IIIT-Bangalore
Probability and Statistics
Problem Set 9

(Expectation II)

1. If X, Y are independent standard normal variates, find the mean value of the greater of $|X|$ and $|Y|$. (Ans. $\frac{2}{\sqrt{\pi}}$)

2. If for any pair of correlated random variables X and Y, we make a linear transformation $(X, Y) \rightarrow (U, V)$ given by the rotation of axes through a constant angle α, i.e.,

$$U = X\cos\alpha + Y\sin\alpha$$

$$V = -X\sin\alpha + Y\cos\alpha$$

then U and V will be uncorrelated if α is given by

$$\tan 2\alpha = \frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}$$

where $\rho = \rho(X, Y)$.

3. If (X, Y) has bivariate normal distribution with parameters $m_x, m_y, \sigma_x, \sigma_y, \rho$, then compute $\rho(X,\ Y)$. (Ans. ρ)

4. If

$$f(x,y) = \begin{cases} x+y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

find (i) m_x, (ii) m_y, (iii) σ_x, (iv) σ_y, (v) $\rho(X, Y)$, (vi) the regression curves, (vii) the least square regression lines.
(Ans. $m_x = \frac{7}{12}, m_y = \frac{7}{12}, \sigma_x = \frac{\sqrt{11}}{12}, \sigma_y = \frac{\sqrt{11}}{12}, \rho = -\frac{1}{11}$)

5. Show that the acute angle θ between the least square regression lines is

$$\tan\theta = \left(\frac{1-\rho^2}{\rho}\right) \cdot \left(\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$

and discuss the cases when $\rho = 0$ and $\rho = \pm 1$.

6. If the regression lines of the distribution of (X, Y) are $x + 6y = 6$ and $3x + 2y = 10$, find (i) the means m_x, m_y and (ii) $\rho(X, Y)$. (Ans. $m_x = 3, m_y = \frac{1}{2}, \rho = -\frac{1}{3}$)

7. The random variables X, Y are connected by the linear relationship $aX + bY + c = 0$. Prove that the correlation coefficient between X and Y is -1 if a, b have the same sign and 1 if a, b have the opposite sign.

8. If for any pair of linearly dependent random variables X, Y we set $U = X \cos \alpha + Y \sin \alpha$ and $V = -X \sin \alpha + Y \cos \alpha$ then prove that V will be constant (i.e. has a one point distribution) if $\tan \alpha = \rho \frac{\sigma_y}{\sigma_x}$

9. The joint p.d.f. of two discrete r.v. X, Y is given by $P(X = i, Y = j) = p_{ij}$, $(i = 0, 1; j = 0, 1)$. Find (i) the joint characteristic function of X and Y, (ii) their individual characteristic functions and (iii) prove that X, Y are independent if $p_{00}p_{11} = p_{01}p_{10}$.

10. Prove the "reproductive property" for the sum of n mutually independent variates $X_1, X_2, \ldots X_n$ in the following cases: each X_i has: (i) Binomial(n_i, p), (ii) Poisson(μ_i), (iii) Gamma(l_i) and (iv) Normal(m_i, σ_i) distribution.