Ch8

Convergence of a Sequence of Random Variables

E : Random Experiment

(S, Δ, P) : Probability Space

$P : \Delta \rightarrow \mathbb{R}$: Probability Function

$X_n : S \rightarrow \mathbb{R} \quad \forall n \in \mathbb{N}$: sequence of random variables.

X:

$E : x_1, x_2, \dots, x_n$

$X_1 \quad X_2 \quad X_n$

$$\frac{x_1 + x_2 + \dots + x_n}{n}$$

(i) Convergence 'In Probability'

$\Downarrow$

(ii) Convergence 'in Distribution'

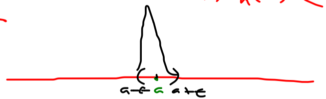
Def^{n}: Convergence in Probability / Stochastic Convergence

A sequence of r.v.s $\{X_n\}$ is said to converge in probability to a constant 'a' if for any $\epsilon > 0$

$$\lim_{n\to\infty} P\big(\underbrace{|X_n - a| < \epsilon}\big) = 1 \qquad (a-\epsilon < X_n < a+\epsilon)$$

$$\equiv \{\omega \in S : a-\epsilon < X_n(\omega) < a+\epsilon\}$$

or $$\lim_{n\to\infty} P\big(|X_n - a| \geq \epsilon\big) = 0$$



and we denote $\boxed{X_n \xrightarrow[\text{in } P]{} a}$ as $n \to \infty$.

Significance: As n increases the probability mass of the distribution of X_n accumulates more and more about the point 'a'.

Note: If $\exists$ a r.v. $X : S \to \mathbb{R}$ s.t.

$X_n - X \xrightarrow[\text{in } P]{} 0$ as $n \to \infty$, then we say

$\{X_n\}$ converges in probability to the r.v. X.

$$\boxed{\begin{aligned} &\lim_{n\to\infty} P\left(|X_n - X| < \epsilon\right) = 1 \\ &\Rightarrow \lim_{n\to\infty} F_n(x) = F(x) \quad \forall x \\ &\qquad\qquad\qquad\qquad \text{(pointwise convergence)} \end{aligned}}$$

Convergence in Distribution

$\{f_n(x)\}$ (i) pointwise conv. $x_0 \in D$ $f_n : D \to R$ $\{f_n(x_0)\}$

(ii) Uniform Convergence.

Let (i) $\{X_n\}$ be a seq. of r.v.s

(ii) $F_n(x)$ is the distribution function of X_n $\forall n$.

If $\exists$ a r.v. X whose distribution function is $F(x)$ such that $\lim_{n \to \infty} F_n(x) = F(x)$ at every point of continuity x of $F(x)$, then (pointwise-convergence)

$\{X_n\}$ is said to be convergent in distribution and we denote $\boxed{X_n \xrightarrow[\text{in } L]{} X}$ as $n \to \infty$.

Th Convergence in probability implies convergence in distribution.

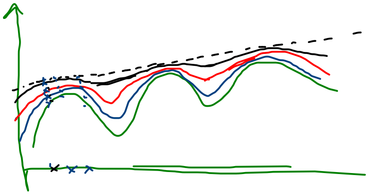
Pointwise Convergence:

$f_n : D \longrightarrow \mathbb{R} \quad \forall n \in \mathbb{N}$

$$\frac{x_0 \in D}{\{f_n(x_0)\} \longrightarrow f(x_0)}$$

If above is true for all $x_0 \in D$:

Then, $\{f_n(x)\}$ converges pointwise to $f(x) \quad \forall\, x \in D$.



Th If $X_n \xrightarrow[\text{in } P]{} a$, $Y_n \xrightarrow[\text{in } P]{} b$ as $n \to \infty$, then

(i) $X_n \pm Y_n \xrightarrow[\text{in } P]{} a \pm b$

(ii) $X_n Y_n \xrightarrow[\text{in } P]{} ab$

(iii) $X_n / Y_n \xrightarrow[\text{in } P]{} a/b$, provided $b \neq 0$.

Pf: Let $\epsilon > 0$ be any no.

$$\left.\begin{aligned} X_n \xrightarrow[\text{in } P]{} a &\Rightarrow \lim_{n\to\infty} P\left(|X_n - a| \geq \epsilon/2\right) = 0 \\ Y_n \xrightarrow[\text{in } P]{} b &\Rightarrow \lim_{n\to\infty} P\left(|Y_n - b| \geq \epsilon/2\right) = 0 \end{aligned}\right\} \text{— (1)}$$

Let A_n denote the event $|X_n - a| < \epsilon/2$

$\hookrightarrow \{\omega \in S : |X_n(\omega) - a| < \epsilon/2\}$

B_n denote . . . . $|Y_n - b| < \epsilon/2$

C_n . . . . . $|(X_n + Y_n) - (a+b)| < \epsilon$.

If A_n and B_n occur simultaneously, then

$|(X_n + Y_n) - (a+b)| \leq |X_n - a| + |Y_n - b| < \epsilon \Rightarrow C_n$ occurs

Thus, $A_n B_n$ implies C_n or $\omega \in A_n B_n \Rightarrow \omega \in C_n$

or, $A_n B_n \subseteq C_n \Rightarrow \overline{A_n B_n} \supseteq \overline{C_n}$

$$\Rightarrow \quad \overline{C_n} \leq \overline{A_n B_n} = \bar{A}_n + \bar{B}_n$$

$$\Rightarrow \quad P(\bar{C}_n) \leq P(\bar{A}_n + \bar{B}_n) = P(\bar{A}_n) + P(\bar{B}_n) - P(\bar{A}_n \bar{B}_n)$$

$$\leq P(\bar{A}_n) + P(\bar{B}_n) \quad \text{---} \ (*)$$

$\bar{A}_n : \ |X_n - a| \geq \epsilon/2$

$\bar{B}_n : \ |Y_n - b| \geq \epsilon/2$

$\bar{C}_n : \ |(X_n + Y_n) - (a+b)| \geq \epsilon$

$\boxed{X_n + Y_n \xrightarrow[\text{in } P]{} a+b}$ $\boxed{\text{as } n \to \infty}$.

$$(*) \Rightarrow \quad 0 \leq P\{|(X_n+Y_n) - (a+b)| \geq \epsilon\} \leq P(|X_n - a| \geq \epsilon/2) + P(|Y_n - b| \geq \epsilon/2)$$

as $n \to \infty$, using (1), $\lim_{n\to\infty} P\{|(X_n+Y_n) - (a+b)| \geq \epsilon\} = 0$.

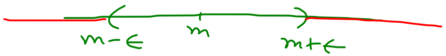
$$\Rightarrow \quad X_n + Y_n \xrightarrow[\text{in } P]{} a+b.$$

Th If $X_n \xrightarrow[\text{in } P]{} a$ as $n \to \infty$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $g(X_n) \xrightarrow[\text{in } P]{} g(a)$ as $n \to \infty$.

Tchebycheff's Inequality

If X is any r.v. having finite variance, then for any $\epsilon > 0$

$$P(|X-m| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$



where m and σ respectively denote the mean and S.D. of X.

Significance: T.I. states the amount of probability mass outside the interval $(m-\epsilon, m+\epsilon)$ is less then or equal to σ^2/ϵ^2, which is small, for a given ϵ, if the variance is small.

Case: X is continuous r.v. with p.d.f. $f_X(x)$

$$P\left(|X-m| \geq \epsilon\right) = \int_{|x-m| \geq \epsilon} f_X(x)\, dx$$

$$\leq \frac{1}{\epsilon^2} \int_{|x-m| \geq \epsilon} (x-m)^2 f_X(x)\, dx$$

$$\leq \frac{1}{\epsilon^2} \int_{-\infty}^{\infty} (x-m)^2 f_X(x)\, dx$$

$$= \frac{\sigma^2}{\epsilon^2}$$

$$\boxed{|X-m| \geq \epsilon \Rightarrow \frac{(X-m)^2}{\epsilon^2} \geq 1.}$$

Pf: HW

Tchebycheff's Theorem

Let $\{X_n\}$ be a sequence of r.v.s s.t.
$E(X_n) = m_n$ and $\sigma(X_n) = \sigma_n$ for $n = 1, 2, \ldots$
(exist finitely).
If $\sigma_n \to 0$ as $n \to \infty$, then $\boxed{X_n - m_n \xrightarrow[\text{in } P]{} 0}$ as $n \to \infty$

Pf: $E(X_n - m_n) = 0 \quad \forall n \in \mathbb{N}$

$Var(X_n - m_n) = E\{(X_n - m_n)^2\} = Var(X_n) = \sigma_n^2$ ✓

By T.I., $0 \le P(|X_n - m_n| \ge \epsilon) \le \dfrac{\sigma_n^2}{\epsilon^2}$, for any $\epsilon > 0$.

Since $\sigma_n^2 \longrightarrow 0$ as $n \longrightarrow \infty$, we get

$$\lim_{n \to \infty} P\left(|X_n - m_n| \geq \epsilon\right) = 0 \quad \text{for any } \epsilon > 0$$

$$\Rightarrow \quad X_n - m_n \xrightarrow[\text{in } P]{} 0 \quad \text{as } n \longrightarrow \infty.$$

③ Bernoulli's Theorem

Let $\{X_n\}$ be a sequence of r.v.s such that

$X_n \sim \text{Binomial}(n, p)$.

$f_i = \binom{n}{i} p^i (1-p)^{n-i}$

$i = 0, 1, 2 - - -, n$

Then $\boxed{\dfrac{X_n}{n}} \xrightarrow[\text{in } P]{} p$ as $n \to \infty$

Pf: $X_n \sim \text{Binomial}(n, p)$.

$$E(X_n) = np, \quad Var(X_n) = npq \quad (q = 1-p).$$

$$E\left(\frac{X_n}{n}\right) = p, \quad Var\left(\frac{X_n}{n}\right) = E\left\{\left(\frac{X_n}{n} - p\right)^2\right\} = \frac{1}{n^2} E\{(X_n - np)^2\}$$

$$= \frac{1}{n^2} Var(X_n) = \frac{pq}{n}$$

$$\sigma\left(\frac{X_n}{n}\right) = \sqrt{\frac{pq}{n}} \longrightarrow 0 \text{ as } n \to \infty$$

Using Tchebycheff's Theorem;

$$\frac{X_n}{n} - p \xrightarrow[\text{in } P]{} 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \boxed{\frac{X_n}{n}} \xrightarrow[\text{in } P]{} p \quad \text{as } n \to \infty.$$

frequency ratio.

probability of 'success'

(Frequency defn of probability).

④ (Law of Large Numbers)

Let $\{X_n\}$ be a seq. of r.v.^{s} s.t. $S_n = X_1 + X_2 + \cdots + X_n$ has finite mean M_n and finite S.D. Σ_n. If $\Sigma_n = o(n)$, i.e. $\frac{\Sigma_n}{n} \to 0$ as $n \to \infty$, then $\boxed{\frac{S_n - M_n}{n} \xrightarrow[\text{in } P]{} 0}$ as $n \to \infty$.

Pf:

$$E\left(\frac{S_n - M_n}{n}\right) = 0$$

$$Var\left(\frac{S_n - M_n}{n}\right) = \frac{1}{n^2} E\left\{(S_n - M_n)^2\right\} = \frac{\Sigma_n^2}{n^2}$$

$$\sigma\left(\frac{S_n - M_n}{n}\right) = \frac{\Sigma_n}{n} \longrightarrow 0 \text{ as } n \to \infty.$$

Hence by T. T.

5 (Law of Large Numbers with equal Components)

Let (i) $\{X_n\}$ be a sequence of r.v.s so that all of them have identical distribution with finite mean m and finite S.D. σ and i.i.d.

(ii) $X_1, X_2, \ldots, X_n$ are mutually independent $\forall n$.

Then $\bar{X} = \dfrac{X_1+X_2+\cdots+X_n}{n} \xrightarrow[\text{in } P]{} m$ as $n \to \infty$

Significance: Sample Mean converges to population mean for large n.

Pf: $E(\bar{X}) = m$

$$Var(\bar{X}) = E\{(\bar{X}-m)^2\} = \frac{1}{n^2}\{Var(X_1) + \cdots + Var(X_n)\}$$

expand ?

since $X_1, X_2, \ldots, X_n$ are mutually independent.

$$= \frac{\sigma^2}{n}$$

$$\sigma(\bar{X}) = \frac{\sigma}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty$$

Using T. T. $\bar{X} \xrightarrow[\text{in } P]{} m$ as $n \to \infty$.