Discrete Distribution: Definition

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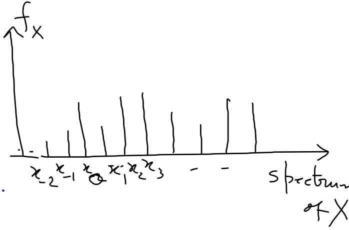
- Let spectrum of X: $T = \{x_i : i = 0, \pm 1, \pm 2, \ldots\}$ with

$$\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots$$

The function $f_X : \mathbb{R} \to [0,1]$, defined as

$$f_X(x) = \begin{cases} P(X = x_i), & x = x_i \in T \\ 0, & \text{elsewhere.} \end{cases}$$

is called the *probability mass function* (p.m.f.) of X.



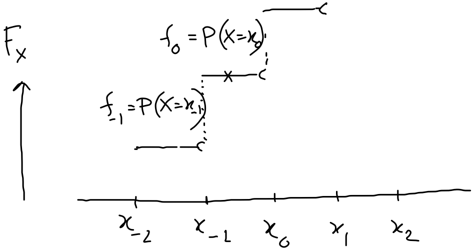
Discrete Distribution: Definition

- $F_X(x) = \sum_{x_j \le x_i} P(X = x_j)$ if $x_i \le x < x_{i+1}$

$$= \sum_{j=-\infty}^{i} f_X(x_j)$$

Discrete Distribution: Definition

- $F_X(x) = \sum_{x_j \le x_i} P(X = x_j)$ if $x_i \le x < x_{i+1}$
- F_X is a step function with steps $f_i = P(X = x_i)$ for $i = 0, \pm 1, \pm 2, \ldots$.



Discrete Distribution: Properties

1. $\sum_{j=-\infty}^{\infty} f_j = 1$

We have,

$$F_X(\infty) = 1$$

1. $\sum_{j=-\infty}^{\infty} f_j = 1$

2. At each non-spectrum point a, $P(X = a) = 0$

$$P(X=a) = F_X(a) - F_X(a-0)$$

let, $\exists\, x_k, x_{k+1} \in$ spectrum of X s.t.

$$\underbrace{x_k < a < x_{k+1}}$$

$$F_X(a) = \sum_{j=-\infty}^{k} f_X(x_j) \quad , \quad F(a-0) = \sum_{j=-\infty}^{k} f_X(x_j)$$

$$\Rightarrow P(X=a) = 0 .$$

Discrete Distribution: Properties

1. $\sum_{j=-\infty}^{\infty} f_j = 1$

2. At each non-spectrum point a, $P(X = a) = 0$

3. $P(a < X \leq b) = \sum_{a < x_i \leq b} f_X(x_i)$

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

1. Binomial (n, p) Distribution

$X \sim \text{Binomial}(n, p)$

Spectrum of $X = \{0, 1, 2, \ldots, n\}$

probability mass fn. (p.m.f)

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots \quad \ldots$$

$$= 0, \quad \text{elsewhere}$$

$(0 < p < 1)$

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = \left(p + (1-p)\right)^n = 1.$$

2. Poisson (μ) Distribution

$X \sim \text{Poisson}(\mu)$

Spectrum of $X = \{0, 1, 2, \ldots\}$

$\mu > 0$

<u>p.m.f.</u> $$f_X(x) = \frac{e^{-\mu}\mu^x}{x!}, \quad x = 0, 1, 2, \ldots$$
$$= 0, \quad \text{elsewhere}$$

$$\sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu} \boxed{\sum_{x=0}^{\infty} \frac{\mu^x}{x!}} = e^{-\mu} e^{\mu} = 1.$$

Continuous Distribution

Definition

The distribution of a random variable X is said to be continuous if

1. the distribution function F_X is continuous
2. $\frac{d}{dx}F_X(x) = F_X'(x)$ is piecewise continuous in $(-\infty, \infty)$

Define: $f_X : \mathbb{R} \to [0,1]$ as $f_X(x) = \frac{d}{dx}F_X(x)$ which is called the *probability density function* (p.d.f.) of X.

Continuous Distribution: Properties

1. $f_X \geq 0$

$f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$

$f_X(x) = \frac{d}{dx} F_X(x) \geq 0$ since $F_X(x)$ is mon. inc. fn.

1. $f_X \geq 0$
2. $P(a < X \leq b) = \int_a^b f_X(x)dx$

$P(a < X \leq b) = F_X(b) - F_X(a)$.

(i) $f_X(x) = \frac{d}{dx} F_X(x)$ (primitive exists)

(ii) $f_X(x)$ has finite no. of jump discontinuities (1st kind) in $[a,b]$ $\Rightarrow$ $f_X(x)$ is R. integrable in $[a,b]$.

$\Rightarrow$ Applying Fundamental Th. of Integral Calculus

$$\int_a^b f_X(x)dx = F_X(b) - F_X(a) = P(a < X \leq b).$$

Continuous Distribution: Properties

1. $f_X \geq 0$
2. $P(a < X \leq b) = \int_a^b f_X(x)dx$
3. $F_X(x) = \int_{-\infty}^x f_X(x)dx$

Continuous Distribution: Properties

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2. $P(a < X \leq b) = \int_a^b f_X(x)dx$
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Continuous Distribution: Properties

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5. $P(X = a) = 0$ for a given constant a

Continuous Distribution: Properties

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4. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
5. $P(X = a) = 0$ for a given constant a
6. **Converse statement:** Every non-negative, real-valued, piecewise-continuous function f that is integrable in $(-\infty, \infty)$ and satisfies $\int_{-\infty}^{\infty} f(x)dx = 1$, is the probability density function of a continuous distribution.

Continuous Distribution: Properties

7. **Probability Differential:** Let X has continuous distribution. In differential notation we write:

$$P(x < X \leq x + dx) = F_X(x + dx) - F_X(x) = dF_X$$

Continuous Distribution: Properties

8. **Density Curve:** The curve $y = f_X(x)$ is called the probability density curve of the corresponding continuous distribution.

Continuous Distribution: Examples

1. Uniform (a, b)

Continuous Distribution: Examples

2. Normal (m, σ)

Continuous Distribution: Examples

3. Cauchy (λ, μ) Distribution

Continuous Distribution: Examples

4. Gamma (Γ) distribution

Continuous Distribution: Examples

5. Beta (l, m) distribution of 1st kind

6. Beta (l, m) distribution of 2nd kind