IIIT-Bangalore
Probability and Statistics
Problem Set 11
(Convergence 'In Probability', Limit Theorems)

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Definition 1. *(*Convergence in Probability*) Let $\{X_n\}$ be a sequence of random variables. $\{X_n\}$ is said to converge in probability to 'a' if given $\epsilon > 0$,*

$$\lim_{n\to\infty} P(|X_n - a| < \epsilon) = 1, \text{ i.e. if } \lim_{n\to\infty} P(|X_n - a| \geq \epsilon) = 0.$$

If $\{X_n\}$ converges in probability to 'a' we write $X_n \xrightarrow[\text{in } P]{} a$ as $n \to \infty$.

Important results to prove: 1-6

1. **(Tchebycheff's inequality)** If X is a random variable having a finite variance, then prove for any $\epsilon > 0$
$$P(|X - m| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$
where m and σ respectively denote the mean and standard deviation of X.

2. **(Tchebycheff's Theorem)** Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables such that mean and S.D. of X_n are m_n and σ_n respectively (both exists finitely). If $\sigma_n \to 0$ as $n \to \infty$ then $X_n \xrightarrow[\text{in P}]{} m_n$ as $n \to \infty$.

3. **(Bernoulli's Theorem)** Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables such that $X_n \sim$ Binomial (n, p). Then $\frac{X_n}{n} \xrightarrow[\text{in P}]{} p$ as $n \to \infty$.

4. **(Law of large numbers)** Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables such that $S_n = X_1 + X_2 + \dots + X_n$ has a finite mean M_n and finite S.D. Σ_n for all n. If $\Sigma_n = o(n)$, i.e. $\frac{\Sigma_n}{n} \to 0$ as $n \to \infty$, then $\frac{S_n - M_n}{n} \xrightarrow[\text{in P}]{} 0$ as $n \to \infty$.

5. **(Law of large numbers with equal components)** Let (i) $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables so that all of them have same distribution with mean m and S.D. σ and (ii) $X_1, X_2, \dots, X_n$ are mutually independent for all n. Then $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow[\text{in P}]{} m$ as $n \to \infty$.

6. If X possess a finite second order moment and c is any fixed number then for any $\epsilon > 0$
$$P(|X - c| \geq \epsilon) \leq \frac{E\{(X - c)^2\}}{\epsilon^2}.$$

7. Show by Tchebycheff's inequality that in 2000 throws of a coin, the probability that the number of heads lies between 900 and 1100 is at least $\frac{19}{20}$.

8. If X is a $\gamma(n)$ variate prove that $P(0 < X < 2n) \geq \frac{n-1}{n}$.

Definition 2. (Asymptotically Normal Distribution.) *Let $\{X_n\}$ be a sequence of random variables and $\{a_n\}$, $\{b_n\}$ be two real sequences. If the distribution function $F_n(x)$ of $\frac{X_n - a_n}{b_n}$ converges pointwise to the distribution function $\Phi(x)$ of standard normal distribution, then we say that X_n is asymptotically normal (a_n, b_n).*

Theorem 1. (Limit theorem of characteristic functions.) *Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of random variables with corresponding distribution functions $F_1(x)$, $F_2(x), \ldots, F_n(x), \ldots$ and characteristic functions $\chi_1(t), \chi_2(t), \ldots, \chi_n(t), \ldots$*

1. *If $F_n(x) \to F(x)$ as $n \to \infty$, $\forall x$, then $\chi_n(t) \to \chi(t)$, the characteristic function determined by $F(x)$.*

2. *Conversely, if $\chi_n(t) \to \chi(t)$ as $n \to \infty$, then $F_n(x) \to F(x)$, $\forall x$, $F(x)$ being the distribution function determined by $\chi(t)$.*

9. (Application) Show that Poisson distribution can be obtained as a limit of Binomial distribution.

Theorem 2. (Central Limit Theorem for the case of equal components) *Let*

1. *$\{X_n\}$ be a sequence of random variables each having same distribution with mean m and standard deviation σ and*

2. *$X_1, X_2, \ldots, X_n$ are mutually independent for all n.*

Then $S_n = X_1 + X_2 + \ldots + X_n$ is asymptotically normal $(nm, \sigma\sqrt{n})$, i.e. the distribution function of $\frac{S_n - nm}{\sigma\sqrt{n}}$ converges pointwise to the distribution function $\Phi(x)$ of standard normal distribution.

Cor. *Then $\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$ is asymptotically normal $(m, \frac{\sigma}{\sqrt{n}})$, i.e. the distribution function of $\frac{\bar{X} - m}{\frac{\sigma}{\sqrt{n}}}$ converges pointwise to the distribution function $\Phi(x)$ of standard normal distribution.*

10. (Application) A random sample of size $n = 81$ is taken from an infinite population with mean $\mu = 128$ and S.D. $\sigma = 6.3$. What is the probability that $\bar{X}$ will not fall between 126.6 and 129.4 if we use Central Limit Theorem?

11. By applying the Central Limit Theorem to a sequence of random variables with Poisson distribution prove that

$$\lim_{n \to \infty} e^{-n} \sum_{r=0}^{n} \frac{n^r}{r!} = \frac{1}{2}.$$

Theorem 3. *(*DeMoivre-Laplace limit theorem.*) Let* X_n *be a binomial* (n, p) *variate* $(0 < p < 1)$*, the corresponding standardised variate being*

$$X_n^* = \frac{X_n - np}{\sqrt{npq}}, \ (q = 1 - p).$$

Then for any fixed numbers $a, b(> a)$

$$\lim_{n \to \infty} P(a < X_n^* \leq b) = \int_a^b \varphi(x)dx$$

where φ *is the probability density function of the standardised normal variate.*

12. (Application): If a die is thrown $1,800$ times, find the probability that the frequency of the event 'multiple of three' lies between 600 ± 50.