CSE 201: Discrete Mathematics
Quiz 1
Maximum Marks: 50

Instructions

1. Any result stated or proved in the class or tutorial can be used without repeating the proof.
2. State all references used other than the class slides, problem sets and the textbooks.

Question 1. A, B, C, D, and E live together in a house. On a particular day, you're trying to figure out who is at home based on the following information:

- If A is at home, then B is also at home.
- Either D or E, or both, are at home.
- Either B or C, but not both, are at home.
- D and C are either both at home or both not at home.
- If E is at home, then A and D are also at home.

Based on these conditions, determine who is at home and who is not.

Solution 1

Let the following propositions represent the individuals being at home:

- A: A is at home.
- B: B is at home.
- C: C is at home.
- D: D is at home.
- E: E is at home.

The given conditions are:

1. $A \to B$ (If A is at home, then B is at home.)
2. $D \vee E$ (Either D or E, or both, are at home.)
3. $\neg(B \wedge C) \wedge (B \vee C)$ (Either B or C is at home, but not both.)
4. $D \leftrightarrow C$ (D and C are either both at home or both not at home.)
5. $E \to (A \wedge D)$ (If E is at home, then A and D are also at home.)

Step 1: Assume E is true (E is at home)

- From condition 5: $E \to (A \wedge D)$, so if E is true, then A and D must both be true.
- Since A is true, from condition 1: $A \to B$, B must also be true.
- Now we have B true, so C must be false because of condition 3: $\neg(B \wedge C) \wedge (B \vee C)$.
- However, from condition 4: $D \leftrightarrow C$, D and C must have the same truth value. This creates a contradiction because D is true while C is false.
- Therefore, E cannot be true.

Step 2: Assume E is false (E is not at home)

- If E is false, then from condition 2: $D \vee E$, D must be true.
- Since D is true, from condition 4: $D \leftrightarrow C$, C must also be true.
- Since C is true, from condition 3: B must be false because $\neg(B \wedge C)$.
- Since B is false, from condition 1: A must be false (because $A \rightarrow B$).

Conclusion

Based on the conditions and the analysis:

- A is not at home.
- B is not at home.
- C is at home.
- D is at home.
- E is not at home.

Question 2. You are a detective and you have three suspects for the murder of Mr. Dave: Alex, Boris, and Carl. Each suspect claims innocence.

- Alex says that Mr. Dave knew Boris and Carl disliked Mr. Dave.
- Boris claims that he did not know Mr. Dave and that he was not in town the day of the murder.
- Carl claims that he saw both Alex and Boris with Mr. Dave on the day of the crime and that one of them must have committed the murder.

Using predicate logic, determine the murderer, knowing that two of the three suspects are telling the truth and the other one is guilty. You can assume that if A knows B, it implies that B knows A.

Solution 2

To represent the statements made by the three suspects, the following predicates can be defined:

- $L(X, Y)$ - X and Y like each other
- $N(X, Y)$ - X and Y know each other
- $T(X)$ - X was in town during the incident
- $S(X)$ - X was seen in the town during the incident
- $K(X, Y)$ - X killed Y

By observation,

$$S(X) \implies T(X)$$

Let us represent Alex as A, Boris as B, Carl as C, Mr.Dave as D. We can represent what Alex, Boris and Carl claimed using the following predicate representation:

- Alex - $N(D, B) \wedge \neg L(C, D)$
- Boris - $\neg N(B, D) \wedge \neg T(B) \equiv \neg N(D, B) \wedge \neg T(B)$
- Carl - $(S(A) \wedge S(B)) \wedge (K(A, D) \vee K(B, D))$

Sherlock Holmes knows that 2 suspects are truthful, but one is telling a lie.
Checking all three pairs of possibly truthful people:

- Alex and Boris - For both of them to be truthful the following has to be true:

  $$(N(D, B) \wedge \neg L(C, D)) \wedge (\neg N(D, B) \wedge \neg T(B))$$

  This will not be true as $N(D, B) \wedge \neg N(D, B) = F$

- Boris and Carl - For both of them to be truthful the following has to be true:

  $$(\neg N(D, B) \wedge \neg T(B)) \wedge ((S(A) \wedge S(B)) \wedge (K(A, D) \vee K(B, D)))$$

  This will not be true as $\neg T(B) \wedge S((B) \equiv \neg T(B) \wedge T(B) = F$

- Alex and Carl - Since we know there are two honest people, Alex and Carl will have to be honest, as both contradict with Boris's claim and do not contradict with each other.

  $$(N(D, B) \wedge \neg L(C, D)) \wedge ((S(A) \wedge S(B)) \wedge (K(A, D) \vee K(B, D)))$$

$\therefore$ Boris is the murderer.

Question 3. You are playing a game with a game-show host where he tells you the following rules for a game:

- There are n keys,$\{k_1, k_2, k_3, ...k_n\}$ and n vaults, $\{v_1, v_2, v_3, ...v_n\}$ present. $(n \geq 1)$
- A key k_i can be used on (to open/close) a vault v_j if the condition $j\%i = 0$ holds, where % denotes the modular operation. Therefore, a key is **valid** for a vault , if it can be used on the vault.
- Initially, all vaults are closed.
- A key when used on a valid vault, will open a vault if it's closed and close a vault if it's open.

The game is that all the keys are used on all of their corresponding valid vaults in succession.
So, Initially k_1 will be used on vaults$\{v_1, v_2, v_3, ...v_n\}$
k_2 will be used on the vaults $\{v_2, v_4, v_6...\}$
k_3 will be used on the vaults $\{v_3, v_6, v_9....\}$
.
.
.
and so on upto k_n

The function $f(n)$ denotes number of vaults that are open after the game is played.
The game show host claims that $f(100) = 10$ and $f(1000) = 31$, Prove/Disprove his claim.

Solution 3

From the rules of the game, we can deduce that a vault v_x remains open if and only if it has been used by an odd number of keys, which implies that x has an odd number of divisors.

Claim: A number has an odd number of divisors if and only if it is a perfect square.

Proof: Let $x \in \mathbb{N}$ be a number with an odd number of divisors.
The prime factorization of x can be written as $p_1^{q_1} \cdot p_2^{q_2} \cdot \cdots \cdot p_k^{q_k}$, where $\{p_1, p_2, \ldots, p_k\}$ are distinct primes and $\{q_1, q_2, \ldots, q_k\} \in \mathbb{N}$.
The total number of divisors of x, denoted $S(x)$, is given by the product $(q_1 + 1)(q_2 + 1) \ldots (q_k + 1)$.
$S(x)$ is odd if and only if each of the terms $(q_1 + 1), (q_2 + 1), \ldots, (q_k + 1)$ is odd, which implies that each q_i is even.
Thus, $q_i = 2e_i$ for some $e_i \in \mathbb{N}$, meaning that $x = (p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_k^{e_k})^2$.
Therefore, x is a perfect square, as $x = y^2$ for some $y \in \mathbb{N}$.

Hence Proved.

Now, let $f(n)$ denote the number of vaults that remain open after the game has been played.
From the claim, a vault v_x is open if and only if x is a perfect square.
Therefore, the game-show host's claim that $f(100) = 10$ and $f(1000) = 31$ is correct, as there are 10 perfect squares between 1 and 100, and 31 perfect squares between 1 and 1000.

Question 4. There are r red and g green blocks for construction of the *red-green* tower. The *red-green* tower can be built following all these rules:

- Let the *red-green* tower consists of n number of levels.
- The first level of this tower should consist of n blocks, the second level should consist of $n-1$ blocks, the third one should consist of $n-2$ blocks, and so on. The last level of such a tower should consist of *one* block. In other words, each successive level should contain one block less than the previous one.
- Each level of the *red-green* tower should contain blocks of the same color.

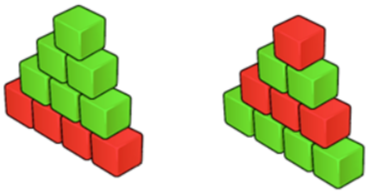

Figure 1: Examples of valid 4 level red-green towers

Prove or disprove that for any non-negative integer values of r and g, if $(r+g) = \frac{n(n+1)}{2}$ holds, then there exists a **valid** *red-green* tower with n levels consisting r red blocks and g green blocks.

Solution 4

Proof. We will prove the statement by induction on the number of red blocks r.

Base Case:
When $r = 0$, the equation $(r+g) = \frac{n(n+1)}{2}$ implies that all the blocks are green. A valid tower can be constructed with n levels where the i-th level consists of $n-i+1$ green blocks for all $i \in \{1, 2, \cdots, n\}$.

The shown construction is a valid red-green tower since each level contains blocks of the same color which is green and the tower uses $r = 0$ red blocks and $g = \frac{n(n+1)}{2}$ green blocks. Therefore, the base case holds.

Induction Hypothesis:
Assume that the statement is true for any $r = x$, where $0 \leq x < \frac{n(n+1)}{2}$. That is, we assume that a valid red-green tower can be constructed using x red blocks and $(r+g) - x$ green blocks.

Inductive Step:
To prove the statement for $r = x + 1$, we consider the following cases:

Case 1: If the n-th level (which consists of a single block) is green, we simply replace it with a red block. This creates a valid tower with $x + 1$ red blocks.

Case 2: If the n-th level is not green, we identify the topmost level that is green, say level k. Such a level must exist since there are non-zero green blocks. Note that $k < n$. Since the topmost block (n-th level is not green.

Also, the level immediately above level k, denoted as $k + 1$, must be red, otherwise, level k wouldn't be the topmost green level.

To construct a valid tower with $x + 1$ red blocks, we swap the blocks in the levels k and $k + 1$, making level k red and level $k + 1$ green. We then remove one green block from level $k + 1$ and add one red block to level k. This adjustment ensures that there are x+1 red blocks and the number of blocks in each level remains the same. Also, all blocks in each level are of the same color.

Hence, the statement holds for $r = x + 1$.

Conclusion:
By the principle of mathematical induction, we have shown that for any non-negative integer values of r and g,

if $(r+g) = \frac{n(n+1)}{2}$ holds, then there exists a **valid** *red-green* tower with n levels consisting r red blocks and g green blocks. ☐

Another solution:

Proof. We will prove the statement by induction on the number of levels n.

Base Case:
When $n = 1$, the equation $(r+g) = \frac{1(1+1)}{2} = 1$ implies that the total number of blocks is 1. Since $r + g = 1$, we either have $r = 1$ and $g = 0$, or $r = 0$ and $g = 1$. A valid tower can be constructed with 1 level where the single block is either red or green, satisfying the condition that each level has blocks of the same color. Therefore, the base case holds.

Induction Hypothesis:
Assume that the statement is true for some $n = k$, where there exists a valid *red-green* tower with k levels, using r red blocks and g green blocks, and satisfying $(r+g) = \frac{k(k+1)}{2}$.

Inductive Step:
We need to prove the statement for $n = k+1$. For $n = k+1$, the total number of blocks is $(r+g) = \frac{(k+1)(k+2)}{2}$.

Now, since $(r+g) = \frac{(k+1)(k+2)}{2}$, we know that either $r \geq k+1$ or $g \geq k+1$. If both r and g were less than or equal to k, then $r + g \leq 2k$, which is strictly less than $\frac{(k+1)(k+2)}{2}$ for all $k \geq 1$. This would contradict the given condition that $(r+g) = \frac{(k+1)(k+2)}{2}$. Therefore, either $r \geq k+1$ or $g \geq k+1$ must hold.

Whichever among r and g is greater than or equal to $k+1$, we use that colored $k+1$ blocks to construct the lowest level. After constructing this level, we are left with $r + g = \frac{k(k+1)}{2}$ blocks, and we need to build k more levels.

By the induction hypothesis, a valid *red-green* tower with k levels can be constructed using these remaining blocks. Thus, we obtain a valid *red-green* tower with $k+1$ levels.

Conclusion:
By the principle of mathematical induction, the statement holds for all non-negative integers n. Therefore, for any non-negative integer values of r and g, if $(r+g) = \frac{n(n+1)}{2}$ holds, then there exists a **valid** *red-green* tower with n levels consisting of r red blocks and g green blocks. ☐

Question 5. Let $k \in \mathbb{N}$ be a natural number. Consider a $2^k \times 2^k$ square board divided into equal square tiles of 1×1 size, like a chess board. (So the $2^k \times 2^k$ board is covered by 2^{2k} tiles.) We now remove one corner tile from the $2^k \times 2^k$ board. Prove by induction that the remaining part of the board can be covered with triomino pieces, i.e. pieces made of three unit tiles with an L-shape. Figure 2 shows an example of a triomino piece.

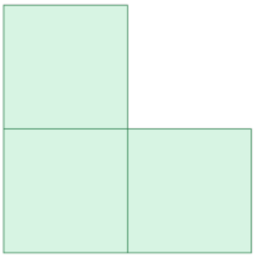

Figure 2: A triomino piece

Solution 5

Claim 5.1. *Let $k \in \mathbb{N}$ be a natural number. Consider a $2^k \times 2^k$ square board divided into equal square tiles of 1×1 size, like a chess board. (So the $2^k \times 2^k$ board is covered by 2^{2k} tiles.) We now remove one corner tile from the $2^k \times 2^k$ board. Then the remaining part of the board can be covered with triomino pieces, i.e. pieces made of three unit tiles with an L-shape.*

Proof. The proof is by induction. Let $P(k)$ =the statement that when a corner tile is removed from a $2^k \times 2^k$ chessboard, then the resulting board can be covered fully by triomino pieces.

Base Case: $k = 1$

A 2×2 chessboard can easily be covered by a triomino piece, leaving one of the corner squares empty, as shown in Figure 3. Hence $P(1)$ is true.

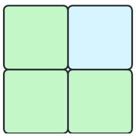

Figure 3: Base Case. Blue Square represents the removed square, and Green Squares represent the tiles covered by triomino pieces.

Induction Hypothesis: Let $P(k)$ be true for some $k = n$

Induction Step: To prove: $P(n+1)$ is also true.

We can divide the $2^{(}n+1) \times 2^{(}n+1)$ board into 4 $2^n \times 2^n$ boards, each of which can be covered using triomino pieces as shown in Figure 4.

From the figure, it is clear that each of the four $2^n \times 2^n$ boards can be covered by triomino pieces, and three remaining tiles in the center can be covered by a single triomino piece, leaving only one tile at the corner, which is the tile to be removed.

Hence, if the board of size $2^n \times 2^n$ can be covered by triomino pieces, then so can the board of size $2^{n+1} \times 2^{n+1}$. Hence, $P(n+1)$ is true whenever $P(n)$ is true.

Hence by induction, $P(k)$ holds true $\forall k \in \mathbb{N}$.

Hence Proved.

$\square$

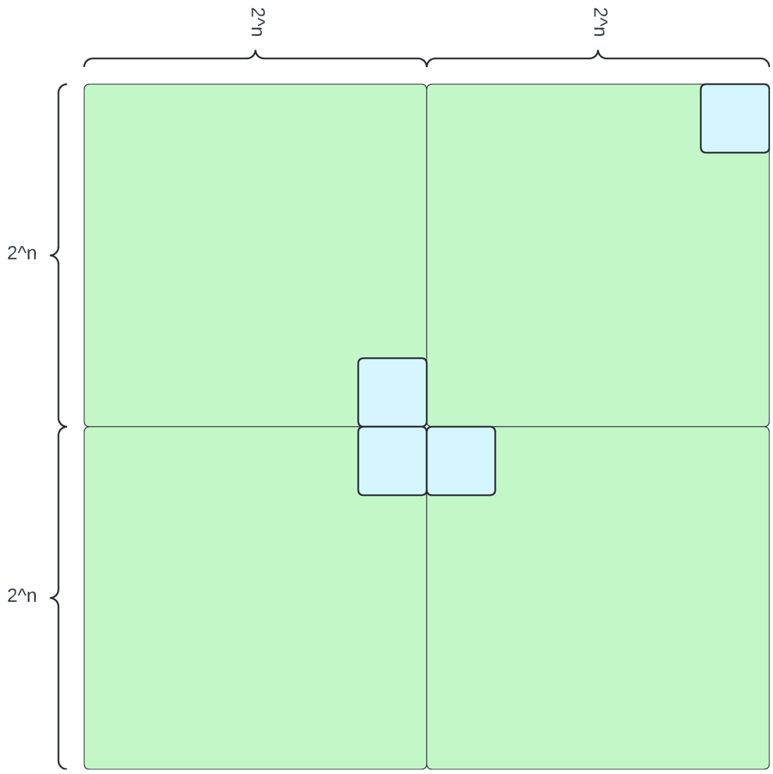

Figure 4: Induction step