z Transform

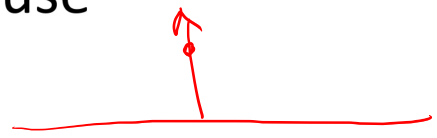
EGC 113

$y[n]$

impulse response

The z-Transform

- Counterpart of the Laplace transform for discrete-time signals. Generalization of the Fourier Transform. Fourier Transform does not exist for all signals
- The z-Transform is often time more convenient to use



$\delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$y[n] = x[n] + \alpha\, y[n-1]$

- Compare to DTFT definition:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- z is a complex variable that can be represented as $z=re^{j\omega}$
- Substituting $z=e^{j\omega}$ will reduce the z-transform to DTFT

$$y[n] = x[n] * h[n]$$

$$= \sum_{k} x[k]\, h[n-k]$$

$$Y(z) = X(z)\, H(z)$$

$$\Big\updownarrow \mathcal{Z}^{-1}$$

$$y(n)$$

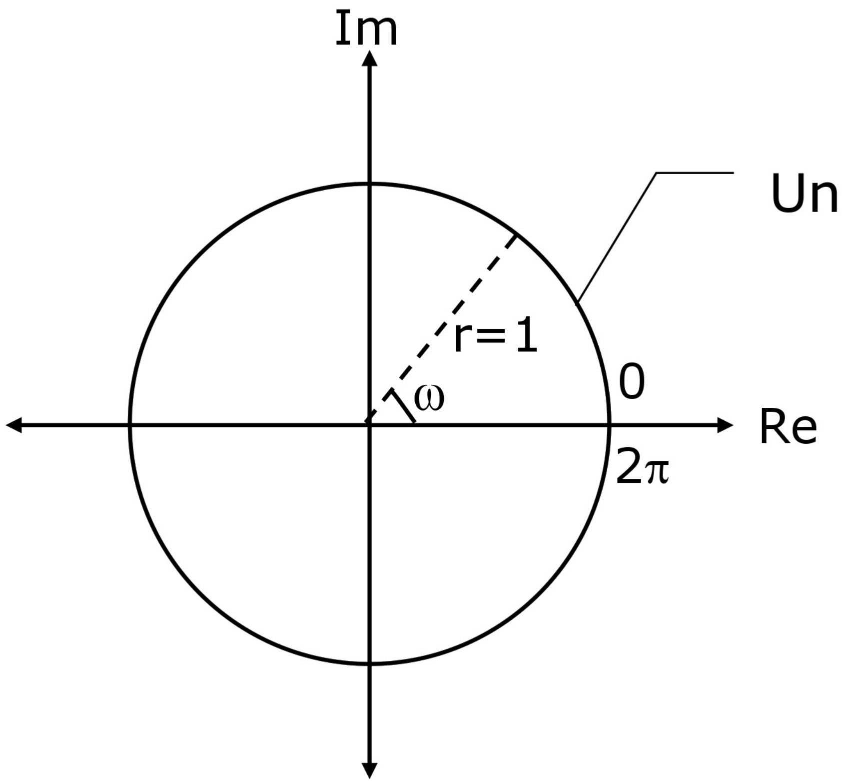
The z-transform and the DTFT

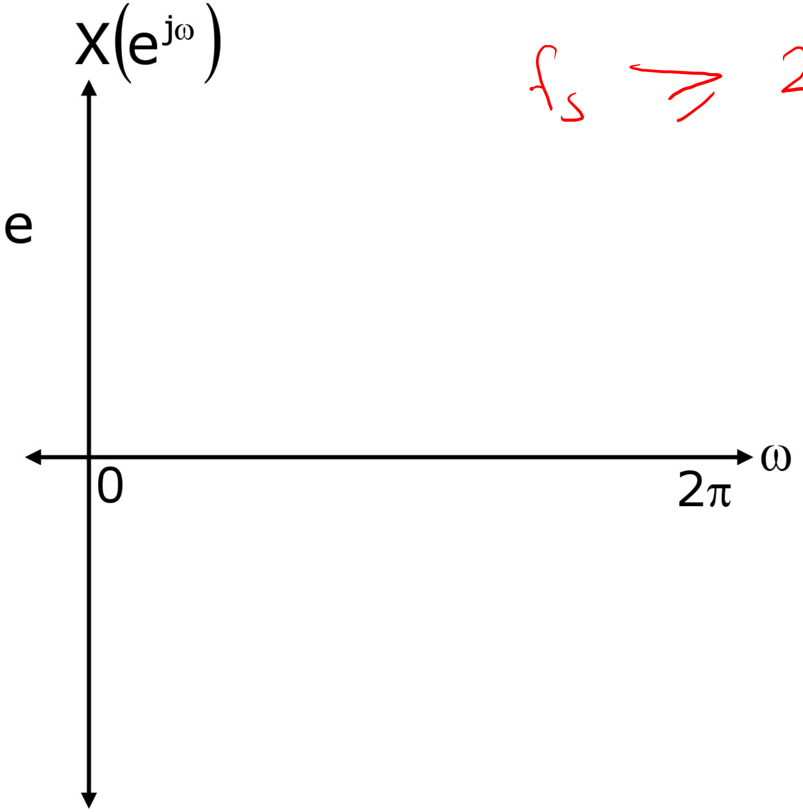
$z = re^{j\omega}$

$\frac{f}{f_s}$

$f_s > 2f$

- The z-transform is a function of the complex variable z.
- Convenient to describe on the complex z-plane
- If we plot $z=e^{j\omega}$ for ω=0 to 2π we get the unit circle





Convergence of the z-Transform

- DTFT does not always converge

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

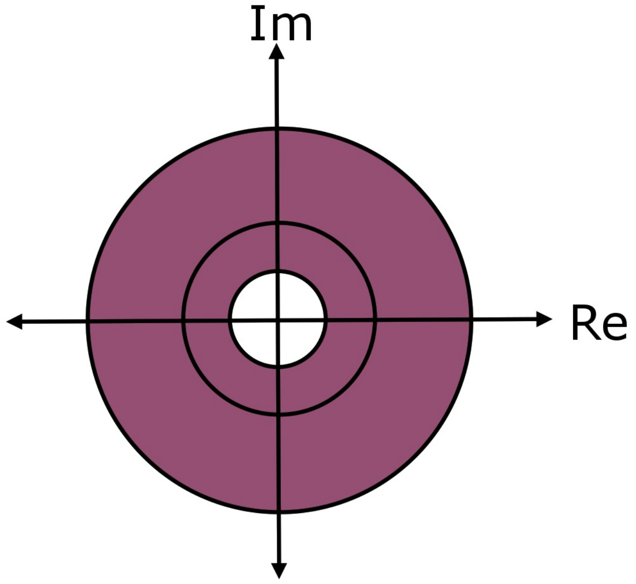
  - Infinite sum not always finite if x[n] no absolute summable
  - Example: $x[n] = a^n u[n]$ for $|a|>1$ does not have a DTFT

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{-j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]\, r^{-n}\right) e^{-j\omega n}$$

- Complex variable z can be written as $re^{j\omega}$ so the z-transform

$$\sum_{n=-\infty}^{\infty} \left|x[n]\, r^{-n}\right| < \infty$$

Region of Convergence



- Example: z-transform converges for values of $a_1<r<a_2$
  - ROC is shown on the left
  - In this example the ROC includes the unit circle, so DTFT exists

- ROC: The set of values of z for which the z-transform converges.
- Each value of r represents a circle of radius r. The region of convergence is made of circles.

Right-Sided Exponential Sequence Example

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

- For Convergence we require

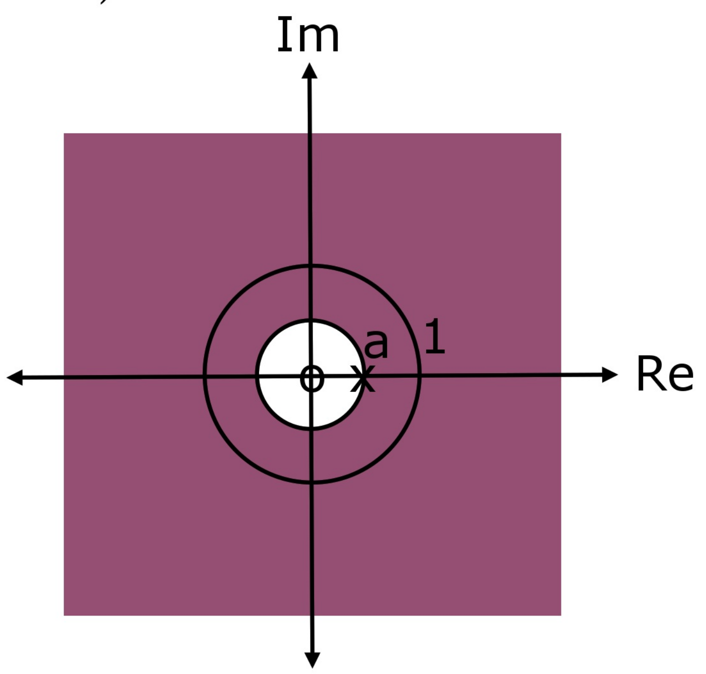
$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

- Hence the ROC is defined as

$$\left|az^{-1}\right|^n < 1 \Rightarrow |z| > |a|$$

- Inside the ROC series converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$



- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

$$x_1[n] = \underline{a^n u[n]} \qquad X(z) = \sum_n x[n] z^{-n}.$$

$$X_1(z) = \sum_n a^n u[n] z^{-n}.$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n. = \boxed{\frac{1}{1 - a z^{-1}}} \qquad |a z^{-1}| < 1.$$

$$|a| < |z|$$

$$\boxed{|z| > |a|.}$$

ROC.

$$x_2[n] = a^{*n} u[-n-1]$$

$$X_2(z) = \sum_{n=-\infty}^{-1} a^{-n} z^{-n}. = \sum_{m=+1}^{\infty} (az)^m$$

$$= \sum_{m=0}^{\infty} (az)^m - 1 = \frac{1}{1-az} - 1 = \frac{1 - 1 + az}{1-az} = \frac{az}{1-az}.$$

$$= \frac{-1}{1 - a^{-1} z^{-1}}$$

$$x_3[n] = a^n u[-n-1]$$

$$= \frac{z}{a-z} = \frac{1}{az^{-1}-1} = \frac{-1}{1-az^{-1}}. \quad |z| < |a|$$

$$x_1[n] = a^n u[n]$$

$$\boxed{\begin{aligned} X_1(z) &= \frac{1}{1-az^{-1}}. \\ ROC_1 &= |z| > |a|. \end{aligned}}$$

$$\boxed{x_4[n] = -a^n u[-n-1].}$$

$$\boxed{\begin{aligned} X_4[z] &= \frac{1}{1-az^{-1}} \\ ROC_4 &\Rightarrow |z| < |a|. \end{aligned}}$$

$$x_4[n] = -a^n u[-n-1]$$

$$X_4(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= -\left[\frac{(az^{-1})^{-\infty} - (az^{-1})^0}{1-az^{-1}}\right]$$

$$(a^{-1}z)^{\infty} \longrightarrow 0$$

$$|a^{-1}z| < 1.$$

$$= \frac{1}{1-az^{-1}}$$

Same Example Alternative Way

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

$$\sum_{n=N_1}^{N_2} \alpha^n = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha} \qquad \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{\left(az^{-1}\right)^0 - \left(az^{-1}\right)^{\infty}}{1 - az^{-1}}$$

$$|z|>a$$

- For the term with infinite exponential to vanish we need

$$\left|az^{-1}\right| < 1 \quad \Rightarrow \quad |a| < |z|$$

- In the ROC the sum converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}}$$

Two-Sided Exponential Sequence Example

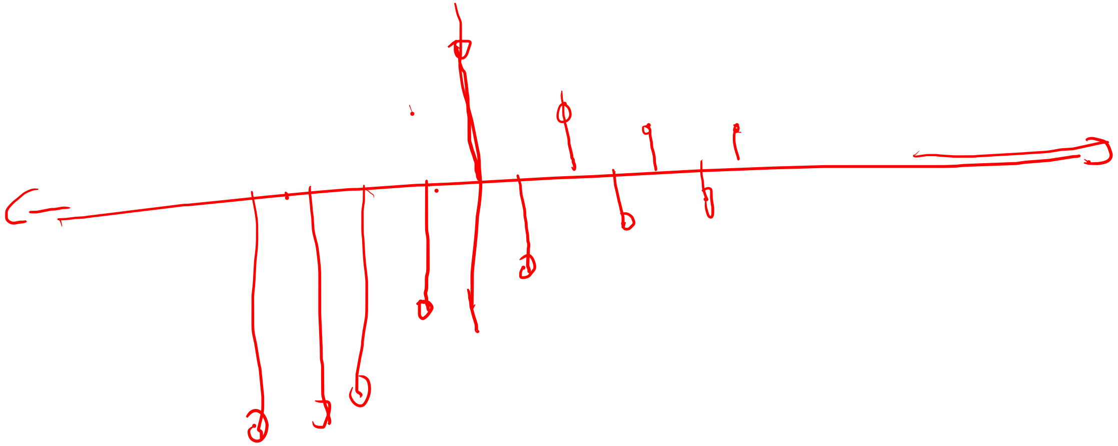
$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-2z^{-1}} \quad \frac{1}{3} < |z| < 2.$

$(2)^n.$

$$x[n] = \underbrace{\left(-\frac{1}{3}\right)^n u[n]} - \left(\frac{1}{2}\right)^n u[-n-1]$$

$x_1[n] + x_2[n]$

$$\boxed{X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}.}$$



$x_1[n] = \left(-\frac{1}{3}\right)^n u[n]$

$X_1(z) = \frac{1}{1+\frac{1}{3}z^{-1}} \quad |z| > \frac{1}{3}$

$x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

$X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad |z| < \frac{1}{2}.$

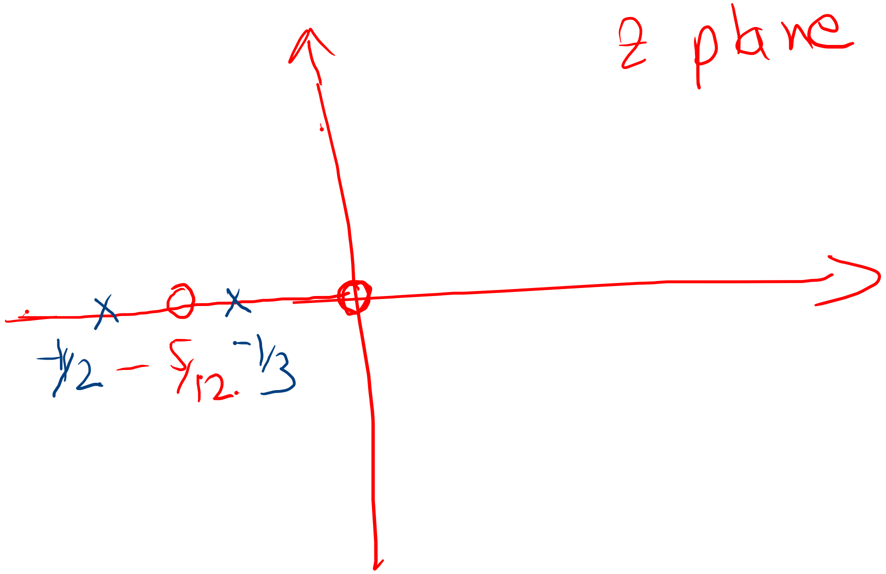
ROC:

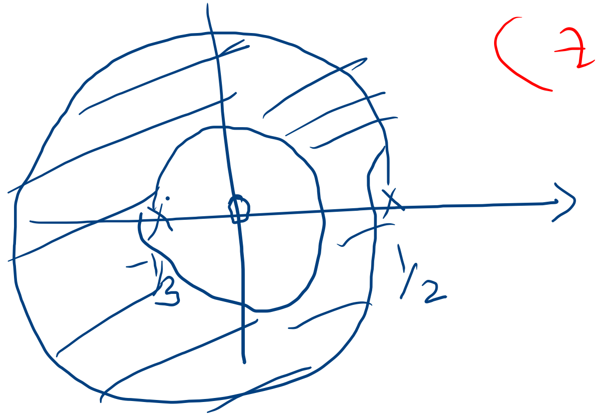
$\frac{1}{3} < |z| < \frac{1}{2}.$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z+\frac{1}{3}} + \frac{z}{z+\frac{1}{2}}$$

$$= \frac{z^2 + \frac{1}{2}z + z^2 + \frac{1}{3}z}{z^2 + \frac{5}{6}z + \frac{1}{6}} = \frac{2z^2 + \frac{5}{6}z}{\left(z+\frac{1}{3}\right)\left(z+\frac{1}{2}\right)}$$

$$= \frac{2z\left(z+\frac{5}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z+\frac{1}{2}\right)}$$





Two-Sided Exponential Sequence Example

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{\left(-\frac{1}{3}z^{-1}\right)^0 - \left(-\frac{1}{3}z^{-1}\right)^{\infty}}{1+\frac{1}{3}z^{-1}} = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n = \frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty} - \left(\frac{1}{2}z^{-1}\right)^0}{1-\frac{1}{2}z^{-1}} = \frac{-1}{1-\frac{1}{2}z^{-1}}$$

Two-Sided Exponential Sequence Example

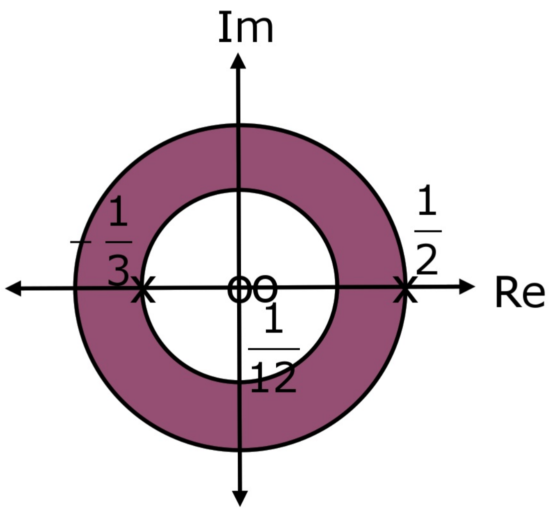
$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$$

$$\text{ROC}: \left| -\frac{1}{3}z^{-1} \right| < 1$$

$$\frac{1}{3} < |z|$$

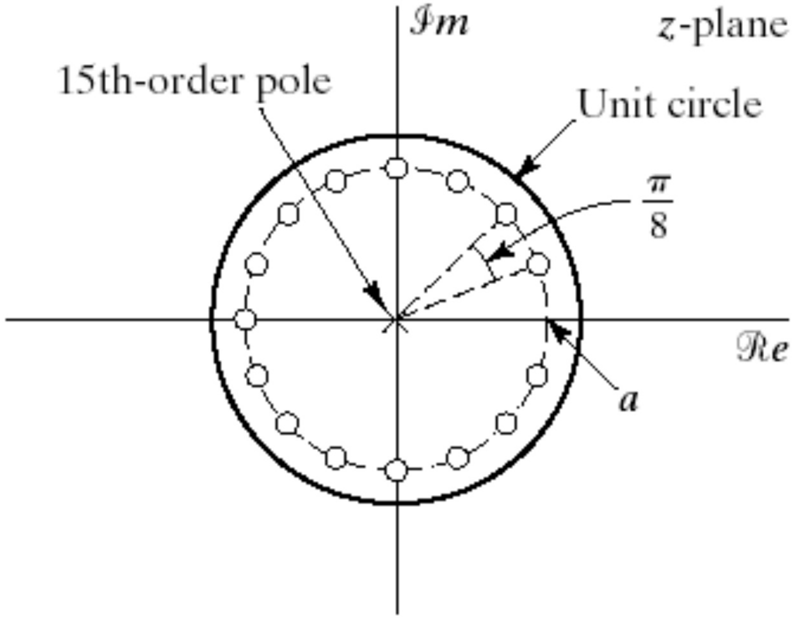
$$\text{ROC}: \left| \frac{1}{2}z^{-1} \right| > 1$$

$$\frac{1}{2} > |z|$$



Finite Length Sequence

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$



$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

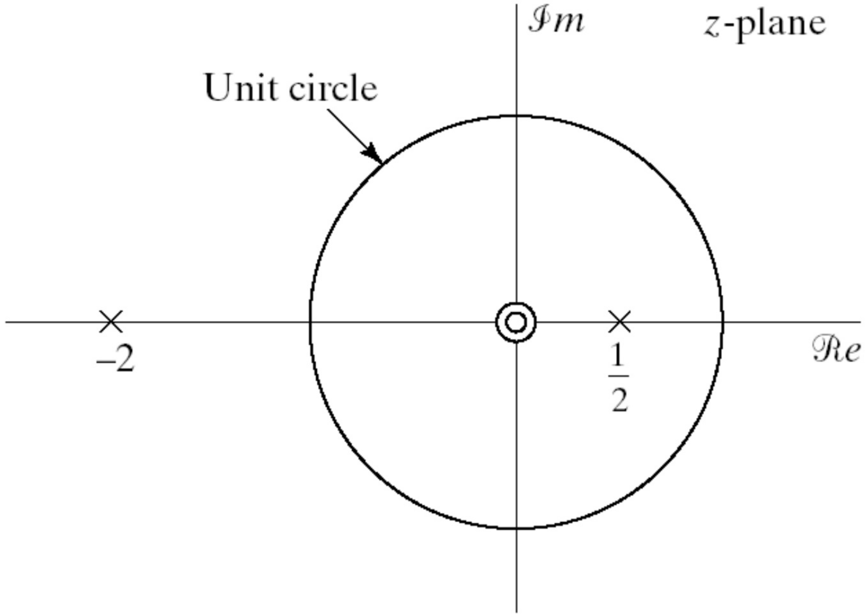
Properties of The ROC of Z-Transform

- The ROC is a ring or disk centered at the origin
- DTFT exists if and only if the ROC includes the unit circle
- The ROC cannot contain any poles
- The ROC for finite-length sequence is the entire z-plane
  - except possibly $z=0$ and $z=\infty$
- The ROC for a right-handed sequence extends outward from the outermost pole possibly including $z=\infty$
- The ROC for a left-handed sequence extends inward from the innermost pole possibly including $z=0$
- The ROC of a two-sided sequence is a ring bounded by poles
- The ROC must be a connected region
- A z-transform does not uniquely determine a sequence without specifying the ROC

Stability, Causality, and the ROC

- Consider a system with impulse response h[n]
- The z-transform H(z) and the pole-zero plot shown below
- Without any other information h[n] is not uniquely determined
  - |z|>2 or |z|<½ or ½<|z|<2
- If system stable ROC must include unit-circle: ½<|z|<2
- If system is causal must be right sided: |z|>2

- $H(Z) = \frac{A}{(1+2z^{-1})(1-0.5z^{-1})}$



Properties of z Transform

Property	Sequence	Transform	ROC
	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R except for the possible addition or deletion of the origin

Initial Value Theorem

If $x[n] = 0$ for $n < 0$, then

$x[0] = \lim_{z\to\infty} X(z)$

Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1} where z is in R)
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$ where z is in R)
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
Differentiation in the z-Domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

$x(n) = \left(\frac{1}{2}\right)^n u[n]$

$x_1[n] = x[-n]$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

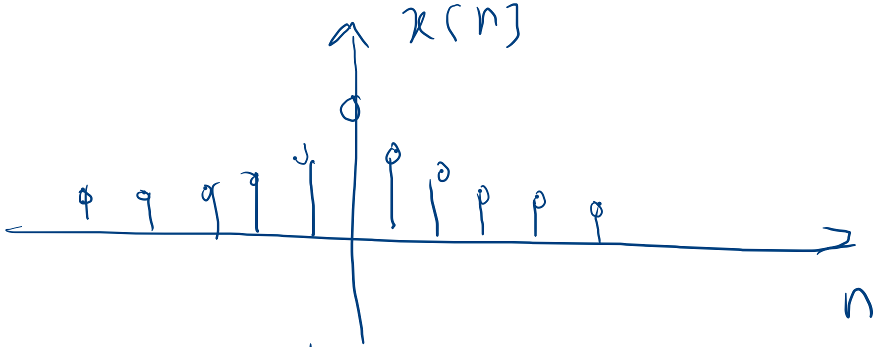
$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \text{ROC: } |z| > \frac{1}{2}.$$

$$x_1[n] = x[-n]$$

$$X(z) = \frac{1}{1-\frac{1}{2}z} \qquad \text{ROC: } |z| < \underline{2}.$$

$x[n] = a^{|n|}$ $\quad a > 0$

$a < 1$



$a > 1$

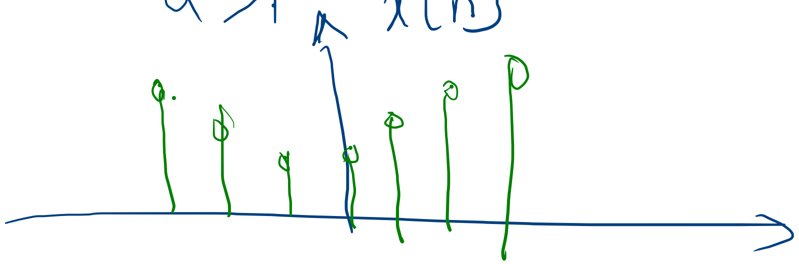


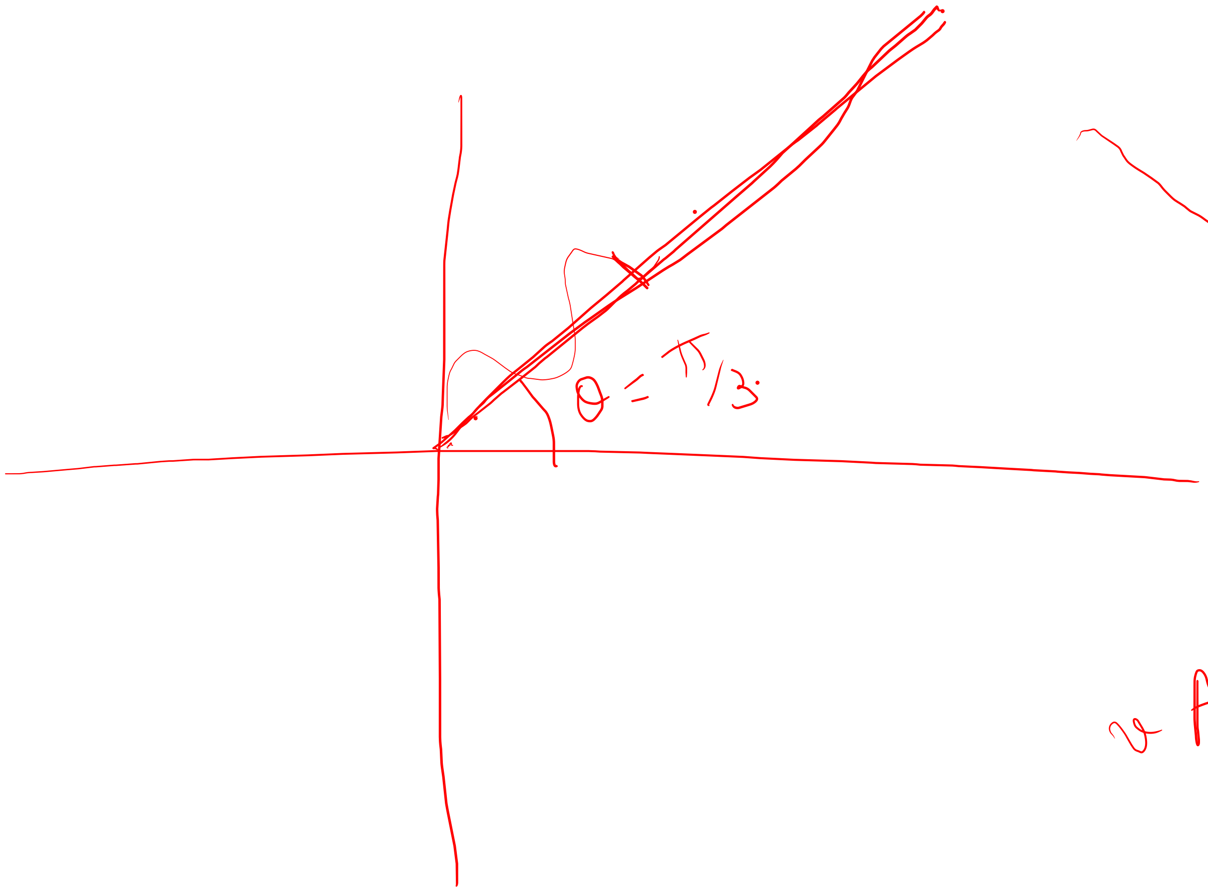
Table 4: **Some Common z-Transform Pairs**

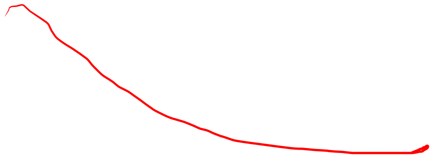
Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \|\alpha\|$

ROC $\quad |a| < |z| < \left|\frac{1}{a}\right|$

$x[n] = a^n u[n] + a^{-n} u[-n-1]$

$$X(z) = \underbrace{\frac{1}{1-az^{-1}}}_{|z| > |a|} - \underbrace{\frac{1}{1-\frac{1}{a}z^{-1}}}_{|z| < \left|\frac{1}{a}\right|}$$





$vA = \lambda v$

$e^{j\omega\theta}$

Examples

- Find the z transform of
  - $x[n] = \boxed{-a^n u[-n-1]}$
  - $x[n] = a^{-n} u[-n-1]$
  - $x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$ ✓
  - $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$
  - $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n-1]$ ✓
  - $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[-n-1]$

$x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$

$X(z) = \sum x[n] z^{-n}$

$= 5z^2 + 3z - 2 + 4z^{-1} - 3z^{-2}$

$= \dfrac{5z^5 + 3z^4 - 2z^3 + 4z - 3}{z^3}$

$x[-2] = 5$

$x[-1] = 3$

$x[0] = -2$

$x[1] = 0$

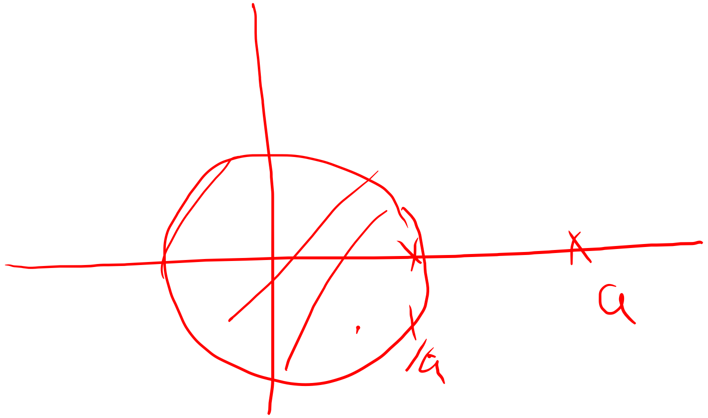
$x[n] = a^{|n|} \qquad a > 0$

$$X(z) = \frac{1}{1 - a z^{-1}} - \frac{1}{1 - \frac{1}{a} z^{-1}}$$

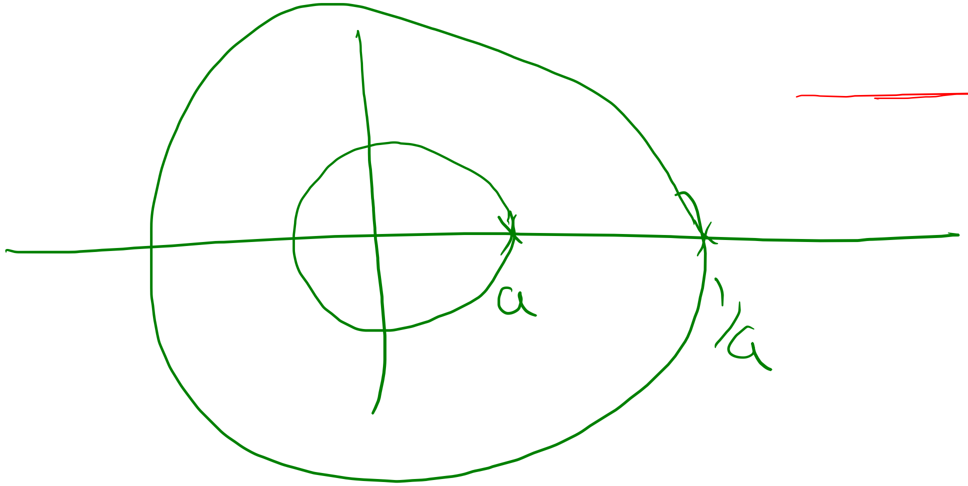
ROC

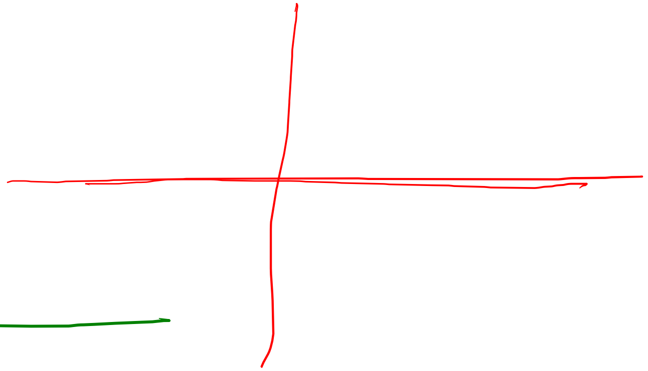
$|a| < |z| < \left|\frac{1}{a}\right|$

for $a > 1$



if $a < 1$





$$x[n] = \underbrace{\left(\frac{1}{3}\right)^n u[n]}_{x_1[n]} + \underbrace{\left(\frac{1}{2}\right)^n u[-n-1]}_{x_2[n]}$$

$$X_1(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \qquad \text{ROC: } |z| > \frac{1}{3}$$

$$X_2(z) = -\frac{1}{1-\frac{1}{2}z^{-1}} \qquad \text{ROC: } |z| < \frac{1}{2}.$$

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{3}} - \frac{z}{z-\frac{1}{2}} = \frac{\cancel{z^2} - \frac{1}{2}z - \cancel{z^2} + \frac{1}{3}z}{(z-\frac{1}{3})(z-\frac{1}{2})}$$

$$= \frac{-\frac{1}{6}z}{(z-\frac{1}{3})(z-\frac{1}{2})}$$