Z Transform

EGC 113

Source: inverse z transform slides from Prof. Deepa Kundur’s slides (UoT)

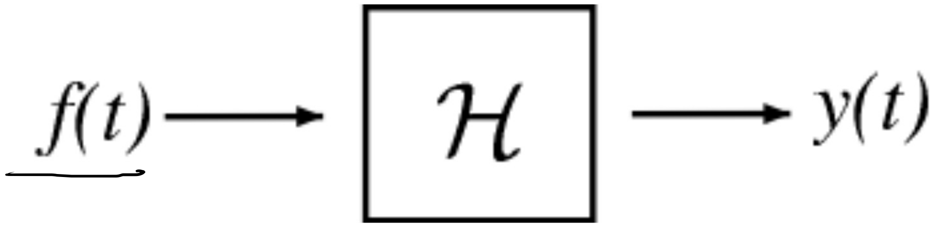
Eigenfunctions

Convolution

$$y[n] = \sum_{k} x[k]\, h[n-k]$$

$$Y(z) = X(z)\, H(z)$$

- A linear time invariant (LTI) system H operating on a continuous input $f(t)$ to produce continuous time output $y(t)$
  - $H[f(t)]=y(t)$



Σ

- is mathematically analogous to an $N\times N$ matrix A operating on a vector $\boldsymbol{x}\in\mathbb{C}^N$ to produce another vector $\boldsymbol{b}\in\mathbb{C}^N$. Just as an eigenvector of $\boldsymbol{A}$ is a $\boldsymbol{v}\in\mathbb{C}^N$ such that $\boldsymbol{Av}=\boldsymbol{\lambda v}$, $\boldsymbol{\lambda}\in\mathbb{C}$,

$$A\underline{v} = \lambda v$$

Eigenfunctions of System

- An input x to a system $\mathcal{H}$ is said to be an **eigenfunction** of the system $\mathcal{H}$ with the **eigenvalue** λ if the corresponding output y is of the form

$$y = \lambda x,$$

  where λ is a complex constant.
- In other words, the system $\mathcal{H}$ acts as an ideal amplifier for each of its eigenfunctions x, where the amplifier gain is given by the corresponding eigenvalue λ.
- Different systems have different eigenfunctions.
- Of particular interest are the eigenfunctions of LTI systems.

Eigenfunctions of LTI System

- As it turns out, every complex exponential is an eigenfunction of all LTI systems.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$e^{st} \longrightarrow \boxed{\mathcal{H}} \longrightarrow H(s)\, e^{st}$$

$$\mathcal{H}\{e^{st}\} = H(s)e^{st},$$

where s is a complex constant and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$

Eigenfunctions of LTI System

- That is, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.
- We refer to H as the **system function** (or **transfer function**) of the system $\mathcal{H}$.
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$.

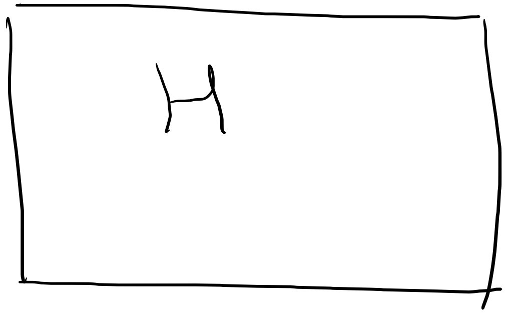
Representation using Eigenfunctions

- Consider a LTI system with input x, output y, and system function H.
- Suppose that the input x can be expressed as the linear combination of complex exponentials

$$x(t) = \sum_k a_k e^{s_k t},$$

  where the a_k and s_k are complex constants.
- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$



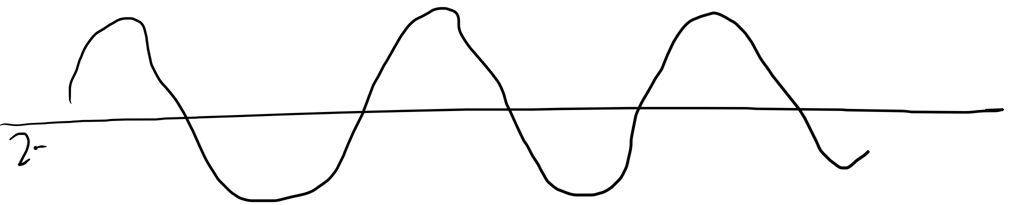
$$y(t) = x(t) * h(t)$$

$$x(t) = \sum_k a_k e^{s_k t}$$

$$y(t) = \sum_k a_k e^{s_k t} H(s_k)$$

$x_1(t) = \sin(2\pi 100 t)$

$f = 100\,Hz$



$x_2(t)$

$f = 200\,Hz$

$x_D(t) = x_1(t) + x_2(t)$

$y_1(t) = \alpha_1 x_1(t - \delta_1)$

$+ \alpha_2 x_2(t - \delta_2)$

Representation using Eigenfunctions

- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(t) = \sum_{k} a_k H(s_k) e^{s_k t}.$$

- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as a linear combination of the *same* complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Summary: Eigen functions of LTI systems

$$x(t) \text{ or } x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow Kx(t) \text{ or } Kx[n]$$

If this happens, the Implication is that all Input properties are preserved

Is there any special family of inputs for which this is true ??

Summary: Eigen Function of LTI systems

- Let the Impulse response of the LTI system be $h(t)$
- If the Input is $x(t)$, then the Output $y(t)$ is given by :

$$y(t) = \int h(\tau)x(t-\tau)d\tau \qquad : \tau \text{ goes from } -\infty\ to\ \infty$$

- Consider the case : $x(t) = e^{(st)}$

Here s is complex in general

Summary: Eigen Function of LTI systems

Consider the case : $x(t) = e^{(st)}$. Here s is complex in general

Then

$$y(t) = \int h(\tau)e^{(s[t-\tau])}d\tau \qquad : \tau \text{ goes from } -\infty \; to \; \infty$$

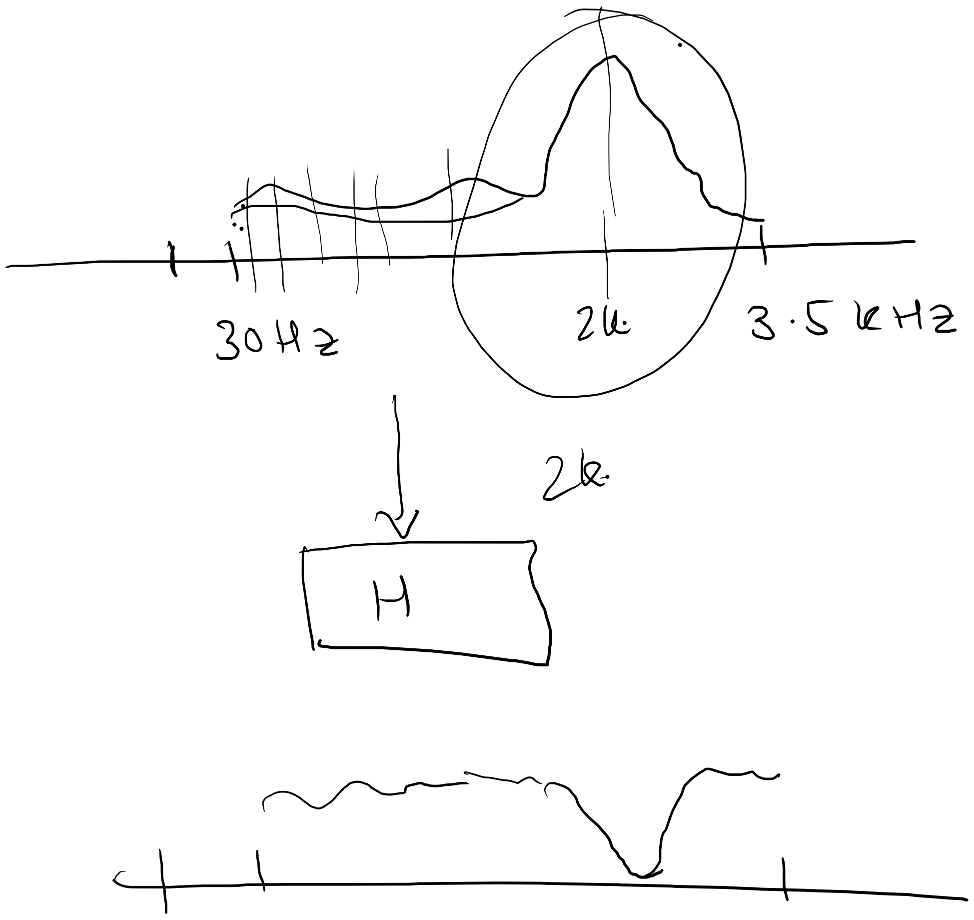
$$y(t) = e^{(st)} \int h(\tau)e^{(-s\tau)}d\tau \qquad : \tau \text{ goes from } -\infty \; to \; \infty$$

Put $H(s) = \int h(\tau)e^{(-s\tau)}d\tau \qquad : \tau$ goes from $-\infty \; to \; \infty$

Then $y(t) = H(s)\, e^{(st)}$

$y(t) \; = \; H(s)\, x(t)$

Hence $e^{(st)}$ is an eigen function of LTI systems speech →



Why are we learning this?

If any arbitrary input x(t) is expressed in terms of the eigen functions

$$x(t) = a_1 e^{(s_1 t)} + a_2 e^{(s_2 t)}$$

Then

$$y(t) = a_1 H(s_1) e^{(s_1 t)} + a_2 H(s_2) e^{(s_2 t)}$$

Expressing $x(t) = \sum a_k \, e^{(s_k t)}$ offers several advantages

This series expression is called ‘Fourier Series’

Discrete-Time Fourier Transform & *Z*-transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}\,d\omega \qquad\qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}$$

The first equation asserts that we can represent any time function $x[n]$ by a linear combination of complex exponentials $e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$

The second equation tells us how to compute the complex weighting factors $X(e^{j\omega})$

In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n.

Generalizing the frequency variable

- In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n
- z^n can be thought of as a generalization of $e^{j\omega n}$
- For an arbitrary z, using polar notation we obtain $z = \rho e^{j\omega}$ so

$$z^n = \rho^n e^{j\omega n}$$

- If both ρ and ω are real, then z^n can be thought of as a complex exponential (*i.e.* sines and cosines) with a real temporal envelope that can be either exponentially decaying or expanding

The inverse *Z*-transform

- We didn't talk about inverse *z*-transforms yet?
- It can be shown that the inverse *z*-transform can be formally expressed as

$$x[n] = \frac{1}{2\pi j}\oint_{c} X(z) z^{n-1} dz$$

- Comments:
  - The integral is over a **complex variable**, z and we need complex residue calculus to evaluate it formally.
  - The contour of integration, c, is a circle around the origin that lies inside the ROC

Inverse z-transform

The inverse z-transform is based on a special case of the Cauchy integral theorem

$$\frac{1}{2\pi j}\oint_C z^{-\ell}\,dz = \begin{cases} 1 & \ell = 1 \\ 0 & \ell \neq 1 \end{cases}$$

where C is a counterclockwise contour that encircles the origin. If we multiply $X(z)$ by z^{n-1} and compute

$$\begin{aligned}
\frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz &= \frac{1}{2\pi j}\oint_C \sum_{m=-\infty}^{\infty} x[m]z^{\underline{-m+n-1}}\,dz \\
&= \sum_{m=-\infty}^{\infty} x[m]\underbrace{\frac{1}{2\pi j}\oint_C z^{-(m-n+1)}\,dz}_{=1 \text{ only when } \underline{m-n+1}=1} \\
&= \sum_{m=-\infty}^{\infty} x[m]\delta(m-n) \\
&= x[n]
\end{aligned}$$

Hence, the inverse z-transform of $X(z)$ is defined as $x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz$ where C is a counterclockwise closed contour in the ROC of $X(z)$ encircling the origin.

Inverse z-transform

Cauchy's residue theorem works, but it can be tedious and there are lots of ways to make mistakes. The Matlab function `residuez` (discrete-time residue calculator) can be useful to check your results.

Other (typically easier) options for computing inverse z-transforms:

1. Inspection (table lookup).
2. Partial fraction expansion (only for rational z-transforms).
3. Power series expansion (can be used for non-rational z-transforms).

Power Series Expansion Method

The idea here is to write $X(z)$ as

$$X(z) = \cdots + c_{-2}z^2 + c_{-1}z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots$$

and recognize that $x[n] = c_n$ by the definition of the z-transform. This can work even for non-rational $X(z)$.

$x[0] = c_0$

$x[1] = c_1$

For example, suppose $a > 0$ and

$$X(z) = e^{az^{-1}}$$

and note the ROC is the whole complex plane except $z = 0$. We can use the series expansion

$$X(z) = \sum_{k=0}^{\infty} \frac{(az^{-1})^n}{n!} = 1 + \frac{a}{1}z^{-1} + \frac{a^2}{2}z^{-2} + \frac{a^4}{6}z^{-3} + \ldots$$

hence $x[n] = \frac{a^n}{n!}u[n]$.

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Partial Fraction Expansion Method

If

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

with all N poles distinct ("first order") then it is possible to express

$$X(z) = \underbrace{\sum_{r=0}^{M-N} B_r z^{-1}}_{\text{only if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - \lambda_k z^{-1}}$$

The inverse z-transform then follows directly from linearity and table lookup (pay attention to the ROC). There are many ways to determine $A_1, \ldots, A_k$, for example

$$A_k = \left[(1 - \lambda_k z^{-1}) X(z)\right]_{z=\lambda_k}$$

$y[n] + a_1 y[n-1] - a_2[y-2] = x[n]$

$+ b_1 x[n-1] + b_2 x[n-2]$

Rational Function

- LCCDE: Linear constant coefficient difference equations

  - $X(z)$ is a rational function iff it can be represented as the ratio of two polynomials in z^{-1} (or z):

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

  - For LTI systems that are represented by LCCDEs, the z-Transform of the unit sample response $h(n)$, denoted $H(z) = \mathcal{Z}\{h(n)\}$, is rational

Poles and Zeros

Let $a_0, b_0 \neq 0$:

$$\begin{aligned}
X(z) = \frac{B(z)}{A(z)} &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \left(\frac{b_0 z^{-M}}{a_0 z^{-N}}\right) \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0) z^{N-1} + \cdots + a_N/a_0} \\
&= \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)} \\
&= G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}
\end{aligned}$$

Poles and Zeros of Rational Function

$$X(z) = Gz^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)} \text{ where } G \equiv \frac{b_0}{a_0}$$

- $X(z)$ has M finite zeros at $z = z_1, z_2, \ldots, z_M$
- $X(z)$ has N finite poles at $z = p_1, p_2, \ldots, p_N$

- For $N - M \neq 0$
  - if $N - M > 0$, there are $|N - M|$ zeros at origin, $z = 0$
  - if $N - M < 0$, there are $|N - M|$ poles at origin, $z = 0$

$$\boxed{\text{Total number of zeros} = \text{Total number of poles}}$$

Poles and Zeros

Example:

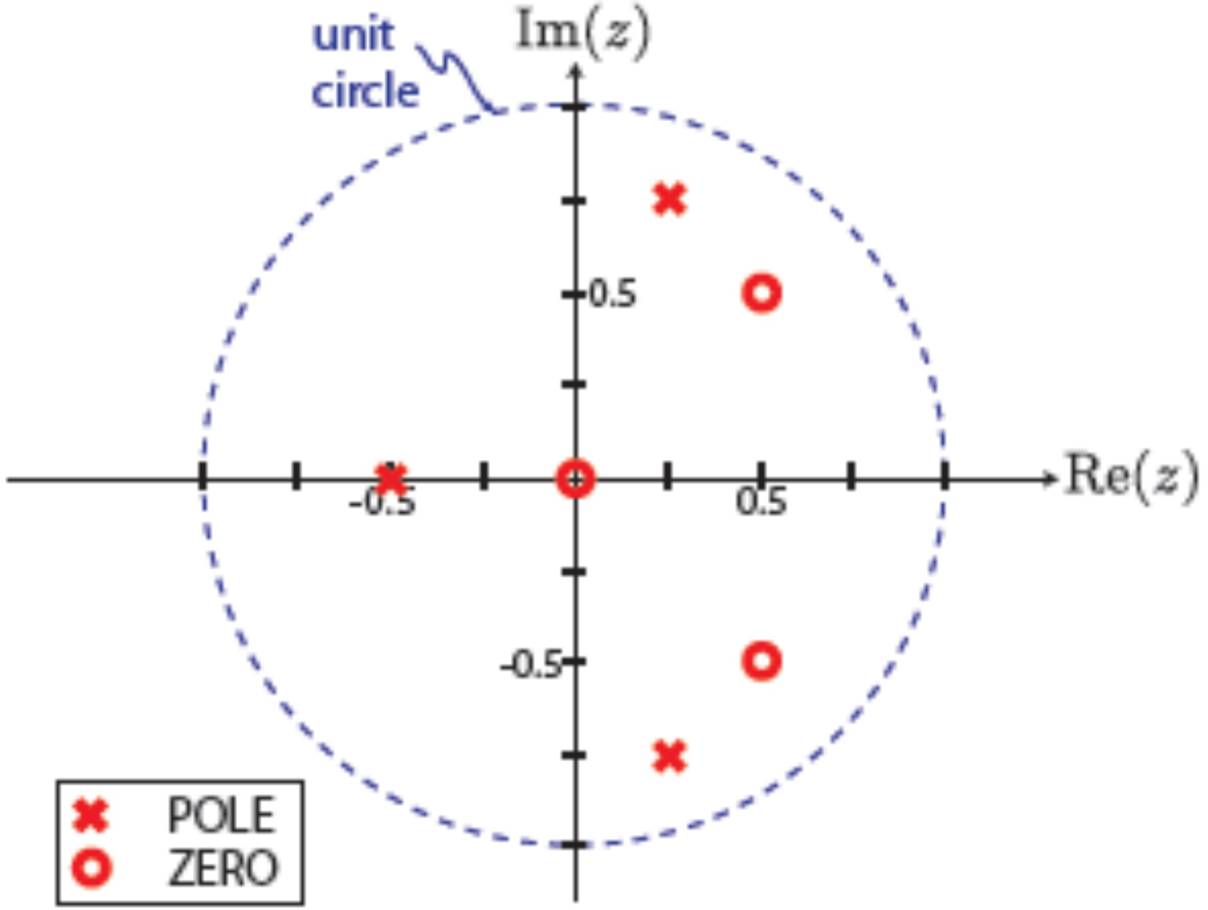
$$
\begin{aligned}
X(z) &= z\frac{2z^2-2z+1}{16z^3+6z+5} \\
&= (z-0)\frac{(z-(\frac{1}{2}+j\frac{1}{2}))(z-(\frac{1}{2}-j\frac{1}{2}))}{(z-(\frac{1}{4}+j\frac{3}{4}))(z-(\frac{1}{4}-j\frac{3}{4}))(z-(-\frac{1}{2}))}
\end{aligned}
$$

$$\boxed{\text{poles: } z=\tfrac{1}{4}\pm j\tfrac{3}{4}, -\tfrac{1}{2}}$$

$$\boxed{\text{zeros: } z=0, \tfrac{1}{2}\pm j\tfrac{1}{2}}$$

Pole-Zero Plot

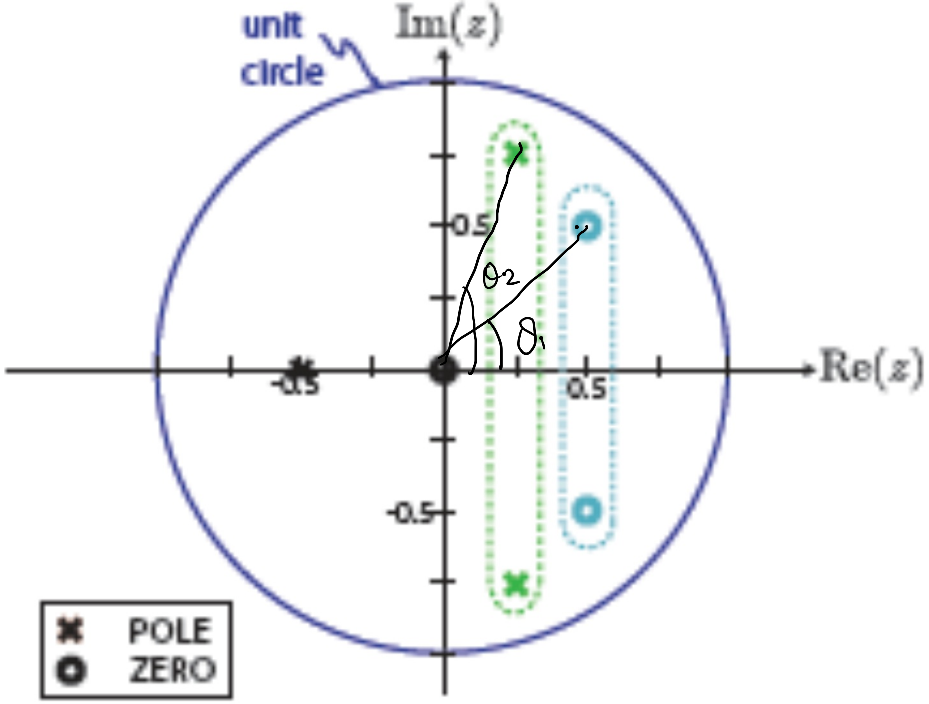
Example: $\boxed{\text{poles: } z = \frac{1}{4} \pm j\frac{3}{4}, -\frac{1}{2}}$, $\boxed{\text{zeros: } z = 0, \frac{1}{2} \pm j\frac{1}{2}}$



Pole-Zero Plot and Conjugate Pairs

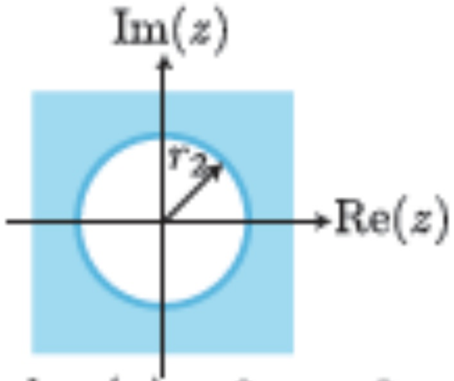
- For real time-domain signals, the coefficients of $X(z)$ are necessarily real
  - complex poles and zeros must occur in conjugate pairs
  - note: real poles and zeros do not have to be paired up

$$X(z) = z\frac{2z^2 - 2z + 1}{16z^3 + 6z + 5} \Longrightarrow$$



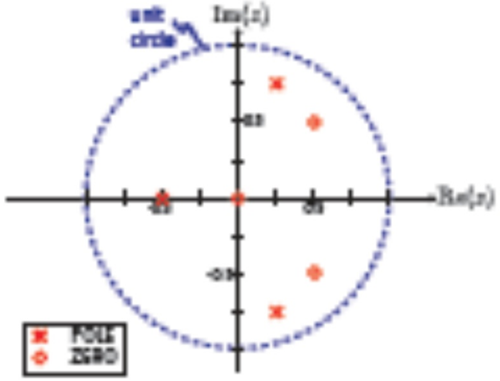
Pole-Zero Plot and the ROC

- Recall, for causal signals, the ROC will be the outer region of a disk



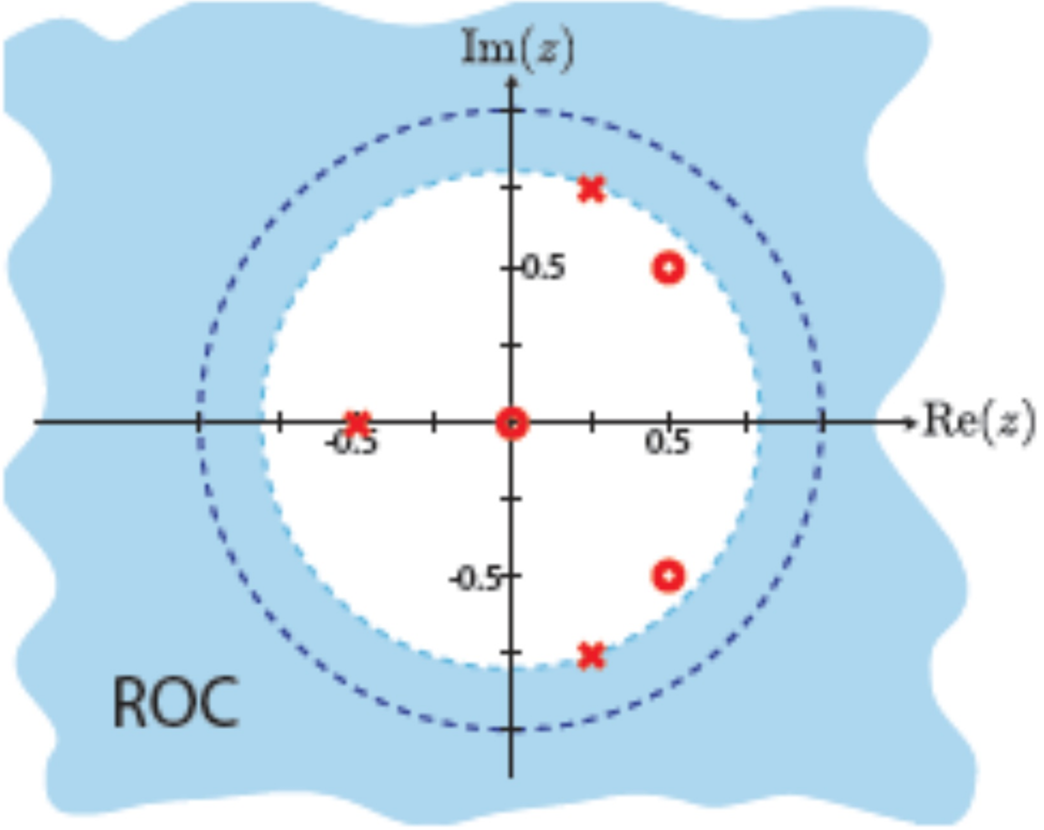
Causal: $x(n) = 0 \ \ n < 0$

- ROC cannot necessarily include poles ($\because X(p_k) = \infty$)



Pole-Zero Plot and the ROC

- Therefore, for a causal signal the ROC is the smallest (origin-centered) circle encompassing all the poles.



Causality and Stability

- Recall,

$$\text{LTI system is stable} \iff \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- Moreover,

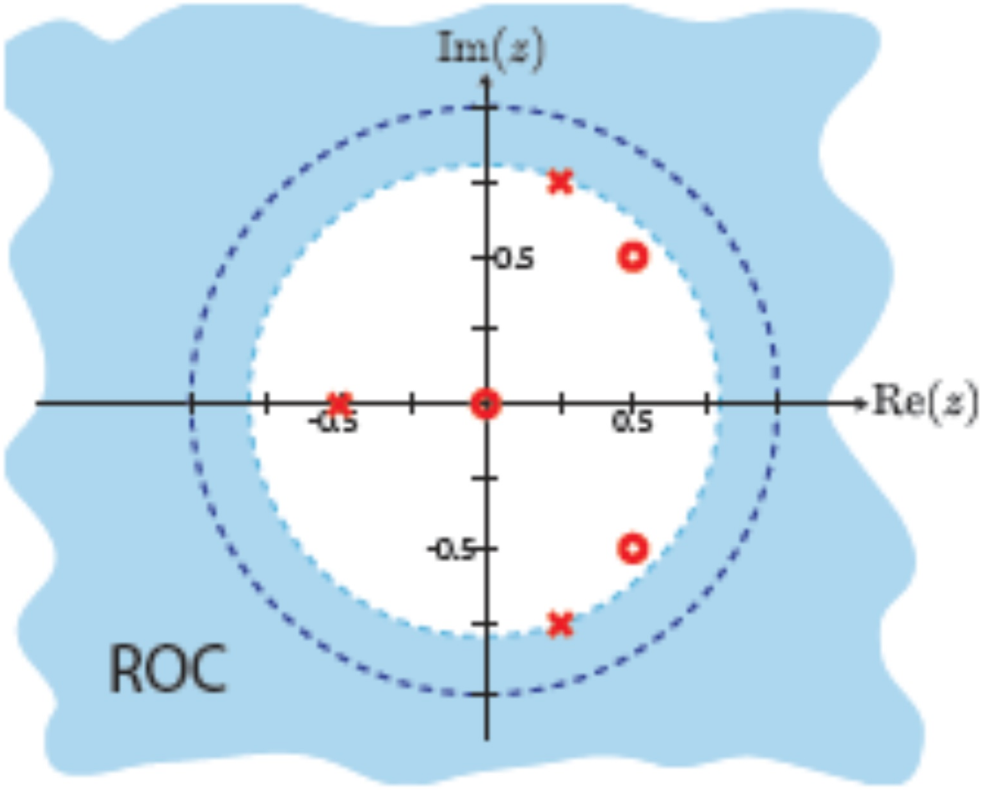
$$\begin{aligned} |H(z)| &= \left| \sum_{n=-\infty}^{\infty} h(n) z^{-n} \right| \le \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| \\ &= \sum_{n=-\infty}^{\infty} |h(n)| \qquad \text{for } |z| = 1 \end{aligned}$$

- It can be shown:

$$\text{LTI system is stable} \iff \sum_{n=-\infty}^{\infty} |h(n)| < \infty \iff \text{ROC of } H(z) \text{ contains unit circle}$$

Pole-Zero Plot, Causality and Stability

- For stable systems, the ROC will include the unit circle.



- For stability of a causal system, the poles must lie inside the unit circle.

System Function or Transfer Function

$x[n] \rightarrow \boxed{h} \rightarrow y[n]$

$$h(n) \overset{z}{\longleftrightarrow} H(z)$$

$$\text{time-domain} \overset{z}{\longleftrightarrow} z\text{-domain}$$

$$\text{impulse response} \overset{z}{\longleftrightarrow} \text{system function}$$

$h_{inv} * y[n] = x[n]$

$$y(n) = x(n) * h(n) \overset{z}{\longleftrightarrow} Y(z) = X(z) \cdot H(z)$$

Therefore,

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

$H_{inv}(z) = \frac{X(z)}{Y(z)}$

System Function

$$
\begin{aligned}
y(n) &= -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \\
\mathcal{Z}\{y(n)\} &= \mathcal{Z}\{-\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)\} \\
\mathcal{Z}\{y(n)\} &= -\sum_{k=1}^{N} a_k \mathcal{Z}\{y(n-k)\} + \sum_{k=0}^{M} b_k \mathcal{Z}\{x(n-k)\} \\
Y(z) &= -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)
\end{aligned}
$$

System Function

$$
\begin{aligned}
Y(z)+\sum_{k=1}^{N} a_k z^{-k} Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) \\
Y(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] &= X(z)\sum_{k=0}^{M} b_k z^{-k} \\
H(z)=\frac{Y(z)}{X(z)} &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\left[1+\sum_{k=1}^{N} a_k z^{-k}\right]}
\end{aligned}
$$

$$\text{LCCDE} \quad \longleftrightarrow \quad \text{Rational System Function}$$

Many signals of practical interest have a rational z-Transform.

Partial Fraction Expansion

1. Find the distinct poles of $X(z)$: $p_1, p_2, \ldots, p_K$ and their corresponding multiplicities $m_1, m_2, \ldots, m_K$.

2. The partial-fraction expansion is of the form:

$$z^{-R}X(z) = \sum_{k=1}^{K} \left(\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \cdots + \frac{A_{m_k k}}{(z - p_k)^{m_k}} \right)$$

   where p_k is an m_kth order pole (i.e., has multiplicity m_k) and R is selected to make $z^{-R}X(z)$ a strictly proper rational function.

3. Use an appropriate approach to compute $\{A_{ik}\}$

Examples

- Find the inverse z-transform of
  - X(z) = $z^2\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})(1+2z^{-1})$
  - $X(z) = log\left(\frac{1}{1-az^{-1}}\right), |z| > |a|$
  - $X(z) = log\left(\frac{1}{1-za^{-1}}\right), |z| < |a|$
  - $X(z) = \frac{z}{2z^2-3z+1}, |z| < \frac{1}{2}$
  - $X(z) = \frac{z}{2z^2-3z+1}, |z| > 1$

$$X(z) = z^{2}\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)\left(1 + 2z^{-1}\right)$$

$$= z^{2}\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1} - 2z^{-2}\right)$$

$$= z^{2}\left(1 + z^{-1} - 2z^{-2} - \frac{1}{2}z^{-1} + \frac{1}{2}z^{2} + z^{-3}\right)$$

$$= z^{2}\left(1 + \frac{1}{2}z^{-1} - \frac{5}{2}z^{-2} + z^{-3}\right)$$

$$= z^{2} + \frac{1}{2}z - \frac{5}{2} + z^{-1}$$

$$x[n] = \delta[n+2] + \frac{1}{2}\delta[n+1] - \frac{5}{2}\delta[n] + \delta[n-1]$$

$$= \left\{ \cdots 0, 1, \frac{1}{2}, \underset{\uparrow}{-5/2}, 1, 0, 0 \cdots \right\}$$

$$X(z) = \log\left(\frac{1}{1-az^{-1}}\right) \qquad |z| > |a|$$

$\boxed{\text{Causal}}$

$$\log(1-r) = -\sum_{n=1}^{\infty} \frac{1}{n} r^n \qquad |r| < 1.$$

$$X(z) = -\log(1-az^{-1}) \qquad |z| > |a|.$$

$$|az^{-1}| < 1.$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} (az^{-1})^n. = \sum_{n=1}^{\infty} \frac{1}{n} a^n z^{-n}$$

$$x[n] = 0 \qquad n \leq 0$$

$$= \frac{1}{n} a^n \qquad n \geq 1$$

$$x[n] = \frac{1}{n} a^n u[n-1]$$

$$X(z) = \log\left(\frac{1}{1-a^{-1}z}\right) \qquad |z| < |a|.$$

anti causal

$$= -\log(1-a^{-1}z) \qquad |z| < |a|.$$

$$\boxed{|a^{-1}z| < 1}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}(a^{-1}z)^{n}.$$

$m = -n$

$$= \sum_{m=-1}^{-\infty} \frac{-1}{m} a^{m} z^{-m}$$

$$x[n] = 0 \qquad n \geq 0$$

$$= -\frac{1}{n}a^{n} \qquad n \leq -1.$$

$$x[n] = -\frac{1}{n}a^{n}u[-n-1]$$

$$X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| < 1/2.$$

$$= \frac{z}{(2z-1)(z-1)} = \frac{1/2\, z}{z^2 - \frac{3}{2}z + 1/2} = \frac{1/2\, z}{(z-1)(z-1/2)}$$

$$= \frac{A_1}{(z-1)} + \frac{A_2}{(z-1/2)}$$

$$A_1 = X(z)(z-1)\Big|_{z=1}$$

$$= \frac{1/2\, z}{(z-1/2)}\Big|_{z=1} = \frac{1/2}{1/2} = 1.$$

$$\frac{X(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(2z-1)}.$$

$$A_2 =$$

$$= \frac{1}{z-1} - \frac{2}{2z-1}. = \frac{2z-1-2z+2}{}$$

$$X(z) = \frac{z}{z-1} - \frac{2z}{2z-1} = \frac{1}{1-z^{-1}} - \frac{1}{1-1/2\, z^{-1}}.$$

$$\boxed{\begin{aligned} x[n] &= -u[-n-1] \\ &+ \left(\tfrac{1}{2}\right)^n u[-n-1] \end{aligned}}$$

$|z| > 1.$

$$H(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$h[n] = u[n] - \left(\frac{1}{2}\right)^n u[n]$$

$\frac{1}{2} < |z| < 1.$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - u[-n-1]$$

Examples

- Power series

- $log(1 - r) = -\sum_{n=1}^{\infty} \frac{1}{n} r^n$