Discrete Mathematics, Tutorial IV

1. Let R_1 and R_2 be two equivalence relations on a non-empty set X. Then which of the following is/are true?
   (a) $R_1 \cup R_2$ is an equivalence relation.
   (b) $R_1 \cap R_2$ is an equivalence relation.
2. Let R_1 and R_2 be two equivalence relations on a non-empty set X. Then prove or disprove the following statement: "$R_1 \cup R_2$ is an equivalence relation if and only if $R_1 \circ R_2 = R_1 \cup R_2$".
3. Let $p(n)$ denote the number of different equivalence relations on a set with n elements. Mr. Bean claims that $p(n)$ satisfies the recurrence relation $p(n) = \sum_{j=0}^{n-1} C(n-1, j)p(n-j-1)$, with the initial condition $p(0) = 1$. Here $C(x, y)$ denotes the number of ways of choosing y elements out of x elements. Is Mr. Bean correct?
4. Determine the number of partial orderings that can be constructed over the set $\{1, 2, 3\}$.
5. Let $(S, \leq)$ be a partially-ordered set. For any subset T of S, an element x is called a minimum element of T if x is in T and $x \leq y$ for all y in T.

   Prove or disprove: If every nonempty subset of S has a minimum element, then S is totally ordered.
6. Prove or disprove: For any set A, and any surjective function $f : A \to A$, f is bijective.
7. Let R be an equivalence relation on a set A, where $|A| = 30$ and let R partition A into equivalence classes A_1, A_2 and A_3. If $|A_1| = |A_2| = |A_3|$, then what is $|R|$?
8. Let X and Y be two sets with $|X| = m$ and $|Y| = n$.
   (a) How many functions are possible from X to Y?
   (b) How many injective functions are possible from X to Y?
   (c) How many bijective functions are possible from X to Y?
   (d) Let $S(r, s)$ denote the number of partitions of an r-element set into s non-empty disjoint subsets. The function $S(r, s)$ is also called as the Stirling number of second kind. Then derive a formula for the number of surjective functions from X to Y in terms of m, n and the notation $S(\star, \star)$.
9. Let $S(m, n)$ be Stirling number of the second kind. Then prove or disprove: for every positive integers m, n, where $1 < n \leq m$:
$$S(m+1, n) = S(m, n-1) + nS(m, n)$$
10. Prove or disprove the following:
    (a) Every non-empty symmetric and transitive relation is also a reflexive relation.
    (b) If gof is an injective function, then f is also an injective function. Here $f : A \to B$ and $g : B \to C$.
    (c) If gof is an injective function, then g is also an injective function. Here $f : A \to B$ and $g : B \to C$.
    (d) If gof is a surjection then f is also a surjection. Here $f : A \to B$ and $g : B \to C$.