① X : population r.v.

population distribution: $P(X=0) = P(X=1) = \frac{1}{2}$

$$\text{spectrum } X = \{0, 1\}$$

random sample of size 4 : (x_1, x_2, x_3, x_4)

$$t = x_1 + x_2 + x_3 + x_4$$

Sol: Random variable corresponding to t is

$$T = X_1 + X_2 + X_3 + X_4$$

where X_1, X_2, X_3, X_4 are mutually independent and have the same distribution as X.

Spectrum of $T = \{0, 1, 2, 3, 4\}$

$$P(T=0) = P(X_1=0, X_2=0, X_3=0, X_4=0)$$
$$= P(X_1=0)\,P(X_2=0)\,P(X_3=0)\,P(X_4=0)$$
$$= \frac{1}{2^4} = \binom{4}{0}\frac{1}{2^4}$$

$$P(T=1) = P(X_1=1, X_2=0, X_3=0, X_4=0) + P(X_1=0, X_2=1, X_3=0, X_4=0)$$
$$+ \ldots$$
$$= \binom{4}{1}\frac{1}{2^4}$$

$P(T=2) = ?$, $P(T=3) = ?$ $P(T=4) = ?$ (check!)

$$T \sim \text{Binomial}\left(4, \tfrac{1}{2}\right)$$

Prob ②: Show that the sample mean is asymptotically normal $(m, \sigma/\sqrt{n})$ when m is the population mean
σ is the ... S.D.

Sol: Consider a random sample of size n:
$$(x_1, x_2, \ldots, x_n).$$

Sample mean: $$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

R.V. corresponding to $\bar{x}$ is
$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$
where $X_1, X_2, \ldots, X_n$ have iid as X.

Note: C.L.Th. $\bar{X}$ is asymptotically normal $(m, \sigma/\sqrt{n})$

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$\Rightarrow X_1 + X_2 + \cdots + X_n \quad \cdots\cdots\cdots \quad \left(mn, \frac{\sigma}{\sqrt{n}} \cdot n\right)$

(check using transformation of r.v.s)

Applying C.L.Th

$\bar{X}$ asymptotically normal $(m, \sigma/\sqrt{n})$.

(3) population r.v. $X \sim \text{normal}(m, \sigma)$

To find distribution of $\overline{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$.

$$= \frac{1}{n} X_1 + \frac{1}{n} X_2 + \cdots + \frac{1}{n} X_n$$

$\frac{1}{n} X_1, \frac{1}{n} X_2, \ldots, \frac{1}{n} X_n$ are mutually independent and have the same distribution as $\frac{1}{n} X$

$$E\left(\frac{1}{n} X_i\right) = \frac{m}{n}, \quad Var\left(\frac{1}{n} X_i\right) = \frac{\sigma^2}{n^2}$$

Using Reproductive Prop, $\overline{X} \sim N(m_0, \sigma_0)$

where $m_0 = \frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n} = m$

$\sigma_0 = \sqrt{\frac{\sigma^2}{n^2} \cdot n} = \sigma / \sqrt{n}$.

i.e. $\overline{X} \sim N(m, \sigma/\sqrt{n})$.

<u>Note:</u> $$U = \frac{\bar{X} - m}{\sigma/\sqrt{n}} \sim N(0,1)$$

If m and σ an known then $u = \frac{\bar{x} - m}{\sigma/\sqrt{n}}$ is a statistic.

Prob 4: To show $\chi^2 = \frac{nS^2}{\sigma^2}$ is a $\chi^2(n-1)$ variate.

Sol. Consider a sample of size n: $(x_1, x_2, \dots, x_n)$

Let, $X_1, X_2, \dots, X_n$ be the r.v.s corr. to the sample values. $x_1, x_2, \dots, x_n$, respectively.

$X_1, X_2, \dots, X_n$ follows i.i.d. as X.

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$\Rightarrow nS^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}\left[(X_i - m) - (\bar{X} - m)\right]^2$$

check! $$= \sum_{i=1}^{n}(X_i - m)^2 - n(\bar{X} - m)^2$$

$$nS^2 = \sum_{i=1}^{n}(X_i-m)^2 - n\left(\frac{X_1+\cdots+X_n}{n} - m\right)^2$$

$$= \sum_{i=1}^{n}(X_i-m)^2 - \left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i-m)\right\}^2 \quad \text{(check!)}$$

$$\frac{nS^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i-m}{\sigma}\right)^2 - \left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i-m}{\sigma}\right)\right\}^2$$

$$= \sum_{i=1}^{n} Y_i^2 - \left\{\sum_{i=1}^{n} a_{1i} Y_i\right\}^2 \quad (*)$$

where $Y_i = \dfrac{X_i-m}{\sigma}$, $a_{1i} = \dfrac{1}{\sqrt{n}}$ $\forall\, i=1,2,\ldots,n$.

each X_i is $N(m, \sigma)$

$\Rightarrow$ each Y_i is $N(0, 1)$

Again $X_1, X_2, \ldots, X_n$ are mutually independent

$\Rightarrow$ (i) $Y_1, Y_2, \ldots, Y_n$ . . . .

(ii) $a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2 = \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1.$

(Using the prop. of χ^2-distn) check!

$\Rightarrow$

$$Q = \sum_{i=1}^{n} Y_i^2 - (a_{11}Y_1 + a_{12}Y_2 + \cdots + a_{1n}Y_n)^2$$

is $\chi^2(n-1)$ variate.

(ii) To show $\bar{X}$ and S^2 are independent.

Consider
$$Z_1 = a_{11}Y_1 + a_{12}Y_2 + \cdots + a_{1n}Y_n$$
$$Z_2 = a_{21}Y_1 + a_{22}Y_2 + \cdots + a_{2n}Y_n$$
$$\vdots$$
$$Z_n = a_{n1}Y_1 + a_{n2}Y_2 + \cdots + a_{nn}Y_n$$

where $(a_{11}, a_{12}, \ldots a_{1n}) = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$

and remain coeff's can be chosen s.t.
$$\sum_{k=1}^{n} a_{ki}a_{kj} = 1 \quad \text{if } i=j$$
$$= 0 \quad \text{if } i \neq j$$

We have : $Z_1^2 + Z_2^2 + \cdots + Z_n^2 = Y_1^2 + Y_2^2 + \cdots + Y_n^2$

$$\Rightarrow Z_2^2 + Z_3^2 + \cdots + Z_n^2 = Y_1^2 + Y_2^2 + \cdots + Y_n^2 - Z_1^2$$

$$= \frac{nS^2}{\sigma^2} \quad (\text{using } *)$$

$$\Rightarrow S^2 = \frac{\sigma^2}{n}\left[Z_2^2 + Z_3^2 + \cdots + Z_n^2\right] \quad \text{——— } 1$$

$$\bar{X} = \frac{(X_1 - m) + (X_2 - m) + \cdots + (X_n - m)}{n} + m$$

$$= \frac{\sigma Y_1 + \sigma Y_2 + \cdots + \sigma Y_n}{n} + m$$

$$= \frac{\sigma}{\sqrt{n}} Z_1 + m \quad \text{——— } 2$$

Note: $\underbrace{Z_1}, \underbrace{Z_2, \ldots, Z_n}$ are mutually independent r.v.s (check!)

$\Rightarrow$ $\bar{X}$ and S^2 are independent.

5. $t = \dfrac{(\bar{x} - m)}{s/\sqrt{n}}$ is t-distributed with $(n-1)$ degrees of freedom

$$s^2 = \frac{n}{n-1} S^2 \Rightarrow (n-1)s^2 = nS^2.$$

(unbiased sample variance) (biased sample variance)

<u>So!</u> $U = \dfrac{\sqrt{n}(\bar{x}-m)}{\sigma} \sim N(0,1)$

$$\chi^2 = \frac{nS^2}{\sigma^2} = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

Also, from 4 (ii), U and χ^2 are independent.

(check! in t-distribution properties)

$$t = \frac{U}{\sqrt{\chi^2/(n-1)}}$$ has t-distribution with (n-1) degrees of freedom

$$= \frac{(\bar{x}-m)\sqrt{n}}{s}$$

⑥ Hints. $\boxed{\frac{nS^2}{\sigma^2}} \sim \underbrace{\chi^2(n-1)}$

To find distribution of $\boxed{S^2}$: use 1D transformation rule

⑦ Hints. Use reproductive prop.