Discrete Mathematics, Tutorial II

1. Determine whether the following argument is valid using predicate logic. "Some math majors left the campus for the weekend. All seniors left the campus for the weekend". Therefore some seniors are math majors.

2. A stamp collector wants to include in her collection exactly one stamp from each country of Africa. If $I(s)$ means that she has the stamp s in her collection, $F(s, c)$ means that stamp s was issued by country c, and if the domain for s is the set of all stamps, and the domain for c is the set of all countries of Africa, then express the statement that her collection satisfies her requirement.

3. Give examples of predicates $P(n), Q(n)$ over the set of non-negative integers, such that:
   (a) $P(0)$ is true, but $\forall n : [P(n) \rightarrow P(n+1)]$ is false.
   (b) $Q(0)$ is false, but $\forall n : [Q(n) \rightarrow Q(n+1)]$ is true.

4. Let $P(x), Q(x)$ be arbitrary predicates, over the same domain. Then prove or disprove the following:
   - $(\exists x : P(x)) \wedge (\exists x : Q(x)) \Rightarrow \exists x : (P(x) \wedge Q(x))$.
   - $\exists x : (P(x) \wedge Q(x)) \Rightarrow (\exists x : P(x)) \wedge (\exists x : Q(x))$.

5. Prove that there are infinite number of primes.

6. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers.

7. Use the previous exercise to show that if the first 10 positive integers are placed around a circle, in any order, there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

8. Let $a_1, \ldots, a_n$ be arbitrary positive real numbers, where $n = 2^k$, for some integer k. Then show that $AM \geq GM$, where $AM \stackrel{def}{=} \frac{a_1+\ldots+a_n}{n}$ and $GM \stackrel{def}{=} (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$.

9. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two.

10. A guest at a party is called a celebrity if the person is known by every other guest, but knows none of them.
    - How many celebrities are guaranteed to exist in a party consisting of n guests ?
    - Suppose we want to find out a celebrity in a party. We are allowed to only ask one type of question—asking a guest, whether it knows a second guest. If there are n guests in the party, then show that asking $3(n-1)$ questions suffice to find out a celebrity.

11. Use strong induction to prove that $\sqrt{2}$ is irrational.

12. Using induction, prove that the number of diagonals for a polygon with n sides is $\frac{n(n-3)}{2}$, for every $n \geq 3$.