IIIT-Bangalore
Probability and Statistics
Problem Set 10

(Special Distributions)

1. If X is a $\gamma(\frac{n}{2})$ variate, then show that $Y = 2X$ has a χ^2-distribution with n-degrees of freedom, and conversely if Y is a $\chi^2(n)$ variate, then X is a $\gamma(\frac{n}{2})$ variate.

2. If $X_1, X_2, \dots, X_n$ be mutually independent standard normal variates then $Y = X_1^2 + X_2^2 + \cdots + X_n^2$ has $\chi^2(n)$ distribution.

3. Compute the m.g.f. of $\chi^2(n)$-distribution. Hence find its (i) mean, (ii) variance and (iii) mode. (Ans. mean $= n$, variance $= 2n$, mode $= n - 2$)

4. If $X_1, X_2, \dots, X_n$ are mutually independent standard normal variates, and $Y_1, Y_2, \dots, Y_n$ are obtained by an orthogonal homogeneous linear transformation:

$$Y_i = \sum_{\alpha=1}^{n} a_{i\alpha} X_\alpha \quad (i = 1, 2, \dots, n)$$

where

$$\sum_{\alpha=1}^{n} a_{i\alpha} a_{j\alpha} = \sum_{\alpha=1}^{n} a_{\alpha i} a_{\alpha j} = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad (i, j = 1, 2, \dots, n)$$

then $Y_1, Y_2, \dots, Y_n$ are also mutually independent standard normal variates.

5. If $X_1, X_2, \dots, X_n$ are mutually independent standard normal variates, and $Y_1, Y_2, \dots, Y_m$ be m $(< n)$ linear combinations given as:

$$Y_i = \sum_{\alpha=1}^{n} a_{i\alpha} X_\alpha \quad (i = 1, 2, \dots, m)$$

where

$$\sum_{\alpha=1}^{n} a_{i\alpha} a_{j\alpha} = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad (i, j = 1, 2, \dots, m)$$

then the quadratic form $Q(X_1, X_2, \ldots, X_n) = \sum_{\alpha=1}^{n} X_\alpha^2 - \sum_{\beta=1}^{m} Y_\beta^2$ is χ^2-distributed with $n - m$ degrees of freedom and Q is independent of the given linear combinations.

6. If X is a standard normal variate, Y has $\chi^2(n)$-distribution, and X, Y are independent then $T = \frac{X}{\sqrt{\frac{Y}{n}}}$ has t-distribution with n degrees of freedom.

7. If X and Y are independent variates having $\chi^2(m)$ and $\chi^2(n)$ distributions respectively, then show that

$$Z = \frac{X/m}{Y/n}$$

is and $F(m, n)$ variate.

8. If $X_1, X_2, \ldots, X_n$ are mutually independent normal variates each having mean 0 and standard deviation σ, Find the distribution of $X_1^2 + X_2^2 + \ldots + X_n^2$.

9. If (X, Y) has the general bivariate normal distribution, show that

$$\frac{\left(\frac{X-m_x}{\sigma_X}\right)^2 - 2\rho\left(\frac{X-m_x}{\sigma_X}\right)\left(\frac{Y-m_Y}{\sigma_Y}\right) + \left(\frac{Y-m_Y}{\sigma_Y}\right)^2}{1-\rho^2}$$

has $\chi^2(2)$-distribution.

10. If X and Y are independent variates, X being χ^2-distributed with m degrees of freedom and their sum $X + Y$ χ^2-distributed with $m + n$ degrees of freedom, then show that Y is χ^2-distributed with n degrees of freedom.