IIIT-Bangalore
Course: Probability and Statistics
Problem Set 6

(Mathematical Expectation I)

1. Compute mean and variance of the following well-known distributions: (i) Binomial (n, p), (ii) Poisson (μ), (iii) Normal (m, σ), (iv) Gamma (l) and (v) Cauchy (λ, μ).

2. A point P is chosen at random on a line segment AB of length $2a$. Find the expected values of (i) $AP \,.\, PB$, (ii) $|AP - PB|$, (iii) $\max\{AP, PB\}$. (Ans. (i) $\frac{2}{3}a^2$, (ii) a, (iii) $\frac{3a}{2}$)

3. If X is uniformly distributed over $(0, \frac{\pi}{2})$, compute the expectation of the random variable $\sin X$. Also find the distribution of $\sin X$, and show that the mean of this distribution is the same as the above expectation. (Ans. mean is $\frac{2}{\pi}$ for both.)

4. Show that the expectation of the number of failures preceeding the first success in an infinite sequence of Bernoulli trials with probability of success p is $\frac{1-p}{p}$.

5. If t is a +ve real number and the probability mass function of a discrete random variable X is given by

$$f(x) = \begin{cases} e^{-t}(1 - e^{-t})^{x-1}, & x = 1, 2, 3, \ldots \\ 0, & \text{elsewhere}, \end{cases}$$

   find the mean and variance of X. (Ans. mean: x^t, variance: $e^t(e^t - 1)$)

6. Given $X \sim N(0, 1)$, find the variance of e^X. (Ans. $e(e-1)$)

7. For binomial (n, p) distribution, prove that

$$\mu_{k+1} = p(1-p)\left(n\,k\,\mu_{k-1} + \frac{d\mu_k}{dp}\right)$$

   and obtain γ_1 and γ_2. (Ans. $\gamma_1 = \frac{1-2p}{\sqrt{np(1-p)}}$, $\gamma_2 = \frac{1-6p(1-p)}{np(1-p)}$)

8. For a Poisson distribution with parameter μ, prove that

$$\mu_{k+1} = \mu\left(k\,\mu_{k-1} + \frac{d\mu_k}{d\mu}\right)$$

   and obtain γ_1 and γ_2. (Ans. $\frac{1}{\sqrt{\mu}}$, (ii) $\frac{1}{\mu}$)

9. Show that the 1st absolute moment about the mean for a normal (m, σ) distribution is $\sqrt{\frac{2}{\pi}}\,\sigma$.

10. If $X \sim N(m, \sigma)$ variate, then prove that

$$\mu_{2r+2} = \sigma^2 \mu_{2r} + \sigma^3 \frac{d\mu_{2r}}{d\sigma}.$$

Hence find the coefficient of curtosis β_2 of this distribution. (Ans. $\beta_2 = 3$)