Systems2

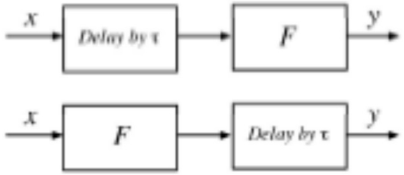
ESS 103

Time invariance

Time-Invariance

- A system is time-invariant if a time shift in the input only produces the same time shift in the output.
- For a system F,
$$y(t) = Fx(t)$$
implies that
$$y(t-\tau) = Fx(t-\tau)$$
for any time shift τ.

- Implies that delay and the system F commute. These block diagrams are equivalent:



Examples

Determine whether system is linear, time invariant or both

1) $y(t) = t^2 x(t-1)$

2) $y[n] = x^2[n-2]$

3) $y[n] = x[n+1] - x[n-1]$

4) $y(t) = Odd\{x(t)\}$

5) y(t) = x(2t)

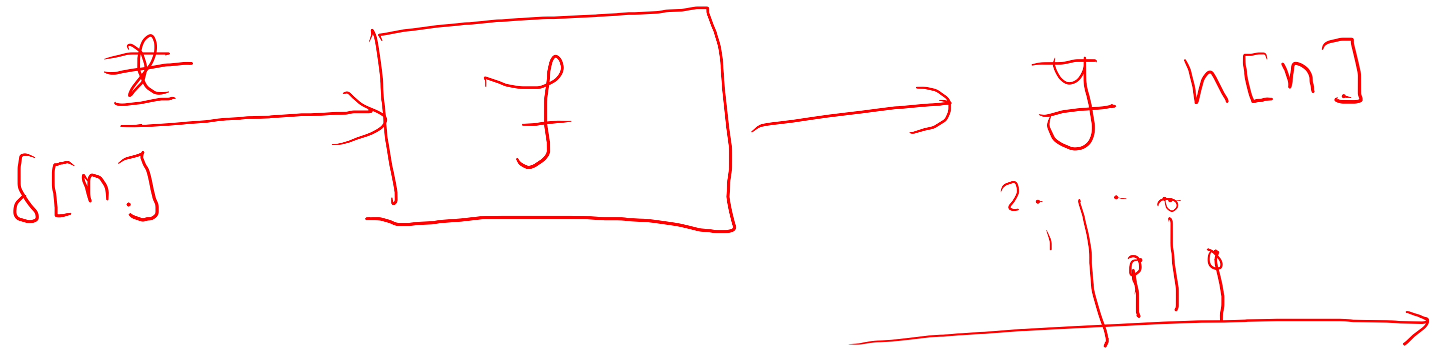
Examples

- y(t) = odd{x(t)}
- $y(t) = \sqrt{x(t)}$
- $y(t) = x(t)z(t), z(t)\ is\ known\ nonzero\ signal$
- $\boxed{y(t) = x(T - t)}$

Linear
Time variant

- Are the systems above,
  - Memoryless, ✗ ✓
  - BIBO stable, ✓
  - invertible, ?
  - causal, ✗
  - Linear, ✓
  - time invariant? ✗

LTI Systems



- Many physical systems can be modeled as linear time-invariant (LTI) systems.
- General signals can be represented as linear combinations of delayed impulses.
- For linear systems, the response y[n] can be represented as the sum of the responses (principle of linearity) the to the individual shifted impulses making up the input signal x[n] (time invariance).

$$x[n] = 1 \cdot \delta[n-1] + 2\delta[n-2] + \delta[n-3]$$

$$y[n] = h[n-1] + 2h[n-2] + h[n-3]$$

LTI

$$\delta[n] \longrightarrow \boxed{T} \longrightarrow h[n]$$

$$\delta[n-1] \longrightarrow h[n-1]$$

$$a\delta[n] \longrightarrow a\,h[n]$$

$$a\delta[n] + b\delta[n-1] \longrightarrow a\,h[n] + b\,h[n-1]$$

The importance of δ function

- Recall Definition of δ
  - Discrete case :
- If we have an arbitrary function $x[n]$, whose sequence values are known…
  Say…. $x[-10], x[-9] \ldots\ldots x[0], x[1] \ldots\ldots$
- Then, we can use $\delta[n]$ to express $x[n]$
- How ??

The importance of δ function

$$x[n] = ..x[-10]\delta[n+10] + x[-9]\delta[n+9] \ldots x[0]\delta[0] + x[1]\delta[n-1] \ldots.$$

In general,

$$x[n] = \Sigma\, x[k]\, \delta[n-k]$$; Index $k = -\infty\ to\ +\infty$

Any arbitrary signal can be written as an addition of weighted Delta functions (Appropriately shifted)

Convolution Sum

A discrete-time signal can be decomposed into a sequence of individual impulses.

Example:

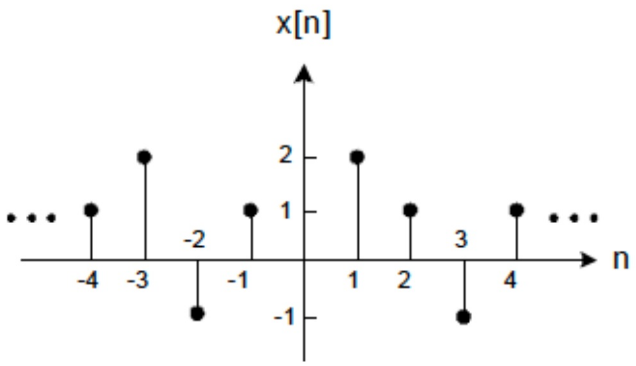


Fig. 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

The signal in Fig. 2.1 can be expressed as a sum of the shifted impulses:

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

- This means an arbitrary signal, $x[n]$ is represented as a linear combination of weighted shifted impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{2.2}$$

This corresponds to the representation of an arbitrary sequence as a linear combination of shifted unit impulse $\delta[n-k]$, where the weights in the linear combination are $x[k]$. Eq. (2.2) is called the ***sifting property*** of the discrete-time unit impulse.

- Why is this important ???

Linear Systems

- If we guarantee Linearity, then we need to know the outputs of some basic signals….This can help us generate the output for the entire span of signals using those basic signals

- But one more thing is essential..

Time-Invariance

Time Invariance

- What does Time – Invariance guarantee ?

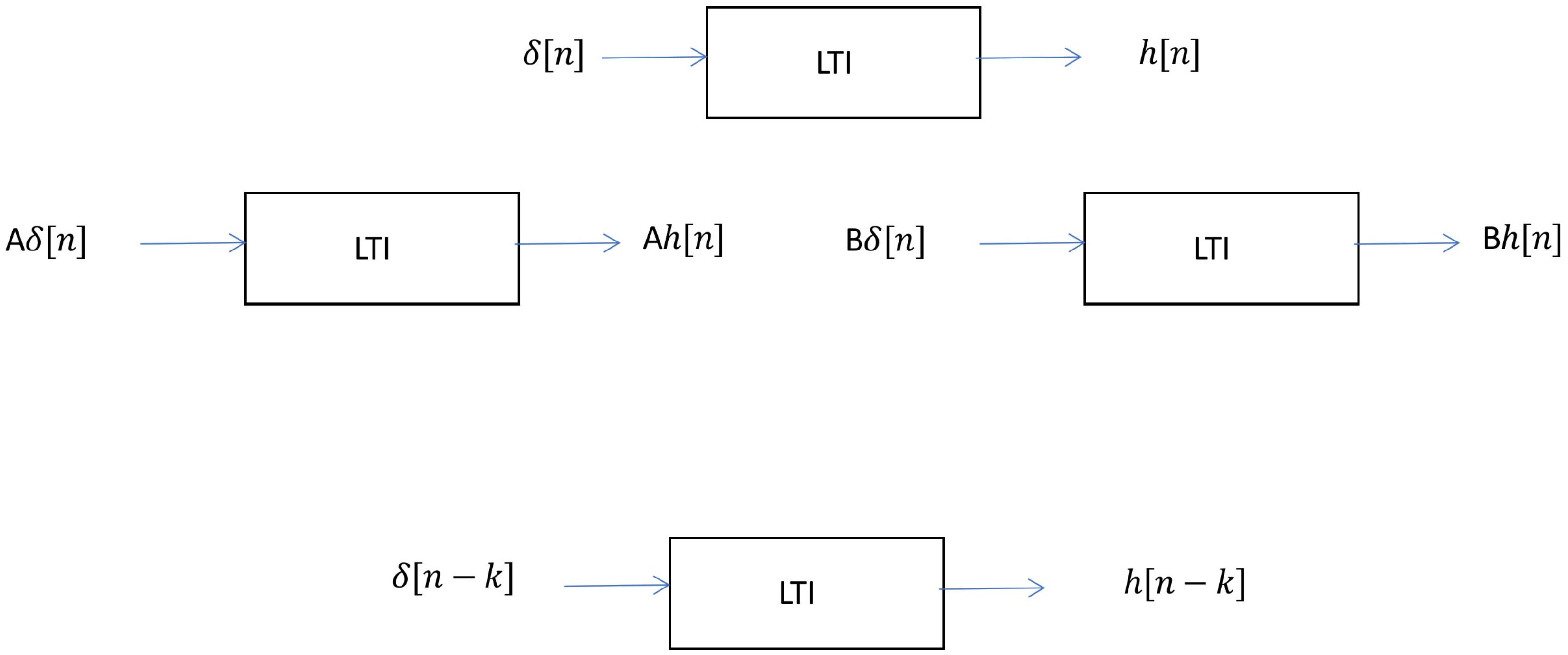
- Output of a Time-shifted signal is same as…

  Time-shifted output

- So now let's try to put Linearity and Time- Invariance together

Linear & Time-Invariant Systems

LTI Systems



Combining : Linearity & Time invariance

$$a_1\delta[n-k_1] \; + \; a_2\delta[n-k_2] \longrightarrow \boxed{\text{LTI}} \longrightarrow a_1\, h[n-k_1] \; + \; a_2\, h[n-k_2]$$

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Recall : any arbitrary signal, $x[n] = \Sigma\, x[k]\, \delta[n-k]$; Index $k = -\infty\ to\ +\infty$

$$x_1\delta[n-k_1] + x_2\delta[n-k_2] + \cdots + x_m\delta[n-km] \longrightarrow \boxed{\text{LTI}} \longrightarrow x_1h[n-k_1] + x_2h\,[n-k_2] + \cdots + x_mh\,[n-km]$$

Combining : Linearity & Time invariance

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$$x[n] = \Sigma\, x[k]\,\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \Sigma\, x[k]\,h[n-k]$$

Index $k = -\infty\ to\ +\infty$

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LTI Systems

$x[n] = \Sigma\, x[k]\, \delta[n-k]$

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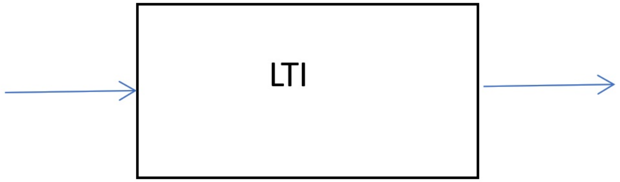

$y[n] = \Sigma\, x[k]\, h[n-k]$

Index $k = -\infty \; to \; +\infty$

Here, $y[n] = \Sigma\, x[k]\, h[n-k]$ is called the Convolution Sum

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LTI Systems

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Here, $y[n] = \Sigma\, x[k]\, h[n-k]$ is called the Convolution Sum

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Convolution Operation, $x[n] * h[n]$

Operator is "$*$"

Defined as , $x[n] * h[n] = \Sigma\, x[k]\, h[n-k]$; Index $k = -\infty\ to\ +\infty$

In Summary, for an LTI System, to determine the O/P of the system to any arbitrary I/P, all we need to know is : Impulse Response, $h[n]$

i.e. LTI System is completely characterized by its Impulse Response

Convolution Sum

$$x[n] * h[n] = \Sigma\, x[k]\, h[n-k] \text{ ; Index } k = -\infty \; to \; +\infty$$

Steps Involved :

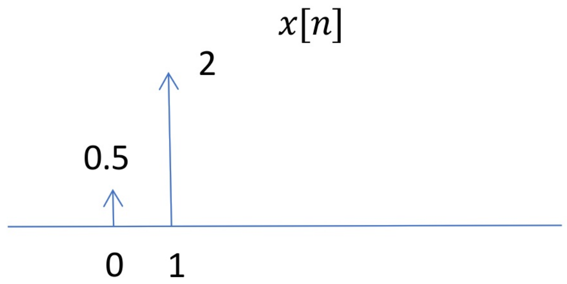
1) Time Reversal of h : $h[k]$ is time-reversed to obtain $h[-k]$

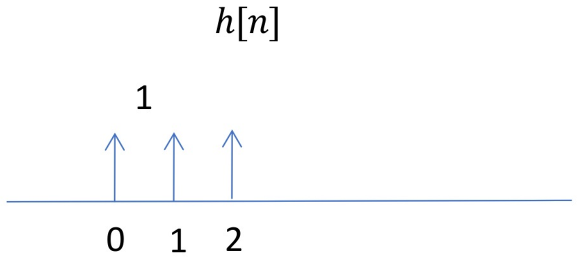
2) Shift it by "n" steps to obtain $h[n-k]$

3) $x[k]$ and $h[n-k]$ are point-wise multiplied for all values of "k" for a given value of "n"

4) Repeat for all values of "n"

Example





$y[n]$ is the convolution of $x[n]$ and $h[n]$

i.e., $y[n] = x[n] * h[n] = \Sigma\, x[k]\, h[n-k]$; Index $k = -\infty\ to\ +\infty$

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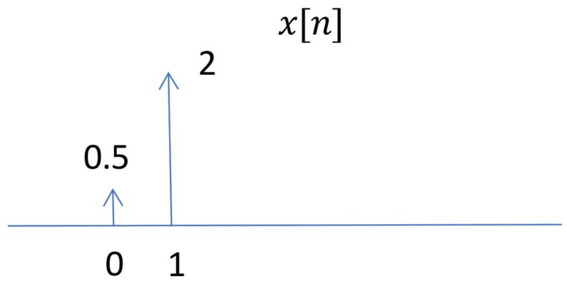

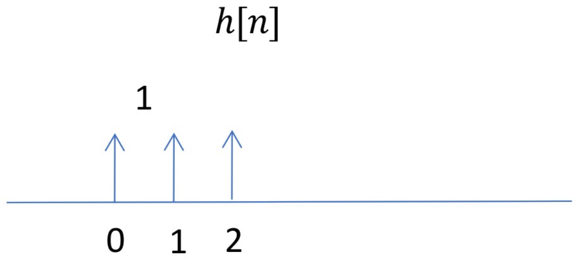

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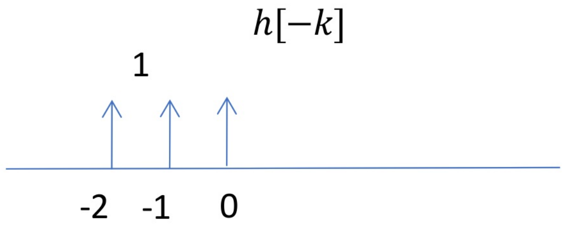

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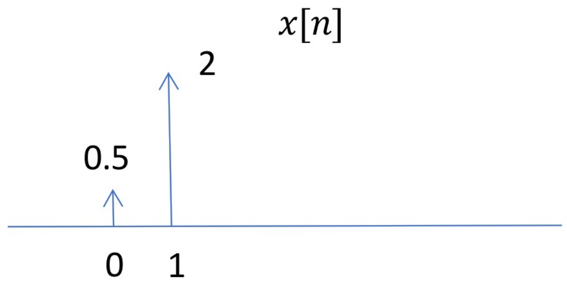

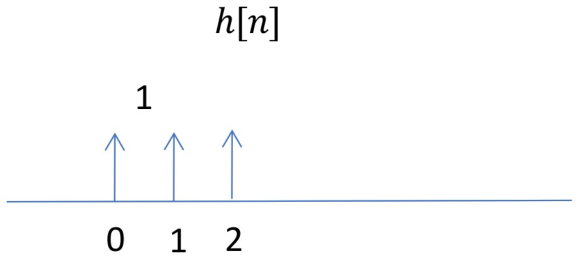

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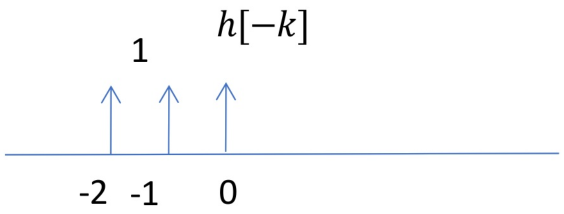

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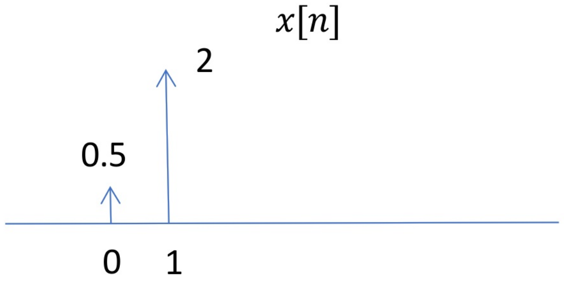

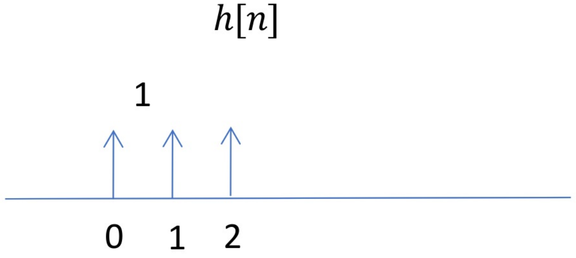

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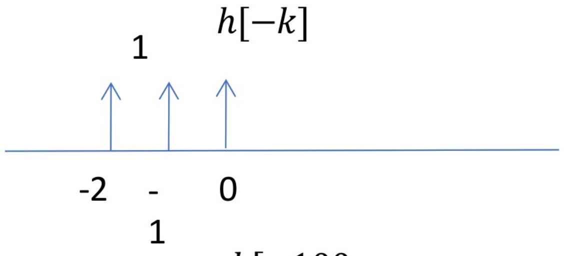
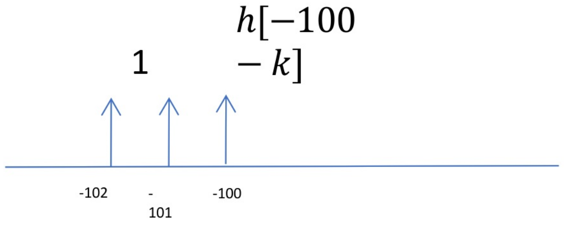


What happens for $n < 0$?

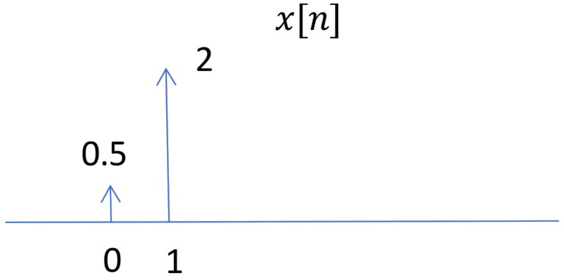
The function moves

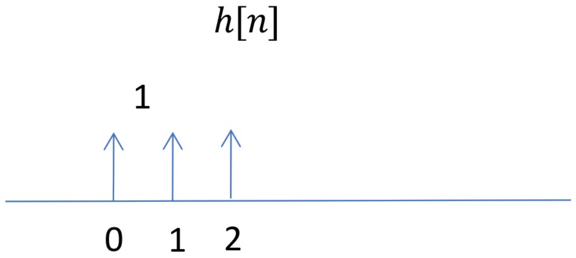


Hence No OVERLAP

Hence, $y[n] = 0$, for $n < 0$

Example





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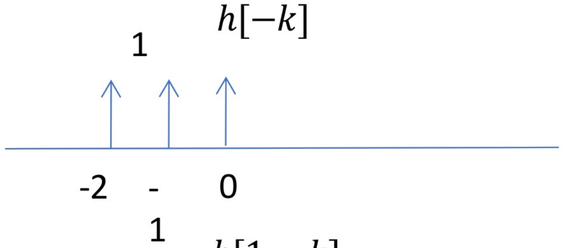
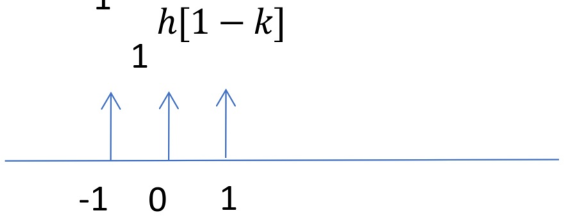




At $n = 0, y[0] = x[0]h[0] = (0.5)(1) = 0.5$

At $n = 1, y[1] = \Sigma\, x[k]\, h[1-k]$; Index $k = -\infty\ to\ +\infty$

What values of k contribute ?

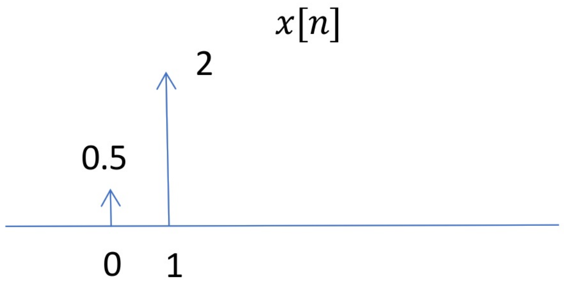
$y[1] = (0.5)(1) + 2(1) = 2.5$

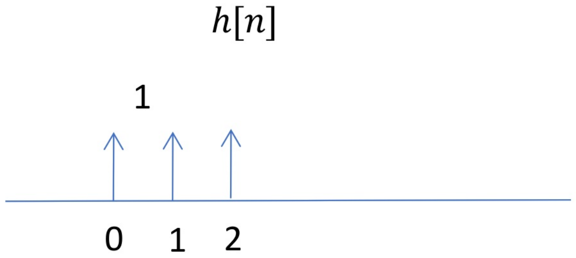
At $n = 2, y[2] = \Sigma\, x[k]\, h[2-k]$; Index $k = -\infty\ to\ +\infty$

What values of k contribute ?

$y[1] = (0.5)(1) + 2(1) = 2.5$

Example





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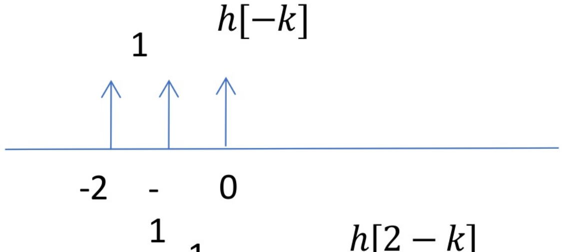
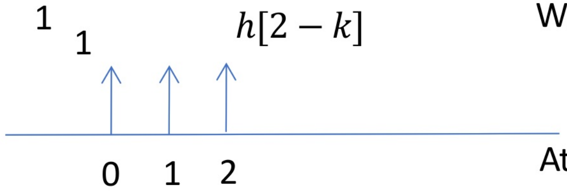
i.e $y[n] \ = \ \ x[n] \ * \ h[n] \ \ = \Sigma\, x[k]\, h[n-k]$; Index $k \ = \ -\infty\ to \ +\infty$





At $n \ = \ 2, y[2] \ = \Sigma\, x[k]\, h[2-k]$; Index $k \ = \ -\infty\ to \ +\infty$

What values of k contribute ?

$$y[1] \ = \ (0.5)(1) \ + \ 2(1) \ = \ 2.5$$

At $n \ = \ 3, y[3] \ = \Sigma\, x[k]\, h[3-k]$; Index $k \ = \ -\infty\ to \ +\infty$

What values of k contribute ?

$$y[3] \ = \ 2(1) \ = \ 2$$

At $n \ = \ 4, y[4] \ = \Sigma\, x[k]\, h[4-k]$; Index $k \ = \ -\infty\ to \ +\infty$

No OVERLAP, hence $y[n] \ = 0, for\, n \ > \ 3$

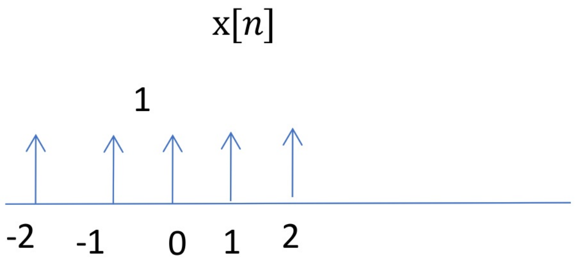
Properties of Convolution

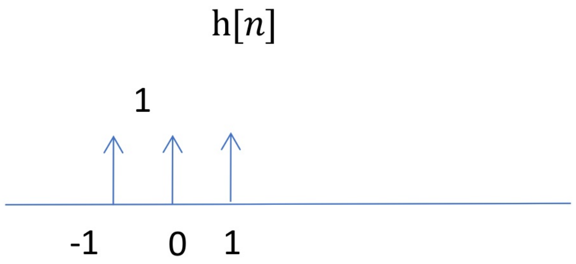
- Commutative : $x[n] * h[n] = h[n] * x[n]$
- Associative : $\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$
- Distributive : $x[n] * \{h_1[n] + h_2[n]\} = \{x[n] * h_1[n]\} + \{x[n] * h_2[n]\}$
- Proof of the Properties
- What is the Identity Element of this Operation ??

Problem

Find $x[n] * h[n]$

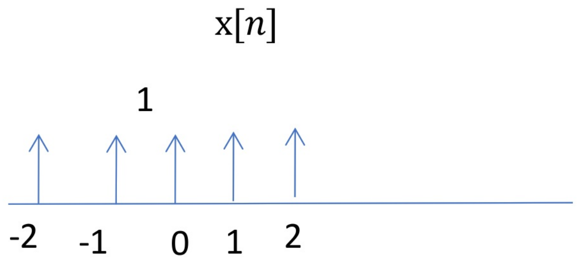
Problem

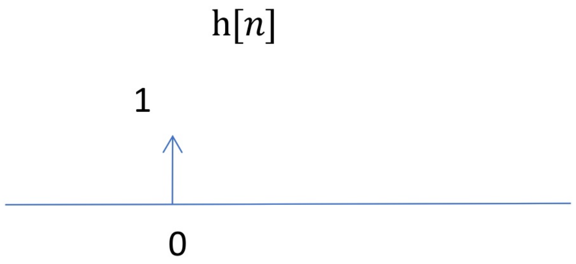




Find $x[n] * h[n]$

Problem





Find $x[n] * h[n]$

Problem

Find $x[n] * h[n]$

Solve Graphically