Kleene's Theorem
&
Properties of Regular Languages–cont'd

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Converting from an FA to an RE: Generalized FA

The FA must be converted to a *generalized FA*, with the following properties:

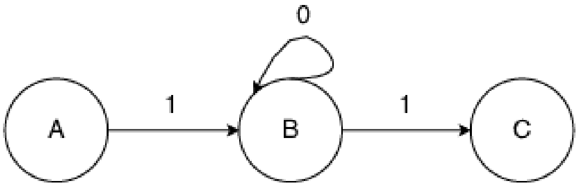
- There is a unique accept state.
- There is no transition out of the accept state.
- There is no transition into the start state (not even a self-loop).

Exercise: convert any arbitrary DFA/NFA to a generalized FA, using ϵ-transitions.

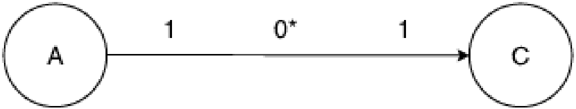
Converting from FA to RE

- First convert the FA to a generalized FA.
- In the generalized FA, remove one state at a time successively, and replace it with the appropriate transition.
- When left with a start state connecting to an accept state, read the RE off the transition.

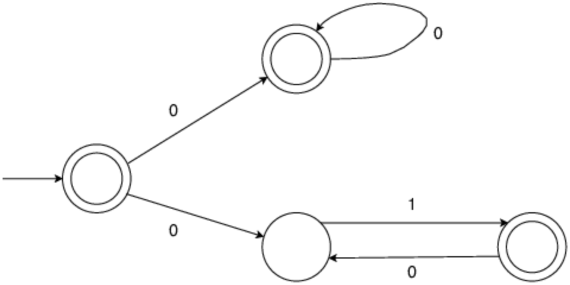
Example of How a State is Removed



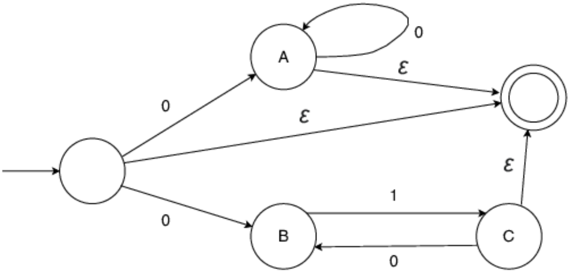
Example of How a State is Removed—cont'd



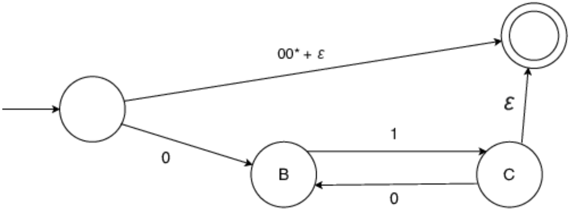
FA



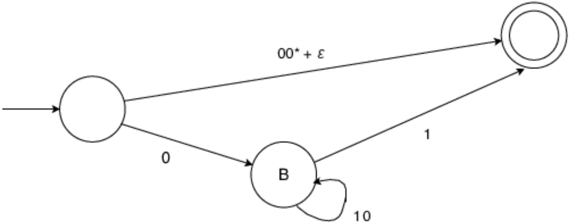
The Corresponding Generalized FA



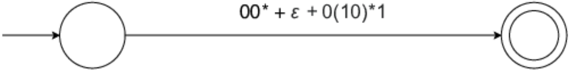
Remove state A from the Generalized FA



Remove state C from the FA



Remove state B from the FA



NB: The resulting RE does not look the same as what we started with, but it describes the same language.

Properties of Regular Languages

- The set of regular languages is closed under the regular operations (union, concatenation, and Kleene-star).
- The complement $\bar{L}$ of a regular language L is also regular. (The set of regular languages is closed under complementation.)
- The set of regular languages is closed under intersection. This can also be proved using De Morgan's Law,

$$L_1 \cap L_2 \equiv \overline{(\overline{L_1} \cup \overline{L_2})}$$

Proving Closure Under Union (Using NFAs)

Let $M_1 = \langle Q_1, \Sigma, \delta_1, q_1, F_1 \rangle$ accept L_1, and
$M_2 = \langle Q_2, \Sigma, \delta_2, q_2, F_2 \rangle$ accept L_2.
Construct $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ to accept $L_1 \cup L_2$.

Proving Closure Under Union (Using NFAs)—cont'd

- $Q = \{q_0\} \cup Q_1 \cup Q_2$, where q_0 is a new start state.
- $F = F_1 \cup F_2$; the accept states of M are all the accept states of M_1 and M_2.
- Define δ so that for $q \in Q$ and any $a \in \Sigma \cup \{\epsilon\}$,

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1, \\ \delta_2(q, a) & \text{if } q \in Q_2, \\ \{q_1, q_2\} & \text{if } q = q_0 \text{ and } a = \epsilon, \\ \emptyset & \text{if } q = q_0 \text{ and } a \neq \epsilon. \end{cases}$$

Proving Closure Under Concatenation

Let $M_1 = \langle Q_1, \Sigma, \delta_1, q_1, F_1 \rangle$ accept L_1, and
$M_2 = \langle Q_2, \Sigma, \delta_2, q_2, F_2 \rangle$ accept L_2.
Construct $M = \langle Q, \Sigma, \delta, q_1, F_2 \rangle$ to accept $L_1 \circ L_2$.

Proving Closure Under Concatenation—cont'd

- $Q = Q_1 \cup Q_2$. The states of M are those of M_1 and M_2 combined.
- The start state of M is the start state of M_1.
- The accept states of M are those of M_2.
- Define δ so that for $q \in Q$ and any $a \in \Sigma \cup \{\epsilon\}$,

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1 \text{ and } q \notin F_1, \\ \delta_1(q, a) & \text{if } q \in F_1 \text{ and } a \neq \epsilon, \\ \delta_1(q, a) \cup \{q_2\} & \text{if } q \in F_1 \text{ and } a = \epsilon, \\ \delta_2(q, a) & \text{if } q \in Q_2. \end{cases}$$

Proving Closure Under Kleene-Star

Let $M_1 = \langle Q_1, \Sigma, \delta_1, q_1, F_1 \rangle$ accept L.
Construct $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ to accept L^*.

Proving Closure Under Kleene-Star—cont'd

- $Q = \{q_0\} \cup Q_1$; the states of M are those of M_1, and a new start state q_0.
- $F = \{q_0\} \cup F_1$; the accept states are the old accept states as well as the new start state.
- For any $q \in Q$ and $a \in \Sigma \cup \{\epsilon\}$, the transition function δ is given by:

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1 \text{ and } q \notin F_1, \\ \delta_1(q, a) & \text{if } q \in F_1 \text{ and } a \neq \epsilon, \\ \delta_1(q, a) \cup \{q_1\} & \text{if } q \in F_1 \text{ and } a = \epsilon, \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \epsilon, \\ \emptyset & \text{if } q = q_0 \text{ and } a \neq \epsilon. \end{cases}$$

Exercises

Consider regular languages $L_1, L_2 \subseteq \Sigma^* \cup \{\epsilon\}$, and some fixed string $w \in \Sigma^*$. Prove that the following languages are also regular:

(1) $L_1 \cdot \overline{L_2}$

(2) $L_1 \cup w\overline{L_2}$

(3) $L_1 \setminus w\overline{L_2}$

Exercise

Create a DFA that accepts the *complement* of the language accepted by the following NFA, and also give an RE for the complement of the language.