Convergence of a Sequence of Random Variables

Limit Theorems

① Convergence in Probability: $\lim_{n\to\infty} P(|X_n - a| \geq \epsilon) = 0$ for any $\epsilon > 0$

then $X_n \xrightarrow[\text{in } P]{} a$ as $n \to \infty$

② Convergence in Distribution: $\lim_{n\to\infty} F_n(x) = F(x)$

then $X_n \xrightarrow[\text{in } L]{} X$

Asymptotically Normal Distribution

$X \sim N(m, \sigma)$
$\Rightarrow \frac{X-m}{\sigma} \sim N(0,1)$

Let $\{X_n\}$ be a sequence of r.v.s and $\{a_n\}, \{b_n\}$ be two real sequences.

Let, $F_n(x)$ be the distribution function of $\dfrac{X_n - a_n}{b_n}$ $\forall n$

If $\lim_{n\to\infty} F_n(x) = \Phi(x)$ (converges pointwise $\forall x$)

$\hookrightarrow$ d.f. of standard normal variate,

then X_n is said to be asymptotically normal (a_n, b_n).

Note: 1. a_n and b_n are not necessarily the mean and S.D. of the variate X_n.

2. X_n is asymptotically normal (a_n, b_n)

$$\Rightarrow \frac{X_n - a_n}{b_n} \sim N(0,1) \text{ for large } n$$

$$\Downarrow$$

check! using transformation of r.v.

or $X_n \sim N(a_n, b_n)$ for large n.

Limit Theorem of Characteristic Functions

Let $X_1, X_2, \ldots, X_n, \ldots$ be a seq. of r.v.s having d.f.s $F_1(x), F_2(x), \ldots F_n(x), \ldots$ and the ch. functions $\psi_1(t), \psi_2(t), \ldots, \psi_n(t), \ldots$, respectively.

A necessary and sufficient condn that $\boxed{\lim_{n\to\infty} F_n(x) = F(x)}$ at "every point of continuity" of F $\Leftrightarrow$ is that for any real t, $\boxed{\lim_{n\to\infty} \psi_n(t) = \psi(t)}$ where $\psi(t)$ is the ch. fn. of the distribution determined by the d.f. $F(x)$ and $\psi(t)$ is continuous at $t=0$.

Prob. 9 (PS 11)

Show that Poisson distribution can be obtained as a limit of Binomial Distribution.

Sol. Let, $X_n \sim \text{Binomial}(n, p)$ where $0 < p < 1$, $n \in \mathbb{N}$

$\psi_n(t) =$ ch. fn. of X_n

$$= \left(pe^{it} + 1 - p\right)^n, \quad i = \sqrt{-1}.$$

$$\lim_{n\to\infty} \psi_n(t) = \lim_{n\to\infty} \left(\frac{\mu}{n} e^{it} + 1 - \frac{\mu}{n}\right)^n$$ where $\boxed{\mu = np\ (>0)}$

⊛ μ being a fixed +ve no.

$$= \lim_{n\to\infty} \left\{1 + \frac{\mu}{n}\left(e^{it} - 1\right)\right\}^n$$

$$= \lim_{n\to\infty} \left[\left\{ 1 + \frac{\mu(e^{it}-1)}{n} \right\}^{\frac{n}{\mu(e^{it}-1)}} \right]^{\mu(e^{it}-1)}$$

$$= e^{\mu(e^{it}-1)}$$

since $\boxed{\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{\frac{n}{x}} = e}$ if $x \neq 0$

where $\psi(t) = e^{\mu(e^{it}-1)}$ is the Ch. fn of a Poisson(μ) variate. Also $\psi(t)$ is continuous at $t=0$.

Using, L. Th. of Ch. fns $\{X_n\}$ converge in distribution to Poisson(μ) variate as $n \to \infty$, if

$\boxed{np = \mu \text{ is a fixed constant.}}$

Note: If p is very small and n is large then np is of moderate magnitude (and fixed in approximate sense) then above approximation holds.

But if p is not small, then we apply DeMoivre Laplace Limit Th., will be discussed later).

Th: (Central Limit Theorem for Equal Components)

Let ① $\{X_n\}$ be a seq. of r.v.s each having same distribution with finite mean m and S.D. σ (>0).

② $X_1, X_2, \ldots, X_n$ are mutually independent $\forall n$.

Then $\dfrac{\overline{X}-m}{\sigma/\sqrt{n}}$ is asymptotically normal $(0,1)$.

or $\overline{X}$ . . . . normal $\left(m, \sigma/\sqrt{n}\right)$

where $\overline{X} = \dfrac{X_1+X_2+\cdots+X_n}{n}$.

i.e. $\lim\limits_{n\to\infty} F_n(x) = \Phi(x)$ $\forall x$

where $F_n(x)$ is the d.f. of $\dfrac{\overline{X}-m}{\sigma/\sqrt{n}}$

Pf: Try yourself!

Apply Limit Th. of Ch. fn.

$\dfrac{X_1-m}{\sigma} \to \psi(t)$

$\dfrac{\overline{X}-m}{\sigma/\sqrt{n}} \to \left\{\psi\left(t/\sigma\right)\right\}^n$

CLTh (for equal components) $\Rightarrow$ Law of Large numbers (for equal components)

(independent & identical distn)

$\{X_n\}$: iid with $E(X_i) = m$, $\sigma(X_i) = \sigma$, $\forall i$

Let, $Y_n = \dfrac{\overline{X} - m}{\sigma/\sqrt{n}}$ has d.f. $F_n(x)$

By C.L.Th, $\lim\limits_{n \to \infty} F_n(x) = \Phi(x) \;\; \forall x.$

To show $\overline{X} \xrightarrow[\text{in } P]{} m$

i.e. For any $\epsilon > 0$:

$$\lim_{n \to \infty} P\left(|\overline{X} - m| \geq \epsilon\right) = 0$$

.u.

$$P\left(|\bar{X}-m| \geq \epsilon\right) = P\left(\left|\frac{\bar{X}-m}{\sigma/\sqrt{n}}\right| \geq \frac{\sqrt{n}\,\epsilon}{\sigma}\right)$$

$$= P\left(|Y_n| \geq \frac{\sqrt{n}\,\epsilon}{\sigma}\right) \leq P\left(|Y_n| > b\right)$$

$$\left(\text{provided } b < \frac{\sqrt{n}\,\epsilon}{\sigma} \Rightarrow n > \frac{b^2\sigma^2}{\epsilon^2}\right)$$

$$\Rightarrow P\left(|\bar{X}-m| \geq \epsilon\right) \leq 1 - P\left(|Y_n| \leq b\right)$$

$$= 1 - P\left(-b \leq Y_n \leq b\right)$$

$$\leq 1 - P\left(-b < Y_n \leq b\right)$$

$$\Rightarrow P\left(|\bar{X}-m| \geq \epsilon\right) \leq 1 - F_n(b) + F_n(-b).$$

taking lim $n \to \infty$

(using CLTh)

$$0 \leq \lim_{n\to\infty} P\left(|\bar{X}-m| \geq \epsilon\right) \leq 1 - \Phi(b) + \Phi(-b) = 2\{1-\Phi(b)\}$$

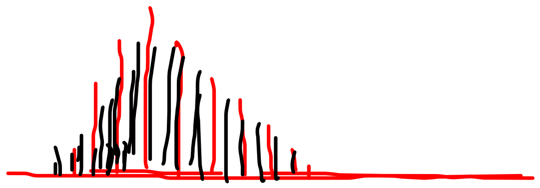
Since this is true for any $b > 0$, taking $b \to \infty$

$$\lim_{n\to\infty} P\left(|\bar{X}-m| \geq \epsilon\right) = 0. \Rightarrow \bar{X} \xrightarrow[\text{in } P]{} m \text{ as } n \to \infty.$$

DeMoivre-Laplace Limit Theorem

If $X_n \sim \text{Binomial}(n,p)$ $\forall n \in \mathbb{N}$ $(0<p<1)$, then

$$\left\{ \frac{X_n - np}{\sqrt{np(1-p)}} \right\}$$

is convergent in distribution to a standard normal variate as $n \to \infty$, 

i.e. $\underset{n\to\infty}{\text{Lt}}\, F_n(x) = \Phi(x) \quad \forall x$

where $F_n(x)$ is the d.f. of $\dfrac{X_n - np}{\sqrt{np(1-p)}}$ $\forall n$.

Proof: $X_n \sim$ Binomial (n, p)

By the reproductive property of Binomial distribution

$$X_n = Y_1 + Y_2 + \cdots + Y_n$$

where $Y_1, Y_2, \ldots, Y_n$ are mutually independent

and $Y_i \sim$ Binomial $(1, p)$ $\forall\, i = 1, 2, \ldots, n$

$$E(Y_i) = p, \quad \sigma(Y_i) = \sqrt{p(1-p)}$$

Applying C.L.Th, $\dfrac{\dfrac{Y_1 + Y_2 + \cdots + Y_n}{n} - p}{\sqrt{p(1-p)}/\sqrt{n}}$ is asymptotically normal $(0, 1)$

$$\Rightarrow \left\{ \frac{X_n - np}{\sqrt{np(1-p)}} \right\} \text{ is asymptotically normal } (0,1)$$

$$\Rightarrow \left\{ \frac{X_n - np}{\sqrt{np(1-p)}} \right\} \text{ converges in distribution to a standard normal variate as } n \to \infty$$

For any $\epsilon > 0$

$$P(|\bar{X}-m| \geq \epsilon) = P\left(\left|\frac{\bar{X}-m}{\sigma/\sqrt{n}}\right| \geq \frac{\sqrt{n}\,\epsilon}{\sigma}\right)$$

$$= P\left(|Y_n| \geq \frac{\sqrt{n}\,\epsilon}{\sigma}\right) \leq P(|Y_n| > b)$$

provided (*) $\underbrace{b < \frac{\sqrt{n}\,\epsilon}{\sigma}}$ or $n > \frac{b^2\sigma^2}{\epsilon^2}$.

$$\Rightarrow P(|\bar{X}-m| \geq \epsilon) \leq 1 - P(|Y_n| \leq b)$$

$$\leq 1 - P(-b < Y \leq b).$$

$$= 1 - \{F_n(b) - F_n(-b)\} \quad \text{if } n > \frac{b^2\sigma^2}{\epsilon^2}$$

Taking $n \to \infty$, $0 \leq P(|\bar{X}-m| \geq \epsilon) \leq \Phi(-b) - \{1 - \Phi(b)\}$

$$= 2\{1 - \Phi(b)\}$$

This is true for any $b > 0$, so taking $b \to \infty$

$$\lim_{n\to\infty} P(|\bar{X} - m| \geq \epsilon) = 0$$

$$\Rightarrow \bar{X} \xrightarrow[\text{in } P]{} m \quad \text{as } n \to \infty$$

Law of Large No (for equal components).