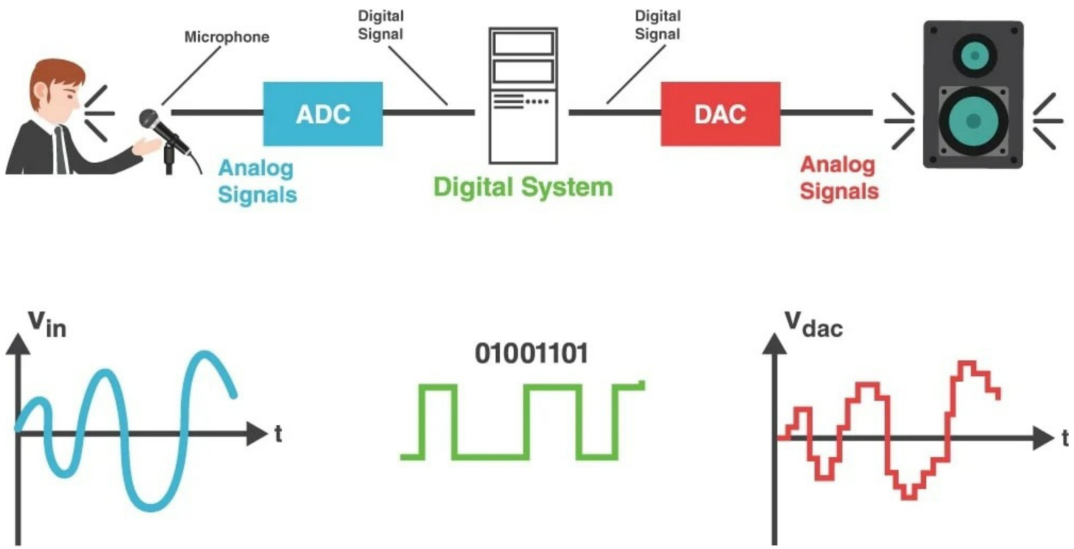
Signals and Systems

Jyotsna Bapat

System Example

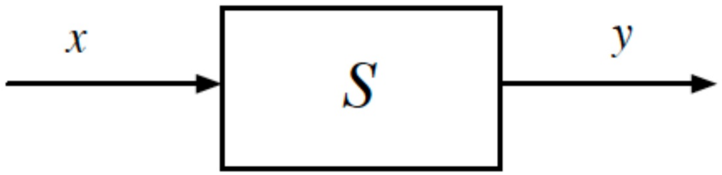


Systems

- A system transforms *input signals* into *output signals*.
- A system is a *function* mapping input signals into output signals.
- We will concentrate on systems with one input and one output *i.e.* *single-input, single-output* (SISO) systems.
- Notation:
    - $y = Sx$ or $y = S(x)$, meaning the system S acts on an input signal x to produce output signal y.
    - $y = Sx$ does not (in general) mean multiplication!

Systems

Systems often denoted by *block diagram*:



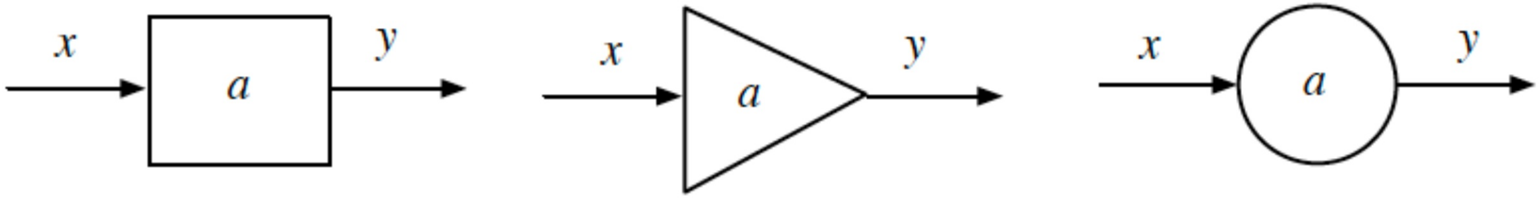
- Lines with arrows denote signals (*not* wires).
- Boxes denote systems; arrows show inputs & outputs.
- Special symbols for some systems.

Examples

(with input signal x and output signal y)

Scaling system: $y(t) = ax(t)$

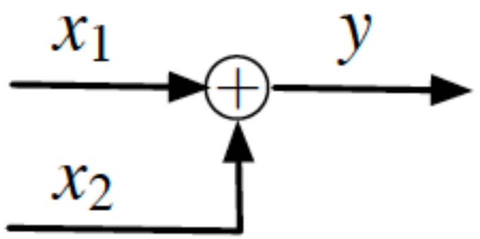
- Called an *amplifier* if $|a| > 1$.
- Called an *attenuator* if $|a| < 1$.
- Called *inverting* if $a < 0$.
- a is called the *gain* or *scale factor*.
- Sometimes denoted by triangle or circle in block diagram:



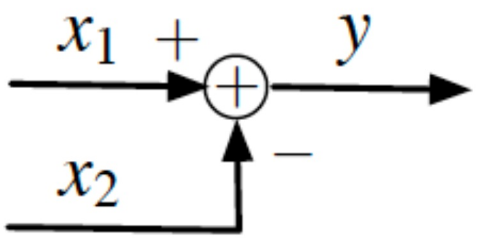
Examples with multiple inputs

Inputs $x_1(t)$, $x_2(t)$, and Output $y(t)$)

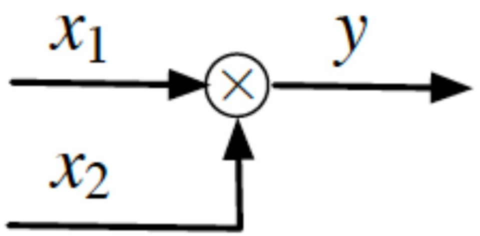
- **summing system:** $y(t) = x_1(t) + x_2(t)$



- **difference system:** $y(t) = x_1(t) - x_2(t)$



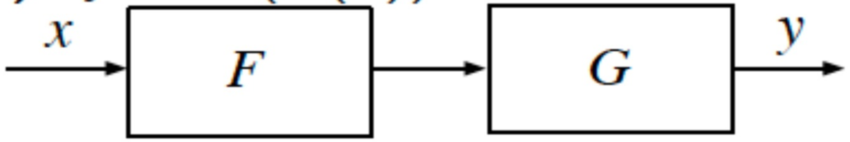
- **multiplier system:** $y(t) = x_1(t)x_2(t)$



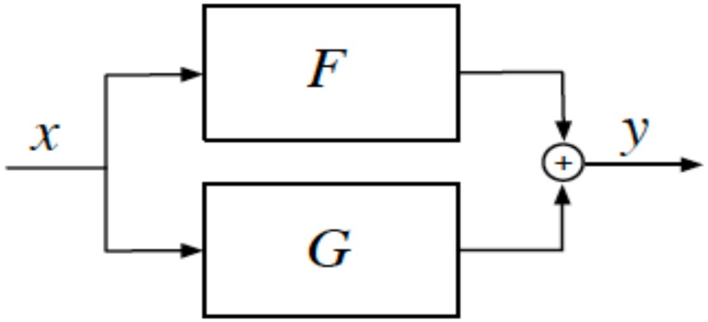
Interconnection of Systems

We can interconnect systems to form new systems,

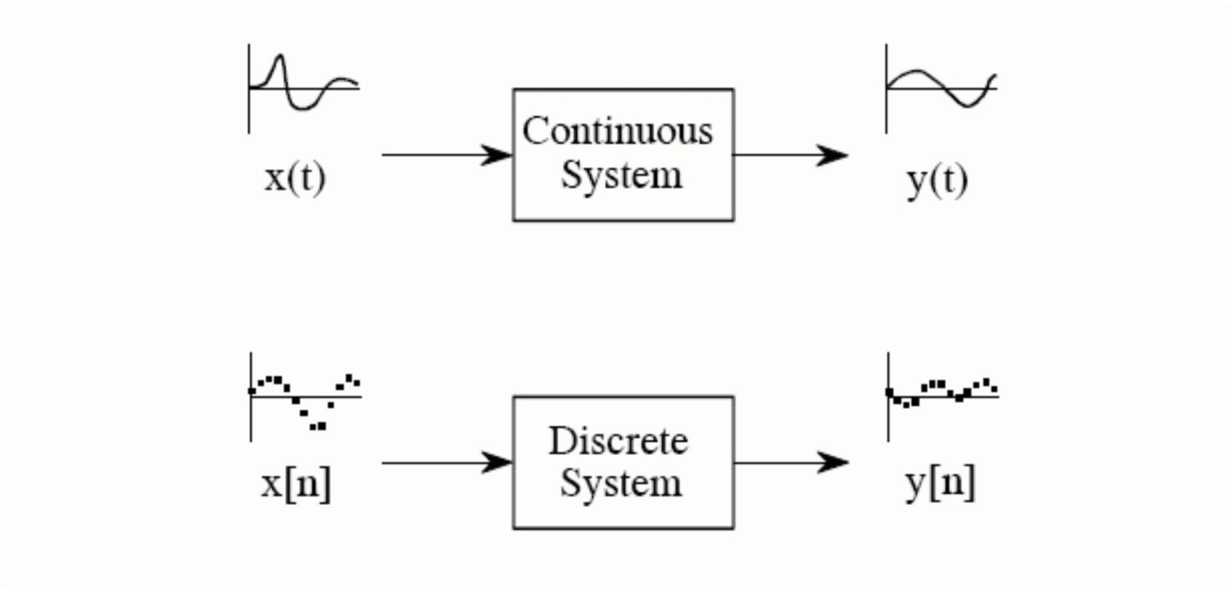
- **cascade (or series):** $y = G(F(x)) = GFx$



  (note that block diagrams and algebra are *reversed*)
- **sum (or parallel):** $y = Fx + Gx$



What is a system ?



Systems

- Systems – Characteristics/Properties
    - Linear
    - Time-Invariance
    - Stability
    - Causal
    - Memory

System Properties

What does the system do to an input?

What all do we need to determine the Output?

If we know what it does to input "x_1", can we say what it does it some other input "x_2" ? (If we know the relation between "x_1" and "x_2")

System Properties - Memory

- Does it have Memory ?
- What does it mean for a system to have Memory ?
- System is memoryless if its output for every value of the independent variable is dependent only on the input at that same time

System Memory

- A system is *memoryless* if the output depends only on the present input.
  - Ideal amplifier
  - Ideal gear, transmission, or lever in a mechanical system
- A *system with memory* has an output signal that depends on inputs in the past or future.
  - Energy storage circuit elements such as capacitors and inductors
  - Springs or moving masses in mechanical systems
- A *causal* system has an output that depends only on past or present inputs.
  - Any real physical circuit, or mechanical system.

System Properties - Memory

Ex. $y(t) = x(t)$

$v(t) = R\, i(t)$

$y[n] = 2x[n] - (x[n])^2$

System Properties - Memory

Ex. $y(t) = x(t)$

$V(t) = R\, i(t)$

$y[n] = 2x[n] - (x[n])^2$

These are all memory-less systems

Examples of Systems with Memory

- Delay Unit :

$$y[n] = x[n-2]$$

- Accumulator :

$$y[n] = \Sigma\, x[k]\,; \quad \text{Index } k = -\infty \; to \; n$$

System Properties - Invertibility

- What does it mean for a system to be Invertible ?

- System is Invertible if distinct inputs lead to distinct outputs.

System Invertibility

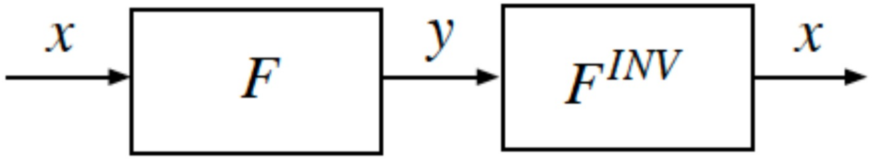
- A system is invertible if the input signal can be recovered from the output signal.
- If F is an invertible system, and

$$y = Fx$$

then there is an inverse system F^{INV} such that

$$x = F^{INV} y = F^{INV} Fx$$

so $F^{INV} F = I$, the identity operator.



Invertibility

- Formal definition: A system T has an *inverse* T_i if when cascaded with T gives the identity system (the output of the two systems is the original input):

$$T_i[T(x[n])] = x[n]$$

- Unit advance and Unit delay are Inverses.

$$\begin{aligned} T &: \quad y[n] = x[n+1] \\ T_i &: \quad x[n] = y[n-1] \end{aligned}$$

Examples:

1. y(t) = 2x(t)

2. Is Accumulator an Invertible system ?

Example

- Accumulator and First Difference are Inverses.

$$
\begin{aligned}
T \quad &: \quad y[n] = \sum_{k=-\infty}^{n} x[k] \\
T_i \quad &: \quad x[n] = y[n] - y[n-1]
\end{aligned}
$$

Systems which are NOT Invertible

- y[n] = C
- $y[n] \;=\; (x[n])^2$

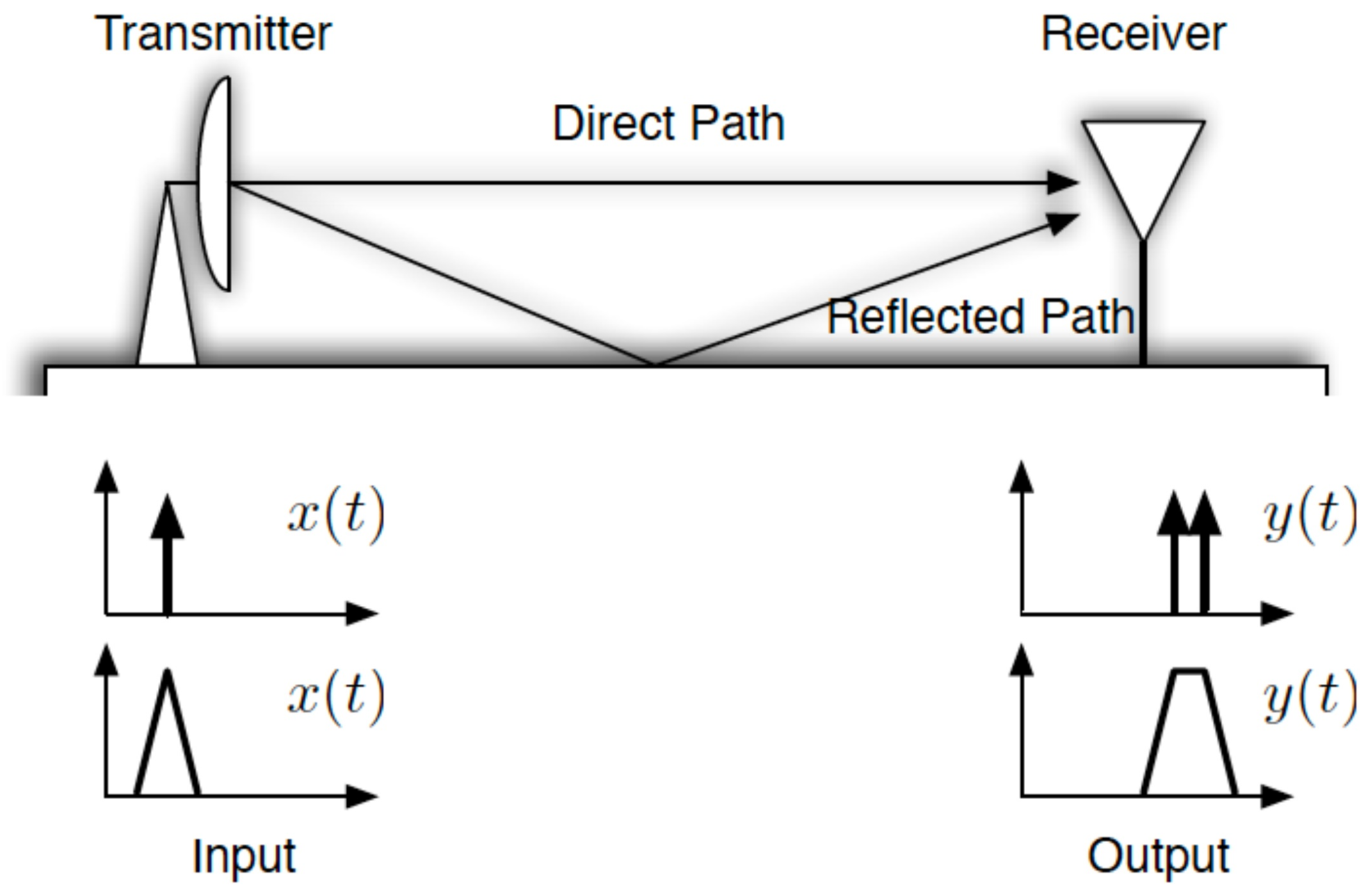
Determine if the system is invertible or not

1. $y(t) = |x(t)|$

2. $y(t) = x(t) sint$

3. $y[n] = x[2n]$

Example: Multipath echo cancelation



Important problem in communications, radar, radio, cell phones.

Generally there will be multiple echoes.

Multipath can be described by a system $y = Fx$

- If we transmit an impulse, we receive multiple delayed impulses.
- One transmitted message gives multiple overlapping messages

We want to find a system F^{INV} that takes the multipath corrupted signal y and recovers x

$$\begin{aligned} F^{INV} y &= F^{INV}(Fx) \\ &= (F^{INV} F)\, x \\ &= x \end{aligned}$$

Often possible if we allow a delay in the output.

System Properties - **Causal**

- What does it mean for a system to be Causal ?
- System is Causal if the output at the current time-stamp depends only on present and past inputs

Causal Systems

Accumulator : $y[n] \; = \; \Sigma \, x[k]$;

index $k \; = \; -\infty \; to \; n$

Delay unit : $y[n] \; = \; x[n-1]$

Examples of Non-causal systems

$$y[n] = x[n] - x[n+1]$$

$$y(t) = x(t+1)$$

What kind of systems are realizable ? Causal or Non-causal ?

Examples

- $y[n] \; = \; x[-n]$
- Is it causal or non-causal ?

- $y(t) = x(t)\cos(t+1)$
- Is it causal or non-causal ?

Stability – Bounded Input Bounded Output (BIBO0

- Is it true that bounded input always => bounded output

- Example : $y(t) = 2x(t)$
- Is this BIBO-stable?

Stability - BIBO

More examples : Consider the Accumulation of Unit-Step

$y[n] \; = \; \Sigma\, u[k]$;

Index $k \; = \; -\infty \; to \; n$

For any n, $y[n] \; = \; (n+1)\, u[n]$

y[0] = 1;

y[1] = 2;

y[2] = 3;

……………………………

y[n] grows without bound

Why Is BIBO stability important ?

IN PRACTICE, WE WOULD NEED ONLY BIBO STABLE SYSTEMS !

Examples

- Characteristics:
- Memory
- Causality
- Stability

Memory

- Consider following systems:
- $y(t) = 10x(t)$
- $y[n] = x[n]x[n-20]$

- How many past sample values do you need to save?

Invertibility

- Either find the inverse or show two inputs that result in same output (or vice versa)

- $y(t) = \cos(x(t))$
- $y(t) = x^3(t-4)$
- $y[n] = (n-1)x[n]$
- $y[n] = x[n] - x[n-1]$

$$y[n] = x[n] - x[n-1]$$

$$x[n] = \sum_{k=-\infty}^{n} y[k] = \underbrace{\sum_{k=-\infty}^{n-1} y[k]}_{x[n-1]} + y[n]$$

$$y[n] = x[n] - x[n-1]$$

Causality

- Output depends on current or past inputs only
- $y[n] = \sum_{k=-\infty}^{n} x[k]$ C
- $y[n] = \sum_{k=-\infty}^{n} x[k] + x[k+1]$ NC
- $y(t) = e^{-2(t+3)}x(t)$ C
- $y(t) = u(-t-4)x(t)$ C
- $y(t) = e^{|x(t)|}$ C

$y(t) = e^{x(1+t)}$

Linearity

A system F is **linear** if the following two properties hold:

1. **homogeneity:** if x is any signal and a is any scalar,

$$F(ax) = aF(x)$$

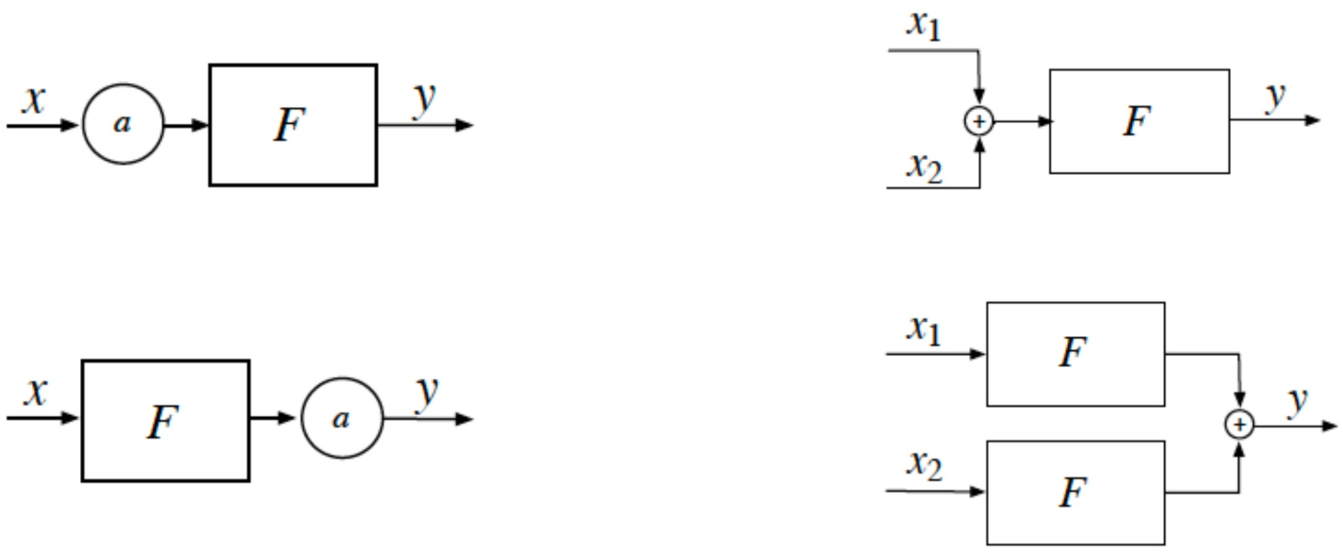
2. **superposition:** if x and $\tilde{x}$ are any two signals,

$$F(x + \tilde{x}) = F(x) + F(\tilde{x})$$

In words, linearity means:

- Scaling before or after the system is the same.
- Summing before or after the system is the same.

Linearity means the following pairs of block diagrams are equivalent, *i.e.*, have the same output for any input(s)



Examples of linear systems: scaling system, differentiator, integrator, running average, time shift, convolution, modulator, sampler.

Examples of nonlinear systems: sign detector, multiplier (sometimes), comparator, quantizer, adaptive filter

To Check Linearity ?

- Let $x(t) = x_1(t)$

Then $y_1(t) = \mathrm{T}[x_1(t)]$-------------------Eq 1

- Let $x(t) = x_2(t)$

Then $y_2(t) = \mathrm{T}[x_2(t)]$––––––––––––––––––––––Eq 2

- Let $x(t) = \mathrm{a}_1 x_1(t) + \mathrm{a}_2 x_2(t)$
- Then $y_3(t) = \mathrm{T}[\mathrm{a}_1 x_2(t) + \mathrm{a}_2 x_2(t)]$––––––––––––––––––––––Eq 3
- Now look at $\mathrm{a}_1\, y_1(t) + \mathrm{a}_2\, y_2(t)$
- Is it the same as $y_3(t)$????
- If yes, then the system is linear.

Examples

- Are these systems linear?

1. $y(t) = kx(t) + c$
2. $y(t) = tx(t)$

$y(t) = t\,x(t)$

$x_1(t) = a\,x(t)$

$y_1(t) = t\,a\,x(t) \overset{\checkmark}{=} a \cdot y(t)$

$x_3(t) = x_1(t) + x_2(t)$

$y_3(t) = t\,x_3(t)$

$= t\,x_1(t) + t\,x_2(t)$

$\checkmark = y_1(t) + y_2(t)$

$$y(t) = k x(t) + C$$

$$x_1(t) = a x(t)$$

$$\xrightarrow{x_1(t)} \boxed{\mathcal{F}} \longrightarrow y_1(t)$$

if $y_1(t) == a y(t)$?

$$y_1(t) = k x_1(t) + C$$

$$= k a x(t) + C \neq a y(t)$$

$$a y(t) = k a x(t) + a \cdot C$$

Time - Invariance

$x(t) \longrightarrow \boxed{} \longrightarrow y(t)$

$x(t-t_0) \qquad\qquad \longrightarrow y(t-t_0)$

- System is Time-Invariant if the behaviour of the system is the same at all time-stamps

- i.e. For input x(t), if output is y(t)

  Then for input $x(t - t_0)$, output should be $y(t - t_0)$

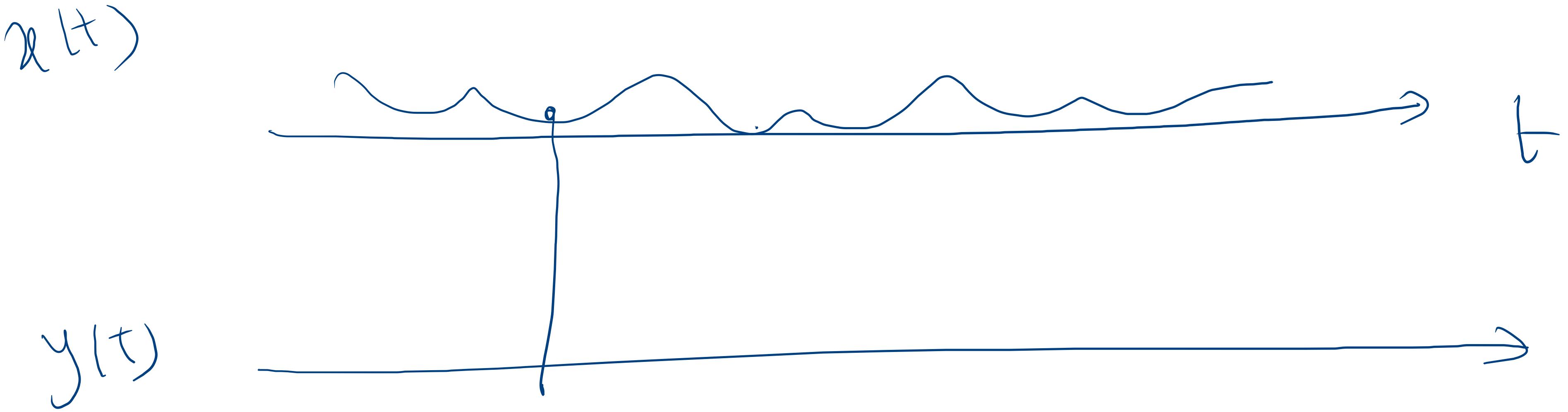
For example : $y(t) = \sin[x(t)]$

$$y(t) = \sin(x(t))$$

$$x_1(t) = x(t - t_0)$$

$$y_1(t) = \sin(x_1(t)) = \sin(x(t-t_0))$$

$$= y(t-t_0)$$



To Check Time-invariance

- Determine if time-Invariance holds for any input and any time-shift !
- Let $x(t) = x_1(t)$

  $y_1(t) = \sin[x(t)] = \sin[x_1(t)]$ --------------------Eq 1

- Let $x(t) = x_2(t) = x_1(t - t_0)$ {Time-shifted version of the first i/p}

  $y_2(t) = \sin[x(t)] = \sin[x_2(t)] = \sin[x_1(t - t_0)]$ -------Eq 2

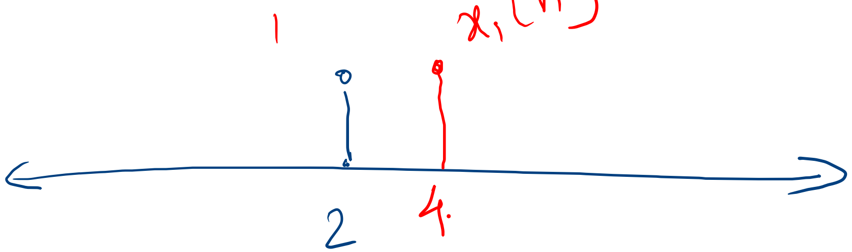
- In Eq (1), Evaluate $y_1(t)$ at $t = (t - t_0)$ {Time−shifted version of the first output}
- What is $y_1(t)$ at $t = (t - t_0)$? It is : $\sin[x_1(t - t_0)] = y_3(t)$ ------Eq 3
- Is $y_2(t)$ = $y_3(t)$
- If yes, then shifted version of output same as output for shifted version of input

Example

$$y[n] = nx[n]$$

$$y[n] = n\,x[n]$$

$$x[n] = \delta[n-2]$$

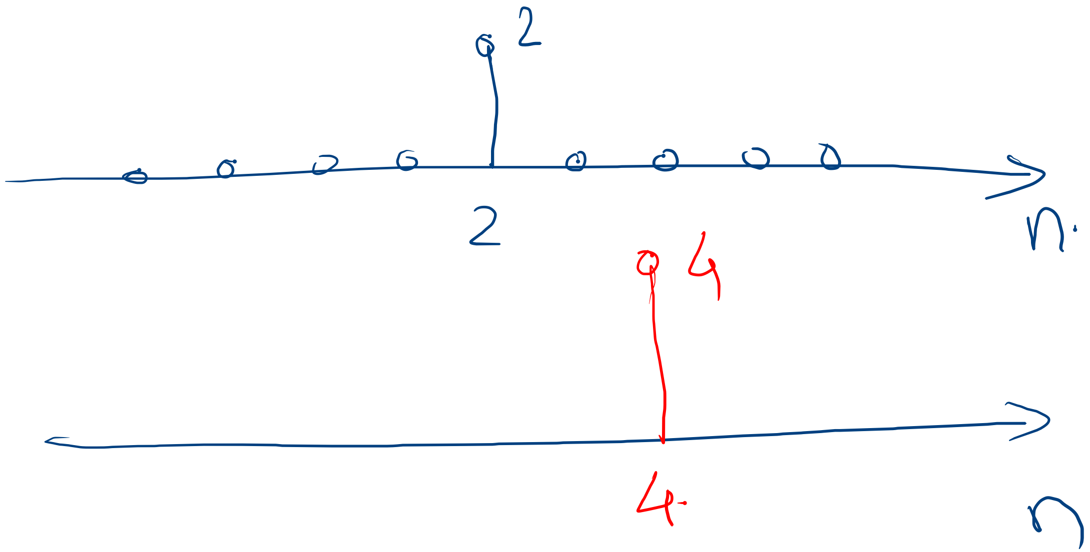


$$x_1[n] = x[n-2]$$

$y_1[n]$

$$y[2] = 2$$

$$y[1] = y[0] = 0$$



$$y[n] = n\,x[n]$$

$$x_1[n] = x[n-n_0]$$

$$y_1[n] = n\,x_1[n] = n\,x[n-n_0] \quad \text{--- (1)}$$

$$y[n-n_0] = (n-n_0)\,x[n-n_0] \quad \text{--- (2)}$$

Example

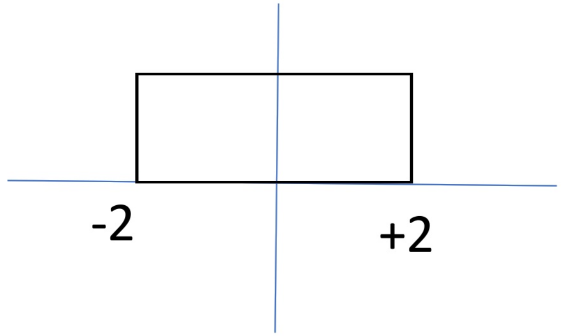
- $y[n] = nx[n]$
- Put $x[n] = x_1[n]$

  Then, $y_1[n] = nx_1[n]$ ----------------(1)
- Put $x[n] = x_2[n] = x_1[n - n_0]$

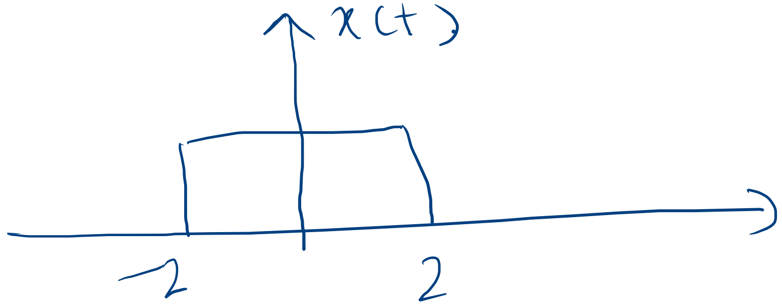
  $y_2[n] = nx_2[n] = nx_1[n - n_0]$ -----------(2)
- Look at $y_3[n] = y_1[n]$ at $n = (n - n_0)$
- We get, $y_1[n]$ at $n = (n - n_0)$ is $(n - n_0)x_1[n - n_0]$ -----(3)
- $y_2[n]$ NOT THE SAME AS $y_3[n]$
- Hence system is NOT Time-Invariant

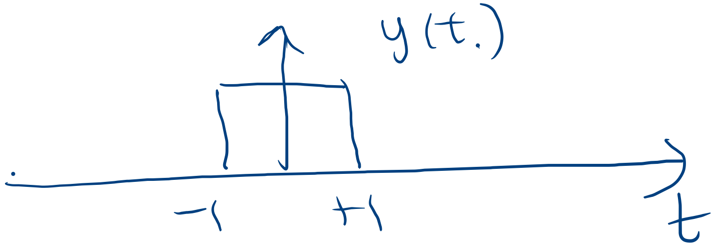
Example

- $y(t) = x(2t)$ ie. Compress by factor of 2
- Is it time invariant ?
- Consider x(t) given :

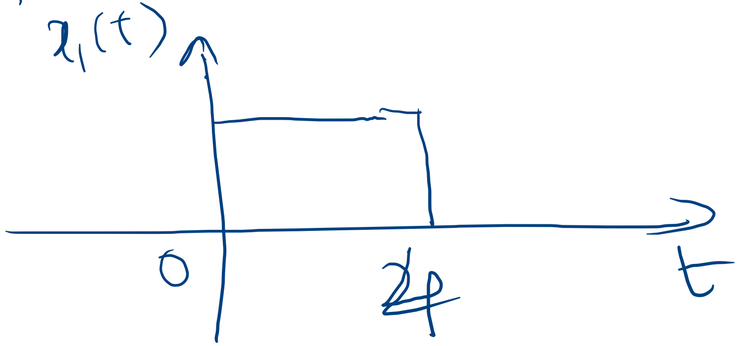


$y(t) = x(2t)$

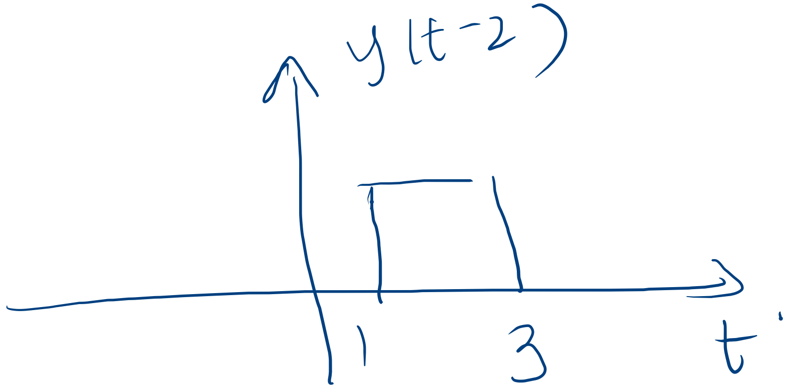




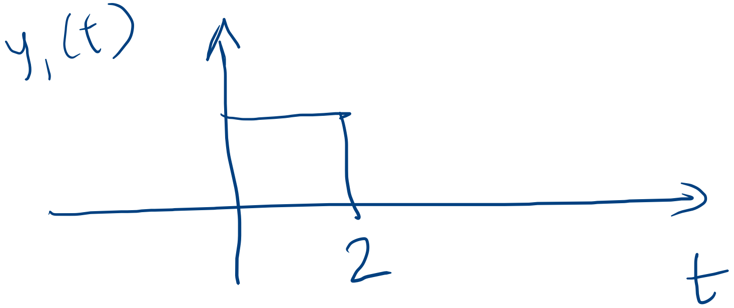
$x_1(t) = x(t-2)$



$y(t-2)$



$y_1(t)$



Compression – NOT TI

Delay the signal **after processing**.

$y(t) = x(2t)$

Replace t by (t-k).

$Y = y(t-k) = x(2(t-k)) = x(2t-2k)$

Delay the signal **before processing**. Delay the input sample alone.

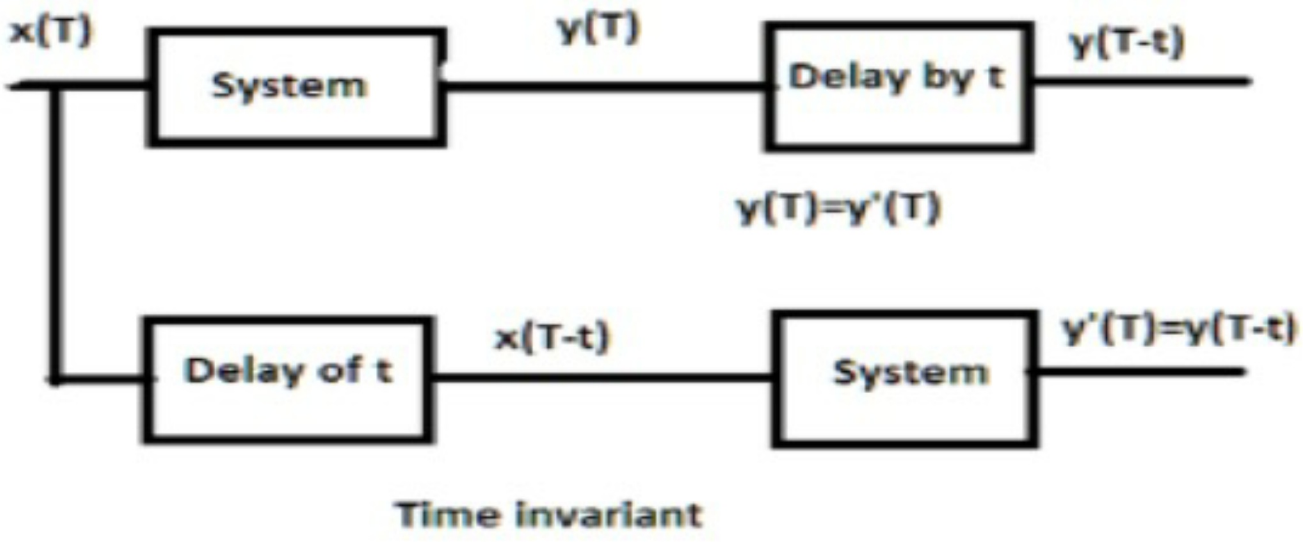
$y(t) = x(2t-k)$

$Y' = x(2t-k)$

Clearly $Y \neq Y'$

Thus, this is a **time-variant system**.

TI-summary



x(T)
System
y(T)
Delay by t
y(T-t)
y(T)=y'(T)
Delay of t
x(T-t)
System
y'(T)=y(T-t)
Time invariant

Problems

Determine whether system is linear, time invariant or both

1) $y(t) = t^2 x(t-1)$ → Linear , TV

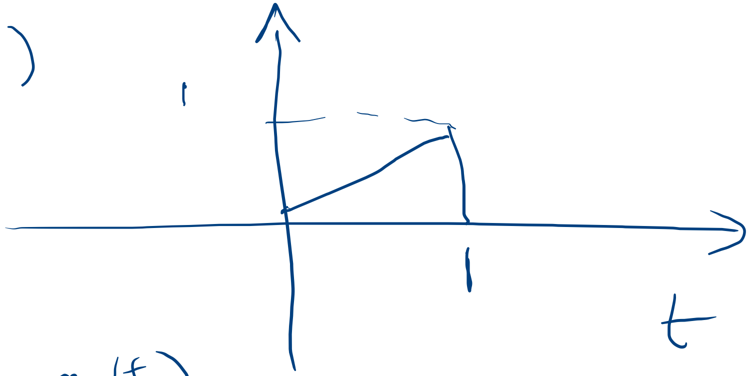
2) $y[n] = x^2[n-2]$ → NL , ~~TV~~ time invariant

3) $y[n] = x[n+1] - x[n-1]$ → invertible, $\boxed{LTI}$ FINV

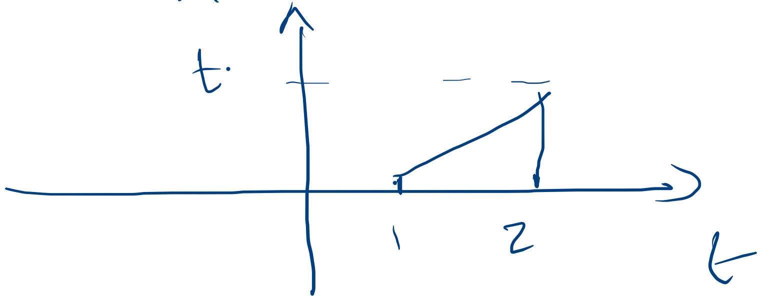
4) $y(t) = Odd\{x(t)\}$

$$y(t) = \frac{x(t) - x(-t)}{2}$$

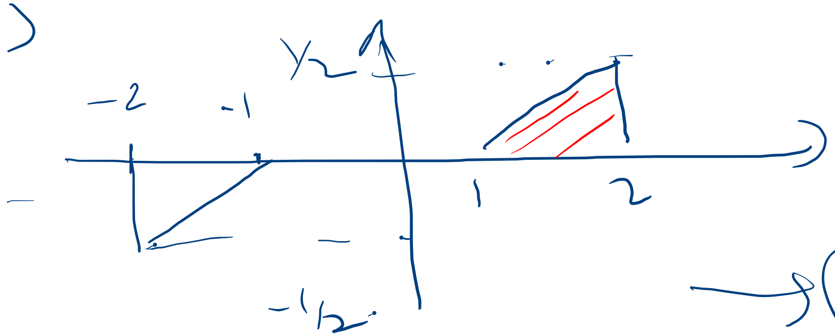
$x(t)$



$x_1(t) = x(t-1)$ $x_1(t)$



$y_1(t)$

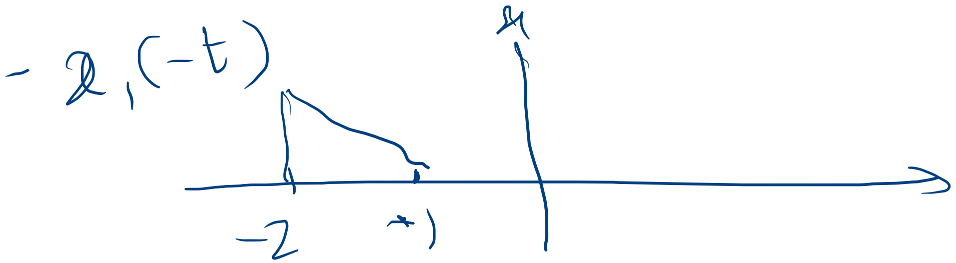


$\rightarrow$ 2

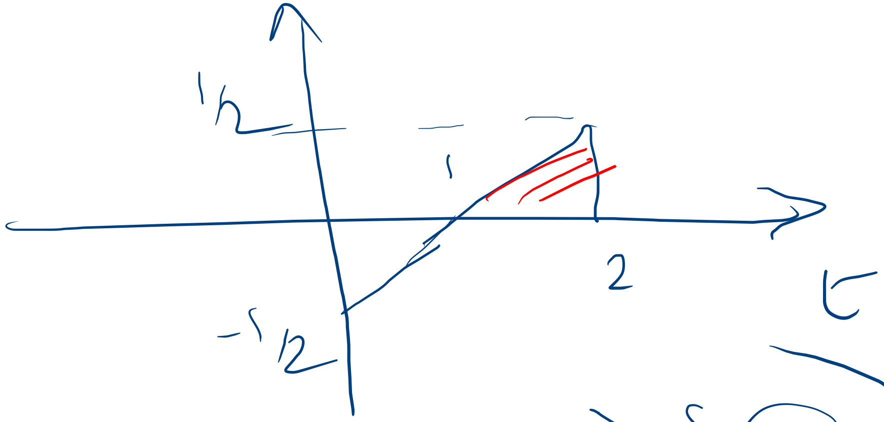
not invertible $\rightarrow$?

$$y_1(t) = \frac{x_1(t) - x_1(-t)}{2}$$

$-x_1(-t)$



$y(t-1)$



$\rightarrow$ 1

$$y(t) = \frac{x(t) - x(-t)}{2}.$$

Consider $x_1(t) = x(t - \tau)$

$$y_1(t) = \frac{x_1(t) - x_1(-t)}{2}$$

$$\underbrace{= \frac{x(t-\tau) - x_1(-t-\tau)}{2}}_{2}$$

$$\underbrace{y(t-\tau) = \frac{x(t-\tau) - x(-t+\tau)}{2}}_{1}$$

$$y(t) = t^2 x(t-1)$$

$$x_1(t) = x(t - T).$$

$$y_1(t) = t^2 x_1(t)$$
$$= t^2 x(t - T)$$

$$y(t - t_0) = (t - T)^2 x(t - T)$$