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Fourier Transform

Time domain signal $\longleftrightarrow$ frequency domain signal

$$x(t) \longleftrightarrow X(\omega)$$

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega}$$

① Time Reversal

$$x(t) \longleftrightarrow X(\omega)$$
$$x(-t) \longleftrightarrow X(-\omega)$$

Proof

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad -t = \tau \quad dt = -d\tau$$

$$X'(\omega) = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt$$

$$X'(\omega) = \int_{\infty}^{-\infty} x(\tau)\, e^{j\omega\tau}(-d\tau) = -\int_{\infty}^{-\infty} x(\tau)\, e^{j\omega\tau}\, d\tau$$

$$e^{-j(-\omega)\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j(-\omega)\tau}\, d\tau = X(-\omega)$$

② Time Scaling

$$x(t) \longleftrightarrow X(\omega)$$
$$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad at = \tau$$

$$X'(\omega) = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt \qquad dt = d\tau / a$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\frac{\omega\tau}{a}}\, d\tau = \underline{\underline{\frac{1}{a}\left(X\left(\frac{\omega}{a}\right)\right)}}$$

③ Frequency Shifting

$$x(t) \longleftrightarrow X(\omega)$$
$$e^{\pm j\omega_0 t} x(t) \longleftrightarrow X(\omega \mp \omega_0)$$

Proof

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X'(\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt = \underline{X(\omega - \omega_0)}$$

④ Convolution

$$x_1(t) \longleftrightarrow X_1(\omega)$$
$$x_2(t) \longleftrightarrow X_2(\omega)$$
$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) X_2(\omega)$$

Proof

$$X_3(\omega) = \int_{-\infty}^{\infty} x_1(t) * x_2(t)\, e^{-j\omega t}\, dt$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$X_3(\omega) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, e^{-j\omega t}\, d\tau \right) dt$$

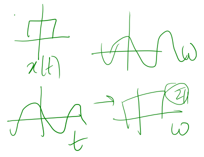
$$e^{-j\omega(\tau + t - \tau)}$$

$$x_1(\tau)\, e^{-j\omega\tau} \cdot x_2(t-\tau)\, e^{-j\omega(t-\tau)}$$

$$= X_1(\omega) \cdot X_2(\omega)$$

Duality

$$x(t) \longleftrightarrow X(\omega)$$
$$X(t) \longleftrightarrow 2\pi\, x(-\omega)$$



Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$x(t) \leftrightarrow X(\omega)$
$X(t) \rightarrow 2\pi x(-\omega)$

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$

[Interchange] $t \leftrightarrow \omega$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

Fourier transform of $X(t)$

$$\boxed{X(t) \rightarrow 2\pi x(-\omega)}$$

⑥ Multiplication

$x_1(t) \rightarrow X_1(\omega)$

$x_2(t) \rightarrow X_2(\omega)$

$$x_1(t) \times x_2(t) \leftrightarrow \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

Proof

$$\mathrm{IFT}\left[\frac{1}{2\pi} X_1(\omega) * X_2(\omega)\right] \rightarrow x_1(t) \times x_2(t)$$

$\mathrm{IFT}\,(X_1(\omega) * X_2(\omega))$

$$\Rightarrow \frac{1}{2\pi}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega) * X_2(\omega) e^{j\omega t} d\omega\right]$$

$$\Rightarrow \frac{1}{4\pi^2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega-\lambda) d\lambda\, e^{j\omega t} d\omega\right]$$

$$\Rightarrow \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega-\lambda) e^{j(\omega-\lambda)t} e^{j\lambda t} d\omega d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda) e^{j\lambda t} d\lambda \; \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega-\lambda) e^{j(\omega-\lambda)t} d\omega$$

$$= x_1(t) \times x_2(t)$$