PS 7 – all questions

PS 8 – Q1, Q2

2 November 2022

3. Find the m.g.f. of a continuous distribution whose density function is $f_X(x) = \frac{1}{2}x^2e^{-x}$, $(0 < x < \infty)$ and compute the values of mean and variance. (Ans. $M_X(t) = \frac{1}{(1-t)^3}$, $m_X = 3$, $var(X) = 3$.)

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} \frac{1}{2} x^2 e^{-x}\, dx = \frac{1}{2}\int_0^\infty x^2 e^{-x(1-t)}\, dx$$


A GUIDE TO
INTEGRATION BY PARTS:
GIVEN A PROBLEM OF THE FORM:
$\int f(x)g(x)\,dx = ?$
CHOOSE VARIABLES u AND v SUCH THAT:
$u = f(x)$
$dv = g(x)\,dx$
NOW THE ORIGINAL EXPRESSION BECOMES:
$\int u\,dv = ?$
WHICH *DEFINITELY* LOOKS EASIER.
ANYWAY, I GOTTA RUN.
BUT GOOD LUCK!

3. Find the m.g.f. of a continuous distribution whose density function is $f_X(x) = \frac{1}{2}x^2 e^{-x}, (0 < x < \infty)$ and compute the values of mean and variance. (Ans. $M_X(t) = \frac{1}{(1-t)^3}$, $m_X = 3$, $var(X) = 3$.)

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}\,\frac{1}{2}x^2 e^{-x}\,dx = \frac{1}{2}\int_0^\infty \underbrace{x^2}_{u}\,\underbrace{e^{-x(1-t)}}_{v'}\,dx$$

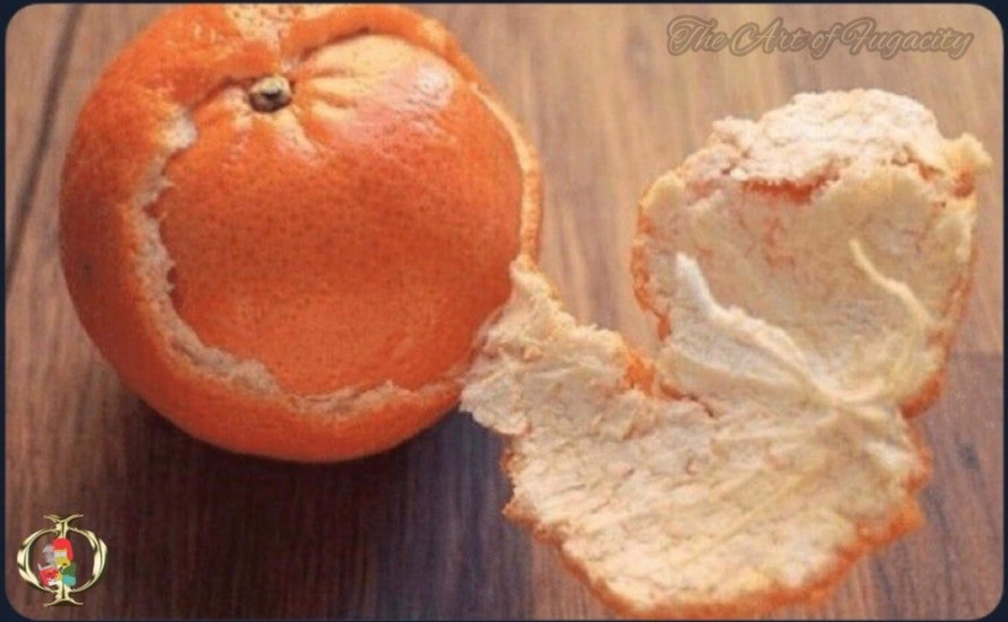
$$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx$$

$$v'(x) = e^{-x(1-t)} \quad \text{so} \quad v(x) = \frac{-1}{1-t}e^{-x(1-t)}$$

$$= \frac{1}{2}\left\{\left[x^2\,\frac{1}{t-1}\,e^{-x(1-t)}\right]_0^\infty + \int_0^\infty \frac{1}{1-t}\,e^{-x(1-t)}\cdot 2x\,dx\right\}$$

$$= \frac{1}{1-t}\int_0^\infty x\,e^{-x(1-t)}\,dx$$

When you integrate by parts then realize you need to integrate by parts again



$$M_X(t) = \frac{1}{1-t}\left\{ \left[ x\,\frac{e^{-x(1-t)}}{t-1} \right]_0^{\infty} + \int_0^{\infty} \frac{1}{1-t}\, e^{-x(1-t)}\, dx \right\}$$

$$= \frac{1}{(1-t)^2}\left[ \frac{-1}{1-t}\, e^{-x(1-t)} \right]_0^{\infty} = \frac{1}{(1-t)^2}\left[ \frac{-1}{1-t}\, e^{-x(1-t)} \right]_0^{\infty}$$

$$M_X(t) = \frac{1}{(1-t)^3}$$

$$M'_X(t) = \frac{3}{(1-t)^4} \Rightarrow M'_X(0) = 3 = m_X$$

$$M''_X(t) = \frac{12}{(1-t)^5} \Rightarrow M''_X(0) = 12 = E(X^2)$$

$$\mathrm{Var}(X) = E\{(X - m_X)^2\} = E(X^2) - (m_X)^2$$

$$\mathrm{Var}(X) = 12 - 3^2 = 3$$

4. A continuous distribution has p.d.f.

$$f_X(x) = ae^{-ax}, \; 0 < x < \infty, \; a > 0.$$

Calculate the m.g.f. and hence obtain $\alpha_k$. (Ans. $M_X(t) = \frac{a}{(a-t)}$, $\alpha_k = \frac{k!}{a^k}$.)

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} a \cdot e^{-ax} \, dx = a \int_0^\infty e^{x(t-a)} \, dx$$

$$M_X(t) = a \left[ \frac{e^{x(t-a)}}{t-a} \right]_0^\infty$$ assuming $t < a$, we get,

$$M_X(t) = \frac{-a}{t-a} = \frac{a}{a-t}$$

$\Downarrow$

$$M'_X(t) = \frac{a}{(a-t)^2} \; ; \quad M^2_X(t) = \frac{2a}{(a-t)^3} \; ; \quad M^3_X(t) = \frac{6a}{(a-t)^4}$$

$$\vdots$$

$$M^k_X(t) = \frac{\underline{|k} \; a}{(a-t)^{k+1}}$$

$\alpha_k = M_x^k(t)$ at $t = 0$

$$M_x^k(t) = \frac{\underline{\lfloor k}\, a}{(a-t)^{k+1}}$$

$$\therefore \alpha_k = \frac{\underline{\lfloor k}}{a^k}$$

5. Prove that m.g.f. of a uniform distribution over the interval $(-a, a)$ is $\frac{\sinh at}{at}$. Hence calculate the central moments. (Ans. $\mu_{2k+1} = 0$, $\mu_{2k} = \frac{a^{2k}}{2k+1}$)

Let the distribution be X. Then $X \sim$ Uniform $(-a, a)$. So, the pdf of X,

$$f_X(x) = \begin{cases} 1/2a, & \text{if } x \in (-a, a) \\ 0, & \text{otherwise} \end{cases}$$

$$M_X(t) = E(e^{tx}) = \int_{-a}^{a} \frac{e^{tx}}{2a}\, dx$$

$\int_{-a}^{a} \frac{e^{tx}}{2a}\, dx$ is convergent $\forall t \in R$: how and why?

$$\int_{-a}^{a} \frac{e^{tx}}{2a}\, dx = \begin{cases} 1, & \text{if } t = 0 \\ \dfrac{e^{at} - e^{-at}}{2at}, & \text{if } t \neq 0 \end{cases} \Rightarrow \sinh at$$

$$\therefore\ M_X(t) = \begin{cases} \dfrac{\sinh at}{at}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0 \end{cases}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

Now, find $M_X'(t)$ . . .

$$\lim_{h \to 0} \frac{M_X(0+h) - M_X(0)}{h} = \frac{\dfrac{\sin ah}{ah} - 1}{h} = \frac{\sin ah - ah}{ah^2}$$

L-Hopital

$$= \frac{a\cosh ah - a}{2ah} = \lim_{h \to 0} \frac{a^2 \sinh ah}{2a} = 0$$

$\therefore\ E(X) = M'(0) = 0$

$$\Rightarrow M_X(t) = \frac{1}{2at}\left(e^{at} - e^{-at}\right)$$

$$M_X(t) = 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \cdots$$ how ?

$\alpha_k$ = coefficient of $t^k$ in the expansion of $M_X(t)$

$$\alpha_k = \begin{cases} 0, & \text{if } k \text{ is a positive odd integer} \\ \dfrac{a^k}{k+1}, & \text{if } k \text{ is a positive even integer} \end{cases}$$

As $E(X) = 0$, $\mu_k = \alpha_k = \begin{cases} 0, & \text{if } k \text{ is a positive odd integer} \\ \dfrac{a^k}{k+1}, & \text{if } k \text{ is a positive even integer} \end{cases}$

$$M_X(t) = 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \cdots$$ how?

$\alpha_k$ = coefficient of $t^k$ in the expansion of $M_X(t)$

6. Find the mean, median and mode of $f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{2} \\ 0, & \text{elsewhere.} \end{cases}$

(Ans. mean: $m_X = 1$, median: $\mu = \frac{\pi}{3}$, mode: $\frac{\pi}{2}$)

Mean

$$\mu = E(X) = \boxed{\int_0^{\pi/2} x \sin x \, dx} = 1$$

how ?

Median

Let $k$ be the median, so,

$$\int_0^{k} \sin x \, dx = \frac{\int_0^{\pi/2} \sin x \, dx}{2}$$

$$k = \pi/3$$

$\boxed{\text{Mode}}$ the value of the random variable with the highest probability.

If $c$ is the mode, the following shall hold:

① $f'(c) = 0$ ② $f''(c) < 0$

$$\frac{d}{dx}\sin x = \cos x = 0 \Rightarrow x = \pi/2$$

$$\frac{d^2}{dx^2}\sin x = -\sin x . \text{ For } x = \pi/2, \; -\sin x = -1 < 0$$

$\therefore c = \pi/2$ is the mode

7. Show that the mode of a Poisson distribution with mean $\mu$ is the integer or integers which are determined by $\mu - 1 \leq M \leq \mu$.

$$\frac{P(X=x)}{P(X=x-1)} = \frac{e^{-\mu} \cdot \mu^{x}}{\lfloor x} \cdot \frac{\lfloor x-1}{e^{-\mu} \mu^{x-1}} = \frac{\mu}{x}$$

$f_i$: frequency

$f_{i-1} < f_i > f_{i+1}$

If $\mu \geqslant x$, $P(X=x) \geqslant P(X=x-1)$

If $\mu = x$, $P(X=x) = P(X=x-1)$

If $\mu < x$, $P(X=x) < P(X=x-1)$

Also, $\frac{P(X=x+1)}{P(X=x)} = \frac{\mu}{x+1}$

If $\mu \leq x+1$, $P(X=x+1) \geqslant P(X=x)$

$$x \leq \mu \leq x+1$$

$$0 \leq \mu - x \leq 1$$

$$-1 \leq x - \mu \leq 0$$

$$\mu - 1 \leq x \leq \mu$$

8. Calculate the first absolute moment about the mean and the semi-interquartile range for Laplace distribution with p.d.f.

$$f_X(x) = \frac{1}{2\lambda} e^{-|x-\mu|/\lambda}, -\infty < x < \infty, \lambda > 0.$$

(Ans. $m_X = \mu$, $E(|X-\mu|) = \lambda$, semi-interquartile range$= \lambda \log 2$ )

$$\boxed{m_X} = E(X) = \frac{1}{2\lambda}\int_{-\infty}^{\infty} x\, e^{-\frac{|x-\mu|}{\lambda}}\, dx$$

$$= \frac{1}{2\lambda}\int_{-\infty}^{\infty} (x-\mu)\, e^{-\frac{|x-\mu|}{\lambda}}\, dx + \frac{\mu}{2\lambda}\int_{-\infty}^{\infty} e^{-\frac{|x-\mu|}{\lambda}}\, dx$$

substitute $x-\mu = z \Rightarrow dx = dz$

$$= \frac{1}{2\lambda}\boxed{\cancel{\int_{-\infty}^{\infty} z \cdot e^{-\frac{|z|}{\lambda}}\, dz}} + \frac{\mu}{2\lambda}\int_{-\infty}^{\infty} e^{-\frac{|z|}{\lambda}}\, dz$$

$\Downarrow$

0 . why ?

$$= \frac{\mu}{\lambda} \int_0^\infty e^{-\frac{z}{\lambda}} dz = \frac{\mu}{z} \lambda \int_0^\infty e^{-t} dt \qquad t = \frac{z}{\lambda}$$

$$M_x = \mu \, \Gamma(1) = \mu$$

$$\boxed{E(|X - m_x|)} = E(|X-\mu|) = \frac{1}{2\lambda} \int_{-\infty}^{\infty} |x-\mu| \, e^{-\frac{|x-\mu|}{\lambda}} dx$$

$$z = x - \mu$$

$$= \frac{1}{2\lambda} \int_{-\infty}^{\infty} |z| \, e^{-\frac{|z|}{\lambda}} dz = 2 \cdot \frac{1}{2\lambda} \int_0^\infty z \cdot e^{-\frac{z}{\lambda}} dz$$

$$= \lambda \int_0^\infty x \cdot e^{-x} dx = \lambda \, \Gamma(2) = \lambda$$

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt$$

$$\boxed{F_X(x)} = \text{df. of } X = \frac{1}{2\lambda}\int_{-\infty}^{x} e^{-\frac{|x-\mu|}{\lambda}}\,dx$$

If $x<\mu$, $F_X(x) = \frac{1}{2\lambda}\int_{-\infty}^{x} e^{\frac{x-\mu}{\lambda}}\,dx = \frac{\lambda}{2\lambda}\left[e^{\frac{x-\mu}{\lambda}}\right]_{-\infty}^{x}$

$$F_X(x) = \frac{1}{2}e^{\frac{x-\mu}{\lambda}}$$

If $x>\mu$, $F_X(x) = \frac{1}{2\lambda}\int_{-\infty}^{\mu} e^{-\frac{x-\mu}{\lambda}}\,dx + \frac{1}{2\lambda}\int_{\mu}^{x} e^{\frac{\mu-x}{\lambda}}\,dx$

$$F_X(x) = \frac{1}{2}\left[e^{\frac{x-\mu}{\lambda}}\right]_{-\infty}^{\mu} + \frac{\lambda}{2\lambda}\left[-e^{\frac{\mu-x}{\lambda}}\right]_{\mu}^{x}$$

$$F_X(x) = \frac{1}{2} - \frac{1}{2}e^{\frac{\mu-x}{\lambda}} + \frac{1}{2} = 1 - \frac{1}{2}e^{\frac{\mu-x}{\lambda}}$$

$$F_X(x) = \begin{cases} \frac{1}{2} e^{\frac{x-\mu}{\lambda}}, & x < \mu \\ 1 - \frac{1}{2} e^{\frac{\mu - x}{\lambda}}, & x \geq \mu \end{cases}$$

$F_X(\mu) = \frac{1}{2} \Rightarrow \mu$ is the median

$\Rightarrow \quad \xi_{1/4} < \mu < \xi_{3/4}$ percentile

$$F_X(\xi_{1/4}) = \frac{1}{4}$$

$$\frac{1}{2} e^{\frac{\xi_{1/4} - \mu}{\lambda}} = \frac{1}{4}$$

$$\xi_{1/4} - \mu = -\lambda \log 2$$

$$F_X(\xi_{3/4}) = 3/4$$

$$1 - \frac{1}{2} e^{\frac{\mu - \xi_{3/4}}{\lambda}} = 3/4$$

$$e^{\frac{\mu - \xi_{3/4}}{\lambda}} = \frac{1}{2}$$

$$-\mu + \xi_{3/4} = \lambda \log 2$$

$\therefore \quad Q_{3/4} - Q_{1/4} = 2\lambda \log 2$

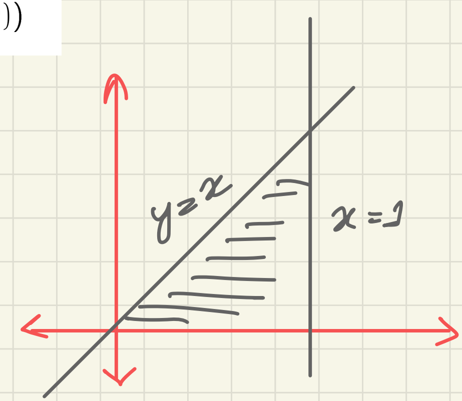
✓ $\boxed{\text{Semi-interquartile range}} = \frac{1}{2}\left[Q_{3/4} - Q_{1/4}\right]$

$= \lambda \log 2$

1. Determine the value of k which makes

$$f(x,y) = \begin{cases} kxy, & 0 < x < 1,\ 0 < y < x \\ 0, & \text{elsewhere.} \end{cases}$$

a joint p.d.f. Calculate the marginal p.d.f. s and show that the variates are dependent. (Ans. $k = 8$, $f_X(x) = 4x^3$, $f_Y(y) = 4y(1-y^2)$)




$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_{x=0}^{1}\int_{y=0}^{x} kxy\,dx\,dy = 1$$

$$k\int_{x=0}^{1} x \int_{y=0}^{x} y\,dy = 1 \Rightarrow k\int_{x=0}^{1} x\cdot\frac{x^2}{2}\,dx = 1 \Rightarrow k = 8$$

$$f(x,y) = 8xy \quad ; \quad 0<x<1, \quad 0<y<x$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{x} 8xy\,dy = 4x^3 , \quad 0<x<1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{y}^{1} 8xy\,dx = 4y\,(1-y^2) , \quad 0<y<1$$

$$f(x,y) \neq f_X(x)\, f_Y(y) \Rightarrow$$

2. If

$$f(x,y) = \begin{cases} 3x^2 - 8xy + 6y^2, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

find $f_X(x|y)$ and $f_Y(y|x)$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 [3x^2 - 8xy + 6y^2]\,dy, \quad 0 < x < 1$$

$$= 3x^2\, y\Big|_0^1 - 4x\, y^2\Big|_0^1 + 2y^3\Big|_0^1$$

$$f_X(x) = 3x^2 - 4x + 2, \quad 0 < x < 1$$

$$f_Y(y) = \int_0^1 f(x,y)\,dx, \quad 0 < y < 1$$

$$f_Y(y) = \int_0^1 (3x^2 - 8xy + 6y^2)\,dx = 1 - 4y + 6y^2, \quad 0 < y < 1$$

$$f_X(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{3x^2 - 8xy + 6y^2}{1 - 4y + 6y^2}, \quad 0 < x < 1$$

&

$y$ fixed in $(0,1)$

$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{3x^2 - 8xy + 6y^2}{3x^2 - 4x + 2}, \quad 0 < y < 1$$

&

$x$ fixed in $(0,1)$