

Estimation of Parameters

✓ Maximum Likelihood Estimation (MLE)

population r.v.

Likelihood Function corresponding to a random sample of size n
is

$$L(x_1, x_2, \dots, x_n; \theta_1, \theta_2, \dots, \theta_k) = \begin{cases} f_{x_1}(\theta_1, \theta_2, \dots, \theta_k) f_{x_2}(\theta_1, \theta_2, \dots, \theta_k) \dots f_{x_n}(\theta_1, \theta_2, \dots, \theta_k) & \text{if } X \text{ has discrete distribution with p.m.f.} \\ f_x(x_1; \theta_1, \theta_2, \dots, \theta_k) f_x(x_2; \theta_1, \theta_2, \dots, \theta_k) \dots f_x(x_n; \theta_1, \theta_2, \dots, \theta_k) & \text{if } X \text{ has continuous distribution with p.d.f } f_x \end{cases}$$

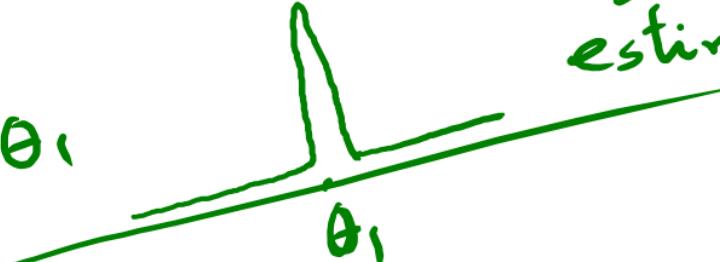
If L is **globally maximum** for $\hat{\theta}_i = \hat{\theta}_i(x_1, x_2, \dots, x_n)$ $i=1, 2, \dots, n$
then $\hat{\theta}_i$ is the maximum likelihood estimate of θ_i
(if they exists).

$\checkmark \boxed{\hat{\theta}_1 = \hat{Q}_1(x_1, x_2, \dots, x_n)}$ $\xrightarrow{\text{good?}} \theta_1$
 ✓ estimate
 .
 .
 ✓ $\hat{\theta}_k = \hat{\theta}_k(x_1, \dots, x_n)$

Sampling distribution

$\hat{\theta}_1(x_1, x_2, \dots, x_n)$
 is a good estimator of θ_1

\checkmark (i) consistency $\hat{\theta}_1 \xrightarrow{\text{irP}} \theta_1$
 \checkmark (ii) Unbiased $E(\hat{\theta}_1) = \theta_1$



Notes :

$$N(m, \sigma) \quad D = \{(m, \sigma) : \begin{array}{l} m \in \mathbb{R} \\ \sigma \in \mathbb{R}^+ \end{array}\}$$

① Let L be continuously differentiable func. of $\theta_1, \theta_2, \dots, \theta_k$

in a domain $D = \{(\theta_1, \theta_2, \dots, \theta_k) : \theta_1 \in R_1, \theta_2 \in R_2, \dots, \theta_k \in R_k\}$
 $R_i \subseteq \mathbb{R}, i=1, 2, \dots, k.$

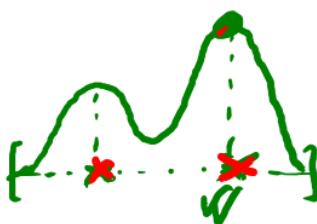
If L has "global maximum" at an interior point

$(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k)$ of D , then

$\left. \frac{\partial L}{\partial \theta_1} = 0, \frac{\partial L}{\partial \theta_2} = 0, \dots, \frac{\partial L}{\partial \theta_n} = 0 \right\} \text{at } (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_n).$

Algorithms (Likelihood eqn^{ns})

[Convex Optimization]



② $L > 0 \Rightarrow \log_e L$ is defined

and $\log_e L$ attains the global maximum at the same point $(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k)$ where L attains global maximum. So $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k$ (if they exist) can be obtained by solving

✓ $\frac{\partial(\log_e L)}{\partial \theta_1} = 0, \frac{\partial(\log_e L)}{\partial \theta_2} = 0, \dots, \frac{\partial(\log_e L)}{\partial \theta_k} = 0$

where $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k$ must satisfy

✓ $L(x_1, x_2, \dots, x_n; \hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k) \geq L(x_1, x_2, \dots, x_n; \theta_1, \theta_2, \dots, \theta_k)$
 $\forall (\theta_1, \theta_2, \dots, \theta_k) \in D$

global max.

③ If L is global max^m at a point which is not
an interior point of D (i.e. at a boundary point)
or some or all the partial derivatives

$\frac{\partial L}{\partial \theta_1}, \dots, \frac{\partial L}{\partial \theta_k}$ do not exist at the points

where L is global max^m, then above method
of calculus cannot be applied.

Prob 6: Find the MLE , say \hat{p} , of p of Binomial (N, p) population. Prove that \hat{p} is consistent and unbiased estimate of p .

Sol. Assume N is known . p is the only unknown parameter where $0 < p < 1$. Let, X be the population

$$P(X=x) = \binom{N}{x} p^x (1-p)^{N-x}, \quad x=0, 1, 2, \dots, N. \quad \text{random variable}$$

Let, (x_1, x_2, \dots, x_n) be a random sample of size n from the population of X .

Likelihood function

$X_i \sim \text{Binomial}(N, p)$
 $i=1, 2, \dots, n$

$$L = L(x_1, x_2, \dots, x_n; p)$$

$$= \binom{N}{x_1} p^{x_1} (1-p)^{N-x_1} \binom{N}{x_2} p^{x_2} (1-p)^{N-x_2} \dots \binom{N}{x_n} p^{x_n} (1-p)^{N-x_n}$$

$$= \binom{N}{x_1} \binom{N}{x_2} \dots \binom{N}{x_n} p^{n\bar{x}} (1-p)^{nN-n\bar{x}}$$

$$\text{where } \bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}.$$

$$\log_e L = \log_e \left\{ \binom{N}{x_1} \dots \binom{N}{x_n} \right\} + (n\bar{x}) \log p + (nN - n\bar{x}) \log (1-p).$$

For a fixed random sample $\log_e L$ is a fn of p .
 $0 < p < 1$

For local maximum :

$$\frac{d}{dp} (\ln_e L) = 0$$

$$\Rightarrow \frac{n\bar{x}}{p} - \frac{nN - n\bar{x}}{1-p} = 0$$

$$\Rightarrow \bar{x}(1-p) - (N-\bar{x})p = 0$$

$$\Rightarrow p = \frac{\bar{x}}{N}$$

$$\frac{d^2(\ln_e L)}{dp^2} = -\frac{n\bar{x}}{p^2} - \frac{nN - n\bar{x}}{(1-p)^2}$$

$$\left. \frac{d^2(\ln_e L)}{dp^2} \right|_{p=\frac{\bar{x}}{N}} = -n \left[\frac{N^2}{\bar{x}} + \frac{N^2}{(N-\bar{x})} \right]$$

Assuming $\bar{x} \neq 0, \bar{x} \neq N$

(critical point of $\ln_e L$)

$\rightarrow p = \frac{\bar{x}}{N}$ is a local maximum . . .

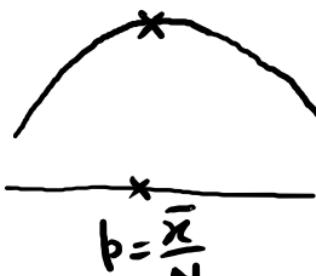
$$\frac{d}{dp} (\ln_e L) = \frac{n\bar{x}}{p} - \frac{nN-n\bar{x}}{1-p} > 0 \text{ if } 0 < p < \frac{\bar{x}}{N}$$

and

$$< 0 \text{ if } \frac{\bar{x}}{N} < p < 1.$$

$\Rightarrow \ln_e L$ is mon. strictly increasing in $0 < p \leq \frac{\bar{x}}{N}$

and mon. strictly decreasing in $\frac{\bar{x}}{N} \leq p \leq 1$.



graph of $\ln_e L$. $\Rightarrow \ln_e L$ has global max.

$$\text{at } p = \frac{\bar{x}}{N}$$

$$\Rightarrow \text{MLE: } \hat{p} = \frac{\bar{x}}{N}$$

Check! ① $\hat{P} \xrightarrow{\text{in P}} p$ (Apply LLNs of equal components)

$$\textcircled{II} E(\hat{P}) = p$$

$$\begin{aligned}\underline{\text{Sol. I}} \quad \hat{P} &= \frac{\bar{X}}{N} = \frac{x_1 + x_2 + \dots + x_n}{n N} \\ &= \frac{x_1}{N} + \frac{x_2}{N} + \dots + \frac{x_n}{N}\end{aligned}$$

① $\frac{x_1}{N}, \frac{x_2}{N}, \dots, \frac{x_n}{N}$ are mutually independent r.v.s

② all have the same distribution as $\frac{X}{N}$

$$\text{so, } E\left(\frac{x_i}{N}\right) = E\left(\frac{X}{N}\right) = \frac{E(X)}{N} = \frac{Np}{N} = p$$

$X \sim \text{Binomial}(N, p)$

$$E(X) = Np$$

Applying LLN (of equal components)

$$\frac{\frac{X_1}{N} + \frac{X_2}{N} + \cdots + \frac{X_n}{N}}{n} \xrightarrow{\text{in P}} \beta \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow \hat{P} = \frac{\bar{X}}{nN} \xrightarrow{\text{in P}} \beta \quad \text{as } n \rightarrow \infty$$

$\Rightarrow \hat{P}$ is a consistent estimate of β .

⑦ Find the MLE of m and σ^2 for a normal (m, σ) population.

$$f_X(x; m, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-m)^2}$$

Sol. (x_1, x_2, \dots, x_n) be a random sample of size n .

$$L = L(x_1, x_2, \dots, x_n; m, \sigma) = \frac{1}{(\sqrt{2\pi})^n \sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - m)^2}$$

for a fixed random sample, L is a fn of m and σ where $m \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$.

To find \hat{m} and $\hat{\sigma}$ for which L is globally maximum.

$$\log_e L = -\frac{n}{2} \log_e 2\pi - n \log_e \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - m)^2$$

Likelihood eqns

$$\frac{\partial \log L}{\partial m} = 0 , \quad \frac{\partial \log L}{\partial \sigma} = 0$$

$$\Rightarrow \frac{2}{2\sigma^2} \sum_{i=1}^n (x_i - m) = 0 \quad \left| \begin{array}{l} -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - m)^2 = 0 \\ \Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - m)^2 = S^2 \end{array} \right.$$

$$\Rightarrow \frac{n}{\sigma^2} (\bar{x} - m) = 0$$

$$\Rightarrow m = \bar{x} \quad \checkmark$$

Check! $\left[\begin{array}{c} \frac{\partial^2 \log L}{\partial m^2} \\ \hline \frac{\partial^2 \log L}{\partial m \partial \sigma} \\ \hline \frac{\partial^2 \log L}{\partial \sigma^2} \end{array} \right]$ is -ve definite

Since above is the only solution of likelihood eqns

MLE $\hat{m} = \bar{x}$, $\hat{\sigma}^2 = s^2$

