Problem Set 3

Solutions

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Contents

Question 2

When a defective die is thrown 10 times the probability that an even face occurs 5 times is twice that of the same event occurring 4 times. Find the probability of an even face never occuring in four throws of the same die.

Solution 2

Let

A: Even face comes up

B: Even face occurs 5 times when given die is thrown 10 times

C: Even face occurs 4 times when given die is thrown 10 times

D: Even face never occurs when given die is thrown 4 times

Given $P(B) = 2 \times P(C)$

$$P(B) = 2 \times P(C)$$
$$\binom{10}{5}(P(A))^5(1 - P(A))^5 = 2 \times \binom{10}{4}(P(A))^4(1 - P(A))^6$$
$$\implies P(A) = \frac{5}{8}$$

We are required to find $P(D)$:

$$
\begin{aligned}
P(D) &= \binom{4}{0}(P(A))^0(1 - P(A))^4 \\
&= \left(\frac{3}{8}\right)^4
\end{aligned}
$$

Question 6

If a die is thrown n times show that the probability of an even number of sixes is $\frac{1}{2}\left(1+\frac{2}{3}^n\right)$.

Solution 3

Let

A: Getting a 6 when a die is rolled

E: Getting even number of '6's when a die is rolled n times

We know that $P(A) = \frac{1}{6}$

$$P(E) = \sum_{i=0}^{\lfloor n \rfloor} \binom{n}{2i} (\frac{1}{6})^{2i} (\frac{5}{6})^{n-2i}$$

We use binomial expansion to obtain the even terms:

$$(\frac{1}{6} + \frac{5}{6})^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{n-i}$$

$$(\frac{5}{6} - \frac{1}{6})^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{-1}{6}\right)^i \left(\frac{5}{6}\right)^{n-i}$$

$$
\begin{aligned}
&(\frac{1}{6} + \frac{5}{6})^n + (\frac{5}{6} - \frac{1}{6})^n \\
&= 2 \times \sum_{i=0}^{\lfloor n \rfloor} \binom{n}{2i} (\frac{1}{6})^{2i} (\frac{5}{6})^{n-2i} \\
&= 2 \times P(A)
\end{aligned}
$$

$$
\implies P(A) = \frac{1}{2}\left(1 + \left(\frac{2}{3}\right)^n\right)
$$

Question 4

Show that the most probable number of heads in $2n$ throws of a coin is n and that the corresponding maximum probability lies between $\frac{1}{2\sqrt{n}}$ and $\frac{1}{\sqrt{2n+1}}$

Solution 4

Let
A_i: i heads when coin in tossed 2n times

$$P(A_i) = \binom{2n}{i}(\frac{1}{2})^i(\frac{1}{2})^{2n-i} \qquad = \binom{2n}{i}(\frac{1}{2})^{2n}$$

The maximum term will be determined by the binomial term, and thus the most probable number of heads will be n.

$$\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} = \frac{1 \times 2 \times 3 \ldots 2n}{n!n!}\left(\frac{1}{2^{2n}}\right)$$

$$= \frac{(1 \times 3 \times 5 \ldots 2n-1) \times 2^n \times (1 \times 2 \times 3 \ldots n)}{n!n!} \times \frac{1}{2^{2n}}$$

$$= \frac{(1 \times 3 \times 5 \ldots 2n-1)}{(2 \times 4 \times 6 \ldots 2n)}$$

We know that $\frac{n}{n+1} < \frac{n+1}{n+2}$

$$\frac{(1 \times 3 \times 5 \ldots 2n-1)}{(2 \times 4 \times 6 \ldots 2n)} < \frac{(2 \times 4 \times 6 \ldots 2n)}{(3 \times 5 \times 7 \ldots 2n+1)}$$

$$\frac{(1\times 3\times 5\ldots 2n-1)}{(2\times 4\times 6\ldots 2n)}\times\frac{(1\times 3\times 5\ldots 2n-1)}{(2\times 4\times 6\ldots 2n)}<\frac{(2\times 4\times 6\ldots 2n)}{(3\times 5\times 7\ldots 2n+1)}$$
$$\times\frac{(1\times 3\times 5\ldots 2n-1)}{(2\times 4\times 6\ldots 2n)}$$
$$\implies (P(A_n))^2<\frac{1}{2n+1}$$
$$\implies (P(A_n))<\frac{1}{\sqrt{2n+1}}$$

Similarly,

$$\frac{1}{2} \times \frac{(3 \times 5 \ldots 2n-1)}{(4 \times 6 \ldots 2n)} > \frac{1}{2} \times \frac{(2 \times 4 \ldots 2n-2)}{(3 \times 5 \ldots 2n-1)}$$

$$\begin{aligned}
\frac{(1 \times 3 \times 5 \ldots 2n-1)}{(2 \times 4 \times 6 \ldots 2n)} \times \frac{(1 \times 3 \times 5 \ldots 2n-1)}{(2 \times 4 \times 6 \ldots 2n)} > \frac{1}{2} \times \frac{(2 \times 4 \ldots 2n-2)}{(3 \times 5 \ldots 2n-1)} \\
\times \frac{(1 \times 3 \times 5 \ldots 2n-1)}{(2 \times 4 \times 6 \ldots 2n)} \\
\implies (P(A_n))^2 > \frac{1}{2 \times 2n} \\
\implies (P(A_n)) > \frac{1}{2\sqrt{n}}
\end{aligned}$$