11. Show that every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

$a_1 \quad a_2 \quad a_3 \quad a_i \cdots \quad a_{n^2+1}$ $\qquad a_i \neq a_j \ \forall i \neq j$

Length

$I_i \Rightarrow$ Longest increasing subseq. starting at a_i

$D_i \Rightarrow$ " dec. " " " a_i

$\sim\left(\exists i \ (I_i > n+1) \vee (D_i > n+1)\right)$

$\forall i\left((I_i \le n) \wedge (D_i \le n)\right)$

$I_i, D_i \in \{1, 2, \ldots, n\}$

$a_i \rightarrow (I_i, D_i)$ $\qquad I_i$

n^2+1

a_1
a_2
a_{n^2+1}

n^2+1

1 1
1 2
1 3
n-1 1
n-1, 2
n, n

n^2

a_i, a_j

$a_i \rightarrow (I_i, D_i)$

$a_j \rightarrow (I_j, D_j)$

$I_i = I_j$

$D_i = D_j$

$\boxed{a_i}$

(I, D)

$I+1, (D+1)$

$a_i < a_j$

a_j

(I, D)

D

$a_i > a_j$

12. There are 9 people, aged from 18 to 58, at a family reunion. Show by pigeonhole principle that it is possible to choose two disjoint groups of these people in such a way that the sums of the ages of the people in each group are equal.

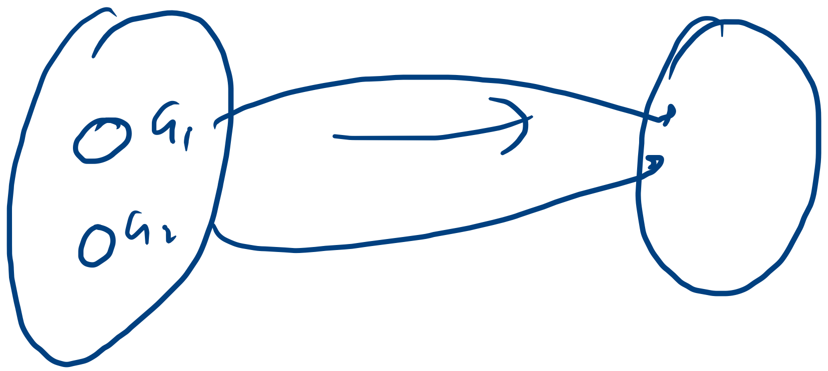
$1 \times 18 = 18$

$9 \times 58 = 522$

$522 - 18 + 1 = 505$

$2^9 - 1 = 511$

Groups (511) Ages (505)



G_1 & G_2 have the same age.

$C = G_1 \cap G_2$

$(G_1 - C)$ $(G_2 - C)$

13. Let $A = \{1, 2, \ldots, 2n\}$ and let $B \subset A$ be any subset of A, such that $|B| = n + 1$. Using pigeon-hole principle, show that there exists two integers $a_i, a_j \in B$, such that either a_i divides a_j or a_j divides a_i.

$$A = \{1, 2, \ldots, 2n\} = O\,\{1, 3, 5, \ldots, 2n-1\} \cup E\,\{2, 4, \ldots, 2n\}$$

$B \subset A$

$|B| = n+1$

$a_i \in A$

$\underline{a_i = 2^{x} \cdot y}$

$y \in O$

$b_i = 2^{x_i} \cdot y_i$ $\quad y_i \in O$

$b_i \to y_i$

$B(n+1)$: $b_1, b_2, \ldots, b_{n+1}$ $\longrightarrow$ O (n)

$n+1$ $\qquad$ n

$\exists$ $b_i = 2^{x_i} \cdot y_i$

$b_j = 2^{x_j} \cdot y_j$ $\quad y_i = y_j$

$\underline{\max(b_i, b_j)} \,\%\, \underline{\min(b_i, b_j)} = 0$

14. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where x_i, $i = 1, 2, 3, 4, 5, 6$, is a non-negative integer such that

(a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?

(b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$, and $x_6 \geq 6$?

(c) $x_1 \leq 5$?

(d) $x_1 < 8$ and $\underline{x_2 > 8}$?

$x_1 + \dots x_n = r$

${}^{n+r-1}C_{n-1}$

r stars. n-1 bars

$\star\star||\star|\star$

$x_1 + x_2 + x_3 + x_4 = 4$

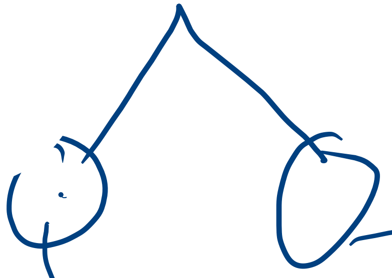
2 0 1 1

$$\boxed{{}^{n+r-1}C_{n-1}}$$

1. A binary tree is called full if every internal vertex has either two children or no children. Let H_n denote the number of full binary trees with $n+1$ leaves. Derive a recurrence equation for H_n with initial conditions.

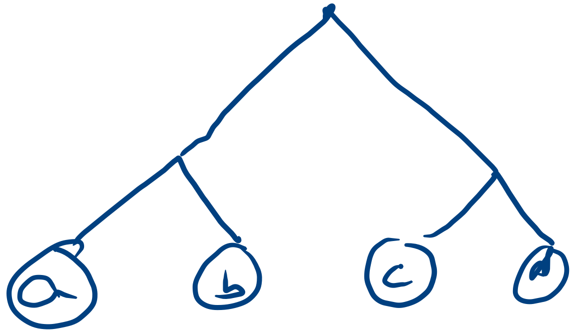
$\boxed{H_n = C_n}$

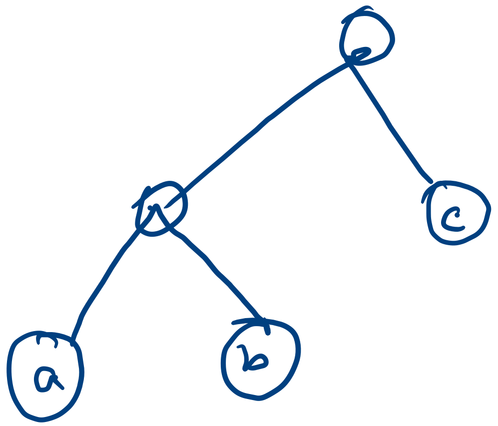
$((a.b).(c.d))$



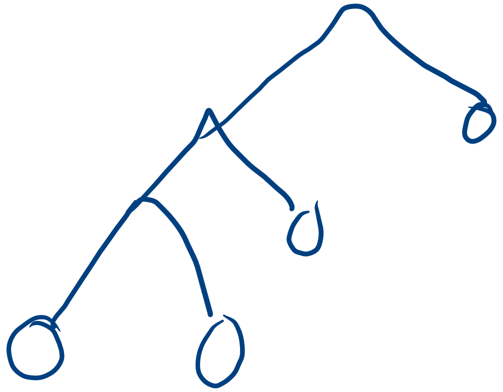
$(n+1)$ leaf nodes.

$a_1 \; a_2 \; \dots \; a_{n+1} \longrightarrow C_n$

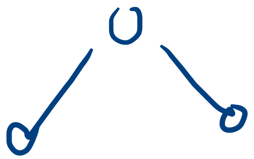




$(a.b).C \longrightarrow \boxed{C_2}$

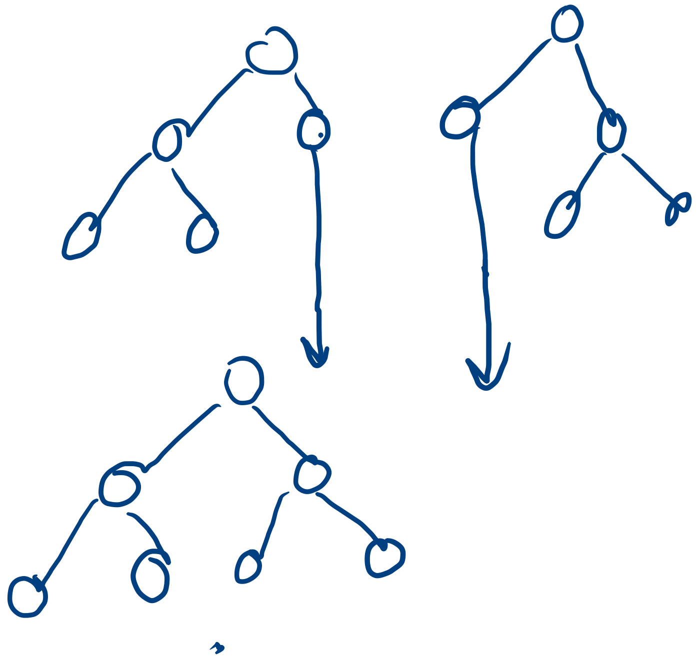


$H_2 \to$



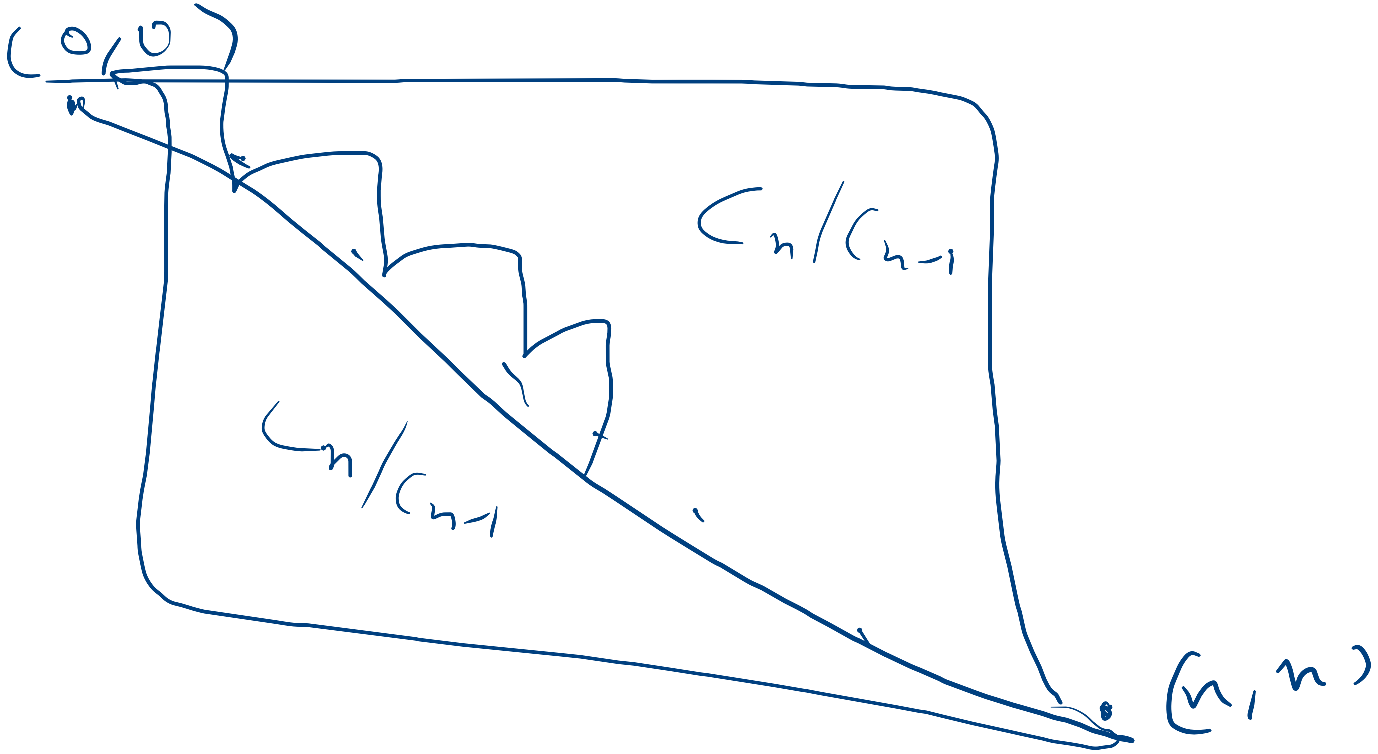
$H_3 \to$

$H_4 \to$



2. Consider an $n \times n$ grid, consisting of n^2 square cells. Suppose you want to travel from the lower left corner to the upper right corner, where you are allowed to move exactly one cell at a time, either to the right or to the top. Then derive a formula for the total number of valid paths possible, satisfying the above constraints.

$\binom{2n}{n}$



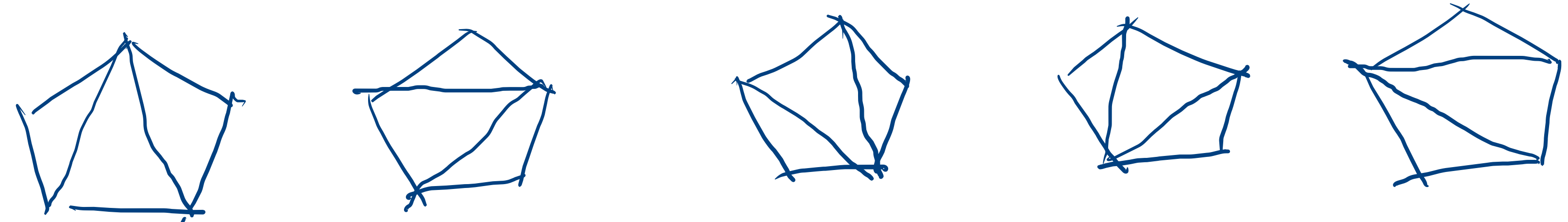
3. How many diagonals does a convex polygon with n sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)

$^{n}C_{2} - n$

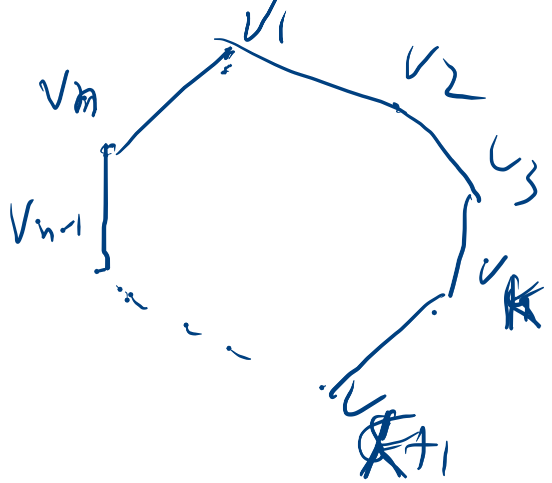
4. By triangulation of a polygon, we mean a way of dividing the polygon into triangles by non-intersecting diagonals. For example, there are two ways to triangulate a rectangle.

(a) Draw all possible triangulation of pentagon.

(b) Derive a recurrence relation for the number of different triangulation of a polygon with n edges.



(b)



$$\sum_{k=0}^{n-3} T_{k+2} \cdot T_{n-k-1}$$

$$T_n = T_2 \cdot T_{n-1} + T_3 \cdot T_{n-2} \cdots + T_{n-1} \cdot T_2$$

$$\boxed{T_3 = 1}$$

$$\boxed{T_n = C_{n-2}}$$

5. Let D_n denote the number of derangement of n distinct elements. Derive a recurrence relation for D_n.

$a_1 \quad a_2 \quad \cdots \quad a_k \quad \cdots \quad a_n$

$1 \quad 2 \quad \cdots \quad k \quad \quad n$

$\forall\ a_i \neq i$

$1 \quad 2 \quad 3 \quad 4 \cdots k \cdots n$

$a_1 \quad a_2 \quad a_3 \quad \quad a_k \quad a_n$

$a_k \cdots \cdots a_i \cdots \cdots$

$k \in \{2 \ldots n\}$

$\boxed{D_{n-2} + D_{n-1}}$

$\boxed{a_k \neq 1}$

$$\boxed{D_n = (n-1)(D_{n-2} + D_{n-1})}$$

$D_1 = 0$

$D_2 = 1$

$$\boxed{D_n = \lfloor n - \sum_{k=1}^{n} D_{n-k}\, C(n,k)}$$