FOURIER SERIES

5.1. We call a set of signals $\{\Psi_n(t)\}$ *orthogonal* on an interval (a, b) if any two signals $\Psi_m(t)$ and $\Psi_k(t)$ in the set satisfy the condition

$$\int_a^b \Psi_m(t)\Psi_k^*(t)\,dt = \begin{cases} 0 & m \neq k \\ \alpha & m = k \end{cases} \tag{5.95}$$

where $*$ denotes the complex conjugate and $\alpha \neq 0$. Show that the set of complex exponentials $\{e^{jk\omega_0 t}: k = 0, \pm 1, \pm 2, \dots\}$ is orthogonal on any interval over a period T_0, where $T_0 = 2\pi/\omega_0$.

For any t_0 we have

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}\,dt = \frac{1}{jm\omega_0} e^{jm\omega_0 t}\Big|_{t_0}^{t_0+T_0} = \frac{1}{jm\omega_0}\left(e^{jm\omega_0(t_0+T_0)} - e^{jm\omega_0 t_0}\right)$$

$$= \frac{1}{jm\omega_0} e^{jm\omega_0 t_0}\left(e^{jm2\pi} - 1\right) = 0 \qquad m \neq 0 \tag{5.96}$$

since $e^{jm2\pi} = 1$. When $m = 0$, we have $e^{jm\omega_0 t}|_{m=0} = 1$ and

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}\,dt = \int_{t_0}^{t_0+T_0} dt = T_0 \tag{5.97}$$

Thus, from Eqs. (*5.96*) and (*5.97*) we conclude that

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}(e^{jk\omega_0 t})^* \, dt = \int_{t_0}^{t_0+T_0} e^{j(m-k)\omega_0 t} \, dt = \begin{cases} 0 & m \neq k \\ T_0 & m = k \end{cases} \tag{5.98}$$

which shows that the set $\{e^{jk\omega_0 t}: k = 0, \pm 1, \pm 2, \dots\}$ is orthogonal on any interval over a period T_0.

5.4. Determine the complex exponential Fourier series representation for each of the following signals:

(a) $x(t) = \cos \omega_0 t$

(b) $x(t) = \sin \omega_0 t$

(c) $x(t) = \cos\left(2t + \dfrac{\pi}{4}\right)$

(d) $x(t) = \cos 4t + \sin 6t$

(e) $x(t) = \sin^2 t$

(a) Rather than using Eq. (5.5) to evaluate the complex Fourier coefficients c_k using Euler's formula, we get

$$\cos \omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\cos \omega_0 t$ are

$$c_1 = \tfrac{1}{2} \qquad c_{-1} = \tfrac{1}{2} \qquad c_k = 0, |k| \neq 1$$

(c) The fundamental angular frequency ω_0 of $x(t)$ is 2. Thus,

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k\, e^{j2kt}$$

Now

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \frac{1}{2}\left(e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)}\right)$$

$$= \frac{1}{2}e^{-j\pi/4}\, e^{-j2t} + \frac{1}{2}e^{j\pi/4}\, e^{j2t} = \sum_{k=-\infty}^{\infty} c_k\, e^{j2kt}$$

Thus, the complex Fourier coefficients for $\cos(2t + \pi/4)$ are

$$c_1 = \frac{1}{2}e^{j\pi/4} = \frac{1}{2}\frac{1+j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1+j)$$

$$c_{-1} = \frac{1}{2}e^{-j\pi/4} = \frac{1}{2}\frac{1-j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1-j)$$

$$c_k = 0 \qquad |k| \neq 1$$

(*e*) From Prob. 1.16(*e*) the fundamental period T_0 of $x(t)$ is π and $\omega_0 = 2\pi/T_0 = 2$. Thus,

$$x(t) = \sin^2 t = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Again using Euler's formula, we get

$$x(t) = \sin^2 t = \left(\frac{e^{jt} - e^{-jt}}{2j} \right)^2 = -\frac{1}{4}(e^{j2t} - 2 + e^{-j2t})$$

$$= -\frac{1}{4}e^{-j2t} + \frac{1}{2} - \frac{1}{4}e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\sin^2 t$ are

$$c_{-1} = -\tfrac{1}{4} \qquad c_0 = \tfrac{1}{2} \qquad c_1 = -\tfrac{1}{4}$$

and all other $c_k = 0$.