PS-4

Problems: 4 to 12

Transformation of Random Variables

Let $X : S \to \mathbb{R}$ be a random variable and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function.

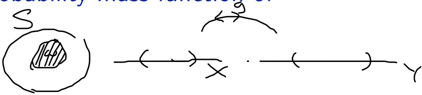
Then $g(X) : S \to \mathbb{R}$, defined as $[g(X)](\omega) = g[X(\omega)]$ for all $\omega \in S$, is a random variable.

Q: Given the distribution of X, how to find the distribution of the transformed random variable $Y = g(X)$?

Theorem

Let (i) X be a discrete random variable whose p.m.f. is $f_X(x)$ and
(ii) $Y = g(X)$ be another r.v. where $\underbrace{g \text{ is a bijective map}}$, so that $X = h(Y)$ (i.e. inverse of g exists). Then the probability mass function of Y is

$$f_Y(y) = f_X[h(Y)].$$



Proof: Let Spectrum of X is $\{x_i : i = 0, \pm 1, \pm 2, \dots\}$

$$P(X = x_i) = f_X(x_i) \quad \forall i = 0, \pm 1, \dots$$

Spectrum of Y : $\{y_i = g(x_i) : i = 0, \pm 1, \pm 2, \dots\}$

$$P(Y = y_i) = P(X = x_i) \qquad (Y = y_i) = (g(X) = g(x_i)) = (X = x_i)$$

$$\Rightarrow f_Y(y_i) = f_X(x_i) = f_X(h(y_i))$$

PS-5,1: Find the distribution of the square of a Poisson (μ) variate.

Sol.

$X \sim \text{Poisson}(\mu)$

$$P(X=i) = f_X(i) = \frac{e^{-\mu}\mu^i}{i!}, \quad i = 0, 1, 2, \ldots$$

$Y = X^2$. spectrum of Y: $\{ i^2 : i = 0, 1, 2, \ldots \}$

$P(Y = i^2) = ?$ $\quad i = 0, 1, 2, \ldots$

$$P(Y = i^2) = P(X^2 = i^2) = P(X = i) = \frac{e^{-\mu}\mu^i}{i!}$$

$$P(Y = 4) = \frac{e^{-\mu}\mu^2}{2!} \qquad i = 0, 1, 2, \ldots$$

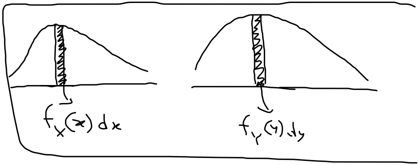
$$f_Y(j) = P(Y = j) = \frac{e^{-\mu}\mu^{\sqrt{j}}}{(\sqrt{j})!}, \quad j = 0, 1, 4, 9, \ldots$$

Theorem

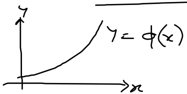
Let (i) X be a continuous random variable with p.d.f. $f_X(x)$ and

(ii) $y = \phi(x)$ be continuously differentiable and either strictly increasing or strictly decreasing throughout, so that $x = \psi(y)$ (i.e. inverse of ϕ exists). Then the p.d.f. of the transformed random variable $Y = \phi(X)$ is

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \; = f_X(\psi(y))\left|\frac{dx}{dy}\right|.$$



Proof: Case I: ϕ is strictly increasing



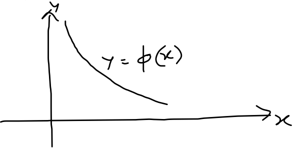
$F_Y(y)$ = d.f. of $Y = P(Y \le y) = P(\phi(X) \le \phi(x))$

Since ϕ is strictly increasing $(\phi(X) \le \phi(x)) \Leftrightarrow (X \le x)$

$F_Y(y) = P(\phi(X) \le \phi(x)) = P(X \le x) = F_X(x)$.

$f_Y(y)$ = p.d.f. of $Y = \frac{d}{dy}F_Y(y) = \frac{d}{dx}F_X(x)\,\frac{dx}{dy} = f_X(x)\frac{dx}{dy}$ - - - - (1)

Case II: ϕ is strictly decreasing



$$F_Y(y) = P(Y \le y)$$
$$= P(\phi(X) \le \phi(x))$$
$$= P(X \ge x) \quad (\text{since } \phi \text{ is strictly dec.})$$
$$= 1 - F_X(x)$$

$$\Rightarrow f_Y(y) = -f_X(x)\frac{dx}{dy} \quad \text{- - - - - (II)}$$

From (I) & (II)
$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X(\psi(y))\left|\frac{dx}{dy}\right|$$

PS-5,2:

$$f_X(x) = 2x e^{-x^2}, \quad x > 0$$
$$= 0, \quad \text{elsewhere}$$

$Y = X^2$

Sol. transformation : $y = x^2$

$$\frac{dy}{dx} = 2x > 0 \quad \text{for } x > 0$$

$\Rightarrow$ transformation is strictly increasing for $x > 0$

$$f_Y(y) = f_X(x) \frac{dx}{dy}$$
$$= \cancel{2x} e^{-x^2} \cdot \frac{1}{\cancel{2x}} \quad \text{for } x > 0$$
$$= \begin{cases} e^{-y} & \text{for } y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

since x varies from 0 to ∞
y - - - 0 to ∞

Prob (10) :

$X \sim N(0,1)$

$Y = \frac{X^2}{2}$ is $\gamma\left(\frac{1}{2}\right)$.

Sol. tranformation (in term of real variables)

$y = \frac{x^2}{2}$

x varies from $-\infty$ to ∞

y - - - - - - - - 0 to ∞

$\frac{dy}{dx} = x$ $\Rightarrow$ neither strictly inc. nor strictly dec throughout the dom.

$F_Y(y) = $ d.f. of $Y = P(Y \le y)$

$y < 0$, $F_Y(y) = 0$, since $(Y \le y) = \left(\frac{X^2}{2} \le y\right)$ is an impossible event.

$\phi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \; -\infty < x < \infty$

For $y > 0$,

$$F_Y(y) = P(Y \le y) = P\left(\frac{X^2}{2} \le y\right)$$
$$= P\left(-\sqrt{2y} \le X \le \sqrt{2y}\right)$$
$$= P\left(-\sqrt{2y} < X \le \sqrt{2y}\right)$$
$$= \Phi\left(\sqrt{2y}\right) - \Phi\left(-\sqrt{2y}\right)$$

since X is cont, $P\left(X = -\sqrt{2y}\right) = 0$

$\frac{d}{dy}\left(\sqrt{2y}\right) = \frac{1}{\sqrt{2y}}$

$$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2y}} \Phi'\left(\sqrt{2y}\right) + \frac{1}{\sqrt{2y}} \Phi'\left(-\sqrt{2y}\right)$$
$$= \frac{1}{\sqrt{2y}} \phi\left(\sqrt{2y}\right) + \frac{1}{\sqrt{2y}} \phi\left(-\sqrt{2y}\right)$$
$$= \frac{y^{-\frac{1}{2}}}{2\sqrt{\pi}}\left[e^{-y} + e^{-y}\right]$$
$$= \begin{cases} \dfrac{e^{-y} y^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)}, & y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$\Rightarrow Y \sim \Gamma\left(\frac{1}{2}\right)$.

Poisson Process

Stochastic Process

A family of random variables $\{X(t) : t \in T\}$ which depends parametrically on time t, is called a stochastic process.

Examples

1. $X(t)$: Total number of customers that have entered in a supermarket at time t.
2. $X(t)$: Number of persons infected by a disease in a given time t.
3. $X(t)$: Number of persons in a queue at time t.

Poisson Process

A particular example of stochastic process which counts **number of changes** in a given time interval. This process obeys two laws:

1. The number of changes during the time interval $(t, t+h)$ is independent of number of changes occurred in $(0, t)$, for all t and h (> 0).
2. (i) The probability of exactly one change in $(t, t+h)$ is $\lambda h + o(h)$ where λ is a positive constant and $o(h)$ is a function of h such that $\frac{o(h)}{h} \to 0$ as $h \to 0$.

   (ii) The probability of more than one change in $(t, t+h)$ is $o(h)$.

Theorem

Number of changes of a stochastic process in a given time interval, satisfying the above two laws, follow the Poisson distribution.
(For proof see Ref. book 1 or 2)

$X(t)$: Number of changes in interval $(0, t)$.

$$\underbrace{P(X(t) = i)}_{= \text{Poisson}(\lambda t)} = e^{-\lambda t} \frac{(\lambda t)^i}{i!}, \; i = 0, 1, 2, \dots$$

λ : rate of the Poisson process
(average number of changes per unit time)

λt : average number of changes in a given time interval $(0, t)$

PS-4, 11:

$\cancel{X}(t)$: no. of wars in $(0,t)$

λ = average no. of changes per unit time

$= \frac{1}{15}$

$X(t) \sim \text{Poisson}\left(\frac{t}{15}\right)$

$X(25) \sim \text{Poisson}\left(\frac{25}{15}\right) \equiv \text{Poisson}\left(\frac{5}{3}\right)$

$P\left(X(25)=0\right) = \dfrac{e^{-\frac{5}{3}}\left(\frac{5}{3}\right)^0}{0!} = e^{-\frac{5}{3}}.$

Prob 12:

$X(t)$: no. of particles emitted in $(0,t)$

λ = no. of . . . . per unit time

$= 2.5$

$X(4) \sim \text{Poisson}(2.5 \times 4)$

$\sim \text{Poisson}(10)$

$$P(X(4) \geq 3) = 1 - P(X(4)=0) - P(X(4)=1) - P(X(4)=2)$$

$= \ldots$ (check!)