Discrete Mathematics, Tutorial I

1. Let p and q be the propositions
   - p: You drive over 65 miles per hour.
   - q: You get a speeding ticket.

   Write the following propositions using p and q and logical connectives.

   (a) You do not drive over 65 miles per hour.
   (b) You will get a speeding ticket if you drive over 65 miles per hour.
   (c) You drive over 65 miles per hour only if you will get a speeding ticket.
   (d) Driving over 65 miles per hour is sufficient for getting a speeding ticket.

2. State the converse, contrapositive and inverse of the following statements:

   (a) If it snows today I will ski tomorrow.
   (b) A positive integer is a prime only if it has no divisors other than 1 and itself.

3. Construct a truth table for each of the following compound propositions.

   (a) $(p \to q) \wedge (\neg p \to q)$.
   (b) $(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$.

4. Express these system specifications using the proposition p "The user enters a valid password", q "Access is granted" and r "The user has paid the subscription fee" and logical connectives.

   (a) "The user has paid the subscription fee but does not enter a valid password".
   (b) "Access is granted whenever the user has paid the subscription fee and enters a valid password".
   (c) "If the user has not entered a valid password but has paid the subscription fee then access is granted".

5. Are the following system system specifications consistent?

   (a) "The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state then it is in interrupt mode. The system is not in interrupt mode".
   (b) "The router can send packets to the edge system only if it supports the new address space. For the router to support the new address space it is necessary that the latest software release be installed. The router can send packets to the edge system if the latest software release is installed. The router does not support the new address space".

6. Identify whether the following statements are logically equivalent or not.

   (a) $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$.
   (b) $(p \to q) \to r$ and $p \to (q \to r)$.
   (c) $(p \vee q) \to r$ and $(p \to r) \vee (q \to r)$.
   (d) $(p \to q) \to (r \to s)$ and $(p \to r) \to (q \to s)$.

7. The **dual** of a compound proposition that contains only the logical operators $\wedge$, $\vee$ and $\neg$ is the compound proposition obtained by replacing each $\wedge$ by $\vee$ and vice-versa and each **T** by **F** and vice-versa. The dual of a statement s is denoted by $s^\star$.

   (a) Find the dual of each of the following compound propositions.
      i. $p \vee \neg q \vee \neg r$.

ii. $(p \vee q \vee r) \wedge s$.

iii. $(p \wedge \mathbf{F}) \vee (q \wedge \mathbf{T})$.

(b) When does $s^\star = s$ where s is a compound proposition?

(c) Show that $(s^\star)^\star = s$, where s is a compound proposition.

(d) Show that the duals of two equivalent compound propositions are also equivalent, where these compound propositions contain only the operators $\wedge$, $\vee$ and $\neg$.

8. A collection of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

   (a) Show that $\wedge$, $\vee$ and $\neg$ forms a functionally complete set of logical operators.

   (b) Show that $\neg$ and $\vee$ form a functionally complete set of logical operators.

   (c) Does $\neg$ and $\wedge$ form a functionally complete set of logical operators?

9. A compound proposition is **satisfiable** if there is an assignment of truth values to the variables in the compound proposition that makes the compound proposition true.

   (a) Which of the following compound propositions are satisfiable?

   i. $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg s) \wedge (\neg p \vee \neg r \vee \neg s) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg r \vee \neg s)$.

   ii. $(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg s) \wedge (q \vee \neg r \vee s) \wedge (\neg p \vee r \vee s) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee s) \wedge (\neg p \vee \neg r \vee \neg s)$.

   iii. Explain how an algorithm for determining whether a compound proposition is satisfiable can be used to determine whether a compound proposition is a tautology.

10. Use rule of inference to show that the hypotheses "Randy works hard", "If Randy works hard then he is a dull boy" and "If Randy is a dull boy then he will not get the job", imply the conclusion "Randy will not get the job".

11. Show that the argument form with premises $p_1, \ldots, p_n$ and conclusion $q \to r$ is valid if the argument form with premises $p_1, \ldots, p_n, q$ and conclusion r is valid.

12. Use resolution to show the following:

    (a) The hypotheses "Allen is a bad boy or Hillary is a good girl" and "Allen is a good boy or David is happy" imply the conclusion "Hillary is a good girl or David is happy".

    (b) The hypotheses "It is not raining or Yvo has his umbrella", "Yvo does not have his umbrella or he does not get wet" and "It is raining or Yvo does not get wet", imply that "Yvo does not get wet".

13. Use resolution to show that the compound proposition $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$ is not satisfiable.

14. Determine whether the following argument is valid or not: If Superman were able and willing to prevent evil, he would do so. If superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent . Superman does not prevent evil. If Superman exists he is neither impotent nor malevolent. Therefore Superman does not exist.