PS-2

Q1. To show ⇒ Sample mean is consistent and unbiased estimate of population mean.

unbiased ⇒ mean = population mean

Consistent → converges

Let $(x_1, \ldots, x_n)$ be a random sample of size n.

Sample mean $\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$.

Sampling distribution $\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$.

~~We know that $\bar{X}$ is asymptotically normal $(m, \frac{\sigma}{\sqrt{n}})$.~~

By law of large numbers with equal components.

$\boxed{\bar{X} \xrightarrow[\text{in } P]{} m}$ where m is mean of $X_1, \ldots, X_n$.

but since each X_i has distribution identical to X

$\Rightarrow$ $\boxed{\bar{X} \xrightarrow[\text{in } P]{} \text{Population Mean } M}$

Date
Page

Also $E(\bar{X}) = E\left(\frac{X_1+X_2+\dots+X_n}{n}\right) = \frac{1}{n} \cdot \left[E(X_1)+E(X_2)+\dots+E(X_n)\right]$

$= \frac{1}{n} \cdot (nm) = \boxed{m}$.

$\therefore$ It is unbiased ($E(\bar{x}) = m$) & consistent ($\bar{x} \xrightarrow{p} m$).

$\therefore$ Hence Proved.

Q4. To show $\Rightarrow$ Sample Variance is consistent but biased estimate of population variance.

$S^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$.

Sampling Distribution

$S^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$

$E(S^2) = \frac{1}{n} \cdot E\left(\sum_{i=1}^{n} (X_i - \bar{x})^2\right)$

$= \frac{1}{n} \cdot E\left(\sum_{i=1}^{n} \{X_i - m - (\bar{X}-m)\}^2\right)$

$= \frac{1}{n} \cdot E\left[\sum_{i=1}^{n} \{(X_i-m)^2 - 2(X_i-m)(\bar{X}-m) + (\bar{X}-m)^2\}\right]$.

$= \frac{1}{n} \cdot E\left[\sum_{i=1}^{n} (X_i-m)^2 - 2(\bar{X}-m) \sum_{i=1}^{n} (X_i-m) + (\bar{X}-m)^2\right]$.

$= \frac{1}{n} \cdot \left[n\sigma^2 - n E[(\bar{X}-m)^2] - 2(\bar{X}-m) \cdot E\left(\cancel{\sum_{i=1}^{n}(X_i-m)}^{\,0}\right)\right]$

as $\sum(X_i - m)$ is 0

$= \sigma^2 - E[(\bar{X}-m)^2]$.

Now $E[(\bar{X}-m)^2] = E\left[\left\{\frac{X_1+X_2+\dots+X_n}{n} - m\right\}^2\right]$

$= \frac{1}{n^2} \cdot E[(X_1-m) + (X_2-m) + \dots + (X_n-m)]^2$

Proven previously

$= \frac{1}{n^2} \cdot (\mathrm{Var}\,X_1 + \mathrm{Var}\,X_2 + \dots + \mathrm{Var}\,X_n)$

$= \boxed{\frac{\sigma^2}{n}}$

$\therefore E(S^2) = \boxed{\sigma^2 - \frac{\sigma^2}{n} \neq \sigma^2}$.

$\therefore \boxed{E(S^2) \neq \sigma^2}$. $\therefore$ Biased estimate with bias $\boxed{\frac{\sigma^2}{n}}$.

* We define $\boxed{s^2 = \frac{n}{n-1} S^2}$ then s^2 is the unbiased estimate of σ^2.

as $E(s^2) = \frac{n}{n-1} \cdot E(S^2) = \frac{n}{n-1} \cdot \left(\frac{n-1}{n}\right)\sigma^2$

$\Rightarrow \boxed{E(s^2) = \sigma^2}$.

Now, to show it is a consistent estimate

$\boxed{S^2 \xrightarrow{inP} \sigma^2}$ $E(S^2) = \sigma^2\left(\frac{n-1}{n}\right)$

~~$Var(S^2) = \frac{\sigma^2}{n} \Rightarrow$ ... $S^2 \to \frac{\sigma}{\sqrt{n}}$~~

$\therefore \boxed{\lim_{n\to\infty} E(S^2) = \sigma^2}$ As $n\to\infty$, $E(S^2)$ converges to σ^2.

To show $S^2 \xrightarrow{inP} \sigma^2$, we can show that $Var(S_n) \to 0$ as $n\to\infty$, then by Tchebychef's Thm we can conclude $X_n - m_n \xrightarrow{inP} 0 \Rightarrow \boxed{X_n \xrightarrow{inP} m_n}$.

~~$Var(S_n) \to Var(S^2) =$~~ $E(S$ Let $Y = S^2$, then $E(Y) = E(S^2) =$ $(n-1)\sigma$

To show that $S^2 \xrightarrow{\text{in P}} \sigma^2$, we will show the following,

$\forall k$, A_k = raw moment of sample. α_k = raw moment of population.

then $A_k \xrightarrow{\text{in P}} \alpha_k$.

$a_k = \frac{1}{n} \sum_{i=1}^{n} x_i^k$, $A_k = \frac{1}{n} \sum_{i=1}^{n} X_k$ $\alpha_k = E(X^k)$

Statistic R.V. $= E(X_i^k)$

(as X & X_i have same distribution).

Using Tchebycheff's Inequality

$$P(|A_k - E(A_k)| \geq \epsilon) \leq \frac{Var(A_k)}{\epsilon^2}, \text{ for any } \epsilon > 0.$$

$$Var(A_k) = E\{(A_k - E(A_k))^2\}$$

$$= E\left\{\left(\frac{1}{n}\sum_{i=1}^{n} x_i^k - \alpha_k\right)^2\right\}.$$ How?

as $A_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k$.

$\Rightarrow E(A_k) = \frac{1}{n} \cdot \underbrace{\sum_{i=1}^{n} E(x_i^k)}_{n\alpha_k}$.

$\Rightarrow E(A_k) = \frac{1}{n} \cdot (n\alpha^k)$

$\Rightarrow E(A_k) = \alpha_k$.

$$= E\left\{\left(\frac{1}{n}\sum_{i=1}^{n} x_i^k - \frac{1}{n}\sum_{i=1}^{n}\alpha_k\right)^2\right\}$$

$$= \frac{1}{n^2} E\left(\left\{\sum_{i=1}^{n}(x_i^k - \alpha_k)\right\}^2\right).$$

$$= \frac{1}{n^2} \cdot \sum_{i=1}^{n} E\{(x_i^k - \alpha_k)^2\}$$

$$+ \frac{2}{n^2} \cancel{\sum\sum_{1 \leq i < j \leq n} E\{(x_i^k - \alpha_k)(x_j^k - \alpha_k)\}} \to 0$$

(as they are independent).

$$= \frac{1}{n^2} \cdot \sum_{i=1}^{n} Var(x_i^k) = \frac{n}{n^2} Var(x^k) = \frac{1}{n} Var(x^k).$$

Since $Var(x^k)$ is finite, $\Rightarrow Var(A_k) \to 0$ as $n \to \infty$!

$$\therefore \lim_{n\to\infty} P(|A_k - E(A_k)| \geq \epsilon) \to 0$$

$\Rightarrow$ A_k converges to $E(A_k)$

$$\Rightarrow \boxed{A_k \xrightarrow{in\ P} \alpha_k}$$

Now since $A_1 \xrightarrow{in\ P} \alpha_1 = m_x$

$A_2 \xrightarrow{in\ P} \alpha_2$

$$\Rightarrow A_2 - A_1^2 \xrightarrow{in\ P} \alpha_2 - m_x^2$$

Sample parameters

$$\Rightarrow \boxed{S^2 \xrightarrow{in\ P} \sigma^2}$$

Population parameters.

Hence S^2 is consistent estimate of σ^2. Hence Proved.

PS-2

Q2. Shown in the proof for showing that S^2 is consistent estimate of σ^2.

Q3. m_k = central moment of Sample. μ_k = central moment of population

$M_k \Rightarrow$ R.V. corresponding to m_k

$$m_k = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^k. \quad \Rightarrow \quad M_k = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^k.$$

$$\text{Now } E(M_k) = \frac{1}{n}\sum_{i=1}^{n} E([x_i-\bar{x}]^k) = \frac{n\mu_k}{n} = \boxed{\mu_k}$$

Using Tchebycheff's inequality

$$P(|M_k - E(M_k)| \geq \varepsilon) \leq \frac{Var(M_k)}{\varepsilon^2} \quad \text{for any } \varepsilon > 0$$

$$Var(M_k) = E[\{M_k - E(M_k)\}^2]$$

$$= E\left(\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^k - \mu_k\right]^2\right)$$

$$= \frac{1}{n^2} E\left[\left(\sum_{i=1}^{n}(X_i-\bar{X})^k - n\mu_k\right)^2\right]$$

$$= \frac{1}{n^2} E\left(\left[\sum_{i=1}^{n}[(X_i-\bar{X})-\mu_k]\right]^2\right)$$

$$\cancel{= \frac{1}{n^2} E\left\{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{i=1}^{n}\mu_k^2\right\}}$$

$$= \frac{1}{n^2} E\left\{\sum_{i=1}^{n}(X_i-\bar{X}-\mu_k)^2 + 2\cancel{\sum_{1\leq i<j\leq n}(X_i-\bar{X}-\mu_k)(X_j-\bar{X}-\mu_k)}^{\,0}\right\}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} E\{((X_i-\bar{X})^k-\mu_k)^2\}$$

$$= \frac{1}{n^2} \cdot n \cdot Var(X_i-\bar{X})^k = \boxed{\frac{Var((X_i-\bar{X})^k)}{n}}$$

$\therefore Var(M_n) = \frac{Var(X_i - \bar{X})}{n} \to 0$ as $n \to \infty$.

$\therefore \lim_{n\to\infty} P(M_n - E(M_n) \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$

$\Rightarrow M_n - E(M_n) \xrightarrow{in\,P} 0.$

$\Rightarrow \boxed{M_n \xrightarrow{in\,P} \mu_n}$ Hence Proved.

Q4. Done in Theory.

Q5. $S^2 = \frac{n}{n-1} \cdot S^2 \Rightarrow$ we have shown it is unbiased.

$s^2 = \left(\frac{n}{n-1}\right) \cdot [A_2 - A_1^2].$

$\lim_{n\to\infty} s^2 = \lim_{n\to\infty} \left(\frac{n}{n-1}\right) [A_2 - A_1^2]$

$= \lim_{n\to\infty} \cancel{\left(\frac{n}{n-1}\right)}^{1} \cdot \lim_{n\to\infty} \underbrace{[A_2 - A_1^2]}_{\sigma^2} \Rightarrow \lim_{n\to\infty} S^2 = \sigma^2$

$\therefore \boxed{S^2 \xrightarrow{in\,P} \sigma^2} \Rightarrow$ It is consistent.

Q6. To find $\Rightarrow$ MLE $(\hat{p})$ of p of Binomial (N, p).

$P(X = x) = \binom{N}{x} \cdot p^x \cdot (1-p)^{N-x}, \quad x = 0, 1, \ldots, N.$

Let $(x_1, x_2, \ldots, x_n)$ be a random sample of size n from the population of X.

Likelihood f^n

$L = L(x_1, x_2, \ldots, x_n; p).$

$= \binom{N}{x_1} p^{x_1} (1-p)^{N-x_1} \binom{N}{x_2} p^{x_2} \cdot (1-p)^{N-x_2} \cdots \binom{N}{x_n} \cdot p^{x_n} (1-p)^{N-x_n}$

$= \binom{N}{x_1} \cdots \binom{N}{x_n} \cdot p^{\underbrace{(x_1 + \cdots + x_n)}_{n\bar{x}}} (1-p)^{Nn - \underbrace{(x_1 + x_2 + \cdots + x_n)}_{n\bar{x}}}$

$= \binom{N}{x_1} \cdots \binom{N}{x_n} \; p^{n\bar{x}} \; (1-p)^{n[N - \bar{x}]}$

$$\therefore \log_e L = \log_e \left\{ \binom{N}{x_1} \cdots \binom{N}{x_n} \right\} + n\bar{x}\log p + (nN - n\bar{x})\log(1-p)$$

$\Downarrow$ const. for N. $\qquad$ ($\boxed{0<p<1}$ variable p.

$\therefore \log_e L$ as a f^n of p.

For local maximum.

$$\frac{d}{dp}(\log_e L) = 0.$$

$$\Rightarrow \frac{n\bar{x}}{p} - \frac{(nN - n\bar{x})}{(1-p)} = 0.$$

$$\Rightarrow n\bar{x}(1-p) = n(N-\bar{x}).p$$

$$\Rightarrow \bar{x} - \cancel{\bar{x}p} = Np - \cancel{\bar{x}p} \Rightarrow \boxed{p = \frac{\bar{x}}{N}}$$

and $\dfrac{d^2(\log_e L)}{dp^2} = \dfrac{-n\bar{x}}{p^2} - \dfrac{n(N-\bar{x})}{(1-p)^2} < 0$

$\therefore$ $\boxed{\left(p = \frac{\bar{x}}{N}\right) \text{ is a local maximum}}$.

Now $\dfrac{d}{dp}(\log_e L) > 0 \;\forall\; p < \dfrac{\bar{x}}{N}$.

$\dfrac{d}{dp}(\log_e L) < 0 \;\forall\; p > \dfrac{\bar{x}}{N}$.

$\therefore$ $\boxed{p = \frac{\bar{x}}{N} \text{ is a global maximum}}$. $\therefore$ $\boxed{M.L.E = \hat{p} = \frac{\bar{x}}{N}}$

Now (1) $E(\hat{p}) = E\left(\dfrac{\bar{x}}{N}\right) = E\left(\dfrac{x_1 + x_2 + \cdots + x_n}{nN}\right) = \dfrac{1}{N} \cdot \dfrac{\cancel{n}}{\cancel{n}} \cdot E(X_1)$

$= \dfrac{1}{N} \cdot Np = \boxed{p}$. $\qquad \therefore \boxed{E(\hat{p}) = p}$.

(2) $\left(\dfrac{X_1 + X_2 + \cdots + X_n}{n}\right) \xrightarrow{inP} Np$. ($\bar{X}_n$ converges to Mu., law of large numbers)

$\Rightarrow \dfrac{\bar{X}}{N} \xrightarrow{inP} p$

$\Rightarrow \boxed{\hat{p} \xrightarrow{inP} p}$

$\therefore \hat{p}$ is an unbiased & consistent estimate of p.
Hence Proved

Q8. To find $\Rightarrow$ MLE of μ in Poisson(μ) Distribution.
Let it be $\hat{\mu}$.

$$P(X=x) = \frac{e^{-\mu} \cdot (\mu)^x}{\lfloor x}, \quad x = 0, 1, \ldots, \infty.$$

Let $(x_1, x_2, \ldots, x_n)$ be a random sample of size n
Likelihood $f^n \Rightarrow$

$$L = L(x_1, x_2, \ldots, x_n; \mu).$$

$$= \frac{e^{-\mu} \cdot (\mu)^{x_1}}{\lfloor x_1} \cdot \frac{e^{-\mu} (\mu)^{x_2}}{\lfloor x_2} \ldots \frac{e^{-\mu} (\mu)^{x_n}}{\lfloor x_n}$$

$$= \frac{e^{-n\mu} \cdot (\mu)^{(x_1+x_2+\ldots+x_n)}}{\lfloor x_1 \lfloor x_2 \ldots \lfloor x_n}$$

$$\therefore \log_e L = \log_e (e^{-n\mu}) + (x_1+x_2+\ldots+x_n) \cdot \log_e(\mu) - \log_e(\lfloor x_1 \lfloor x_2 \ldots \lfloor x_n).$$

$$= -n\mu + (x_1+x_2+\ldots+x_n) \log_e(\mu) - \log_e(\lfloor x_1 \lfloor x_2 \ldots \lfloor x_n)$$

$$\frac{d(\log_e L)}{d\mu} = -n + \frac{(x_1+x_2+\ldots+x_n)}{(\mu)} = 0;$$

$$\Rightarrow \frac{x_1+x_2+\ldots+x_n}{n} = \mu \Rightarrow \boxed{\mu = \bar{x}}.$$

$$\frac{d^2(\log_e L)}{d^2\mu} = \frac{-1}{\mu^2} (x_1+x_2+\ldots+x_n)$$

$$= \frac{-1}{\mu^2}(n\bar{x}) = \frac{-1}{(\bar{x})^2} \cdot (n\bar{x}) = \frac{-n}{\bar{x}} < 0$$

$\therefore \boxed{\hat{\mu} = \bar{x}}$ is local maxima

$$\frac{d(\log_e L)}{d\mu} = -n + \frac{x_1+x_2+\ldots+x_n}{\mu}$$

$$> -n + \frac{\overbrace{x_1+x_2+\ldots+x_n}^{n\bar{x}}}{\bar{x}} \quad (\text{when } \mu < \bar{x})$$

$$> 0 \quad (\text{when } \mu < \bar{x}).$$

and

$$\frac{d \log_e L}{d\mu} = -n + \frac{n\bar{x}}{\mu}$$

$$< -n + \frac{n\bar{x}}{\mu} \quad (\text{when } \mu > \bar{x})$$

$$< 0. \quad \text{when } (\mu > \bar{x})$$

$\therefore$ $\boxed{\text{It is a global maxima}}$. $\Rightarrow$ $\boxed{\hat{\mu} = \bar{x}}$

Finally, ① $E(\hat{\mu}) = E(\bar{x}) = E\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right)$

$$= \frac{1}{n} \cdot n \cdot \underbrace{E(X_1)}_{\mu \text{ for poisson}} \to \boxed{\mu}.$$

and ② $\frac{x_1 + x_2 + \dots + x_n}{n} \xrightarrow{\text{in } P} \mu$ (using LLN)

$$\Rightarrow \bar{x} \xrightarrow{\text{in } P} \mu \Rightarrow \boxed{\hat{\mu} \xrightarrow{\text{in } P} \mu}.$$

$\therefore$ $\boxed{\hat{\mu} \text{ is a consistent unbiased estimate of } \mu.}$

Q9. $f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ $(\theta > 0, x > 0)$. $= P(X = x)$.

To find $\Rightarrow$ MLE of θ, denoted by $\hat{\theta}$.

Let $(x_1, x_2, \dots, x_n)$ be a sample of size n.
Likelihood f^n

$$L = L(x_1, x_2, \dots, x_n; \theta).$$

$$= \frac{1}{\theta} \cdot e^{-\frac{x_1}{\theta}} \cdot \frac{1}{\theta} \cdot e^{-\frac{x_2}{\theta}} \dots \frac{1}{\theta} \cdot e^{-\frac{x_n}{\theta}}.$$

$$L = \frac{1}{(\theta)^n} e^{-\left(\frac{x_1 + x_2 + \dots + x_n}{\theta}\right)}.$$

① $\therefore \log_e L = -n \cdot \log_e(\theta) - \left(\frac{x_1 + x_2 + \dots + x_n}{\theta}\right)$

$$\frac{d \log_e L}{d\theta} = -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2} = 0 \Rightarrow \boxed{\bar{x} = \theta}.$$

$$\frac{d^2 \log_e L}{d\theta^2} = \frac{n}{\theta^2} - \frac{2 \cdot n\bar{x}}{\theta^3} \underset{\boxed{\bar{x} = \theta}}{\Rightarrow} \frac{n}{\theta^2} - \frac{2n}{\theta^2} = \boxed{-\frac{n}{\theta^2} < 0},$$

classmate

Date
Page

$\therefore$ $\boxed{\bar{x} = \theta \text{ is a local maxima}}$.

② Also. $\dfrac{d\log_e L}{d\theta} = \dfrac{-n}{\theta} + \dfrac{n\bar{x}}{\theta^2}$ when $\bar{x} > \theta$.

$< \dfrac{-n}{\theta} + \dfrac{n \cdot \theta}{\theta^2}$ $\Rightarrow$ $\boxed{\dfrac{d\log_e L}{d\theta} < 0}$

when $\bar{x} < \theta$

$\Rightarrow \dfrac{d\log_e L}{d\theta} > \dfrac{-n}{\theta} + \dfrac{n\theta}{\theta^2}$ $\Rightarrow$ $\boxed{\dfrac{d\log_e L}{d\theta} > 0}$.

$\therefore$ $\bar{x} = \theta$ is a ~~loco~~ global maxima. $\boxed{\hat{\theta} = \bar{x}}$.

③ Finally $E(\hat{\theta}) = E\left(\dfrac{X_1 + X_2 + \dots + X_n}{n}\right) = \dfrac{1}{\cancel{n}} \cdot \cancel{n} \cdot E(X_1)$

$= \boxed{E(X_1)}$.

$f_X(x) = \dfrac{1}{\theta} e^{-\frac{x}{\theta}}$

$E(X) = \displaystyle\int_0^\infty \frac{x}{\theta} e^{-\frac{x}{\theta}}\, dx$; Let $\dfrac{x}{\theta} = u \Rightarrow dx = \theta\, du$

$= \theta \displaystyle\int_0^\infty u e^{-u}\, du = \theta \cdot \left. e^{-u}(1+u) \right|_\infty^0$

$= \boxed{\theta}$.

$\therefore$ $\boxed{E(\hat{\theta}) = E(X) = \theta}$

and.

$\dfrac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow{\text{inp}} \theta$ (using LLN).

$\Rightarrow \bar{X} \xrightarrow{\text{inp}} \theta$

$\Rightarrow \boxed{\hat{\theta} \xrightarrow{\text{inp}} \theta}$.

$\therefore$ $\hat{\theta} = \bar{X}$ is a consistent & unbiased estimate of θ.

$\boxed{MLE(\theta) = \hat{\theta} = \bar{X}}$.

Q10. $f_X(x;\alpha) = \dfrac{2(\alpha - x)}{\alpha^2}$ $(0 < x < \alpha)$

We are taking a sample of size 1 $\boxed{x_1}$.

Likelihood f^n

$L = L(x_1; \alpha)$

$= \frac{2(\alpha - x_1)}{\alpha^2} \quad (0 < x < \alpha).$

① $\log_e L = \log_e\left(\frac{2}{\alpha^2}\right) + \log_e(\alpha - x_1).$

$\Rightarrow \log_e L = \log_e 2 - 2\log_e \alpha + \log_e(\alpha - x_1).$

$\frac{d(\log_e L)}{d\alpha} = \frac{-2}{\alpha} + \frac{1}{(\alpha - x_1)} = 0$

$\Rightarrow \frac{2}{\alpha} = \frac{1}{(\alpha - x_1)} \Rightarrow 2\alpha - 2x_1 = \alpha \Rightarrow \boxed{\alpha = 2x_1}$

$\frac{d^2(\log_e L)}{d\alpha^2} = \frac{2}{\alpha^2} - \frac{1}{(\alpha - x_1)^2} \Rightarrow \frac{2}{\alpha^2} - \frac{1}{\left(\frac{\alpha}{2}\right)^2} \Rightarrow \boxed{\frac{-2}{\alpha^2}} < 0$

$\left(x_1 = \frac{\alpha}{2}\right)$

∴ It is a local maximum.

② also $\frac{d(\log_e L)}{d\alpha} = \frac{-2}{\alpha} + \frac{1}{(\alpha - x_1)}$

when $\alpha < 2x_1 \Rightarrow \frac{\alpha}{2} < x_1 \Rightarrow \alpha - \frac{\alpha}{2} > (\alpha - x_1)$

$\Rightarrow \frac{2}{\alpha} < \frac{1}{(\alpha - x_1)} \Rightarrow \boxed{\frac{-2}{\alpha} + \frac{1}{(\alpha - x_1)} > 0}$

when $\alpha > 2x_1 \Rightarrow \frac{\alpha}{2} > x_1 \Rightarrow \boxed{\frac{-2}{\alpha} + \frac{1}{(\alpha - x_1)} < 0}$

∴ $\alpha = 2x_1$ is a global maximum.

∴ $\boxed{\hat{\alpha} = 2x_1}$

③ Finally $E(\hat{\alpha}) = E(2x_1) = 2E(x_1).$

Now $E(X_1) = \int_0^\alpha \frac{2(\alpha - x)}{\alpha^2} \cdot x \, dx = \frac{2}{\alpha^2}\int_0^\alpha (\alpha x - x^2)\, dx.$

$= \frac{2}{\alpha^2}\left[\alpha \cdot \left(\frac{\alpha^2}{2}\right) - \frac{\alpha^3}{3}\right] = \frac{2}{\alpha^2} \cdot \left(\frac{\alpha^3}{6}\right) = \boxed{\frac{\alpha}{3}}$

∴ $\boxed{E(\hat{\alpha}) = \frac{2 \cdot \alpha}{3} \neq \alpha}$

∴ It is a biased estimate.

Hence Proved.

Q11. $f(y;\theta) = \dfrac{2\theta^2}{y^3}, \quad 0 < \theta \le y < \infty.$

$(y_1, y_2, \ldots, y_n)$ = Sample of size n. $\boxed{\text{To find} \Rightarrow \hat{\theta}}$

① Likelihood fn

$$L = L(y_1, y_2, \ldots, y_n ; \theta)$$

$$= \frac{2\theta^2}{y_1^3} \cdot \frac{2\theta^2}{y_2^3} \cdots \frac{2\theta^2}{y_n^3}$$

$$L = \frac{[2\theta^2]^n}{[y_1 y_2 \ldots y_n]^3}$$

$$\cancel{\log_e L = n \log_e(2\theta^2) - 3 \log_e [y_1 y_2 \ldots y_n]}$$

$$\cancel{= n \log_e 2 + 2n \log_e(\theta) - 3 \log_e(y_1 \ldots y_n)}$$

$$\cancel{\frac{d \log_e L}{d\theta} = \frac{2n}{\theta}}$$

We can clearly see that L is maximised at boundary condn. $\theta \uparrow, L \uparrow$. ∴ Calculus cannot be applied here.

Hence MLE of θ is the max available value of θ.

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Date

Page

PS-(2)

Q7. $X \sim \text{Normal}(m, \sigma)$. $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$

Likelihood fn L =

$$L = \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{(\sigma)^n} e^{-\frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i - m}{\sigma}\right)^2}$$

$$\log_e L = -\frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - m}{\sigma}\right)^2.$$

(1) $\frac{\partial \log_e L}{\partial m} = 0$ and $\frac{\partial \log_e L}{\partial \sigma} = 0$.

$$\Rightarrow -\frac{1}{2} \times \frac{(-2)}{\sigma^2} \cdot \sum_{i=1}^{n}\left(x_i - m\right) = 0$$

$$\Rightarrow \frac{n\bar{x} - nm}{\sigma} = 0 \Rightarrow \frac{n}{\sigma^2}(\bar{x} - m) \Rightarrow \boxed{\bar{x} = m}.$$

$$\frac{\partial \log_e L}{\partial \sigma} = 0 \Rightarrow -\frac{n}{\sigma} - \frac{1}{2}\cdot(-2)\cdot\left[\sum_{i=1}^{n}(x_i - m)^2\right]\frac{1}{\sigma^3} = 0$$

$$\Rightarrow \frac{n}{\sigma} = \frac{1}{\sigma^3}\cdot\sum_{i=1}^{n}(x_i - m)^2$$

$$\Rightarrow \boxed{\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - m)^2}$$

For local maximum.

$$H_f(a,b) = \begin{array}{cc} D_1 & D_2 \\ \left[\begin{array}{c|c} f_{xx}(a,b) & f_{xy}(a,b) \\ \hline f_{yx}(a,b) & f_{yy}(a,b) \end{array}\right] & \end{array}$$

from maths (1)

$D_1 = f_{xx}(a,b)$

$D_2 = \det(H_f(a,b))$.

(1) $f_x(a,b) = f_y(a,b) = 0 \Rightarrow (a,b)$ is a critical pt of f.

(2) If $D_1 < 0$, $D_2 > 0$: Maximum.

$D_1 > 0$, $D_2 > 0$: Minimum.

$D_2 < 0$: Saddle pt.

$$\frac{\partial \log_e L}{\partial m} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - m).$$

$$\Rightarrow \frac{\partial^2 \log_e L}{\partial m^2} = \frac{1}{\sigma^2} \cdot \sum_{i=1}^{n} (-1) = \boxed{\frac{-n}{\sigma^2}} \qquad \therefore \boxed{A<0}.$$

$$\frac{\partial^2 \log_e L}{\partial m \partial \sigma} = \frac{-2}{\sigma^3} \sum_{i=1}^{n} (x_i - m)$$

$$\frac{\partial \log_e L}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i - m)^2.$$

$$\Rightarrow \frac{\partial^2 \log_e L}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} (x_i - m)^2.$$

putting $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - m)^2$

$$H_f(a,b) = \begin{bmatrix} \frac{-n}{\sigma^2} & \frac{-2}{\sigma^3} n(\bar{x} - m) \\ \frac{-2}{\sigma^3} n(\bar{x} - m) & \frac{n}{\sigma^2} - \frac{3}{\sigma^{\cancel{4}2}} \cdot (n \cancel{\sigma^2}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-n}{\sigma^2} & \frac{-2}{\sigma^3} n(\bar{x} - m) \\ \frac{-2}{\sigma^3} n(\bar{x} - m) & \frac{-2n}{\sigma^2} \end{bmatrix}$$

$$= \frac{2n^2}{\sigma^4} - \frac{4n^2}{\sigma^6} (\bar{x} - m)^2$$

$\boxed{\bar{x} = m}$

$$\Rightarrow H_f(a,b) = \frac{2n^2}{\sigma^4} > 0$$

$$\therefore m, \sigma^2 = \left(\bar{x}, \frac{1}{n} \sum_{i=1}^{n} (x_i - m)^2\right)$$

$\boxed{\text{is a local maxima.}}$

Since it is a local maxima and is the only critical pt.

$\Rightarrow \left(m = \bar{x},\ \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - m)^2\right)$ is a global maxima

$$\therefore \boxed{\hat{m} = \bar{x}} \qquad \boxed{\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - m)^2}.$$