Risk-Neutral Pricing via Moment Generating Functions

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1. Abstract

This paper presents a comprehensive derivation of risk-neutral drift adjustment using moment generating functions (MGFs). We establish the fundamental relationship between MGFs and martingale conditions in derivative pricing, demonstrating how this approach provides a unified framework for both Gaussian and non-Gaussian asset price models. The methodology is extended to multi-asset systems with correlated returns, and connections to fundamental theorems (Girsanov, Radon-Nikodym) are rigorously explored. Practical implementation considerations, including quadrature methods and Monte Carlo techniques, are discussed alongside limitations of traditional tree-based approaches. Applications to various option types demonstrate the versatility of the MGF framework.

2. Fundamentals of Risk-Neutral Pricing

In arbitrage-free pricing theory, derivative valuation requires the existence of a risk-neutral measure $\mathbb{Q}$ under which discounted asset prices become martingales. For an asset price process S_t, this fundamental condition is expressed as:

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT} S_T \mid \mathcal{F}_t] = S_t$$

where r denotes the risk-free rate, T the maturity, and $\mathcal{F}_t$ the filtration representing available information at time t.

For models with a log-price representation $S_T = S_0 e^{X_T}$ where $X_T \sim f_X(x; \mu)$, we require:

$$\mathbb{E}^{\mathbb{Q}}[e^{X_T}] = e^{rT}$$

The moment generating function provides a powerful tool to enforce this condition through its definition:

$$M(u) = \mathbb{E}[e^{uX_T}] = \int_{-\infty}^{\infty} e^{ux} f_X(x) dx$$

Theorem 1: Risk-Neutral Drift Adjustment

For a log-asset price $X_T \sim f_X(x; \mu)$ where μ represents the physical drift, the risk-neutral drift μ_{RN} must satisfy:

$$M(1; \mu_{RN}) = e^{rT}$$

This condition ensures the discounted asset price is a martingale under $\mathbb{Q}$.

Proof

Starting from the martingale condition for the discounted asset price:

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T] = S_0$$

Substituting $S_T = S_0 e^{X_T}$:

$$e^{-rT} S_0 \mathbb{E}^{\mathbb{Q}}[e^{X_T}] = S_0$$

Simplifying and recognizing $\mathbb{E}^{\mathbb{Q}}[e^{X_T}] = M(1)$:

$$M(1) = e^{rT} \quad \blacksquare$$

3. MGF Derivation and Examples

3.1 General Properties

The MGF provides a complete characterization of the distribution when it exists. Key properties include:

- $M(0) = 1$ for any random variable

- $\frac{d^n M}{du^n}(0) = \mathbb{E}[X^n]$ (when derivatives exist)
- For independent random variables: $M_{X+Y}(u) = M_X(u) M_Y(u)$
- Affine transformation: For $Y = aX + b$, $M_Y(u) = e^{bu} M_X(au)$

Example 1: Gaussian Distribution

For $X_T \sim \mathcal{N}(\mu, \sigma^2)$:

$$M(u) = \exp\left(\mu u + \frac{1}{2}\sigma^2 u^2\right)$$

Enforcing the martingale condition:

$$e^{\mu_{RN} + \frac{1}{2}\sigma^2} = e^{rT} \Rightarrow \mu_{RN} = rT - \frac{1}{2}\sigma^2$$

This recovers the classic Black-Scholes drift adjustment. Note that σ here denotes the standard deviation of returns, i.e., volatility multiplied by $\sqrt{T}$.

Example 2: Poisson Jump Diffusion

For a compound Poisson process $X_T = \sum_{i=1}^{N_T} Y_i$ where $N_T \sim \mathrm{Poisson}(\lambda T)$ and $Y_i \sim \mathcal{N}(\mu_J, \sigma_J^2)$:

$$M(u) = \exp\left[T\left(\lambda(e^{\mu_J u + \frac{1}{2}\sigma_J^2 u^2} - 1\right)\right]$$

The risk-neutral drift satisfies:

$$\lambda_{RN}\left(e^{\mu_{J,RN} + \frac{1}{2}\sigma_J^2} - 1\right) = r - \frac{1}{2}\sigma^2 - \lambda\left(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1\right)$$

where σ represents the diffusion volatility.

Example 3: Variance Gamma Process

For the variance gamma model with parameters θ, σ, ν:

$$M(u) = \left(1 - u\theta\nu - \frac{1}{2}\sigma^2\nu u^2\right)^{-T/\nu}$$

The risk-neutral condition requires solving:

$$\left(1 - \theta_{RN}\nu - \frac{1}{2}\sigma_{RN}^2\nu\right)^{-T/\nu} = e^{rT}$$

which involves simultaneous adjustment of multiple parameters.

4. Theoretical Foundations

4.1 Girsanov's Theorem and Radon-Nikodym Derivative

The MGF approach provides a computational implementation of Girsanov's theorem, which describes measure transformations:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta(t)dW_t^{\mathbb{P}} - \frac{1}{2}\int_0^T \theta(t)^2 dt\right)$$

The Radon-Nikodym derivative connects to the MGF through exponential moments. For exponential Lévy models, the density is directly related to the MGF.

Connection Proof

Consider the Radon-Nikodym derivative for a Lévy process with characteristic exponent $\psi(u)$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(uX_T - T\psi(u)\right)$$

The risk-neutral condition requires:

$$\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}e^{X_T}\right] = e^{rT}$$

Substituting and simplifying:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left((u+1)X_T - T\psi(u)\right)\right] = e^{rT}$$

Recognizing the MGF form:

$$e^{-T\psi(u)}M(u+1) = e^{rT} \quad \blacksquare$$

4.2 Itô Calculus and Feynman-Kac Theorem

For diffusion models, the MGF approach provides an alternative to solving PDEs derived from Itô's lemma. See Section 10 for more information. The Feynman-Kac theorem establishes the equivalence between:

- Expectation representations (MGF approach)
- Partial differential equations (PDE approach)

$$\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - rV = 0$$

4.3 Moment Adjustment Interpretation (Taylor Series Expansion)

The MGF encodes all moments of the distribution through its series expansion:

$$M(u) = \sum_{n=0}^{\infty}\frac{u^n}{n!}\mathbb{E}[X^n]$$

This reveals that the risk-neutral adjustment incorporates adjustments to all moments, not just the mean:

$$\mu_{RN} = rT - \log M(1) = rT - \left(\mu T + \frac{1}{2}\sigma^2 T + \frac{\mathbb{E}[X^3]}{3!} + \cdots\right)$$

5. Multi-Asset Generalization

5.1 Gaussian Framework

For a system of n assets with correlated returns $\vec{X} = (X_1, \ldots, X_n) \sim \mathcal{N}(\vec{\mu}, \Sigma)$, the multivariate MGF is:

$$M(\vec{u}) = \exp\left(\vec{\mu}^T \vec{u} + \frac{1}{2}\vec{u}^T \Sigma \vec{u}\right)$$

The martingale condition requires for each asset i:

$$\mathbb{E}^{\mathbb{Q}}[e^{X_i}] = M(0, \ldots, 1_i, \ldots, 0) = e^{rT}$$

This generates a system of equations:

$$\mu_{i,RN} + \frac{1}{2}\Sigma_{ii} + \sum_{j \neq i} \Sigma_{ij}\lambda_j = rT$$

where λ_j are market prices of risk. The complete adjustment requires solving:

$$\vec{\mu}_{RN} = rT\vec{1} - \frac{1}{2}\text{diag}(\Sigma) - \Sigma\vec{\lambda}$$

5.2 Copula-Based Extensions

To model joint distributions beyond Gaussian assumptions, we can use copulas to construct multivariate distributions with flexible dependence structures.

> **Multivariate t-Copula**
>
> Each asset has its own marginal distribution (e.g., t-distribution), and the t-copula captures tail dependence. The joint MGF is approximated via Monte Carlo:
>
> $$M(\vec{u}) = \mathbb{E}\left[e^{\vec{u}^T \vec{X}}\right]$$
>
> Steps:
>
> 1. Define marginals $F_i(x)$

2. Transform to uniform: $u_i = F_i(x_i)$
3. Apply t-copula to model joint dependence
4. Sample $\vec{X}$ and compute MGF

Empirical Copula

Constructed from historical data, it captures real-world dependencies without parametric assumptions:

1. Rank-transform historical returns
2. Construct empirical copula from joint ranks
3. Simulate joint samples
4. Estimate MGF numerically:

$$M(\vec{u}) \approx \frac{1}{N} \sum_{i=1}^{N} e^{\vec{u}^T \vec{X}^{(i)}}$$

6. Numerical Implementation

6.1 Quadrature Methods

For distributions without closed-form MGFs (e.g., Weibull, generalized hyperbolic), numerical integration is essential:

$$M(1) \approx \sum_{k=1}^{N} w_k e^{x_k} f_X(x_k)$$

Gauss-Hermite quadrature is particularly efficient for approximately Gaussian densities, while adaptive quadrature handles heavy-tailed distributions.

6.2 Monte Carlo Simulation

When analytical and quadrature methods fail, Monte Carlo provides a flexible alternative:

$$M(1) \approx \frac{1}{N} \sum_{i=1}^{N} e^{X_T^{(i)}}$$

Variance reduction techniques dramatically improve convergence:

- **Importance sampling:** Adjust sampling density to focus on critical regions
- **Control variates:** Use correlated random variables with known expectations
- **Antithetic variates:** Exploit negative dependence to reduce variance

6.3 Limitations of Lattice Methods

Traditional binomial/trinomial trees face fundamental limitations:

- Exponential growth in multi-dimensional settings ($O(m^n)$ complexity)
- Difficulty matching higher moments (skewness > 3, kurtosis > 4)
- Lattice artifacts in barrier option pricing (barrier position mismatch)

Advanced alternatives include:

- **Heptanomial trees:** Match up to 4 moments but with $O(n^4)$ complexity
- **Edgeworth expansions:** Adjust lattice probabilities to match moments
- **Markov chain approximations:** Preserve distributional properties

7. Option Pricing Applications

7.1 European Options

For European calls, the risk-neutral expectation becomes:

$$C = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] = e^{-rT}\int_K^{\infty}(S_0e^x - K)f_{X_T}(x)dx$$

When the MGF is known, this can be efficiently computed using Fourier transform methods.

7.2 Exotic Options

For barrier options, the MGF of the first passage time can be derived for Brownian motion:

$$M_{\tau_b}(u) = \mathbb{E}[e^{-u\tau_b}] = e^{-b(\sqrt{2u+\theta^2}-\theta)}$$

where $\tau_b = \inf\{t > 0 : W_t = b\}$ and $\theta = \mu/\sigma$. This enables analytical pricing of barrier options under the Black-Scholes framework.

7.3 Stochastic Volatility Models

In the Heston model, the characteristic function (Fourier transform of MGF) has a closed form:

$$\phi(u) = \exp\left(C(u,T) + D(u,T)v_0 + iu\ln S_0\right)$$

where $C(\cdot)$ and $D(\cdot)$ solve Riccati equations:

$$\frac{dD}{dt} = \alpha D^2 + \beta D + \gamma, \quad \frac{dC}{dt} = rD$$

This enables efficient option pricing via Fourier inversion:

$$C = \frac{e^{-rT}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\ln K}\phi(u-i)}{iu\phi(-i)}\right] du$$

This method achieves high accuracy with computational complexity $O(N\log N)$ using FFT techniques.

8. Conclusion

The moment generating function provides a powerful unified framework for risk-neutral drift adjustment across diverse asset price models. This approach:

- Generalizes beyond Gaussian assumptions to Lévy processes and jump diffusions
- Provides explicit connections to fundamental theorems of asset pricing
- Enables efficient computation via quadrature when closed-form solutions exist
- Extends naturally to multi-asset systems with correlations
- Integrates with Fourier methods for efficient option pricing

Future research directions include developing MGF-based methods for rough volatility models, machine learning accelerated quadrature techniques, and quantum computing approaches for high-dimensional problems. The mathematical elegance and practical versatility of MGFs ensure their continued relevance in derivative pricing theory.

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Appendix

9. Moment-Generating Functions of Common Distributions

Here are eight frequently-used families that possess a closed or semi-closed MGF.

Distribution	Density $f_X(x)$	MGF $M(u)$	Risk-neutral drift μ_{RN}
Weibull (scale λ, shape k)	$f(x) = \frac{k}{\lambda}(x/\lambda)^{k-1}\exp[-(x/\lambda)^k]$	$M(u) = \int_0^\infty e^{ux} f(x)\,dx \sim \frac{1}{\Gamma(1+1/k)}\,{}_1F_1\big(1+\frac{u\lambda}{k}, 1+\frac{1}{k}; -u^k u^k\big)$	$\mu_{RN} = rT - \log M(1)$
Truncated normal (lower truncation a)	$f(x) = \frac{\phi((x-\mu)/\sigma)}{\sigma\,[1-\Phi((a-\mu)/\sigma)]}\mathbf{1}_{\{x\geq a\}}$	$M(u) = e^{\mu u+\frac{1}{2}\sigma^2u^2}\frac{1-\Phi(\frac{a-\mu}{\sigma}-\sigma u)}{1-\Phi(\frac{a-\mu}{\sigma})}$	$\mu_{RN} = rT - \frac{1}{2}\sigma^2 T - \log\Big(\frac{1-\Phi((a-\mu)/\sigma-\sigma}{1-\Phi((a-\mu)/\sigma)}$
Skew normal (α,ω,ξ)	$f(x) = \frac{2}{\omega}\phi\Big(\frac{x-\xi}{\omega}\Big)\Phi\Big(\alpha\frac{x-\xi}{\omega}\Big)$	$M(u) = \exp\big(\xi u + \frac{1}{2}\omega^2u^2\big)\frac{\Phi\big(\alpha\omega u/\sqrt{1+\alpha^2}\big)}{\Phi(0)}$	$\mu_{RN} = rT - \log M(1)$
Student-t (ν,μ,σ)	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}\sigma}\Big(1+\frac{(x-\mu)^2}{\nu\sigma^2}\Big)^{-\frac{\nu+1}{2}}$	$M(u) = \exp\big(\frac{\nu\sigma^2u^2}{2}\big)U\big(\frac{1}{2},\frac{\nu}{2},\frac{(\sigma u)^2}{2}\big) \quad (\lvert u\rvert < \sqrt{\nu}/\sigma)$	$\mu_{RN} = rT - \log M(1)$
Laplace (double-exponential) (λ,μ)	$f(x) = \frac{\lambda}{2}\exp[-\lambda\lvert x-\mu\rvert]$	$M(u) = \frac{\exp(\mu u)}{1-u^2/\lambda^2} \quad (\lvert u\rvert<\lambda)$	$\mu_{RN} = rT - \log\Big(\frac{e^{\mu}}{1-1/\lambda}$
Lévy-stable $(\alpha,\beta,\gamma,\delta)$	Characteristic exponent $\psi(u) = i\delta u - \gamma\lvert u\rvert^\alpha\Big(1 - i\beta\,\mathrm{sgn}(u)\tan\frac{\pi\alpha}{2}\Big)$	$M(u) = \exp[-\psi(-iu)],\ \lvert\Im u\rvert < \gamma^{-1/\alpha}$	$\mu_{RN} = rT - \log M(1)$
Gamma (k,θ)	$f(x) = \frac{1}{\Gamma(k)\theta^k}x^{k-1}e^{-x/\theta}$	$M(u) = (1-\theta u)^{-k} \quad (u < 1/\theta)$	$\mu_{RN} = rT + k\log(1-\theta)$
Inverse Gaussian (λ,μ)	$f(x) = \frac{\lambda}{\sqrt{2\pi x^3}}\exp\Big(-\frac{\lambda(x-\mu)^2}{2\mu^2x}\Big)$	$M(u) = \exp\Big(\lambda\big(\sqrt{1-2\mu u/\lambda}-1\big)\Big) \quad (u<\frac{\lambda}{2\mu})$	$\mu_{RN} = rT - \log M(1)$

10. Itô Calculus, Feynman–Kac PDE and the MGF

The following short proof exhibits the precise equivalence of the three approaches.

Proposition. Let X_t satisfy the Itô SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \qquad X_0 = x.$$

Define the contingent claim value

$$V(t, x) = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}\big[\Phi(X_T) \mid X_t = x\big].$$

Then the backward Kolmogorov PDE

$$-V_t + \mu V_x + \tfrac{1}{2}\sigma^2 V_{xx} - rV = 0, \quad V(T, x) = \Phi(x) \tag{10.1}$$

has the explicit solution

$$V(t, x) = \int_{\mathbb{R}} e^{-r(T-t)+ux}\, p(t, x \to y)\, \Phi(y)\, dy$$

where p is the transition density of X. The Laplace transform in time of p is precisely the MGF of X_T. Hence the PDE solution can be written as the expectation value

$$V(t, x) = e^{-r(T-t)}\, \mathbb{E}\big[e^{uX_T} \mid X_t = x\big] = e^{-r(T-t)}\, M(u, t, x)$$

with $u = 1$ if $\Phi(x) = e^x$.

Inserting $V(t, x) = e^{-r(T-t)} M(u, t, x)$ into PDE (10.1) eliminates the time derivative and yields the ordinary differential equation that the MGF satisfies. Conversely, inserting the MGF into the expectation representation immediately recovers the PDE solution. Thus Itô calculus, Feynman–Kac and the MGF are mathematically equivalent.