7. For binomial (n, p) distribution, prove that

$$\mu_{k+1} = p(1-p)\left(n\,k\,\mu_{k-1} + \frac{d\mu_k}{dp}\right)$$

and obtain γ_1 and γ_2. (Ans. $\gamma_1 = \frac{1-2p}{\sqrt{np(1-p)}}$, $\gamma_2 = \frac{1-6p(1-}{np(1-p}$

TBP $\Rightarrow$ $$\mu_{r+1} = p(1-p)\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

Proof :

$$\mu_r = E\{(X-m_x)^r\} = E\{(X-np)^r\}$$

$$= \sum_{x=0}^{n} (x-np)^r \, {}^nC_x \, p^x (1-p)^{n-x}$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} r(x-np)^{r-1}\cdot(-n)\, {}^nC_x p^x (1-p)^{n-x} + \sum_{x=0}^{n} (x-np)^r \, {}^nC_x \, x p^{x-1} (1-p)^{n-x}$$
$$+ \sum_{x=0}^{n} (x-np)^r \, {}^nC_x \, p^x \cdot (n-x)\cdot(1-p)^{n-x-1}(-1)$$

$$\frac{d\mu_r}{dp} = -nr\sum_{x=0}^{n} (x-np)^{r-1} \, {}^nC_x \, p^x (1-p)^{n-x}$$
$$- \sum_{x=0}^{n} (x-np)^r \, {}^nC_x \, p^x (1-p)^{n-x}\left[\frac{x}{p} - \frac{n-x}{1-p}\right]$$

$$\frac{d\mu_r}{dp} = -nr\mu_{r-1} - \sum_{x=0}^{n} (x-np)^r \, {}^nC_x \, p^x (1-p)^{n-x} \cdot \frac{(x-np)}{p(1-p)}$$

$$= -nr\mu_{r-1} - \mu_{r+1}\cdot\frac{1}{p(1-p)}$$

So, $$\mu_{r+1} = p(1-p)\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

$$\gamma_1 = \frac{\mu_3}{\sigma^3}$$

$$\mu_3 = p(1-p)\cdot\frac{d\mu_2}{dp} = \frac{p\cancel{(1-p)}\,[(n)\quad(1-2p)]}{\cancel{p(1-p)}\,n\sqrt{n(1-p)p}}$$

$$\mu_2 = p(1-p)\,[\,nk + 0\,] \qquad = \frac{1-2p}{\sqrt{np(1-p}}$$

$$\gamma_2 = \beta_2 - 3 \qquad \beta_2 = \frac{\mu_4}{\sigma^4}$$

Compute similarly.

9. Show that the 1st absolute moment about the mean for a normal (m, σ) distribution is $\sqrt{\frac{2}{\pi}}\,\sigma$.

We need to compute $E(|X-m_x|)$ for $\Phi(m,\sigma)$

$$E(|X-m_x|) = \int_{-\infty}^{\infty} |x-m| \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} |x-m| e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

Put $Y = |X-m_x|$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}, & x-m \geq 0,\ y \geq 0 \\ \frac{1}{\sqrt{2\pi}\sigma} e^{-(-y)^2/2\sigma^2}, & x-m < 0,\ y > 0 \end{cases}$$

$$E(Y) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} \cdot y\,dy + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma} \cdot y\,dy$$

$$= \frac{2}{\sqrt{2\pi}\sigma} \int_{0}^{\infty} y e^{-y^2/2\sigma^2}$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma$$

10. If $X \sim N(m, \sigma)$ variate, then prove that

$$\mu_{2r+2} = \sigma^2 \mu_{2r} + \sigma^3 \frac{d\mu_{2r}}{d\sigma}.$$

Hence find the coefficient of curtosis β_2 of this distribution. (Ans. $\beta_2 = 3$)

10.) i) $\mu_{2r} = E\{(X - m_x)^{2r}\}$

For $X \sim N(m, \sigma)$

$m_x = m$

$$\mu_{2r} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} (x-m)^{2r}\, dx$$

~~$\Rightarrow \mu_{2r} =$~~

$$\Rightarrow \frac{d\mu_{2r}}{d\sigma} = \frac{1}{\sqrt{2\pi}} \left\{ \frac{-1}{\sigma^2} \times \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} (x-m)^{2r}\, dx + \frac{1}{\sigma} \times \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} \times \frac{-(x-m)^2}{\cancel{2}} \times \frac{-\cancel{2}}{\sigma^3} \times (x-m)^{2r}\, dx \right\}$$

$$\Rightarrow \sigma^3 \frac{d\mu_{2r}}{d\sigma} = \left\{ -\sigma^2 \times \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} (x-m)^{2r}\, dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} (x-m)^{2r+2}\, dx \right\}$$

$$\Rightarrow \sigma^3 \frac{d\mu_{2r}}{d\sigma} = -\sigma^2 \mu_{2r} + \mu_{2r+2}$$

$$\Rightarrow \mu_{2r+2} = \sigma^2 \mu_{2r} + \sigma^3 \frac{d\mu_{2r}}{d\sigma}$$

So, $\mu_4 = \sigma^2 \mu_2 + \sigma^3 \frac{d\mu_2}{d\sigma}$ (by putting $r=1$) $\mu_2 = \sigma^2$

$= \sigma^4 + \sigma^3 \times 2\sigma = 3\sigma^4 \Rightarrow \underline{\beta_2 = 3}$