eg-① Find $100\times(1-\epsilon)\%$ CI for the parameter m of a normal (m, σ) distribution.

Case ①:- σ is known.

① Choose statistic $\left(\frac{\bar{X}-m}{\sigma/\sqrt{n}}\right) = u$, where sampling distribution of this distribution is $N(0,1)$.

~~We know th~~ $\therefore$ $\boxed{u \sim \text{Normal}(0,1)}$

② Choose pts $\pm u_\epsilon$ symmetrically about the origin s.t. $P(-u_\epsilon < U < u_\epsilon) = 1-\epsilon$.

symmetric abt origin coz we need least length of interval.

$\therefore P\left(-u_\epsilon < \frac{\bar{x}-m}{\sigma/\sqrt{n}} < u_\epsilon\right) = 1-\epsilon$.

this will constitute the C.I.

$\Rightarrow P\left(\bar{x} - \frac{\sigma u_\epsilon}{\sqrt{n}} < m < \bar{x} + \frac{\sigma u_\epsilon}{\sqrt{n}}\right) = 1-\epsilon$.

sample mean.

$\therefore 100\times(1-\epsilon)\%$ C.I. $= \left(\bar{x} - \frac{\sigma u_\epsilon}{\sqrt{n}},\ \bar{x} + \frac{\sigma u_\epsilon}{\sqrt{n}}\right)$

sample size.

Case ② σ unknown.

① Choose statistic $t = \frac{\bar{x}-m}{S/\sqrt{n}}$ where $S^2 = \frac{n s^2}{n-1}$

s^2 sample variance.

Sampling distribution of t is t-distribution with $(n-1)$ degrees of freedom.

To find $100 \times (1-\epsilon)\%$ confidence interval for m, we find two real nos $\pm t_\epsilon$ (from the table of t-distributions) s.t.

$$P\left(-t_\epsilon < \frac{\bar{x}-m}{S/\sqrt{n}} < t_\epsilon\right) = 1-\epsilon.$$

$$\Rightarrow P\left(\bar{x} - \frac{St_\epsilon}{\sqrt{n}} < m < \bar{x} + \frac{St_\epsilon}{\sqrt{n}}\right) = 1-\epsilon.$$

$$\therefore \boxed{C.I. = \left(\bar{x} - \frac{St_\epsilon}{\sqrt{n}},\ \bar{x} + \frac{St_\epsilon}{\sqrt{n}}\right)}.$$

eg. ② Find $100 \times (1-\epsilon)\%$ C.I. for the parameter σ of a normal (m, σ).

$X \sim$ Normal (m, σ).

$(x_1, x_2, \ldots, x_n) \Rightarrow$ random sample of size n.

Choose statistic $\chi^2 = \frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$.

So we choose $\chi^2_{\epsilon_1}$, $\chi^2_{\epsilon_2}$ from table of distributions such that,

$$P(\chi^2_{\epsilon_1} < \chi^2 < \chi^2_{\epsilon_2}) = 1-\epsilon.$$

$$\Rightarrow P\left(\chi^2_{\epsilon_1} < \frac{nS^2}{\sigma^2} < \chi^2_{\epsilon_2}\right) = 1-\epsilon.$$

$$\Rightarrow P\left(\frac{1}{\chi^2_{\epsilon_1}} > \frac{\sigma^2}{nS^2} > \frac{1}{\chi^2_{\epsilon_2}}\right) = 1-\epsilon.$$

$$\Rightarrow P\left(\frac{nS^2}{\chi^2_{\epsilon_1}} > \sigma^2 > \frac{nS^2}{\chi^2_{\epsilon_2}}\right) = 1-\epsilon$$

$$\therefore \boxed{C.I. = \left(\sqrt{n}\cdot\frac{S}{\sqrt{\chi^2_{\epsilon_2}}},\ \sqrt{n}\cdot\frac{S}{\sqrt{\chi^2_{\epsilon_1}}}\right)}$$

* Now how to choose $\chi^2_{\epsilon_1}$ & $\chi^2_{\epsilon_2}$?

In normal, we chose symmetric abt origin to reduce the interval length.

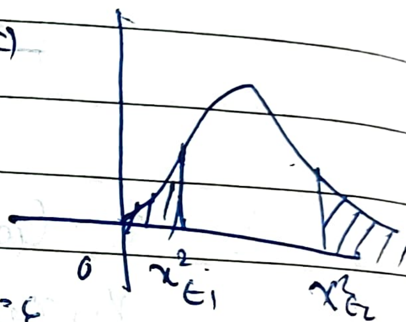
We follow that here also.

Choose as close to ~~mean~~ median as possible, and probabilities are symmetric abt. it.

∴ We choose $\chi^2_{\epsilon_1}$, $\chi^2_{\epsilon_2}$ s.t.

$$P(0<\chi^2<\chi^2_{\epsilon_1}) = P(\chi^2_{\epsilon_2}<\chi^2<\infty)$$

symmetric.



$$P(\chi^2_{\epsilon_1}<\chi^2<\chi^2_{\epsilon_2}) = 1-\epsilon$$

$$\Rightarrow P(0<\chi^2<\chi^2_{\epsilon_1}) + P(\chi^2_{\epsilon_2}<\chi^2<\infty) = \epsilon$$

$$\Rightarrow \boxed{P(0<\chi^2<\chi^2_{\epsilon_1}) = \frac{\epsilon}{2}} \quad \text{and} \quad \boxed{P(\chi^2>\chi^2_{\epsilon_2}) = \frac{\epsilon}{2}}$$

$$\Downarrow$$

$$\boxed{P(0<\chi^2<\chi^2_{\epsilon_2}) = 1-\frac{\epsilon}{2}}$$

~~eg. 3 Find an approximate 100 × (1 − ε) % C.I. for p for a binomial (N, p) population.~~

Continuing the notes (Interval Estimation).

eg. ② Find an approximate $100 \times (1-\varepsilon)\%$ C.I. for p for a binomial (N, p) population.

$X \sim \text{Binomial}(N, p)$.

Consider a random sample of size $1 : \boxed{x_1}$.

From DeMoivre Laplace Limit Thm.

$$U = \frac{X - np}{\sqrt{npq}}$$ is approximately normal.

We have shown previously that
$\hat{p} = \frac{\bar{x}}{N}$ is a good estimate (MLE) of p.

$\therefore$ $\hat{p} = \frac{x_1}{N}$ is a good estimate.

Thus $\sqrt{Npq} = \sqrt{N \frac{x_1}{N}\left(1-\frac{x_1}{N}\right)} = \sqrt{\frac{x_1(N-x_1)}{N}}$.

$\therefore$ $\frac{X-np}{\sqrt{\frac{x_1(N-x_1)}{N}}}$ is approximately normal.

$\therefore$ For finding C.I., we choose points $\pm u_\epsilon$ from the table of Normal (0,1) s.t.

$$P\left(-u_\epsilon < \frac{X-Np}{\sqrt{\frac{x_1(N-x_1)}{N}}} < u_\epsilon\right) \simeq 1-\epsilon$$

$$\Rightarrow P\left(\frac{X}{N} - u_\epsilon\sqrt{\frac{x_1(N-x_1)}{N^3}} < p < \frac{X}{N} + u_\epsilon\sqrt{\frac{x_1(N-x_1)}{N^3}}\right) =$$

$\therefore$ $100\times(1-\epsilon)\%$ C.I. $\Rightarrow$

$$\boxed{\left(\frac{x_1}{N} - u_\epsilon\sqrt{\frac{x_1(N-x_1)}{N^3}},\ \frac{x_1}{N} + u_\epsilon\sqrt{\frac{x_1(N-x_1)}{N^3}}\right)}$$

P.S. 3

Q1,2,3 are the eg.s discussed in theory.

Q4. To find $\Rightarrow$ 95% C.I. for the mean of Normal Distribution.
In Q1 we showed that C.I. is

$\left(\bar{x} - \frac{\sigma u_\epsilon}{\sqrt{n}},\ \bar{x} + \frac{\sigma u_\epsilon}{\sqrt{n}}\right)$ s.t. $\bar{x}$ = sample mean
σ = Population Variance
n = Sample size.

$$\bar{x} = \frac{2.3-0.2-0.4-0.9}{4} = \boxed{0.2}.$$

u_ε is chosen s.t. $P\left(-u_\varepsilon < \frac{\bar{x}-m}{\sigma/\sqrt{n}} < u_\varepsilon\right) = 1-\varepsilon$.

$\Rightarrow \Phi(u_\varepsilon) - \Phi(-u_\varepsilon) = 0.95$

$\Rightarrow 2\Phi(u_\varepsilon) - 1 = 0.95$

$\Rightarrow \boxed{\Phi(u_\varepsilon) = 0.975}$

$\therefore \boxed{u_\varepsilon = 1.96}$ $\Rightarrow$ from the ques^n.

$\therefore$ C.I. $\Rightarrow \left(0.2 - \frac{3\times1.96}{2},\ 0.2 + \frac{3\times1.96}{2}\right)$

$\boxed{\text{C.I.} \Rightarrow (-2.74, 3.14)}$

Q5. Let p be the true proportion of defectives in the population.

Population R.V. $X \sim$ Binomial (n, p). $\boxed{N = 400}$.

We have one sample. 40 out of 400 are defective.

$\therefore \hat{p} = \frac{x_1}{N}$ is a good estimate. $\boxed{x_1 = 40}$

$\therefore \hat{p} = \frac{40}{400} = \boxed{0.1}$. The no. of defectives.

The confidence interval exp^n is

$$\left(\frac{x_1}{N} - u_\varepsilon\sqrt{\frac{x_1(N-x_1)}{N^3}},\ \frac{x_1}{N} + u_\varepsilon\sqrt{\frac{x_1(N-x_1)}{N^3}}\right).$$

$x_1 = 40$, $N = 400$, $u_\varepsilon = 1.96$ (similar to prev.)

$$\therefore \text{C.I.} \Rightarrow \left(0.1 - 1.96\sqrt{\frac{40\cdot360}{400\cdot400\cdot400}},\ 0.1 + 1.96\sqrt{\frac{40\cdot360}{400\cdot400\cdot400}}\right)$$

$$\Rightarrow \left(0.1 - 1.96\times\frac{6}{400},\ 0.1 + 1.96\times\frac{6}{400}\right)$$

$\Rightarrow (0.1 - 0.0294,\ 0.1 + 0.0294)$

$\Rightarrow \boxed{(0.0706,\ 0.1294)}$ Ans

Q7. Population is normal $(m, 5.2)$. σ known.
Sample size = 20. Observed mean $\bar{x} = 16.9$.

C.I. $\Rightarrow \left(\bar{x} - \frac{\sigma u_\epsilon}{\sqrt{n}}, \bar{x} + \frac{\sigma u_\epsilon}{\sqrt{n}}\right)$

Plugging values, $u_\epsilon = 1.96$.

C.I. $\Rightarrow \left(16.9 - \frac{5.2(1.96)}{\sqrt{20}}, 16.9 + \frac{5.2 \times 1.96}{\sqrt{20}}\right)$

$\Rightarrow (16.9 - 2.279, 16.9 + 2.279)$

$\Rightarrow \boxed{(14.621, 19.179)}$ Ans.

Q8. ① Normal(m, σ) distribution. m, σ are unknown.

$\therefore$ C.I. when σ is unknown $\Rightarrow \left(\bar{x} - \frac{s u_\epsilon}{\sqrt{n}}, \bar{x} + \frac{s u_\epsilon}{\sqrt{n}}\right)$

where $s^2 = \left(\frac{n}{n-1}\right) S^2$.

$\boxed{u_\epsilon \text{ is from } t(n-1) \text{ distribution}}$

Sample size $n = 10$.
Sample mean $\bar{x} = 68$.
Sample Variance $(S^2) = \frac{25+4+25+1+0+1+4+9+16+9}{10}$

$= \frac{94}{10} = 9.4$.

$\therefore s^2 = \left(\frac{n}{n-1}\right) S^2 \Rightarrow s^2 = \frac{10 \times 9.4}{9} = 10.444\ldots$

u_ϵ is for the t-distribution. Symmetric abt origin. $\therefore$ follows properties similar to normal distribution.

$\Rightarrow T(u_\epsilon) - T(-u_\epsilon) = 1 - \epsilon$

$\Rightarrow T(u_\epsilon) - T(-u_\epsilon) = 0.95$

$\Rightarrow 2T(u_\epsilon) - 1 = 0.95$

$\Rightarrow \boxed{T(u_\epsilon) = 0.975}$ From ques^n, $u_\epsilon = 2.262$.

Plugging values, C.I. = $\left(68 - \sqrt{\frac{94}{9}} \frac{2.262}{\sqrt{10}}, 68 + \sqrt{\frac{94}{9}} \times \frac{2.262}{\sqrt{10}}\right)$

~~$= (68 - 7.875, 68 + 7.875)$~~

$$= \cancel{(60.125, 75.875)}$$

$$= \left(68 - \frac{\sqrt{94} \times 2.262}{\sqrt{90}}, 68 + \frac{\sqrt{94} \times 2.262}{\sqrt{90}}\right)$$

$$\cancel{= 68 - 2.436}$$

$$= (68 - 2.312, 68 + 2.312)$$

$$= (65.688, 70.312) \text{ Ans.}$$

(2) For parameter σ, derived C.I. is.

$$C.I. = \left(\frac{\sqrt{n} \cdot S}{\chi^2_{\epsilon_2}}, \frac{\sqrt{n} \cdot S}{\chi^2_{\epsilon_1}}\right)$$

where $P(0 < \chi^2 < \chi^2_{\epsilon_1}) = \frac{\epsilon}{2} \Rightarrow \boxed{P(\chi^2 > \chi^2_{\epsilon_1}) = 1 - \frac{\epsilon}{2}}$

$P(0 < \chi^2 < \chi^2_{\epsilon_2}) = 1 - \frac{\epsilon}{2} \Rightarrow \boxed{P(\chi^2 > \chi^2_{\epsilon_2}) = \frac{\epsilon}{2}}$

$\epsilon = 0.05 \Rightarrow \boxed{\epsilon/2 = 0.025}$.

$P(\chi^2 > \chi^2_{\epsilon_1}) = 0.975$. $\therefore \boxed{\chi^2_{\epsilon_1} = \cancel{19.023}\ 2.7}$

$P(\chi^2 > \chi^2_{\epsilon_2}) = 0.025$. $\therefore \boxed{\chi^2_{\epsilon_2} = 19.023}$

$$\therefore C.I. = \left(\frac{\sqrt{10} \cdot S}{\sqrt{19.023}}, \frac{\sqrt{10} \cdot S}{\sqrt{2.7}}\right)$$

where $\boxed{S = \sqrt{9.4}}$

$\Rightarrow \cancel{C.I. = (1.5626, 11.01)}$ $\Rightarrow \boxed{C.I. = 2.223, 5.9}$.

$\cancel{\Rightarrow C.I. = 0.509}$

Q9. Weights $\Rightarrow$ 7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17.

$\boxed{n = 12}$.

$\boxed{\sigma = 5}$.

$\boxed{\bar{x} = 15}$.

$$C.I. \Rightarrow \left(\bar{x} - \frac{\sigma u_\epsilon}{\sqrt{n}}, \bar{x} + \frac{\sigma u_\epsilon}{\sqrt{n}}\right) \quad \text{where } u_\epsilon = 1.96.$$

$$\Rightarrow (15 - 2.829, 15 + 2.829) \Rightarrow \boxed{(12.171, 17.829)}$$

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Q10. ~~[illegible]~~

Sample mean = 57, = $\bar{x}$. Sample variance = 64.
S^2

$$s^2 = \frac{n}{n-1} \cdot S^2 \rightarrow s^2 = \frac{17}{16} \cdot 64^4 \Rightarrow \boxed{68 = s^2}.$$

$$C.I. = \left(\bar{x} - \frac{s t_\epsilon}{\sqrt{n}}, \bar{x} + \frac{s t_\epsilon}{\sqrt{n}} \right)$$

We want $\Phi(t_\epsilon) - \Phi(-t_\epsilon) = 1 - \epsilon = 0.99$.

$$\Rightarrow \Phi(t_\epsilon) = \frac{1.99}{2}$$

$$\Rightarrow \boxed{\Phi(t_\epsilon) = 0.995}.$$

From tables, $t_\epsilon = 2.921$.

$$\therefore C.I. = \left(57 - \frac{\sqrt{68} \cdot 2.921}{\sqrt{17}}, 57 + \frac{\sqrt{68} \cdot 2.921}{\sqrt{17}} \right)$$

$$= (57 - 5.842, 57 + 5.842)$$

$$\boxed{C.I. = (51.158, 62.842)}.$$

Q6. $f(x;\alpha) = \begin{cases} \frac{2}{\alpha^2}(\alpha - x), & 0<x<\alpha. \\ 0, & \text{elsewhere}. \end{cases}$

We have to use the statistic $y = \left(\frac{\alpha - x}{\alpha}\right)$.

$\therefore$ Find distribution of $\boxed{Y = \frac{\alpha - X}{\alpha}}$.

$$\frac{dy}{dx} = \frac{-1}{\alpha} < 0 \qquad \therefore f_Y(y) = f_X(x) \cdot \left| \frac{dx}{dy} \right|$$

$$= \frac{2(\alpha - x)}{\alpha^2} \cdot \alpha$$

$$\boxed{f_Y(y) = 2y}. \qquad \leftarrow \qquad = \frac{2}{\alpha} \underbrace{(\alpha - x)}_{y}$$

Limits $\Rightarrow$ $0<x<\alpha \Rightarrow 0 < \alpha - x < \alpha$

$$\Rightarrow \boxed{0<y<1}.$$

$$\therefore \boxed{f_Y(y) = \begin{cases} 2y, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}}$$

Since Y contains α in the exp^n and is independent of α, we can use this as a Statistic.

$\boxed{1-\varepsilon = 0.95}$ ~~we now need to choose y_1 & y_2 s.t.~~

Let the confidence interval is (Y_1, Y_2). It should be chosen s.t. $P(Y_1 < Y < Y_2) = 0.95$

& $P(Y \le y_1) = 0.025 = P(Y \ge y_2)$.

$$P(Y \le y_1) = \int_0^{y_1} 2y\, dy \Rightarrow y_1^2 = 0.025$$
$$\Rightarrow \boxed{y_1 = \sqrt{0.025}}$$

$$P(Y \ge y_2) = 1 - \int_0^{y_2} 2y\, dy = 1 - y_2^2 = 0.025$$
$$\Rightarrow y_2^2 = 0.975$$
$$\Rightarrow \boxed{y_2 = \sqrt{0.975}}$$

$$\therefore P(\sqrt{0.025} < Y < \sqrt{0.975}) = 0.95$$
$$\Rightarrow P\left(\sqrt{0.025} < \frac{\alpha - x}{\alpha} < \sqrt{0.975}\right) = 0.95$$
$$\Rightarrow P\left(\frac{X}{1-\sqrt{0.025}} < \alpha < \frac{X}{1-\sqrt{0.975}}\right) = 0.95.$$

$$\therefore \boxed{95\%\ \text{C.I. of } \alpha = \left(\frac{x}{1-\sqrt{0.025}}, \frac{x}{1-\sqrt{0.975}}\right)}$$