PS-4

Problems: 4 to 12

Transformation of Random Variables

Let $X : S \to \mathbb{R}$ be a random variable and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function.

Then $g(X) : S \to \mathbb{R}$, defined as $[g(X)](\omega) = g[X(\omega)]$ for all $\omega \in S$, is a random variable.

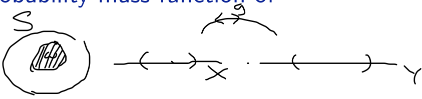
Q: Given the distribution of X, how to find the distribution of the transformed random variable $Y = g(X)$?

Transformation of Random Variables: Discrete Case

Theorem

Let (i) X be a discrete random variable whose p.m.f. is $f_X(x)$ and
(ii) $Y = g(X)$ be another r.v. where g is a bijective map, so that $X = h(Y)$ (i.e. inverse of g exists). Then the probability mass function of Y is

$$f_Y(y) = f_X[h(Y)].$$



Proof: Let spectrum of X is $\{x_i : i = 0, \pm 1, \pm 2, \dots\}$

$$P(X = x_i) = f_X(x_i) \quad \forall i = 0, \pm 1, \dots$$

Spectrum of Y : $\{y_i = g(x_i) : i = 0, \pm 1, \pm 2, \dots\}$

$P(Y = y_i) = P(X = x_i)$ $\quad (Y = y_i) = (g(X) = g(x_i)) = (X = x_i)$

$$\Rightarrow f_Y(y_i) = f_X(x_i) = f_X(h(y_i))$$

PS-5,1: Find the distribution of the square of a Poisson (μ) variate.

Sol.

$X \sim \text{Poisson}(\mu)$

$P(X=i) = f_X(i) = \dfrac{e^{-\mu}\mu^i}{i!}, \quad i = 0, 1, 2, \ldots$

$Y = X^2$. spectrum of Y : $\{ i^2 : i = 0, 1, 2, \ldots \}$

$P(Y = i^2) = ? \quad i = 0, 1, 2, \ldots$

$P(Y = i^2) = P(X^2 = i^2) = P(X = i) = \dfrac{e^{-\mu}\mu^i}{i!}$

$P(Y = 4) = \dfrac{e^{-\mu}\mu^2}{2!} \quad i = 0, 1, 2, \ldots$

$f_Y(j) = P(Y = j) = \dfrac{e^{-\mu}\mu^{\sqrt{j}}}{(\sqrt{j})!}, \quad j = 0, 1, 4, 9, \ldots$

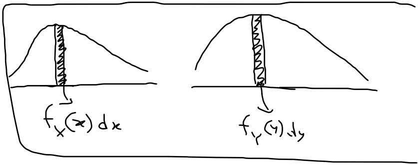
Theorem

Let (i) X be a continuous random variable with p.d.f. $f_X(x)$ and

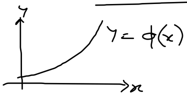
(ii) $y = \phi(x)$ be continuously differentiable and either strictly increasing or strictly decreasing throughout, so that $x = \psi(y)$ (i.e. inverse of ϕ exists). Then the p.d.f. of the transformed random variable $Y = \phi(X)$ is

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| .$$

$$= f_X(\psi(y))\left|\frac{dx}{dy}\right|$$



Proof: Case I: ϕ is strictly increasing



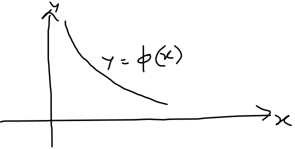
$F_Y(y) =$ d.f. of $Y = P(Y \le y) = P(\phi(X) \le \phi(x))$

Since ϕ is strictly increasing $(\phi(X) \le \phi(x)) \Leftrightarrow (X \le x)$

$F_Y(y) = P(\phi(X) \le \phi(x)) = P(X \le x) = F_X(x).$

$f_Y(y) =$ p.d.f. of $Y = \frac{d}{dy} F_Y(y) = \frac{d}{dx} F_X(x)\,\frac{dx}{dy} = f_X(x)\,\frac{dx}{dy}$ ---- (1)

Case II: ϕ is strictly decreasing



$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(\phi(X) \le \phi(x)) \\
&= P(X \ge x) \quad (\text{since } \phi \text{ is strictly dec.}) \\
&= 1 - F_X(x)
\end{aligned}
$$

$$\Rightarrow f_Y(y) = -f_X(x)\frac{dx}{dy} \quad \text{- - - - (ii)}$$

From (i) & (ii)

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X(\psi(y))\left|\frac{dx}{dy}\right|$$

PS-5,2:

$$f_X(x) = 2x e^{-x^2}, \quad x > 0$$
$$= 0, \quad \text{elsewhere}$$

$Y = X^2$

Sol. transformation: $y = x^2$

$$\frac{dy}{dx} = 2x > 0 \quad \text{for } x > 0$$

$\Rightarrow$ transformation is strictly increasing for $x > 0$

$$f_Y(y) = f_X(x) \frac{dx}{dy}$$
$$= \cancel{2x} e^{-x^2} \cdot \frac{1}{\cancel{2x}} \quad \text{for } x > 0$$
$$= \begin{cases} e^{-y} & \text{for } y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

since x varies from 0 to ∞
Y ... 0 to ∞

Prob 10 :

$X \sim N(0,1)$

$Y = \frac{X^2}{2}$ is $\gamma\left(\frac{1}{2}\right)$.

Sol. tranformation (in term of real variables)

$y = \frac{x^2}{2}$

x varies from $-\infty$ to ∞

y - - - - - - - - 0 to ∞

$\frac{dy}{dx} = x$ $\Rightarrow$ neither strictly inc. nor strictly dec throughout the dom.

$F_Y(y) = $ d.f. of $Y = P(Y \leq y)$

$y < 0$, $F_Y(y) = 0$, since $(Y \leq y) = \left(\frac{X^2}{2} \leq y\right)$ is an impossible event.

$$\phi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

For $y > 0$,

$$F_Y(y) = P(Y \le y) = P\left(\frac{X^2}{2} \le y\right)$$
$$= P\left(-\sqrt{2y} \le X \le \sqrt{2y}\right)$$
$$= P\left(-\sqrt{2y} < X \le \sqrt{2y}\right)$$
$$= \Phi\left(\sqrt{2y}\right) - \Phi\left(-\sqrt{2y}\right)$$

since X is cont, $P\left(X = -\sqrt{2y}\right) = 0$

$$\boxed{\frac{d}{dy}\left(\sqrt{2y}\right) = \frac{1}{\sqrt{2y}}}$$

$$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2y}} \Phi'\left(\sqrt{2y}\right) + \frac{1}{\sqrt{2y}} \Phi'\left(-\sqrt{2y}\right)$$
$$= \frac{1}{\sqrt{2y}} \phi\left(\sqrt{2y}\right) + \frac{1}{\sqrt{2y}} \phi\left(-\sqrt{2y}\right)$$
$$= \frac{y^{-\frac{1}{2}}}{2\sqrt{\pi}} \left[e^{-y} + e^{-y}\right]$$
$$= \begin{cases} \dfrac{e^{-y} y^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)}, & y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$\Rightarrow Y \sim \Gamma\left(\frac{1}{2}\right)$.

Poisson Process

Stochastic Process

A family of random variables $\{X(t) : t \in T\}$ which depends parametrically on time t, is called a stochastic process.

Examples

1. $X(t)$: Total number of customers that have entered in a supermarket at time t.
2. $X(t)$: Number of persons infected by a disease in a given time t.
3. $X(t)$: Number of persons in a queue at time t.

Poisson Process

A particular example of stochastic process which counts **number of changes** in a given time interval. This process obeys two laws:

1. The number of changes during the time interval $(t, t+h)$ is independent of number of changes occurred in $(0, t)$, for all t and h (> 0).
2. (i) The probability of exactly one change in $(t, t+h)$ is $\lambda h + o(h)$ where λ is a positive constant and $o(h)$ is a function of h such that $\frac{o(h)}{h} \to 0$ as $h \to 0$.

   (ii) The probability of more than one change in $(t, t+h)$ is $o(h)$.

Theorem

Number of changes of a stochastic process in a given time interval, satisfying the above two laws, follow the Poisson distribution.

(For proof see Ref. book 1 or 2)

$X(t)$: Number of changes in interval $(0, t)$.

$$P(X(t) = i) = e^{-\lambda t} \frac{(\lambda t)^i}{i!}, \; i = 0, 1, 2, \ldots$$

$X(t) = \text{Poisson}(\lambda t)$

λ : rate of the Poisson process
(average number of changes per unit time)

λt : average number of changes in a given time interval $(0, t)$

PS-4, 11:

$\cancel{X}(t)$: no. of wars in $(0, t)$

λ = average no. of changes per unit time

$= \frac{1}{15}$

$X(t) \sim \text{Poisson}\left(\frac{t}{15}\right)$

$X(25) \sim \text{Poisson}\left(\frac{25}{15}\right) \equiv \text{Poisson}\left(\frac{5}{3}\right)$

$P\left(X(25) = 0\right) = \dfrac{e^{-\frac{5}{3}} \left(\frac{5}{3}\right)^0}{0!} = e^{-\frac{5}{3}}.$

Prob 12:

$X(t)$: no. of particles emitted in $(0,t)$

λ = no. of . . . . per unit time

$= 2.5$

$$X(4) \sim \text{Poisson}(2.5 \times 4)$$

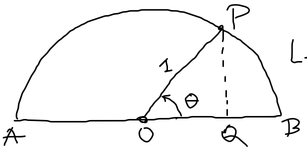
$$\sim \text{Poisson}(10)$$

$$P(X(4) \geq 3) = 1 - P(X(4)=0) - P(X(4)=1) - P(X(4)=2)$$

$= \cdots$ (check!)

PS-5

(6)



Let, $\theta = \angle POQ$

The r.v. Θ corr. to θ has uniform distribution

$$f_\Theta(\theta) = \frac{1}{\pi}, \quad 0 < \theta < \pi$$
$$= 0, \quad \text{elsewhere.}$$

$\boxed{OQ = \cos\theta}$, $\boxed{y = \cos\theta}$ (*), $\frac{dy}{d\theta} = -\sin\theta < 0 \Rightarrow y(\theta)$ is mon. dec. fn in $(0, \pi)$

To find distribution of $Y = \cos\Theta$

$$f_Y(y) = \text{p.d.f. of } Y = f_\Theta(\theta)\left|\frac{d\theta}{dy}\right|$$

(*) since θ varies from 0 to π

Y ------- -1 to 1

$$= \frac{1}{\pi}\frac{1}{\sin\theta}, \quad \theta \in (0, \pi)$$

$$= \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}, & y \in (-1, 1) \\ 0, & \text{elsw} \end{cases}$$

(8)

$$t^2 + 2t - X = 0$$

$$\Rightarrow t = \frac{-2 \pm \sqrt{4+4X}}{2}$$

$$= -1 \pm \sqrt{1+X}$$

$f_X(x) = \frac{1}{2}, \quad x \in (0,2)$
$= 0, \quad$ else

$Y = -1 + \sqrt{1+X}$ is the larger root.

transformation: $y = -1 + \sqrt{1+x}$

if x varies from 0 to 2
y - - - - - - - 0 to $-1+\sqrt{3}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x}} > 0 \text{ for } x \in (0,2)$$

$\Rightarrow y(x)$ is a mon. inc. fn of x

$$\int_0^{\sqrt{3}-1} (1+y)\,dy$$
$$= \left[y + \frac{y^2}{2} \right]_0^{\sqrt{3}-1}$$
$$= \sqrt{3} - 1 + \frac{1}{2}(\sqrt{3}-1)^2$$
$= 1.$ (check!)

$$f_Y(y) = \text{p.d.f. of } Y = f_X(x) \cdot \left| \frac{dx}{dy} \right| = \frac{1}{\cancel{2}} \cancel{2}\sqrt{1+x} = \begin{cases} (1+y), & y \in (0, \sqrt{3}-1) \\ 0, & \text{else} \end{cases}$$

(9) $X \sim N(m, \sigma)$

$$P(a < X < b) = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)$$

Φ : std. normal d.f.

Sol. $f_X(x) = $ p.d.f. of X $= \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-m)^2}, \quad -\infty < x < \infty$

Let, $Y = \dfrac{X-m}{\sigma}$. transformation: $y = \dfrac{x-m}{\sigma}$

since x varies from $-\infty$ to ∞

y - - - - $-\infty$ to ∞

$\dfrac{dy}{dx} = \dfrac{1}{\sigma} > 0 \Rightarrow y(x)$ is a mon. inc. fn of x

$f_Y(y) = $ p.d.f of Y $= f_X(x) \dfrac{dx}{dy} = \dfrac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-\frac{1}{2\sigma^2}(x-m)^2} \cdot \cancel{\sigma}$

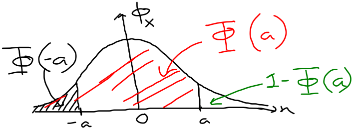
$\Rightarrow Y \sim N(0,1)$ with d.f. Φ $= \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, \quad -\infty < y < \infty$

$$P(a<X<b) = P(a<X\le b)$$

$$= P\left(\frac{a-m}{\sigma} < \underbrace{\frac{X-m}{\sigma}}_{=Y} \le \frac{b-m}{\sigma}\right)$$

$$= \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right) \qquad \text{—— (i)}$$

II. $P(|X-m| > a\sigma) = 2\left[1-\Phi(a)\right]$.



$$LHS = 1 - P\left(|X-m| \le a\sigma\right)$$

$$= 1 - P\left(-a\sigma \le X-m \le a\sigma\right)$$

$$= 1 - P\left(-a < \underbrace{\frac{X-m}{\sigma}}_{=Y} \le a\right)$$

$$= 1 - \left[\Phi(a) - \Phi(-a)\right]$$

$$= 2\left[1-\Phi(a)\right]$$

(*) $\Phi(-a) = 1 - \Phi(a)$

$$\Phi(-a) = \int_{-\infty}^{-a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Put $x = -y$

$= - - - - -$

$= - - - -$

$= 1 - \Phi(a)$.