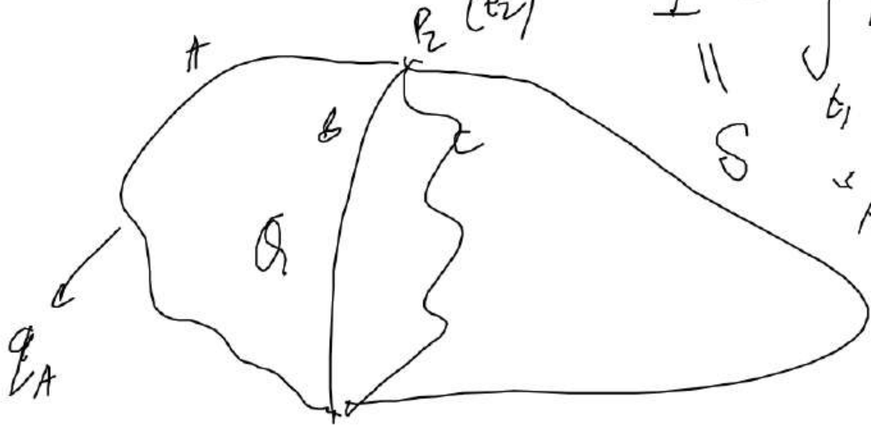
$\delta S = 0$

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

$= S$ "Action"



$q_A(t) = Q + \epsilon \eta(t)$

LANDAU & LIFSHITZ

Classical Mechanics

$$I \text{ or } S = \int L(q, \dot{q}, t)\, dt$$

Lagrangian

q : generalized coord.

$\dot{q}$: generalized vel.

$$\frac{dq}{dt} = \dot{q}$$

Principle of Least Action

(Hamilton's Principle)

$$\delta S = 0$$

(or δI)

Landau & Lifshitz

Rana & Jog

And this yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Herbert Goldstein

Euler-Lagrange Eqn.

(Eqns of motion)

Lagrangian — not unique.

We can construct a different Lagrangian

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

"Gauge function" $\leftarrow$ $f(x,t)$ — some fn. of q, t

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad q \equiv q_i \quad i = 1,2,3$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

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The Lagrangian is defined (only to within) upto an additive total time derivative of a function of co-ordinates & time.

<u>L for a simple pendulum</u> $L = T - V$
KE PE

The Lagrangian is not unique.
One can add a total time derivative of a function of coordinates & time to it.
And still satisfy the Euler-Lagrange eqns

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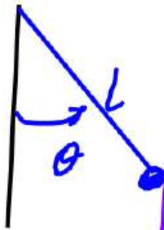
$$L = T - V$$

KE − PE



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$KE = \frac{1}{2} m v^2 = \frac{1}{2} m (l\dot{\theta})^2$$

$$v = \frac{ds}{dt}$$

$$\frac{s}{l} = \theta$$

$$s = l\theta$$

$$\frac{ds}{dt} = l\dot{\theta}$$

$$PE = mgl - mgl\cos\theta$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - (mgl - mgl\cos\theta)$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta - mgl$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(m l^2 \dot{\theta}\right) + mgl\sin\theta = 0$$

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$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

For small angle approximation, $\sin\theta \approx \theta$

$$ml^2\ddot{\theta} + mgl\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\omega^2 = g/l \quad \text{or} \quad \omega = \sqrt{g/l}$$

$$\text{or } f = \frac{1}{2\pi}\sqrt{g/l} \qquad \omega \equiv 2\pi f$$

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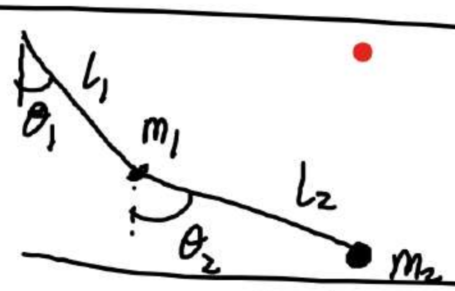
$$f(x, \dot{x}, t) = Ax^2 + Bx + C\dot{x} + D$$

$$\frac{df}{dx} \qquad \frac{\partial f}{\partial x}$$

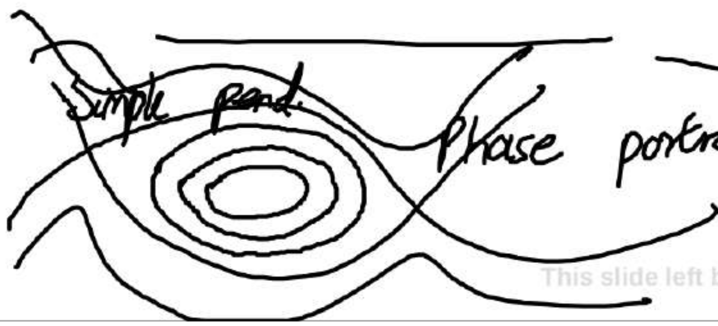
E. Kreyszig

Advanced C

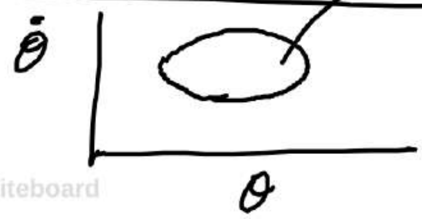
Double pendulum.



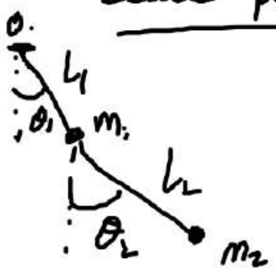
for $\sin\theta \approx \theta$



Phase portrait



Double pendulum



For mass m_1

$$KE = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$

$$PE = \overset{m_1 g l_1}{-m_1 g l_1 \cos\theta_1}$$

For m_2

$$KE = \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2) \equiv T_2$$

$$PE = -m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

$$L = (T_1 + T_2) - (PE_1 + PE_2)$$

$$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2$$

$$\dot{y}_2 = -l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin\theta_2 \dot{\theta}_2$$

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + (2 l_1 l_2 \cos\theta_1 \cos\theta_2 + 2 l_1 l_2 \sin\theta_1 \sin\theta_2) \dot{\theta}_1 \dot{\theta}_2$$

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 \right)$$
$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

Find eqns of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \,, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{d}{dt}\left[m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_2 \right]$$

etc. etc. ...

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + 2 l_1 l_2 m_2 \frac{d}{dt}\left(\cos(\theta_1 - \theta_2) \dot{\theta}_2\right) - (m_1 + m_2) g l_1 (-\sin\theta_1) = 0$$

$$\& \quad m_2 l_2^2 \ddot{\theta}_2 + 2 l_1 l_2 m_2 \frac{d}{dt}\left(\cos(\theta_1 - \theta_2) \dot{\theta}_1\right) - m_2 g l_2 (-\sin\theta_2) = 0$$

Finally:

$$m_1 l_1^2 \ddot{\theta}_1 + m l_1^2 \ddot{\theta}_1 + 2 m_2 l_1 l_2 \left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\right) + (m_1 + m_2) g\, l_1 \sin\theta_1 = 0$$

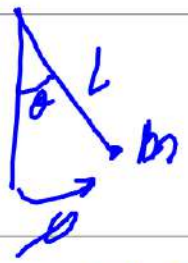
&

$$m_2 l_2^2 \ddot{\theta}_2 + 2 m_2 l_1 l_2 \left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\right) + m_2 g\, l_2 \sin\theta_2 = 0$$

are the eqns of motion.

Spherical pendulum of Length L (fixed).

BUT NOW mass can move in any way.



$$x = l \sin\theta \cos\phi$$

$$y = l \sin\theta \sin\phi$$

$$z = -l\cos\theta$$

$$T = \frac{1}{2} m l^2 \left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right)$$

$$V = -mgl\cos\theta$$

$$\boxed{L = \frac{1}{2} m l^2 \left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + mgl\cos\theta}$$

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The importance of correct choice of generalized co-ords:

Simple pendulum

In cartesian coords. $x = l\sin\theta$

$v = l\dot{\theta}$

$\dot{x} = l\cos\theta\,\dot{\theta} \quad \Rightarrow \quad v = \frac{\dot{x}}{\cos\theta} = \frac{l\dot{x}}{\sqrt{l^2-x^2}}$

$\therefore \quad T = \frac{1}{2} m \frac{l^2\dot{x}^2}{l^2-x^2} \qquad V = -mg\sqrt{l^2-x^2}$

$L = T - V = \frac{1}{2} m \frac{l^2\dot{x}^2}{l^2-x^2} + mg\sqrt{l^2-x^2}$

Final eqns of motion:

$$m\ddot{x} = -\frac{mx\dot{x}^2}{(l^2-x^2)} - \frac{mgx}{l^2}\left(l^2-x^2\right)^{1/2}$$

$\therefore$ Cartesian coords are **NOT** necessarily the best coords for a system.

In general, we can define generalized coords as the independent coords sufficient to completely specify the configuration of a dynamical system.

These need NOT be rectangular Cartesian Co-ords.

In analogy to Newtonian mechanics

momentum $p_i = m v_i = \frac{\partial \overbrace{T}^{KE}}{\partial \dot{x}_i}$ $\quad v_i = \dot{x}_i$ (Cartesian)

generalized momentum p_i corresp. to a generalized co-ord q_i is $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ $\quad \dot{q}_i \equiv$ generalized velocity

For simple (plane) pendulum

$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$. $\quad p_i \equiv \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$ (corresp to θ coord.)

$$L = \frac{1}{2} m l^2 (\dot\theta^2 + \dot\varphi^2 \sin^2\theta) + mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} = m l^2 \dot\varphi^2 \sin\theta\cos\theta - mgl\sin\theta$$

$$\boxed{\frac{\partial L}{\partial \varphi} = 0}$$

$$\frac{\partial L}{\partial \dot\theta} = m l^2 \dot\theta$$

$$\frac{\partial L}{\partial \dot\varphi} = m l^2 \dot\varphi \sin^2\theta.$$

Momentum corresp. to a cyclic coord. is always conserved

$\therefore$ E-L eqns are:

$$m l^2 \ddot\theta - m l^2 \dot\varphi^2 \sin\theta\cos\theta + mgl\sin\theta = 0$$

$$\cancel{m l^2 \ddot\varphi \sin^2\theta = 0} \quad \text{or} \quad \frac{d}{dt}\Big(\underbrace{m l^2 \dot\varphi \sin^2\theta}_{\text{angular momentum}}\Big) = 0$$

$\varphi \equiv$ "cyclic coord." ie. φ is missing.

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