Topic: Discrete Random Variables

1. Letting X denote the random variable that is defined as the sum of two fair dice; then find the PMF. Check whether it is a valid PMF. Draw it.

$S = \{(x,y) \; 1 \le x, y \le 6\}$

$x = 2 \quad (1,1) \longrightarrow 1$

$x = 3 \quad (1,2), (2,1) \longrightarrow 2$

$x = 4 \quad (1,3), (3,1), (2,2) \longrightarrow 3$

$x = 5 \quad (1,4), (4,1), (2,3), (3,2) \longrightarrow 4$

$x = 6 \quad (1,5), (5,1), (2,4), (4,2), (3,3) \longrightarrow 5$

$x = 7 \quad (1,6), (6,1), (2,5), (5,2), (4,3), (3,4) \longrightarrow 6$

$x = 8 \quad (2,6), (6,2), (3,5), (5,3), (4,4) \longrightarrow 5$

$x = 9 \quad (3,6), (6,3), (5,4), (4,5) \longrightarrow 4$

$x = 10 \quad (4,6), (6,4), (5,5) \longrightarrow 3$

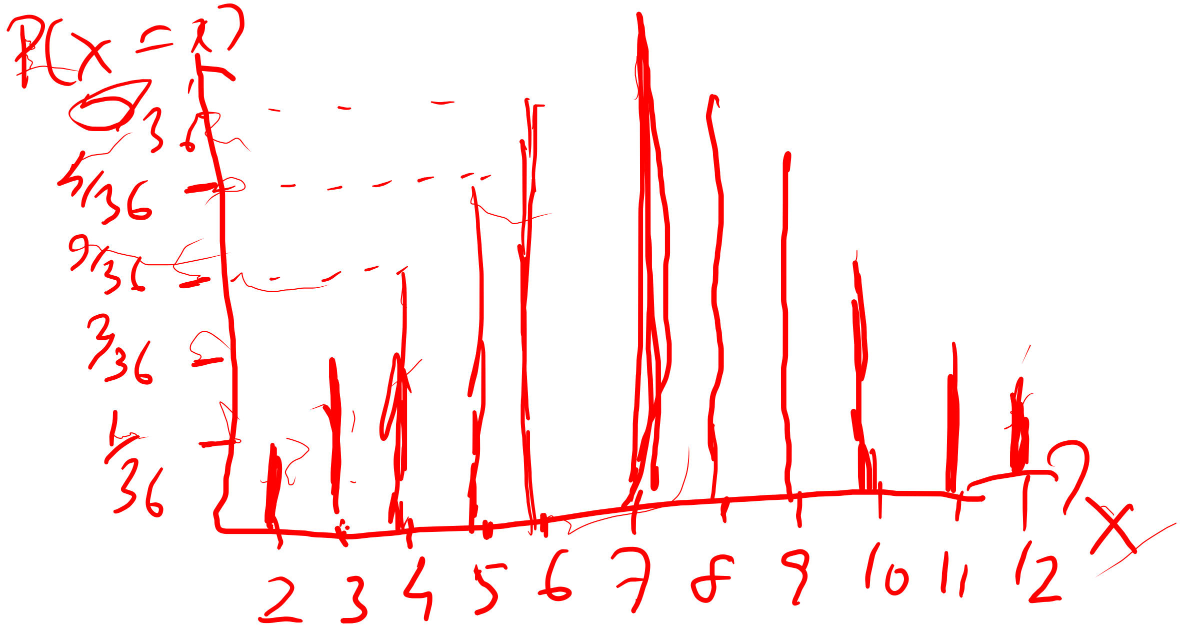
$x = 11 \quad (5,6), (6,5) \longrightarrow 2$

$X = 12$ $\quad (6,6) \to 1$

$P(X=2) \to 1/36$ $\quad P(X=3) = 2/36$ $\quad P(X=4) = \frac{3}{36}$ $\quad P(X=5) = \frac{4}{36}$

$P(X=6) = \frac{5}{36}$ $\quad P(X=7) = 6/36$ $\quad P(X=8) = 5/36$ $\quad P(X=9) = 4/36$

$P(X=10) = \frac{3}{36}$ $\quad P(X=11) \to 2/36$ $\quad P(X=12) \to 1/36$



$$\sum_{x=2}^{12} P(X=x) \to 1$$

$$P(X=x) \geq 0$$

$$\forall x \in 2 \text{ to } 12 \;//$$

. Suppose that an airplane engine will fail, when in flight, with probability 1 - p independently from engine to engine; suppose that the airplane will make a successful flight if at least 50 percent of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane?

Probability of failure $= 1-p$

For 4-engine:

$P(\text{success}) = P(2 \text{ success}) + P(3 \text{ success}) + P(4 \text{ success})$

$= {}^4C_2\, p^2(1-p)^2 + {}^4C_3\, p^3(1-p) + p^4$

For 2-engines:

$P(\text{success}) = P(1 \text{ success}) + P(2 \text{ success})$

$= {}^2C_1\, p(1-p) + p^2$

Check $6p^2(1-p)^2 + 4p^3(1-p) + p^4 \geq 2p(1-p) + p^2$

$P(\text{fail}) \to 1-p$

$P(\text{don't fail}) \to 1 - P(\text{fail}) = p$

. Suppose that an airplane engine will fail, when in flight, with probability 1 - p independently from engine to engine; suppose that the airplane will make a successful flight if at least 50 percent of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane?

Plane with 4 engines: $X \to$ No of engines that don't fail

$P(X \geq 2) = P(X=2) + P(X=3) + P(X=4)$

$\Rightarrow {}^4C_2\, p^2(1-p)^2 + {}^4C_3\, p^3(1-p) + {}^4C_4\, p^4(1-p)^0$ — (I)

Plane with 2 engines:

$P(X \geq 1) \Rightarrow$ ~~${}^2C_0$~~ $P(X=1) + P(X=2)$

$= {}^2C_1\, p(1-p) + {}^2C_2\, p^2$ — (II)

(I) $\geq$ (II)

$$6p^2(1-p)^2 + 4p^3(1-p) + p^4 \geq p^2 + 2p(1-p)$$

$$\Rightarrow 6p^2 + 6p^4 - 12p^3 + 4p^3 - 3p^4 \geq \cancel{p^2 + 2p} \; 2p - p^2$$

$$\Rightarrow 3p^3 - 8p^2 + 7p - 2 \geq 0$$

$$\Rightarrow (p-1)^2(3p-2) \geq 0$$

$$\Rightarrow 3p - 2 \geq 0$$

$$\Rightarrow \boxed{p \geq \frac{2}{3}}$$

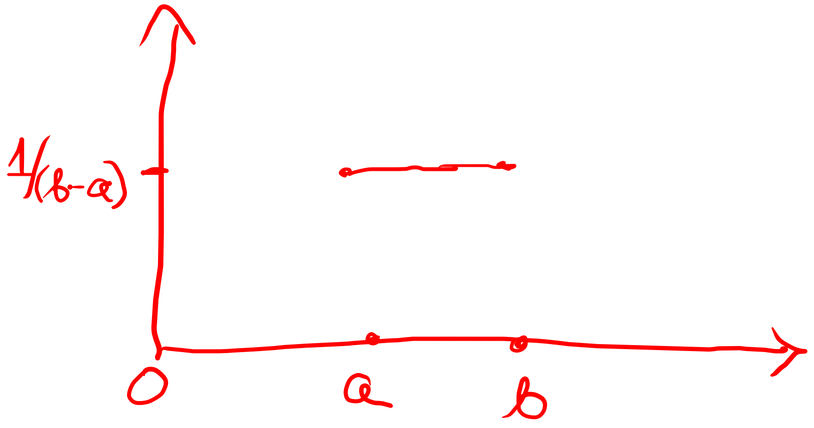
Uniform Distribution:

A r.v. X is called a *uniform* r.v. over (a, b) if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Determine the CDF, mean and variance of this r.v. Plot the PDF and CDF.

Graph of pdf:



$$\text{mean} = E(X) = \int_{-\infty}^{+\infty} x\, p(x)\, dx = \int_a^b \frac{x}{(b-a)}\, dx = \frac{1}{2(b-a)} \left[x^2\right]_a^b = \frac{(b^2-a^2)}{2(b-a)} = \frac{a+b}{2}$$

$$\text{Variance} = E(X^2) - (E(X))^2$$

$$\text{for } E(X^2) = \int_{-\infty}^{+\infty} x^2\, p(x)\, dx = \int_a^b \frac{x^2}{(b-a)}\, dx = \frac{1}{(b-a)} \left[\frac{x^3}{3}\right]_a^b = \frac{(b^3-a^3)}{3(b-a)} = \left(\frac{a^2+ab+b^2}{3}\right)$$

$$\text{So, } \operatorname{var}(X) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(a-b)^2}{12}$$

Uniform Distribution:

A r.v. X is called a *uniform* r.v. over (a, b) if its pdf is given by

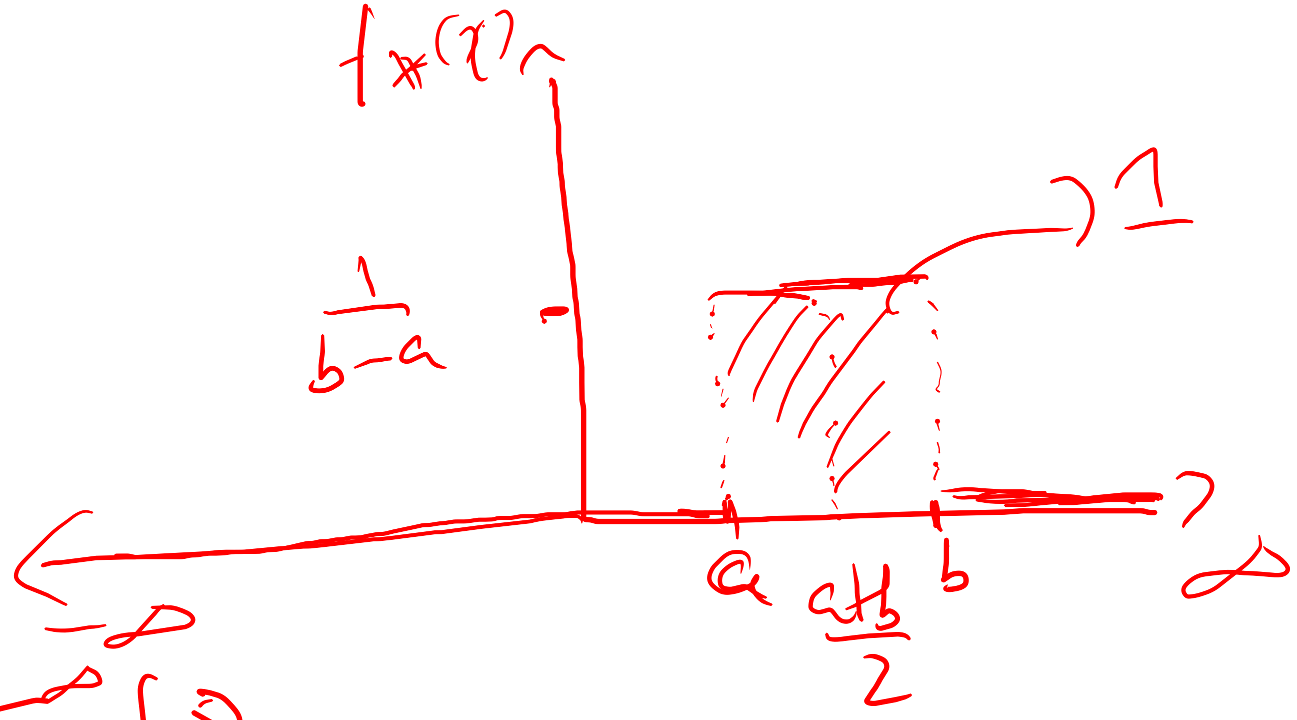
$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Determine the CDF, mean and variance of this r.v. Plot the PDF and CDF.

$$E(x) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

$$= \int_{-\infty}^{a} 0 + \int_{a}^{b} \frac{x \, dx}{b-a} + \int_{b}^{\infty} 0$$

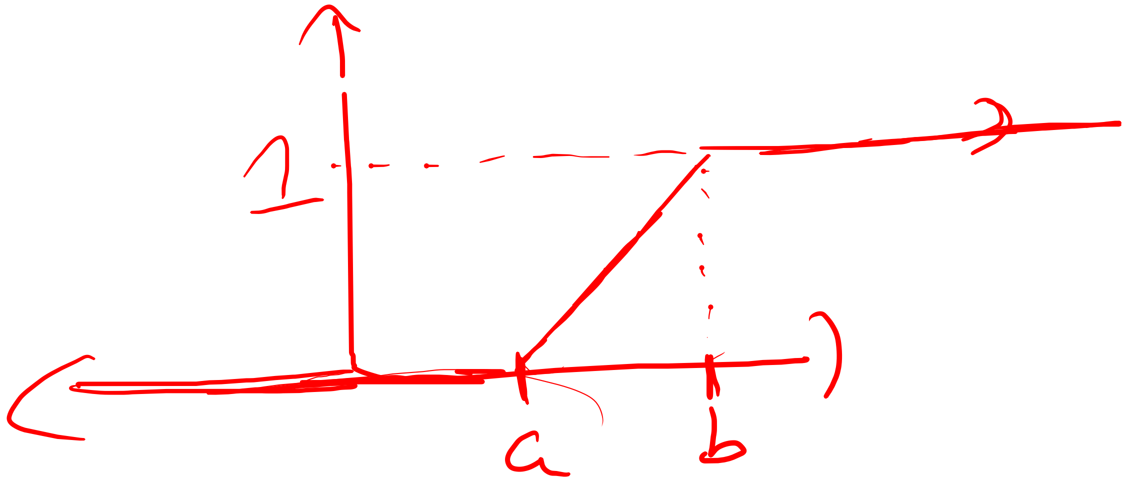
$$= \frac{x^2}{2(b-a)} \Big|_a^b = \frac{a+b}{2}$$



Variance $\rightarrow \sigma^2 = E(x^2) - (E(x))^2$

$$= \int_{-\infty}^{\infty} x^2 f_x \, dx - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

$$F_X(x) = P(X \le x)$$



$$= \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_{-\infty}^{a} 0\,dx + \int_{a}^{x} f_X(x)\,dx$$

$$= \frac{x-a}{b-a}$$

$$F_X(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$

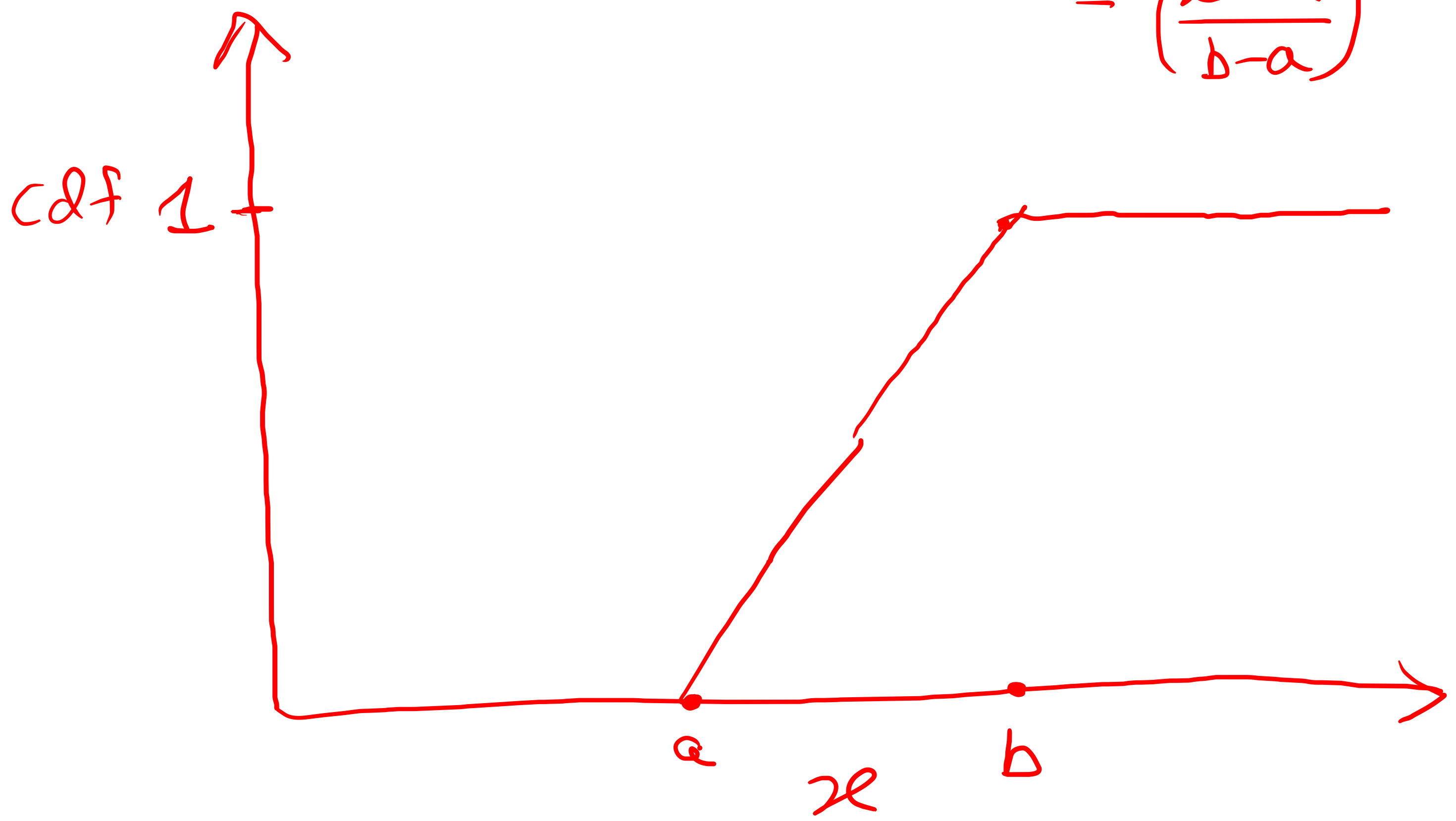
for cdf :

$$P(X \le x) = \int_{-\infty}^{x} p(x)\,dx$$

$$= \int_{-\infty}^{a} 0\,dx + \int_{a}^{x} \frac{1}{(b-a)}\,dx$$

$$= \left(\frac{x-a}{b-a}\right)$$

$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a} & a < x \le b \\ 1, & x > b \end{cases}$$



4.

An information source generates symbols at random from a four-letter alphabet $\{a, b, c, d\}$ with probabilities $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{4}$, and $P(c) = P(d) = \frac{1}{8}$. A coding scheme encodes these symbols into binary codes as follows:

a	0
b	10
c	110
d	111

Let X be the r.v. denoting the length of the code, that is, the number of binary symbols (bits).

(a) What is the range of X?

(b) Assuming that the generations of symbols are independent, find the probabilities $P(X = 1)$, $P(X = 2)$, $P(X = 3)$, and $P(X > 3)$.

i) Range of $X = 1, 2, 3$

ii) $P(X=1) = \frac{1}{2}$

$P(X=2) = \frac{1}{4}$

$P(X=3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, $P(X>3) = 0$

5.

Let X denote the number of heads obtained in the flipping of a fair coin twice.

(*a*) Find the pmf of X.

(*b*) Compute the mean and the variance of X.

$S = \{HH, HT, TH, TT\}$

$R(X) = \{0, 1, 2\}$

$P(X=0) = \frac{1}{4}$ (TT)

$P(X=1) = \frac{1}{4} + \frac{1}{4}$ (HT, TH)

$P(X=2) = \frac{1}{4}$

$$\text{Mean} = \sum_{x=0}^{2} x\,p_x$$

$$= 0\left(\tfrac{1}{4}\right) + 1\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{4}\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2} = \underline{\underline{1}}$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$\downarrow$$

$$E(x^2)$$

know this

$$\text{Ans} = \frac{1}{2}$$

6.

Consider the function given by

$$p(x) = \begin{cases} \dfrac{k}{x^2} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Find the value of k such that $p(x)$ can be the pmf of a discrete r.v. X.

$$k \cdot \underbrace{\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)}_{\frac{\pi^2}{6}} = 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots -$$

Taylor's expansion

$$k \times \frac{\pi^2}{6} = 1 \Rightarrow k = \frac{6}{\pi^2}$$

7.

A digital transmission system has an error probability of 10^{-6} per digit. Find the probability of three or more errors in 10^6 digits by using the Poisson distribution approximation.

$$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\lambda = np$$
$$= 10^6 \times 10^{-6}$$
$$\lambda = 1$$

$$P(x \geq 3) = 1 - P(X=2) - P(X=1) - P(X=0)$$
$$= 1 - \frac{1^2 e^{-1}}{2!} - \frac{1^1 e^{-1}}{1!} - \frac{1^0 e^{-1}}{0!}$$
$$1 - e^{-1}\left[\frac{1}{2} + 1 + 1\right]$$
$$= 1 - e^{-1}\left[\frac{5}{2}\right]$$

8.

Suppose that independent trials, each having probability $p, 0 < p < 1$, of being a success are performed until a total of r successes is accumulated. If we let X equal the number of trials required, then

Prove that

$$P\{X = n\} = \binom{n-1}{r-1} p^r (1-p)^{n-r} \quad n = r, r+1, \ldots$$

Find the mean and variance of this RV.

$n \rightarrow r$ successes

$P(X = n) = {}^{n-1}C_{r-1}\, p^{r-1} \cdot (1-p)^{n-r} \cdot p$

$1, 2 \ldots \ldots n-1, n$

$\underbrace{1, 2 \ldots n-1}_{(r-1)\text{ success}}$ $\boxed{\text{success}}$

${}^{n-1}C_{r-1} \cdot p^{r-1} (1-p)^{n-r} \cdot p$

$$E[X^k] = \sum_{n=r}^{\infty} P(X=n) \cdot n^k$$

$n \rightarrow$ number of trials

$$= \sum_{n=r}^{\infty} n^k \cdot \underline{\binom{n-1}{r-1}} \cdot p^r (1-p)^{n-r}$$

$$= r \sum_{n=r}^{\infty} n^{k-1} \cdot \underline{\left(\frac{n}{r}\right)} p^r (1-p)^{n-r} \underline{\binom{n-1}{r-1}}$$

$$= \frac{r}{p} \sum_{n=r}^{\infty} n^{k-1} \underline{\binom{n}{r}} \cdot p^{r+1} (1-p)^{n-r}$$

$m = n+1$ $\boxed{n = m-1}$ $n \to (r, \infty)$ $m \to (r+1, \infty)$

$$E[X^k] = \frac{r}{p} \sum_{m=r+1}^{\infty} (m-1)^{k-1} \binom{m-1}{r} \cdot p^{r+1} \cdot (1-p)^{m-(r+1)}$$

Probability of getting $(r+1)$ successes in m trails

$Y(r+1, p)$ $m \to Y$

$$= \frac{r}{p} E\left[(Y-1)^{k-1}\right]$$

$$E[X] = \frac{r}{p} \qquad \boxed{k = 1}$$

$$E[X^2] = \left(\frac{r}{p}\right)\left[E[Y-1]\right] \qquad \frac{r+1}{p}$$

$$= \left(\frac{r}{p}\right)\left[E[Y] - 1\right]$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{r(1-p)}{p^2}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \cdots$$

$$= k'(x - z_0)(x - z_1)(x - z_2) \cdots$$

$$= k'(x - \pi)(x + \pi)(x - 2\pi)(x + 2\pi) \cdots$$

$$= k'(x^2 - \pi^2)(x^2 - (2\pi)^2) \cdots$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \Rightarrow \lim_{x \to 0} \frac{\cos x}{1} = 1$$

$$\Rightarrow 1 = k'(-\pi^2)(-4\pi^2)(-9\pi^2) \cdots$$

$$\Rightarrow k' = \frac{1}{-\pi^2(-4\pi^2)(-9\pi^2) \cdots}$$

$$\frac{\sin x}{x} = \frac{1}{(-\pi^2)(-4\pi^2)\cdots}(x^2-\pi^2)(x^2-(2\pi)^2)\cdots$$

$$= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{(2\pi)^2}\right)\cdots$$

Coefficient of $x^2 = -\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} \cdots$

$$= -\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots\frac{1}{3^2}\cdots\right)$$

$$-\frac{1}{6} = -\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots\right)$$

$$\Rightarrow \frac{\pi^2}{6}$$