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Probability Distribution - 1

Motivation

① Developing mathematical theory of probability using the probability f^n $P: \Delta \to \mathbb{R}$ is difficult because its domain Δ consists of coll^n of events.

② An event corresponding to a random ~~variable~~ exp. consists of objects which are not necessarily real no's.

③ Random Variable :- Finds a correspondence b/w events in Δ and numbers.

Distribution Function :- Replaces probability f^n by a real valued f^n defined on real line.

→ Random Variable :-

A real valued f^n $X: S \to \mathbb{R}$ is called a random variable if for any $x \in \mathbb{R}$

$$\{ w \in S : -\infty < X(w) \leq x \} \in \Delta .$$

① Spectrum of X : The range of the random variable X.

② X is said to be discrete or continuous if the spectrum is discrete or cont., respectively.

eg. Let E = A coin is tossed twice.

$S = \{ w_1 = (H,H),\ w_2 = (H,T),\ w_3 = (T,H),$
$w_4 = (T,T) \}.$

① Define $X_1 : S \to \mathbb{R}$ as

$X_1(w_i)$ = number of heads.

∴ Spectrum of $X_1 = \{0, 1, 2\}$.

② Define $X_2 : S \to \mathbb{R}$ as

$X_2(w_i)$ = Sq. of number of heads

∴ Spectrum of $X_2 = \{0, 1, 4\}$.

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③ Define $X_3 : S \to \mathbb{R}$ as
$X_3(w_i)$ = no. of heads − no. of tails.
Spectrum of X = $\{-2, 0, 2\}$.

Q. what are the events $\{w : X_i(w) \leq 1\}$ for $i = 1, 2, 3$?
From the prev. def^ns?

Ans. $E_1 = \{w : X_1(w) \leq 1\}$ = Events for 0, 1.

$\therefore \boxed{E_1 = \{w_2, w_3, w_4\}}$.

$E_2 = \{w : X_2(w) \leq 1\}$ = Events for 0, 1.

$\boxed{E_2 = \{w_2, w_3, w_4\}}$

$E_3 = \{w : X_3(w) \leq 1\}$ = Events for −2, 0.

$\boxed{E_3 = \{w_2, w_3, w_4\}}$

→ <u>Expressing Events using Random Variables:-</u>

By properties of σ-algebra the following are events:-

① $\{w \in S : a \leq X(w) \leq b\}$.
$\{w \in S : X(w) = a\}$.

② For brevity, we use following notations:-
$\{w \in S : a < X(w) \leq b\} \equiv (a < X \leq b)$
$\{w \in S : -\infty < X(w) \leq b\} \equiv (X \leq b)$
$\{w \in S : X(w) = a\} \equiv (X = a)$ ...

* **σ-Algebra** → ① $S \in \Delta$ ② If $A \in \Delta$ then $A^c \in \Delta$.
③ If $A_1, A_2, \dots, A_k \in \Delta$ then $\bigcup_{i=1}^{k} A_i \in \Delta$.

To prove :- $\bigcap_{i=1}^{k} A_i \in \Delta$.

Proof :- If $A_i \in \Delta$ then $A_i^c \in \Delta$. (②).

$\Rightarrow \bigcup_{i=1}^{k} A_i^c \in \Delta$ (③).

$\Rightarrow \left(\bigcup_{i=1}^{k} A_i^c\right)^c \in \Delta$ (②).

$\Rightarrow \bigcap_{i=1}^{k} A_i \in \Delta$. Hence Proved.

→ **Distribution f^n** :-

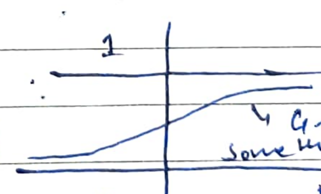
Let (S, Δ, P) be a prob. space with the probability f^n $P : \Delta \to \mathbb{R}$ and let $X : S \to \mathbb{R}$ be a random variable. Then the distribution f^n of X is a real valued f^n $F_X : \mathbb{R} \to \mathbb{R}$ defined as

$$\boxed{F_X(x) = P(-\infty < X \leq x) \quad \forall x \in (-\infty, \infty)}$$

Properties :-

① $0 \leq F_X \leq 1$. as $F_X(x) = P(-\infty \leq X \leq x)$ and P for any event has to be b/w 0 & 1.

② F_X is monotonically increasing.



Graph is something like this.

Proof :- M.I. when $(x_1 < x_2)$, $x_1, x_2 \in \mathbb{R}$ then $f(x_1) \leq f(x_2)$.

Here $f = F_X$.

$F_X(x_1) = P(-\infty \leq X \leq x_1)$ this is subset of.

$F_X(x_2) = P(-\infty \leq X \leq x_2)$

mutually exclusive set

$(-\infty \leq X \leq x_2) = (-\infty \leq X \leq x_1) \cup (x_1 < X \leq x_2)$ of events.

$\therefore P(-\infty \le X \le x_2) = P(-\infty \le X \le x_1) + P(x_1 < X \le x_2)$

$\therefore P(-\infty \le X \le x_2) \ge P(-\infty \le X \le x_1)$ $\because \ge 0.$

$\therefore F_X(x_2) \ge F_X(x_1)$ when $x_1 < x_2$. Hence M.I.

Another imp rel^n :- $\boxed{P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)}$ ★★.

③ F_X is continuous from right at every pt, i.e.
$F_X(a+0) = F_X(a)$ $\forall a \in \mathbb{R}$.

From continuity theorem of prob.
if $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \ldots$ → M.I. seq. of events.

$\lim_{n\to\infty} A_n = \sum_{i=1}^{\infty} A_i$

if $A_1 \supseteq A_2 \supseteq A_3 \ldots$
then $\lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i$

then $\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right)$.

Proof :- Let $A_n = \left(a < X \le a + \frac{1}{n}\right)$, $\forall n \in \mathbb{N}$.

$\therefore A_1 = (a < X \le a+1)$
$A_2 = \left(a < X \le a + \frac{1}{2}\right) \ldots$

$\therefore A_1 \supseteq A_2 \supseteq A_3 \ldots$

then $\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right) = P(\phi) = 0$.

use formula.

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$$\lim_{n\to\infty} P\left(a < X \le a+\frac{1}{n}\right) = 0 \Rightarrow \boxed{\lim_{n\to\infty} F_X\left(a+\frac{1}{n}\right) = F_X(a)}$$

As when $n\to\infty$, $a < x \le a$, x can't be both $> a$ and $\le a$.

$\therefore$ It has to be ϕ.

& $\boxed{P(\phi)=0}$.

Q

Proper proof for this

Claim :- To show $\bigcap_{n=1}^{\infty} A_n = \phi$. where $A_n = \left(a \le x \le a+\frac{1}{n}\right)$

If possible, let $w \in \bigcap_{n=1}^{\infty} A_n$ for some $w \in S$.

$\Rightarrow w \in A_n \ \forall n \in N$. (as for our defn of A_n, $A_1 \cap A_2 \cap \dots \cap A_n = A_n$).

$\Rightarrow a < X(w) \le a+\frac{1}{n} \ \forall n \in N$.

Let $X(w) = b$.

$\therefore (b-a) > 0$.

Now applying Archimedian principle for $(b-a)$ & 1.

$\exists m \in N$ s.t. $m(b-a) > 1$.

$\Rightarrow b-a > \frac{1}{m} \Rightarrow b > a+\frac{1}{m}$.

But we had assumed that $X(w) \le a+\frac{1}{n} \ \forall n \in N$.

But we have got $m \in N$ s.t. $b > a+\frac{1}{m}$.

$\therefore$ Contradiction. Hence $w \notin \bigcap_{n=1}^{\infty} A_n$. $\therefore \boxed{\bigcap_{n=1}^{\infty} A_n = \phi}$

Hence Proved.

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④ $\boxed{F_X(a) - F_X(a-0) = P(X=a)}$ $\forall a \in \mathbb{R}$.

⑤ $\boxed{F_X(\infty) = 1}$ $\Rightarrow$ $\boxed{F_X(\infty) = P(-\infty < X < \infty) = 1 \text{ obv.} \\ = P(S)}$

↳ Informal proof.

⑥ $\boxed{F_X(-\infty) = 0}$

⑦ The set of points of discontinuity of F_X is atmost enumerable (finite ~~&~~ or countable infinite).

Note :-

~~Any f^n F with domain $(-\infty, \infty)$ & range $[0, 1]$ is a distribution f^n of a random variable~~

* Maths 3.

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* How is $\{w \in S : a \leq X(w) \leq b\}$ an event?

A. $\{w \in S ; -\infty < x \leq a\}$ $\Rightarrow A^c$. $\{w \in S$, $a < x < \infty\}$
B. $\{w \in S$: $-\infty < x \leq b\}$.

We have shown that intersection is an event.
$\therefore B \cap A^c \in \Delta$
$\Rightarrow \{a < x < \infty\} \cap \{-\infty < x \leq b\}, \in \Delta$.
$\Rightarrow \{a < x \leq b\} \in \Delta$ — ①
Hence Proved.

Now how is $\{w \in S : X(w) = a\}$ an event?
Now $\left\{a - \frac{1}{n} < x \leq a\right\}$ $\forall n \in N$.
from prev statement.

$\therefore \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n} < x \leq a\right) \in \Delta$
Intersection $\in \Delta$.

$\Rightarrow (X = a) \in \Delta$. — ②

$\therefore$ By these ① ② eq^n s $\{a \leq x \leq b\} \in \Delta$.

Q. Spectrum of random variable X consists of the points $\{1, 2, \ldots, n\}$ and $P(X=i) \propto \frac{1}{i(i+1)}$.

(i) Determine distribution f^n of X.
(ii) Compute $P(3 < x \leq n)$ & $P(X > 5)$.

Ans. $P(X=i) \propto \frac{1}{i(i+1)}$ $\therefore$ Let $P(X=i) = \frac{k}{i(i+1)}$

$\therefore k \sum_{i=1}^{n} \frac{1}{i(i+1)} = k \sum_{i=1}^{n} \frac{(i+1) - i}{i(i+1)}$

$= k \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{(i+1)}$

$$= \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots - \frac{1}{n+1}\right) k$$

$$= \left(1 - \frac{1}{n+1}\right) k \Rightarrow \boxed{\frac{nk}{n+1}}$$

The summation should be 1. $\Rightarrow \boxed{k = \frac{n+1}{n}}$

$$\therefore \boxed{P(X=i) = \left(\frac{n+1}{n}\right) \cdot \frac{1}{i(i+1)}}$$

(i) $\therefore F_X(x) = P(-\infty < X \le x) = 0$ when $x < 1$

$= 1$ when $x \geq n$.

Direct P when in this range. $\rightarrow$ $\left[\left(\frac{n+1}{n}\right) \cdot \frac{1}{r(r+1)}\right]$, $r \le x < r+1$, $\boxed{r = 1, 2, \dots n-1}$

(ii) $P(3 < X \le n) = 1 - P(X=1) - P(X=2) - P(X=3)$.

$$= 1 - \left(\frac{n+1}{n}\right)\left[\frac{1}{2} + \frac{1}{6} + \frac{1}{12}\right]$$

$$= 1 - \left(\frac{n+1}{n}\right) \cdot \left(\frac{9}{12 \times 4}\right)$$

$$= \boxed{\frac{n-3}{4n}}$$

$P(X > 5) = 1 - P(X \le 5)$

$$= 1 - \left(\frac{n+1}{n}\right)\left[\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}\right]$$

$$= 1 - \left(\frac{n+1}{n}\right)\left[\frac{\cancel{30 + 10 + 5 + 3 + 2}\ 5}{\cancel{60}\ 6}\right] = \boxed{\frac{n-5}{6n}}$$

Q. Number chosen from random from $\{0,1,2,3\}$, $\{0,1,2,3\}$. X = random var denoting sum. Find $F_X(x)$.

Ans.

	0	1	2	3
0	0	1	2	3
1	1	2	3	4
2	2	3	4	5
3	3	4	5	6

$P(x<0) = 0$
$P(0 \le x < 1) = 1/16$
P

Let E = random expt of choosing numbers from the sets.

S = Sample space corr. to E.
$= \{(x,y) : x,y = 0,1,2,3\}$. , $|S| = 16$.

$X : S \to \mathbb{R}$, : $X(x,y) = x+y$. $\forall (x,y) \in S$.

$\therefore$ Spectrum of $X = \{0,1,2,3,4,5,6\}$.

Now $P(X=0) = \frac{1}{16}$, $P(X=1) = \frac{2}{16}$, $P(X=2) = \frac{3}{16}$

$P(X=3) = \frac{4}{16}$, ... $P(X=6) = \frac{1}{16}$

Q. Find the probability distribution of the no. of failures preceding the first success in an ∞ seq. of Bernoulli trials with prob. of success p.

Ans. E = infinite seq of Bernoulli trials.

$\therefore X : S \to \mathbb{R}$. : $X(\omega)$ = no. of failures preceding the first success.

$\therefore$ Spectrum of $X = \{0,1,2,\ldots\infty\}$.

Now, $P(X=i) \Rightarrow i$ failures, $(i+1)^{th}$ trial is success.

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$\Rightarrow (1-p)^i \cdot p \quad \forall i = 0, 1, \ldots, n.$

* Check sum is 1.

$P(X=0) + P(X=1) + \ldots \infty \Rightarrow p + (1-p)p + \ldots \infty$

$\Rightarrow \dfrac{p}{1-(1-p)} = \dfrac{p}{p} = 1.$

* This kind of distribution when the terms are in a GP is known as Geometric Distribution.

→ **Discrete Distribution:-**

All distributions discussed so far were discrete.

Defn :-

If the spectrum of a random variable X is finite or countably infinite then the distribution of X is called a discrete distribution.

Let spectrum of X : $T = \{x_i : i = 0, \pm 1, \pm 2, \ldots\}$ with $\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots$

The fn $f_X : \mathbb{R} \to [0, 1]$ defined as (obvious).

$$f_X(x) = \begin{cases} P(X = x_i), & x = x_i \in T, \\ 0, & \text{elsewhere.} \end{cases}$$

is called the probability mass fn (p.m.f.) of X.

Only when $X = x_i$, it has prob of occuring, else it doesn't. Defn of spectrum.

* Properties of pmf

① $F_X(x) = \sum_{x_j \le x_i} P(X = x_j)$ if $x_i \le x < x_{i+1}$.

Taking $P(x_j)$ $\forall x_j \le x_i$.

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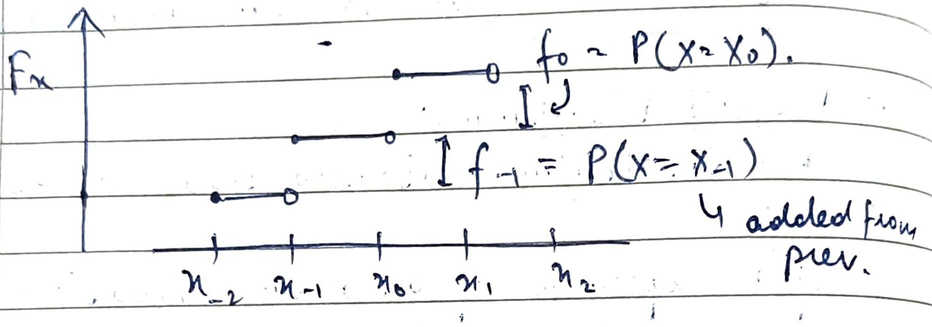
This is used to get the probability distribution f^n from pmf.

② F_X is a step f^n with steps $f_i = P(X = x_i)$ for $i = 0, \pm 1, \pm 2, \dots$

↳ The step heights will be equal to the added probabilities $P(x = x_i)$.



③ $\sum_{j=-\infty}^{\infty} f_j = 1$. → All points in spectrum.

④ At any $a \notin$ spectrum of X, $\boxed{P(X = a) = 0}$

↳ Def^n.

⑤ $P(a < X \leq b) = \sum_{a < x_i \leq b} f_X(x_i)$.

→ Discrete Distribution Examples :-

① Binomial (n, p) Distribution :-

$X \sim$ Binomial (n, p).

Spectrum of $X = \{0, 1, 2, \dots, n\}$.

↳ obv. in n trials, no. of successes can be 0 to n.

Probability mass f^n (p.m.f.)

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$$f_X(x) = \begin{cases} \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}, & x = 0,1,2,\dots \\ 0, & \text{elsewhere.} \end{cases}$$

Here $(0<p<1.)$.

and. $\sum_{x=0}^{n} \binom{n}{x} p^x \cdot (1-p)^{n-x} = [p+(1-p)]^n = 1.$

② Poisson (μ) Distribution:

$X \sim \text{Poisson}(\mu)$

Spectrum of $X = \{0,1,2,\dots\}$. Here $\mu > 0$.

p.m.f.

$\hookrightarrow f_X(X) = \dfrac{e^{-\mu} \cdot (\mu)^x}{\lfloor x}, \quad x = 0,1,\dots$

$= 0, \quad$ elsewhere.

$$\sum_{x=0}^{\infty} \frac{e^{-\mu} \cdot (\mu)^x}{\lfloor x} = e^{-\mu} \cdot \overbrace{\sum_{x=0}^{\infty} \frac{(\mu)^x}{\lfloor x}}^{\text{Expn of } e^{\mu}}$$

$$= e^{-\mu} \cdot e^{\mu}$$

$$= \boxed{1}$$

③ ~~Continue~~

→ Continuous Distribution

The distribution of a random variable X is said to be continuous if

(i) the distribution f^n F_X is continuous.

(ii) $\frac{d}{dx} F_X(x) = F'_X(x)$ is piecewise continuous in $(-\infty, \infty)$.

Finite pts of discontinuity.

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$f_X : \mathbb{R} \to [0,1]$ as $f_X(x) = \frac{d}{dx} F_X(x)$ which is called the probability density function (p.d.f.) of X.

__Properties__

① $f_X \geq 0$. $F_X(x)$ is monotonic increasing.
Hence derivative +ve.

② $P(a < X \leq b) = F_X(b) - F_X(a)$. Same thing.
$= \int_a^b f_X(x)\,dx$

③
(i) $f_X(x) = \frac{d}{dx} F_X(x)$ (primitive exists.)

(ii) $f_X(x)$ has finite no of jump discontinuities (1st kind) in $[a,b]$ $\Rightarrow$ $f_X(x)$ is R. integrable in $[a,b]$.

Applying Fundamental th. of integral Calculus
$$\int_a^b f_X(x)\,dx = F_X(b) - F_X(a) = P(a < x \leq b).$$

③ $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx. \to \boxed{P(-\infty < X \leq x)}$

④ $F(-\infty) = 0$.

⑤ $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

⑥ $P(X = a) = 0$ for a given constant a. → at one particular point, $P(X = a) = 0$.

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This is proved using $P(X=a) = F_X(a) - F_X(a-0^+)$.
Continuous ⇒ equal.

⑦ Converse statement :- Every non negative, real valued, piecewise continuous f^n f that is integrable in $(-\infty, \infty)$ and satisfies $\int_{-\infty}^{\infty} f(x)\,dx = 1$, is the probability density f^n of a continuous distribution.

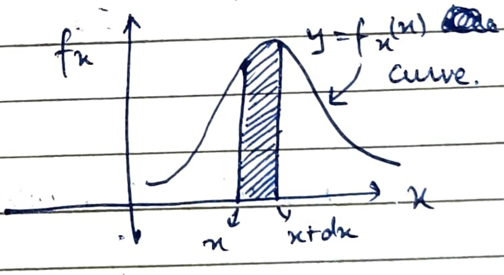
⑧ Probability Differential :- Let X has continuous distribution - In differential notation, we write :-

$$P(x < X \leq x+dx) = F_X(x+dx) - F_X(x) = dF_X.$$

$$F_X'(x) = \frac{dF_X}{dx}$$

This quantity = $F_X'(x)\,dx = f_X(x)\,dx$.

Geometrically.



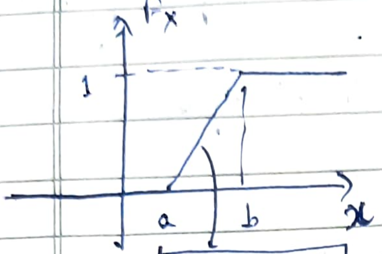
⑨ Density Curve :- The curve $y = f_X(x)$ is called the probability density curve of the corresponding continuous distribution.

→ Examples of continuous distribution

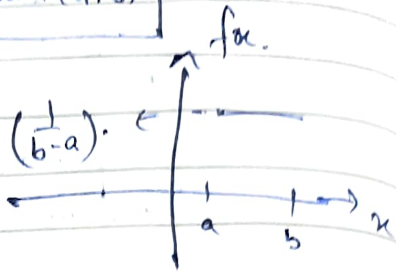
eg. ① $f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{elsewhere.} \end{cases}$

This is a distribution f^n as
① $f_X(x) \geq 0$. ② $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

This distribution is $\boxed{\text{Uniform}(a,b)}$



$\Leftarrow F_X(x)$



$\boxed{\text{slope} = \left(\frac{1}{b-a}\right)}$ → use formula $\boxed{F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx}$

$F_X(x) = 0$ when $x < a$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{a}^{x} \frac{1}{b-a}\,dx = \left(\frac{x-a}{b-a}\right)$$ when $a \leq x \leq b$.

$F_X(x) = 1$, when $x > b$.

eg. ② <u>Normal (m, σ) Distribution</u>
↳ $X \sim \text{Normal}(m, \sigma)$ → to denote X has normal distribution.

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}}$$ $-\infty < x < \infty$

when $\boxed{\sigma > 0}$.

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This is a distribution f^n as

① $f_X(x) \geq 0. \quad \forall x.$ obv.

② $\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$

$$\underbrace{\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx}_{\boxed{I.}} \Rightarrow \text{Using } \gamma\text{-}f^n.$$

Assuming $\frac{x-\mu}{\sqrt{2}\sigma} = z$

then $\frac{1}{\sqrt{2}\sigma}\,dx = dz$

$$\therefore I = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\,dz \Rightarrow \frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-z^2}\,dz$$

limits change.

~~Assuming~~ $z^2 = t$ $\cancel{\Rightarrow 2z\,dz = dt}$.

Assuming $z^2 = t \quad 2z\,dz = dt$

$$\Rightarrow 2\,dz = \frac{dt}{\sqrt{t}}$$

$$\Rightarrow I = \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-t}\cdot\frac{dt}{\sqrt{t}} = \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-t}\cdot(t)^{1/2-1}\,dt$$

$$\Rightarrow I = \frac{\gamma(1/2)}{\sqrt{\pi}} \quad \therefore \boxed{I = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1} \quad \text{as } \boxed{\gamma(1/2) = \sqrt{\pi}}$$

Hence this is a probability distribution.

* In particular, if $\mu = 0,\ \sigma = 1$ then.

normal distribution will be called Standard Normal distribution.

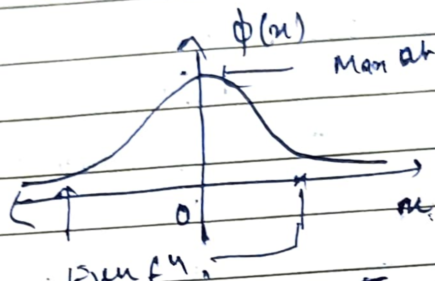
Density f^n $\boxed{\phi_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$, $-\infty < x < \infty$.

Distribution f^n $\boxed{\Phi_x(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx}$

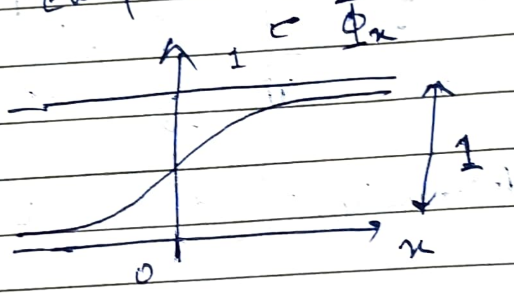
Capital phi.

Graph.

for $f_X(x)$.



for $\Phi_x(x)$.



③ <u>Cauchy Distribution (λ, μ)</u>

$$f_x(x) = \frac{1}{\pi} \cdot \frac{\lambda}{\lambda^2 + (x-\mu)^2}, \quad -\infty < x < \infty, \text{ and } \lambda > 0.$$

① $f_x(x) \geq 0 \quad \forall x.$

② $\int_{-\infty}^{\infty} f_x(x) \cdot dx = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{1}{\lambda^2 + (x-\mu)^2}$

$$= \frac{\lambda}{\pi} \cdot \left[\tan^{-1}\left(\frac{x-\mu}{\lambda}\right) \right] \cdot \frac{1}{\lambda} \Bigg|_{-\infty}^{\infty}$$

$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$.

$\Rightarrow \frac{1}{\Lambda}\left[\frac{\Lambda}{2} - \left(-\frac{\Lambda}{2}\right)\right] \Rightarrow \boxed{1}$.

④ Gamma (l) Distribution $\quad X \sim \gamma(l)$.

$$f_X(x) = \frac{1}{\Gamma(l)} e^{-x} \cdot x^{l-1}, \quad 0 < x < \infty.$$
$$= 0 \quad , \text{ elsewhere}, \quad \boxed{l>0}.$$

① $f_X(x) \geq 0$.

② $\int_{-\infty}^{\infty} \frac{1}{\Gamma(l)} e^{-x} \cdot x^{l-1} = \frac{\Gamma(l)}{\Gamma(l)} = \boxed{1}$.
↳ this only is $\Gamma(l)$

⑤ Beta (l,m) distribution of 1st kind

$x \sim \beta_1(l,m)$. Here $l,m>0$.
↳ 1st kind β.

$\cancel{f_X(x) = \frac{1}{\beta(l,m)} \cdot x^{(l-1)} \cdot (1-x)^{(m-1)}}$
~~↳ β f^n ~~only~~ terms only.~~

$\cancel{① f_X(x) \geq 0}$. $\quad \cancel{② \int_{-\infty}^{\infty} \frac{1}{\beta(l,m)} \cdot x^{(l-1)} (1-x)^{m-1}}$.

$$f_X(x) = \begin{cases} \frac{1}{\beta(l,m)} \cdot x^{(l-1)} \cdot (1-x)^{m-1}, & 0<x<1, \\ 0, & \text{elsewhere.} \end{cases}$$

① $f_X(x) \geq 0$. $\quad$ ② $\int_{-\infty}^{\infty} \frac{1}{\beta(l,m)} x^{(l-1)} (1-x)^{m-1}$

$$= \int_0^1 \frac{1}{\beta(l,m)} x^{(l-1)} (1-x)^{m-1}$$
↳ β f^n.

$\Rightarrow \frac{\beta(l,m)}{\beta(l,m)} = \boxed{1}$.

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⑥ Beta (l,m) distribution of 2nd kind.

$X \sim \beta_2(l,m)$, $(l,m>0)$.

$$f_X(x) = \begin{cases} \frac{1}{\beta(l,m)} \frac{x^{(l-1)}}{(1+x)^{l+m}} & , 0<x<\infty. \\ 0 & , \text{elsewhere}. \end{cases}$$

Problem Set 4

First 3 ques^n s done.

Q4. $f(x) = ke^{-|x-\alpha|}$, $-\infty < x < \infty$.

This should be a probability f^n.

$\therefore$ ① $f(x) \geq 0 \Rightarrow ke^{-|x-\alpha|} \geq 0$

$\Rightarrow \boxed{k>0}$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{\infty} ke^{-|x-\alpha|}\,dx$ ~~sep~~

$\Rightarrow k\int_{-\infty}^{\infty} e^{-|x-\alpha|}\,dx$ ~~= 1~~

Taking $x-\alpha = z$ $\quad dx = dz$

$\Rightarrow k\int_{-\infty}^{\infty} e^{-|z|}\,dz \Rightarrow 2k\int_{0}^{\infty} e^{-|z|}\,dz$

$\Rightarrow 2k\int_{0}^{\infty} e^{-z}\,dz \Rightarrow -2k \cdot \left[e^{-z}\right]_0^{\infty}$

$\Rightarrow -2k[0-1] \Rightarrow \boxed{2k=1} \Rightarrow \boxed{k=\frac{1}{2}}$

Distribution f^n.

$$\frac{1}{2}\int_{-\infty}^{x} e^{-|x-\alpha|}\,dx$$

Case-① when $x<\alpha$.

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{x} e^{-|x-\alpha|}\,dx$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{x} e^{-(\alpha-x)}\,dx \Rightarrow \frac{1}{2}\left[e^{x-\alpha}\right]_{-\infty}^{x}$$

$$\Rightarrow \frac{1}{2}\left[e^{x-\alpha}\right]_{-\infty}^{x} \Rightarrow \boxed{\frac{e^{x-\alpha}}{2}}$$

Case-② when $x\geq\alpha$.

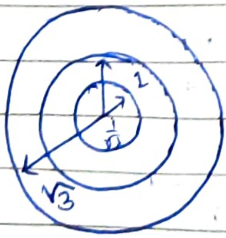
$$\Rightarrow \frac{1}{2}\int_{-\infty}^{\alpha} e^{-|x-\alpha|}\,dx + \frac{1}{2}\int_{\alpha}^{x} e^{-|x-\alpha|}\,dx$$

$$\Rightarrow \frac{1}{2}\left(e^{\alpha-\alpha}\right) + \frac{1}{2}\int_{\alpha}^{x} e^{\alpha-x}\,dx$$

$$\Rightarrow \frac{1}{2} + \frac{1}{-2}\left[e^{\alpha-x}\right]_{\alpha}^{x}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{2}\left[e^{\alpha-x}-1\right]$$

$$\Rightarrow \boxed{1-\frac{e^{\alpha-x}}{2}}$$

Q8. 

Probability Density f^n of x

$$\left(\frac{2}{\pi}\cdot\frac{1}{1+x^2}\right)$$

~~P(Score 3)~~ : Spectrum of $X = \{0, 1, 2, 3\}$.

~~$P(X=3) = P(R \text{ lies between } 0, 1/\sqrt{3})$.~~

~~$$= \int_0^{1/\sqrt{3}} \frac{2}{\pi} \cdot \frac{1}{(1+x^2)} \, dr$$~~

Let R be the random variable corr. to the distance of hit from O.

$$\therefore f_R(r) = \begin{cases} \frac{2}{\pi} \cdot \left(\frac{1}{1+r^2}\right), & 0 \le r < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

$$P(X=0) = P(R > \sqrt{3})$$

$$= 1 - P(R \le \sqrt{3})$$

$$= 1 - \int_{-\infty}^{\sqrt{3}} \frac{2}{\pi} \cdot \frac{1}{1+r^2} \, dr$$

$$= 1 - \frac{2}{\pi} \cdot \left[\tan^{-1}(r)\right]_0^{\sqrt{3}}$$

$$= 1 - \frac{2}{\pi} \cdot \left[\frac{\pi}{3} + 0\right]$$

$$= 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

$$P(X=1) = P(1 \le R \le \sqrt{3})$$

$$= P(R \le \sqrt{3}) - P(R \le 1)$$

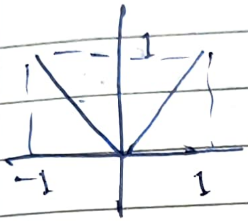
$$= \int_1^{\sqrt{3}} \frac{2}{\pi} \left(\frac{1}{1+r^2}\right)$$

$$= \frac{2}{\pi} \left[\tan^{-1}(r)\right]_1^{\sqrt{3}} = \frac{2}{\pi} \cdot \left[\frac{\pi}{3} - \frac{\pi}{4}\right]$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{12} = \boxed{\frac{1}{6}}.$$

|||ly $\boxed{P(X=2) = 1/6}$ $\boxed{P(X=3) = 1/3}$

Q5. $f(x) = \begin{cases} |x| , & x \in (-1,1). \\ 0 , & \text{elsewhere.} \end{cases}$



$$\int_{-\infty}^{\infty} |x|\,dx$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx + \int_{1}^{\infty} 0\,dx$$

$$= -\frac{x^2}{2}\Big]_{-1}^{0} + \frac{x^2}{2}\Big]_{0}^{1} \Rightarrow \boxed{1}$$

$$F_X = \int_{-\infty}^{x} f(x)\,dx = 0, \quad x < -1.$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{x} -x\,dx = -\frac{x^2}{2}\Big]_{-1}^{x}$$

$$= -\left(\frac{x^2}{2} - \frac{1}{2}\right) = \left(\frac{1-x^2}{2}\right), \quad -1 \le x \le 0.$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} -x\,dx + \int_{0}^{x} x\,dx.$$

$$= \frac{1}{2} + \left(\frac{x^2}{2}\right) \to \left(\frac{1+x^2}{2}\right), \quad 0 < x \le 1.$$

$$= 1, \quad x > 1.$$

$$\therefore \boxed{F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{1-x^2}{2}, & -1 \le x < 0 \\ \frac{1+x^2}{2} & 0 \le x < 1 \\ 1 & x \ge 1. \end{cases}}$$

Q6. $f(x) = \begin{cases} x, & x \in (0,1) \\ k-x, & x \in (1,2) \\ 0, & \text{elsewhere.} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} (k-x)\,dx + \int_{2}^{\infty} 0\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_0^1 + [kx]_1^2 - \left[\frac{x^2}{2}\right]_1^2 + 0$$

$$= \frac{1}{2} + k - \frac{3}{2} \Rightarrow \boxed{k-1}$$

$\therefore k-1 = 1 \Rightarrow \boxed{k=2}$

Now $P(1/2 < x < 3/2)$

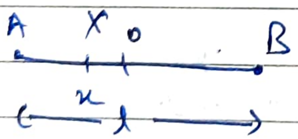
$$= \int_{1/2}^{1} x\,dx + \int_{1}^{3/2} (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{1/2}^{1} + [2x]_1^{3/2} - \left[\frac{x^2}{2}\right]_1^{3/2}$$

$$= \frac{3}{8} + 1 - \frac{5}{8} = \boxed{\frac{3}{4}}$$

~~Q7 Poisson Distribution~~

Q9.



Let length of AB be l.
length of $AX = x$.

$\therefore f_X(x) = \begin{cases} 0, & x < 0 \\ k, & 0 \le x \le l \\ 0, & x > l. \end{cases}$

Now we want AX, BX and AO to make a Triangle
$AX = x$, $BX = l-x$, $AO = l/2$

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Now $\int_{-\infty}^{\infty} f_X(x) = 1 \Rightarrow \int_{-\infty}^{\infty} k\,dx$

$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{l} k\,dx + \int_{l}^{\infty} 0\,dx$

$\Rightarrow kl = 1 \Rightarrow \boxed{k = \frac{1}{l}}$

$\therefore f_X(x) = \begin{cases} 0, & x \in (-\infty, 0) \cup (l, \infty) \\ \frac{1}{l}, & x \in [0, l]. \end{cases}$

Now for these to be sides of a Δ

(i) $\frac{l}{2} + x > l - x \Rightarrow 2x > \frac{l}{2} \Rightarrow \boxed{x > \frac{l}{4}}$

(ii) $l - x + x > \frac{l}{2} \Rightarrow l > \frac{l}{2}$ which is always true.

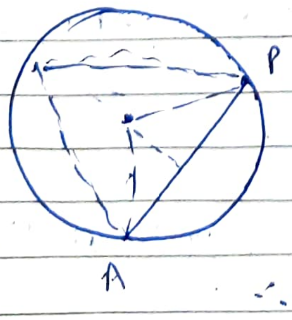
(iii) $l - x + \frac{l}{2} > x \Rightarrow 2x < \frac{3l}{2} \Rightarrow \boxed{x < \frac{3l}{4}}$

$\therefore P\left(\frac{l}{4} < x < \frac{3l}{4}\right) = \int_{l/4}^{3l/4} \frac{1}{l}\,dx$

$= \left[\frac{x}{l}\right]_{l/4}^{3l/4} = \left[\frac{3l}{4} - \frac{l}{4}\right] \cdot \frac{1}{l}$

$\boxed{P\left(\frac{l}{4} < x < \frac{3l}{4}\right) = \frac{1}{2}}$

Q10.

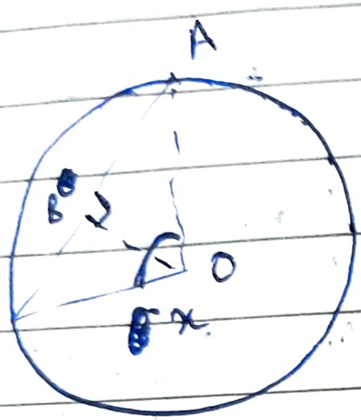


If AP is ~~an equita~~ equal to side of eq. Δ in this circle, then

$\boxed{AP = \sqrt{3}a}$

$\therefore$ We need to find $P(AP > \sqrt{3}a)$.

$$\cancel{P(x > \sqrt{3}a) = \int_{\sqrt{3}a}^{2a} \frac{1}{2a}\,dx}$$



Let X be the random variable denoting the angle subtended by the chord at the centre.

$$\therefore f_X(x) = \begin{cases} k, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$$

For $AP > \sqrt{3}a$,

$$\boxed{\angle AOP > 2\pi/3}$$

$$\Downarrow$$

Side length $AB = AO \sin\left(\frac{x}{2}\right)$ $\quad \therefore AP = 2AO \sin\left(\frac{x}{2}\right)$

$$= 2\cancel{a} \sin\left(\frac{x}{2}\right) > \sqrt{3}\cancel{a}$$

$$\Rightarrow \sin\left(\frac{x}{2}\right) > \frac{\sqrt{3}}{2} \Rightarrow \frac{x}{2} > \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \frac{x}{2} > \frac{\pi}{3} \Rightarrow \boxed{x > \frac{2\pi}{3}}$$