Problem Set 4

Solutions

September 28, 2021

Contents

Question 4.7

If X is poisson distribution with parameter μ, prove that
$P(X \leq n) = \frac{1}{n!} \cdot \int_{\mu}^{\infty} e^{-x} \cdot x^n \, \mathrm{d}x$

Solution 4.7

$X \sim Poisson(\mu)$,

$$f(x) = \begin{cases} \frac{e^{-\mu}\mu^x}{x!}, & \text{for } x = 0, 1, 2, 3, \cdots \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{aligned} P(x \leq n) &= P(x - 0) + P(x - 1) + \cdots + P(x = n) \\ &= e^{-\mu} + e^{-\mu}\mu + \frac{e^{-\mu}\mu^2}{2!} + \cdots + \frac{e^{-\mu}\mu^n}{n!} \end{aligned}$$

Solution 4.7

$$
\begin{aligned}
I_n &= \frac{1}{n!}\int_{\mu}^{\infty} e^{-x}x^n dx \\
&= \frac{1}{n!}\left[-e^{-x}x^n\right]_{\mu}^{\infty} + \frac{n}{n!}\int_{\mu}^{\infty} x^{n-1}e^{-x}dx \\
\Rightarrow I_n &= \frac{e^{-\mu}\mu^n}{n!} + I_{n-1} \\
I_1 &= \frac{e^{-\mu}\mu^1}{1!} + I_0 \\
I_2 &= \frac{e^{-\mu}\mu^2}{2!} + I_1 \\
&\vdots \\
I_n &= \frac{e^{-\mu}\mu^n}{n!} + I_{n-1}
\end{aligned}
$$

Solution 4.7

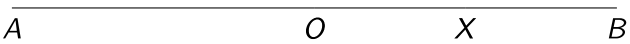
$$
\begin{aligned}
I_n &= I_0 + \sum_{i=1}^{n} \frac{e^{-\mu}\mu^i}{i!} = e^{-\mu} + \sum_{i=1}^{n} e^{-\mu}\frac{\mu i}{i!} \\
&= P(X \leq n)
\end{aligned}
$$

Question 4.9

A point X is chosen at random on a line segment AB whose middle point is O. Find the probability that AX, BX and AO form the sides of a triangle.

Solution 4.9

Let X be a random point on the straight line,



Given AX, BX and AO form the sides of a triangle, we'll take the help of triangle inequality.

Solution 4.9

As it's obvious to notice, $AX + BX > OB$.

Solution 4.9

Next we've, $BX + OB > AX$,

$$\begin{aligned} BX + OB &= AB - AX + \frac{AB}{2} \\ &= \frac{3AB}{2} - AX \\ \Rightarrow AX &< \frac{3AB}{4} \end{aligned}$$

Solution 4.9

Next we've, $AX + OB > BX$,

$$AX + OB = AB - BX + \frac{AB}{2}$$
$$= \frac{3AB}{2} - BX$$
$$\Rightarrow BX < \frac{3AB}{4}$$

Using $AX < \frac{3AB}{4}$ & $BX < \frac{3AB}{4}$, we can deduce the length of region where X can lie, to be $\frac{AB}{2}$.

$\therefore P(triangle) = \frac{\frac{AB}{2}}{AB} = \frac{1}{2}$

Question 4.10

A point P is chosen at random on a circle of radius 'a' and A is a fixed point on the circle. Find the probability that the chord AP will exceed the length of an equilateral triangle inscribed in the circle.

Solution 4.10

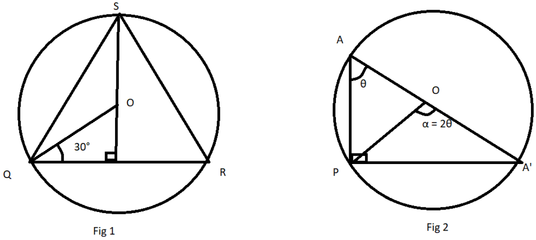


Figure: Circles

Solution 4.10

Let length of an equilateral triangle be a, and a can be computed as $2r\cos 30^\circ = \sqrt{3}r$
From the figure, $AP = AA'\cos\theta$, where θ lies between 0 and $\frac{\pi}{2}$. An observation can be made, as θ increases, length of AP decreases, and $AA' = 2r$.

Solution 4.10

$\Rightarrow 2r\cos\theta > \sqrt{3}r$
$\Rightarrow \alpha < \frac{\pi}{3}$

Solution 4.10

Thus, probability is $\frac{2\alpha}{2\pi} = \frac{1}{3}$