1. (a) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, ..., a_k$, where $a_1 = 1, a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, ..., k-1$.

$S_n = \sum_{i=1}^{n-1} S_i$

$\boxed{S_n = 2S_{n-1}}$

$S_{n-1} = \sum_{i=1}^{n-2} S_i$

(b) What are the initial conditions?

S_1 S_2

(c) How many sequences of the type described in (a) are there when n is an integer with $n \geq 2$?

S_k

1 k

n $\sum_{k=1}^{n-1} S_{n-1} = S_n$

$r - 2 = 0$

$r = 2$

$S_n = \alpha_1 (2)^n$

$S_2 = \alpha_1 \cdot (2)^2 = 1$

$\alpha_1 = \frac{1}{4}$

$\boxed{S_n = 2^{n-2}}$

2. (a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

(b) What are the initial conditions?

(c) How many bit strings of length seven contain three consecutive 0s?

$$\boxed{R_n = 2R_{n-1} + \cancel{R}\, 2^{n-4} - R_{n-4}}$$

$1\,000\;0$

n^{th}

$$\boxed{R_n = 2^n - \left(\underbrace{R_{n-1}}_{T_{n-1}} + \underbrace{R_{n-2}}_{T_{n-2}} + \underbrace{R_{n-3}}_{T_{n-3}}\right)}$$

$R_{n=1,2} = 0$

$R_3 = 1$

000

$T_n =$

1 T_{n-1}

$0\;1$ T_{n-2}

$0\;0\;1$ T_{n-3}

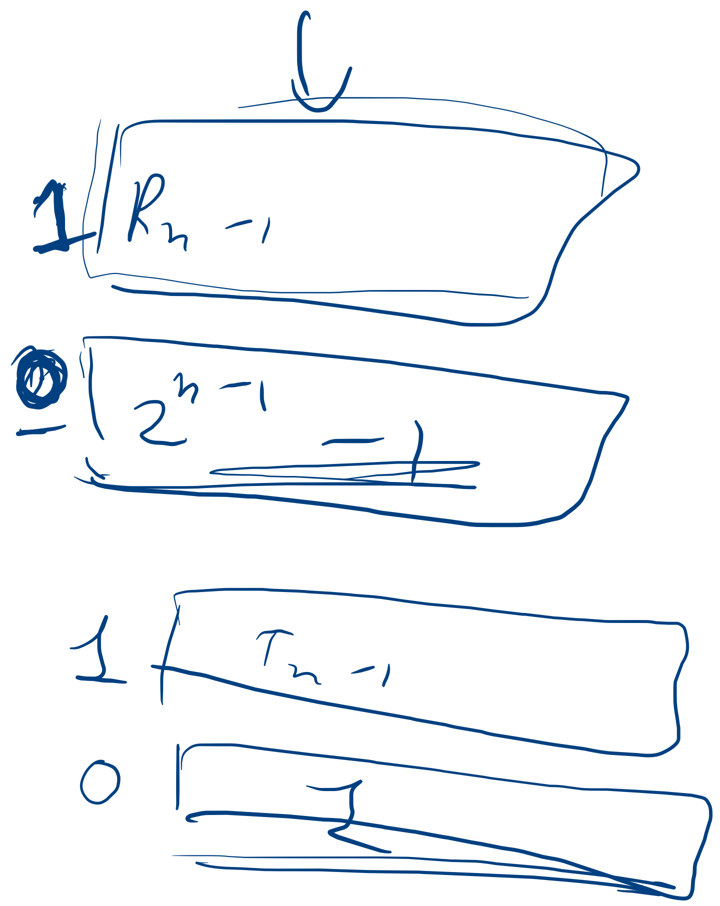
$$R_n = R_{n-1} + R_{n-2} + R_{n-3} + 2^{n-3}$$

1	R_{n-1}
01	R_{n-2}
001	R_{n-3}
000	2^{n-3}

3. (a) Find a recurrence relation for the number of bit strings of length n that contain the string 01.
(b) What are the initial conditions?
(c) How many bit strings of length seven contain the string 01?

01

$$R_n = R_{n-1} + 2^{n-1} - 1$$



M-II

$$R_n = 2^n - T_n$$

$$T_n = T_{n-1} + 1$$

$$2^n - R_n = 2^{n-1} - R_{n-1} + 1$$

$$R_n = 2^{n-1} - 1 + R_{n-1}$$

M-III

$$1^k 0^{n-k}$$

M-IV

$$R_n = 2R_{n-1} + 2^{n-2} - R_{n-2}$$

4. A string that contains only 0s, 1s, and 2s is called a **ternary string**.

(a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive symbols that are the same.

(b) What are the initial conditions?

(c) How many ternary strings of length six contain consecutive symbols that are the same? R_6

n.

$3(n-1)\times$

$$\boxed{R_n = 3R_{n-1} + 3^{n-1} - R_{n-1}}$$

$$= 2R_{n-1} + 3^{n-1}$$

Case 1

$\underbrace{\qquad\qquad}_{n-1}$

R_{n-1}

0, 1, 2.

Case 2

00 / 11 / 22

$\boxed{3^{n-1} - R_{n-1}}$

copy last bit

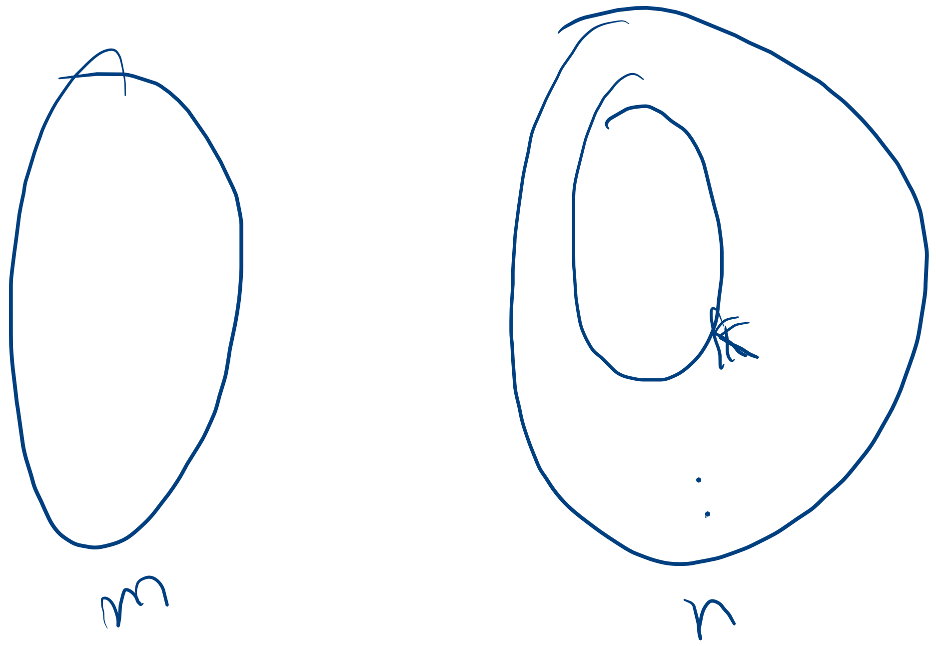
5. Let $T(m, n)$ denote the number of onto functions from a set with m elements to a set with n elements. Show that $T(m, n)$ satisfies the recurrence relation

$$T(m,n) = n^m - \sum_{k=1}^{n-1} C(n,k)T(m,k)$$

whenever $m \geq n$ and $n > 1$, with the initial condition $T(m, 1) = 1$.

$$\text{Total} = n^m$$

$$\text{Into} = \sum_{k=1}^{n-1} {}^{n}C_{k}\, T(m,k)$$



6. (a) Let $S(n, k)$ denote the number of ways of partitioning n distinct elements into k disjoint non-empty subsets. Give a combinatorial proof that $S(n, k) = S(n-1, k-1) + kS(n-1, k)$.

(b) What will be the value of $S(n+1, n)$?

7. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = \boxed{n\binom{2n-1}{n-1}}$.

$^nC_k \cdot {}^nC_k = {}^nC_k \cdot {}^nC_{n-k}$

$$\sum_{k=1}^{n} k \cdot {}^nC_k \cdot {}^nC_{n-k}$$

n Math Profs

n CS Profs

Committee of n People.

with one head being a Math Prof.

n

$n \leftarrow 1$ $\quad n-1$

n

$n \leftarrow 2$ $\quad n-2$

3

8. Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

$(x_i, y_i) \longrightarrow$ $\boxed{e, e}$ $\boxed{e, o}$

o, o o, o

$\left(\frac{e+e}{2} \quad \frac{o+o}{2}\right)$

pigeon hole principle

9. (a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.

(b) Is the conclusion in part (a) true if four integers are selected rather than five?

1 – 8

2 – 7

3 – 6

4 – 5

10. Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

$a_1 = 1$

$a_2 = 11$

$a_3 = 111$

$\vdots$

$a_{n+1} = 111\ldots1$ $(n+1)$ times.

$(0 \ldots n-1)$

$$\begin{cases} r_1 = a_1 \bmod n \\ r_2 = a_2 \bmod n \\ \vdots \\ r_{n+1} = a_{n+1} \bmod n \end{cases}$$

$(i > j)$

$a_i \quad a_j$

$a_i - a_j$

$(a_i - a_j) \bmod n \equiv (r_i - r_i) \bmod n$

same remainder.

$\equiv 0 \bmod n.$

$\underbrace{111\ldots1}_{i \text{ times}}$

$\underbrace{111\ldots1}_{j \text{ times}}$

$\underbrace{111\ldots1}_{i-j \text{ times}}\underbrace{00\ldots0}_{j \text{ times}}$

$$^{n+1}C_k = {}^nC_k + {}^nC_{k-1}$$

$$\underline{C(n+1, k)} = \underline{C(n, k) + C(n, k-1)}$$

$$^nC_k = \frac{\underline{n}}{\underline{k}\;\underline{n-k}}$$