=i}(1-\mu_i)^{x_n\ne i}$$

$$L(\mu_i|x)=\prod_{n=1}^{N}\mu_i^{x_n=i}(1-\mu_i)^{x_n\ne i}$$

$$=\mu_i^{\sum_{n=1}^{N}x_n=i}(1-\mu_i)^{\sum_{n=1}^{N}x_n\ne i}$$

$$\frac{\partial\log L(\mu_i|x)}{\partial\mu_i}=\frac{\partial\{\sum_{n=1}^{N}x_n=i\log\mu_i+\sum_{n=1}^{N}x_n\ne i\log(1-\mu_i)\}}{\partial\mu_i}$$

$$=\frac{1}{\mu_i}\sum_{n=1}^{N}x_n=i-\frac{1}{1-\mu_i}\sum_{n=1}^{N}x_n\ne i$$

$$=0$$

$$(1-\mu_i)\sum_{n=1}^{N}x_n=i=\mu_i\sum_{n=1}^{N}x_n\ne i$$

$$\mu_i=\frac{1}{N}\sum_{n=1}^{N}x_n=i\quad(1\le i\le 6)$$

2.9 $$P(x_{1n},x_{2n}|\rho)=\frac{1}{2\pi(1-\rho^2)^{\frac12}}\exp\left(-\frac{x_{1n}^2-2\rho x_{1n}x_{2n}+x_{2n}^2}{2(1-\rho^2)}\right)$$

$$L(\rho|x_1,x_2)=\prod_{n=1}^{N}P(x_{1n},x_{2n}|\rho)$$

$$=\left(\frac{1}{2\pi(1-\rho^2)^{\frac12}}\right)^N\exp\left(-\sum_{n=1}^{N}\frac{x_{1n}^2-2\rho x_{1n}x_{2n}+x_{2n}^2}{2(1-\rho^2)}\right)$$

$$\frac{\partial\log L(\rho|x_1,x_2)}{\partial\rho}=\frac{\partial\left\{-N\log(2\pi(1-\rho^2)^{\frac12})-\sum_{n=1}^{N}\frac{x_{1n}^2-2\rho x_{1n}x_{2n}+x_{2n}^2}{2(1-\rho^2)}\right\}}{\partial\rho}$$

$$\frac{\partial}{\partial\rho}\{-N\log(2\pi(1-\rho^2)^{\frac12})\}=-N\cdot\frac{1}{\cancel{2\pi}(1-\rho^2)^{\frac12}}\cdot\frac12(1-\rho^2)^{-\frac12}\cdot-2\rho$$

$$=\frac{N\rho}{\cancel{2\pi}(1-\rho^2)}$$

$$\frac{\partial}{\partial\rho}\left\{-\sum_{n=1}^{N}\frac{x_{1n}^2-2\rho x_{1n}x_{2n}+x_{2n}^2}{2(1-\rho^2)}\right\}=-\sum_{n=1}^{N}\frac{\partial}{\partial\rho}\frac{x_{1n}^2-2\rho x_{1n}x_{2n}+x_{2n}^2}{2(1-\rho^2)}$$

$$=-\sum_{n=1}^{N}\frac{-2x_{1n}x_{2n}\cdot 2(1-\rho^2)-(x_{1n}^2+x_{2n}^2-2\rho x_{1n}x_{2n})\cdot(-4\rho)}{4(1-\rho^2)^2}$$

$$=-\sum_{n=1}^{N}\frac{-x_{1n}x_{2n}(1-\rho^2)+\rho(x_{1n}^2+x_{2n}^2-2\rho x_{1n}x_{2n})}{(1-\rho^2)^2}$$

$$\frac{\partial\log L(\rho|x_1,x_2)}{\partial\rho}=\frac{N\rho}{(1-\rho^2)}-\sum_{n=1}^{N}\frac{-x_{1n}x_{2n}(1-\rho^2)+\rho(x_{1n}^2+x_{2n}^2-2\rho x_{1n}x_{2n})}{(1-\rho^2)^2}=0$$

$$N\rho^3-\sum_{n=1}^{N}x_{1n}x_{2n}\rho^2+\left(\sum_{n=1}^{N}(x_1^2+x_2^2)-N\right)\rho-\sum_{n=1}^{N}x_{1n}x_{2n}=0$$

解即为ρ的估计

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2.11 $P(\mu|x) = \frac{P(x|\mu)P(\mu)}{P(x)}$

$$= \frac{1}{P(x)} \prod_{n=1}^{N} \frac{1}{(2\pi)^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}(x_n-\mu)^T\Sigma^{-1}(x_n-\mu)) \frac{1}{(2\pi)^{\frac{1}{2}}|\Sigma_0|^{\frac{1}{2}}} \exp(-\frac{1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)$$

$$= \frac{1}{P(x)} \left(\frac{1}{(2\pi)^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}}\right)^N \frac{1}{(2\pi)^{\frac{1}{2}}|\Sigma_0|^{\frac{1}{2}}} \exp(-\frac{1}{2}(x_n-\mu)^T\Sigma^{-1}(x_n-\mu) - \frac{1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0))$$

$$= C\cdot \exp(-\frac{1}{2}\{\sum_{n=1}^{N}(x_n^T\Sigma^{-1}x_n - x_n^T\Sigma^{-1}\mu - \mu^T\Sigma^{-1}x_n + \mu^T\Sigma^{-1}\mu) + \mu^T\Sigma_0^{-1}\mu$$
$$- \mu^T\Sigma_0^{-1}\mu_0 - \mu_0^T\Sigma_0^{-1}\mu + \mu_0^T\Sigma_0^{-1}\mu_0\})$$

$$= C\cdot \exp(-\frac{1}{2}\{\mu^T(N\Sigma^{-1}+\Sigma_0^{-1})\mu - \mu^T(\Sigma^{-1}\sum_{n=1}^{N}x_n - \Sigma_0^{-1}\mu_0) - (\sum_{n=1}^{N}x_n^T\Sigma^{-1} - \mu_0^T\Sigma_0^{-1})\mu$$
$$+ \sum_{n=1}^{N}x_n^T\Sigma^{-1}x_n - \mu_0^T\Sigma_0^{-1}\mu_0\})$$

$$\mu^T(N\Sigma^{-1}+\Sigma_0^{-1})\mu = \mu^T N\Sigma^{-1}(\Sigma_0 + \frac{1}{N}\Sigma)\Sigma_0^{-1}\mu$$
$$= \mu^T(\Sigma_0(\Sigma_0+\frac{1}{N}\Sigma)\frac{1}{N}\Sigma)^{-1}\mu$$
$$= \mu^T\Sigma_N^{-1}\mu$$

$$-\mu^T(\Sigma^{-1}\sum_{n=1}^{N}x_n - \Sigma_0^{-1}\mu_0) = -\mu^T(N\Sigma^{-1}(\Sigma_0+\frac{1}{N}\Sigma)\Sigma_0^{-1}\Sigma_0(\Sigma_0+\frac{1}{N}\Sigma)^{-1}\frac{1}{N}\sum_{n=1}^{N}x_n$$
$$+ \Sigma^{-1}(\Sigma_0+\frac{1}{N}\Sigma)\Sigma_0^{-1}\Sigma(\Sigma_0+\frac{1}{N}\Sigma)^{-1}\mu_0)$$
$$= -\mu^T(N\Sigma^{-1}(\Sigma_0+\frac{1}{N}\Sigma)\Sigma_0^{-1})(\Sigma_0(\Sigma_0+\frac{1}{N}\Sigma)^{-1}\frac{1}{N}\sum_{n=1}^{N}x_n$$
$$+ \frac{1}{N}\Sigma(\Sigma_0+\frac{1}{N}\Sigma)^{-1}\mu_0))$$

$$= -\mu^T\Sigma_N^{-1}\mu_N$$

同理可得 $-(\sum_{n=1}^{N}x_n^T\Sigma^{-1} - \mu_0^T\Sigma_0^{-1})\mu = -\mu_N^T\Sigma_N^{-1}\mu$

$$\therefore P(\mu|x) = C'\cdot\exp(-\frac{1}{2}(\mu^T\Sigma_N^{-1}\mu - \mu^T\Sigma_N^{-1}\mu_N - \mu_N^T\Sigma_N^{-1}\mu + \mu^T\Sigma_N^{-1}\mu))$$
$$= C'\exp(-\frac{1}{2}(\mu-\mu_N)^T\Sigma_N^{-1}(\mu-\mu_N))$$

$$\therefore P(\mu|x) = N(\mu|\mu_N, \Sigma_N)$$

2.12 $p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2}\right) \exp\left(\frac{\mu x}{\sigma^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\sigma^2}{2}\left(\frac{\mu}{\sigma^2}\right)^2\right) \exp\left(-\frac{x^2}{2\sigma^2}\right) \exp\left(\frac{\mu}{\sigma^2} x\right)$$

$$= g\left(\frac{\mu}{\sigma^2}\right) h(x) \exp\left(\frac{\mu}{\sigma^2} x\right)$$