Signals 3

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Disclaimer

- Source : Prof. Paul Cuff’s (Princeton University) Lecture Notes#2
  ELE 301 (Signals and Systems Fall 2011)

Tutorial session

- Rescheduled to tomorrow same time.

$\cos\left(\frac{\pi}{2} n\right)$

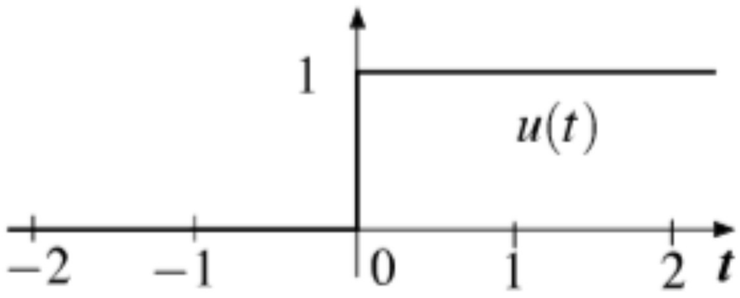
$\cos\left(\left(\frac{\pi}{2} + 2\pi\right) n\right)$

Unit Step Functions

- The *unit step function* $u(t)$ is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Also known as the *Heaviside step function*.
- Alternate definitions of value exactly at zero, such as $1/2$.



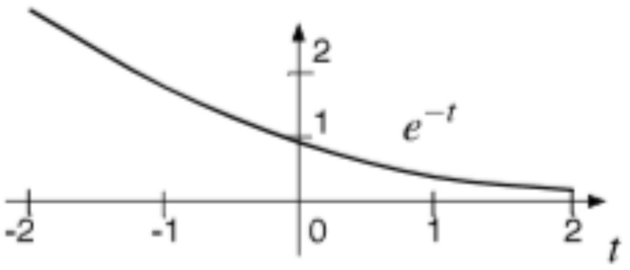
Uses for the unit step:

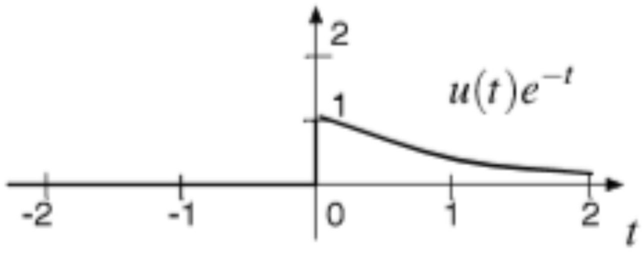
- Extracting part of another signal. For example, the piecewise-defined signal

$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

can be written as

$$x(t) = u(t)e^{-t}$$



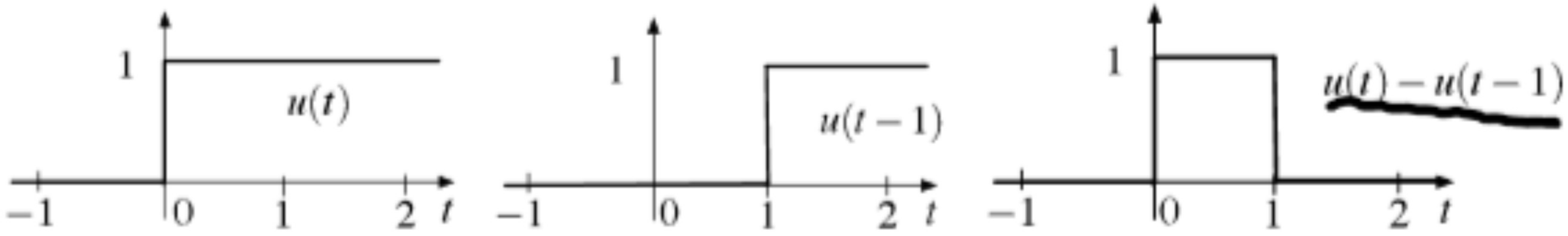


- Combinations of unit steps to create other signals. The offset rectangular signal

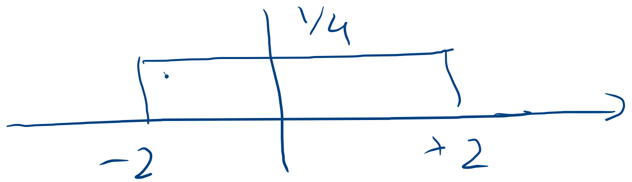
$$x(t) = \begin{cases} 0, & t \geq 1 \\ 1, & 0 \leq t < 1 \\ 0, & t < 0 \end{cases}$$

can be written as

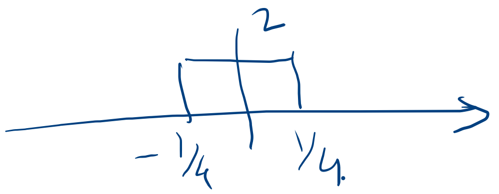
$$x(t) = u(t) - u(t-1).$$



$a = 4$



$a = 1/2$

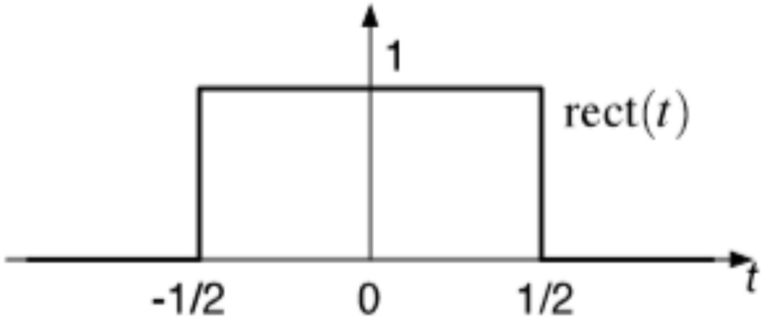


Unit Rectangle

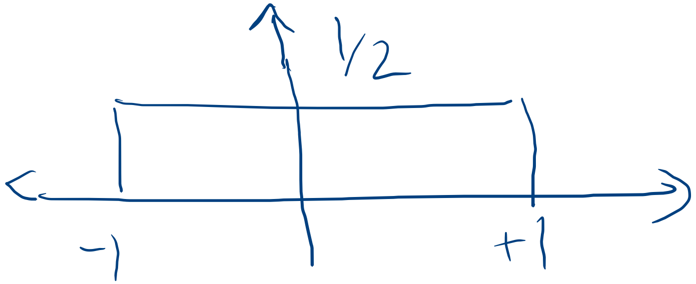
$$\text{rect}_a(t) = \frac{1}{a}\,\text{rect}\left(\frac{t}{a}\right)$$

Unit rectangle signal:

$$\text{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$



$a = 2$



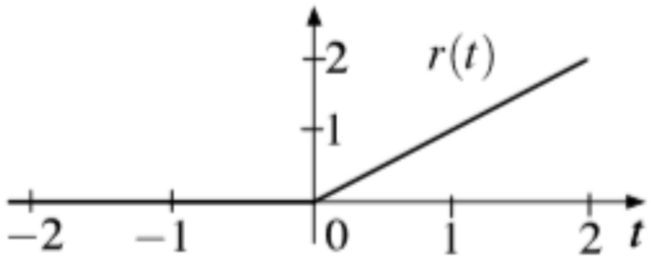
Unit Ramp

- The *unit ramp* is defined as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- The unit ramp is the integral of the unit step,

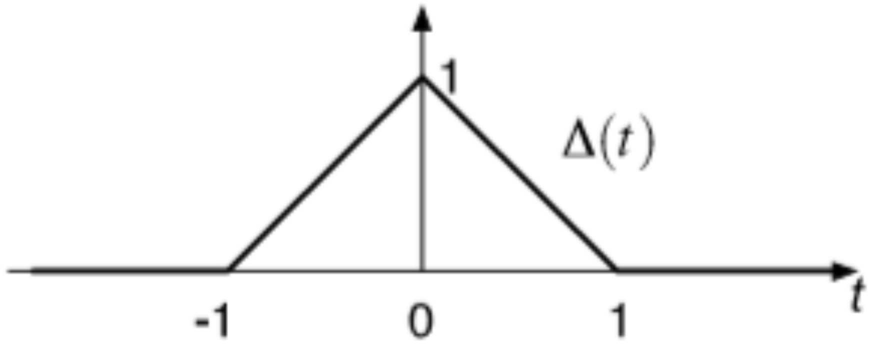
$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$



Unit Triangle

Unit Triangle Signal

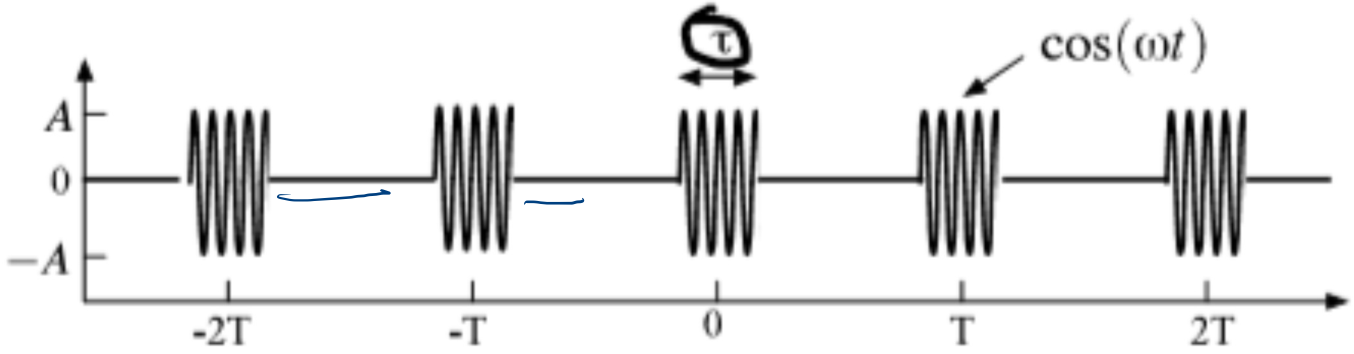
$$\Delta(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



More Complex Signals

Many more interesting signals can be made up by combining these elements.

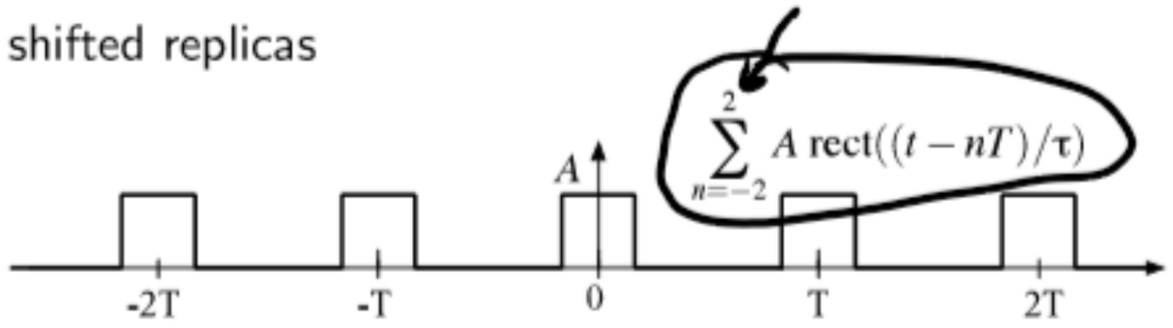
Example: Pulsed Doppler RF Waveform (we'll talk about this later!)

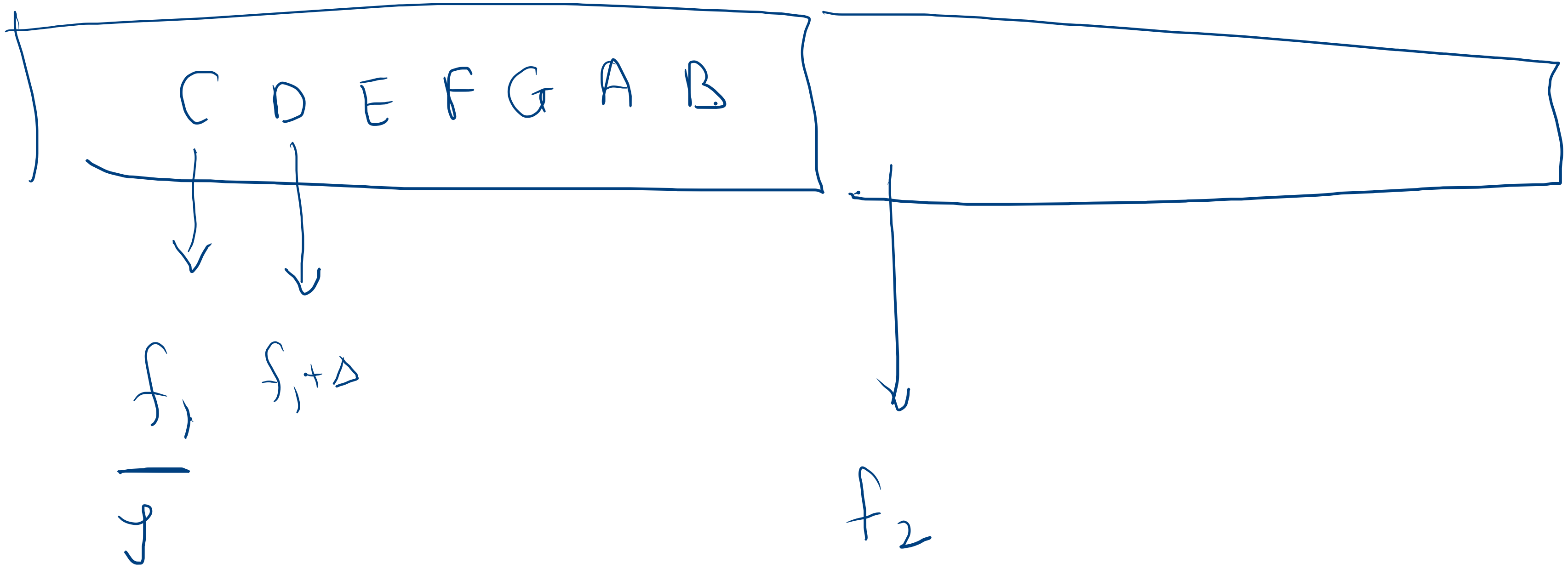


Start with a simple rect(t) pulse

Scale to the correct duration and amplitude for one subpulse

Combine shifted replicas



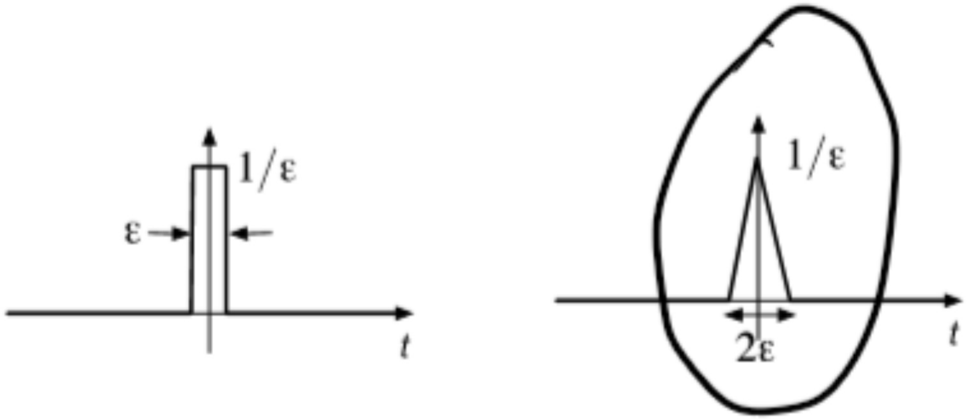

C D E F G A B
f_1
$f_1+\Delta$
f_2

Impulsive signals

(Dirac's) **delta function** or **impulse** δ is an *idealization* of a signal that

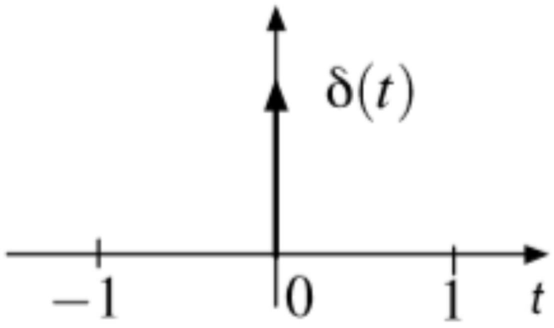
- is very large near $t = 0$
- is very small away from $t = 0$
- has integral 1

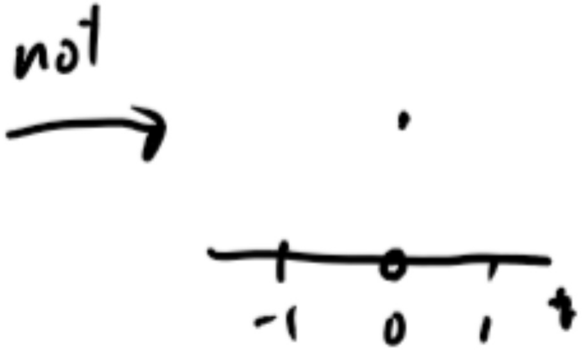
for example:

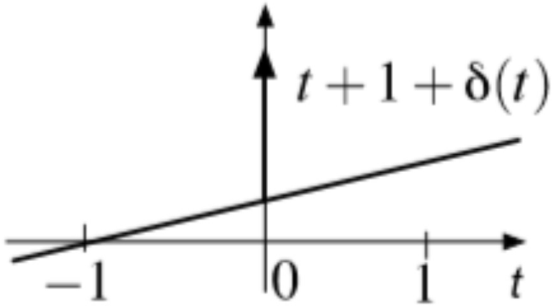


- the exact shape of the function doesn't matter
- ϵ is small (which depends on context)

On plots δ is shown as a solid arrow:







Formal properties

$\delta^2(t)$

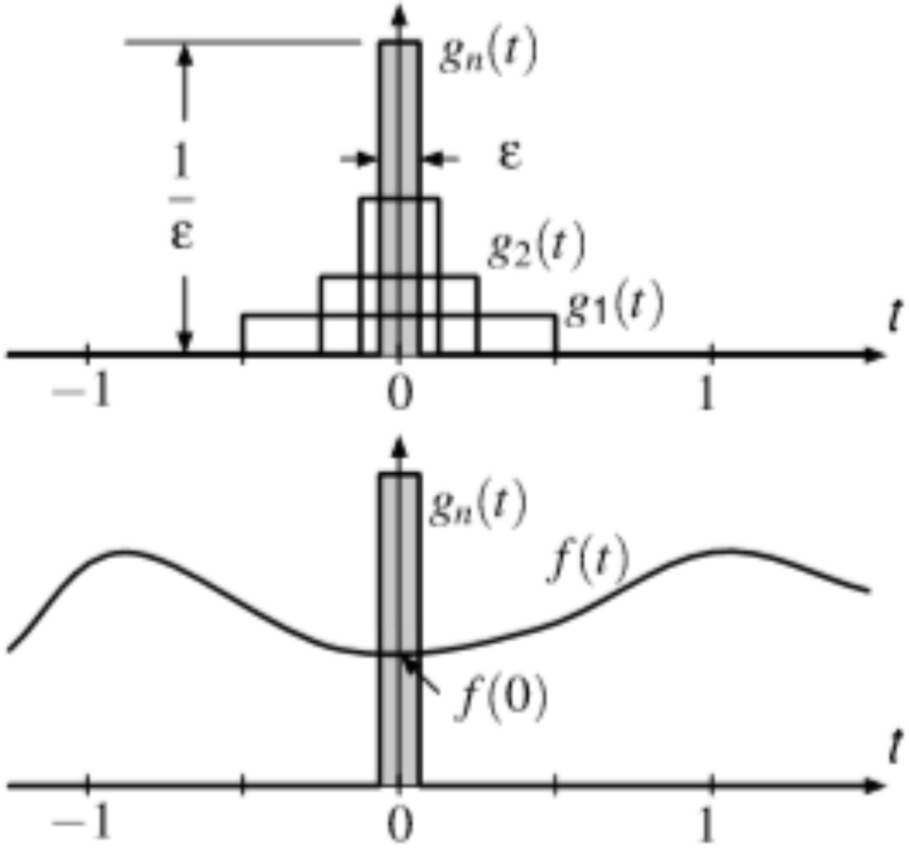
Formally we **define** δ by the property that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0)$$

provided f is continuous at $t = 0$

idea: δ acts over a time interval very small, over which $f(t) \approx f(0)$

- $\delta(t)$ is not really defined for any t, only its behavior in an integral.
- Conceptually $\delta(t) = 0$ for $t \neq 0$, infinite at $t = 0$, but this doesn't make sense mathematically.



$g_n(t)$
ε
$\frac{1}{\varepsilon}$
$g_2(t)$
$g_1(t)$
t
-1
0
1
$g_n(t)$
$f(t)$
$f(0)$
t
-1
0
1

Scaled impulses

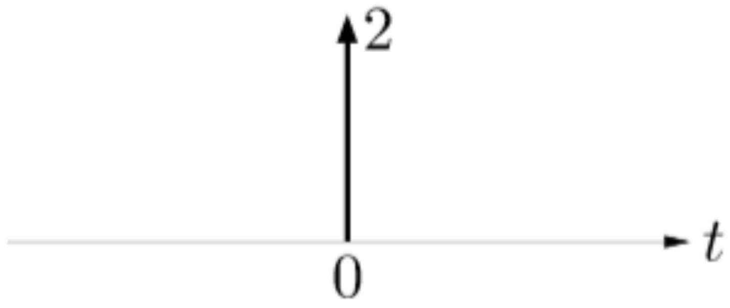
$\alpha\delta(t)$ is an impulse at time T, with *magnitude* or *strength* α

We have

$$\int_{-\infty}^{\infty} \alpha\delta(t)f(t)\, dt = \alpha f(0)$$

provided f is continuous at 0

On plots: write area next to the arrow, *e.g.*, for $2\delta(t)$,



Sifting property

- The signal $x(t) = \delta(t - T)$ is an impulse function with impulse at $t = T$.
- For f continuous at $t = T$,
$$\int_{-\infty}^{\infty} f(t)\delta(t - T)\, dt = f(T)$$
- Multiplying by a function $f(t)$ by an impulse at time T and integrating, extracts the value of $f(T)$.
- This will be important in modeling sampling later in the course.

Limits of Integration

The integral of a δ is non-zero only if it is in the integration interval:

- If $a < 0$ and $b > 0$ then

$$\int_a^b \delta(t)\; dt = 1$$

$$= \int_{-\infty}^{\infty} \delta(t) \left(u(t-a) - u(t-b) \right) dt$$

  because the δ is within the limits.

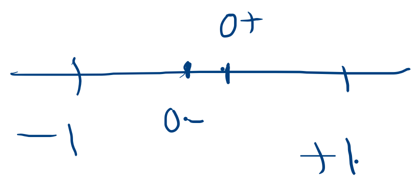
- If $a > 0$ or $b < 0$, and $a < b$ then

$$\int_a^b \delta(t)\; dt = 0$$

  because the δ is outside the integration interval.

- **Ambiguous** if $a = 0$ or $b = 0$

Our convention: to avoid confusion we use limits such as $a-$ or $b+$ to denote whether we include the impulse or not.



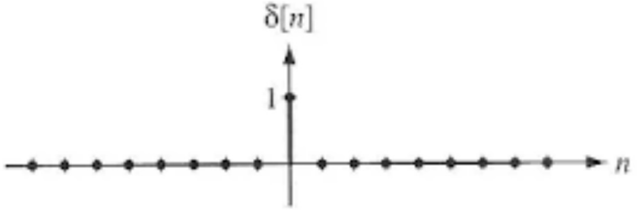
$$\int_{0+}^{1} \delta(t)\, dt = 0, \quad \int_{0-}^{1} \delta(t)\, dt = 1, \quad \int_{-1}^{0-} \delta(t)\, dt = 0, \quad \int_{-1}^{0+} \delta(t)\, dt = 1$$

example:

$$\begin{aligned}
&\int_{-2}^{3} f(t)(2 + \delta(t+1) - 3\delta(t-1) + 2\delta(t+3))\, dt \\
&= 2\int_{-2}^{3} f(t)\, dt + \int_{-2}^{3} f(t)\delta(t+1)\, dt - 3\int_{-2}^{3} f(t)\delta(t-1)\, dt \\
&\quad + 2\int_{-2}^{3} f(t)\delta(t+3))\, dt \\
&= 2\int_{-2}^{3} f(t)\, dt + f(-1) - 3f(1)
\end{aligned}$$

Discrete-time unit Impulse

$$\delta[n] = \begin{cases} 1\ , n = 0 \\ 0\ , n \neq 0 \end{cases}$$



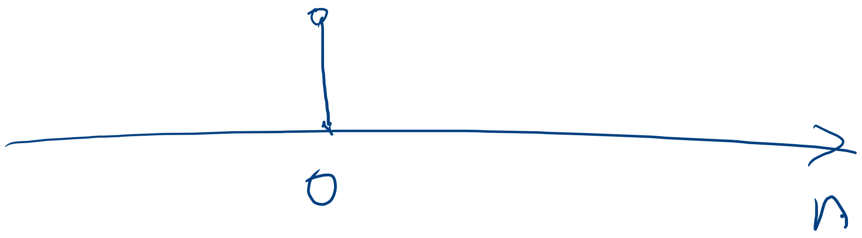
The signal is Non-zero (takes the value 1) only when the argument to the function is 0

Plot δ [n-10], δ [n+100]

What happens when we multiply x[n] with δ [n] ???

What happens when we multiply x[n] with δ [n-50] ???

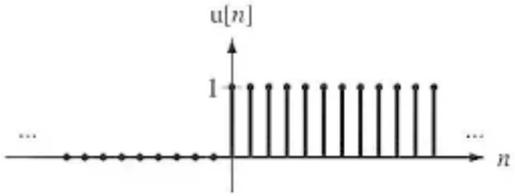
Unit-step



$\delta[n] = 1 \quad n=0$

$\quad\quad\;\; 0 \quad o/w$

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



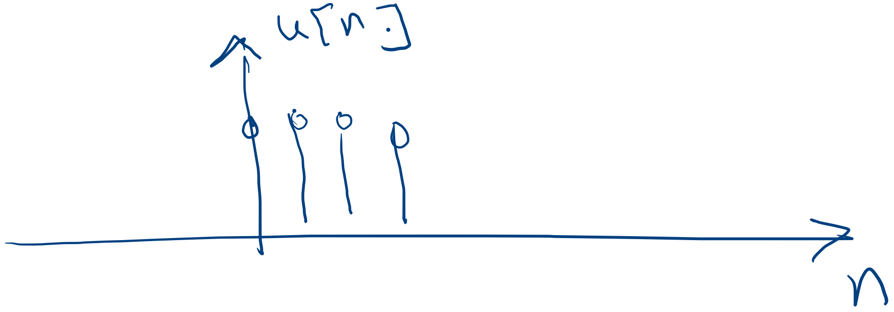
The signal is Non-zero (1) only when the argument to the function is NOT NEGATIVE

$u[n] = \sum_{k} \delta[n-k]$

Plot u [n-10], u [n+100]

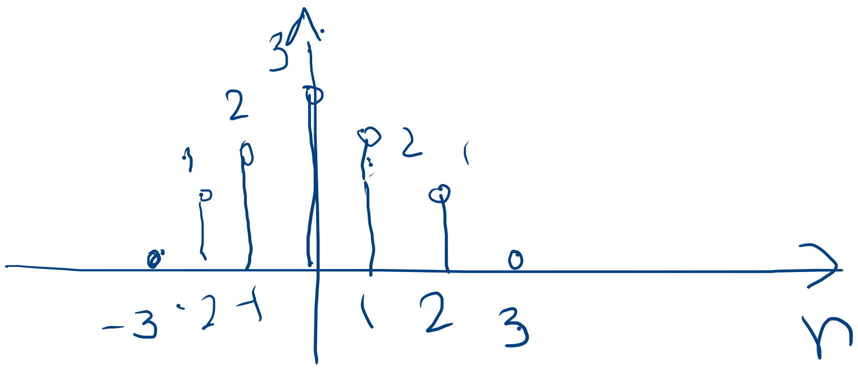
What happens when we multiply x[n] with u [n] ???

What happens when we multiply x[n] with u [n-50] ???



$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \cdots$$

$$= \sum_{k=0}^{\infty} \delta[n-k]$$



$$x[n] = 1\,\delta[n+2] + 2\,\delta[n+1] + 3\,\delta[n] + 2\,\delta[n-1] + \delta[n-2]$$

Relation between u[n] and δ[n]

- Plot u[n]
- Plot u[n-1]
- Now look at u[n]-u[n-1] $= \delta[n]$
- What do we get ?
- $\delta[n] = u[n]-u[n-1]$

- Conversely, we can write, $u[n] = \Sigma\ \delta[n - k]$

Limits on k ??

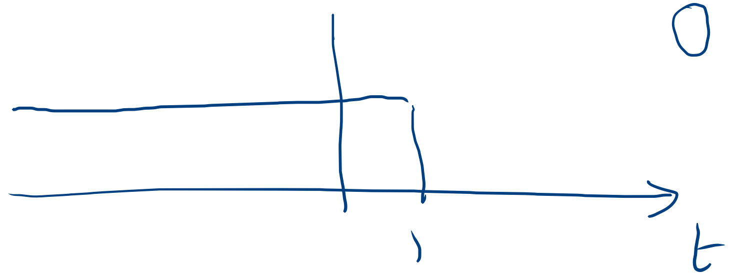
Problems

$u(t) = 1 \quad t \geq 0$

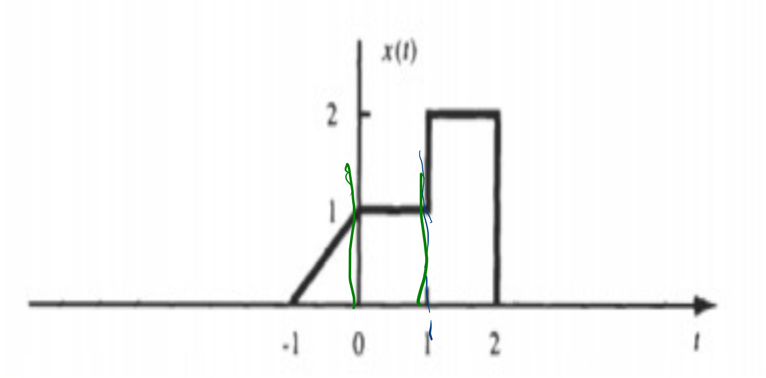
$\quad\;\; = 0 \quad t < 0$

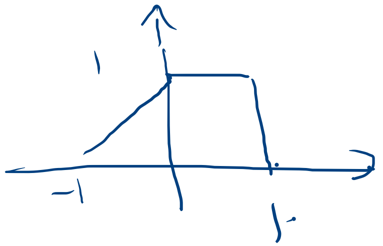
$u(1-t) = 1 \quad t \leq 1$

$\qquad\quad\;\;\; 0 \quad t > 1$



1. A continuous-time signal $x(t)$ is shown below. Sketch the following signals:



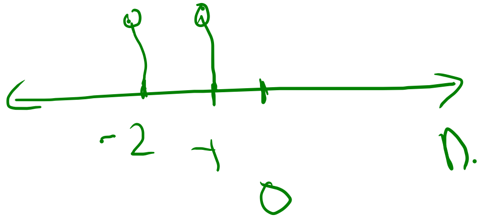


a) $x(t)u(1-t)$

b) $x(t)[u(t) - u(t-1)]$

Problems

$u[n+2] - u[n]$



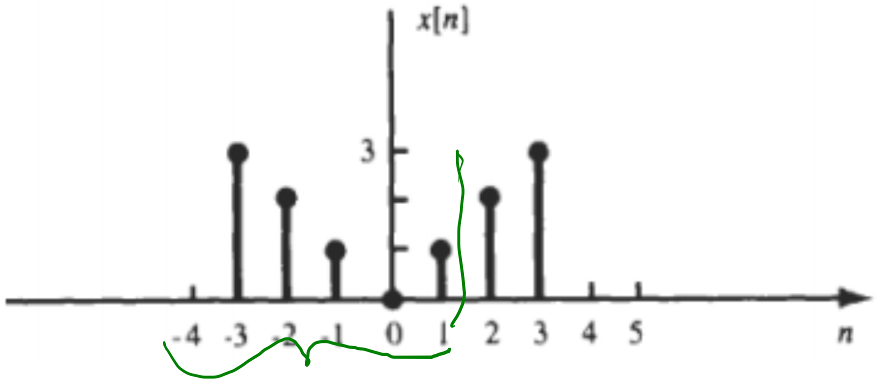
$n = 0$

$u[2] - u[0] = 1 - 1 = 0$

$n = -1$

$u[1] - u[-1] = 1 - 0 = 1$

2. A discrete-time signal $x[n]$ is shown below. Sketch the following signals:



a) $x[n]u[1-n]$
b) $x[n]\{u[n+2] - u[n]\}$
c) $x[n]\delta[n-1]$

What have we covered so far ?

- Signals
    - Representation
    - Classification
        - Cts-time, Discrete
        - Energy, Power
        - Even, Odd
        - Periodic, Non-periodic
- Special signals
    - Complex exponentials
    - Unit step
    - Impulse

What Next ?

- Systems – Characteristics
  - Linear
  - Time-Invariance
  - Stability
  - Causal
  - Memory