Problem Set 2

Solutions

September 9, 2022

Contents

< solved by a student in the class >

Question 1

Let A, B be two independent events. Prove that:

1. $A, \bar{B}$ are independent. ✓
2. $\bar{A}, \bar{B}$ are independent. ✓

$P(A|B) = P(A) \longrightarrow P(A \cap B) = P(A) \cdot P(B) \quad (1)$

$P(A) = P(A \cap (\underbrace{B \cup \bar{B}}_{S})) = P(A \cap B) + P(A \cap \bar{B}) \Rightarrow \boxed{P(A \cap \bar{B})}$

~~ind~~ mutually exclusive

$= P(A) - P(A \cap B)$

$= P(A) \cdot P(\bar{B}) = P(A)(1 - P(B))$

$\bar{B} = C$

$P(A \cap C) = P(A) P(C)$

Hence proved

$P(\bar{A} \cap C) = P(\bar{A}) \cdot P(C) \longrightarrow P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$

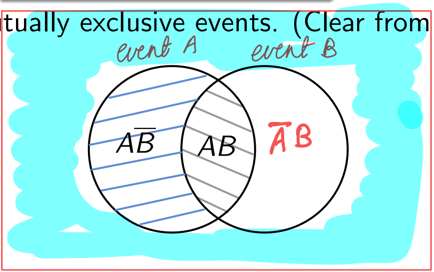
Solution 1.1

Question 1

Let A, B be two independent events. Prove that:

1. $A, \bar{B}$ are independent.
2. $\bar{A}, \bar{B}$ are independent.

$A\bar{B}$ and AB are mutually exclusive events. (Clear from Venn Diagram)



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$A\bar{B} + AB = A$

$$\implies P(A\bar{B} + AB) = P(A)$$

$$\implies P(A\bar{B}) + P(AB) = P(A) \qquad (A\bar{B}, AB \text{ are mutually exclusive events})$$

$$\implies P(A\bar{B}) = P(A) - P(AB)$$

$$\implies P(A\bar{B}) = P(A) - P(A)P(B) \qquad (A, B \text{ are independent events})$$

as given

$$\implies P(A\bar{B}) = P(A)[1 - P(B)]$$

$$\implies P(A\bar{B}) = P(A)P(\bar{B})$$

Solution 1.2

$$
\begin{aligned}
P(\bar{A}\bar{B}) &= P(\overline{A+B}) \\
&= 1 - P(A+B) \\
&= 1 - [P(A) + P(B) - P(AB)] \\
&= 1 - P(A) - P(B) + P(AB) \\
&= 1 - P(A) - P(B) + P(A)P(B) \quad (A, B \text{ are independent events}) \\
&= 1 - P(A) - P(B)[1 - P(A)] \\
&= P(\bar{A}) - P(B)P(\bar{A}) \\
&= P(\bar{A})[1 - P(B)] \\
&= P(\bar{A})P(\bar{B})
\end{aligned}
$$

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Question 2

Let A, B, C be three events such that A, B are independent; B, C are independent. Does it imply that A, C are independent?

A = getting Head $= \{H\}$

C = getting Tail $= \{T\}$

B = getting even no on a die $= \{2, 4, 6\}$

$P(B) = 1/2$, $P(A) = 1/2$, $P(AB) = P(A).P(B) = 1/4$

$P(BC) = P(B).P(C) = 1/4$

but

$P(AC) = 0 \neq P(A).P(C)$

Solution 2

Question 2

Let A, B, C be three events such that A, B are independent; B, C are independent. Does it imply that A, C are independent?

Let
E: Tossing a coin experiment
A: Head on 1st toss
B: Head on 2nd toss
C: Tail on 1st toss
1st and 2nd toss are independent events.

$$P(AB) = P(A)P(B) = \frac{1}{4}$$

$$P(BC) = P(B)P(C) = \frac{1}{4}$$

But,

$$P(AC) = 0 \neq P(A)P(C) = \frac{1}{4}$$

< solved by a student in the class >

Question 3

An urn contains 4 white and 6 black balls. Two balls are drawn successively without replacement. If the first ball is seen to be white, what is the probability that the second ball is also white?

10 B   4 W   6 B

$$P(E) = \frac{{}^4C_1 \times {}^3C_1}{{}^4C_1 \times {}^3C_1 + {}^4C_1 \times {}^6C_1} = \frac{1}{3}$$

Solution 3

Question 3

An urn contains 4 white and 6 black balls. Two balls are drawn successively without replacement. If the first ball is seen to be white, what is the probability that the second ball is also white?

Let

A: Event that first ball is white

B: Event that second ball is white

$$P(B/A) = \frac{P(AB)}{P(A)} = \frac{\frac{4}{10} \times \frac{3}{9}}{\frac{4}{10}} = \frac{3}{9} = \frac{1}{3}$$

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Question 4

Two urns contain respectively 2 white, 1 black balls and 1 white, 5 black balls. One ball is transferred from the first urn to the second urn and then a ball is drawn from the second urn. What is the probability that the ball drawn is white?

White: Black

$$\frac{2}{3} \times \frac{2}{7} + \frac{1}{3} \times \frac{1}{7} = \frac{5}{21}$$

Solution 4

Question 4

Two urns contain respectively 2 white, 1 black balls and 1 white, 5 black balls. One ball is transferred from the first urn to the second urn and then a ball is drawn from the second urn. What is the probability that the ball drawn is white?

Let

A: Event that ball drawn is white

B: Event that transferred ball is white

C: Event that transferred ball is black

B and C are mutually exclusive events.

LaTex

Beamer

$$
\begin{aligned}
P(A) &= P(B)P(A|B) + P(C)P(A|C) \\
&= \frac{2}{3} \times \frac{2}{7} + \frac{1}{3} \times \frac{1}{7} \\
&= \frac{4}{21} + \frac{1}{21} \\
&= \frac{5}{21}
\end{aligned}
$$