IIIT-Bangalore
Probability and Statistics
Problem Set 7

(Mathematical Expectation I: Moment Generating Function, Characteristic Functio, Median, Quantile, Mode)

1. Compute moment generating functions (m.g.f.) and characteristic functions of the following well-known distributions: (i) Binomial (n, p), (ii) Poisson (μ), (iii) Normal (m, σ) and (iv) Gamma (l).

2. Find the m.g.f. for $N(m, \sigma)$ variate and hence calculate the central moments.

3. Find the m.g.f. of a continuous distribution whose density function is $f_X(x) = \frac{1}{2}x^2e^{-x}, (0 < x < \infty)$ and compute the values of mean and variance. (Ans. $M_X(t) = \frac{1}{(1-t)^3}$, $m_X = 3$, $var(X) = 3$.)

4. A continuous distribution has p.d.f.
$$f_X(x) = ae^{-ax},\ 0 < x < \infty,\ a > 0.$$
Calculate the m.g.f. and hence obtain α_k. (Ans. $M_X(t) = \frac{a}{(a-t)}$, $\alpha_k = \frac{k!}{a^k}$.)

5. Prove that m.g.f. of a uniform distribution over the interval $(-a,\ a)$ is $\frac{\sinh at}{at}$. Hence calculate the central moments. (Ans. $\mu_{2k+1} = 0$, $\mu_{2k} = \frac{a^{2k}}{2k+1}$)

6. Find the mean, median and mode of $f(x) = \begin{cases} \sin x, & 0 \leq x \leq \frac{\pi}{2} \\ 0, & \text{elsewhere.} \end{cases}$
(Ans. mean: $m_X = 1$, median: $\mu = \frac{\pi}{3}$, mode: $\frac{\pi}{2}$)

7. Show that the mode of a Poisson distribution with mean μ is the integer or integers which are determined by $\mu - 1 \leq M \leq \mu$.

8. Calculate the first absolute moment about the mean and the semi-interquartile range for Laplace distribution with p.d.f.
$$f_X(x) = \frac{1}{2\lambda}e^{-|x-\mu|/\lambda}, -\infty < x < \infty, \lambda > 0.$$
(Ans. $m_X = \mu$, $E(|X - \mu|) = \lambda$, semi-interquartile range$= \lambda \log 2$)