Mathematics 3 (SM 211): Probability and Statistics

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Ch. 4: Mathematical Expectation - I

Syllabus: Outline

Probability:

1. The Concept of Probability
2. Compound or Joint Experiment
3. Probability Distributions-I
4. Mathematical Expectation-I
5. Probability Distributions-II
6. Mathematical Expectation-II
7. Some Important Continuous Univariate Distributions
8. Convergence of a Sequence of Random Variables and Limit Theorems

Statistics:

1. Random Samples
2. Sampling Distributions
3. Estimation of Parameters
4. Testing of Hypothesis
5. Regression

Reference Books

1. Mathematical Probability by A. Banerjee, S.K. De and S. Sen

2. Mathematical Statistics by S.K. De and S. Sen

3. Groundwork of Mathematical Probability and Statistics by Amritava Gupta

4. Introduction to Probability and Statistics for Engineers and Scientists by S.M. Ross

5. Introduction to Probability Models, by S.M. Ross

6. Probability and Statistics, (Schaum's Outlines) by Murray R Spiegel, John J Schiller and R Alu Srinivasan

Mathematical Expectation - I

Objective

- Expectation - Properties
- Variance, SD, Moments, Skewness, Kurtosis
- Moment Generating Function, Characteristic Function

Motivation: An example of a Game of Chance

Rules:

1. A player needs to pay Rs. $M = 10$ to participate (for each play)
2. The player draws a card from a well-shuffled pack of 52 cards. He receives the following amounts on occurrence of the following events:

   - If A_1 : 'King'. Then receives $a1 = 14$.
   - If A_2 : 'Queen' of 'Spade'. Then receives $a2 = 13$.
   - If A_3 : Ace of 'Spade' or 'Club'. Then receives $a3 = 12$.
   - If A_4 : 'Queen' of 'Heart' or 'Diamond' or 'Club'. Then receives $a4 = 11$.
   - If A_5 : Ace of 'Heart' or 'Diamond'. Then receives $a5 = 10$.
   - If A_6 : Card which is not 'King' or 'Queen' or 'Ace'. Then receives $a6 = 9$.

Q: Should the player participate in the game?

Motivation: An example of a Game of Chance

Note:

If the player participate only once, it is difficult to give a definite answer. But if the player participates large number of times, we can give an answer.

- The events $A_1, A_2, \ldots, A_6$ are mutually exclusive and exhaustive set of events.
- Let the player participate N times and $N(A_1), N(A_2), \ldots, N(A_6)$ be the number of occurrences of the corresponding events
- Then the total amount received by the player $= a_1 N(A_1) + a_2 N(A_2) + \ldots + a_6 N(A_6)$
- Then the *average amount* received per trial $= \frac{a_1 N(A_1) + a_2 N(A_2) + \ldots + a_6 N(A_6)}{N}$

Motivation: An example of a Game of Chance

- When $N \to \infty$, LHS $\to a_1 P(A_1) + a_2 P(A_2) + \ldots + a_6 P(A_6)$. This quantity is called the expectation of a random variable $X : S \to \mathbb{R}$ which is defined as:

$$X = i \text{ if event } A_1 \text{occurs } (i = 1, 2, \ldots, 6).$$

- We write:

$$E(X) = a_1 P(X = 1) + a_2 P(X = 2) + \ldots + a_6 P(X = 6).$$

- $P(A_1) = P(X = 1) = \frac{4}{52}$,
  $P(A_2) = P(X = 2) = \frac{1}{52}$,
  $P(A_3) = P(X = 3) = \frac{2}{52}$,
  $P(A_4) = P(X = 4) = \frac{3}{52}$,
  $P(A_5) = P(X = 5) = \frac{2}{52}$,
  $P(A_6) = P(X = 6) = \frac{40}{52}$.
- $E(X) \approx 9.73 \quad (< 10)$

Expectation: Definition

Discrete Case

Let X be a discrete random variable with spectrum $\{x_0, x_{\pm 1}, x_{\pm 2}, \ldots\}$. Then

$$E(X) = \sum_{i=-\infty}^{i=\infty} x_i P(X = x_i) = \sum_{i=-\infty}^{i=\infty} x_i f_X(x_i)$$

provided the infinite series is absolutely convergent. Here f_X is the p.m.f. of X.

Expectation: Interpretation

Discrete Case

1. Expectation of a r.v. X, in the long run (when the experiment E is repeated large number of times), is the average of the outcomes of X.
   Ex. E : Throwing a die
2. Position of **centre of mass** of the probability mass distribution on a straight line.

Expectation: Definition

Continuous Case

$E(X) = \int_{-\infty}^{\infty} x\, f_X(x) dx$

provided the improper integration is absolutely convergent. Here, f_X is p.d.f. of X.

- Note, $E(X)$ is also called the 'mean of X'

Expectation of Functions of Random Variable

Discrete Case

Ex. Let X be a random variable with spectrum $\{-1, 0, 1\}$.
$P(X = -1) = 0.2$, $P(X = 0) = 0.5$, $P(X = -1) = 0.3$.
Compute $E(X^2)$.

Expectation of Functions of Random Variable

Property

Let X be a discrete random variable with spectrum $\{x_i : i = 1, 2, \ldots, n\}$ and p.m.f. f_X. Then for any real-valued function g

$$E(g(X)) = \sum_i g(x_i) f_X(x_i).$$

Continuous Case

Ex. Let X be a random variable with p.d.f

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Compute $E(e^X)$.

Expectation of Functions of Random Variable

Continuous Case

Let X be a random variable with p.d.f. f_X. Then for any real-valued function g we have

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

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5. If $g(x) \geq 0$, then $E\{g(X)\} \geq 0$
6. If $g(x) \geq 0$ and $E\{g(X)\} = 0$, then $g(X) = 0$. That is, the random variable $g(X)$ has a one-point distribution at $g(x) = 0$.