Lecture - 19/11/24

Phase and Group velocity

consider 2 sinusoidal waves of slightly different wavelengths on a string

$$y_1 = A \sin 2\pi (K_1 x - \nu_1 t)$$

$$y_2 = A \sin 2\pi (K_2 x - \nu_2 t)$$

characteristic speeds are $v = \frac{\omega}{K} = \frac{2\pi\nu}{K}$

→ superposition of these 2 waves yields

$$y = A\left(\sin[2\pi(K_1 x - \nu_1 t)] + \sin[2\pi(K_2 x - \nu_2 t)]\right)$$

$$= 2A \cos \pi \left((K_1 - K_2)x - (\nu_1 - \nu_2)t\right) \times \sin\left(2\pi\left(\frac{K_1 + K_2}{2}x - \frac{\nu_1 + \nu_2}{2}t\right)\right)$$

↳ 2 wavelengths - very similar

$K_1 - K_2 = \Delta K$

$\nu_1 - \nu_2 = \Delta\nu$

$\circ\ y = 2A \cos\pi(x\Delta k - t\Delta\nu)\sin 2\pi(kx - \nu t)$

where $k = \frac{k_1 + k_2}{2}$, $\nu = \frac{\nu_1 + \nu_2}{2}$

2 characteristics velocities here:

One - the speed at which a crest belonging to the average wave no. 'k' moves along.

this is phase velocity

$$\boxed{V_p = \frac{\omega}{k} = \nu\lambda}$$

Other - Is the velocity at which the modulating envelope moves. This envelope encloses a group of short waves.

Group velocity = $\boxed{V_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}}$

consider wave in deep water "gravity waves"

These are strongly dispersive - wave speed (i.e phase velocity) for a well-defined λ

$\propto$ proportional to $\lambda^{1/2}$

C: constant

$$V_p = C\lambda^{1/2} \sim C' K^{-1/2}$$

But $V_p = \frac{\omega}{K}$ $\therefore$ $\omega \sim C K^{1/2}$

$$\frac{d\omega}{dK} = \frac{1}{2} C K^{-1/2} = V_g$$

ie $\boxed{V_g = \frac{V_p}{2}}$

component wave crests, seem to run rapidly through the group 1st growing in amplitude and disappearing again

NOTE: Sound waves in gases (like other elastic vibration) — are "non-dispersive"

⊛ Review of gradient, divergence and curl:

→ If we have a function of one variable $f(x)$ then $\frac{df}{dx}$ tells us how rapidly $f(x)$ varies when we change the arguement by a tiny amount dx.

$$df = \left(\frac{df}{dx}\right) dx$$

Geometrical interpretation
– $\frac{df}{dx}$ is the slope of the curve $f(x)$

Gradient:

Now, if we have a function of 3 variables eg. Temperature $T(x,y,z)$ in a room

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

– the partial derivatives being along each of the 3 coordinate direction i.e $dT = \left(\frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j} + \frac{\partial T}{\partial z} \hat{k} \right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right)$

$= \underbrace{\vec{\nabla} T}_{\text{gradient of } T} \cdot \underbrace{d\vec{l}}_{\text{infinitesimal displacement vector}}$

$dT = \vec{\nabla} T \cdot d\vec{l} = |\vec{\nabla} T|\, |d\vec{l}| \cos\theta$

→ if we keep $|d\vec{l}|$ fixed - i.e fix its magnitude and search around in various directions, max change in T occurs when $\theta = 0$

∴ for a fixed distance $|d\vec{l}|$, dT is greatest when moved in the same direction as $\vec{\nabla} T$

i.e $\vec{\nabla} T$ points in the direction of maximum increase of the function T.

→ Should gradient vanish

i.e $\vec{\nabla} T = 0$ at (x, y, z)

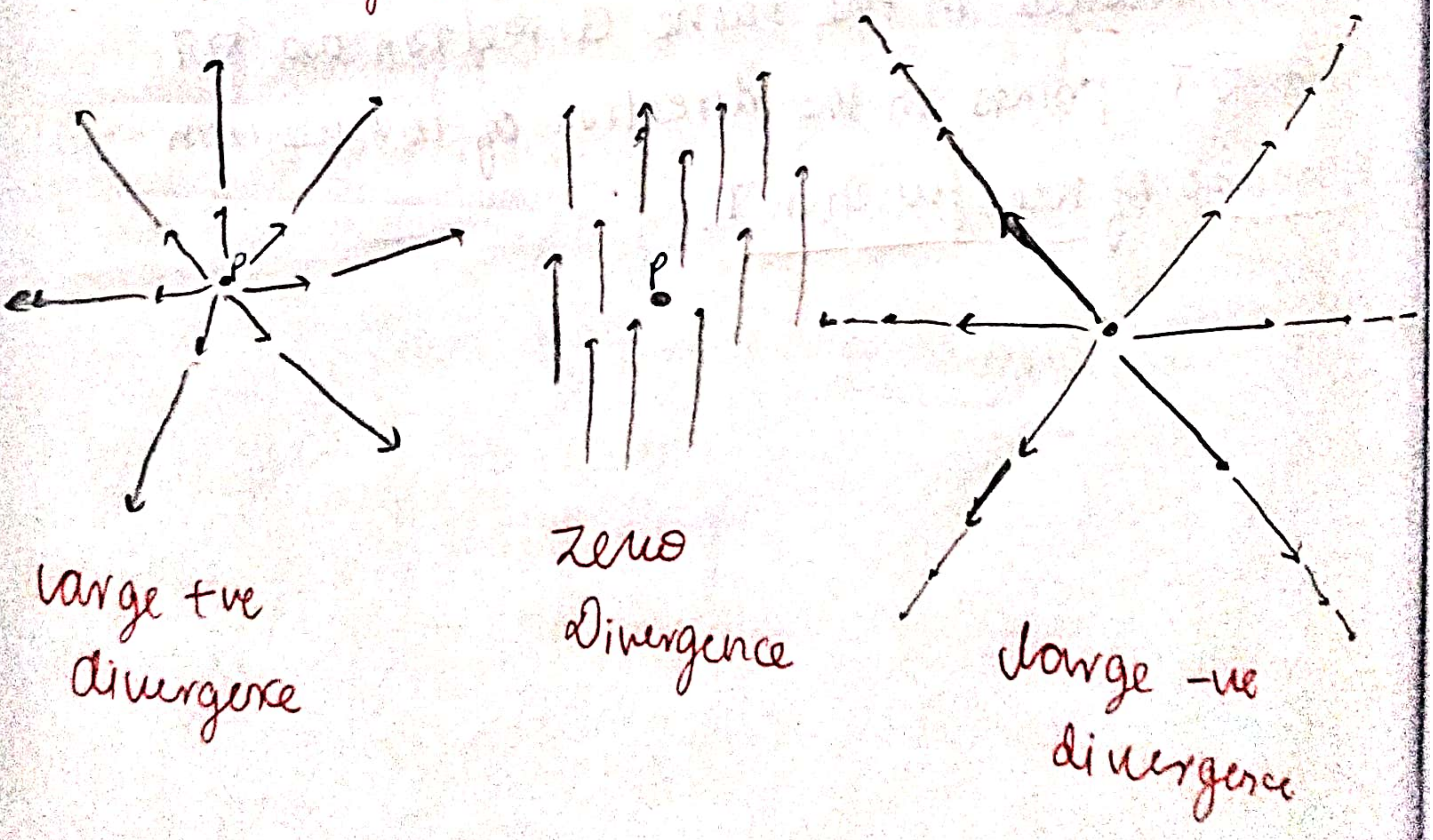
$\Rightarrow$ $dT = 0$ for small points about (x, y, z)

$\Rightarrow$ This is a stationary point of the function $T(x, y, z)$

- i.e it could be a maximum (summit)
- or could be a minimum (valley)
- or a saddle point (a pass)

④ Divergence:

- a measure of how much the vector spreads out (diverges) from a point

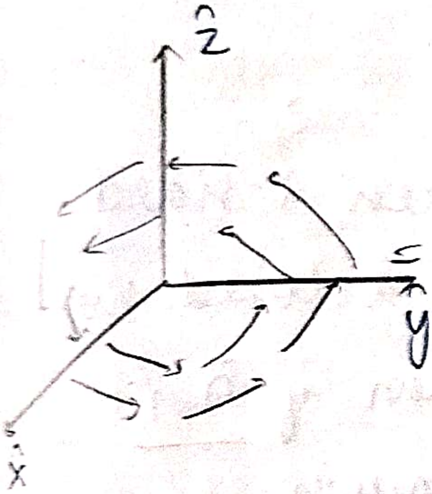


$$\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

Indicative of rotation

Curl:

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$



$$= \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Divergence:

- If a large +ve divergence – source
- If a large –ve divergence – sink or a drain

Consider the vector $\vec{V}$ – velocity of surface of water

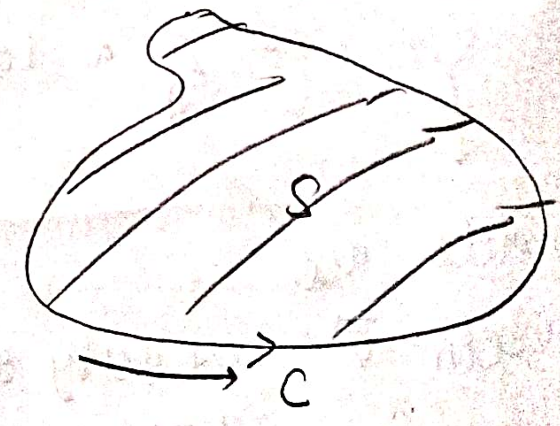
Gauss Theorem or divergence theorem

$$\int_{Vol} (\vec{\nabla} \cdot \vec{V})\, d\tau = \oiint_{surface} \vec{V} \cdot d\vec{a}$$

Stokes theorem : If S is an open 2-sided surface bounded by a closed non-intersecting curve C, then if $\vec{A}$ is continuous

$$\oint_C \bar{A} \cdot d\bar{l} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, ds$$

c is traversed in the +ve direction



⊛ Green's Theorem in a plane:

→ If R is a closed region of the xy plane bounded by a simple closed curve C and if M & N are countinuous functions of x & y having continuous derivatives in R then

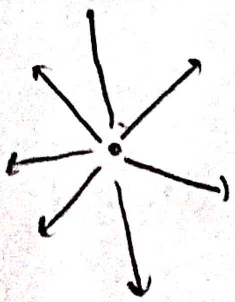
$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

→ C being traversed in the +ve counterclockwise direction

⊛ Divergence of $\frac{\hat{r}}{r^2}$:

consider $\vec{v} = \frac{\hat{r}}{r^2}$

The source of problem is the singularity at $r=0$
$\vec{\nabla} \cdot \vec{v} = 0$ everywhere except at the origin



→ In spherical coordinates

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, A_\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} A_\phi$$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0 \;??$$

Now $$\int \vec{v}\cdot d\vec{a} = \int\left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2 \sin\theta \, d\theta \, d\phi\right)\hat{r}$$

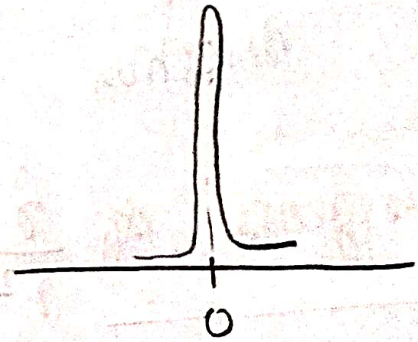
$$= \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi$$

But this

$\int \vec{\nabla}\cdot\vec{v} \, da$ (gauss's theorem)

→ such a function is the dirac delta fn

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$



$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$$\delta(x-a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

In 3 D

$$\delta^3(r) = \delta(x)\,\delta(y)\,\delta(z) \;..$$

$$\int_{\text{all surface}} \delta^3(r)\, \cancel{dx}\, dy\, dz = 1$$

$$\boxed{\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\, \delta^3(\vec{r})}$$