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$\therefore$ Reqd. prob $= \displaystyle\int_{2\pi/3}^{\pi} k\,dx.$

Now $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_0^{\pi} k\,dx = 1$

$\Rightarrow k\pi = 1 \Rightarrow \boxed{k = \frac{1}{\pi}}$

Now Reqd. prob $= \displaystyle\int_{2\pi/3}^{\pi} \frac{1}{\pi}\,dx$

$= \frac{1}{\pi}\left(\pi - \frac{2\pi}{3}\right) \Rightarrow \boxed{\frac{1}{3}}$

Poisson Process

→ **Stochastic Process** :- A family of random variables $\{X(t) : t \in T\}$ which depends parametrically on time t, is called a stochastic process.

eg. $X(t)$:- (i) No. of people who entered in a supermarket at time t.

(ii) No. of people affected by disease at time t.

(iii) No. of persons in a queue at time t.

Here, we consider a particular eg. of stochastic process which counts no. of changes in a given time interval. This process obeys 2 laws :-

(i) The number of changes during the time interval $(t, t+h)$ is independent of no. of changes occurred in $(0,t)$ for all t and h ($h>0$).

(ii) The probability of exactly one change in $(t, t+h)$ is $\lambda h + o(h)$, where λ is a +ve const, $o(h)$ is a f^n of h s.t. $\frac{o(h)}{h} \to 0$ as $h \to 0$.

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The probability of more than one change in $(t, t+h)$ is $o(h)$.

Thm :- No. of changes in a stochastic process in a given time interval satisfying the above laws, follows the Poisson distribution.

$X(t)$: No. of changes in interval $(0,t)$:-

$$P(X(t)=i) = \frac{e^{-\lambda t} \cdot (\lambda t)^i}{\lfloor i}, \quad i=0,1,\ldots$$

Poisson (λt)

$\boxed{\mu = \lambda t}$.

Proof not reqd.

λ :- average rate of Poisson process (avg no. of changes per unit time).

λt :- avg no. of changes in a given time interval $(0,t)$.

P. 3.4

Q11. $X(t)$: No. of wars in $(0,t)$.

There is war every 15 years on avg.

$\therefore \boxed{\lambda = \frac{1}{15}}$.

$X(t) \sim \text{Poisson}\left(\frac{t}{15}\right) \rightarrow \lambda t.$

$\because$ We need no wars in 25 years.

$\therefore i=0$ & $t=25$.

$X(25) \sim \text{Poisson}\left(\frac{25}{15}\right) = \text{Poisson}\left(\frac{5}{3}\right)$

$\therefore P(X(2t)=0) = \dfrac{e^{-\frac{5}{3}} \cdot \left(\frac{5}{3}\right)^0}{\underline{|0}} = \boxed{e^{-5/3}}$.

Q12. 2.5 particles per second.

$\therefore \lambda = \dfrac{1}{2.5} = 0.4$ ↳ in 1 time unit, how much?

$\therefore$ we need prob $(i \geq 3)$, and $t = 4$ sec.

$X(t)$ = no. of particles emitted in $(0,t)$.

$P(X(t)=i) = e$

$\therefore X(t) = \text{Poisson}\left(\frac{2t}{5}\right)$.

$X(4) \sim \text{Poisson}\left(\frac{8}{5}\right)$.

$\therefore P(X(4) \geq 3) = 1 - P(X(4)=0) - P(X(4)=1) - P(X(4)=2)$

$= 1 - e^{-8/5}\Big[$

~~$\lambda = 2.5$ 2.5 particles emitted~~

$X(t)$ = no. of particles emitted in $(0,t)$.

2.5 particles per second.

$\therefore \lambda = 2.5$.

Here $i \geq 3$, and $t = 4$.

$\therefore X(t) \sim \text{Poisson}(2.5t)$ $\therefore X(4) \sim \text{Poisson}(10)$

$P(X(4) \geq 3) = 1 - P(X(4)=0) - P(X(4)=1) - P(X(4)=2)$.

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$$= 1 - e^{-10}\left[\frac{(10)^0}{\lfloor 0} + \frac{(10)^1}{\lfloor 1} + \frac{(10)^2}{\lfloor 2}\right]$$

$\lfloor 0 \to 1$, $\lfloor 1 \to 10$, $\lfloor 2 \to 50$

$$\Rightarrow \boxed{1 - 61e^{-10}}.$$

→ **Transformations of Random Variables:-**

Let $X : S \to \mathbb{R}$ be a random var. and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function.
Then $g(X) : S \to \mathbb{R}$, defined as $[g(X)](\omega) = g[X(\omega)]$, for all $\omega \in S$ is a random variable.

Given the distribution of X, how to find distribution of the transformed variable $Y = g(X)$?

① **Discrete Case**

Let
(i) X be a discrete random variable whose pmf is $f_X(x)$ and.
(ii) $Y = g(X)$ be another r.v. where g is a bijective map, so that $X = h(Y)$ exists. (i.e $h = g^{-1}$). Then the pmf of Y is

$$\boxed{f_Y(y) = f_X[h(Y)]}$$

Proof :- Let spectrum of X is $\{x_i : i = 0, \pm 1, \pm 2, \ldots\}$

Now $P(X = x_i) = f_X(x_i)$ $\forall i = 0, \pm 1, \ldots$
↳ Defn is this only.

$\therefore$ Spectrum of $Y = \{ g(x_i) = y_i, \; i = 0, \pm 1, \pm 2, \dots \}$

$\therefore$ We need to find $P(Y = y_i) \; \forall i = 0, \pm 1, \dots$

$P(Y = y_i) = ?$

$(Y = y_i) \Rightarrow (g(X) = g(x_i)) \Rightarrow (X = x_i)$

obv as $Y = y_i$ only when $X = x_i$
(It is a bijective f^n).

$\therefore P(Y = y_i) = P(X = x_i).$

$\Rightarrow$

$f_Y(y_i) = f_X(\underbrace{x_i}_{h(y_i)}) \rightarrow \boxed{f_Y(y_i) = f_X(h(y_i))}$

PS5

Q1. To find $\rightarrow$ Prob. distribution of square of Poisson(μ) variate.

Ans. $X = \text{Poisson}(\mu)$.

$P(X = i) = f_X(i) = \dfrac{e^{-\mu} \cdot (\mu)^i}{\lfloor i}; \; i = 0, 1, 2, \dots$

$Y = X^2$

$P(Y = i^2) = ?$ when $i = 0, 1, 2, \dots$

This is a ~~one one f^n~~ bijective f^n as we are only considering +ve i's, otherwise it would have been many one.

only +ve, not -ve.

$\therefore P(Y = i^2) = P(X^2 = i^2) = P(X = i).$

$$\therefore \boxed{P(Y = i^2) = \dfrac{e^{-\mu} \cdot (\mu)^i}{\lfloor i}, \; i = 0, 1, 2, \dots}$$

∴ Only the spectrum points have changed. The masses themselves have not changed.

$$\boxed{P(Y=4) = \frac{e^{-\mu}\cdot(\mu)^2}{\lfloor 2} \quad \text{etc.}}$$

$$\therefore \boxed{P(Y=j) = \frac{e^{-\mu}\cdot(\mu)^{\sqrt{j}}}{\lfloor(\sqrt{j})}}$$

② <u>Continuous Case</u>

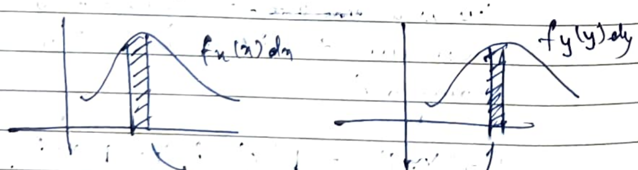
let

(i) X be a continuous random variable with p.d.f. $f_X(x)$ and

(ii) $y=\phi(x)$ be continuously differentiable & either strictly increasing or strictly decreasing (basically bijective) throughout, so that $x=\psi(y)$ ($\psi=\phi^{-1}$) exists. Then the p.d.f. of the transformed random variable

$Y=\phi(X)$ is

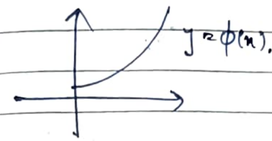
$$\boxed{f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|}$$



indicates that area under the curve is same.

Proof:-

(i) Case ① → ϕ = strictly increasing.



$F_Y(y)$ = d.f. of $Y = P(Y\le y)$
↳ Probability distribution fⁿ.

$\Rightarrow P(\phi(X)\le\phi(x))$
↳ here $\boxed{\phi(x)=y}$.

Since ϕ is strictly increasing ∴ it's a bijection.

$\therefore \phi(X)\le\phi(x)$ only when $X\le x$.

$$[\phi(X)\le\phi(x)] \iff [X\le x].$$

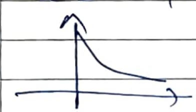
$\therefore F_Y(y) = P(Y\le y) = P(X\le x) = F_X(x)$.

Derivative wrt y.

$$\Rightarrow \boxed{f_Y(y) = f_X(x)\cdot\frac{dx}{dy}} \quad \boxed{+\text{ve.}}$$

P.D.F $= \frac{d}{dy}$(D.F).

(ii) Case 2 → ϕ = strictly decreasing



$F_Y(y) = P(Y\le y) = P(\phi(X)\le\phi(x))$

Since ϕ is strictly decreasing ⇒ $\phi(X)\le\phi(x)$ only when $X\ge x$.

$\therefore [\phi(X)\le\phi(x)] \iff [X\ge x]$.

$\therefore F_Y(y) = P(Y\le y) = P(X\ge x)$
$= 1-P(X<x)$

$$\boxed{F_Y(y) = 1-F_X(x)}$$

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Taking derivative

$$f_Y(y) = f_X(x) \cdot \left(-\frac{dx}{dy}\right)$$

↳ +ve as $\frac{dx}{dy} < 0$.

∴ We can combine the 2 results using mod.

$$\therefore \boxed{f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|}$$

Hence Proved.

PS 5

Q2. $f_X(x) = \begin{cases} 2xe^{-x^2} & , x>0 \\ 0 & , \text{otherwise.} \end{cases}$

Find distribution of $Y = X^2$

Ans. $\cancel{f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|}$

Transformation : $y = x^2$

$\Rightarrow \frac{dy}{dx} = 2x > 0 \quad \forall x>0.$

∴ Strictly increasing.

$$f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|$$

$$= 2xe^{-x^2} \cdot \frac{1}{2x} \;, x>0$$

$$0 \qquad , x \le 0$$

$\Rightarrow f_Y(y) = e^{-x^2}, \; x>0$

Since $y = x^2$

$\Rightarrow \boxed{f_Y(y) = e^{-y} \;, \; y>0}$

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Q10. X :- Standard Normal Distribution.

$X \sim N(0,1)$.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \;, \; -\infty < x < \infty.$$

↳ Strictly decreasing.

$Y = \frac{X^2}{2}$.

$\cancel{f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|}$

∵ x varies from $-\infty$ to ∞.

∴ y varies from 0 to ∞.

$\frac{dy}{dx} = x$ ∴ Neither strictly increasing nor decreasing.

$F_Y(y) = \text{d.f. of } Y = P(Y \le y)$.

↳ $y<0$ is impossible.

∴ $F_Y(y) = 0$, since $(Y \le y) = \left(\frac{x^2}{2} \le 0\right)$ is an impossible event.

$\boxed{\text{For } y<0}$.

For $y \ge 0$,

$F_Y(y) = P(Y \le y) \Rightarrow P\left(\frac{x^2}{2} \le y\right)$

$\Rightarrow P(-\sqrt{2y} \le x \le \sqrt{2y})$

$\Rightarrow P(-\sqrt{2y} < x \le \sqrt{2y})$

↳ as $P(-\sqrt{2y}) = 0$.

↳ as X is continuous.

$\Rightarrow \Phi(\sqrt{2y}) - \Phi(-\sqrt{2y})$.

$$\therefore f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right).$$

$\Phi' = \phi$.

$$= \phi(\sqrt{2y})\cdot\sqrt{2}\cdot\frac{1}{2\sqrt{y}} + \phi(-\sqrt{2y})\cdot\sqrt{2}\cdot\left(\frac{1}{2\sqrt{y}}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2y}}\left[\phi(\sqrt{2y}) + \phi(-\sqrt{2y})\right]$$

Putting $\phi_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\Rightarrow \frac{1}{\sqrt{2y}}\cdot\left[e^{-y} + e^{(-y)}\right]\cdot\frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \frac{1}{2\sqrt{y\pi}}\cdot(e^{-y})\times 2$$

$$\Rightarrow \frac{y^{-1/2}\cdot e^{-y}}{\sqrt{\pi}}$$

$$\Rightarrow \boxed{\Gamma(1/2)}$$

$$\therefore \boxed{f_y(y) = \begin{cases} \Gamma(1/2), & y>0 \\ 0, & \text{elsewhere.} \end{cases}}$$

Q3. $X \sim \text{Normal}(0,1)$ to find distribution of $Y = e^X$.

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

Now $y = e^x$

$\therefore \frac{dy}{dx} = e^x > 0 \ \forall x$.

$\therefore$ It is an increasing f^n.

$\therefore f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|$

$$\cancel{= \frac{1}{\sqrt{2\pi}}\cdot e^{-x^2}\cdot\frac{1}{e^x}}$$

$$\cancel{= \frac{1}{\sqrt{2\pi}} e^{-x/2}}$$

$$\cancel{= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{e^x}\right)^{1/2}}$$

~~Putting $y = e^x$~~

$$\cancel{\Rightarrow f_x(y) = \frac{1}{\sqrt{2\pi}} y^{-1/2}}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}\cdot e^{-x}$$

$\therefore$ Putting $e^x = y$

$$\Rightarrow \boxed{f_y(y) = \frac{1}{\sqrt{2\pi}}\cdot\frac{e^{-\frac{1}{2}(\log y)^2}}{y}}$$

Q4. $X \sim \gamma(l)$.

To find :- prob :- $Y = \sqrt{X}$, $X>0$, $y>0$.

Now $y = \sqrt{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} > 0 \ \forall x$. $\therefore y = \sqrt{x}$ is S.I. f^n.

Now $f_x(x) = \frac{1}{\Gamma(l)} e^{-x}\cdot x^{l-1}$.

$$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{1}{\Gamma(l)}\cdot e^{-x}\cdot x^{l-1}\cdot 2\sqrt{x}$$

$$f_y(y) = \frac{1}{\Gamma(l)}\cdot e^{-(x)}\cdot(x)^{l-1}\cdot 2(x)^{1/2}.$$

Putting value $y=\sqrt{x}$ in this eqⁿ.

$$\Rightarrow f_Y(y) = \frac{1}{\gamma(l)} e^{-y^2} \cdot (y^2)^{l-1} \cdot 2 \cdot y$$

$$\Rightarrow f_Y(y) = \frac{1}{\Gamma(l)} \cdot e^{-y^2} \cdot y^{2l-2+1} \cdot 2$$

$$\Rightarrow \boxed{f_Y(y) = \frac{2e^{-y^2}\cdot(y)^{2l-1}}{\Gamma(l)}}$$

Q5. $X \sim \beta_1(l,m)$

To find :- Distribution of $Y = \frac{X}{1-X}$

Now $y = \frac{x}{1-x}$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x)\cdot 1 - x\cdot(-1)}{(1-x)^2} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2} > 0$$

Strictly increasing fⁿ.

Now $f_X(x) = \begin{cases} \frac{1}{\beta(l,m)} \cdot x^{l-1}\cdot(1-x)^{m-1}, & 0<x<1 \\ 0, & \text{elsewhere.} \end{cases}$

Now, range of $Y = [0,\infty)$. [from the range of values of x, elsewhere, there will be no distribution. only when $0<x<1$ there is a distribution.

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

$$= \frac{1}{\beta(l,m)} \cdot x^{l-1}\cdot(1-x)^{m-1}\cdot(1-x)^2$$

$$= \frac{1}{\beta(l,m)} \cdot x^{l-1}\cdot(1-x)^{m+1} \quad \text{—(1)}$$

Now $y = \frac{x}{1-x} \Rightarrow y - yx = x$

$$\Rightarrow x(1+y) = y$$

$$\Rightarrow \boxed{x = \frac{y}{1+y}}$$

Plugging value $x = \frac{y}{1+y}$ in eqⁿ (1).

$$\Rightarrow \frac{1}{\beta(l,m)} \cdot \frac{(y)^{l-1}}{(1+y)^{l-1}} \cdot \left(1-\frac{y}{1+y}\right)^{m+1}$$

$$\Rightarrow \frac{1}{\beta(l,m)} \cdot \frac{(y)^{l-1}\cdot(1)^{m+1}}{(1+y)^{l+m}}$$

$\therefore \boxed{Y \sim \beta_2(l,m)}$ Hence Proved.

PS-4

Q7. $X \sim \text{Poisson}(\mu)$.

$$\therefore f(x) = \begin{cases} \frac{e^{-\mu}\cdot(\mu)^x}{\lfloor x}, & x=0,1,2,\dots \\ 0, & \text{elsewhere.} \end{cases}$$

$\therefore P(X\le n) = P(X=0)+P(X=1)+\dots+P(X=n).$

$$\boxed{P(X\le n) = e^{-\mu}\left[\frac{1}{\lfloor 0}+\frac{\mu}{\lfloor 1}+\frac{\mu^2}{\lfloor 2}+\dots+\frac{\mu^n}{\lfloor n}\right]}$$

To show: RHS is also this.

$$I_n = \frac{1}{\lfloor n} \int_{\mu}^{\infty} \underbrace{e^{-x}}_{g(x)} . \underbrace{x^n}_{f(x)} \, dx$$

$$= \frac{1}{\lfloor n} \left[-e^{-x} . x^n \right]_{\mu}^{\infty} - \frac{1}{\lfloor n} \int_{\mu}^{\infty} (-e^{-x}) . x^{n-1} . n \, dx$$

$$= \frac{1}{\lfloor n} \left[e^{-x} . x^n \right]_{\mu}^{\infty} + \frac{1}{\lfloor n-1} \underbrace{\int_{\mu}^{\infty} e^{-x} . (x)^{n-1} \, dx}_{I_{n-1}}$$

$$\therefore I_n = \frac{e^{-\mu} . (\mu)^n}{\lfloor n} + \cancel{I_{n-1}}$$

$\cancel{I_{n-1}}$

$$\cancel{I_1} = \frac{e^{-\mu} . (\mu)^1}{\lfloor 1} + I_0 .$$

Adding.

$$\Rightarrow I_n = I_0 + \sum_{i=1}^{n} \frac{e^{-\mu} . (\mu)^i}{\lfloor i}$$

Now $I_0 = \int_{\mu}^{\infty} e^{-x} dx = \left[-e^{-x} \right]_{\mu}^{\infty}$

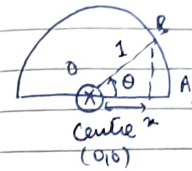
$$= e^{-\mu} = \frac{e^{-\mu} . (\mu)^0}{\lfloor 0}$$

$$\therefore I_n = \sum_{i=0}^{\infty} \frac{e^{-\mu} . (\mu)^i}{\lfloor i} = R.P(X \le n)$$

Hence Proved

P55

Q6.



$x = \cos\theta$

Let θ denote the Random variable indicating the angle made by the point with radius OA.

$\therefore \theta = \angle BOA$.

$\theta \in [0, \pi]$ and $f_{\theta}(\theta) = \begin{cases} k, & 0 \le \theta \le \pi \\ 0, & \text{otherwise.} \end{cases}$

Uniform Distribution.

$\int_{-\infty}^{\infty} f_{\theta}(\theta) \, d\theta = 1 \Rightarrow \int_0^{\pi} k \, d\theta = 1 \Rightarrow k\pi = 1$

$\Rightarrow \boxed{\pi = \frac{1}{k}}$

Now let X denote random variable indicating distance of pt of projⁿ from O in dirⁿ of A.

$\therefore X = \cos(\theta)$.

Now $f_X(x) = f_{\theta}(\theta) . \left| \frac{d\theta}{dx} \right|$ → if it is M.D. or M.I. fⁿ.

$\frac{dx}{d\theta} = -\sin\theta \Rightarrow \frac{d\theta}{dx} = -\operatorname{cosec}\theta \Rightarrow \left| \frac{d\theta}{dx} \right| = \operatorname{cosec}\theta$

< 0 when $0 < \theta < \pi$.

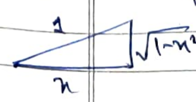
$\therefore \boxed{M.D.}$

as $\operatorname{cosec} > 0$ when $\theta \in [0, \pi]$.

$\therefore f_X(x) = \begin{cases} k \operatorname{cosec}\theta, & 0 \le \theta \le \pi \\ 0, & \text{otherwise.} \end{cases}$

$\therefore$ Putting $\theta = \cos^{-1}(x)$

$\Rightarrow f_X(x) = \begin{cases} \frac{1}{\pi} . \operatorname{cosec}(\cos^{-1} x), & 0 \le \theta \le \pi \\ 0, & \text{otherwise} \end{cases}$



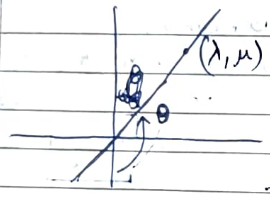
when $\theta\in(0,\pi)$ $x\in(-1,1)$

$$\Rightarrow f_X(x) = \begin{cases} \dfrac{1}{\pi\sqrt{1-x^2}}, & -1<x<1 \\ 0, & \text{otherwise.} \end{cases}$$

Cosec is undefined for $-1, 1$.

$$\therefore \boxed{f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad -1<x<1}$$

Q7.



$\boxed{\lambda>0}$

Let Θ → capital theta be the random variable denoting angle with the X-axis in ACW dirn.

$\therefore$ Slope of line $= \tan\left(\theta - \frac{\pi}{2}\right)$

$= -\cot\theta$

$\therefore$ eqn $\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - \mu = -\cot\theta(x-\lambda)$.

$\Rightarrow y = (-\cot\theta)x + (\lambda\cot\theta + \mu)$

be R.V. to

$\therefore$ Let Y denote the intercept on the Y axis.

$$\therefore \boxed{Y = \lambda\cot(\Theta) + \mu}$$

Now $f_\Theta(\theta) = \begin{cases} 1/\pi, & 0<x<\pi \\ 0, & \text{elsewhere} \end{cases}$

as $\lambda>0$ $\therefore$ It can only make an angle b/w 0 and π with the X axis.

Now $f_Y(y) = f_\Theta(\theta)\left|\dfrac{d\theta}{dy}\right|$

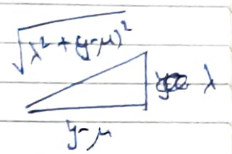
$\dfrac{dy}{d\theta} = \underbrace{-\lambda}_{>0}\underbrace{\operatorname{cosec}^2\theta}_{>0} < 0$ $\therefore$ MD f^n.

and $\left|\dfrac{d\theta}{dy}\right| = \dfrac{1}{\lambda}\cdot\dfrac{1}{\operatorname{cosec}^2\theta} = \dfrac{\sin^2\theta}{\lambda}$.

$$\therefore f_Y(y) = \begin{cases} \dfrac{1}{\pi}\cdot\dfrac{\sin^2\theta}{\lambda}, & 0<x<\pi \\ 0, & \text{elsewhere.} \end{cases}$$

Since $y = \lambda\cot\theta + \mu$

$\Rightarrow \dfrac{y-\mu}{\lambda} = \cot\theta$



$\therefore \sin\theta = \dfrac{\lambda}{\sqrt{\lambda^2+(y-\mu)^2}} \Rightarrow \sin^2\theta = \dfrac{\lambda^2}{\lambda^2+(y-\mu)^2}$.

$$\therefore \boxed{f_Y(y) = \left\{ \frac{1}{\pi}\cdot\frac{\lambda}{\lambda^2+(y-\mu)^2}, \quad -\infty<x<\infty, \ \lambda>0 \right.}$$

Hence Proved.

Q8. $t^2+2t-X=0$.

X → random variable, uniform over $(0,2)$.

$\therefore f_X(x) = \begin{cases} \frac{1}{2}, & 0<x<2 \\ 0, & \text{elsewhere.} \end{cases}$

Now eqn $\Rightarrow t^2+2t-X=0$

$\Rightarrow t = \dfrac{-2\pm\sqrt{4+4X}}{2}$

$= -1\pm\sqrt{1+X}$ Larger root → $-1+\sqrt{1+X}$.

$\therefore$ Transformation $\boxed{Y = -1+\sqrt{1+X}}$.

X varies from $(0,2) \Rightarrow Y$ varies from 0 to $-1+\sqrt{3}$.

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Now $\frac{dy}{dx} = \frac{1}{2\sqrt{1+x}} > 0$ $\therefore$ M-I.

$\therefore f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$

$$= \begin{cases} \frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{1+x}, & 0<x<2 \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \sqrt{1+x}, & 0<x<2. \\ 0, & \text{elsewhere.} \end{cases}$$

Taking $y = -1+\sqrt{1+x} \Rightarrow \sqrt{1+x} = 1+y$.

$$\therefore \boxed{f_Y(y) = \begin{cases} 1+y, & 0<y<\sqrt{3}-1 \\ 0, & \text{elsewhere.} \end{cases}}$$

Q9. $X \sim$ Normal (m, σ).

To prove:- $P(a<x<b) = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)$.

X is Normal(m,σ) but Φ denotes the distribution f^n for standard normal f^n.

We can see that we need a $\left(\frac{x-m}{\sigma}\right)$ kind of transformation.

$f_X(x) =$ p.d.f of $X = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-m)^2}$, $-\infty<x<\infty$, $\sigma>0$.

Let $Y = \frac{X-m}{\sigma}$.

$\frac{dy}{dx} = \frac{1}{\sigma} > 0$ as $\sigma>0$. $\therefore$ $\boxed{\text{M-I.}}$

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$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}\cdot\cancel{\sigma}} \cdot \cancel{\sigma} \cdot e^{-\frac{1}{2}\cdot\left(\frac{x-m}{\sigma}\right)^2}, & -\infty<x<\infty. \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\left(\frac{x-m}{\sigma}\right)^2}, & -\infty<x<\infty. \end{cases}$$

Putting $y = \frac{x-m}{\sigma}$.

$$\therefore f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, & -\infty<y<\infty. \end{cases}$$

$\Rightarrow \boxed{f_Y(y) = \phi_Y(y)} \Rightarrow Y$ is a Normal $(0,1)$ variate with distribution Φ.

$\therefore$ $a<X<b \Rightarrow \frac{a-m}{\sigma} < Y < \frac{b-m}{\sigma}$.

$\therefore P(a<X<b) = \cancel{\Phi_X(b) - \Phi_X(a)} =$

$P\left(\frac{a-m}{\sigma} < Y < \frac{b-m}{\sigma}\right) = \Phi_Y\left(\frac{b-m}{\sigma}\right) - \Phi_Y\left(\frac{a-m}{\sigma}\right)$

$\therefore \boxed{P(a<X<b) = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)}$ Hence Proved.

And $P(|X-m| > a\sigma) = 1 - P(|X-m| \leq a\sigma)$.
$= 1 - P(-a\sigma < X-m < a\sigma)$
$= 1 - P(m-a\sigma < X < m+a\sigma)$.

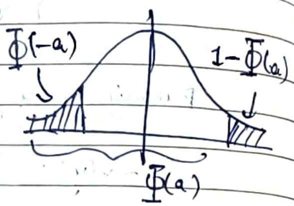
From prev. formula.

$$P = 1 - \left[\Phi\left(\frac{m+a\sigma-m}{\sigma}\right) - \Phi\left(\frac{m-a\sigma-m}{\sigma}\right)\right].$$

$= P(|X-m| > a\sigma) = 1 - [\Phi(a) - \Phi(-a)]$

$\therefore \boxed{P(|X-m| > a\sigma) = 1 - [\Phi(a) - \Phi(-a)]}$

Now $\Phi(a) - \Phi(-a) = ?$



From graph

$\boxed{\Phi(-a) = 1 - \Phi(a)}$

V-Imp.

Can be proven mathematically also.

$\therefore \Phi(a) - \Phi(-a) = 2\Phi(a) - 1.$

$\therefore \boxed{P(|X-m| > a\sigma) = 2[1 - \Phi(a)]}$

Mathematically:-

$\cancel{\Phi(a) - \Phi(-a) = \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx}$

$$\Phi(-a) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-x^2/2} dx - \int_{-a}^{\infty} e^{-\frac{x^2}{2}} dx\right]$$

Take $u = -x$, $du = -dx$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-x^2/2} dx + \int_{a}^{-\infty} e^{-\frac{u^2}{2}} du\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-x^2/2} dx - \underbrace{\int_{-\infty}^{a} e^{-\frac{u^2}{2}} du}\right]$$

$\Phi(a)$, when multiplied with $\frac{1}{\sqrt{2\pi}}$

Now $\int_{-\infty}^{\infty} e^{-x^2/2} dx$ take $v = \frac{x}{\sqrt{2}}$ $\Rightarrow dx = \frac{dv}{\sqrt{2}}$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-v^2} \frac{dv}{\sqrt{2}}$$

$$\Rightarrow \frac{2}{\sqrt{2}} \int_{0}^{\infty} e^{-v^2} dv$$

$\Gamma(1/2) = \sqrt{\pi}$.

$\Rightarrow \sqrt{2\pi}.$

$$\therefore \Phi(-a) = \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\right] - \Phi(a)$$

$\Rightarrow \boxed{\Phi(-a) = 1 - \Phi(a)}$ Hence Proved.

Q11. $f_X(x) = \begin{cases} k, & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \Rightarrow 2k = 1 \Rightarrow \boxed{k = 1/2}$

$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$ $\boxed{Y = |X|}$

$\frac{dy}{dx} = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ → Neither increasing nor decreasing.

$\cancel{F_Y(y) = P(Y \leq y) \Rightarrow P(|X| \leq x)}$

For $y < 0$

$F_Y(y) = P(Y \leq y) = 0$ as $|x| \nless 0$.

$\therefore$ Impossible event.

For $y \geq 0$

$F_Y(y) = P(Y \leq y) = P(|x| \leq y)$

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~~Mathematical Expectation~~

$\Rightarrow \quad P(-y \le x \le y)$

$= F_X(y) - F_X(-y)$

$= \int_{-y}^{y} \frac{1}{2}\, dx = \frac{1}{2}\left[y-(-y)\right] \cdot \boxed{y}$

$\therefore \quad F_Y(y) = \begin{cases} y, & 0<y<1 \\ 1, & y \ge 1 \\ 0, & y \le 0 \end{cases}$

$\therefore \quad \boxed{f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 1, & 0<y<1 \\ 0, & \text{elsewhere.} \end{cases}}$

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Mathematical Expectations -1

→ **Expectation**

① **Discrete** :- If X is a discrete random var. with spectrum $\{x_0, x_{\pm1}, x_{\pm2}, \dots\}$. Then

$$\boxed{E(X) = \sum_{i=-\infty}^{\infty} x_i P(X=x_i) = \sum_{i=-\infty}^{\infty} x_i f_X(x_i)}$$

provided infinite series is convergent, and here $f_X(x_i)$ is the pmf of X.

① Expectation of a r.v. X in the long run is the average of the outcomes of X. eg. E: Throwing a die.

② Position of COM of P.m.f on a straight line is the expectation of X.

② **Continuous**

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Provided the Improper integration is absolutely convergent. Here f_X is pdf of X.
$E(X)$ is also called the mean of 'X', and is denoted by μ_X.

→ **Expectation of f^ns of R.V.** :-

eg. let X be a R.V. with spectrum $\{-1, 0, 1\}$.
$P(X=-1) = 0.2$, $P(X=0) = 0.5$, $P(X=1) = 0.3$.
Compute $E(X^2)$.

$Y = X^2$, Spectrum of $Y = \{0, 1\}$.
$E(Y) = 0 \cdot P(Y=0) + 1 \cdot P(Y=1)$

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$P(Y=0) = P(X=0) = 0.5$

$P(Y=1) = P(X=1) + P(X=-1) = 0.2 + 0.3 = 0.5$

$$\boxed{E(Y) = 0.5}$$

<u>Thm</u> :- Let X be a discrete random variable with spectrum $\{x_i : i = 1, 2, \ldots, n\}$, and pmf f_X. Then for any real valued f^n g.

$$\boxed{E(g(X)) = \sum_i g(x_i) \cdot f_X(x_i)}$$