Risk-Neutral Pricing via Moment Generating Functions

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1. Abstract

This paper presents a comprehensive derivation of risk-neutral drift adjustment using moment generating functions (MGFs). We establish the fundamental relationship between MGFs and martingale conditions in derivative pricing, demonstrating how this approach provides a unified framework for both Gaussian and non-Gaussian asset price models. The methodology is extended to multi-asset systems with correlated returns, and connections to fundamental theorems (Girsanov, Radon-Nikodym) are explored. Practical implementation considerations, including quadrature methods and Monte Carlo techniques, are discussed alongside limitations of traditional tree-based approaches.

2. Fundamentals of Risk-Neutral Pricing

In arbitrage-free pricing theory, derivative valuation requires the existence of a risk-neutral measure $\mathbb{Q}$ under which discounted asset prices become martingales. For an asset price process S_t, this fundamental condition is expressed as:

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T|\mathcal{F}_t] = S_t$$

where r denotes the risk-free rate, T the maturity, and $\mathcal{F}_t$ the filtration representing available information at time t.

For models with a log-price representation $S_T = S_0 e^{X_T}$ where $X_T \sim f_X(x;\mu)$, we require:

$$\mathbb{E}^{\mathbb{Q}}[e^{X_T}] = e^{rT}$$

The moment generating function provides a powerful tool to enforce this condition through its definition:

$$M(u) = \mathbb{E}[e^{uX_T}] = \int_{-\infty}^{\infty} e^{ux} f_X(x) dx$$

Theorem 1: Risk-Neutral Drift Adjustment

For a log-asset price $X_T \sim f_X(x; \mu)$ where μ represents the physical drift, the risk-neutral drift μ_{RN} must satisfy:

$$M(1; \mu_{RN}) = e^{rT}$$

This condition ensures the discounted asset price is a martingale under $\mathbb{Q}$. The explicit solution for μ_{RN} is:

$$\mu_{RN} = \log\left(\frac{e^{rT}}{M(1; \mu = 0)}\right)$$

where $M(1; \mu = 0)$ denotes the MGF evaluated at $u = 1$ with zero drift.

3. MGF Derivation and Examples

3.1 General Properties

The MGF provides a complete characterization of the distribution when it exists. Key properties include:

- $M(0) = 1$ for any random variable
- $\frac{d^n M}{du^n}(0) = \mathbb{E}[X^n]$ (when derivatives exist)
- For independent random variables: $M_{X+Y}(u) = M_X(u)M_Y(u)$

Example 1: Gaussian Distribution

For $X_T \sim \mathcal{N}(\mu, \sigma^2)$:

$$M(u) = \exp\left(\mu u + \frac{1}{2}\sigma^2 u^2\right)$$

Enforcing the martingale condition:

$$e^{\mu_{RN} + \frac{1}{2}\sigma^2} = e^{rT} \Rightarrow \mu_{RN} = rT - \frac{1}{2}\sigma^2$$

This recovers the classic Black-Scholes drift adjustment.

Example 2: Poisson Jump Diffusion

For a compound Poisson process $X_T = \sum_{i=1}^{N_T} Y_i$ where $N_T \sim \mathrm{Poisson}(\lambda T)$ and $Y_i \sim \mathcal{N}(\mu_J, \sigma_J^2)$:

$$M(u) = \exp\left[T\left(\lambda(e^{\mu_J u + \frac{1}{2}\sigma_J^2 u^2} - 1\right)\right]$$

The risk-neutral drift satisfies:

$$\lambda_{RN}\left(e^{\mu_{J,RN}+\frac{1}{2}\sigma_J^2}-1\right)=r-\frac{1}{2}\sigma^2-\lambda\left(e^{\mu_J+\frac{1}{2}\sigma_J^2}-1\right)$$

where σ represents the diffusion volatility.

Example 3: Variance Gamma Process

For the variance gamma model with parameters θ, σ, ν:

$$M(u)=\left(1-u\theta\nu-\frac{1}{2}\sigma^2\nu u^2\right)^{-T/\nu}$$

The risk-neutral condition requires solving:

$$\left(1-\theta_{RN}\nu-\frac{1}{2}\sigma_{RN}^2\nu\right)^{-T/\nu}=e^{rT}$$

which involves simultaneous adjustment of multiple parameters.

4. Connections to Fundamental Theorems

4.1 Girsanov Theorem

The MGF approach provides a computational implementation of Girsanov's theorem, which describes measure transformations:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}=\exp\left(-\int_0^T\theta(t)dW_t^{\mathbb{P}}-\frac{1}{2}\int_0^T\theta(t)^2dt\right)$$

The Radon-Nikodym derivative connects to the MGF through exponential moments. For exponential Lévy models, the density is directly related to the MGF.

4.2 Itô Calculus

For diffusion models, the MGF approach provides an alternative to solving PDEs derived from Itô's lemma. The Feynman-Kac theorem establishes the equivalence between:

- Expectation representations (MGF approach)
- Partial differential equations (PDE approach)

$$\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - rV = 0$$

4.3 Taylor Series Expansion

The MGF encodes all moments of the distribution through its series expansion:

$$M(u) = \sum_{n=0}^{\infty} \frac{u^n}{n!} \mathbb{E}[X^n]$$

This reveals that the risk-neutral adjustment incorporates adjustments to all moments, not just the mean:

$$\mu_{RN} = rT - \log M(1) = rT - \left(\mu T + \frac{1}{2}\sigma^2 T + \frac{\mathbb{E}[X^3]}{3!} + \cdots\right)$$

5. Multi-Asset Generalization

For a system of n assets with correlated returns $\vec{X} = (X_1, \ldots, X_n) \sim \mathcal{N}(\vec{\mu}, \Sigma)$, the multivariate MGF is:

$$M(\vec{u}) = \exp\left(\vec{\mu}^T\vec{u} + \frac{1}{2}\vec{u}^T\Sigma\vec{u}\right)$$

The martingale condition requires for each asset i:

$$\mathbb{E}^{\mathbb{Q}}[e^{X_i}] = M(0, \ldots, 1_i, \ldots, 0) = e^{rT}$$

This generates a system of equations:

$$\mu_{i,RN} + \frac{1}{2}\Sigma_{ii} + \sum_{j\neq i}\Sigma_{ij}\lambda_j = rT$$

where λ_j are market prices of risk. The complete adjustment requires solving:

$$\vec{\mu}_{RN} = rT\vec{1} - \frac{1}{2}\text{diag}(\Sigma) - \Sigma\vec{\lambda}$$

This framework extends to non-Gaussian multivariate models through copula-based MGF constructions.

6. Implementation Considerations

6.1 Quadrature Methods

For distributions without closed-form MGFs (e.g., Weibull, generalized hyperbolic), numerical integration is essential:

$$M(1) \approx \sum_{k=1}^{N} w_k e^{x_k} f_X(x_k)$$

Gauss-Hermite quadrature is particularly efficient for approximately Gaussian densities, while adaptive quadrature handles heavy-tailed distributions.

6.2 Monte Carlo Simulation

When analytical and quadrature methods fail, Monte Carlo provides a flexible alternative:

$$M(1) \approx \frac{1}{N} \sum_{i=1}^{N} e^{X_T^{(i)}}$$

with variance reduction techniques (importance sampling, control variates) dramatically improving convergence.

6.3 Limitations of Lattice Methods

Traditional binomial/trinomial trees face fundamental limitations:

- Exponential growth in multi-dimensional settings
- Difficulty matching higher moments (skewness > 3, kurtosis > 4)
- Lattice artifacts in barrier option pricing

Advanced alternatives include:

- **Heptanomial trees:** Match up to 4 moments but with $O(n^4)$ complexity
- **Edgeworth expansions:** Adjust lattice probabilities to match moments
- **Markov chain approximations:** Preserve distributional properties

7. Applications in Option Pricing

7.1 European Options

For European calls, the risk-neutral expectation becomes:

$$C = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] = e^{-rT}\int_K^{\infty}(S_0e^x - K)f_{X_T}(x)dx$$

When the MGF is known, this can be efficiently computed using Fourier transform methods.

7.2 Exotic Options

For barrier options, the MGF of the first passage time can be derived for Brownian motion:

$$M_{\tau_b}(u) = \mathbb{E}[e^{-u\tau_b}] = e^{-b(\sqrt{2u+\theta^2}-\theta)}$$

where $\tau_b = \inf\{t > 0 : W_t = b\}$ and $\theta = \mu/\sigma$.

7.3 Stochastic Volatility Models

In the Heston model, the characteristic function (Fourier transform of MGF) has a closed form:

$$\phi(u) = \exp\left(C(u,T) + D(u,T)v_0 + iu\ln S_0\right)$$

where $C(\cdot)$ and $D(\cdot)$ solve Riccati equations. This enables efficient option pricing via Fourier inversion.

8. Conclusion

The moment generating function provides a powerful unified framework for risk-neutral drift adjustment across diverse asset price models. This approach:

- Generalizes beyond Gaussian assumptions to Lévy processes and jump diffusions
- Provides explicit connections to fundamental theorems of asset pricing
- Enables efficient computation via quadrature when closed-form solutions exist

- Extends naturally to multi-asset systems with correlations
- Integrates with Fourier methods for efficient option pricing

Future research directions include developing MGF-based methods for rough volatility models, machine learning accelerated quadrature techniques, and quantum computing approaches for high-dimensional problems.