

Z Transform

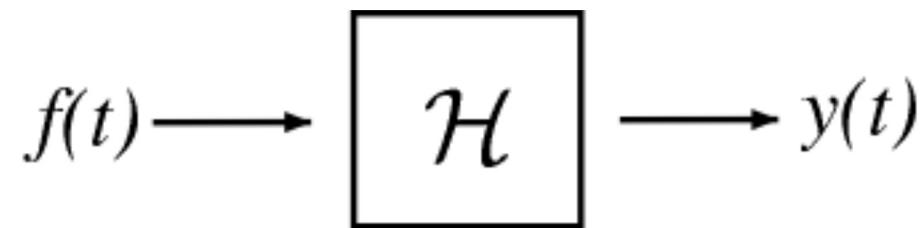
EGC 113

Source: inverse z transform slides from Prof. Deepa Kundur's slides (UoT)

Eigenfunctions

- A linear time invariant (LTI) system H operating on a continuous input $f(t)$ to produce continuous time output $y(t)$

- $H[f(t)] = y(t)$



- is mathematically analogous to an $N \times N$ matrix A operating on a vector $x \in \mathbb{C}^N$ to produce another vector $b \in \mathbb{C}^N$. Just as an eigenvector of A is a $v \in \mathbb{C}^N$ such that $Av = \lambda v$, $\lambda \in \mathbb{C}$,

Eigenfunctions of System

- An input x to a system \mathcal{H} is said to be an **eigenfunction** of the system \mathcal{H} with the **eigenvalue** λ if the corresponding output y is of the form

$$y = \lambda x,$$

where λ is a complex constant.

- In other words, the system \mathcal{H} acts as an ideal amplifier for each of its eigenfunctions x , where the amplifier gain is given by the corresponding eigenvalue λ .
- Different systems have different eigenfunctions.
- Of particular interest are the eigenfunctions of LTI systems.

Eigenfunctions of LTI System

- As it turns out, every complex exponential is an eigenfunction of all LTI systems.
- For a LTI system \mathcal{H} with impulse response h ,

$$\mathcal{H}\{e^{st}\} = H(s)e^{st},$$

where s is a complex constant and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt.$$

Eigenfunctions of LTI System

- That is, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.
- We refer to H as the **system function** (or **transfer function**) of the system \mathcal{H} .
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$.

Representation using Eigenfunctions

- Consider a LTI system with input x , output y , and system function H .
- Suppose that the input x can be expressed as the linear combination of complex exponentials

$$x(t) = \sum_k a_k e^{s_k t},$$

where the a_k and s_k are complex constants.

- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$

Representation using Eigenfunctions

- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$

- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as a linear combination of the *same* complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Summary: Eigen functions of LTI systems



If this happens, the Implication is that all Input properties are preserved

Is there any special family of inputs for which this is true ??

Summary: Eigen Function of LTI systems

- Let the Impulse response of the LTI system be $h(t)$
- If the Input is $x(t)$, then the Output $y(t)$ is given by :

$$y(t) = \int h(\tau)x(t - \tau)d\tau \quad : \tau \text{ goes from } -\infty \text{ to } \infty$$

- Consider the case : $x(t) = e^{(st)}$

Here s is complex in general

Summary: Eigen Function of LTI systems

Consider the case : $x(t) = e^{(st)}$. Here s is complex in general

Then

$$y(t) = \int h(\tau) e^{(s[t-\tau])} d\tau \quad : \tau \text{ goes from } -\infty \text{ to } \infty$$

$$y(t) = e^{(st)} \int h(\tau) e^{(-s\tau)} d\tau \quad : \tau \text{ goes from } -\infty \text{ to } \infty$$

Put $H(s) = \int h(\tau) e^{(-s\tau)} d\tau \quad : \tau \text{ goes from } -\infty \text{ to } \infty$

Then $y(t) = H(s) e^{(st)}$

$$y(t) = H(s) x(t)$$

Hence $e^{(st)}$ is an eigen function of LTI systems

Why are we learning this?

If any arbitrary input $x(t)$ is expressed in terms of the eigen functions

$$x(t) = a_1 e^{(s_1 t)} + a_2 e^{(s_2 t)}$$

Then

$$y(t) = a_1 H(s_1) e^{(s_1 t)} + a_2 H(s_2) e^{(s_2 t)}$$

Expressing $x(t) = \sum a_k e^{(s_k t)}$ offers several advantages

This series expression is called ‘Fourier Series’

Discrete-Time Fourier Transform & Z-transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

The first equation asserts that we can represent any time function $x[n]$ by a linear combination of complex exponentials $e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$

The second equation tells us how to compute the complex weighting factors $X(e^{j\omega})$

In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n

Generalizing the frequency variable

- In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n
- z^n can be thought of as a generalization of $e^{j\omega n}$
- For an arbitrary z , using polar notation we obtain $z = \rho e^{j\omega}$ so
$$z^n = \rho^n e^{j\omega n}$$
- If both ρ and ω are real, then z^n can be thought of as a complex exponential (*i.e.* sines and cosines) with a real temporal envelope that can be either exponentially decaying or expanding

The inverse Z-transform

- We didn't talk about inverse z-transforms yet?
- It can be shown that the inverse z-transform can be formally expressed as

$$x[n] = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

- Comments:
 - The integral is over a **complex variable**, z and we need complex residue calculus to evaluate it formally.
 - The contour of integration, c , is a circle around the origin that lies inside the ROC

Inverse z-transform

The inverse z -transform is based on a special case of the Cauchy integral theorem

$$\frac{1}{2\pi j} \oint_C z^{-\ell} dz = \begin{cases} 1 & \ell = 1 \\ 0 & \ell \neq 1 \end{cases}$$

where C is a counterclockwise contour that encircles the origin. If we multiply $X(z)$ by z^{n-1} and compute

$$\begin{aligned} \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz &= \frac{1}{2\pi j} \oint_C \sum_{m=-\infty}^{\infty} x[m] z^{-m+n-1} dz \\ &= \sum_{m=-\infty}^{\infty} x[m] \underbrace{\frac{1}{2\pi j} \oint_C z^{-(m-n+1)} dz}_{=1 \text{ only when } m-n+1=1} \\ &= \sum_{m=-\infty}^{\infty} x[m] \delta(m-n) \\ &= x[n] \end{aligned}$$

Hence, the inverse z -transform of $X(z)$ is defined as $x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$ where C is a counterclockwise closed contour in the ROC of $X(z)$ encircling the origin.

Inverse z-transform

Cauchy's residue theorem works, but it can be tedious and there are lots of ways to make mistakes. The Matlab function `residuez` (discrete-time residue calculator) can be useful to check your results.

Other (typically easier) options for computing inverse z -transforms:

1. Inspection (table lookup).
2. Partial fraction expansion (only for rational z -transforms).
3. Power series expansion (can be used for non-rational z -transforms).

Power Series Expansion Method

The idea here is to write $X(z)$ as

$$X(z) = \dots + c_{-2}z^2 + c_{-1}z + c_0 + c_1z^{-1} + c_2z^{-2} + \dots$$

and recognize that $x[n] = c_n$ by the definition of the z -transform. This can work even for non-rational $X(z)$.

For example, suppose $a > 0$ and

$$X(z) = e^{az^{-1}}$$

and note the ROC is the whole complex plane except $z = 0$. We can use the series expansion

$$X(z) = \sum_{k=0}^{\infty} \frac{(az^{-1})^n}{n!} = 1 + \frac{a}{1}z^{-1} + \frac{a^2}{2}z^{-2} + \frac{a^4}{6}z^{-3} + \dots$$

hence $x[n] = \frac{a^n}{n!} u[n]$.

TABLE 3.1 SOME COMMON z -TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$ z > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$ z > a $
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$ z < a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z > a $
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z < a $
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$ z > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$ z > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$ z > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$ z > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$ z > 0$

Partial Fraction Expansion Method

If

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

with all N poles distinct ("first order") then it is possible to express

$$X(z) = \underbrace{\sum_{r=0}^{M-N} B_r z^{-1}}_{\text{only if } M \geq N} + \sum_{k=1}^N \frac{A_k}{1 - \lambda_k z^{-1}}$$

The inverse z -transform then follows directly from linearity and table lookup (pay attention to the ROC). There are many ways to determine A_1, \dots, A_k , for example

$$A_k = [(1 - \lambda_k z^{-1}) X(z)]_{z=\lambda_k}$$

Rational Function

- LCCDE: Linear constant coefficient difference equations
 - ▶ $X(z)$ is a rational function iff it can be represented as the ratio of two polynomials in z^{-1} (or z):
$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$
 - ▶ For LTI systems that are represented by LCCDEs, the z -Transform of the unit sample response $h(n)$, denoted $H(z) = \mathcal{Z}\{h(n)\}$, is rational

Poles and Zeros

Let $a_0, b_0 \neq 0$:

$$\begin{aligned} X(z) = \frac{B(z)}{A(z)} &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\ &= \left(\frac{b_0 z^{-M}}{a_0 z^{-N}} \right) \frac{z^M + (b_1/b_0)z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0)z^{N-1} + \cdots + a_N/a_0} \\ &= \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)} \\ &= G z^{N-M} \frac{\prod_{k=1}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)} \end{aligned}$$

Poles and Zeros of Rational Function

$$X(z) = G z^{N-M} \frac{\prod_{k=1}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)} \text{ where } G \equiv \frac{b_0}{a_0}$$

- ▶ $X(z)$ has M finite zeros at $z = z_1, z_2, \dots, z_M$
- ▶ $X(z)$ has N finite poles at $z = p_1, p_2, \dots, p_N$
- ▶ For $N - M \neq 0$
 - ▶ if $N - M > 0$, there are $|N - M|$ zeros at origin, $z = 0$
 - ▶ if $N - M < 0$, there are $|N - M|$ poles at origin, $z = 0$

Total number of zeros = Total number of poles

Poles and Zeros

Example:

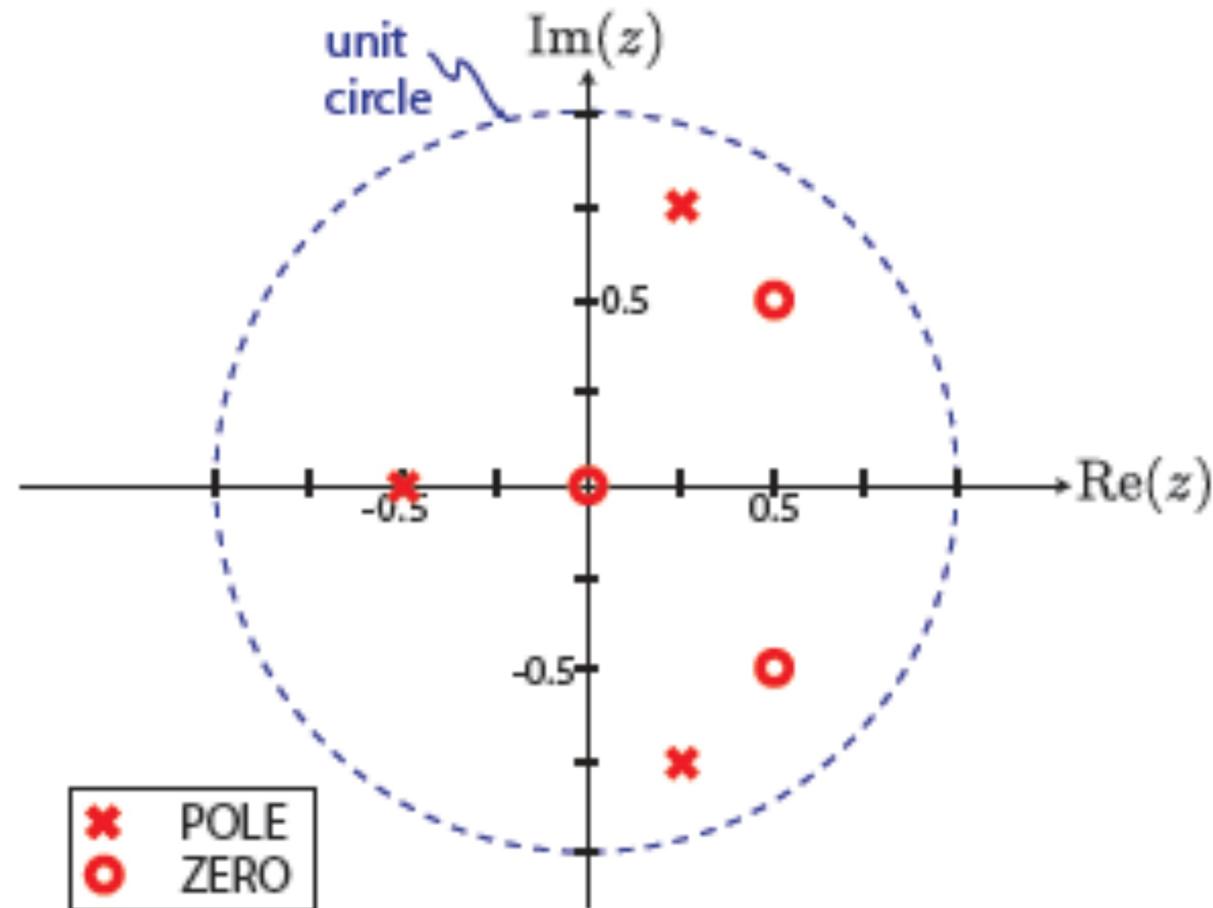
$$\begin{aligned} X(z) &= z \frac{2z^2 - 2z + 1}{16z^3 + 6z + 5} \\ &= (z - 0) \frac{(z - (\frac{1}{2} + j\frac{1}{2})) (z - (\frac{1}{2} - j\frac{1}{2}))}{(z - (\frac{1}{4} + j\frac{3}{4})) (z - (\frac{1}{4} - j\frac{3}{4})) (z - (-\frac{1}{2}))} \end{aligned}$$

poles: $z = \frac{1}{4} \pm j\frac{3}{4}, -\frac{1}{2}$

zeros: $z = 0, \frac{1}{2} \pm j\frac{1}{2}$

Pole-Zero Plot

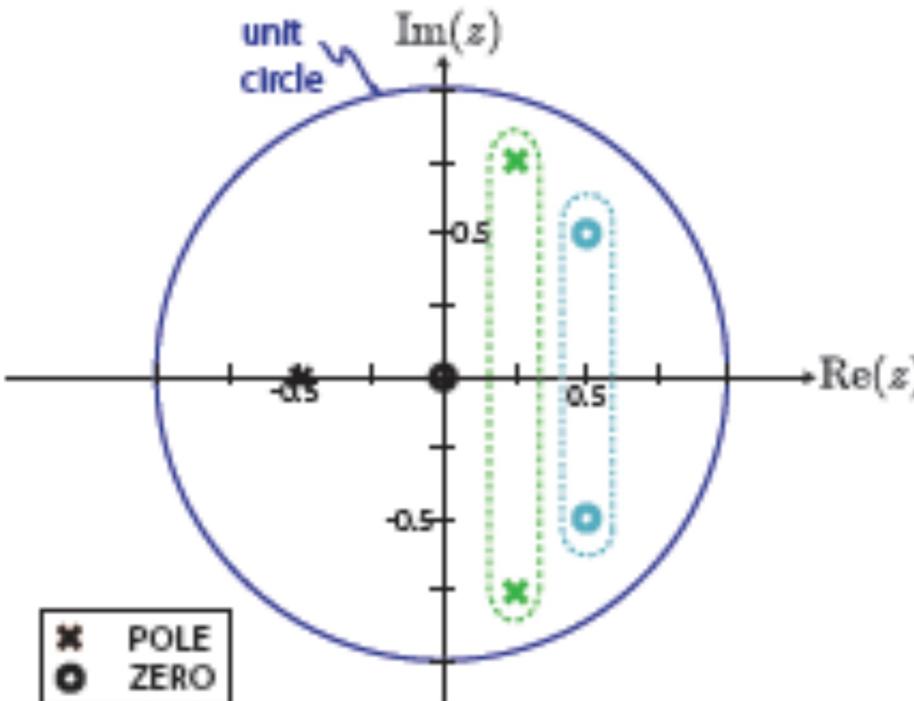
Example: poles: $z = \frac{1}{4} \pm j\frac{3}{4}, -\frac{1}{2}$, zeros: $z = 0, \frac{1}{2} \pm j\frac{1}{2}$



Pole-Zero Plot and Conjugate Pairs

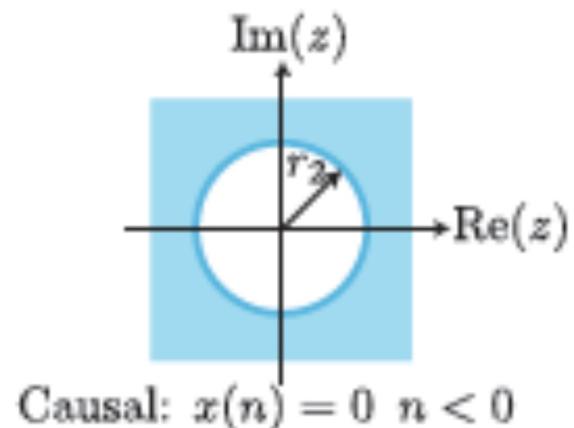
- ▶ For **real** time-domain signals, the coefficients of $X(z)$ are necessarily **real**
 - ▶ complex poles and zeros must occur in **conjugate pairs**
 - ▶ note: real poles and zeros do not have to be paired up

$$X(z) = z \frac{2z^2 - 2z + 1}{16z^3 + 6z + 5} \Rightarrow$$

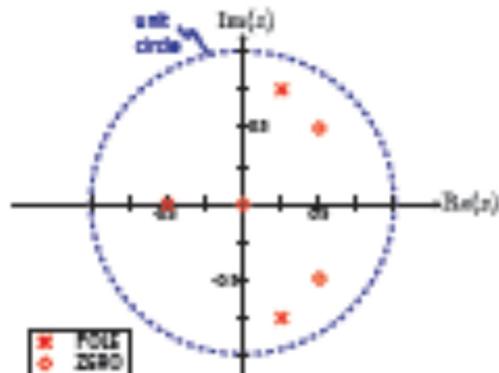


Pole-Zero Plot and the ROC

- Recall, for causal signals, the ROC will be **the outer region of a disk**

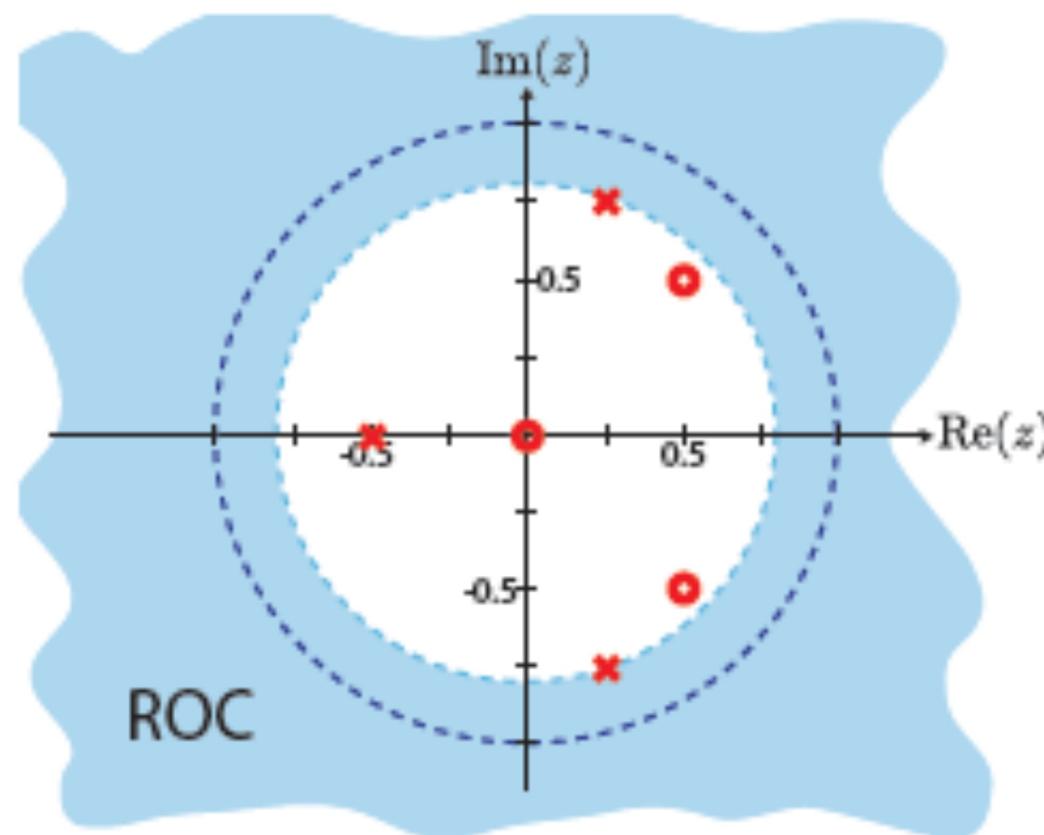


- ROC cannot necessarily include poles ($\because X(p_k) = \infty$)



Pole-Zero Plot and the ROC

- Therefore, for a **causal** signal the ROC is the smallest (origin-centered) circle encompassing all the poles.



Causality and Stability

- ▶ Recall,

$$\text{LTI system is stable} \iff \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- ▶ Moreover,

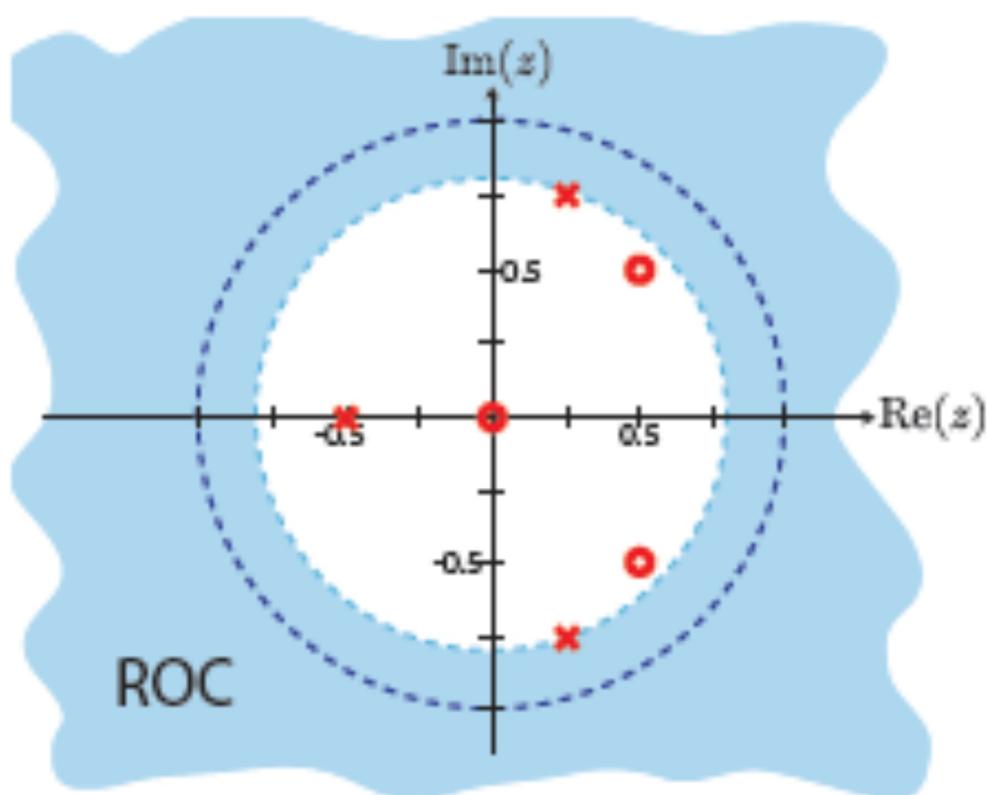
$$\begin{aligned}|H(z)| &= \left| \sum_{n=-\infty}^{\infty} h(n)z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |h(n)z^{-n}| \\ &= \sum_{n=-\infty}^{\infty} |h(n)| \quad \text{for } |z|=1\end{aligned}$$

- ▶ It can be shown:

$$\text{LTI system is stable} \iff \sum_{n=-\infty}^{\infty} |h(n)| < \infty \iff \text{ROC of } H(z) \text{ contains unit circle}$$

Pole-Zero Plot, Causality and Stability

- For stable systems, the ROC will include the unit circle.



- For stability of a **causal** system, the poles must lie **inside** the unit circle.

System Function or Transfer Function

$$\begin{array}{ccc} h(n) & \xleftrightarrow{z} & H(z) \\ \text{time-domain} & \xleftrightarrow{z} & z\text{-domain} \\ \text{impulse response} & \xleftrightarrow{z} & \text{system function} \end{array}$$

$$y(n) = x(n) * h(n) \xleftrightarrow{z} Y(z) = X(z) \cdot H(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)}$$

System Function

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k)$$

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\left\{- \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k)\right\}$$

$$\mathcal{Z}\{y(n)\} = - \sum_{k=1}^N a_k \mathcal{Z}\{y(n-k)\} + \sum_{k=0}^M b_k \mathcal{Z}\{x(n-k)\}$$

$$Y(z) = - \sum_{k=1}^N a_k z^{-k} Y(z) + \sum_{k=0}^M b_k z^{-k} X(z)$$

System Function

$$Y(z) + \sum_{k=1}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

$$Y(z) \left[1 + \sum_{k=1}^N a_k z^{-k} \right] = X(z) \sum_{k=0}^M b_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\left[1 + \sum_{k=1}^N a_k z^{-k} \right]}$$

LCCDE \longleftrightarrow Rational System Function

Many signals of practical interest have a rational z -Transform.

Partial Fraction Expansion

1. Find the distinct poles of $X(z)$: p_1, p_2, \dots, p_K and their corresponding multiplicities m_1, m_2, \dots, m_K .
2. The partial-fraction expansion is of the form:

$$z^{-R}X(z) = \sum_{k=1}^K \left(\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \cdots + \frac{A_{mk}}{(z - p_k)^{m_k}} \right)$$

where p_k is an m_k th order pole (i.e., has multiplicity m_k) and R is selected to make $z^{-R}X(z)$ a strictly proper rational function.

3. Use an appropriate approach to compute $\{A_{ik}\}$

Examples

- Find the inverse z-transform of

$$\bullet X(z) = z^2 \left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})(1 + 2z^{-1})$$

$$\bullet X(z) = \log\left(\frac{1}{1-az^{-1}}\right), |z| > |a| \quad \checkmark$$

$$\bullet X(z) = \log\left(\frac{1}{1-zA^{-1}}\right), |z| < |a|$$

$$\bullet X(z) = \frac{z}{2z^2-3z+1}, |z| < \frac{1}{2}$$

$$\bullet X(z) = \frac{z}{2z^2-3z+1}, |z| > 1$$

$$x(z) = -\log(1-\alpha z^{-1})$$

$$|z| > |\alpha|.$$

$$nx[n] \longleftrightarrow -z \frac{d}{dz} x(z)$$

$$\frac{d x(z)}{d z} = -\left(\frac{\alpha z^{-2}}{1-\alpha z^{-1}}\right)$$

$$\frac{1}{n} \alpha^n u[n-1]$$

$$-z \frac{d x(z)}{d z} = \alpha \underline{(z^{-1})} \left(\underbrace{\frac{1}{1-\alpha z^{-1}}} \right)$$

$$= \alpha^n u[n]$$

$$\alpha \cdot \alpha^{n-1} u[n-1]$$

$$= \overset{n}{\overbrace{\alpha u[n-1]}} \longleftrightarrow n x[n]$$

$$x[n] = \frac{1}{n} \alpha^n u[n-1]$$

Examples

- Power series
- $\log(1 - r) = - \sum_{n=1}^{\infty} \frac{1}{n} r^n$

Properties of z Transform

Property	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
Time shifting	$x[n - n_0]$	$z^{-n_0} X(z)$	R except for the possible addition or deletion of the origin

Initial Value Theorem
If $x[n] = 0$ for $n < 0$, then
 $x[0] = \lim_{z \rightarrow \infty} X(z)$

Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1} where z is in R)
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$ where z is in R)
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
First difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $ z > 0$
Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least the intersection of R and $ z > 1$
Differentiation in the z -Domain	<u>$nx[n]$</u>	<u>$-z \frac{dX(z)}{dz}$</u>	R

$$x[n] \longleftrightarrow X(z)$$

$$x[-n] \longleftrightarrow X(z^{-1})$$

even signal.

odd signal

$$x_1[n] = x[-n]$$

$$X_1(z) = -X(z^{-1})$$

$$x[n] \xleftrightarrow{z} x(z)$$

$$x_1[n] = x[-n]$$

① $x[n]$ is even

$$x_1[n] = x[n]$$

$$x_1(z) = x(z)$$

ROC. \rightarrow remain the same.

n

② $x[n]$ is odd

$$x_1[n] = -x[n]$$

$$x_1(z) = -x(z)$$

Example

- Determine the z transform of a digital ramp signal
- $x[n] = nu[n]$
- Idea: Use the derivative property of the z transform.
- $x[n] = nu[n] \rightarrow X(z) = -z \frac{dU(z)}{dz}$
- $U(z) = \frac{z}{z-1}, |z| > 1$
- $X(z) = -z \frac{d}{dz} \left[\frac{z}{z-1} \right] = -z \frac{(z-1)(1)-z(1)}{(z-1)^2} = \frac{z}{(z-1)^2}, |z| > 1$

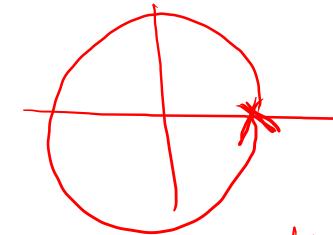
$$x[n] = u[n]$$

$$X(z) = \frac{1}{1-z} = \frac{z}{z-1} \quad (z > 1)$$

$$x_1[n] = n u[n]$$

$$x_1(z) = -z \frac{d}{dz} X(z) = -z \left[\frac{(z-1) - z}{(z-1)^2} \right]$$

$$= \frac{z}{(z-1)^2}$$



$$\frac{u}{v}$$

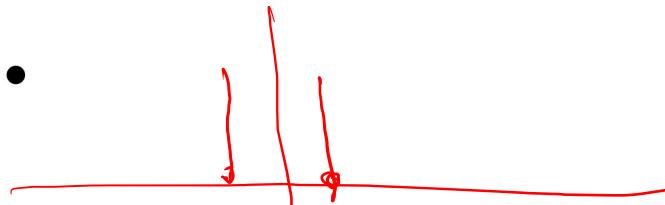
$$\frac{u'v - uv'}{v^2}$$

Example

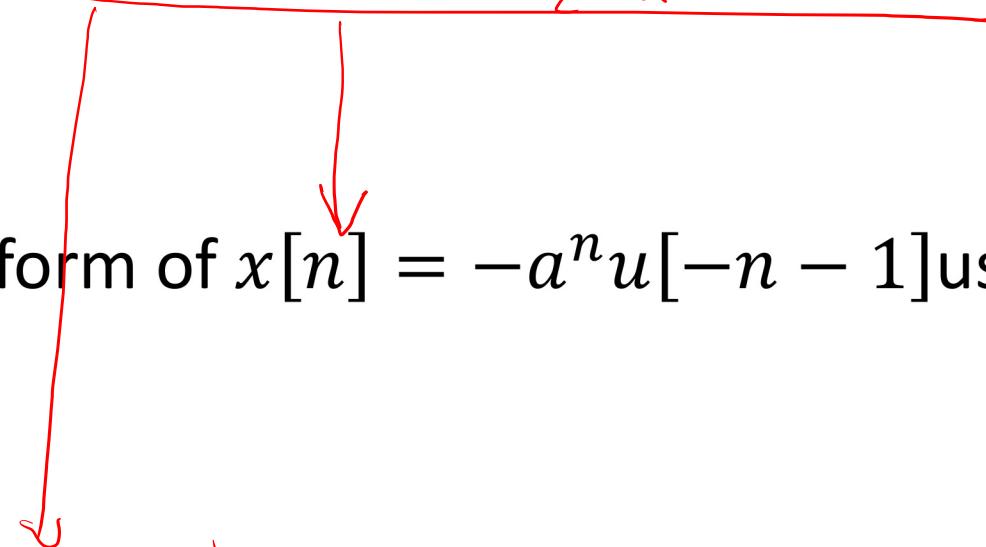
- Use the differentiation property to determine the inverse z transform
- $X(z) = \log\left(\frac{1}{1-az^{-1}}\right), |z| > |a|$

Example

- Determine the z transform of $x[n] = -a^n u[-n - 1]$ using time reversal property



$$a^n u[n] \longleftrightarrow \frac{z}{z-a} \quad |z| > |a|$$



$$a^{n+1} u[n-1] \longleftrightarrow \frac{1}{z-a} \quad |z| > |a|.$$

Example

$$x[n] = u[n]$$

$$X(z) = \frac{1}{1-2z}$$

- The **unit step response** of an LTI system is $y[n] = 2\left(\frac{1}{3}\right)^n u[n]$.
- Find the impulse response of the system and output $y[n]$ when input is $x[n] = \left(\frac{1}{2}\right)^n u[n]$



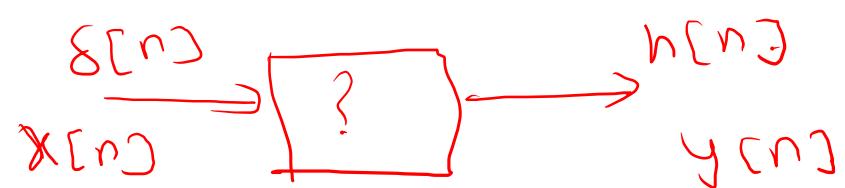
$$Y(z) = \frac{2 \cdot z}{z - 1/3} \quad H(z) = \frac{z}{z - 1/3}$$

$$H(z) = \frac{2z}{z - 1/3} \cdot \frac{z-1}{z} = \frac{2z^2 - 2}{z^2 - 1/3}$$

$$H(z) = 2 \frac{z}{z - 1/3} - 2z^2 \frac{z}{z^2 - 1/3}$$

$$h[n] = 2\left(\frac{1}{3}\right)^n u[n] - 2\left(\frac{1}{3}\right)^{n-1} u[n-1] \quad |z| > 1/3$$

$h[n] \rightarrow$ impulse response.



$$H(z) = \frac{Y(z)}{X(z)} = \frac{H(z)}{1}$$



$h[n], H(z)$

$$u[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$
$$= (\frac{1}{2})^n u[n]$$

Example

$$H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{C_1}{1 - \frac{1}{4}z^{-1}} + \frac{C_2}{1 - \frac{1}{2}z^{-1}}$$

- A causal discrete time system is described as

$$\cdot y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

- Determine the impulse response, transfer function and step response of the system.

\downarrow
 $y[n]$

\downarrow
 $H(z)$

Example

$$y[n] = w[n] - 2w[n-1].$$

$$w[n] = x[n] + \frac{1}{4}w[n-1]$$

$$Y(z) = W(z) [1 - 2z^{-1}]$$

$$W(z) [1 - \frac{1}{4}z^{-1}] = X(z).$$

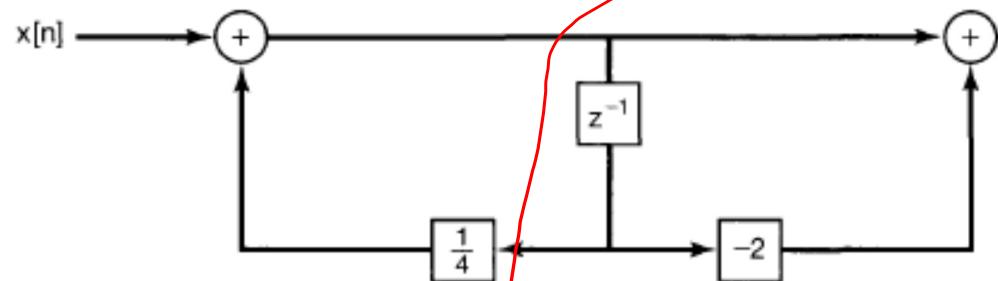
$$w[n] = x[n] + \frac{1}{4}w[n-1] + \frac{1}{8}u[n]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

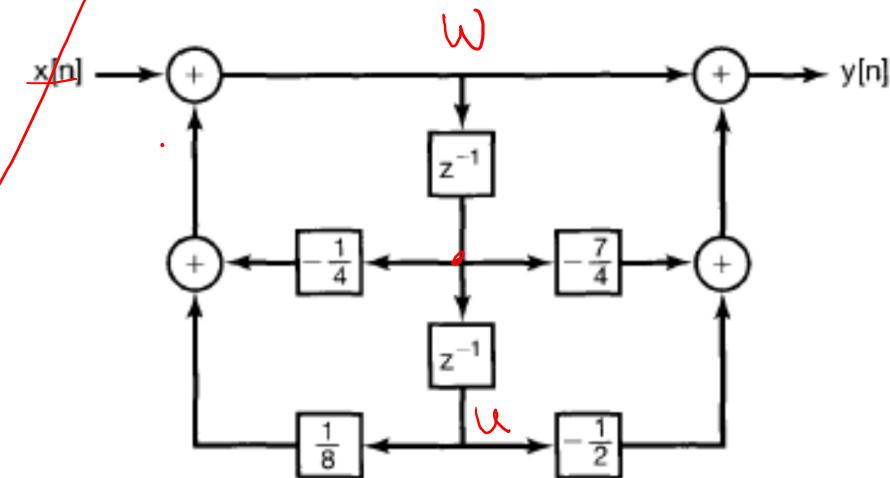
- Determine the transfer function and impulse response of the system below

~~$$u[n] = w[n-2]$$~~

$$y[n] = w[n] - \frac{7}{4}w[n-1] - \frac{1}{2}w[n-2]$$



$$= \frac{1}{1 - \frac{1}{4}z^{-1}} - 2z^{-1} \frac{1}{1 - \frac{1}{4}z^{-1}}$$



$$h[n] = (\frac{1}{4})^n u[n] - 2(\frac{1}{4})^n u[n-1]$$

$x[n] \rightarrow$ has a pole at $\frac{1}{2}$ & is absolutely summable.

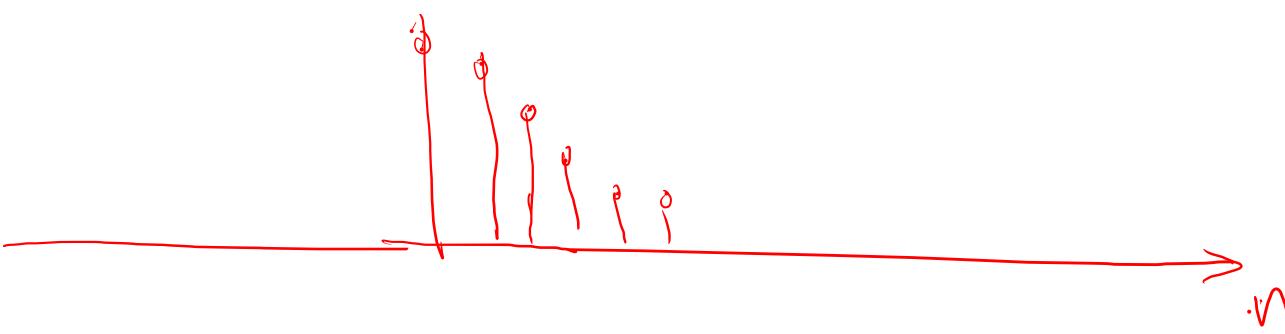
$\left(\frac{1}{4}\right)^n x[n] \rightarrow$ is absolutely summable.

$\left(\frac{1}{8}\right)^n x[n] \rightarrow$ is NOT " "

$x[n] \rightarrow ?$

rt
left
Both

$$\left(\frac{1}{4}\right)^n$$



$$\left(\frac{1}{8}\right)^n$$



$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 6^n u[-n-1]$$

$$\begin{aligned} \left(\frac{1}{8}\right) x[n] &= \left(\frac{1}{8}\right) \left(\frac{1}{2}\right)^n u[n] + 6^n \cdot \left(\frac{1}{8}\right)^n u[-n-1] \\ &= \underbrace{\left(\frac{1}{16}\right)^n u[n]}_{\text{Term 1}} + \underbrace{(0.125)^n u[-n-1]}_{\text{Term 2}} \end{aligned}$$