CS 201: Discrete Mathematics	October 6, 2022

Assignment 1

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Solution 1

$S(x_i)$ is true if the student x_i is present in the campus. $P(x_i)$ is true if the parent of the student x_i is present in the campus. Number of students in IMT2021 = n

(a) $A(k)$: There are atleast k students in the campus where $k \geq 0$

$\Rightarrow \exists a \geq k : [S(x_{b_1}) \wedge S(x_{b_2}) \cdots S(x_{b_a})]$ ($b_i \in \mathbb{N}_0$ and $b_i \leq n$)

(b) $E(k)$: There are exactly k students in the campus.

$\Rightarrow \exists k : A(k) \wedge \neg A(k+1)$

(c) $Match$: All the students whose parents have come to campus are also in the campus.

$\Rightarrow \forall i : (P(x_i) \to S(x_i))$

(d) $Single$: There is exactly one student in the campus such that their parent is also in the campus.

$\Rightarrow \exists i : [S(x_i) \wedge P(x_i) \wedge (\forall i \neq j : \neg(S(x_j) \wedge P(x_j)))]$

(e) $Surprise$: While not all parents where able to come to the campus. None of the students whose parents came to the campus, were in the campus.

$\Rightarrow \forall i : [(P(x_i) \to \neg S(x_i)) \wedge (\exists j : \neg P(x_j))]$

Solution 2

(a) **Given:** $A \cup B = C$

To Prove/Disprove: $A \times B \in P(C \times C)$

The given statement is true. To prove this statement, it is sufficient to prove that $A \times B \subset C \times C$

Proof. $A \times B = \{(a, b) : a \in A \wedge b \in B\}$

But $(a \in A) \to (a \in C)$ and $(b \in B) \to (b \in C)$ [follows from given]

$\Rightarrow (a, b) \in A \times B \to (a, b) \in C \times C$

$\Rightarrow A \times B \subset C \times C$ (follows from definition of subset)

$\Rightarrow A \times B \in P(C \times C)$ (follows from definition of power set)

Hence Proved $\square$

(b) **Given:** $A \cup B = C$

To Prove/Disprove: $(A \times C) \cap (C \times B) = (C \times C)$

Let us take an example.

$A = \{a\}, B = \{b\}$

$\Rightarrow C = \{a, b\}$

$A \times C = \{(a, a), (a, b)\}$, $C \times B = \{(a, b), (b, b)\}$

$C \times C = \{(a, a), (b, b), (a, b), (b, a)\}$

$(A \times C) \cap (C \times B) = \{(a, b)\}$

Clearly, $(A \times C) \cap (C \times B) \neq (C \times C)$

$\Rightarrow$ the given statement is false

Solution 3

Given: A finite universal set U and a partition of set U into various sets (s_1, s_2,..., s_n), where $s_i \cap s_j = \phi\ \forall$ i, j $\in N$ and i, j $\leq$ n and $s_1 \cup s_2 \cup \ldots s_n$ = U

R: U $\rightarrow$ U = { (a, b) | (a, b) $\in$ U $\times$ U and if $a \in s_i$ and $b \in s_j$ then i $\neq$ j }

3.1

(a) for R to be reflexive, $(a, a) \in R, \forall a \in U$

Since s_i are all disjoint sets, $a \in s_i \Rightarrow a \notin s_j$, $\forall j \neq i$

$\Rightarrow (a, a) \notin R, \forall a \in U$

$\Rightarrow$ R is not reflexive

(b) for R to be symmetric, $(a, b) \in R \rightarrow (b, a) \in R$

let $(a, b) \in R \Rightarrow a \in s_i$ and $b \in s_j$, $i \neq j$

also $a, b \in U \Rightarrow (a, b)$ and $(b, a) \in U \times U$

$\Rightarrow (b, a) \in R$

$\Rightarrow R$ is symmetric

(c) for R to be transitive, $[(a, b) \in R$ and $(b, c) \in R] \rightarrow (a, c) \in R$

Let (a, b) and $(b, c) \in R$

$\Rightarrow a \in s_i, b \in s_j$ and $c \in s_k$, where $i \neq j$ and $j \neq k$

but this does not imply that $i \neq k$

$\Rightarrow$ the relation is not transitive

3.2

Claim 3.1. *$|R|$ is maximised when $\forall$ i, j $\in N$ and i, j$\leq$ n we have $||s_i|$ - $|s_j|| \leq$ 1*

Proof. Let cardinality of the set $s_i = x_i, \forall i$

$$|R| = \sum x_i x_j = \frac{1}{2}[(x_1 + x_2 + \ldots + x_n)^2 - (x_1^2 + x_2^2 + \ldots + x_n^2)] \tag{1}$$

in order to maximise $|R|$, we need to minimise $X = (x_1^2 + x_2^2 + \ldots + x_n^2)$ as the first term in the above equation is constant

using the AM-GM inequality, X will be minimised when all the x_i are equal (AM $\geq$ GM and AM = GM when all the elements are equal)

thus we distribute the elements into the various partitions as equally as possible.

ie, we put $\frac{\sum x_i}{n}$ in each of the partitions. Let the y be the number of remaining elements. Clearly, $y < n$. Distribute these y elements to the first y partitions.

In such a distribution, $||s_i| - |s_j|| \leq 1 \ \forall i, j$

Hence proved $\square$

Solution 4

$a, b, c \in \mathbb{N} \ni a + b = c.$

$R = \{(x, y) | (ax + by) \mod c = 0\}$ on $\mathbb{Z}$.

(a) <u>Reflexive</u>

Choose an arbitrary $x \in \mathbb{Z}$

$(ax + bx) \mod c = x(a + b) \mod c$

$=$ xc $\mod c = 0$

$\Rightarrow (x, x) \in R$

$\Rightarrow$ R is reflexive

<u>Symmetric</u>

let $(x, y) \in R$

$\Rightarrow$

$$ax + by = kc \tag{2}$$

put $a = c - b$ and $b = c - a$ in (1)

$$(c - b)x + (c - a)y = kc \tag{3}$$

on simplification, we get

$$ay + bx = (x + y - k)c \tag{4}$$

$\Rightarrow ay + bx \mod c = 0$

$\Rightarrow (y, x) \in R$

$\Rightarrow$ R is symmetric

<u>Transitive</u>

let (x, y) and $(y, z) \in R$

$\Rightarrow$

$$ax + by = k_1 c \tag{1}$$

$$ay + bz = k_2 c \tag{2}$$

(for some integers k_1 and k_2)

putting $b = c - a$ in (1)

$$ax + cy - ay = k_1 c \tag{3}$$

(2) + (3)

$$ax + cy + bz = (k_2 + k_1)c \tag{4}$$

$$ax + bz = (k_2 + k_1 - y)c \tag{5}$$

$\Rightarrow (ax + bz) \mod c = 0$

$\Rightarrow (x, z) \in R$

$\Rightarrow$ R is transitive

R is reflexive, symmetric and transitive

$\Rightarrow$ R is an equivalence relation

(b) $[24] = \{y : (24, y) \in R\}$

$(24, y) \in R$

$\Rightarrow (24a + yb) \mod c = 0$

$(24a + y(c - a)) \mod c = 0$

$(24a - ya) \mod c = 0$

$\frac{a(24-y)}{c} = k$

on simplification, we get

$y = 24 - \frac{kc}{a}$

since y, c and 24 are integers, $\frac{k}{a}$ must be an integer

$\Rightarrow y$ is of the form:

$y = 24 - cn, n \in \mathbb{Z}$

$[24] = \{24 - cn : n \in \mathbb{Z}\}$

$[24] = \{\cdots, 24 - 3c, 24 - 2c, 24 - c, 24, 24 + c, 24 + 2c, 24 + 3c, \cdots\}$

Proof. $(24a + b(24 - cn))$
$= 24(a + b) - bcn$
$= 24c - bcn$
$= c(24 - bn)$, which is divisible by c

$\Rightarrow (24a + b(24 - cn)) \mod c = 0$

$(24, 24 - cn) \in R, \forall n \in \mathbb{Z}$

Hence $\{24 - cn : n \in \mathbb{Z}\}$ is the equivalence class of the integer 24

□

(c) let us take an arbitrary integer p.

$[p] = \{p - nc : n \in \mathbb{Z}\}$

$\Rightarrow$ relation R partitions $\mathbb{Z}$ into c equivalence classes as the equivalence class corresponding to every cth integer is the same

let the equivalence classes be $S_1, S_2, \cdots, S_c$

$$S_1 = \{\ldots, -2c, -c, 0, c, 2c, \ldots\} \text{ representative element} = 0$$
$$S_2 = \{\ldots, 1 - 2c, 1 - c, 1, 1 + c, 1 + 2c, \ldots\} \text{ representative element} = 1$$
$$\vdots$$
$$S_c = \{\ldots, -1 - 2c, -1 - c, -1, c - 1, 2c - 1, \ldots\} \text{ representative element} = c - 1$$

Solution 5

$g = f^k = f \circ f \circ \cdots \circ f(k \text{ times})$

$h = f^{-k} = f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}(k \text{ times})$

$\Rightarrow g^a = f \circ f \circ \cdots \circ f(ka \text{ times})$ and $h^b = f^{-k} = f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}(kb \text{ times})$

$\gcd(a, b) = 1$

$\Rightarrow (g^a \circ h^b)(x) = f^{k(a-b)}$

$$(g^a \circ h^b)(x) = f^{k(a-b)}(x) = \begin{cases} f \circ f \circ \cdots \circ f(k(a-b) \text{ times}) & \text{if } a > b \\ x & \text{if } a = b \\ f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}(k(b-a) \text{ times}) & \text{if } a < b \end{cases} \quad (1)$$

We know that the inverse of bijection is a bijection and composition of two bijections is a bijection. Also Identity function is a bijection.

$\Rightarrow (g^a \circ h^b)(x)$ is a bijection $\forall a, b \in \mathbb{Z}$

$p(x, g, h) = \{(g^a \circ h^b)(x) \mid a, b \in \mathbb{Z}\} = \{\ f^{nk}\text{: n} \in \mathbb{Z}\} = \{\cdots, f^{-3k}, f^{-2k}, f^{-k}, f^0, f^k, f^{2k}, f^{3k}, \cdots\}(a, b \in \mathbb{N})$

Case 1. $\underline{x_1 = x_2}$

Since each $f^{nk}(x)$ is a bijective function, for $x_1 = x_2$, each $f^{nk}(x_1) = f^{nk}(x_2)$

$\Rightarrow p(x_1, g, h) = p(x_2, g, h)$

Case 2. $\underline{x_1 \neq x_2}$

Case 2.1. $\underline{\exists r, q : f^{rk}(x_1) = f^{qk}(x_2)}$

$\Rightarrow f^k(f^{rk}(x_1)) = f^k(f^{qk}(x_2))$

$\Rightarrow f^{k(r+1)}(x_1) = f^{k(q+1)}(x_2)$

$\Rightarrow f^{k(r+2)}(x_1) = f^{k(q+2)}(x_2)$

$\Rightarrow f^{k(r+3)}(x_1) = f^{k(q+3)}(x_2)$

$\vdots$

ie, all the elements in $p(x_1, g, h)$ and $p(x_2, g, h)$ have one to one correspondence

$\Rightarrow$ *in this case, $p(x_1, g, h) = p(x_2, g, h)$*

Case 2.2. $\underline{\forall r, q : f^{rk}(x_1) \neq f^{qk}(x_2)}$

$\Rightarrow f^k(f^{rk}(x_1)) \neq f^k(f^{qk}(x_2))$

$\Rightarrow f^{k(r+1)}(x_1) \neq f^{k(q+1)}(x_2)$

$\Rightarrow$ *in this case, $p(x_1, g, h)$ and $p(x_2, g, h)$ will have completely different elements as f^{nk} are all bijective*

$\Rightarrow$ *They are completely disjoint*

$\Rightarrow$ in all cases, either $p(x_1, g, h)$ and $p(x_2, g, h)$ are the same or disjoint

Solution 6

(a)

Claim 6.1. *If the current cumulative XOR = 0, then any choice of move will make the cumulative XOR non-zero.*

Proof. the i^{th} deck have a_i cards

X = initial cumulative XOR

$$\Rightarrow X = a_1 \oplus a_2 \oplus \cdots \oplus a_n = 0 \quad (1)$$

let in the next move, p cards are removed from the $j^{th} deck$

Y = new cumulative XOR

$$\Rightarrow Y = a_1 \oplus a_2 \oplus \cdots \oplus (a_j - p) \oplus \cdots a_n \quad (2)$$

taking XOR of both sides of (1) with a_j

$$X \oplus a_j = a_1 \oplus a_2 \oplus \cdots \oplus a_j \oplus \cdots a_n \oplus a_j \tag{3}$$

but $X = 0$, $a_j \oplus a_j = 0$ and $0 \oplus a_j = a_j$

$\Rightarrow$ (1) becomes

$$a_j = a_1 \oplus a_2 \oplus \cdots \oplus a_{j-1} \oplus a_{j+1} \oplus \cdots a_n \tag{4}$$

putting this value in (2)

$$Y = a_j \oplus (a_j - p) \tag{5}$$

Now, Y can be 0 only if either $p = 0$ or $a_j = 0 = p$. But since the player is forced to remove at least one card and the decks from which the player might remove a card must be non-empty, neither a_j nor p can be zero.

$\Rightarrow Y \neq 0$

$\Rightarrow$ any choice of move will make the cumulative XOR non-zero. □

(b)

Claim 6.2. *If the current cumulative XOR $\neq 0$, then there is always a move to make the cumulative XOR $= 0$*

Proof. X = initial cumulative XOR

$$\Rightarrow X = a_1 \oplus a_2 \oplus \cdots \oplus a_n \neq 0 \tag{6}$$

X has at least one set bit. Let the positon of the leftmost set bit be k from the left.

$\Rightarrow$ binary representation of X = 000...1XXXX...

this can be true only $\exists a_j$ which has k^{th} bit set.

$\Rightarrow a_j$ =XXX...1XXX...

let in the next move, p cards are removed from the j^{th} deck and it makes the new cumulative XOR (Y) equal to zero.

$$Y = a_1 \oplus a_2 \oplus \cdots \oplus (a_j - p) \oplus \cdots a_n = 0 \tag{7}$$

from (3)

$X \oplus a_j = a_1 \oplus a_2 \oplus \cdots \oplus a_j \oplus \cdots a_n \oplus a_j$

but $a_j \oplus a_j = 0$ and $0 \oplus a_j = a_j$

$\Rightarrow X \oplus a_j = a_1 \oplus a_2 \oplus \cdots \oplus a_{j-1} \oplus a_{j+1} \oplus \cdots a_n \oplus a_j$

putting this value in (7)

$$Y = (a_j - p) \oplus (X \oplus a_j) = 0 \tag{8}$$

$$\Rightarrow (a_j - p) = (X \oplus a_j) \tag{9}$$

$X \oplus a_j$ will have the leftmost k-1 bits same as a_j, while k^{th} bit will be flipped from 1 to 0

$\Rightarrow a_j > X \oplus a_j$

$\Rightarrow$ We can always find a value p = $a_j - X \oplus a_j$ to make the new cumulative XOR equal to 0.

$\Rightarrow$ there is always a move to make the cumulative XOR = 0. □

(c)

Claim 6.3. *If the initial cumulative XOR = 0,(at the start of the game) then Rachna can always win the game.*

Proof. Optimal Strategy: For a player to win, the optimal strategy is to make the cumulative XOR = 0. The last move of the game would make the total number of cards = 0, hence the cumulative XOR would become 0. The player making this move would win the game. If a player, say player 1 makes the cumulative XOR = 0 at any point in the game, the next move would make the cumulative XOR non-zero (follows from 6.1). Now the player 1 can again make the cumulative XOR = 0 (follows from 6.2). This process is repeated till all the decks are empty. Hence it is assured that player 1 will win the game.

Assumption: both the players play optimally.

Since Karthik makes the first move and the initial XOR was zero, his move will make the cumulative XOR non-zero. (follows from 6.1)

Now Rachna plays optimally and makes the cumulative XOR = 0 (follows from 6.2)

Now Karthik's move will again make the XOR non-zero.

Hence we can see in each move, Rachna makes the XOR equal to zero and Karthik makes the XOR non-zero.

The game ends when there are no more cards left. At this point, the cumulative XOR = 0 and it is clear that Rachna made the last move as it made the XOR = 0.

⇒Therefore, if the initial cumulative XOR = 0,(at the start of the game) then Rachna can always win the game. □

Claim 6.4. *If the initial cumulative XOR = 0,(at the start of the game) then Karthik can always win the game.*

Proof. Assumption: both the players play optimally.

Since Karthik makes the first move and the initial XOR was non-zero, his optimal move will make the cumulative XOR zero. (follows from 6.2)

similar to 6.3, we can see that in the given starting conditions, each of Karthik's moves will make the XOR = 0 while each of Rachna's moves will make it non-zero.

Hence, Karthik will make the last move.

⇒ If the initial cumulative XOR = 0,(at the start of the game) then Karthik can always win the game. Therefore □