# Basic Signals

Signals and Systems

Prof. Jyotsna Bapat

# Recap

- Signal Properties:
  - Periodic vs aperiodic
  - Power vs energy
  - Causal vs noncausal
  - Odd vs Even

- Signal Operations:
  - Amplitude scaling
  - Time Scaling
  - Shifting
  - Folding

## Periodic Signals

- Very important in this class.

- Continuous time signal is periodic if and only if there exists a $T_0 > 0$ such that

$$x(t + T_0) = x(t) \qquad \text{for all t}$$

  $T_0$ is the period of $x(t)$ in time.

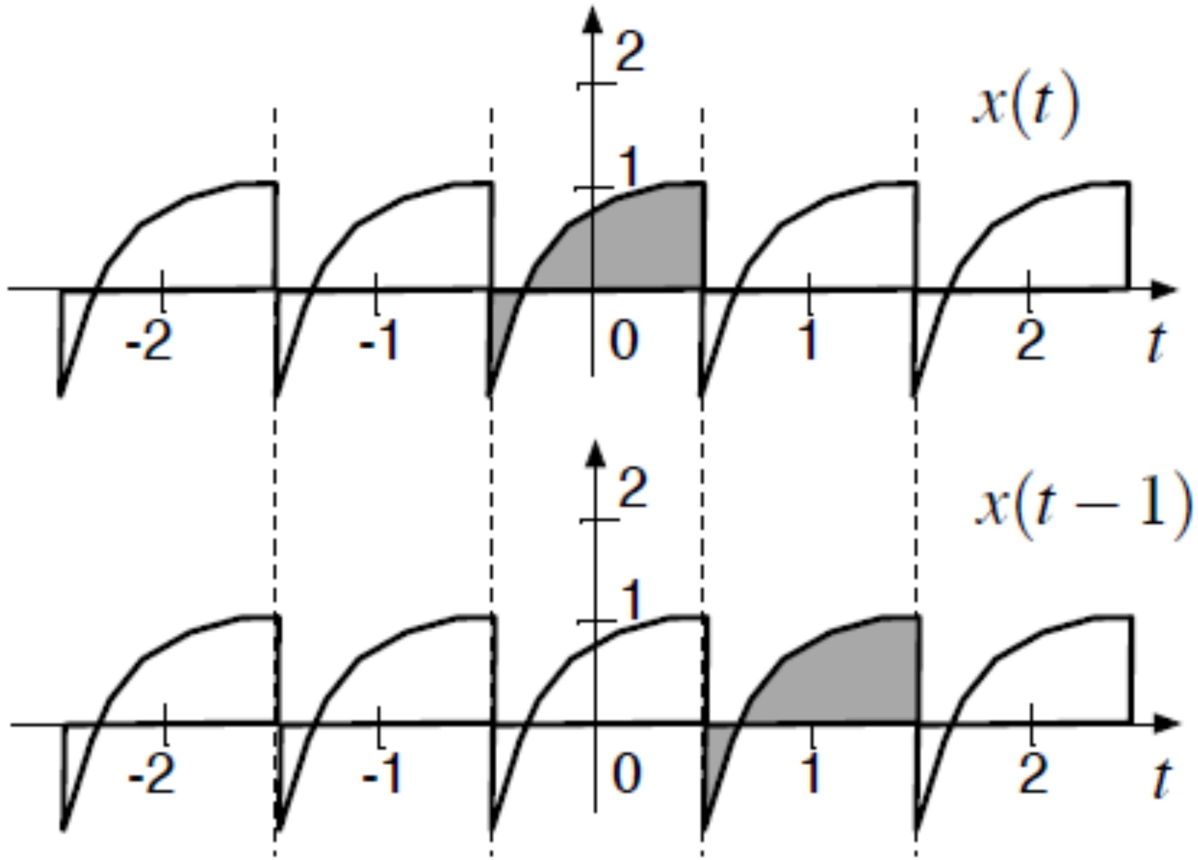
- A discrete-time signal is periodic if and only if there exists an integer $N_0 > 0$ such that

$$x[n + N_0] = x[n] \qquad \text{for all } n$$

  $N_0$ is the period of $x[n]$ in sample spacings.

- The smallest $T_0$ or $N_0$ is the *fundamental period* of the periodic signal.

**Example:**



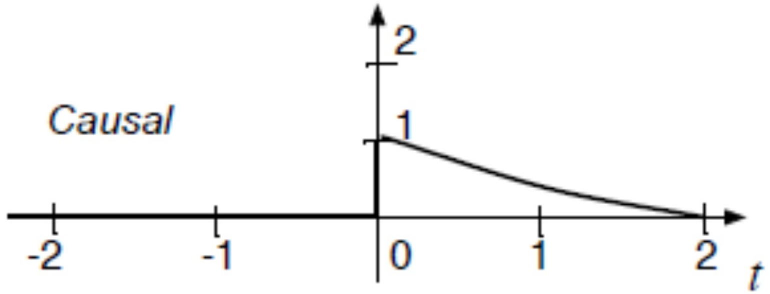
# Periodicity

- What is the span of a Periodic signal ?
- Find whether the given signal is periodic and find fundamental period :

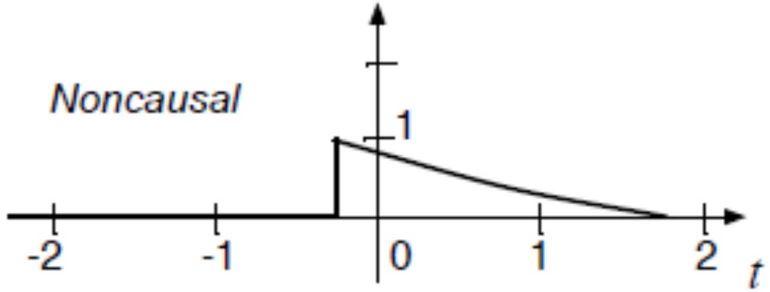
1. $x(t) = sin^2(4\pi t)$
2. $x(t) = \sin(6\pi t) + \cos(5\pi t)$
3. $x[n] = \ e^{j2n}$
4. $x[n] = \cos(\frac{3\pi}{4}n)$
5. $x[n] = \sin(\frac{3\pi}{4}n) + \cos(\frac{5\pi}{7}n)$

## Causal Signals

- *Causal signals* are non-zero only for $t \geq 0$ (starts at $t = 0$, or later)




- *Noncausal signals* are non-zero for some $t < 0$ (starts before $t = 0$)




- *Anticausal signals* are non-zero only for $t \leq 0$ (goes backward in time from $t = 0$)

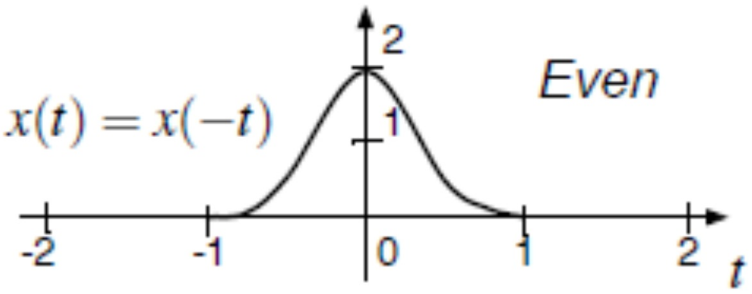
# Even and Odd Symmetry

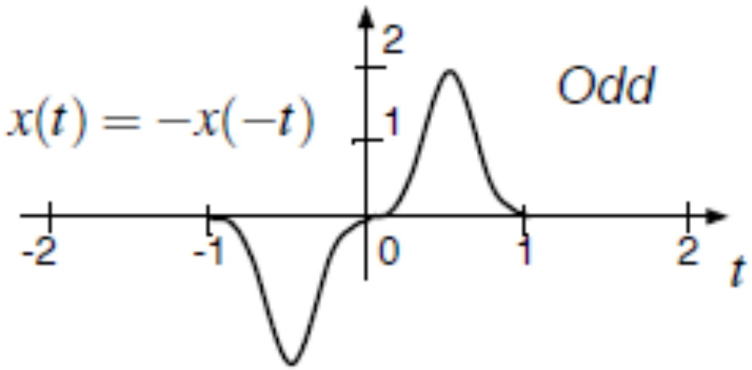
- An *even* signal is symmetric about the origin

$$x(t) = x(-t)$$

- An *odd* signal is antisymmetric about the origin

$$x(t) = -x(-t)$$



- Any signal can be decomposed into even and odd components

$$\begin{aligned} x_e(t) &= \frac{1}{2}[x(t)+x(-t)] \\ x_o(t) &= \frac{1}{2}[x(t)-x(-t)]. \end{aligned}$$

Check that

$$\begin{aligned} x_e(t) &= x_e(-t), \\ x_o(t) &= -x_o(-t), \end{aligned}$$

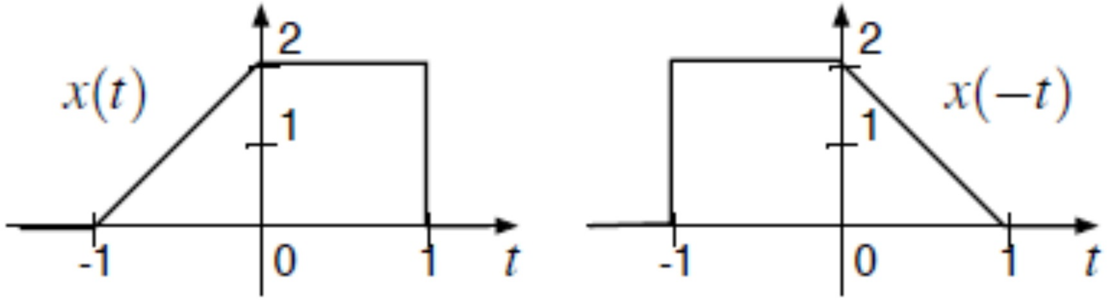
and that

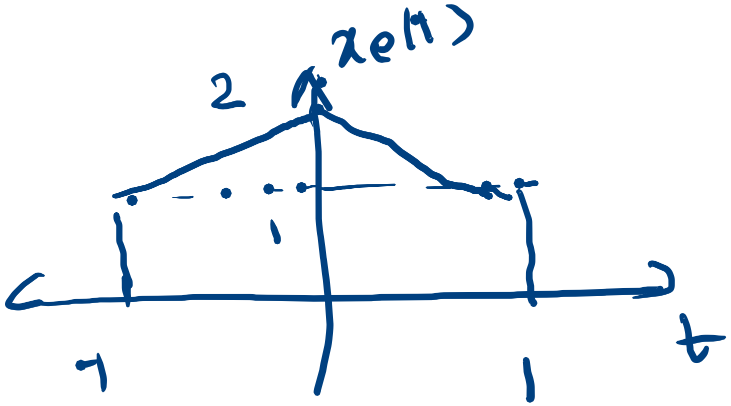
$$x_e(t)+x_o(t)=x(t).$$

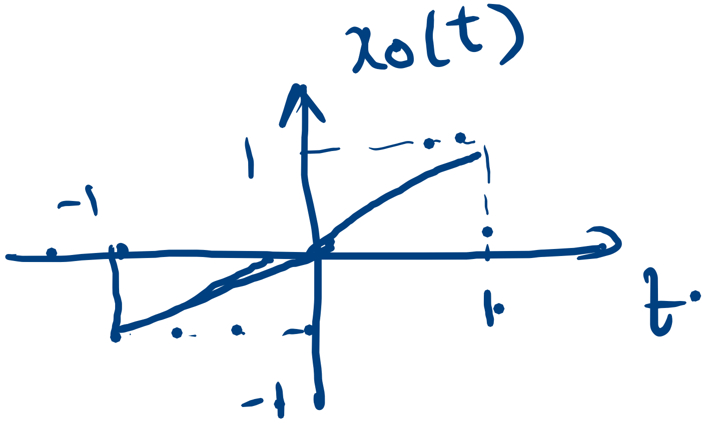
- Example

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

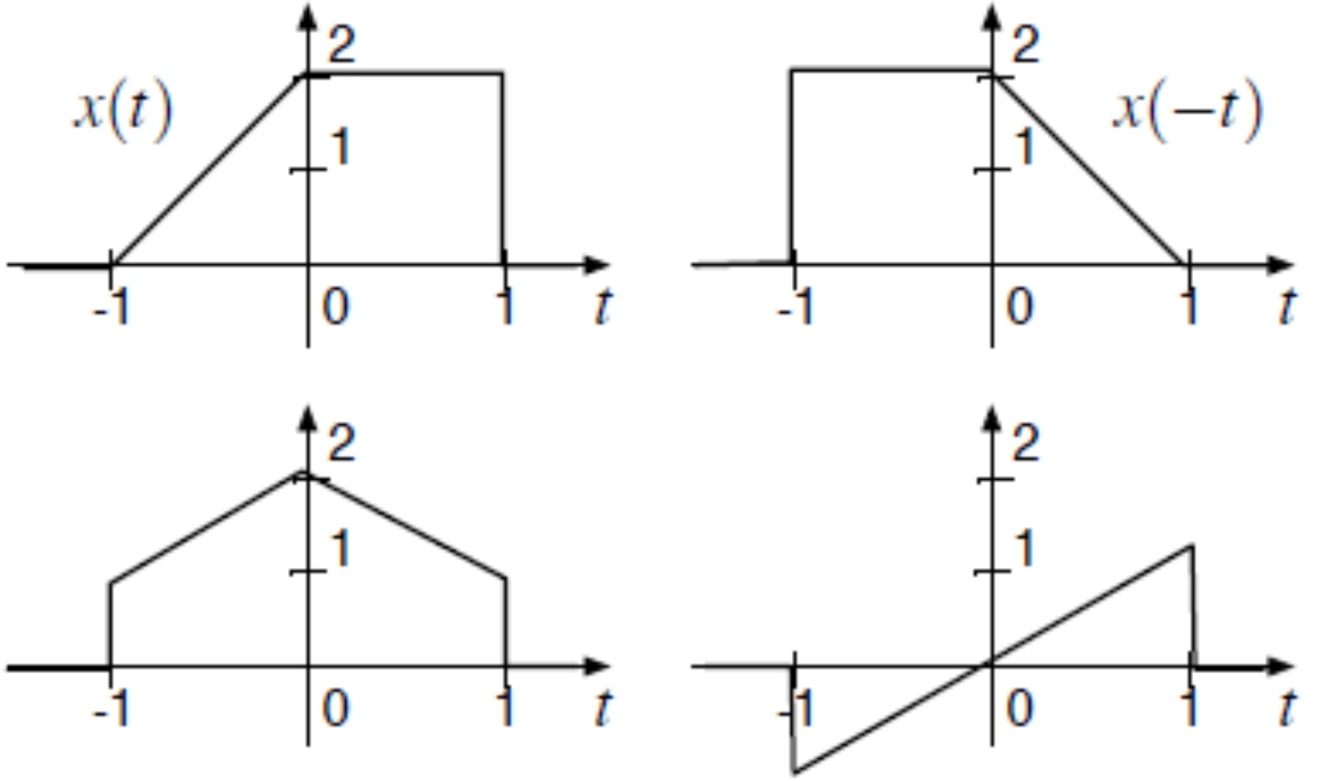
$$x_o(t) = \frac{x(t) - x(-t)}{2}.$$



- Example



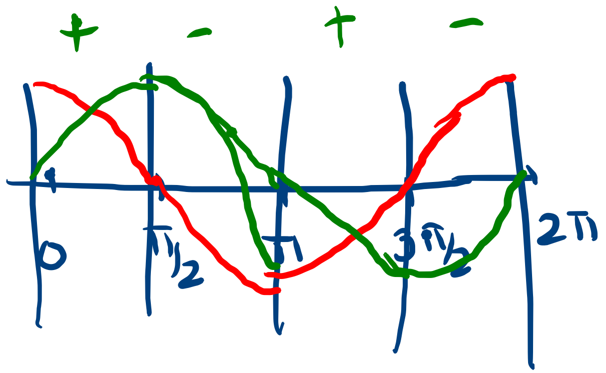

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \qquad x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

# Problems

$$\int_0^{2\pi} \cos\theta \sin\theta \, d\theta$$

$\cos\theta$

$\sin\theta$




1. Find the even and odd components of $e^{jt}$

$x_1(t) = a\cos\theta$

$x_2(t) = b\sin\theta$

$$\int_0^{2\pi} (x_1 + x_2)\cos\theta \, d\theta$$

Let $x_e(t)$ and $x_o(t)$ be the even and odd components of $e^{jt}$, respectively.

$$e^{jt} = x_e(t) + x_o(t)$$

$$x_e(t) = \cos(t) = \frac{e^{jt} + e^{-jt}}{2}$$

$$x_o(t) = j\sin(t)$$

# Problems

1. Find the even and odd components of $e^{jt}$

Let $x_e(t)$ and $x_o(t)$ be the even and odd components of $e^{jt}$, respectively.

$$e^{jt} = x_e(t) + x_o(t)$$

$$x_e(t) = \tfrac{1}{2}(e^{jt} + e^{-jt}) = \cos t$$

$$x_o(t) = \tfrac{1}{2}(e^{jt} - e^{-jt}) = j \sin t$$

Show that the product of two even signals or of two odd signals is an even signal and that the product of an even and an odd signal is an odd signal.

Show that the product of two even signals or of two odd signals is an even signal and that the product of an even and an odd signal is an odd signal.

Let $x(t) = x_1(t)x_2(t)$. If $x_1(t)$ and $x_2(t)$ are both even, then

$$x(-t) = x_1(-t)x_2(-t) = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even.

If $x_1(t)$ and $x_2(t)$ are both odd, then

$$x(-t) = x_1(-t)x_2(-t) = -x_1(t)[-x_2(t)] = x_1(t)x_2(t) = x(t)$$

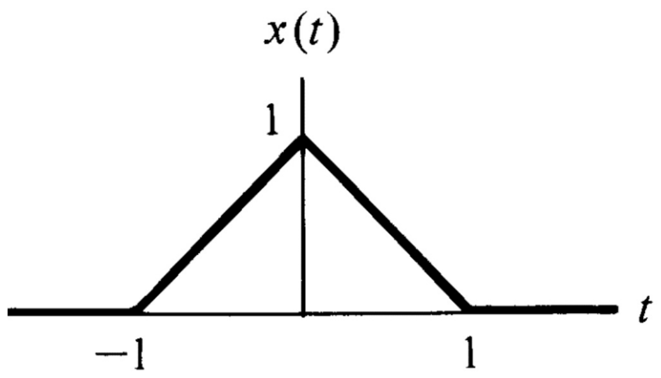
and $x(t)$ is even.

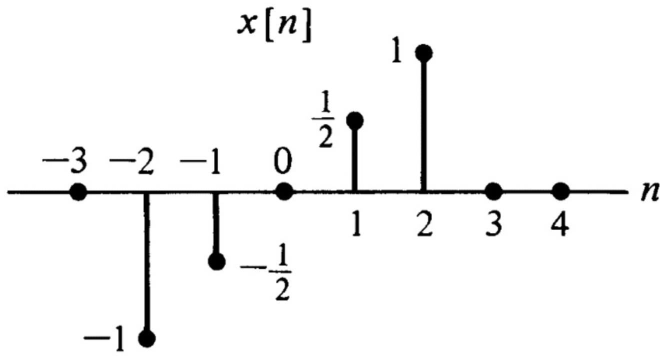
If $x_1(t)$ is even and $x_2(t)$ is odd, then

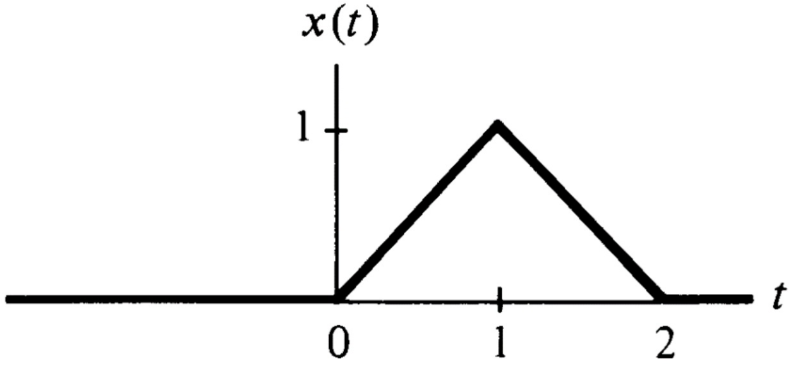
$$x(-t) = x_1(-t)x_2(-t) = x_1(t)[-x_2(t)] = -x_1(t)x_2(t) = -x(t)$$

and $x(t)$ is odd.

For each of the following signal determine if it is even or odd or neither even nor odd

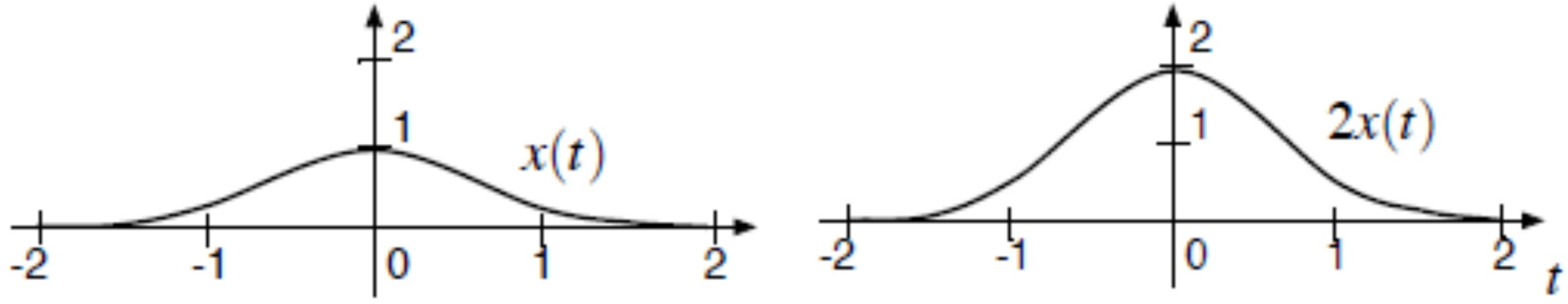


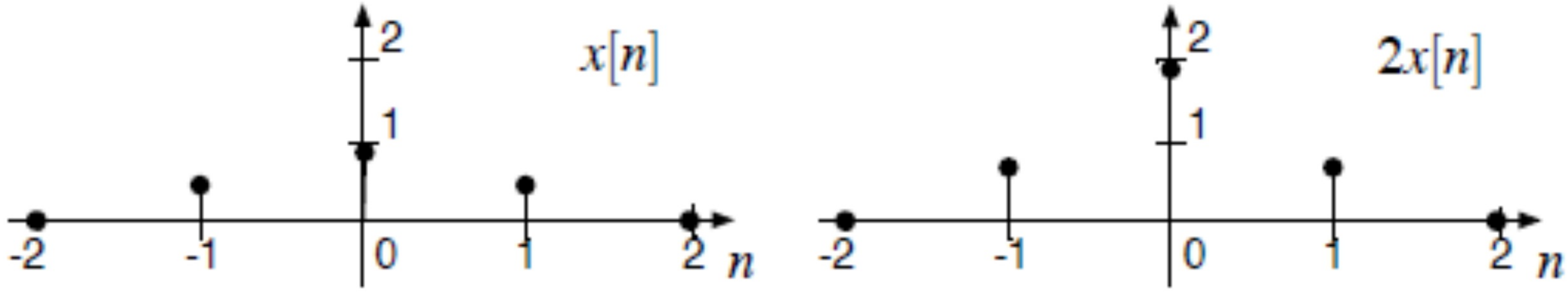



# Amplitude Scaling

- The scaled signal $ax(t)$ is $x(t)$ multiplied by the constant $a$

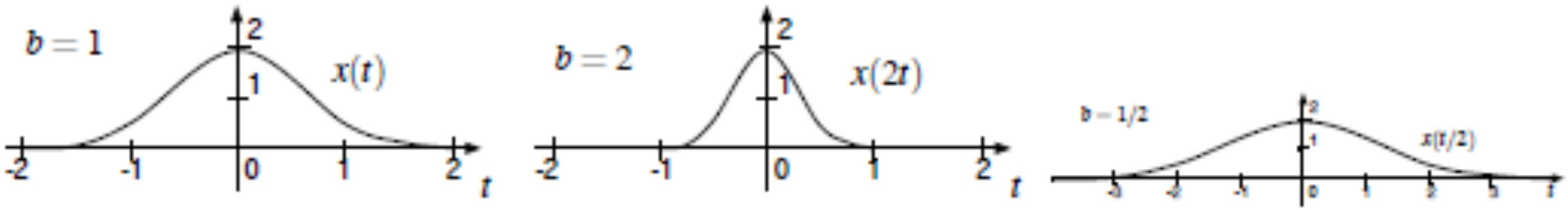



- The scaled signal $ax[n]$ is $x[n]$ multiplied by the constant $a$



# Time Scaling, Continuous Time

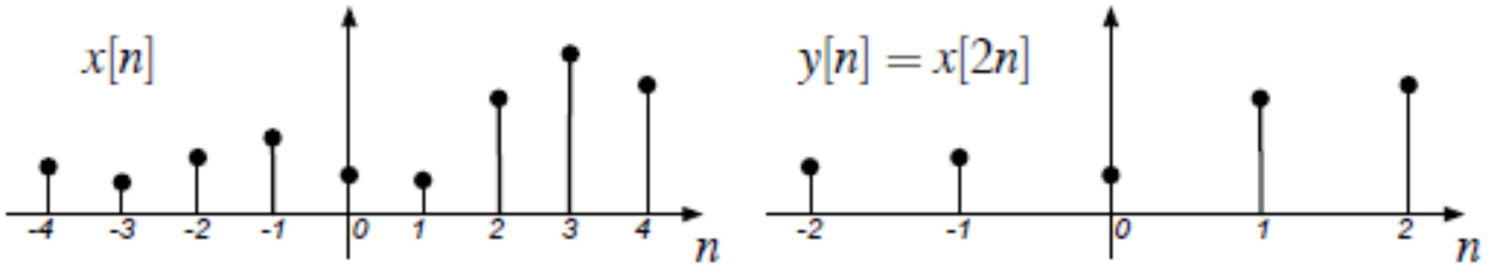
A signal $x(t)$ is scaled in time by multiplying the time variable by a positive constant $b$, to produce $x(bt)$. A positive factor of $b$ either expands $(0 < b < 1)$ or compresses $(b > 1)$ the signal in time.



## Time Scaling, Discrete Time

The discrete-time sequence $x[n]$ is *compressed* in time by multiplying the index $n$ by an integer $k$, to produce the time-scaled sequence $x[nk]$.

- This extracts every $k^{th}$ sample of $x[n]$.
- Intermediate samples are lost.
- The sequence is shorter.




Called *downsampling*, or *decimation*.

The discrete-time sequence $x[n]$ is *expanded* in time by dividing the index $n$ by an integer $m$, to produce the time-scaled sequence $x[n/m]$.

$n = 1:100$

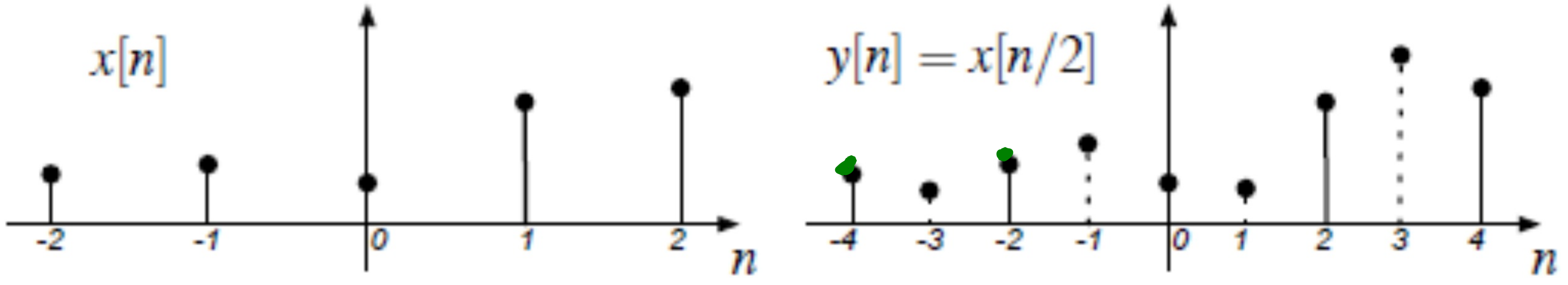
$x(n) = (0.1).\hat{}n$

- This specifies every $m^{th}$ sample.
- The intermediate samples must be synthesized (set to zero, or interpolated).
- The sequence is longer.

$x[n] = \cdot$

$x_1(n) = x(2n)$

$x_1 = x(1:2:\text{length}(x))$




Called *upsampling*, or *interpolation*.

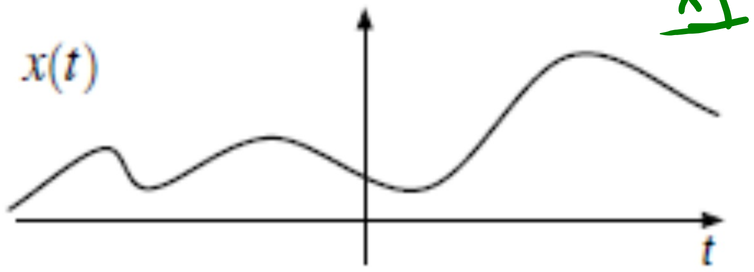
# Time Reversal

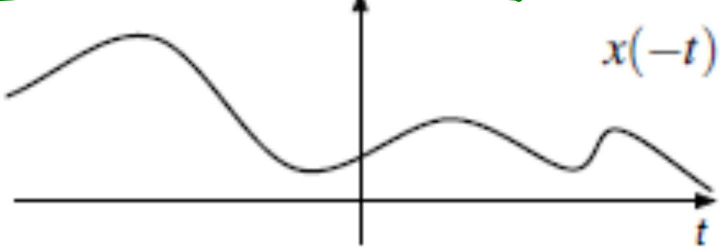
$x \longrightarrow 1000 \quad x(1000)$

$x$ $\boxed{1}$ ... $\boxed{1000}$

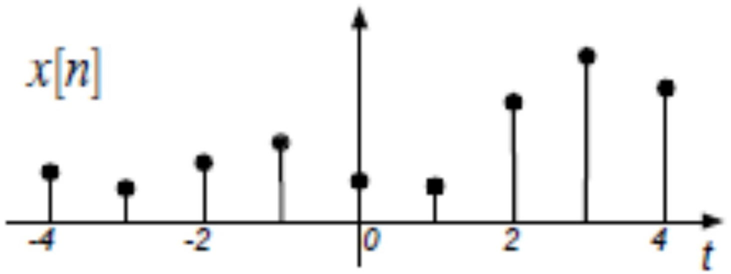
$x_1$

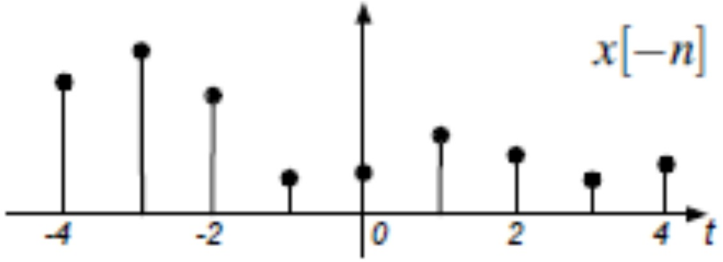
- Continuous time: replace $t$ with $-t$, time reversed signal is $x(-t)$

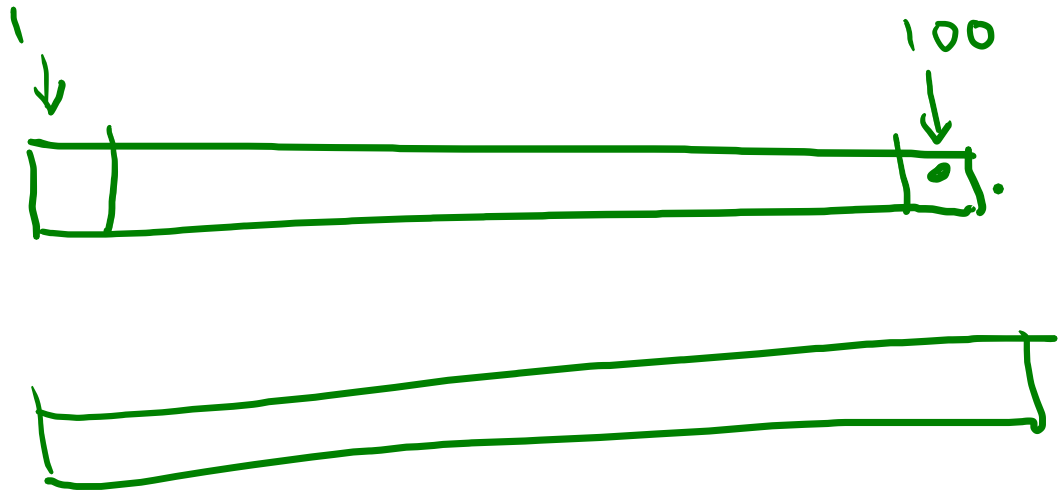



- Discrete time: replace $n$ with $-n$, time reversed signal is $x[-n]$.



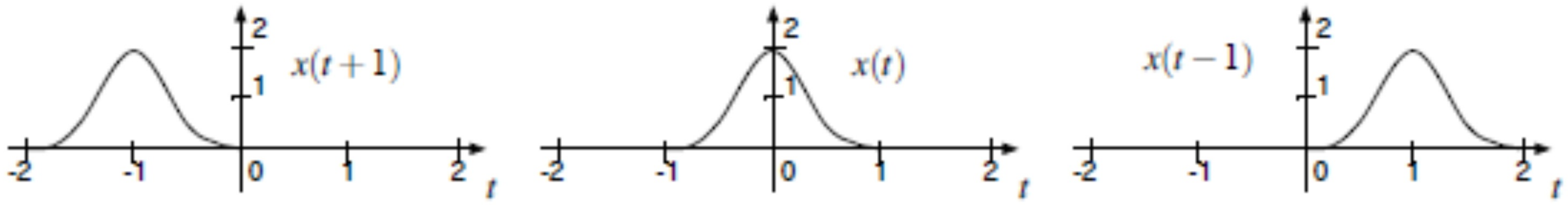



index

# Time Shift

For a continuous-time signal $x(t)$, and a time $t_1 > 0$,

- Replacing $t$ with $t - t_1$ gives a *delayed* signal $x(t - t_1)$
- Replacing $t$ with $t + t_1$ gives an *advanced* signal $x(t + t_1)$



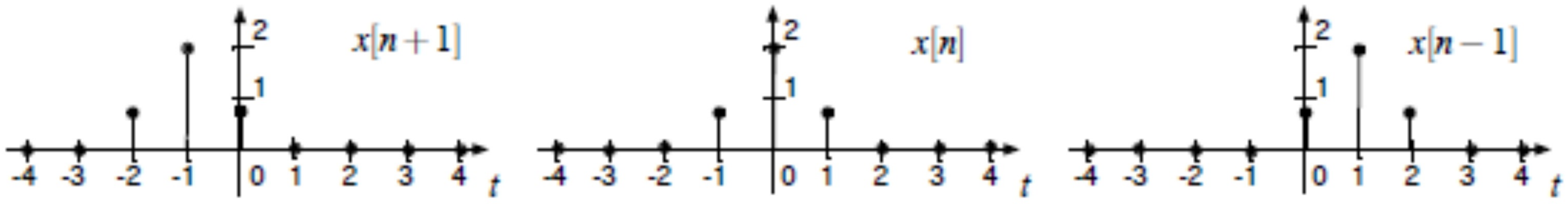
$x(t)$

$\rightarrow x(1-t)$

$x(-1-t)$

For a discrete time signal $x[n]$, and an integer $n_1 > 0$

- $x[n - n_1]$ is a delayed signal.
- $x[n + n_1]$ is an advanced signal.
- The delay or advance is an integer number of sample times.



## Complex Signals

- So far, we have only considered real (or integer) valued signals.
- Signals can also be complex

$$z(t) = x(t) + jy(t)$$

where $x(t)$ and $y(t)$ are each real valued signals, and $j = \sqrt{-1}$.

- Arises naturally in many problems
  - Convenient representation for sinusoids
  - Communications
  - Radar, sonar, ultrasound

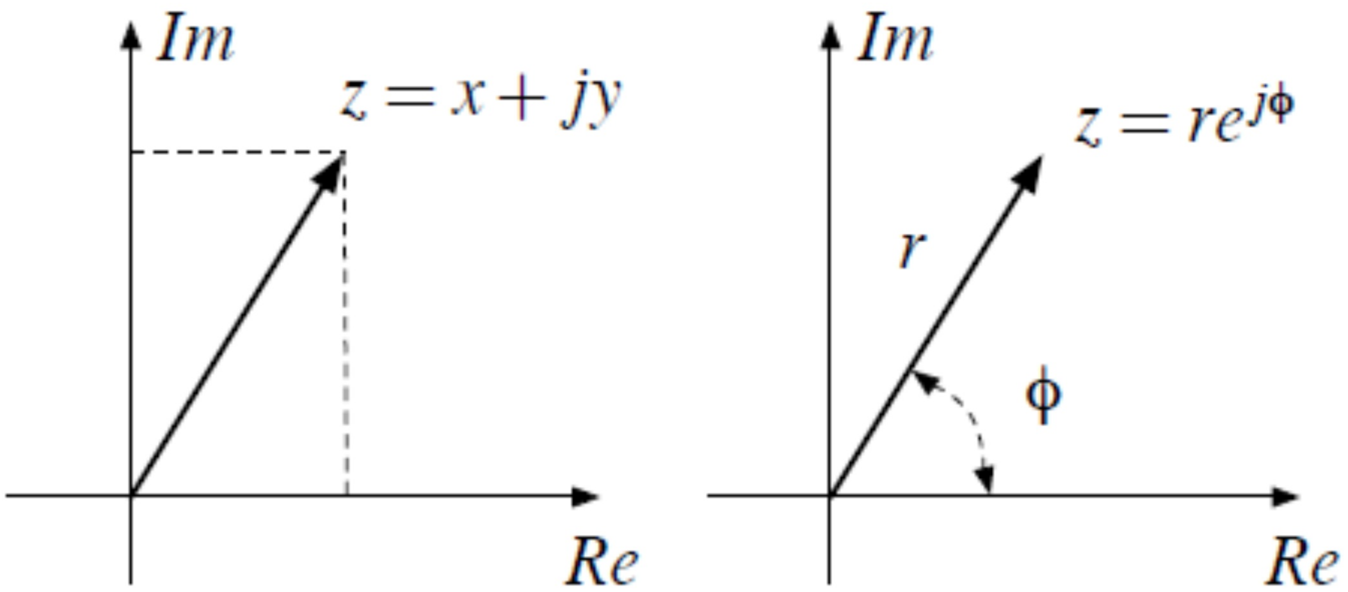
# Review of Complex Numbers

Complex number in Cartesian form: $z = x + jy$

- $x = \Re z$, the *real part* of $z$
- $y = \Im z$, the *imaginary part* of $z$
- $x$ and $y$ are also often called the *in-phase* and *quadrature* components of $z$.
- $j = \sqrt{-1}$ (engineering notation)
- $i = \sqrt{-1}$ (physics, chemistry, mathematics)

Complex number in polar form: $z = re^{j\phi}$

- $r$ is the *modulus* or *magnitude* of $z$
- $\phi$ is the *angle* or *phase* of $z$
- $\exp(j\phi) = \cos\phi + j\sin\phi$




- complex exponential of $z = x + jy$:

$$e^z = e^{x+jy} = e^x e^{jy} = e^x(\cos y + j\sin y)$$

$x(t) = C e^{at}$

$C, a$ are both real.

# Some Important Signals

- Complex exponential

$$x(t) = C\, e^{at}$$

In general, both "C" and "a" are complex

Let's examine the signal for the various values that "C" and "a" can take

# Case (i) Both “C” and “a” are Real

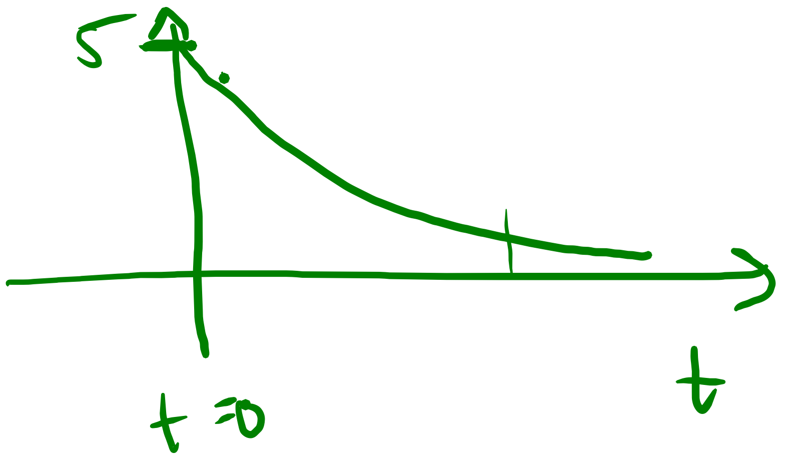
- $x(t) = C\, e^{at}$ → Scaling factor is Real

  → Exponent is also Real
- The signal becomes a Real-exponential
  - If $a > 0$, then the signal increases exponentially
  - If $a < 0$, then the signal decreases exponentially
- What will be the Signal value at $a = 0$?

# Case (i) Both “C” and “a” are Real

$x(t) = C\,e^{at}$ → Scaling factor is Real

→ Exponent is also Real




Lets plot for some specific values :

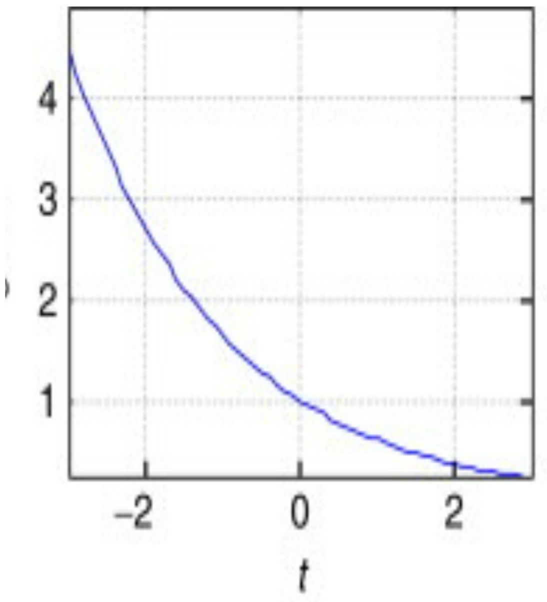
C = 5, a = 0.5

C = -5, a = 1

C = 5, a = -0.5

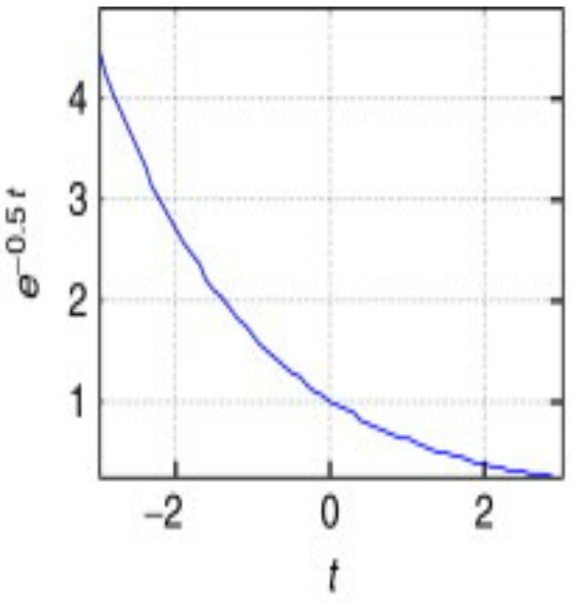
C = -5, a = -1

# Plots



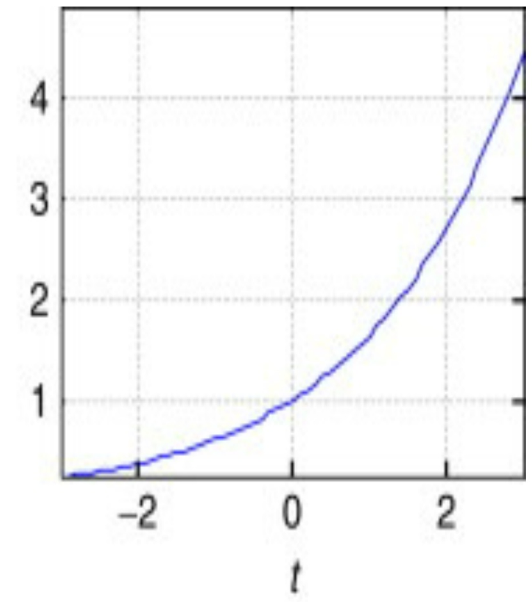

(a)

# Plots



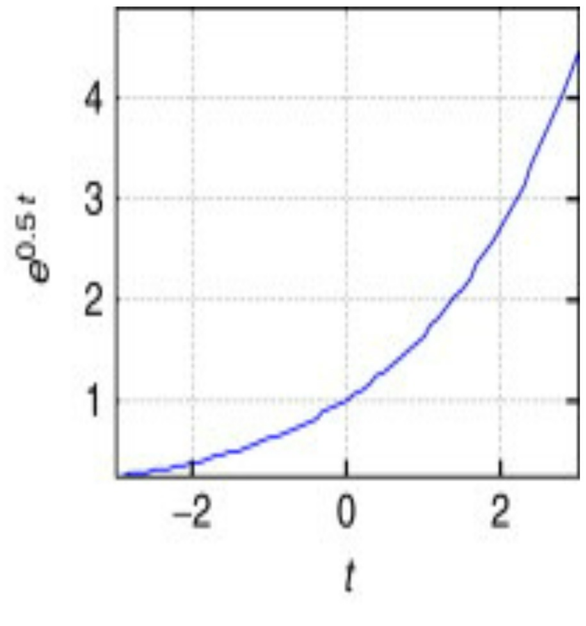

(a)

# Plots




(b)

# Plots



(b)

# Case (ii) Let “a” be purely imaginary

$x(t) = C\, e^{at}$

$a = j\omega_0$

Then, $x(t) = C\, e^{at} = C\, e^{j\omega 0t}$

$x(t) = C\, [\cos(\omega_0 t) + j \sin(\omega_0 t)]$

$x(t)$ is periodic with period $2\pi/\omega_0$

$C\, e^{j\omega_0 t}$

$C = \alpha e^{j\beta}$

$= \alpha e^{j\beta} e^{j\omega_0 t} = \alpha\, e^{j(\omega_0 t + \beta)}$

$= \alpha \left[\cos(\omega_0 t + \beta) + j \sin(\omega_0 t + \beta)\right]$

# Periodicity of $\cos(\omega_0 t)$

$\omega_0 T_0 = 2\pi$

- Let $x(t) = \cos(\omega_0 t)$

- Then $x(t + T_0) = \cos(\omega_0 (t + T_0))$

$$x(t + T_0) = \cos(\omega_0 t + \omega_0 T_0) = \cos(\omega_0 t + \underline{2\pi})$$

$$\omega_0 T_0 = 2\pi$$

- Hence $\boxed{T_0 = 2\pi/\omega_0}$

# Role of “C”

- If “C” is Real, then the amplitude of the sinusoid is scaled by “C”

  Say, C = 2, then the plot of $2\sin(\omega_0 t)$
- If “C” is Complex, then the phase of the sinusoid gets influenced

# Role of "C"

- Say, $C = \alpha e^{j\beta}$
- And we have already set, $a = j\omega_0$

$x(t) = C\ e^{at} = \alpha e^{j\beta} e^{at} = \alpha e^{j\beta} e^{j\omega 0t} = \alpha e^{j(\beta+\omega 0t)}$

- Hence, $x(t) = \alpha[\cos(\omega_0 t + \beta) + j\ \sin(\omega_0 t + \beta)\ ]$

# Case (iii) Both "C" and "a" are complex

$x(t) = C e^{at}$

$C = \alpha e^{j\beta}$

$a = r + j\omega_0$

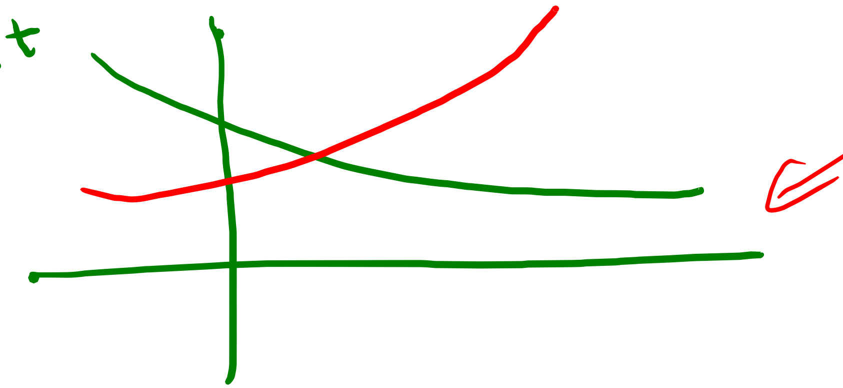
?

$= \alpha e^{j\beta} e^{(r+j\omega_0)t}$

$= \alpha e^{rt} e^{j(\omega_0 t + \beta)}$

$= \underline{\alpha e^{rt}} \underbrace{[\cos(\omega_0 t + \beta) + j \sin(\omega_0 t + \beta)]}_{T_0 = \frac{2\pi}{\omega_0}}$

$\alpha e^{rt}$



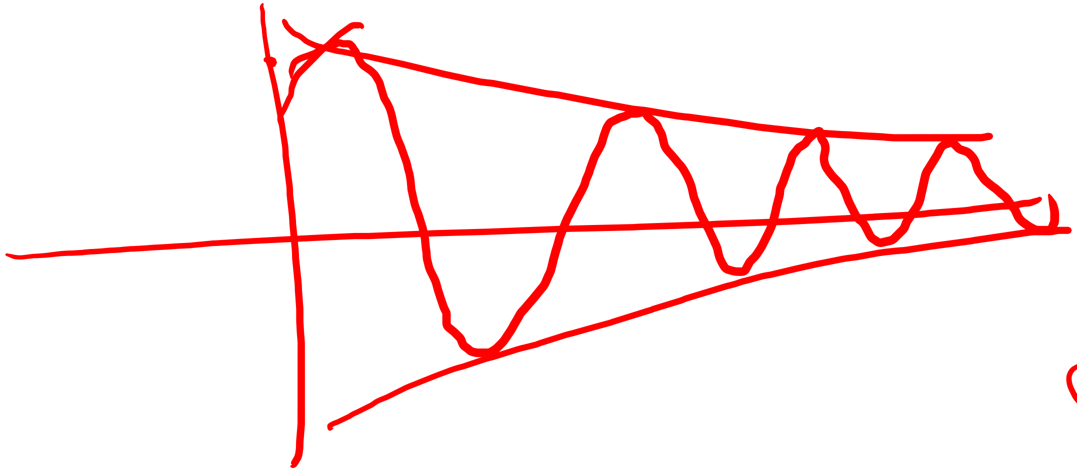
$r = -1/2.$

$r = -0.01$

$r = 1$

$r = 0$

$\gamma = \pi/2$

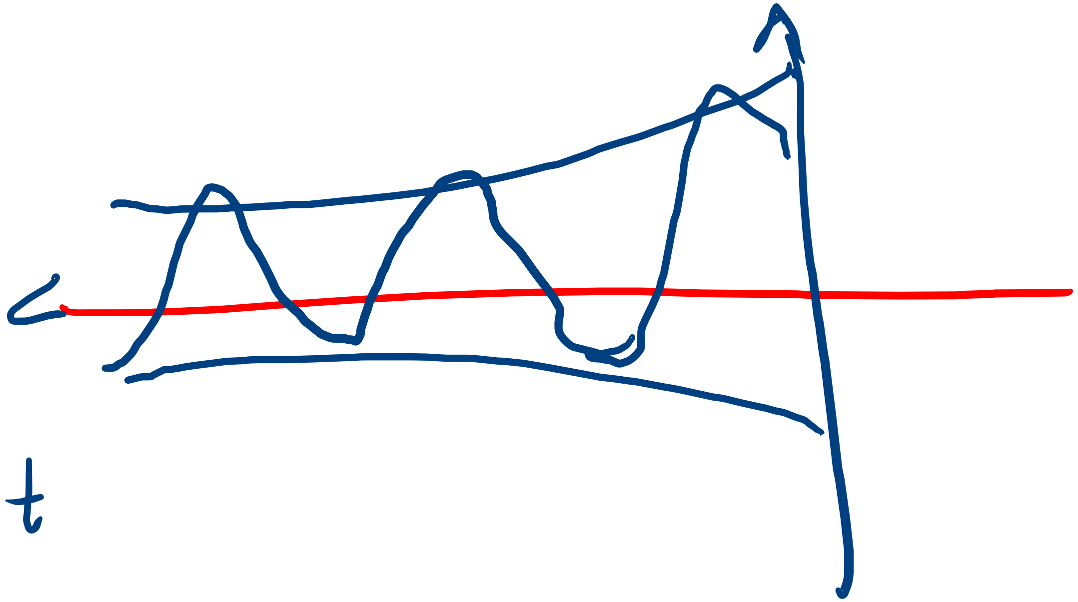


Causal Signals

anticausal signal

$-\sigma$

$t$



# Problem

- Given $x(t) = 5\cos(6\pi t)$, find amplitude, frequency and period.

$$Frequency: \quad f = \frac{\omega}{2\pi} = \frac{6\pi}{2\pi} = 3 \quad (cycles/sec)$$

$$Period: \quad T = \frac{1}{f} = \frac{1}{3} \quad (seconds)$$

What does "ω" denote ?
What is its relation to frequency "f" ?

# Problems

$x(t) = C e^{at}$

- Find parameters of these signals, plot them

$\rightarrow$ C & a.

$1)\, y(t) = 12\, e^{(j5+2)t}$ $\longrightarrow$ $C = 12, \quad a = 2 + j5$

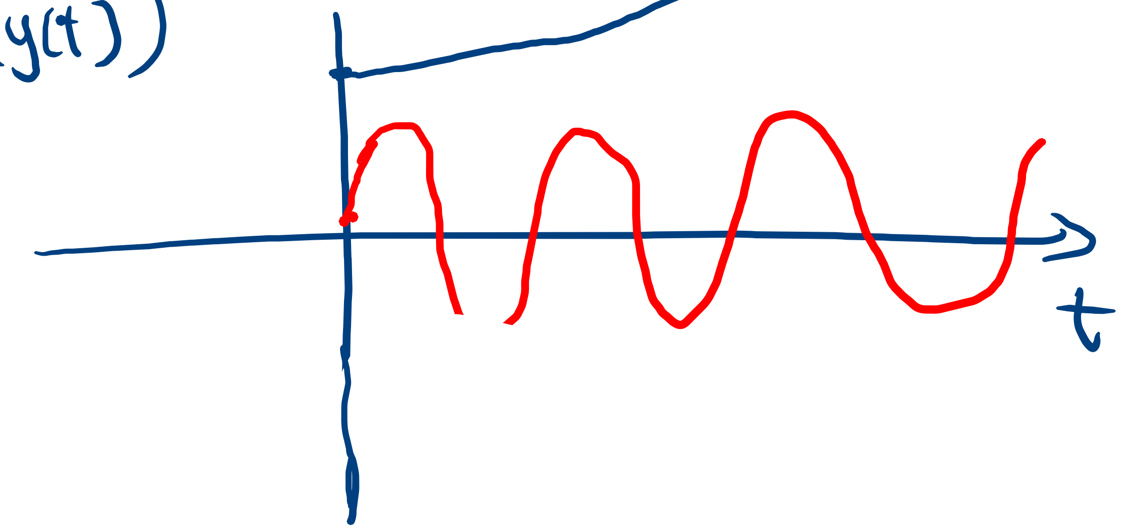
$y(t) = 12 e^{2t}\left[\cos(5t) + j \sin(5t)\right]$

imag(y(t))

$2)\, x(t) = \underline{7 e^{3j} e^{j2t}}$

$C = 7e^{j3}$

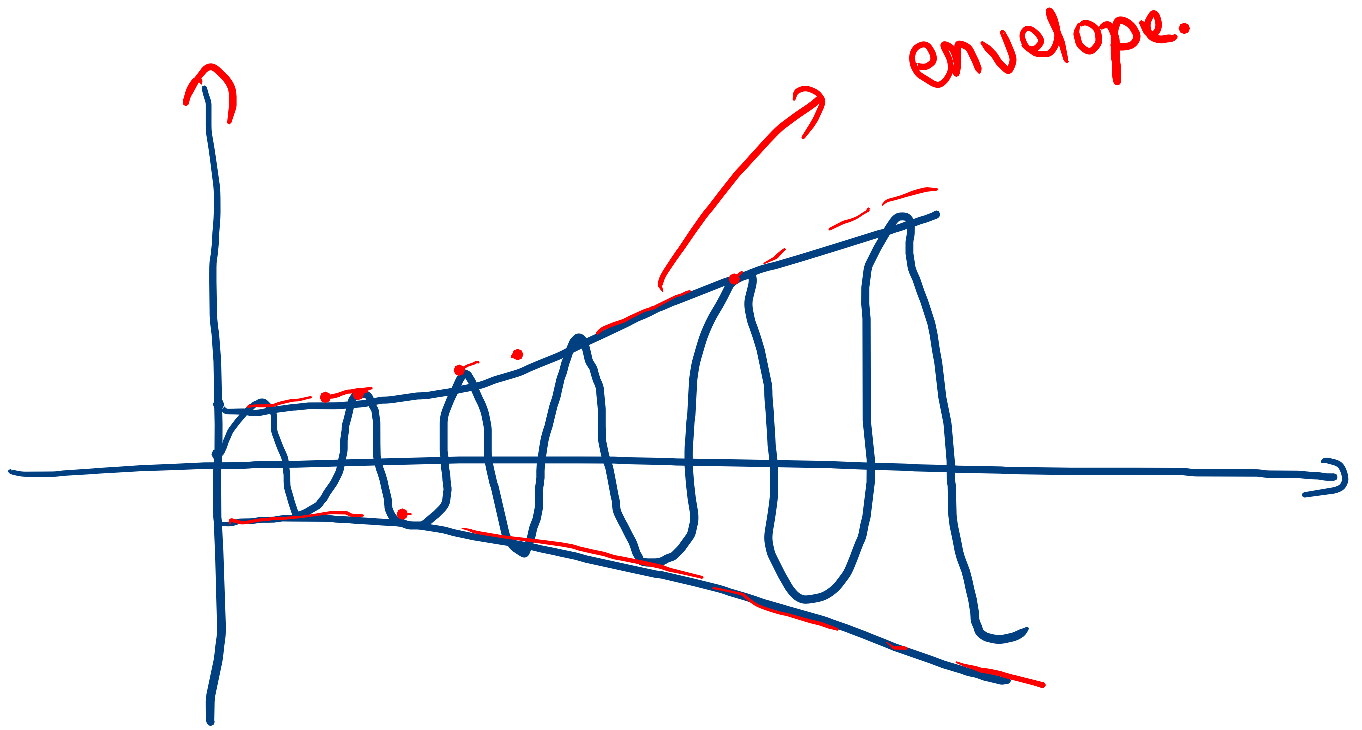
$a = j2$



$C = 7e^{j3}$ $\qquad a = j2$

$$x(t) = 7e^{j3}\, e^{j2t}$$

$$= 7\, e^{j(2t+3)} = 7\left[\cos(2t+3) + j\sin(2t+3)\right]$$


envelope.

# Discrete –time complex exponentials

- Recap : The Cts-time complex exponential , $x(t) = C\ e^{[at]}$
- Discrete-time complex exponential is defined as : $x[n] = C\alpha^n$

# Case (i) : C and α are Real

- If C and α are Real, then for

(i) $|\alpha| > 1$

The sequence $x[n] = C\alpha^n$ grows Exponentially

(ii) $|\alpha| < 1$

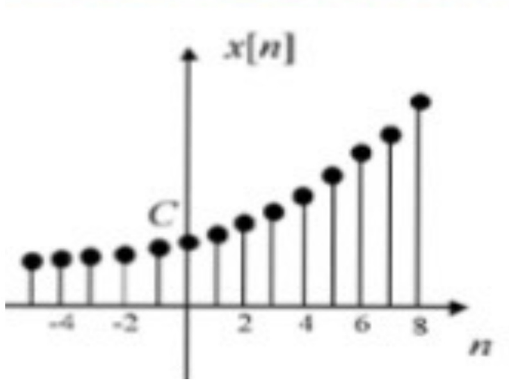
The sequence $x[n] = C\alpha^n$ decays Exponentially

(iii) $\alpha = 1$, Then the sequence is Constant

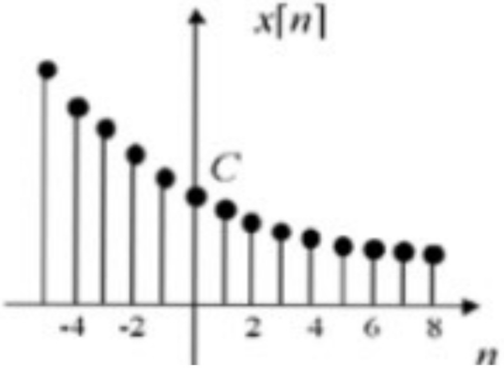
If α is POSITIVE, all values have the same sign

If α is NEGATIVE, the sign of x[n] alternates

# Plots




(i) |α| > 1
The sequence
$x[n] = C\alpha^n$ grows
Exponentially

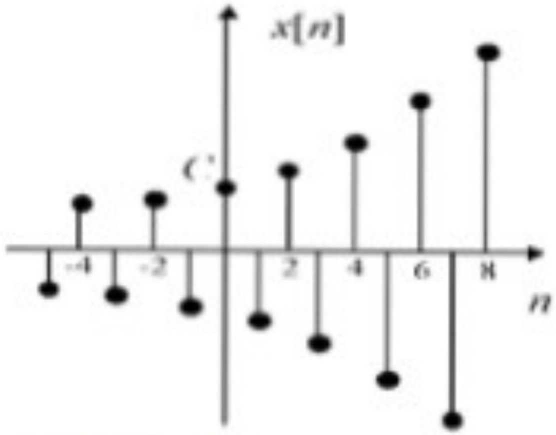



(ii) |α| < 1
The sequence
$x[n] = C\alpha^n$ decays
Exponentially

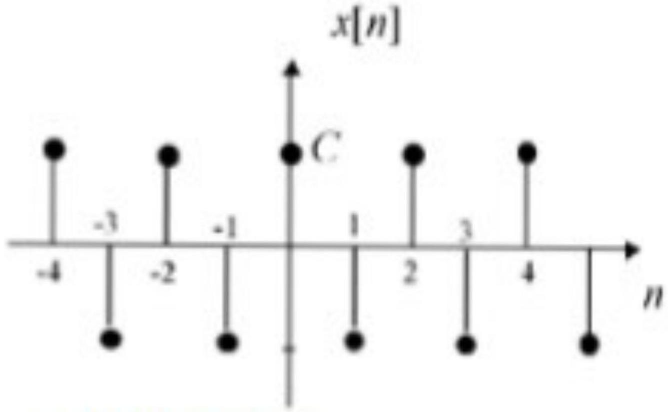
# Plots

$C\alpha^n$

$\alpha = -1$




If α is NEGATIVE, the sign of x[n] alternates



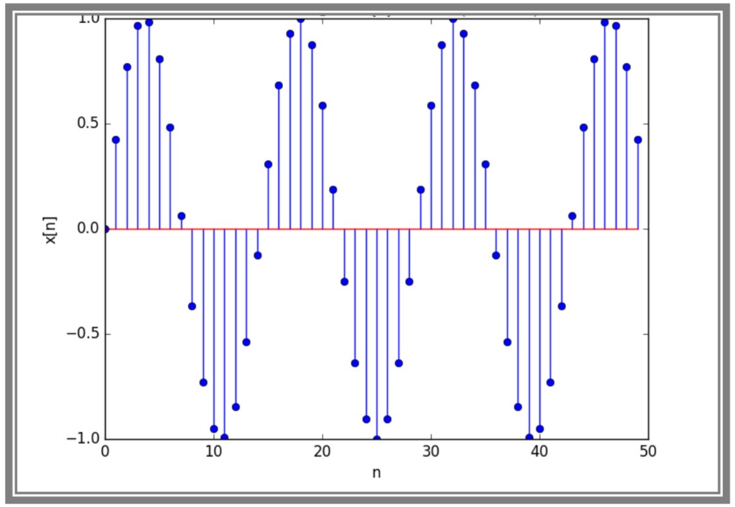

What sequence is this ?

# Case (ii) : C is Real; Let “$\alpha = e^{\beta}$” where $\beta$ is purely imaginary

Since $\beta$ is purely imaginary , $\beta = j\omega_0$

Then $x[n] = Ce^{[j\omega_0]n}$

i.e. $x[n] = C[\cos(\omega_0 n) + jsin(\omega_0 n)]$



# Case (iii) : C is Complex; α is also complex

- Let $C = |C|e^{j\theta}$
- Let $\alpha = |\alpha|e^{j\omega_0}$
- $x[n] = |C|\,|\alpha|^n\,\{C[\cos(\omega_0 n + \theta) + j\sin(\omega_0 n + \theta)\}$

What are the different cases to consider ??

# Case (iii) : C is Complex; α is also complex

- Let $C = |C|e^{j\theta}$
- Let $\alpha = |\alpha|e^{j\omega_0}$
- $x[n] = |C|\,|\alpha|^n\,\{[\cos(\omega_0 n + \theta) + j\sin(\omega_0 n + \theta)\}$

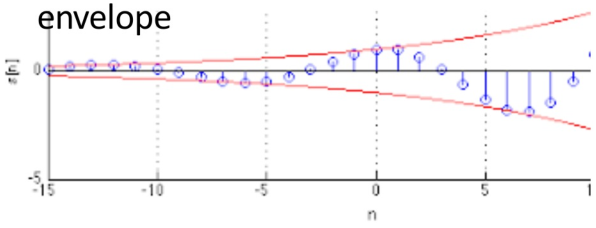
For $|\alpha| = 1$, we get pure sinusoids

For $|\alpha| < 1$, we get decaying envelope

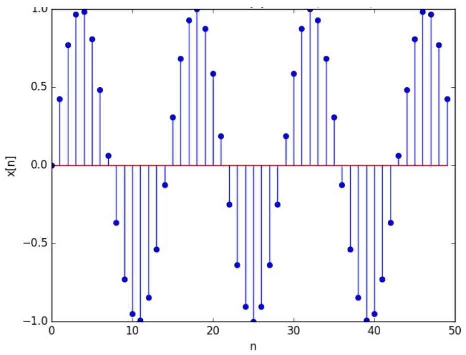
For $|\alpha| > 1$, we get growing envelope

# Plots

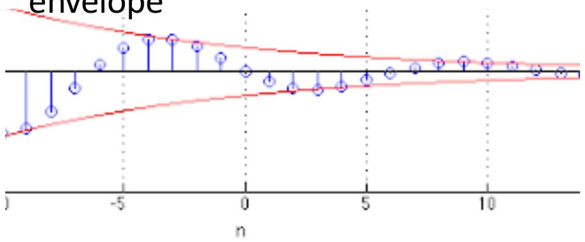
For |α| > 1, we get growing envelope



For |α| =1, we get pure sinusoids



For |α| < 1, we get decaying envelope



# Do signal Properties change ?
# Cts ➔ Discrete-time

Consider x(t) = t

Properties : Monotonic Increasing, Non-periodic

Consider its Discrete version x[n] = n

Properties : Monotonic Increasing, Non-periodic

Are signal properties preserved for all signals..??

# Do signal Properties change ? Cts ➔ Discrete-time

Consider $x(t) = e^{j\omega t}$

For $\omega = \omega_1$, we get $x_1(t) = e^{j\omega_1 t}$

$= \cos(\omega_1 t) + j\sin(\omega_1 t)$

Period = $\frac{2\pi}{\omega_1}$ $= T_1$

For $\omega = \omega_2$, we get $x_2(t) = e^{j\omega_2 t}$

$= \cos(\omega_2 t) + j\sin(\omega_2 t)$

Period = $\frac{2\pi}{\omega_2}$ $= T_2$

If $\omega_1 \neq \omega_2$, then $x_1(t) \neq x_2(t)$

Is this always true..?? YES..

Distinct values of "$\omega$" lead to Distinct signals

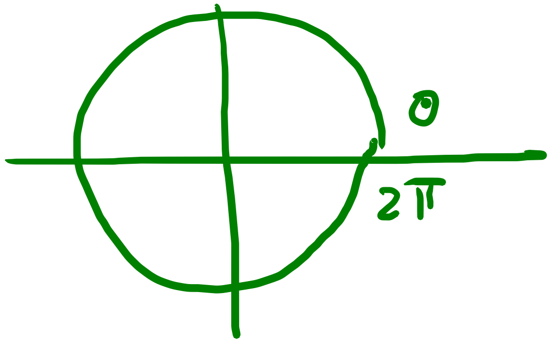
- Consider $x[n] = e^{j\omega n}$

For $\omega = \omega_1$, we get $x_1[n] = e^{j\omega_1 n}$

For $\omega = \omega_2 = \omega_1 + 2\pi$

Then $x_2[n] = e^{j\omega_2 n} = e^{j(\omega_1 + 2\pi)n} = e^{j\omega_1 n} e^{j2\pi n} = e^{j\omega_1 n}$

So we get $x_1[n] = x_2[n]$




i.e. Distinct sinusoids exist only in the interval 0 to $2\pi$

# Periodicity

- Let's say 'N' (an Integer) is the period of $x[n] = e^{j\omega n}$
- Then it follows that, $x[n] = x[n+N]$

- i.e. $e^{j\omega n} = e^{j\omega (n+N)} = e^{j\omega n} e^{j\omega N}$
- This can only be true if $\boxed{e^{j\omega N} = 1}$

- i.e. $e^{j\omega N} = e^{j(2\pi)k}$, where k is an Integer
- $\omega N = (2\pi)k$
- $\omega$ should be $\underline{2\pi k/N}$

$\omega$ should be a rational Multiple of $(2\pi)$

# Varying values of $\omega$

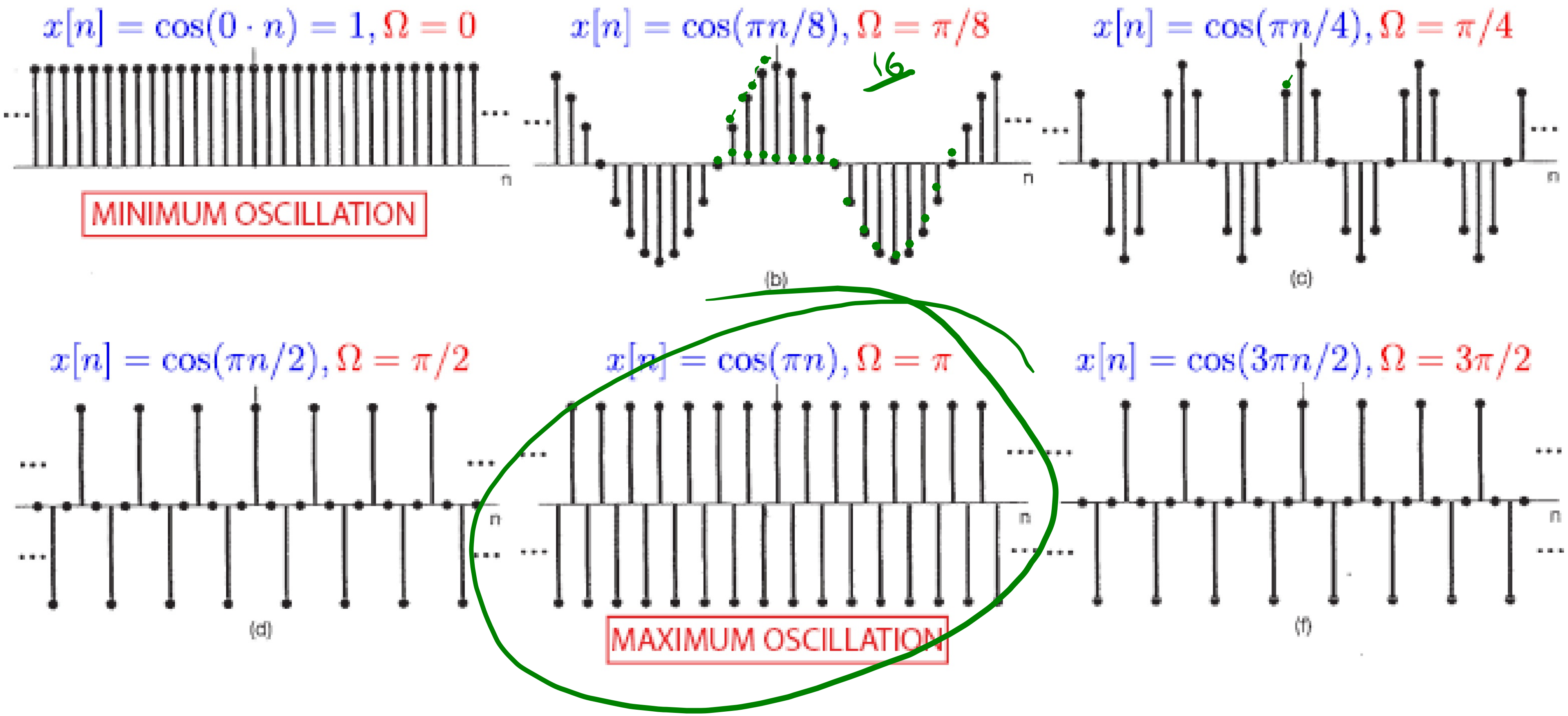
Look at the signal as $\omega$ increases from 0 to $2\pi$:

$x[n] = e^{j\omega n}$

At $\omega = 0$, it is a constant signal with "0" rate of oscillation

As $\omega$ increases, the rate of oscillation also increases

# Plots : Varying ω



# Varying values of $\omega$

Look at the signal as $\omega$ increases from 0 to $2\pi$ :

$x[n] = e^{j\omega n}$

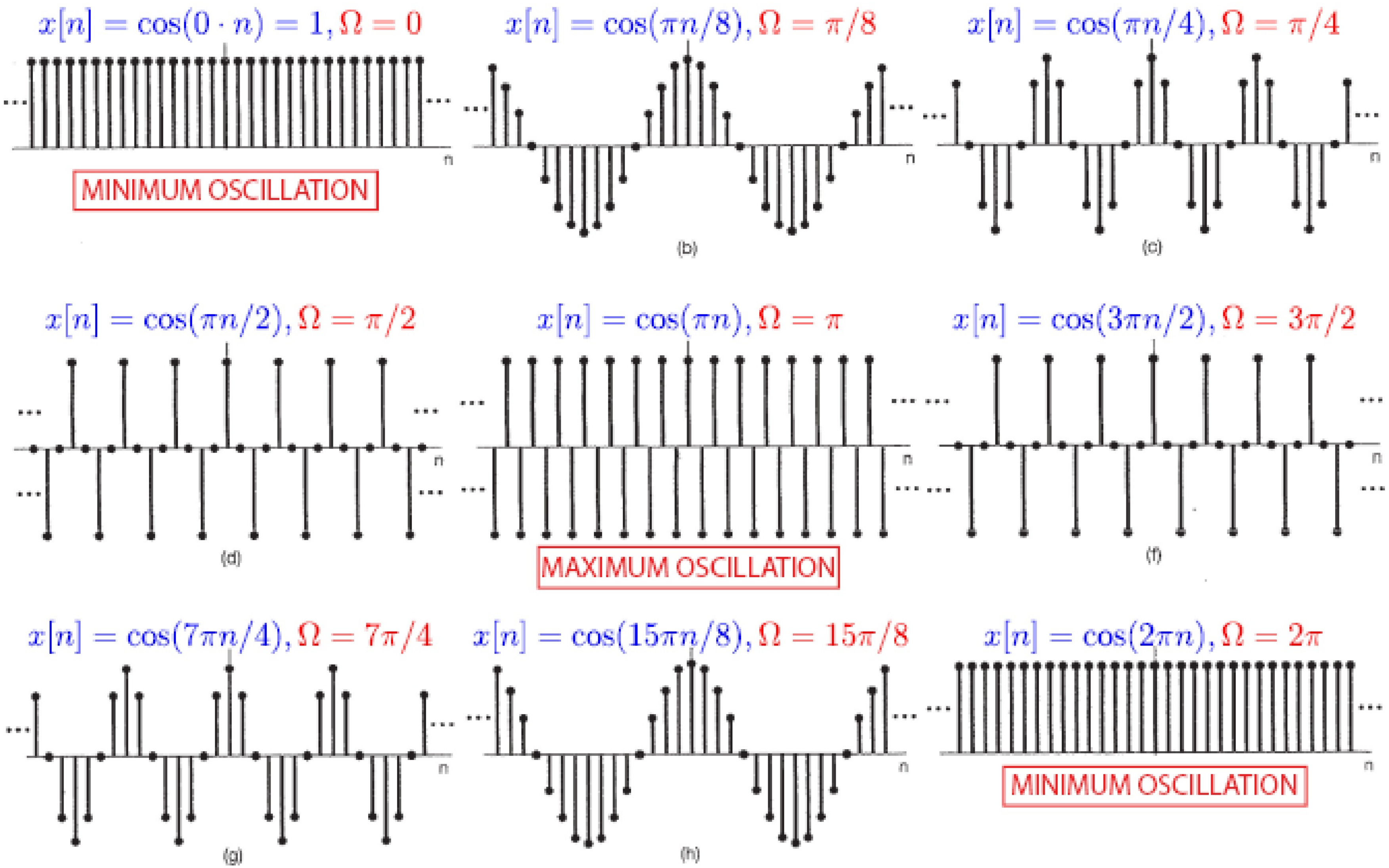
At $\omega = 0$, it is a constant signal with "0" rate of oscillation

As $\omega$ increases, the rate of oscillation also increases

At $\omega = \pi$, the rate of oscillation is maximum

At $\omega$ greater than $\pi$, the rate of oscillation starts to decrease

As $\omega$ reaches $2\pi$, we get back the same constant sequence as $\omega = 0$


$x[n] = \cos(0 \cdot n) = 1, \Omega = 0$
MINIMUM OSCILLATION
$x[n] = \cos(\pi n/8), \Omega = \pi/8$
(b)
$x[n] = \cos(\pi n/4), \Omega = \pi/4$
(c)
$x[n] = \cos(\pi n/2), \Omega = \pi/2$
(d)
$x[n] = \cos(\pi n), \Omega = \pi$
MAXIMUM OSCILLATION
$x[n] = \cos(3\pi n/2), \Omega = 3\pi/2$
(f)
$x[n] = \cos(7\pi n/4), \Omega = 7\pi/4$
(g)
$x[n] = \cos(15\pi n/8), \Omega = 15\pi/8$
(h)
$x[n] = \cos(2\pi n), \Omega = 2\pi$
MINIMUM OSCILLATION

# Features of $e^{j\omega t}$

- $\{e^{j\omega t}\}$ Distinct signals for distinct values of $\omega$

- Periodic for any choice of $\omega$

# Comparison: $\{e^{j\omega t}\}$ vs $\{e^{j\omega n}\}$

- $\{e^{j\omega t}\}$ Distinct signals for distinct values of $\omega$

- Periodic for any choice of $\omega$

$\{e^{j\omega n}\}$ Identical signals for values of $\omega$ that are separated by multiples of $2\pi$

Periodic only if $\omega$ is a rational multiple of $2\pi$