CS 201: Discrete Mathematics
Assignment 1
Maximum Marks: 55

Instructions

1. Any result stated or proved in the class or tutorial can be used without repeating the proof.
2. State all references used other than the class slides, problem sets and the textbooks.
3. Everyone is required to typeset their assignments in latex.
4. Handwritten and poorly typesetted solutions won't be accepted.
5. You are not allowed to discuss the solutions of assignment with anyone.
6. You are not allowed to use the internet. Plagiarism will be punished heavily.

Question 1. Let there be n students in the batch of IMT2021. The parents of IMT2021 batch have been called on a surprise visit to campus. The propositional variable $S(x_i)$ is true if the student x_i is present in the campus. While the propositional variable $P(x_i)$ is true if the parent of the student x_i is present in the campus. (Assume there are no twins in this version of IMT2021).
Give the compound statements to express the following:

1. $A(k)$: There are atleast k students in the campus where $k \geq 0$
2. $E(k)$: There are exactly k students in the campus.
3. $Match$: All the students whose parents have come to campus are also in the campus.
4. $Single$: There is exactly one student in the campus such that their parent is also in the campus.
5. $Surprise$: While not all parents where able to come to the campus. None of the students whose parents came to the campus, were in the campus.

[2 + 2 + 2 + 2 + 2 = 10 marks]

Question 2. If $A \cup B = C$, then prove or disprove the following:

1. $A \times B \in P(C \times C)$
2. $(A \times C) \cap (C \times B) = (C \times C)$

[5 marks]

Question 3. Suppose we have a finite universal set U and a partition of set U into various sets (s_1, s_2, ... s_n), where $s_i \cap s_j = \phi$ $\forall$ i, j $\in N$ and i, j $\leq$ n and $s_1 \cup s_2 \cup$ ..... s_n = U. Suppose we define a relation R : U $\rightarrow$ U = {(a, b) | (a, b) $\in$ U $\times$ U and if $a \in s_i$ and $b \in s_j$ then i $\neq$ j }

1. Is R a) reflective b) symmetric c) transitive?
2. Prove that |R| is maximised when $\forall$ i, j $\in N$ and i, j$\leq$ n we have $||s_i| - |s_j|| \leq 1$

[3+7=10 marks]

Question 4. Let $a, b, c \in \mathbb{N} \ni a + b = c$. Let R be the relation $R = \{(x, y) | (ax + by) \mod c = 0\}$ on $\mathbb{Z}$.

a) Prove that the relation R is a equivalence relation.

b) What is the equivalence class of the integer 24 with respect to the relation R over $\mathbb{Z}$. Prove that your result is indeed the equivalence class of integer 24.

c) The relation R partitions $\mathbb{Z}$ into how many equivalence classes? Give a representative element for each equivalence class.

[4 + 3 + 3 = 10 marks]

Question 5. Let X be a set and $f : X \to X$ be a bijection. Define f^0 as the identity function. For some integer $k \geq 1$, define functions g and h:

$$g = f^k = f \circ f \circ \cdots \circ f(k \text{ times})$$

$$h = f^{-k} = f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}(k \text{ times})$$

For any $x \in X$, let $p(x, g, h) = \{(g^a \circ h^b)(x) \mid a, b \in \mathbb{Z}\}$. Prove that for any $x_1, x_2 \in X$, $p(x_1, g, h)$ and $p(x_2, g, h)$ are either the same or disjoint.

[10 marks]

Question 6. Kartik and Rachna were bored from correcting Discrete math assignments so they decided to play a game. They have n decks of cards lined up such that 1^{st} deck has a_1 cards, 2^{nd} deck has a_2 cards and so on, such that $a_i \geq 0$. Both of them move in turns, alternatively. In one move, a player can choose exactly one deck and remove some cards from it and the player is forced to remove at least one card. The first player to not be able to make a move, loses the game. Karthik moves first.

A bitwise XOR is a binary operation on two numbers say a and b where a, b $\in I^+ \cup \{0\}$. We convert a and b into their binary representation and make them of equal length by adding leading 0s to the smaller number. Then we take the two bit patterns and perform the logical exclusive OR operation on each pair of corresponding bits. The result in each position is 1 if only one of the bits is 1, but will be 0 if both are 0 or both are 1. For example, if we have 10(1010 in binary) and 6(0110 in binary), then 10 XOR 6 = 12 (1100 in binary).

The cumulative XOR at any point of time is the XOR of the number of cards in each deck, that is, a_1 XOR a_2 XOR a_3 .... XOR a_n.

Prove the following claims:

1. If the current cumulative XOR = 0, then any choice of move will make the cumulative XOR non-zero. [1 marks]
2. If the current cumulative XOR $\neq$ 0, then there is always a move to make the cumulative XOR = 0. [2 marks]
3. If the initial cumulative XOR = 0,(at the start of the game) then Rachna can always win the game. [4 marks]
4. If the initial cumulative XOR $\neq$ 0,(at the start of the game) then Karthik can always win the game. [3 marks]