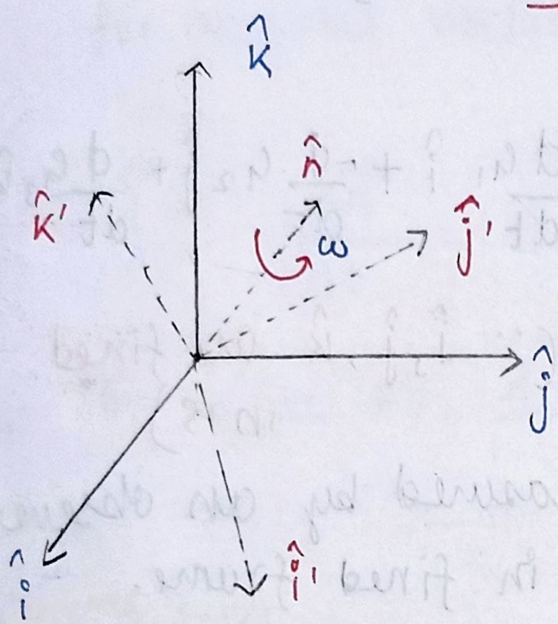
(A) Rotating frames of Reference :-

consider a fixed frame 'S' with $(\hat{i}, \hat{j}, \hat{k})$ unit vectors.



→ Frame S' originally coincident with S is rotating about the common origin with velocity $\vec{\omega} = \hat{n}\,\omega$

Now, consider some vector $\vec{G}$.

physically, a change in coordinate system should not change anything.

i.e $\boxed{\vec{G}_{fixed} = \vec{G}_{rot}}$ (but individual components in S, S' might differ)

$$\vec{G}_{fixed} = G_1\hat{i} + G_2\hat{j} + G_3\hat{k}$$

$$\vec{G}_{rot} = G_1'\hat{i} + G_2'\hat{j} + G_3'\hat{k}$$

→ Similarly time derivatives of $\vec{G}_{fixed}$ & $\vec{G}_{rot}$ should be the same.

i.e $\frac{d}{dt}[\vec{G}_{fixed}] = \frac{d}{dt}[\vec{G}_{rot}]$

LHS $= \left[\frac{d\vec{G}}{dt}\right]_{fixed} = \frac{dG_1}{dt}\hat{i} + \frac{d}{dt}G_2\hat{j} + \frac{dG_3}{dt}\hat{k}$

($\because \hat{i}, \hat{j}, \hat{k}$ are fixed in S)

$\because$ measured by an observer in fixed frame.

RHS

$= \frac{d}{dt}[\vec{G}_{rot}]$

$= \frac{d}{dt}(G_1'\hat{i}' + G_2'\hat{j}' + G_3'\hat{k}')$

$= \left(\frac{dG_1'}{dt}\hat{i}' + \frac{dG_2'}{dt}\hat{j}' + \frac{dG_3'}{dt}\hat{k}'\right) +$

$G_1'\frac{d}{dt}\hat{i}' + G_2'\frac{d}{dt}\hat{j}' + G_3'\frac{d}{dt}\hat{k}'$

$$= \left(\frac{d\vec{G}}{dt}\right)_{rot} + G_1' \frac{d\hat{i}'}{dt} + G_2' \frac{d\hat{j}'}{dt} + G_3' \frac{d\hat{k}'}{dt}$$

Recall,

for an orbit vector $\vec{A}$: $\delta\vec{A} = (\delta\theta\, \hat{n}) \times \vec{A}$

OR : $$\frac{\Delta\vec{A}}{\Delta t} = \hat{n}\frac{\Delta\theta}{\Delta t} \times \vec{A}$$

So, as $\Delta t \to 0$, $\frac{d}{dt}\vec{A} = \vec{\omega} \times \vec{A}$

$$\vec{\omega} = \hat{n}\frac{d\theta}{dt}$$

general form :

$$\frac{d(\ )}{dt} = \vec{\omega} \times (\ )$$

Example ke liye

$$\frac{d\hat{i}'}{dt} = \vec{\omega} \times \hat{i}'$$

$$\therefore \left(\frac{d\vec{G}}{dt}\right)_{fixed} = \left(\frac{d\vec{G}}{dt}\right)_{rot} + \vec{\omega} \times \underbrace{(G_1'\hat{i}' + G_2'\hat{j}' + G_3'\hat{k}')}_{\vec{G}' \equiv \vec{G}}$$

$$= \left(\frac{d\vec{G}}{dt}\right)_{rot} + \vec{\omega} \times \vec{G}$$

if $\vec{G} = \vec{r}$, we have :

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}$$

OR $\underbrace{\vec{v}_{fixed}}_{\equiv \vec{v}_0} = \underbrace{\vec{v}_{rot}}_{\equiv \vec{v}} + \vec{\omega} \times \vec{r}$

If $\vec{G} = \vec{v}_0$,

$$\frac{d}{dt}\vec{v}_0 = \left(\frac{d}{dt}\vec{v}_0\right)_{rot} + (\vec{\omega} \times \vec{v}_0)$$

$$= \frac{d}{dt}(\vec{v} + \vec{\omega} \times \vec{r}) + \vec{\omega} \times (\vec{v} + \vec{\omega} \times \vec{r})$$

$$\frac{d\vec{v}_0}{dt} = \frac{d}{dt}(\vec{v} + \vec{\omega} \times \vec{r}) + \vec{\omega} \times (\vec{v} + \vec{\omega} \times \vec{r})$$

$$= \frac{d\vec{v}}{dt} + \vec{\omega} \times \vec{r} + 2(\vec{\omega} \times \vec{r}) + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

measured in s'

So, for a particle of mass m undergoing motion:

$$m\frac{d\vec{v}}{dt} = m\frac{d\vec{v}_0}{dt} + m(\vec{r}\times\dot{\vec{\omega}}) + 2m(\vec{v}\times\vec{\omega}) + m(\vec{\omega}\times\vec{r})\times\vec{\omega}$$

measured in ~~[illegible]~~ S'

True Inertial force

Euler force

Coriolis force

Centrifugal force

fictitious force or pseudo force

Lecture - 06 04-11-2024

Alternatively, Langrangian formalism

L to be evaluated in an intertial frame.

$$L = \frac{1}{2}m|v_0|^2 - V$$

$$\frac{\partial}{\partial\bar{v}}(m\bar{v}\cdot\bar{v}) = m$$

$$L = \frac{1}{2}m|(\bar{v}+(\bar{\omega}\times\bar{r}))|^2 - V$$

$$L = \frac{1}{2}mv^2 + \cancel{\tfrac{1}{2}}m\bar{v}\cdot(\bar{\omega}\times\bar{r}) + \frac{1}{2}m(\bar{\omega}\times\bar{r})\cdot(\bar{\omega}\times\bar{r}) - V$$

$$\frac{\partial L}{\partial \vec{v}} = m\vec{v} + m(\vec{\omega} \times \vec{r}) = m(\vec{v} + \vec{\omega} \times \vec{r})$$

$$= m\vec{v}_0$$

$$= \vec{p} \text{ (momentum corresponding to } \bar{r})$$

This is true momentum & not $m\vec{v}$

foucault Pendulum: proof that the earth is a rotatory frame.

$$L = \frac{1}{2} m |(\bar{v} + \bar{\omega} \times \bar{r})|^2 - V$$

$$= \frac{1}{2} m v^2 + m\bar{v} \cdot (\bar{\omega} \times \bar{r}) + \frac{1}{2} m (\bar{\omega} \times \bar{r}) \cdot (\bar{\omega} \times \bar{r}) - V$$

$$\frac{\partial L}{\partial \vec{r}} = m(\vec{v} \times \bar{\omega}) - m(\vec{\omega} \times \bar{r}) \times \bar{\omega} - \vec{\nabla} V$$

E-L equations

$$m\frac{d\vec{v}}{dt} + m(\dot{\vec{\omega}} \times \vec{r}) + m(\vec{\omega} \times \bar{v}) - m(\bar{v} \times \bar{\omega}) + m(\bar{\omega} \times \bar{r}) \times \vec{\omega} - \vec{\nabla} V = 0$$

OR

$$\boxed{m\frac{d\vec{v}}{dt} = -\vec{\nabla} V + m(\vec{r} \times \dot{\vec{\omega}}) + 2m(\vec{v} \times \vec{\omega}) + m(\vec{\omega} \times \vec{r}) \times \vec{\omega}}$$

How $(\bar{\omega} \times \bar{r}) \cdot (\bar{\omega} \times \bar{r})$

$= \bar{r} \cdot ((\bar{\omega} \times \bar{r}) \cdot \bar{\omega})$

$\bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \vec{c}$

If you write down E-L eqns - the equation are identical to what was obtained earlier.

Energy in rotating frame (related to that in a stationary frame)

Recall

$$\left.\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \equiv \text{Jacobi Integral}\right\} = E$$

i.e $\underline{E = \vec{p}\cdot\vec{v} - L}$

$$= m\vec{v}_0(\bar{v}_0 - \bar{\omega}\times\bar{r}) - L$$

$$E = m\bar{v}_0(\bar{v}_0 - \bar{\omega}\times\bar{r}) - \frac{1}{2}mv_0^2 + V$$

$$E = \underbrace{\frac{1}{2}mv_0^2 + V}_{E_0} - \underbrace{mv_0(\bar{\omega}\times\bar{r})}_{\bar{p}\cdot(\bar{\omega}\times\bar{r})}$$

$\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c}$

$$\boxed{E = E_0 - \vec{\omega}\cdot(\vec{r}\times\vec{p}) = E_0 - \vec{\omega}\cdot\vec{L}}$$

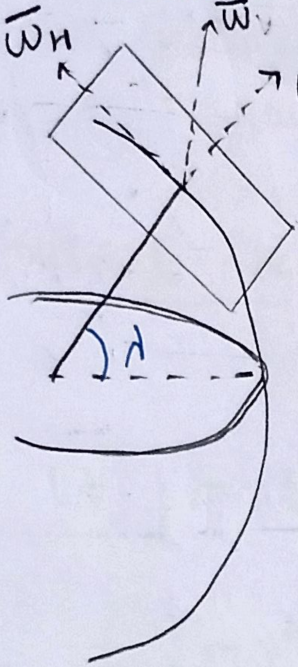
$$E = E_0 - \vec{\omega}\cdot\vec{L}$$

angular momentum

(A) Coriolis Force Effects : on river flow

Geographical latitude 'λ'

→ choose direction of flow $\hat{x}$ direction



$\vec{V} = v\hat{i}$ (choosing direction of flow to be along $\hat{x}$)

$\omega_V = \omega \cos\left(\frac{\pi}{2} - \lambda\right) = \omega \sin\lambda$

$\omega_H = \omega \sin\left(\frac{\pi}{2} - \lambda\right) = \omega \cos\lambda$

$$\vec{F}_c = 2m\left((\vec{V} \times \vec{\omega}_H) + (\vec{V} \times \vec{\omega}_V)\right)$$

$$= 2m\left(v\omega \cos\lambda\,(\hat{i} \times (-\hat{j})) + v\omega \sin\lambda\,(\hat{i} \times \hat{k})\right)$$

$$= -2mv\omega\,(\sin\lambda\,\hat{j} + \cos\lambda\,\hat{k})$$

→ along $\hat{j}$ direction there is also a component of the coriolis force ie along $\hat{j}$: $\underline{-2v\omega\sin\lambda}$ is the "coriolis acceleration"

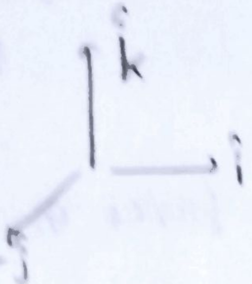
latitude dependence

- If $\lambda > 0$ - i.e in the northern hemisphere acc^n is $-2v\omega\sin\lambda$ (−ve)
- If $\lambda < 0$ - i.e in the southern hemisphere acc^n is $2v\omega\sin\lambda$ (+ve)

→ Assuming flow in $v\hat{i}$

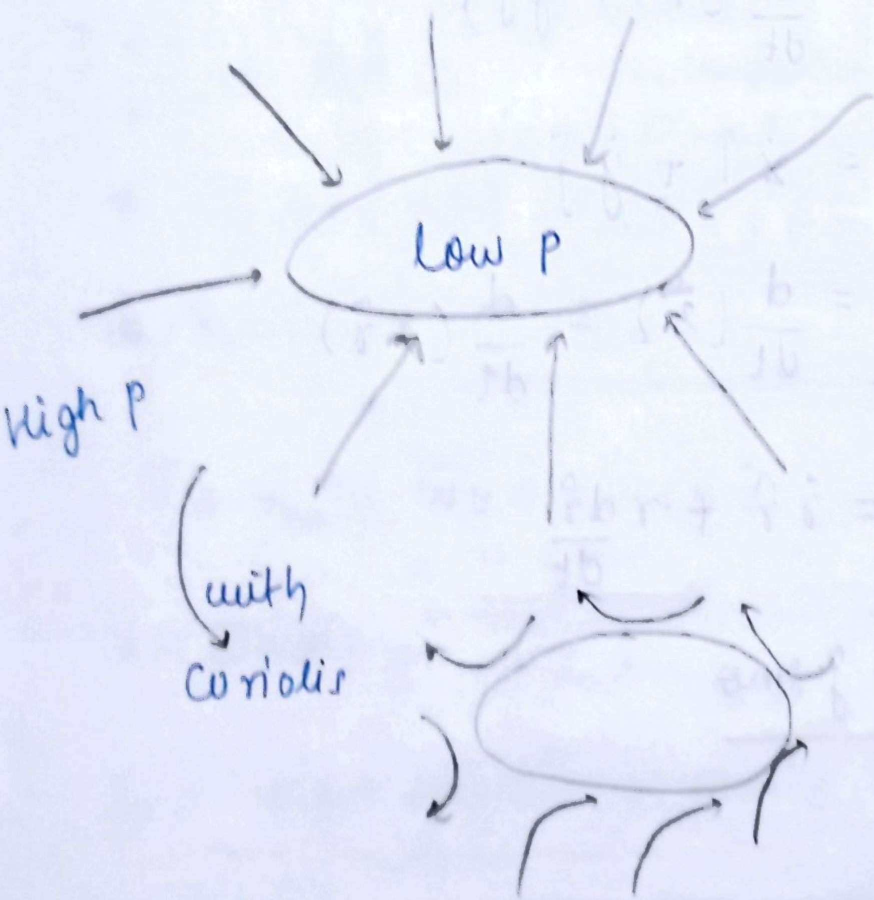
→ In Northern hemisphere – force is along $-\hat{j}$

to the right of the flow



→ Southern hemisphere – force is along $\hat{j}$

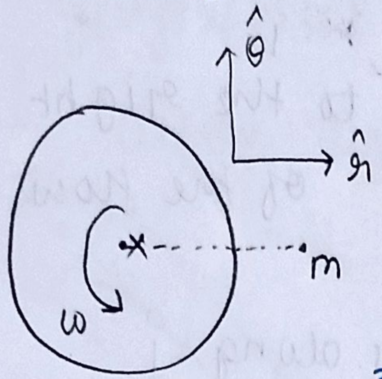
to the left of the flow



→ ACW in Northern Hemisphere.

→ CW in Southern hemisphere

→ Deflection of a falling stone dropped from a height 'h' on the equator (r, θ) - fixed to the earth



Apparent force on m is

$$\vec{F} = -mg\hat{r} - 2m(\vec{\omega} \times \vec{v}_{rot}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Velocity,

$$\vec{v} = \frac{d}{dt}(x\hat{i} + y\hat{j})$$

$$= \dot{x}\hat{i} + \dot{y}\hat{j}.$$

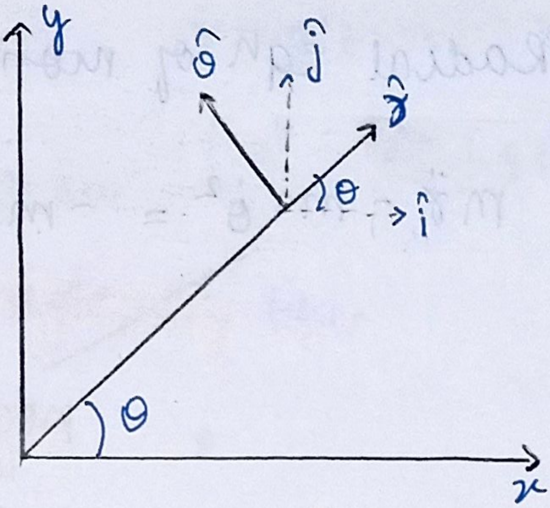
$$= \frac{d}{dt}(\vec{r}) = \frac{d}{dt}(r\hat{r})$$

$$= \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$

$\cancel{\hat{v} - \dot{r}\hat{r} +}$:

$$\hat{v} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$



$$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\frac{d\hat{r}}{dt} = (-\hat{i}\sin\theta + \hat{j}\cos\theta)\dot{\theta}$$

$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

$$\therefore \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\Rightarrow \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} := v\hat{r} + r\dot{\theta}\hat{\theta}$$

$\underline{\hat{\omega} \times \hat{r} = \hat{\theta}}$ OR $\underline{\hat{\omega} \times \hat{\theta} = -\hat{r}}$

so $\vec{\omega} \times \vec{v}_{rot} = \omega v\hat{\theta} - \omega r\dot{\theta}\hat{r}$

and

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2 r\,\hat{r}$$

$$\therefore \underline{F_r = -mg + 2m\omega\dot{\theta}r + m\omega^2 r}$$

and

$$\underline{F_\theta = -2m\dot{r}\omega}$$

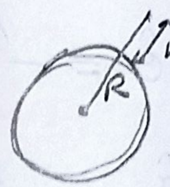
$\therefore$ Radial eqn of motion is

$$\underbrace{m\ddot{r} - mr\dot{\theta}^2 = -mg + 2m\omega\dot{\theta}r}_{\text{neglecting this}} + m\omega^2 r$$

$m\omega^2 r$ is the dominant term - neglect

$\dot{\theta} << \omega$

$$\boxed{\ddot{r} \approx -g + \omega^2 r}$$



Tangential eqn of motion is

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = -2m\dot{r}\omega$$

Now assume that $\vec{r}$ changes only slightly from $R_\oplus + h$ to $R_\oplus$ $\quad (R_\oplus + h \approx R_\oplus)$

$\therefore$ Let $r \approx R_\oplus$ & $g \approx$ const

$$\ddot{r} \approx -g + \omega^2 R_\oplus = g' \Rightarrow \dot{r} = -g't$$

$$\text{or } r = r_0 - \frac{1}{2} g t^2$$

Tangential eq^n of motion :

$\dot{r} = -g't$

$r = r_0 - \frac{1}{2} g't^2$

$$m r \ddot{\theta} + \underbrace{2m\dot{r}\dot{\theta}}_{} = -2m\dot{r}\omega$$

neglecting this

$$r\ddot{\theta} = -2\dot{r}\omega$$

$$= -2(-g't)\omega$$

$$\text{or } \ddot{\theta} = \frac{+2g'\omega t}{R} \qquad \text{or } \dot{\theta} = \frac{g'\omega t^2}{R_\oplus}$$

$$\text{or } \boxed{\theta = \frac{g'\omega t^3}{3R_\oplus}}$$

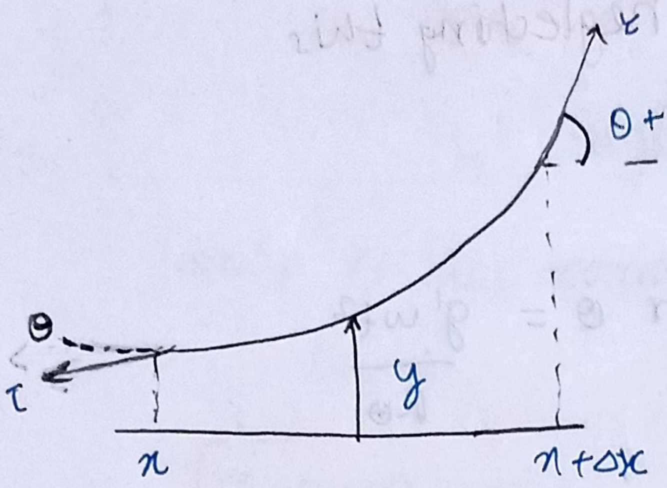
Deflection,

$$y \approx R_\oplus \theta$$

$$= \frac{g'\omega t^3}{3}$$

Lecture- 11-11-24

(A) Free vibrations of a streched string :



- String of Length L held fixed at $x = 0$ and $x = L$ uniform linear density "μ" streched with tension T.

$$F_y = T\sin(\theta + \Delta\theta) - T\sin\theta$$

$$F_x = T\cos(\theta + \Delta\theta) - T\cos\theta$$

→ Assume transverse displacement is small,

$$\therefore \; F_y \approx T\Delta\theta \qquad F_x \approx 0$$

$\theta, \theta + \Delta\theta$ are small angles

- Transverse motion of the string is governed by F_y

$$\underline{T\Delta\theta \approx (\mu\Delta x)\, a_y}$$

$\rightarrow$ $T\Delta\theta = (\mu\Delta x)\, a_y$

Now $\tan\theta = \dfrac{\partial y}{\partial x} \Rightarrow \sec^2\theta\, \Delta\theta = \dfrac{\partial^2 y}{\partial x^2}\Delta x$

• For a small angle $\sec\theta \simeq 1$

So,
$$\Delta\theta \approx \frac{\partial^2 y}{\partial x^2}\Delta x$$

Now acceleration $a_y = \dfrac{\partial^2 y}{\partial x^2}$

$$T\Delta\theta \simeq (\mu\Delta x)\, a_y = \mu\Delta x \cdot \frac{\partial^2 y}{\partial x^2}$$

OR
$$T\Delta\theta = T\frac{\partial^2 y}{\partial x^2}\Delta x = \mu\Delta x \frac{\partial^2 y}{\partial t^2}$$

or
$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

$v \equiv \sqrt{T/\mu}$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$ wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 y}{\partial t^2} \qquad V = \sqrt{T/\mu} = \text{velocity}$$

→ Since each part of the string is moving sinusoidally with time and amplitude of motion is a function of distance 'x' of the point along the string.

Let us assume.

$$y(x,t) = f(x)\cos(\omega t)$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x)\cos(\omega t)$$

$$\& \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}\cos(\omega t)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{-\omega^2}{V^2} f$$

$$f(x) = A\sin\left(\frac{\omega x}{V}\right)$$

Now B.C. requires $y = 0$ at $x = 0$ and at $x = L$

$$\Rightarrow \frac{\omega L}{V} = n\pi \longrightarrow \frac{\omega}{V} = \frac{n\pi}{L}$$

$$\frac{\omega}{V} = \frac{n\pi}{L}$$

$$f(x) = A\sin\left(\frac{\omega x}{V}\right) = A\sin\left(\frac{n\pi}{L}x\right)$$

$\nu \equiv \frac{\omega}{2\pi} \equiv$ no. of cycles per unit time.

$\therefore \nu = \frac{\omega}{2\pi} = \frac{nv}{2L} = \frac{n}{2L}\sqrt{T/\mu}$ velocity

frequency

fundamental frequency

$\therefore \frac{2L}{n}$ defines a wavelength

$$\lambda_n \equiv \frac{2L}{n}$$

associated with the nth mode.

→ "normal mode" of vibration

$\frac{\omega}{v} = \frac{n\pi}{L} = \frac{2\pi}{\lambda_n} = k_n$ also called the 'wave number'

$\therefore$ shape of the string in mode 'n' is given by $f_n(x) = A_n \sin\left(\frac{2\pi x}{\lambda_n}\right) = A_n \sin\left(\frac{n\pi x}{L}\right)$

→ The complete description of motion is given by

$$y(x,t) = f(x)\cos(\omega t)$$

i.e $y_n(x,t) = A_n \sin\left(\frac{2\pi}{\lambda_n} x\right)\cos(\omega t)$

$$= A_n \sin(k_n x)\cos(\omega t)$$

(with $\omega_n = \frac{n\pi}{L} v = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} = n\omega_1$

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{\mu}}$$

"Fundamental ~~mode~~ mode"

So, given the length of the string, its diameter and density, you will know its frequencies of vibrations.

In a prism - dispersion - wave speed varies with wavelength

$$\boxed{v_g = \text{group velocity} = \frac{d\omega}{dk}}$$

energy transport

$\frac{\omega}{k} = v_p$ = phase velocity

$$\Rightarrow y_n(x,t) = A_n \sin\left(\frac{2\pi x}{\lambda_n}\right)\cos(\omega_n t)$$

$$= A_n \sin(k_n x)\cos(\omega_n t)$$

This can be rewritten as

$$y_n = \frac{A_n}{2}\left(\sin(k_n x - \omega_n t) + \sin(k_n x + \omega_n t)\right)$$

→ Since $k_n = \frac{2\pi}{\lambda} = \frac{n\pi}{L}$ & $\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$,

we can write:

$$y_n(x,t) = \frac{A_n}{2}\sin\left(\frac{n\pi}{L}\left(x - \sqrt{\frac{T}{\mu}}\,t\right)\right) +$$

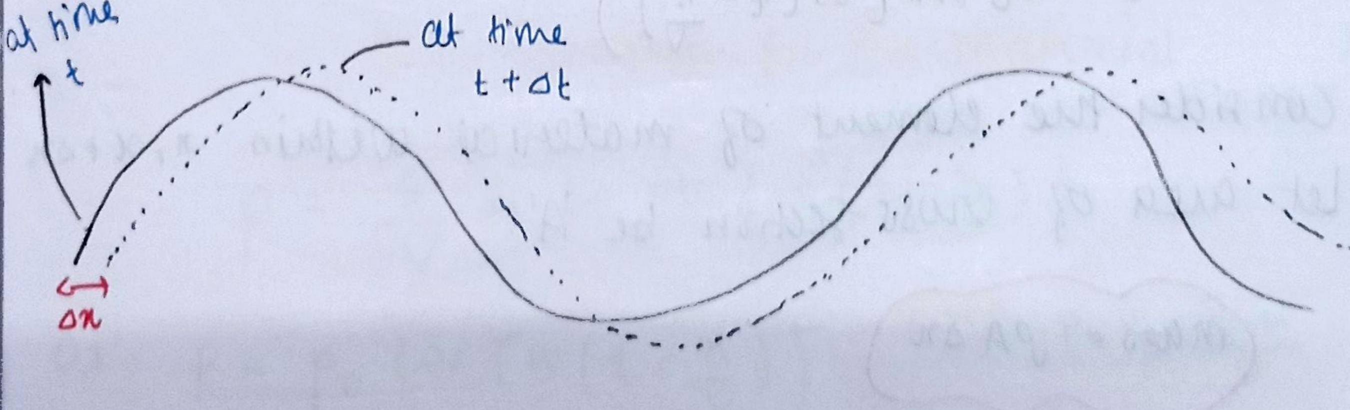
$$\frac{A_n}{2}\sin\left(\frac{n\pi}{L}\left(x + \sqrt{\frac{T}{\mu}}\,t\right)\right)$$

→ This is actually the eqn. of 2 travelling waves going in opposite directions, whose superposition has given us a STANDING WAVE.

- Consider the 1st part of the wave:

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

this is a disturbance - at some instant of time, consider a value of y and see where we find it after $(t+\Delta t)$ - say at $(x+\Delta x)$



$$y(x,t) = y(x+\Delta x, t+\Delta t)$$

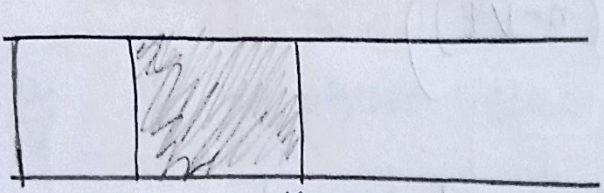
i.e $$\sin\left(\frac{2\pi}{\lambda}(x - vt)\right) = \sin\frac{2\pi}{\lambda}(x + \Delta x - v(t+\Delta t))$$

$$\Rightarrow \Delta x - v\Delta t = 0$$

$$\text{or } \frac{\Delta x}{\Delta t} = v$$

→ the wave is moving in the +ve x direction with velocity v

ⓐ Displacement wave, Pressure wave



consider the motion of a wave going in the x-direction in a fluid with displacement 's' in the x-direction

$$s = s_0 \sin\left(\omega\left(t - \frac{x}{v}\right)\right)$$

→ consider the element of material within $x, x+\Delta x$
Let area of cross-section be 'A'.

$$\text{mass} = \rho A \Delta x$$

$\therefore$ Increase in volume as wave passes is

$$\boxed{\Delta V = A \Delta s}$$

$$\therefore \quad \Delta V = A s_0 \left(\frac{-w}{v}\right) \cos\left(w\left(t - \frac{x}{v}\right)\right) \Delta x$$

i.e there is a volume strain

$$\frac{\Delta V}{V} = \frac{-A s_0 \frac{w}{v} \cos\left(w\left(t - \frac{x}{v}\right)\right) \Delta x}{A \Delta x}$$

$$= -\frac{s_0}{v} w \cos\left(w\left(t - x/v\right)\right)$$

$$\frac{\Delta V}{V} = -\frac{s_0}{v} w \cos\left(w\left(t - \frac{x}{v}\right)\right)$$

The stress corresponding to this

= excess pressure developed in the element under consideration is

$$p = B\left(-\frac{\Delta V}{V}\right)$$

Bulk modulus for the material

$$\therefore \quad p = \frac{B}{v} s_0 w \cos\left(w\left(t - \frac{x}{v}\right)\right)$$

or $\quad p = p_0 \cos\left(w\left(t - \frac{x}{v}\right)\right)$ with $p_0 = \frac{B s_0 w}{v}$

$$= B s_0 k$$

i.e the pressure wave differs in phase by $\pi/2$ from the displacement wave. 7

Ex:

Sound wave. $\lambda = 40$ cm travels in air

$P_{max} - P_{min} = 10^{-3}$ N/m^2 at some point

What is the amplitude of vibrations of particles of the medium?

$$B = 1.4 \times 10^{5} \ N/m^2$$

Lecture - 12/11/24 S.H.O in last pages

$$F = m\ddot{x} = -kx - b\dot{x}$$

spring force

frictional force (viscous)

$$\ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} = 0$$

$\gamma \equiv \frac{b}{m}$ $\omega_0^2 \equiv \frac{k}{m}$

$$\ddot{x} + \omega_0^2 x + \gamma\dot{x} = 0$$

$$\ddot{y} + \omega_0^2 y + \gamma\dot{y} = 0$$

$$z \equiv x + iy$$

$$\ddot{z} + \omega_0^2 z + \gamma\dot{z} = 0$$

→ if $z = z_0 e^{\alpha t}$ is a solution then,

$$(\alpha^2 + \alpha\gamma + \omega_0^2) z_0 e^{\alpha t} = 0$$

or $\alpha = \left(-\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)$

so there are ~~4~~ 2 roots α_1, α_2

so that

$$z = z_1 e^{\alpha_1 t} + z_2 e^{\alpha_2 t}$$

is the solution

z_1, z_2 ~~roots~~ constants

→

$$\alpha = \frac{-\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

There are 3 distinct possibilities

- $\gamma^2 << 4\omega_0^2$

light dampling

$\sqrt{(\,)}$ is complex

$$\alpha = \frac{-\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$$= \frac{-\gamma}{2} \pm i\omega_1 \quad \text{where} \quad \omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

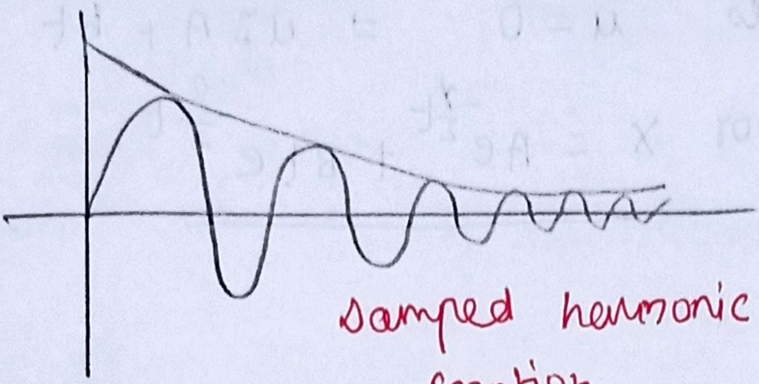
$$z = x + iy = e^{-\frac{\gamma}{2}t}\Big((x_1 + iy_1)(\cos\omega_1 t + i\sin\omega_1 t)$$

$$+ (x_2 + iy_2)(\cos\omega_1 t - i\sin\omega_1 t)\Big)$$

$$x = e^{-\gamma/2\, t}\left[B\cos\omega_1 t + C\sin\omega_1 t\right]$$

(taking only the real terms on RHS)

$= A e^{-\gamma/2 \, t} \cos(\omega_1 t + \phi)$.



damped harmonic motion

- $\gamma^2 = 4\omega_0^2$

Critical damping

So only 1 solution

$x = A e^{-\gamma/2 \, t}$

is the incomplete solⁿ

(a 2nd order ODE - 2 constants needed to specify both initial push and initial velocity)

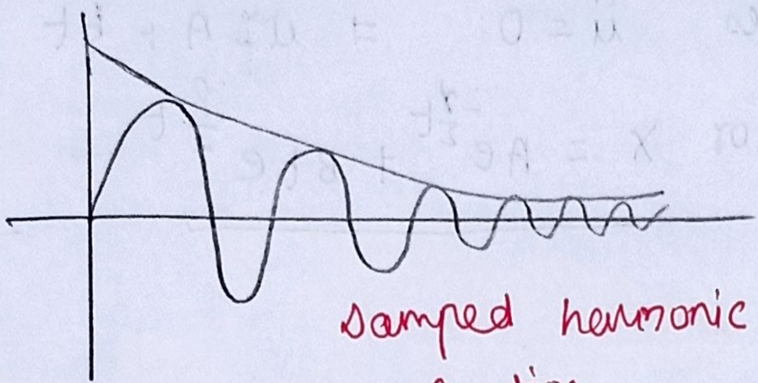
Extra Info

$\ddot{y} = ax$ → Rewrite as 2 1st order ODE

$\dot{y} = q$

$\dot{q} = ax$

$$= A e^{-\gamma/2 t} \cos(\omega_1 t + \phi).$$



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Extra Info

$\ddot{y} = ax$ → Rewrite as 2 1^{st} order ODE

$$\dot{y} = q$$
$$\dot{q} = ax$$

let $x = u(t)e^{-\gamma/2\, t}$

using this gives $\ddot{u} = 0 \Rightarrow u = A + bt$

or $x = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$

- $\gamma^2 > 4\omega_0^2$

Heavy damping

$$\alpha = -\frac{\gamma}{2} \pm \frac{\gamma}{2}\sqrt{1 - \frac{4\omega_0^2}{\gamma^2}}$$

Both roots are negative

$$\rightarrow \quad z = z_1 e^{-|\alpha_1|t} + z_2 e^{-|\alpha_2|t}$$

look for the real parts

$$x = Ae^{-|\alpha_1|t} + Be^{-|\alpha_2|t}$$

No oscillatory behaviour here.

exponentially decaying fun

Overdamped case

- Energy at some time t — $E(t)$

$E(t) = E(0) +$ work done due to friction

$$W_{friction} = \int_{x_0}^{x(t)} f\, dx = \int_0^t f\, v\, dt = \int_0^t -b v^2 dt < 0$$

$\therefore$ E(t) decreases with time

For light damping:

$$x(t) = A e^{-\frac{\gamma}{2}t} \cos(\omega_1 t + \phi)$$

or $v = -A e^{-\frac{\gamma}{2}t} \left(\omega_1 \sin(\omega_1 t + \phi) + \frac{\gamma}{2} \cos(\omega_1 t + \phi)\right)$

For very light damping:

$$\frac{\gamma}{\omega_1} << 1$$

$$\Rightarrow v \approx -\omega_1 A e^{-\frac{\gamma}{2}t} \sin(\omega_1 t + \phi)$$

$$\Rightarrow KE = \frac{1}{2} m v^2 \approx \frac{1}{2} m \omega_1^2 A^2 e^{-\gamma t} \sin^2(\omega_1 t + \phi)$$

$$\Rightarrow PE = \frac{1}{2} k x^2 = \frac{1}{2} k A^2 e^{-\gamma t} \cos^2(\omega_1 t + \phi)$$

Now,

$\omega_0^2 = \omega_1^2 + \frac{\gamma^2}{4}$ $\Rightarrow$ For very light damping

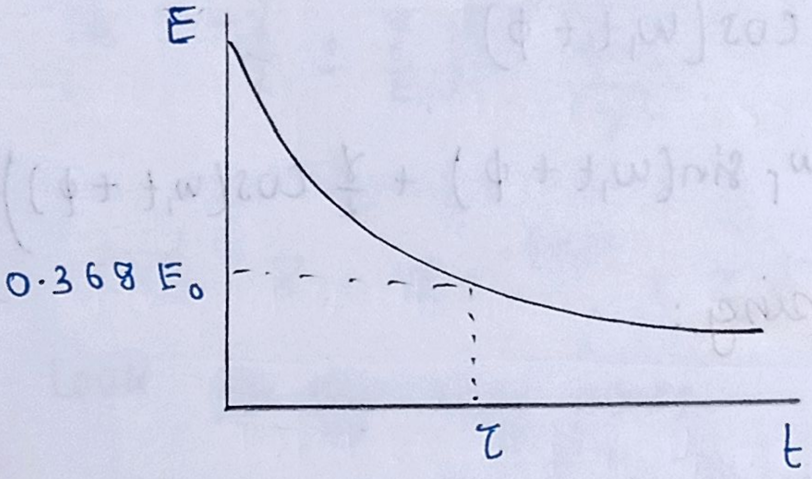
$$\omega_0^2 \approx \omega_1^2$$

$$\therefore \quad E(t) = KE + PE$$
$$= \frac{1}{2} k A^2 e^{-\gamma t}$$

$$\left[\because \ \frac{k}{m} = \omega_0^2 \simeq \omega_1^2\right]$$

At $t = 0$, $E(0) = E_0 = \frac{1}{2} k A^2$

$$E(t) = E_0 e^{-\gamma t}$$ — decay characterize by time τ, require for E to drop by e^{-1}



i.e. $\tau = \frac{1}{\gamma}$

$$E(\tau) = E_0 e^{-\gamma \tau} = e^{-1} E_0$$
$$= 0.368 E_0$$

$\tau = \frac{1}{\gamma} = \frac{m}{b} =$ "Damping Time"

or Time Constant

or "Characteristic time" for the system

In the limit of very light damping :

$\gamma \to 0$

$\tau \to \infty$

and E is effectively constant

→ Degree of damping is also quantified by the quality factor Q.

$$Q = \frac{\text{energy stored in oscillator}}{\text{energy dissipated per radian}}$$

i.e energy lost during the time the system oscillates through 1 radian

$$E(t) = E_0 e^{-\gamma t}$$

$$\frac{dE}{dt} = -\gamma E_0 e^{-\gamma t} = -\gamma E(t)$$

∴ Energy dissipated $\Delta E \approx \left|\frac{dE}{dt}\right| \Delta t = \gamma E \Delta t$

1 radian of oscillation

$\Rightarrow \Delta t = \frac{1}{\omega_1}$

$\therefore$ Energy lost is $\frac{\gamma E}{\omega_1} \Rightarrow Q = \frac{E}{\gamma E/\omega_1}$

$$= \frac{\omega_1}{\gamma} \approx \frac{\omega_0}{\gamma}$$

→ Undamped oscillation : $\gamma = 0$, $Q \to \infty$

→ Lightly damped : $\gamma \ll \omega_1$, $Q \gg 1$

→ Heavily damped : $\gamma \gg \omega_1$, Q – very small

Damped oscillations in an LCR circuit :

Total Energy $= U_E + U_S$

$$= \frac{Q^2}{2C} + \frac{1}{2} L I^2$$

First consider,

an LCR circuit with current $I(t)$

- pdf across inductor is $L\dot{I}$
- pdf across capacitor $V = Q/C$

$$L\dot{I} + \frac{Q}{C} = 0 \quad \text{or} \quad L\ddot{I} + \frac{I}{C} = 0$$

i.e $\ddot{I} + \omega^2 I = 0$, $\omega = \frac{1}{\sqrt{LC}}$

This is a sample harmonic motion eqn without damping.

→

$I(t) = I_0 \cos(\omega t - \phi)$

Now $V = -L\dot{I}$ across inductor

$= L I_0 \omega \sin(\omega t - \phi)$

$= \sqrt{\frac{L}{C}} I_0 \sin(\omega t - \phi)$

$= \sqrt{\frac{L}{C}} I_0 \cos(\omega t - \phi - \frac{\pi}{2})$

i.e b/w V & I - there is a phase shift of $\frac{\pi}{2}$

i.e V is max when I is minimum

and vice-a-verse

LCR circuit:

$$L\dot{I} + RI + \frac{Q}{C} = 0 \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

or $\ddot{I} + \gamma\dot{I} + \omega_0^2 I = 0 \qquad \gamma = R/L$

(damped harmonic oscillation

$b \rightarrow R$

$L \rightarrow m$

$$\frac{dE}{dt} = -RI^2$$

$$E = \frac{1}{2}LI^2 + \frac{1}{2}Q^2/C$$

Recall Q factor for damped Harmonic Oscillation was $\frac{\omega_1}{\gamma}$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \qquad \omega_0^2 = \omega^2 + \frac{\gamma^2}{4}$$

For light damping $Q = \frac{\omega_0}{\gamma} = \frac{1}{\sqrt{LC}}\frac{1}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$