Discrete Distribution: Definition

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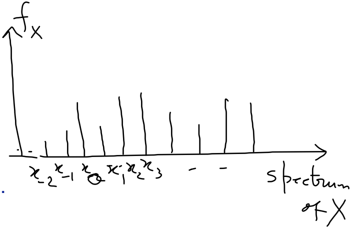
- Let spectrum of X: $T = \{x_i : i = 0, \pm 1, \pm 2, \dots\}$ with

$$\dots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \dots$$

The function $f_X : \mathbb{R} \to [0,1]$, defined as

$$f_X(x) = \begin{cases} P(X = x_i), & x = x_i \in T \\ 0, & \text{elsewhere.} \end{cases}$$

is called the *probability mass function* (p.m.f.) of X.



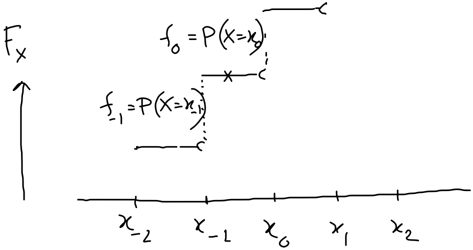
Discrete Distribution: Definition

- $F_X(x) = \sum_{x_j \leq x_i} P(X = x_j)$ if $x_i \leq x < x_{i+1}$

$$= \sum_{j=-\infty}^{i} f_X(x_j)$$

Discrete Distribution: Definition

- $F_X(x) = \sum_{x_j \le x_i} P(X = x_j)$ if $x_i \le x < x_{i+1}$
- F_X is a step function with steps $f_i = P(X = x_i)$ for $i = 0, \pm 1, \pm 2, \ldots$.



Discrete Distribution: Properties

1. $\sum_{j=-\infty}^{\infty} f_j = 1$

We have,

$$F_X(\infty) = 1$$

1. $\sum_{j=-\infty}^{\infty} f_j = 1$

2. At each non-spectrum point a, $P(X = a) = 0$

$$P(X=a) = F_X(a) - F_X(a-0)$$

let, $\exists\, x_k, x_{k+1} \in$ spectrum of X s.t.

$$\underbrace{x_k < a < x_{k+1}}$$

$$F_X(a) = \sum_{j=-\infty}^{k} f_X(x_j) \quad , \quad F(a-0) = \sum_{j=-\infty}^{k} f_X(x_j)$$

$$\Rightarrow P(X=a) = 0 .$$

Discrete Distribution: Properties

1. $\sum_{j=-\infty}^{\infty} f_j = 1$

2. At each non-spectrum point a, $P(X = a) = 0$

3. $P(a < X \le b) = \sum_{a < x_i \le b} f_X(x_i)$

$P(a < X \le b) = F_X(b) - F_X(a)$

1. Binomial (n, p) Distribution

$X \sim \text{Binomial}(n, p)$

Spectrum of $X = \{0, 1, 2, \ldots, n\}$

probability mass fn. (p.m.f)

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots$$

$$= 0, \quad \text{elsewhere}$$

$(0 < p < 1)$

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = 1.$$

2. Poisson (μ) Distribution

$X \sim \text{Poisson}(\mu)$

Spectrum of $X = \{0, 1, 2, \ldots\}$

$\mu > 0$

p.m.f.

$$f_X(x) = \frac{e^{-\mu}\mu^x}{x!}, \quad x = 0, 1, 2, \ldots$$
$$= 0, \quad \text{elsewhere}$$

$$\sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu} \boxed{\sum_{x=0}^{\infty} \frac{\mu^x}{x!}} = e^{-\mu} e^{\mu} = 1.$$

Continuous Distribution

Definition

The distribution of a random variable X is said to be continuous if

1. the distribution function F_X is continuous
2. $\frac{d}{dx}F_X(x) = F_X'(x)$ is piecewise continuous in $(-\infty, \infty)$

Define: $f_X : \mathbb{R} \to [0,1]$ as $f_X(x) = \frac{d}{dx}F_X(x)$ which is called the *probability density function* (p.d.f.) of X.

Continuous Distribution: Properties

1. $f_X \geq 0$

$f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$

$f_X(x) = \frac{d}{dx} F_X(x) \geq 0$ since $F_X(x)$ is mon. inc. fn.

Continuous Distribution: Properties

1. $f_X \geq 0$

2. $P(a < X \leq b) = \int_a^b f_X(x)dx$

$P(a < X \leq b) = F_X(b) - F_X(a)$.

(i) $f_X(x) = \frac{d}{dx} F_X(x)$ (primitive exists)

(ii) $f_X(x)$ has finite no. of jump discontinuities (1st kind) in $[a,b]$ $\Rightarrow$ $f_X(x)$ is R. integrable in $[a,b]$.

$\Rightarrow$ Applying Fundamental Th. of Integral Calculus

$$\boxed{\int_a^b f_X(x)dx = F_X(b) - F_X(a)} = P(a < X \leq b).$$

1. $f_X \geq 0$
2. $P(a < X \leq b) = \int_a^b f_X(x)dx$
3. $F_X(x) = \int_{-\infty}^x f_X(x)dx$

$F(-\infty) = 0 \Rightarrow F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx.$

Continuous Distribution: Properties

1. $f_X \geq 0$
2. $P(a < X \leq b) = \int_a^b f_X(x)dx$
3. $F_X(x) = \int_{-\infty}^x f_X(x)dx$
4. $\int_{-\infty}^\infty f_X(x)dx = 1$ Since $F_X(\infty) = 1$.

Continuous Distribution: Properties

1. $f_X \geq 0$
2. $P(a < X \leq b) = \int_a^b f_X(x)dx$
3. $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$
4. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
5. $P(X = a) = 0$ for a given constant a

$P(X=a) = F_X(a) - F_X(a-0) = 0$ since F_X is continuous.

Continuous Distribution: Properties

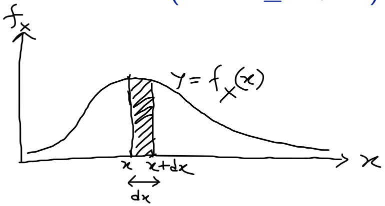
1. $f_X \geq 0$
2. $P(a < X \leq b) = \int_a^b f_X(x)dx$
3. $F_X(x) = \int_{-\infty}^x f_X(x)dx$
4. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
5. $P(X = a) = 0$ for a given constant a
6. **Converse statement:** Every non-negative, real-valued, piecewise-continuous function f that is integrable in $(-\infty, \infty)$ and satisfies $\int_{-\infty}^{\infty} f(x)dx = 1$, is the probability density function of a continuous distribution.

7. **Probability Differential:** Let X has continuous distribution. In differential notation we write:

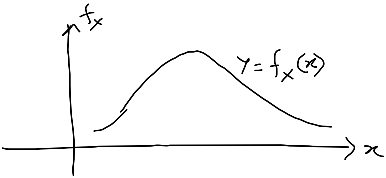
$$P(x < X \leq x + dx) = F_X(x + dx) - F_X(x) = dF_X$$

$$= F_X'(x)\, dx$$

$$= f_X(x)\, dx$$



8. **Density Curve:** The curve $y = f_X(x)$ is called the probability density curve of the corresponding continuous distribution.

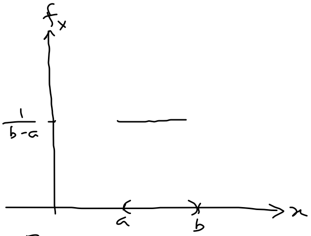


1. Uniform (a, b)

$$f_X(x) = \frac{1}{b-a}, \quad a < x < b$$
$$= 0, \quad \text{elsewhere}$$

Note: (i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$

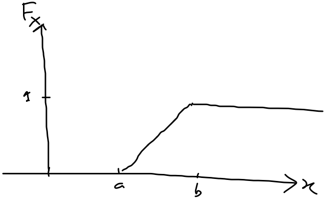


$\underline{x \leq a}$ $F_X(x) = 0$

$\underline{a < x < b}$ $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{a}^{x} \frac{1}{b-a}\,dx$

$$= \frac{x-a}{b-a}$$

$x \geq b$, $F_X(x) = 1.$



2. Normal (m, σ) , $X \sim \text{Normal}(m, \sigma)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

where $\sigma > 0$

Note: (i) $f_X(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$$

put, $\frac{x-m}{\sigma\sqrt{2}} = z$

$\Rightarrow dx = \sigma\sqrt{2}\,dz$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-z^2} \sigma\sqrt{2}\,dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= \frac{2}{\sqrt{\pi}} \boxed{\int_0^{\infty} e^{-z^2} dz} =$$

$z^2 = t \Rightarrow 2z\,dz = dt$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi}$$

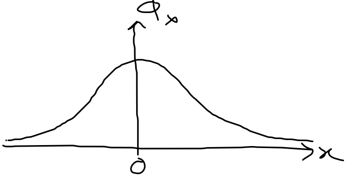
$$= 1.$$

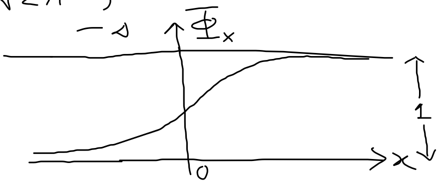
In particular if $m=0$, $\sigma=1$

normal distribution will be called Standard Normal Distribution.

density fn : $\phi_X(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $-\infty < x < \infty$

distribution fn : $\Phi_X(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-\frac{x^2}{2}} \, dx$.





3. Cauchy (λ, μ) Distribution

$X \sim$

$$f_X(x) = \frac{1}{\pi} \cdot \frac{\lambda}{\lambda^2 + (x-\mu)^2}, \quad -\infty < x < \infty$$

$\lambda > 0$, μ are the parameters

(i) $f_X(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$ (check!)

4. Gamma (l) distribution

$X \sim$

or, $X \sim \gamma(l)$, $l > 0$

$$f_X(x) = \frac{1}{\Gamma(l)} e^{-x} x^{l-1}, \quad 0 < x < \infty$$
$$= 0 \quad , \text{elsewhere.}$$

Note: (i) $f_X(x) \geq 0$ & (ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

5. Beta (l, m) distribution $\underbrace{\text{of 1st kind}}$

$$X \sim \beta_1(l, m) \qquad (l, m > 0)$$

$$f_X(x) = \frac{1}{B(l,m)} x^{l-1}(1-x)^{m-1}, \qquad 0 < x < 1.$$
$$= 0, \quad \text{elsewhere}.$$

6. Beta (l, m) distribution of 2nd kind

$$X \sim \beta_2(l, m) \quad , \quad (l, m > 0)$$

$$f_X(x) = \frac{1}{B(l, m)} \frac{x^{l-1}}{(1+x)^{l+m}} \quad , \quad 0 < x < \infty$$

$$= 0 \quad , \quad \text{elsewhere.}$$

PS-4

Prob. 4 :

$$f(x) = k e^{-|x-a|}, \quad -\infty < x < \infty$$

(i) For what k, f is a possible p.d.f.?
(ii) Calculate the d.f.

Sol :

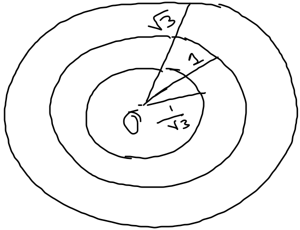
$$\int_{-\infty}^{\infty} k e^{-|x-a|} \, dx = 1. \quad , \quad k = \frac{1}{2} \quad (\text{check!})$$

$$f(x) = \frac{1}{2} e^{-|x-a|}, \quad -\infty < x < \infty$$

$x \leq a$, $F(x) = \frac{1}{2} \int_{-\infty}^{x} e^{+(x-a)} \, dx$

$x > a$, $F(x) = \frac{1}{2} \int_{-\infty}^{a} e^{(x-a)} \, dx + \frac{1}{2} \int_{a}^{x} e^{-(x-a)} \, dx$.

Prob (8)



Let X: corr. to score

Spectrum of X : $\{0, 1, 2, 3\}$

To find $P(X=0)$, $P(X=1)$, ...

Let R be the r.v. corr to distance of hit from O

$$f_R(r) = \frac{2}{\pi}\frac{1}{1+r^2}, \quad 0 \le r < \infty$$

$$= 0, \text{ elsewhere}$$

$$P(X=0) = P(R > \sqrt{3}) = 1 - P(R \le \sqrt{3}) = 1 - \frac{2}{\pi}\int_0^{\sqrt{3}} \frac{1}{1+r^2}\,dr$$

$$= 1 - \frac{2}{\pi}\Big[\tan^{-1} r\Big]_0^{\sqrt{3}}$$

$$= 1 - \frac{2}{\pi}\frac{\pi}{3} = \frac{1}{3}$$

$$P(X=1) = \frac{1}{6}, \quad P(X=2) = \frac{1}{6}, \quad P(X=3) = \frac{1}{3}$$