Mathematics 3 (SM 211): Probability and Statistics

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Ch. 2: Compound Experiment

Syllabus: Outline

Probability:

1. The Concept of Probability
2. Compound or Joint Experiment
3. Probability Distributions-I
4. Mathematical Expectation-I
5. Probability Distributions-II
6. Mathematical Expectation-II
7. Some Important Continuous Univariate Distributions
8. Convergence of a Sequence of Random Variables and Limit Theorems

Statistics:

1. Random Samples
2. Sampling Distributions
3. Estimation of Parameters
4. Testing of Hypothesis
5. Regression

Reference Books

1. Mathematical Probability by A. Banerjee, S.K. De and S. Sen
2. Mathematical Statistics by S.K. De and S. Sen
3. Groundwork of Mathematical Probability and Statistics by Amritava Gupta
4. Introduction to Probability and Statistics for Engineers and Scientists by S.M. Ross
5. Introduction to Probability Models, by S.M. Ross
6. Probability and Statistics, (Schaum's Outlines) by Murray R Spiegel, John J Schiller and R Alu Srinivasan

Compound or Joint Experiment

Objective

- Bernoulli Trials
- Poisson Trials
- Binomial and Multinomial Laws

Compound Experiment

Definition

Let E_1 and E_2 be two random experiments with sample spaces $S_1 = \{u_i^{(1)} : i = 1, 2, \ldots, m\}$ and $S_2 = \{u_j^{(2)} : j = 1, 2, \ldots, n\}$, respectively. The joint performance of E_1 and E_2 is called the compound experiment $E^{'}$ (say) of E_1 and E_2 with sample space:

$$S_1 \times S_2 = \{(u_i^{(1)}, u_j^{(2)}) : 1, 2, \ldots, m; j = 1, 2, \ldots, n\}.$$

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Example

E_1 : throwing a die with $S_1 = \{1, 2, 3, 4, 5, 6\}$
E_2 : tossing a coin with $S_2 = \{H, T\}$
Then the compound experiment $E^{'}$ of E_1 and E_2 has the sample space $S_1 \times S_2 = \{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T),$ (3, T), (4,T), (5, T), (6,T)}.

Stochastically Independent Random Experiments

Definition

The random experiments E_1 and E_2 are called stochastically independent if the assignment of probabilities to the elementary events of their compound experiment $E^{'}$ are:

$$P\{(u_i^{(1)}, u_j^{(2)})\} = P\{u_i^{(1)}\}P\{u_j^{(2)}\}$$

for all i, j.

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Theorem

If A and B are two events connected to the random experiments E_1 and E_2 respectively and if E_1 and E_2 are independent, then

$$P\{(A, B)\} = P(A)P(B).$$

$A = \{u_i : i \in I\} \subseteq S_1$, $B = \{u_j' : j \in J\} \subseteq S_2$,

$(A,B) = \sum_{j \in J} \sum_{i \in I} \{(u_i, u_j')\} \Rightarrow P\{(A,B)\} = \sum_{j \in J} \sum_{i \in I} P\{(u_i, u_j')\} = \sum_{j \in J} \sum_{i \in I} P\{u_i\} P\{u_j'\}$

$= \sum_{i \in I} P\{u_i\} \sum_{j \in J} P\{u_j'\}$

$= P(A)P(B)$

Successive performance of some experiment is called repeated trials of the experiment

E , S

E is repeated n times which will give a compound experiment E^n with sample space

$$S^n = \underbrace{S \times S \times \cdots \times S}_{n \text{ times}} = \left\{ (U_{i_1}, U_{i_2}, \ldots, U_{i_n}) : \begin{array}{l} i_1, i_2, \ldots, i_n \text{ are the} \\ \text{indices connected with } S \end{array} \right\}$$

Repeated trials are independent if

$$P\{(U_{i_1}, U_{i_2}, \ldots, U_{i_n})\} = P(U_{i_1}) P(U_{i_2}) \ldots P(U_{i_n})$$

$\forall \; i_1, i_2, \ldots, i_n$.

Bernoulli Trials

Let E be the random experiment when the event space S consists of two outcomes: 'success' (denoted by 's') and 'failure' (denoted by 'f').

We consider n repeated independent trials of E. These trials are called Bernoulli trials if

the probability of 'success' remains same throughout the trials.

Binomial Law

Th: Let A_i denote the event that there are exactly i successes in a sequence of n ~~independent~~ Bernoulli trials with probability of 'success' p. Then

$$P(A_i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad 0 < p < 1$$

Pf: Sample points of A_i are of type $(s, s, f, s, \ldots, s)$ in which there are exactly i 's's and $(n-i)$ 'f's

$$P\{(s,s,f,s,\ldots,s)\} = p^i (1-p)^{n-i}$$

there are $\binom{n}{i}$ such sample points in A_i

$$\Rightarrow P(A_i) = \binom{n}{i} p^i (1-p)^{n-i}.$$

$$P(S^n) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= (p + 1 - p)^n = 1 .$$

Poisson Approximation to Binomial Law

Poisson Trials

Multinomial Law