FOURIER SERIES

5.1. We call a set of signals $\{\Psi_n(t)\}$ orthogonal on an interval (a, b) if any two signals $\Psi_m(t)$ and $\Psi_k(t)$ in the set satisfy the condition

$$\int_{a}^{b} \Psi_{m}(t) \Psi_{k}^{*}(t) dt = \begin{cases} 0 & m \neq k \\ \alpha & m = k \end{cases}$$
 (5.95)

where * denotes the complex conjugate and $\alpha \neq 0$. Show that the set of complex exponentials $\{e^{jk\omega_0t}: k=0, \pm 1, \pm 2, ...\}$ is orthogonal on any interval over a period T_0 , where $T_0 = 2\pi/\omega_0$.

For any t_0 we have

$$\int_{t_0}^{t_0 + T_0} e^{jm\omega_0 t} dt = \frac{1}{jm\omega_0} e^{jm\omega_0 t} \Big|_{t_0}^{t_0 + T_0} = \frac{1}{jm\omega_0} \left(e^{jm\omega_0 (t_0 + T_0)} - e^{jm\omega_0 t_0} \right)$$

$$= \frac{1}{jm\omega_0} e^{jm\omega_0 t_0} \left(e^{jm2\pi} - 1 \right) = 0 \qquad m \neq 0$$
(5.96)

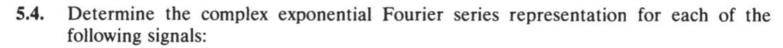
since $e^{jm2\pi} = 1$. When m = 0, we have $e^{jm\omega_0 t}|_{m=0} = 1$ and

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} dt = \int_{t_0}^{t_0+T_0} dt = T_0$$
 (5.97)

Thus, from Eqs. (5.96) and (5.97) we conclude that

$$\int_{t_0}^{t_0 + T_0} e^{jm\omega_0 t} (e^{jk\omega_0 t})^* dt = \int_{t_0}^{t_0 + T_0} e^{j(m-k)\omega_0 t} dt = \begin{cases} 0 & m \neq k \\ T_0 & m = k \end{cases}$$
 (5.98)

which shows that the set $\{e^{jk\omega_0t}: k=0,\pm 1,\pm 2,\ldots\}$ is orthogonal on any interval over a period T_0 .



(a)
$$x(t) = \cos \omega_0 t$$

(b)
$$x(t) = \sin \omega_0 t$$

$$(c) \quad x(t) = \cos\left(2t + \frac{\pi}{4}\right)$$

$$(d) x(t) = \cos 4t + \sin 6t$$

(e)
$$x(t) = \sin^2 t$$

(a) Rather than using Eq. (5.5) to evaluate the complex Fourier coefficients c_k using Euler's formula, we get

$$\cos \omega_0 t = \frac{1}{2} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right) = \frac{1}{2} e^{-j\omega_0 t} + \frac{1}{2} e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\cos \omega_0 t$ are

$$c_1 = \frac{1}{2}$$
 $c_{-1} = \frac{1}{2}$ $c_k = 0, |k| \neq 1$

(c) The fundamental angular frequency ω_0 of x(t) is 2. Thus,

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \sum_{k = -\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k = -\infty}^{\infty} c_k e^{j2kt}$$

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \frac{1}{2} \left(e^{j(2t + \pi/4)} + e^{-j(2t + \pi/4)}\right)$$

$$= \frac{1}{2} e^{-j\pi/4} e^{-j2t} + \frac{1}{2} e^{j\pi/4} e^{j2t} = \sum_{k = -\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\cos(2t + \pi/4)$ are

$$c_{1} = \frac{1}{2}e^{j\pi/4} = \frac{1}{2}\frac{1+j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1+j)$$

$$c_{-1} = \frac{1}{2}e^{-j\pi/4} = \frac{1}{2}\frac{1-j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1-j)$$

$$c_{k} = 0 \qquad |k| \neq 1$$

(e) From Prob. 1.16(e) the fundamental period T_0 of x(t) is π and $\omega_0 = 2\pi/T_0 = 2$. Thus,

$$x(t) = \sin^2 t = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Again using Euler's formula, we get

$$x(t) = \sin^2 t = \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^2 = -\frac{1}{4}(e^{j2t} - 2 + e^{-j2t})$$
$$= -\frac{1}{4}e^{-j2t} + \frac{1}{2} - \frac{1}{4}e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\sin^2 t$ are

$$c_{-1} = -\frac{1}{4}$$
 $c_0 = \frac{1}{2}$ $c_1 = -\frac{1}{4}$

and all other $c_k = 0$.