Problem Set ~ 8

9. If the cartesian coordinate of a random point are independent standard normal variates, show that its polar coordinates are also independent and find their distributions.

Consider the random variables X, Y, R and θ s.t (X,Y) represents the cartesian co-ordinates and (R,θ) represents polar co-ordinates.

Given that X and Y are independent :

$$\boxed{\begin{aligned} f_X &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty \\ f_Y &= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad -\infty < y < \infty \end{aligned}}$$

$$f_{X,Y} = f_X \cdot f_Y$$

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2}, \quad -\infty < x < \infty, \; -\infty < y < \infty$$

We know that :

$X = R\cos\theta$

$Y = R\sin\theta$

$x = r\cos\theta$, $y = r\sin\theta$ where $0 < r < \infty$, $0 \le \theta < 2\pi$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r > 0$$

$$f_{R,\theta} = f_{X,Y}(x,y) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right|$$

$$= \frac{r}{2\pi} e^{-r^2/2}$$

$$f_R(r) = \int_0^{2\pi} \frac{r}{2\pi} e^{-r^2/2} \, d\theta = \underline{\underline{r e^{-r^2/2}}}, \quad 0 < r < \infty$$

$$f_\theta(\theta) = \int_0^{\infty} \frac{r}{2\pi} e^{-r^2/2} \, dr$$

$$= \frac{1}{2\pi} \underbrace{\int_0^{\infty} e^{-r^2/2} \, d\left(\frac{r^2}{2}\right)}_{\Gamma(1)}$$

$$= \underline{\underline{\frac{1}{2\pi}}}$$

Clearly, $f_{R,\theta}(r,\theta) = f_R(r) \cdot f_\theta(\theta)$

$\Rightarrow$ R and θ are independent.

10. If X_1 and X_2 and independent random variates each having density functions $2xe^{-x^2}$, $(0 < x < \infty)$, find the density function of $\sqrt{X_1^2 + X_2^2}$.

Given that X_1 and X_2 are independent,

Let $X_1 = R\cos\theta$ and $X_2 = R\sin\theta$

So, $R = \sqrt{X_1^2 + X_2^2}$

$0 < r < \infty$; $0 < \theta < \pi/2$

$x_1 = r\cos\theta$, $x_2 = r\sin\theta$

$f_{X_1,X_2} = f_{X_1} \cdot f_{X_2}$

$f_{X_1,X_2}(x_1,x_2) = 4x_1x_2\, e^{-(x_1^2+x_2^2)}$

$0 < x_1 < \infty$

$0 < x_2 < \infty$

$$\frac{\partial(x_1,x_2)}{\partial(r,\theta)} = r > 0$$

$$f_{R,\theta}(r,\theta) = f_{X_1,X_2}(x_1,x_2)\left|\frac{\partial(x_1,x_2)}{\partial(r,\theta)}\right|$$

$$= 4r^2\sin\theta\cos\theta\, e^{-r^2}\cdot r$$

$$= 2r^3e^{-r^2}(2\sin\theta\cos\theta)$$

$$= 2r^3e^{-r^2}\sin 2\theta \qquad \begin{array}{l} 0 < r < \infty \\ 0 < \theta < \pi/2 \end{array}$$

$$f_R(r) = \int_0^{\pi/2} 2r^3e^{-r^2}\sin 2\theta\, d\theta$$

$$= 2r^3e^{-r^2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$\boxed{= 2r^3e^{-r^2} \qquad 0 < r < \infty}$$

11. Consider the random experiment of throwing a pair of dice. Let X denote the number of sixes and Y denote the number of fives that turn up. Find the joint p.m.f. of the two-dimensional random variable (X, Y) and the marginal p.m.f.s of X and Y.

Draw the probability table

$X \backslash Y$	0	1	2	$P(X=x_i)$
0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$	$25/36$
1	$\frac{8}{36}$	$\frac{2}{36}$	0	$\frac{10}{36}$
2	$\frac{1}{36}$	0	0	$\frac{1}{36}$
$P(Y=y_j)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$	1

$$P(X+Y\geq 2) = P(X=0, Y=2) + P(X=1, Y=2) + P(X=2, Y=2) + P(X=1, Y=1) + P(X=2, Y=1) + P(X=2, Y=0)$$

$$= \frac{1}{36} + 0 + 0 + \frac{2}{36} + 0 + \frac{1}{36}$$

$$= \underline{\underline{\frac{1}{9}}}$$