Continuous random variable

- Can take any num. val within some interval
- A continuous distribut^n is characterized by its probability density func^n

$$P(a < X \leq b) = \int_a^b f(x)\,dx$$

Probability density func^n

~~just like PMF in discrete R.V.~~

Every PDF $f(x)$ must satisfy

$$f(x) \geq 0 \ \forall x \quad \& \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Mean

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

Variance

$$Var(X) = E[(X - \mu)^2]$$

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The normal distribution

The PDF $f(x)$ of $X \xrightarrow{\nearrow C.R.V}$ aka Normal R.V

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\mu \rightarrow$ mean of X $\sigma \rightarrow$ S.D of X

To calculate the mean & variance of this distribut
we use a moment generating funct^n:

$$\varphi(t) = E(e^{tx}) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$E(x) = \varphi'(0)$ $E(x^2) = \varphi''(0)$

$Var(x) = \varphi''(0) - (\varphi'(0))^2$

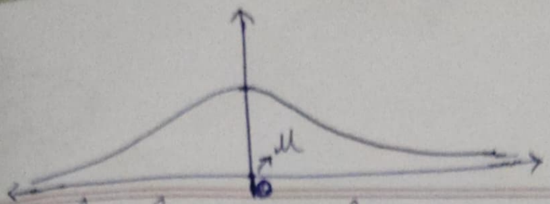
This works because

$$\varphi(t) = E[e^{tx}] = \int e^{tx} f(x) dx$$

$$\varphi'(t) = \frac{d}{dt} \int e^{tx} f(x) dx$$

$$\varphi'(t) = \int x e^{tx} f(x) dx$$

$$\varphi'(0) = \int x f(x) dx \rightarrow E[x]$$



Standard normal R.V

$X \to$ Normally distributed with param μ, σ^2

then $\boxed{Z = \dfrac{X - \mu}{\sigma}}$ ~~with~~ be normal R.V with param 0, 1

$$E[Z] = \frac{E[X] - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0$$

$$E[Z^2] = E\left(\frac{(X-\mu)^2}{\sigma^2}\right)$$

$$= \frac{1}{\sigma^2} E[(X-\mu)^2] = \frac{\sigma^2}{\sigma^2} = 1$$

Z is called standard normal R.V

Note: From here on X represents a standard normal random variable

$$P(X > a) = \int_a^\infty f(x)\,dx = Q(a)$$

$\hookrightarrow$ special func^n

$$\therefore P(X < a) = \underbrace{1 - Q(a)}_{\text{aka } \Phi(a)}$$

$$\boxed{\Phi(a) = 1 - Q(a)}$$

$$P(X > -a) = 1 - P(X < -a)$$

$$P(X < -a) = P(X > a)$$

since ~~graph~~ distribut^n is symmetric

$$Q(-a) = 1 - P(X > a)$$
$$= 1 - Q(a)$$

$$\therefore \boxed{Q(a) + Q(-a) = 1}$$

Cumulative distribution fun^{n}

$$F(a) \overset{def}{=} \int_{-\infty}^{a} P[X \le a]$$

$\hookrightarrow$ CDF

$$= \int_{-\infty}^{a} f(x)\,dx$$

Eg: X be continuous R.V with PDF $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$ what is CDF?

Sol^{n}:

$$F(a) = \int_{-\infty}^{a} f(x)\,dx$$

$$= \cancel{\int_{-\infty}^{0} f(x)\,dx} + \int_{0}^{a} f(x)\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_{0}^{a}$$

$$= 1 - e^{-\lambda a}$$

$$\therefore F(a) = \begin{cases} 0 & a < 0 \\ 1 - e^{-\lambda a} & a \ge 0 \end{cases}$$

or

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

Properties of CDF :

X be continuous / Discrete R.V, then CDF has following prop

(i) CDF is non-decreasing

(ii) Max CDF when $x = +\infty$: $F(+\infty) = 1$

(iii) Min CDF when $x = -\infty$ - $F(-\infty) = 0$

Theorem: Let $X \rightarrow$ continuous R.V. If CDF F is continuous at any $a \le x \le b$

$$P[a \le X \le b] = F(b) - F(a)$$

Left and Right continuous

if $F(b) = \lim_{h \to 0} F(b+h) \rightarrow$ Right continuous

if $F(b) = \lim_{h \to 0} F(b-h) \rightarrow$ Left continuous

if both left and right cont then it's called continuous at b.

Theorem: CDF must be right continuous.

Theorem: For any R.V

$$P[X = b] = \begin{cases} F(b) - F(b^-) & \text{if } F \text{ discont at } x = b \\ 0 & \text{otherwise} \end{cases}$$

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Retreiving PDF from CDF

Theorem: The probability density funct^n (PDF) is the derivative of CDF

$$f(x) = \frac{d(F(x))}{dx}$$

Provided $F(x)$ is differentiable at x.

if $F(x)$ is not differentiable at x then

$$f(x) = P[X = x]$$
$$= F(x) - \lim_{h \to 0} F(x-h)$$

~~Jointly distributed RV~~

Markov inequality

$$\boxed{P[X \geq a] \leq \frac{E[X]}{a}}$$ for X non-negative

$$\boxed{P[|x-\mu| \geq a] \leq \frac{\sigma^2}{a^2}}$$

Moment generating funcⁿ

$$M_X(s) = E[e^{sX}]$$

Transform to moments

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$$

$$\frac{d}{ds}(M_X(s)) = \frac{d}{ds}\int_{-\infty}^{+\infty} e^{sx} f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{sx}\, x f_X(x)\, dx$$

$\therefore$ At $s=0$ $$\frac{d}{ds}(M_X(s)) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= E[X]$$

$$\boxed{\frac{d^n}{ds^n}(M(s)) = E[X^n]}$$

at $s=0$

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Joint random variable

$S \to$ Sample space of random experiment. let X, Y be 2 R.V's. Then the pair (X, Y) is called a bivariate R.V.

$$R_{xy} = \{(x,y);\ \omega \in S \text{ and } X(\omega) = x,\ Y(\omega) = y\}$$

Eg: $F_X(x) = P[X \le x]$

$F_{X,Y}(x,y) = P[X \le x, Y \le y]$

$\to$ ~~Bivariate~~ joint CDF

The event $[X \le x,\ Y \le y]$ is equivalent to the event $A \cap B$

$$A = \{\omega \in S;\ X(\omega) \le x\}$$
$$B = \{\omega \in S;\ Y(\omega) \le y\}$$

$$\therefore P(A) = F_X(x)$$
$$P(B) = F_Y(y)$$

$$F_{XY}(x,y) = P(A \cap B)$$

If for particular values of x & y, A and B were independant:

$$F_{XY}(x,y) = P(A \cap B) = P(A)P(B)$$
$$= F_X(x)\, F_Y(y)$$

Prop of $F_{XY}(x,y)$

1. $\geq 0 \;\&\; \leq 1$

2. $F_{XY}(x,\infty) = F_X(x)$
$F_{XY}(\infty,y) = F_Y(y)$
$F_{XY}(\infty,\infty) = 1$

$F_{XY}(x,-\infty) = 0$
$F_{XY}(-\infty,y) = 0$
$F_{XY}(-\infty,-\infty) = 0$

3. $P(x_1 < X \leq x_2, Y \leq y) = F_{XY}(x_2,y) - F_{XY}(x_1,y)$

$P(X \leq x, y_1 \leq Y \leq y_2) = F_{XY}(x,y_2) - F_{XY}(x,y_1)$

4. $P(x_1 \leq X \leq x_2 ;\; y_1 \leq Y \leq y_2)$

$= F_{XY}(x_2,y_2) - F_{XY}(x_1,y_1) - F_{XY}(x_2,y_1) + F(x_1,y_1)$