Balanced Innovation: Nonlinear optimal patent breadth in an interaction of demand and technology networks

Dario Marino

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Abstract

We study the interaction of demand and technology networks to derive the optimal patent breadth for all the relevant structures representable in a 3-Firms setting. Each firm is connected at the technological level depending on the intersection of their patent sets. At the demand level firms are connected depending on the substitutability of their respective goods. Each edge is weighted on the demand network by its corresponding substitution parameter and on the technological network depending on the number of technologies in common. The 3-Firms compete à la Cournot. We compute the quantities and investments which maximize firms' profits. Firms can invest in all the technologies available in their patent sets, at a convex cost. This investment returns on average a linear cost reduction which will be transmitted to other connected firms depending on the weight of their technology links with the other firms and the externality level γ set exogenously by the regulator, which in our setting coincides with patent breadth. Finally, optimal investments will depend only on patent breadth γ. We will plug them back into the optimal quantities of the corresponding network to compute the optimal externality level γ^* which maximizes the general welfare for that network. The general welfare is the sum of consumer utility and firms' profits. Expanding to more complex structures with eight firms, such as Circle and Core-Periphery networks, confirms the non-linearities of optimal patent breadth. We diverge from the standard idea of increasing patent protection for increasing substitution levels between goods, giving evidence for a welfare balance with strong or full externality for most substitution levels and weak or no externality for some intermediate values.

Keywords: Network Economics, Intellectual Property, Industrial Organization

Introduction

The desired effect of patent regulation is to generate welfare benefits by rewarding technical progress. An important number of relevant papers examine a single innovation process. Nordhaus 1969 set the stage for this analysis by advocating that the length of patents should balance off two factors. First, the reward for innovation, thus patent breadth and length should be strong enough to stimulate innovation. Second, deadweight losses, thus the conditions for monopoly pricing should be reduced.

A crucial article to understand the idea behind my paper is Arora, Belenzon, et al. 2019. This economic history review identifies 3 phases of American innovation. The characterization is not completely new since we can find similar historical distinctions, for example, in Gertner 2012 and Khan and Sokoloff 2001. However, this is the first paper that I have found that characterizes the change in the innovation process and the need for a new approach towards intellectual property.

The paper starts with the old innovation ecosystem, which extends from 1850 to 1940. At the beginning of this period, firms relied majorly on individual investors, with the idea that it was not profitable to invest in corporate research. However, from 1900 we start to observe a transition toward the second phase, which will kick off in the postwar period and ends in 1980. In this period, antitrust pressures, the growing but still relatively small contribution of public universities, and economies of scale all contributed to the creation of corporate centers of research, such as Xerox's Palo Alto Research Center, DuPont Central Research, and Bell Telephone Laboratories (which then became AT&T Bell Laboratories). The beginning of this new period preceded an expansion of intellectual property rights (Henry and Turner 2006).

The last shift is the "New Innovation Ecosystem", which the study identifies as the period between 1980 to 2016. My model wants to contribute to a credible representation of this new ecosystem. In this way, it will be possible to envision adequate IP laws that will balance the two factors that we have seen before. At the beginning of the current innovation chapter, the specialization of universities disrupted the corporate research environment. Research faced increasing specialization, thus each branch has been divided into subgroups at the academic level, where researchers specialized in one of these subgroups. Companies could not sustain all the necessary investments in the vastly diversified work profiles and research objectives. Therefore there is a sort of comeback of firms relying on outside inventors. However, this time inventors are not isolated skilled figures but more groups of individuals closely related to academia who decided to convert some scientific findings into something marketable. Usually, if the idea is successful will be later sold to large companies. Sometimes companies are fundamental already before the commercialization step because they give the funding and intermediate products necessary to make an idea marketable. An example of the current widespread symbiosis between universities and companies can be observed in the interplay between computer science experts, who have an understanding of the research frontier, and companies such as Google which can offer funding and large datasets. OpenAI is only one of the last companies which have focused on a research frontier helped by a larger corporation (Microsoft in this case) at the funding and dataset level.

In synthesis, the key contribution of this paper to my study is the characterization of the recent innovation ecosystem which exhibits a deepening division of labor between universities that specialize in basic research, small start-ups converting promising new findings into inventions, and larger, more established firms specializing in product development and commercialization (Arora and Gambardella 1994).

This recent phase of American Innovation gave me a reason to speculate on two phenomena that will be integrated extensively into the paper. First, the fact that general-purpose technology such as ChatGPT did not require patent protection to be developed. This can be attributed to the fact that this technology requires a certain level of openness to other firms' and universities' contributions to be fully developed (Chesbrough 2003). Another reason that makes patent protection unnecessary, in this case, is that most of the firms which will use this technology freely are not competitors at the demand level for Microsoft and OpenAI. The second recent phenomenon that I want to represent in my model is the fact that novel technologies precede large company investments, which fund and adopt technologies mostly from during an intermediate stage of development. For this reason, I characterized the firm as endowed with a certain set of patent sets, which are just the technologies available to this firm to invest in to produce their single good at a lower marginal cost. Firms do not own the technologies because they are the result of the specialized academic process. However they can invest in these technologies with convex costs, and they can protect their improvements with a level of patent breadth set by the planner. The planner will choose this level of protection depending on the general welfare, which is the sum of firms' profits and consumers' utilities.

As we will see in the results of this paper, the dominant incentive in most cases will lead to a Balanced Innovation setting, where firms will invest ex-ante more for a less homogeneous market which yields higher profit and requires no protection due to low substitutability while investing ex-ante less for a high homogeneous market which also does not require protection due to its smaller level of investment at any point of the market. We will extend the model from three to eight firms, from lines and triangles to circles and core-periphery lattices, to observe the nonlinear trend within a single setting instead of conjecturing it through a comparison of simple networks. The conclusions remain the same, but they will be more immediate since we will see the nonlinear trend in each graph which will relate substitution and patent breadth parameters. For minor substitution levels, firms are protected by the extensive difference between their goods. While for high substitution levels, the need for protection is lower for two reasons. First, on the profit side, firms' optimal quantity is limited for any externality level due to consumers' market power which leads to lower profit for any patent breadth setting. Secondly, on the utility side, consumers are less harmed by a quantity reduction due to the high substitution between goods.

Section 1 will introduce the literature on intellectual property at the Industrial and Network Economics level. Section 2 will characterize the model. In section 3 we move on to the results chapter where we will derive optimal investments for all the network structures in the 3-Firms setting. Optimal investments will be derived to depend only on γ so that we can compute the optimal patent breadth γ^* for each network structure subject to intermediate and high levels of substitution. In section 4 we comment the results. Section 5 restates the conclusions from the comparison of the 3-firms networks with five more complex networks. Within these lattices, we can find the nonlinearities that, with three firms, were visible only by comparing all the different smaller structures. In section 6 we delve into a few extensions of the model to investigate the scope of our findings' generalization possibilities. In the final section 7 we will give the basis for a future econometrics study which will be necessary to test the conclusion reached in this study.

1 Literature review

The increasing importance of human capital in modern economic analysis sparked also interest to the topic of intellectual property. The increased interest, both from academia and from the lawmakers, generated some controversial articles such as Boldrin and Levine 2008. To put in prospect the reasons behind a clear conceptualization of firm entry and intellectual property, we need to outline how academia approached this topic. This literature review will divide the contributions between general innovation economic articles (more conceptual, usually from business schools) and the industrial organization articles, which have been more focused on the modelization of firm behavior when they can employ patents. The last subsection covers the network economics literature, to introduce the necessary tools that I will make use of. After this literature review, we will see the model, which is mainly based on network and applied IO. However, the focus of the model will be narrower, mainly targeted at the verification of the theoretical findings of innovation economics, and rationalization of them in a more structured framework.

1.1 Innovation economics articles

One topic of increasing importance in innovation economics articles is the market for technology. This concept broadly encompasses the transactions that happen thanks to the existence of intellectual property rights. A complete literature review on this topic has been done by Arora and Gambardella 2010. The main contributions of this concept have been the identification of investment incentives for firms. For example, tech based firms can gain from trade between technologies and can specialize in some research objective, thus developing a comparative advantage. Some firms can even specialize in the creation of technologies for third parties. Cognitive limitations could also incentivize basic research to better understand the competitors' technologies. A certain degree of this market could still exist with lower or no patents at all since the know-how cannot be easily replicated. However, looking at the historical context in Teece 2010, it is possible to recognize that the interest in innovation investment has been more linked to the perceived return of human capital, rather than linked to the development of a stable market for technologies. The funding of R&D during the Cold War was strongly directed by the state, while in the 90s firms started to expand their department and even outsource them. This change was probably driven by the increased return of human capital and by a reduction of state crowding out in R&D. Moreover, looking at firm behavior historically, it is possible to see that competencies, resources, and dynamic capabilities are not easily traded but rather are acquired or created "in-house". Lamoreaux and Sokoloff 2002 instead sustained the opposite, namely that patent laws created a stable market for technologies. They look at patent data to find evidence of comparative advantages, such as the increasing time devoted to innovation for intermediaries and talented inventors. However, this analysis mainly covered the nineteenth and the beginning of the twentieth century. Thus it analyzed more the effect of the introduction of a patent system, not an increase in patent breadth in a well-established IP framework. The historical increase in private research in the 90s also highlights the minor role that individual inventors have in modern economies. Recent high-tech patents do not show a peculiar behavior, even if they have some sector-specific characteristics. Webb et al. 2018 documented an expected spike in software protection demand, with a 60.2% increase in successful filings between 2000 and 2013, and a 168.6% increase in applications. The increase rate is steeper for many of the emerging technologies, such as drones, cloud computing, and machine learning. These new technological fields follow a similar path to past avant-garde industrial niches. They are characterized by rapid bursts of innovation from a relatively small group of inventors, followed by a slowing down in productivity per inventor since more competitors enter the market. American, Japanese, and Korean inventors took the lead. A wide proportion of them is employed in large firms with a robust patenting history.

For every discourse around IP, the elephant in the room is always pharmaceutical firms, which often argue for a special IP status derived from their type of product. Scherer 2010 highlights the risk costs and revenues for pharmaceutical firms. The principal difference concerns R&D expenditures as a share of total sales. In 2005 pharmaceutical R&D as a percentage of sales was around 20% while for all manufacturers was around 4%. However, the cost-sharing with public research is also different, even if more in line with other manufacturers. DiMasi 2000 estimates that 38% of the 691 new chemical ingredients approved in the US between 1963 and 1999 originated from sources other than the inventor applying to FDA. 1980s interviews with R&D managers and executives find that the role of R&D is perceived differently in this specific industry. For example, Mansfield, Schwartz, and Wagner 1981 found that the share of patents that would not have been developed in the absence of patent protection for pharmaceuticals had an average of 60%, while for all industries it was 14%. The ease of imitation in this industry seems to be the reason behind this important role of patents, even if other factors, like first-mover advantage, are still valid and should be counted to find an optimal patent breadth/length. These factors have been advanced recently by Boldrin and Levine 2008. In their model they assume a perfectly competitive economy, if a firm develops a product it means that the costs are lower than the revenue, and even if other producers can develop and sell the same product, they can just reduce the market share, and they will not drive the price down to a level that is profitable for second movers and unprofitable for first movers. This is different from what happens in a more advanced model of competition or even in a Cournot setting. However, the intuition and additional factors like the first-mover advantage or reverse engineering can be used to improve other models. There is confirmation for reverse engineering costs in the literature, such as Levin et al. 1987 and Mansfield, Schwartz, and Wagner 1981. Using survey evidence, they have found that imitating a new invention in a manufacturing firm was not free, but could cost as much as fifty to seventy-five percent of the cost of the original invention. The case studies highlighted by the authors are interesting to look at, such as the Red Hat software company.

A further refinement on the usefulness of open source has been done by Hippel 2010. Users may innovate if they want something that is not available on the market. Many users do not find exactly what they want on the market. Thus this could bring requested improvements to a product, and the only cost would be that other users can freely use the program, while the know-how and the management structure would remain to the creator. The rationale behind it is that there are low-cost innovation niches that can be exploited, it is just a question of giving the right incentives for the revelation of these niche preferences. An empirical analysis by Lerner and Tirole 2003 of the SourceForge database gave some interesting findings. End-users projects were more likely to have restrictive licenses, while developer's projects were less likely to have them. Projects designed to run on commercial operating systems with English as the primary language were less likely to have restrictive licenses. Projects with unrestricted licenses attract more contributors.

A synthesis between the positive externalities of open source and the protection from patents is the concept of Patent Pools. Studied by Quint 2006, Layne-Farrar and Lerner 2011 and Lerner and Tirole 2004, a patent pool contains the essential patents needed to produce a number of technologies by different firms. To be a Pareto improvement, a patent pool cannot contain nonessential patents because it could reduce the price of one technology, but may increase the prices of others. In the case of inclusion of nonessential patents, it tends to increase total producer surplus but hurts some individual participants. It increases also the profits of few outsiders patent holders. Pro-competitive pools are usually composed of complementary patents that lower license fees for manufacturers, in addition to lowering the transaction costs of obtaining licenses. Vertically integrated firms have similar efficiency gains inside themselves since every division has similar behavior. However, even for a large vertically integrated firm, it could still be beneficial to join a patent pool to reduce transaction costs with cross-licensing without carrying the same cost of expansion and coordination in production. We can find more on the management of innovation in vertically integrated firms in Aghion and Tirole 1994. In general, the overall welfare effect is likely to be positive but can be harmful if substitute technologies already exist, because they could introduce transaction cost inefficiencies. A pool containing only substitute patents decreases producer surplus and is likely to decrease total welfare.

A review of the legal studies will help also to reconcile the role and inefficiencies of patents. Starting from Lemley 2013 it is possible to recognize the problem of excessive patent breadth. To reduce excessive protection the author focuses directly on the patent office. After he discards some ineffectual proposals, like increasing or reducing entry fees, he proposes some improvements that had already been successful in reducing excessive patent breadth. For example, a double check before the approval of a patent by two examiners. Jaffe and Lerner 2006 argue that the increase in cost and time will be partly offset by a downward adjustment in patent demand. However, the authors suggest that the system must live with patent litigation, thus the cost of legal hurdles must be taken into account alongside patent breadth. The reforms that can be taken to reduce these inefficiencies are part of the law literature. For example, a reformulation of the presumption that approved patents have a standard of validity so elevated, that even the literature that the examiner did not cite could be ineffective in overturning a patent. The novelty requirement is another instrument of the patent office. However, the model developed by Scotchmer and Green 1990 shows that exists a bottom level of novelty where it is not profitable to disclose innovation any more because it could help competitors without giving to the patenting firm enough protection to increase revenues. Thus at some point, the novelty requirement becomes useless because firms will not patent small improvements to reduce information for other firms. A recent reform changed the framework that Jaffe and Lerner 2006 analyzed. In 2013 the USA abandoned its "first to invent" system (FTI), to move towards the "first-to-file" (FTF) patent priority requirement. In a first-to-file system, the right to the grant of a patent for a given invention lies with the first person to file a patent application for the protection of that invention, regardless of the date of the actual invention. The first to invent instead has been kept due to its intrinsic justice of rewarding the person or organization which first discovered the patented invention. However, taking into account the workload of the patent office and the increasing filling rate (also discussed thoroughly in Jaffe and Lerner 2006) the legislation moved to a first-to-file system which requires fewer verification steps and it is also more stable in terms of investment planning. There is a trade-off between the two, but in an environment with a specialized group of inventors accompanied by legal teams, and with patent offices already in work overload, it is clear that the most adequate balance is reached with a first-to-file legal framework.

We can cite the last study by Gilbert and Shapiro 1990 which argues for the equivalence between patent breadth and patent length. The Gilbert and Shapiro 1990 article arrives at the opposite conclusion of Klemperer 1990. In Klemperer 1990 we observe that patent breadth creates a trade-off for consumers between a patented good with a higher price and a non-patented good at a lower price. The scope of patent breadth increases the surplus of the firm which owns the patent while it leaves a narrower "product space" to the firm without the patent. The firms are disciplined by the consumer, which suffers a deadweight loss of substituting. The firm which owns the patent cannot set a price that surpasses the substitution cost. The optimal suggested patent is a short-lived patent with a wide breadth to immediately reward the innovating firm for the investment and then move to the unconstrained competition. A similar discipline happens in Gilbert and Shapiro 1990 but in this case regarding patent length. We have a similar substitution process, however, for the proposed infinite-lived patent with a low level of patent breadth, the trade-off is minimal, and even if infinitely repeated the deadweight loss is smoothed and discounted in time. While for the case of large and finite patent breadth, the deadweight loss is experienced immediately. It is clear that this confrontation does not settle the debate, since different factors from the innovation externalities of cumulative innovation, along with changing tastes and costs, are not taken into account. However, we can see that there is an equality of the two tools in the sense that they are both a way to compensate innovating firms by creating a monopoly privilege with subsequent deadweight loss. Therefore, by taking into account discounting, an increase in patent length can have the same effects, both on the rewarding side and on the evaluation of deadweight losses.

After having seen the theoretical contributions, we move to the quantitative models. The background given will be helpful since we want to test theoretical contribution in a more structured framework. Our framework will focus on patent breadth.

1.2 IO, competition policies, and patent strategies

A different prospect on the problem has been developed by Benoit, Galbiati, and E. Henry 2013. They have used both game theory and experimental evidence to show that, in a setting with dominant and follower firms, the so-called "parasites" will limit themselves to maintain alive the leader firm. The problem is that this setting does not take into account that competition is unstable and every firm could be both leader and competitor, and the level of competition between every firm could be different so that every player has different gains from the existence or the failure of a certain firm. Moreover, there is a clear problem with free riding in this setting. To take into account more complex interactions we could look both at international trade and industrial organization. In Akcigit and M. Melitz 2021, they develop a model of endogenous innovation for incumbent firms, where a new product of better quality partially displaces existing products, even the ones already commercialized by the firm. They see both cases of discrete-time maximization of one step ahead profits, and continuous-time maximization with forward-looking firms. The firms have at their disposal another tool to escape competition, apart from different typologies, which is product quality.

Their findings are in line with the past literature. They have found that liberalization leads to increased innovation and more patenting for exporting firms. The import competition effect is also positive in terms of productivity and innovation, however Autor et al. 2020 finds a negative effect on patenting in the US due to Chinese competition. This does not mean that innovation decreased, but rather that increased competition displaced few lower type of productivity firms, as advanced by Sampson 2016. The substantial positive effect of trade liberalization highlights some mechanisms that will be important to bear in mind even in a generic model of competition. Foreign competitors reduce the monopoly power of a firm (like a reduction in patent coverage would), thus increasing welfare, but also incentivizing firms to innovate to escape the increased competition. The two views that this model wanted to synthesize are the Schumpeterian higher innovation efficiency of monopolies, and the Arrow escape competition, which states that is useful to innovate when you have competitors. Other important trade models to cite are M. J. Melitz 2003 Eaton and Kortum 2002 and Caliendo and Rossi-Hansberg 2011 because they introduce the concept of heterogenous productivity in a competitive setting, where firms draw their productivity type from a given distribution and then competition outcome creates the subsequent productivity distribution.

Other contributions targeted specifically to patent decisions come from IO. For instance, Fan, Jun, and Wolfstetter 2018 develop a model to examine optimal patent decisions for an incumbent firm that has developed superior technology and needs to understand how to make use of this right in a downstream oligopoly market. They have found that restricting the number of royalties is never profitable. This is in line with Katz and Shapiro 1986, namely the "chutzpah mechanism" based on the threat of revealing for free the new technology to all the other competitors. To depart from this model the authors compute a take-it-or-leave-it offer in the second period, which will be accepted by every firm. Furthermore, it will not suffer from the credible threat problem, as the chutzpah mechanism did.

Different firms with compatible technologies could also make us of cumulative innovation. Following Clancy 2018 we can see a model of innovation where new technologies are derived by combining technological components already existing. The main findings are that knowledge is exhaustible, namely ideas are a finite number. However, as firms learn which technologies can be combined, new ideas become feasible. It means that science follows the diminishing returns mechanism, however, changes in paradigms and new prospects can be thought of as productivity improvements that increase constantly output in a diminishing returns environment. The model is then tested with around 80 years of US patent data. The econometric part shows that an increase in patenting in one sector is correlated with a subsequent increase in patents that share the component in question. Meanwhile, there is a decrease in identical patents. This means that productivity increase due to knowledge is aggregate and positive, while the technology in question falls victim of diminishing returns. These results suggest that the positive impact of learning on an existing patent is larger than the negative impact of fishing out.

The parallel between productivity and paradigm change can be linked with the established concept of "learning by doing" developed in Arrow 1962. A recent review by Thompson 2010 on this topic shows interesting developments. The microeconomic development of the concept is interesting for its flexibility of use. Passive learning in fact can be applied not only to the general process of increasing productivity but also to market inefficiencies like monopoly pricing. Imperfect monopoly pricing in fact could become "better" with time, in the sense that the monopoly could learn better how to increase profits, meanwhile, deadweight losses could also increase. Thus passive learning does not carry inefficiencies with itself but could exacerbate existing inefficiencies like the suboptimality of monopoly output. Passive learning will also put at stake the existence of price-taking equilibrium like the ones used by Boldrin and Levine 2008. If firms are heterogeneous in productivity and in marginal costs, they will produce different quantities and they will also adjust with time by better estimate the other players' distribution for narrower characteristics. For example by focusing exclusively on estimating the productivity distribution within their good typology. The fixed number of firms, industry concentration, and other concepts of stability have been proven flawed when tested rigorously.

Another IO adaptation mechanism for patent behavior has been studied in Takalo and Kanniainen 2000. In this paper, the authors develop a Cournot competition setting where firms are uncertain if they will make use of the technology patented to commercialize a product. The main mechanism that they highlight is the fact that an increase in technology value, due to patent protection, will also incentivize to take into account a larger period for strategic entry. This evaluation will result in aggregate entry delays and a decrease in welfare. Another behavior that contrasts with the idea of infinite monopoly has been observed in Lanjouw 1998, where the author observed that over half of the patents, commercialized or not, are useless after ten years. The trend could have been a sporadic spike in the software industry, but still, the fact that some patents could not be used should be taken into account, since it is relevant for every sector.

1.3 Network Literature

The idea of a spatial representation for different good typologies is used in the current IO and network economics debate, as we can see in Ederer and Pellegrino 2022 the spatial representation of product differentiation can be interacted both with common ownership, industry concentration, and innovation spillovers. For an exhaustive depiction of innovation spillovers in a spatial context, we can look at Bloom, Schankerman, and Reenen 2005. In this paper they represent firms both in the technology space and in the product market space. To make a real-world example, IBM, Apple, Samsung, and Intel are all close in the technology space but not all in the product space, for the most part, Intel is near IBM and Apple near to Samsung. Older examples of product and technology spatial models can be found in Jaffe 1998 and Branstetter and Nakamura 2003.

1.3.1 Cournot Competition

In Bimpikis, Ehsani, and İlkılıç 2019 we can see an example of Cournot competition in a network setting. The paper examines competition among firms in a networked environment, where an undirected graph determines which markets a firm can supply. Firms compete using the Cournot model and allocate their production output to connected markets. In my 3-Firms example, we will also have an oligopoly competition where firms compete on quantity. However, I do not characterize different markets, rather my edges connect technology and demand characteristics. This paper also characterizes the production quantities at the unique equilibrium for any network and identifies a connection between the equilibrium outcome and technology paths. It also studies the impact of changes in competition structure on profits and consumer welfare. However, here the planner establishes the optimal link between firm and markets (whether certain firms should expand in other markets) instead of the optimal technology externality γ (patent breadth).

Another contribution to the network structure of production comes from Oberfield 2018. This paper develops a theory where the network structure of production, which in this case is who buys inputs from whom, forms endogenously. Entrepreneurs produce using labor and one intermediate input and must decide which other entrepreneur's good to use as input. Their choices determine the economy's equilibrium input-output structure, generating substantial differences in size and shaping productivity. When the output elasticity to intermediate inputs is high, star suppliers emerge endogenously, raising aggregate productivity as more supply chains are routed through higher-productivity technologies. In our case instead, we do not have intermediate goods but rather our main differentiation comes from the different technologies which will receive different levels of investment depending on the intersection of patent sets and network structures. Also, the network structure in the cited paper is a result of its production function, which in the case of firm f_i that uses technology τ and produces good g is:

$$y_g = \frac{z_\tau \boldsymbol{x}_{\boldsymbol{f_i}}^{\alpha} l_{f_i}^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \tag{1}$$

Where y_g is the amount of good g generated with technology τ. x_{f_i} is the sum of goods acquired by f_i from the other firms to produce good g, that is $x_{f_i} = \sum_{g=a}^{G} x_{f_i}(g)$.

In our case, the network structure precedes production choices. In this way, we can observe the patterns between different networks, and compute the levels of patent breadth which maximize welfare. We will also compare the network structures to conjecture the driving force for a specific optimal patent breadth outcome.

1.3.2 Common Ownership

A network game of oligopoly with common ownership has been developed in Ederer and Pellegrino 2022. This paper studies the welfare implications of the rise of common ownership in the United States from 1994 to 2018. The authors build a general equilibrium model with hedonic demand and firms competing in a network game of oligopoly. Specifically, this network builds on the demand matrix using a Generalized Hedonic-Linear (GHL) Demand System. This demand system is based on the assumption that there is a representative consumer who has quadratic preferences over product characteristics (as opposed to plain product quantity used in the usual Cournot setting)

Common ownership induces unilateral incentives to soften competition, with the magnitude depending on network overlap. The model is estimated using firm financials, investor holdings, and text-based product similarity data. Counterfactual calculations show that the welfare cost of common ownership has increased nearly tenfold between 1994 and 2018, with a significant reallocation of surplus from consumers to producers.

In the model, two firms are connected through ownership overlap and product market rivalry. Ownership overlap here is not exclusively a technological improvement sharing, rather it has to do more with financial data on the holdings of each firm. Therefore the findings of this paper regarding the increase in welfare due to common ownership are not analogous to my setting with no patent breadth where firms share their technological improvement without sharing any of their profits.

The main idea behind this paper is that an underlying increase in the benefit of collaboration has incentivized the spike in common ownership. It is interesting to connect this with the study cited before about the new structure of American innovation since this is an example of micro theory and firm-level microeconometrics which gives a sense of the process behind a change that can only be detected in the historical series.

2 Model

2.1 Demand Network

The vector of goods is:

$$\mathbb{G} = \{1, 2, \ldots, N\}$$

Each firm produces one good:

$$||f_1 \Leftrightarrow 1||f_2 \Leftrightarrow 2||\ldots||f_n \Leftrightarrow n||\ldots||f_N \Leftrightarrow N||$$

The utility function of the consumers is:

$$U = \alpha \sum_n q_n - \sum_n \frac{q_n^2}{2} - \sum_{n \neq \eta} b_{n,\eta} q_n q_\eta - \sum_n p_n q_n \tag{2}$$

Its derivative with respect to quantity would give us the prices:

$$\frac{\partial U}{\partial q_n} = \alpha - q_n - \sum_{n \neq \eta} b_{n\eta} q_\eta = P_n \tag{3}$$

In the single good case, the competition effects of each good can be summarized in the following competition matrix.

$$\mathbf{B} = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,N} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N,1} & b_{N,2} & \cdots & b_{N,N} \end{pmatrix}$$

The matrix $\mathbf{B}$ captures substitutability and complementarity patterns between goods, and it is a positive definite matrix (a necessary condition for $U(g)$ to be strictly concave). In the $\mathbf{B}$ matrix the diagonal terms $b_{g,g} = b_g$ correspond to $\frac{-\partial^2 U}{\partial g}$, capturing the decreasing marginal utility. The off diagonal terms, $b_{g,j}$ $g \neq j$ correspond to $\frac{-\partial^2 U}{\partial g \partial j}$ and they capture the pattern of complementarity and substitutability between the goods. If $b_{g,j}$ is positive the two goods are substitute, if $b_{g,j}$ is negative the two goods are complements.

In the demand network, the nodes are the firms, which are linked depending on the substitutability/complementarity elements introduced in matrix $\mathbf{B}$. The links are weighted by the substitutability/complementarity values $b_{i,i'}$ between the goods that the linked firms produce.

2.2 Technology Network

The vector of technologies is:

$$\mathbb{T} = \{a, b, \dots, T\}$$

Each firm f_n is endowed with a certain number of technologies t_n from the vector of technologies $\mathbb{T}$. Each firm has its subset of $\mathbb{T}$:

$$\mathbb{T}_n = \{a, b, \dots, T_n\}$$

Firms can invest only in the technologies in their subset, and the value $z_{n,t}$ represents the amount invested by firm f_n on technology t.

Now we want to characterize a connection between firms on the technology side. Since we are talking about intellectual property, firms should be linked depending on how much the technologies that they use are similar to each other. We could use a matrix similar to the **B** matrix used in the demand network. However, it is not straightforward to see how much two technologies are "substitutable/complementary" between each other, since we don't have a characterization to derive how two technologies are considered substitutable or complementary to each other. We would need an external index that measures the similarity between technologies at the level of patents, like the Google "Patent Phrase Similarity Dataset". However, this is a theoretical section, thus we need to characterize similarity inside the model that we are developing.

Therefore, to take into account this link between technologies we define a vector $\mathbb{P}$ of patent sets:

$$\mathbb{P} = \{\mathbb{P}_a, \dots, \mathbb{P}_t, \dots, \mathbb{P}_T\}$$

Each patent set $\mathbb{P}_t$ contains the firms which can capture the improvements on that technology from the other firms in that set.

$$\mathbb{P}_t = \{[n, \eta, \dots]\}$$

To be more precise, the patented cost reduction improvements coming from the investments in technology t will be shared between the members of the patent set depending on the externality parameter γ, which represents patent breadth. The patent breadth parameter can go from no externality $\gamma = 0$ to full externality with $\gamma = 1$. Patent breadth is stronger at $\gamma = 0$ because firms cannot convert, even partly, outside improvement, while is absent with

$\gamma = 1$ because all the improvements are experienced in the same magnitude by the firm that invests and the firms that are in the same patent set.

It is also possible that separate groups of firms can invest in the same technologies without being allowed to receive and also share the benefits of the patented improvements. In that case, the patent vector will contain two or more patent sets for the same technology, depending on the characteristics of the technology network.

$$\mathbb{P}_t = \{[n, \eta, \ldots]\}$$
$$\mathbb{P}'_t = \{[n', \eta', \ldots]\}$$

In the technology network, the nodes are the firms. The edges are undirected and weighted by the number of patent sets shared by the two firms. In other words, firms are connected depending on the technologies in which both firms invest, with the feature to absorb and share the improvements found for the technologies in the patent set.

2.3 3-Firms Example: Prices and Costs

Now we move to the 3-Firms example since we already conveyed an idea of the general model. The same process that I described before and that I will describe here can be used to represent the interaction between an arbitrary number of firms. We have three firms, they choose their investment levels and compete in quantity in 2 periods. We will represent the case for the complete network. The variations from one network to another can suppress certain links to observe different behaviors. However, when two links are operative in the demand or technology network, they will have the same attributes that we will see for the complete network. We have the following inverse demands:

$$\begin{aligned} P_1 &= \alpha - q_1 - b_{1,2}q_2 - b_{1,3}q_3 \\ P_2 &= \alpha - q_2 - b_{2,1}q_1 - b_{2,3}q_3 \\ P_3 &= \alpha - q_3 - b_{3,1}q_1 - b_{3,2}q_2 \end{aligned} \tag{4}$$

The value of $b_{g,j}$ can go from -1 in case of perfect complements to 1 in case of perfect substitutes. The value $b_{g,j}$ comes from the matrix $\mathbf{B}$ with a non-uniform distribution $b \in [-1, 1]$. We set the same $b_{g,j}$ between all goods, with two variations. We will have demand networks with intermediate substitution for $b_{1,2} = b_{1,3} = 0.5$, and high substitution for $b_{1,2} = b_{1,3} = 0.8$. The substitution parameters are symmetric. We will compute optimal values for 5 types of network interactions with both types of substitution levels. Indeed we will have 10 cases to analyze in the end.

In the first period, each firm decides its level of R&D investment. The research will start immediately and will be paid at the second period, whether the improvement happens or not.

In the second period, the firms compete in quantity. The R&D direct and indirect improvements are experienced through a linear reduction of the unit production cost.

The vector of technologies in our 3-Firms case is:

$$\begin{aligned} \mathbb{T}_1 &= \{a, b\} \\ \mathbb{T}_2 &= \{a, b, c\} \\ \mathbb{T}_3 &= \{a, b\} \end{aligned} \tag{5}$$

The vector of patent sets in our 3-Firms setting is:

$$\begin{aligned} \mathbb{P}_a &= \{[1, 2, 3]\} \\ \mathbb{P}_b = \{[1, 2]\} \; ; \; \mathbb{P}'_b &= \{[3]\} \\ \mathbb{P}_c &= \{[2]\} \end{aligned} \tag{6}$$

The patent sets vector has been selected to be as interesting as possible by capturing general behavior from an interaction between 3 firms. The general characterization would require to estimate what are the technologies in which each firm can invest and its opportunity to convert in its productive process other firms' improvements.

By observing the patent sets we can see that technology a can be improved by all the firms, which can also absorb all the patented improvements depending on γ. For technology b instead, we have that all firms can invest in it, but the improvements of Firm 3 cannot be absorbed by the other firms, and Firm 3 cannot absorb the improvements of firms 1 and 2. The latter firms instead can share their patented cost reduction depending on γ. Technology c is available only to Firm 2, therefore its improvements will be experienced only by Firm 2, for any γ.

In the technology network, the nodes are the firms. The edges are undirected and weighted by the number of patent sets that the two firms share. We can call this weight ϕ and thus, in the 3-Firms setting, we have:

$$\{\phi_{1,2} = 2; \phi_{2,3} = 1; \phi_{3,1} = 1\}$$

The weight is symmetric since represents how many technologies the two firms share. As we have seen in the patent sets vector, Firm 3 is connected on only one technology a with the other 2 firms, while the other 2 firms have also technology b in common.

$\phi_{n,n} * 2z_{n,t}^2$ is the fixed cost of investment chosen by each firm which on average yields z_t in terms of linear marginal cost reduction. Firms are risk-neutral so their decisions are based on the average return on investment. We choose separate choices for each investment to allow firms to invest in technologies that are not in common with other firms. However, their return is linear and the cost of each one is convex, so there is still an incentive to invest a positive amount in all technologies. The cost also depends on the number of technologies owned by each firm. In this way, we want to capture the fact that a firm with a larger number of patents will also face larger fixed costs for each patent, due to the increasing complexity of its production structure. This higher fixed cost should reinforce the need for IP protection for these technologically intensive firms. With this specification, we want also to capture a low-hanging fruit mechanism (Cowen 2011), in the sense that a larger share of firms will invest in safer technologies while the more experimental firms will have to spend more since they are investing in less explored combinations of technologies. We assign the subsequent number of technologies to each firm, making Firm 2 the technology-intensive one. We can see the number of technologies available to each firm in the following set:

$$\{\phi_{1,1} = 2; \phi_{2,2} = 3; \phi_{3,3} = 2\}$$

$\phi_{n,\eta}$ is the number of patents in common between two firms. It has the same value of the weight that the technology network assigns to each pair of firm. Again this value is exogenous because it comes from the assigned patent sets that each firm has.

The five networks will activate or suppress different edges but whenever two firms are connected these will be the weight from the common patent sets. Same idea goes for the demand case from the section before. We have shown the values for complete networks and the only change in the structure will be the suppression of some of these links (namely they will be set to 0).

The investment on technology will be experienced at different levels by the connected firms depending on γ. In the extreme cases we have that with $\gamma = 0$ they both invest in the same technology without sharing their improvements, while with $\gamma = 1$ their investment in the same technology are fully shared.

γ is the exogenous level of patent breadth set by the planner. In the end, this is the optimal value that we have to choose to maximize welfare. We must think of patent breadth not as a litigation measure, but more as a degree of positive externality from innovation that is exogenously set by the planner. The planner must balance the social return of innovation with the incentive to invest.

The investment decisions can be summarized in the following vectors:

$$\mathbb{Z}_n = \{z_{n,a}, z_{n,b}, \ldots, z_{n,t}\}$$

In our 3-Firms case:

$$\begin{aligned} \mathbb{Z}_1 &= \{z_{1,a}, z_{1,b}\} \\ \mathbb{Z}_2 &= \{z_{2,a}, z_{2,b}, z_{2,c}\} \\ \mathbb{Z}_3 &= \{z_{3,a}, z_{3,b}\} \end{aligned} \tag{7}$$

Finally, the unit production cost for all firms experienced at the second period is:

$$C_n = c - \sum_{t_n}^{T_n} z_{n,t_n} - \gamma \sum_{t_\eta \neq t_n}^{T_{\phi n,\eta}} z_{\eta,t_\eta}.$$

Explicitly:

$$\begin{aligned} C_1 &= c - \sum_{t_1}^{T_1} z_{1,t_1} - \gamma \sum_{t_2}^{T_{\phi_{1,2}}} z_{2,t_2} - \gamma \sum_{t_3}^{T_{\phi_{1,3}}} z_{3,t_3} \\ C_2 &= c - \sum_{t_2}^{T_2} z_{2,t_2} - \gamma \sum_{t_1}^{T_{\phi_{2,1}}} z_{1,t_1} - \gamma \sum_{t_3}^{T_{\phi_{2,3}}} z_{3,t_3} \\ C_3 &= c - \sum_{t_3}^{T_3} z_{3,t_3} - \gamma \sum_{t_1}^{T_{\phi_{3,1}}} z_{1,t_1} - \gamma \sum_{t_2}^{T_{\phi_{3,2}}} z_{2,t_2} \end{aligned} \tag{8}$$

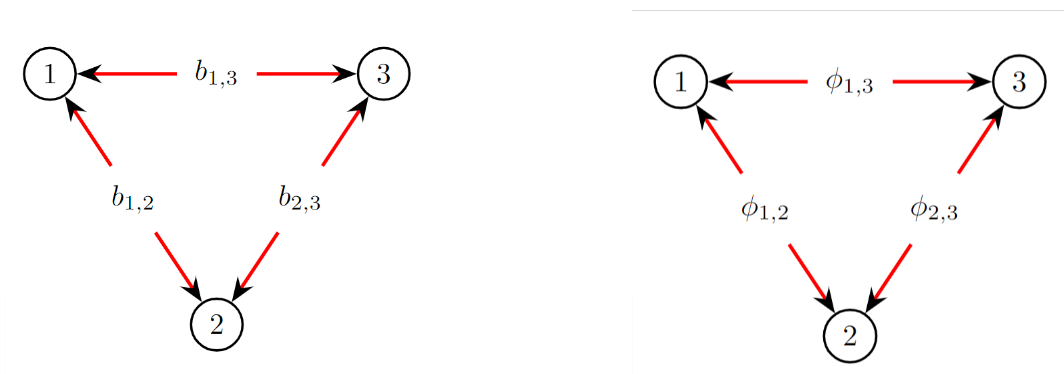
$$\begin{aligned} C_1 &= c - z_{1,a} - z_{1,b} - \gamma z_{2,a} - \gamma z_{2,b} - \gamma z_{3,a} \\ C_2 &= c - z_{2,a} - z_{2,b} - z_{2,c} - \gamma z_{1,a} - \gamma z_{1,b} - \gamma z_{3,a} \\ C_3 &= c - z_{3,a} - z_{3,b} - \gamma z_{1,a} - \gamma z_{2,a} \end{aligned} \tag{9}$$

γ is a value between 0 and 1 which represent how much the patent is protected. There is no licensing process here, if $\gamma = 0$ it means that the admitted externality set by the regulator is equal to 0. Therefore no one can use the technological improvement outside of the firm which discovered it. Instead with $\gamma = 1$ everyone in the same patent set can use the technology improvement of every firm discovered with the R&D investment, because there is no IP law which forbids it. As we will see later the intermediate case is rare for this setting, however for intermediate values it means that the firm can absorb in its process the technological improvement of another firm up to the point set by the regulator, otherwise it will incur in a costly litigation. Therefore even in the intermediate case there is no licensing process.

To continue with our example we now represent how the firm are linked in both networks.

2.4 Comparison between 5 different network structures

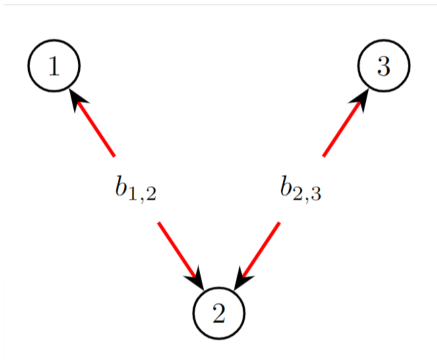
We now show 5 networks, labeled from A to E. The combination of two network makes possible to have several interesting cases resulting from the interaction between the two. With 3-Firms the interesting cases come from lines and triangle networks' shapes. Some results of this comparison can be generalized to higher nodes structures, while certain conclusion are visible only due to the low interconnection possibilities of 3-nodes networks.



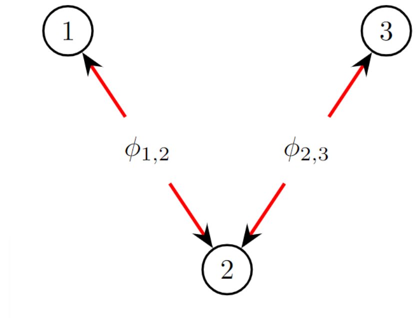
(a) Demand

(b) Technology

Figure 1: Network A

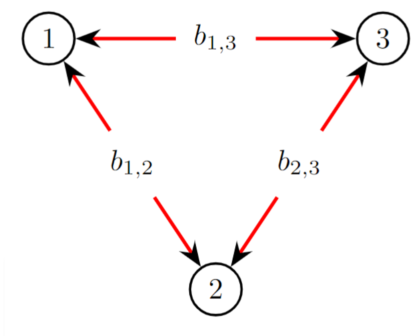


(a) Demand

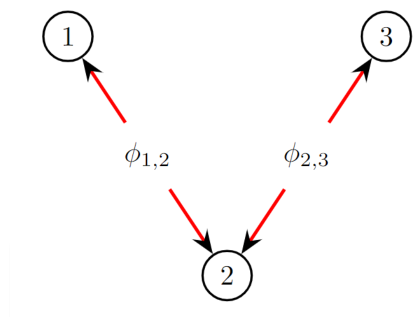


(b) Technology

Figure 2: Network B

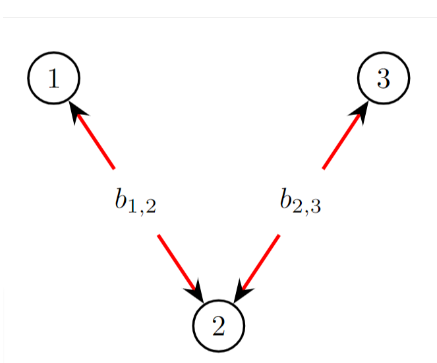


(a) Demand

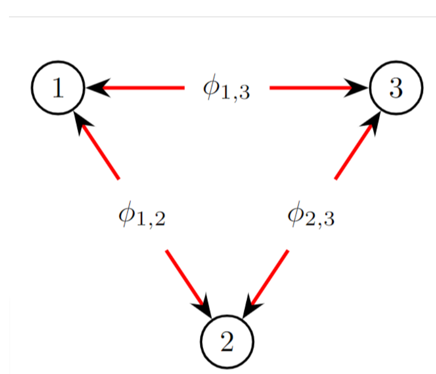


(b) Technology

Figure 3: Network C



(a) Demand



(b) Technology

Figure 4: Network D

(a) Demand (b) Technology

Figure 5: Network E

3 Results

3.1 Optimal Quantities

We compute the optimal quantities in the usual Cournot fashion. The first derivative, with respect to one good, of the consumers' utility function returns the prices set by the firm that produces that good. Prices minus costs multiplied by the quantity produced by that firm give us the profits for that firm. Since all the prices depend on the other firm quantities we have to solve a system of 3 equations to find the optimal quantities of each firm. The 3 equations of the system that we have to solve to find the optimal quantities are the first derivatives of the profits, with respect to quantity, set to 0. With our setting we have 3-Firms $1 \neq 2 \neq 3$ which produce one good each.

Prices and Costs are:

$$\begin{aligned} P_1 &= \alpha - q_1 - b_{1,2}q_2 - b_{1,3}q_3 \\ P_2 &= \alpha - q_2 - b_{2,1}q_1 - b_{2,3}q_3 \\ P_3 &= \alpha - q_3 - b_{3,1}q_1 - b_{3,2}q_2 \end{aligned} \tag{10}$$

$$\begin{aligned} C_1 &= c - z_{1,a} - z_{1,b} - \gamma z_{2,a} - \gamma z_{2,b} - \gamma z_{3,a} \\ C_2 &= c - z_{2,a} - z_{2,b} - z_{2,c} - \gamma z_{1,a} - \gamma z_{1,b} - \gamma z_{3,a} \\ C_3 &= c - z_{3,a} - z_{3,b} - \gamma z_{1,a} - \gamma z_{2,a} \end{aligned} \tag{11}$$

When two firms are not demand-connected we can set $b_{n,\eta} = 0$. When two firms are not technology-connected we can set the value $\sum_{t_\eta}^{T_{\phi_{n,\eta}}} z_{\eta,t_\eta} = 0$. Throughout all networks for the 3-Firms case, we have never found a negative optimal investment value for all the patent breadth levels. Negative investments values $z_{n,t_n} < 0$ could be also interpreted as a disinvestment procedure, where a firm can decide to obtain immediate returns by selling its discovered improvements at the cost of increasing its linear costs in the second period. If negative investments stay at a reasonable proportion with respect to the market parameter α and c, it makes sense to interpret $z_{n,t_n} < 0$ in such a way. Besides, it would have been possible to set negative investments to 0. In that case, the system of equations should be derived for all the levels of γ, at least 101 times with 0.01 steps from 0 to 1, because each time the optimal level of investment for one firm is set to 0, the other firms should readjust their investment decisions too.

We remind that we set:

$$\alpha = 10, c = 5, \phi_{1,2} = 2, \phi_{1,3} = 1, \phi_{2,3} = 1, \phi_{1,1} = 2, \phi_{3,3} = 2, \phi_{2,2} = 3$$

The patents held by each firm and the patent sets intersections are set so that we can observe the most interesting cases in a 3-Firms setting. α is the utility per quantity and c is the marginal production cost. These two parameters only need to have a reasonable proportion between each other to depict the behavior of a representative market.

Profits then are $\Pi_n = (P_n - C_n)q_n - \phi_{n,n}2z_{n,t}^2$. In our numeric example:

$$\begin{aligned} \Pi_1 &= (P_1 - C_1)q_1 - 4(z_{1,a})^2 - 4(z_{1,b})^2 \\ \Pi_2 &= (P_2 - C_2)q_2 - 6(z_{2,a})^2 - 6(z_{2,b})^2 - 6(z_{2,c})^2 \\ \Pi_3 &= (P_3 - C_3)q_3 - 4(z_{3,a})^2 - 4(z_{3,b})^2 \end{aligned} \tag{12}$$

To derive optimal quantities we derive each profit function with respect to quantity (fixed costs disappear). We will have then to solve with respect to quantities this system of 3 first derivatives set to 0. For $\Pi_n(q_n)$:

$$\begin{aligned} 0 &= (\alpha - b_{1,2}q_2 - b_{1,3}q_3 - c + z_{1,a} + z_{1,b} + \gamma z_{2,a} + \gamma z_{2,b} + \gamma z_{3,a})/2 - q_1 \\ 0 &= (\alpha - b_{2,1}q_1 - b_{2,3}q_3 - c + z_{2,a} + z_{2,b} + z_{2,c} + \gamma z_{1,a} + \gamma z_{1,b} + \gamma z_{3,a})/2 - q_2 \\ 0 &= (\alpha - b_{3,a}q_1 - b_{3,b}q_2 - c + z_{3,a} + z_{3,b} + \gamma z_{1,a} + \gamma z_{2,a})/2 - q_3 \end{aligned} \tag{13}$$

Solving this system returns the optimal quantities produced that we will plug back into the profits functions of each firm to find the optimal investments. After we plug optimal quantities in, the profits $\Pi_n(z_{n,t_n}, \gamma)$ will depend only on the externality parameter γ and all the investment levels z_{n,t_n}.

3.2 Optimal Investments

Before moving to profit maximization with respect to quantity, we need to check the concavity of this function with respect to the investment levels of every technology that each firm can choose to invest in. We have to do this because for a profit function to have a maximum point, its first derivative must be set equal to zero, and its second derivative must be negative. Therefore, when the profit function is concave, the marginal revenue decreases faster than the marginal cost, otherwise, we would not reach an optimal level of investment because increasing returns would be everlasting and it would be always convenient for the firm to spend more on R&D for any level of investment.

For revenues equal to $R_n(z_{n,t_n}, \gamma)$, and total costs $C_n(z_{n,t_n}, \gamma)$, the first derivative of the profit function is:

$$\Pi'(z_{n,t_n}, \gamma) = R'_n(z_{n,t_n}, \gamma) - C'_n(z_{n,t_n}, \gamma)$$

Setting this equal to zero, we have:

$$R_n'(z_{n,t_n},\gamma) = C_n'(z_{n,t_n},\gamma)$$

This means that at the maximum point, the marginal revenue is equal to the marginal cost.

The second derivative of the profit function is:

$$\Pi''(z_{n,t_n},\gamma) = R_n''(z_{n,t_n},\gamma) - C_n''(z_{n,t_n},\gamma)$$

For the profit function to be concave, this result must be negative:

$$R_n''(z_{n,t_n},\gamma) - C_n''(z_{n,t_n},\gamma) < 0$$

Therefore, when the profit function is concave, the second moment of the revenues decreases faster than the second moment of marginal costs. Thus the function starts at a point where marginal revenue with respect to investments is above costs of investments but converges to a point where marginal revenue is equal to marginal costs.

Our profit function has a linear and convex return on investment (linear cost reduction, convex interaction terms with γ and the adversarial effect on competitors) while the fixed costs $\phi_{n,n}2z_{n,t}^2$ are convex. In our second derivative, we find a negative drift that comes from the convex costs, while the positive drift comes from the second-order interactions both at the $b_{n,\eta}$ demand level and at the γ technology level.

The γ second-order effects have negative drifts because they help competitors reduce their costs and thus increase their quantities. However, they also have positive second-order effects if they help competitors of competitors because in that case, they reduce the quantity produced by competitors. When the patent set is composed only of one firm, the second derivative will always be negative since it is just the second derivative of the convex cost (for example, in the case of Firm 2 investing in technology c or Firm 3 investing in technology b).

All the second derivatives are checked for concavity with a Python code available on GitHub. The other codes implemented for all the computations cited in this paper are available in the same repository, with a branch for each stage. The link to the GitHub repository is available in the Appendix I.

In summary, for our example we need to check that:

$$\frac{\partial^2 \Pi_n(z_{n,t_n},\gamma)}{\partial z_{n,t_n}^2} < 0$$

For every firm, we have that each profit function depends on all the other firms' investment decisions. We partially (second) derive each function for the choices available by the single firm because the investment needs to be maximized only for the levels that are under the control of the firm in question.

$$
\begin{gathered}
\frac{\partial^2 \Pi_1(z_{n,t_n}, \gamma)}{\partial z_{1,a}^2} < 0 \\
\frac{\partial^2 \Pi_1(z_{n,t_n}, \gamma)}{\partial z_{1,b}^2} < 0 \\
\frac{\partial^2 \Pi_2(z_{n,t_n}, \gamma)}{\partial z_{2,a}^2} < 0 \\
\frac{\partial^2 \Pi_2(z_{n,t_n}, \gamma)}{\partial z_{2,b}^2} < 0 \\
\frac{\partial^2 \Pi_2(z_{n,t_n}, \gamma)}{\partial z_{2,c}^2} < 0 \\
\frac{\partial^2 \Pi_3(z_{n,t_n}, \gamma)}{\partial z_{3,a}^2} < 0 \\
\frac{\partial^2 \Pi_3(z_{n,t_n}, \gamma)}{\partial z_{3,b}^2} < 0
\end{gathered} \tag{14}
$$

After we check for concavity we come back to the first derivatives of profits set equal to 0. We have already optimal quantities, so at this step the system that we solve will give us the optimal investments for every firm which maximize each one profit. We have seven version of the following derivative for our system:

$$
\frac{\partial \Pi_n(z_{n,t_n}, \gamma)}{\partial z_{n,t_n}} = 0
$$

For every firm, we have that each profit function depends on all the other firms' investment decisions. We partially derive each function for the choices available by the single firm because the investment needs to be maximized only for the levels that are under the control of the firm in question.

$$\begin{aligned}
\frac{\partial \Pi_1(z_{n,t_n},\gamma)}{\partial z_{1,a}} &= 0 \\
\frac{\partial \Pi_1(z_{n,t_n},\gamma)}{\partial z_{1,b}} &= 0 \\
\frac{\partial \Pi_2(z_{n,t_n},\gamma)}{\partial z_{2,a}} &= 0 \\
\frac{\partial \Pi_2(z_{n,t_n},\gamma)}{\partial z_{2,b}} &= 0 \\
\frac{\partial \Pi_2(z_{n,t_n},\gamma)}{\partial z_{2,c}} &= 0 \\
\frac{\partial \Pi_3(z_{n,t_n},\gamma)}{\partial z_{3,a}} &= 0 \\
\frac{\partial \Pi_3(z_{n,t_n},\gamma)}{\partial z_{3,b}} &= 0
\end{aligned} \tag{15}$$

Solving this system will give us the optimal level of investment for each technology by each firm. Due to the investments' interactions, each first derivative of the profit function depends on the other amounts of technology investments. However, this system finds the values of all the z such that all the first partial derivatives with respect to investments are equal to 0. At this point, the investment levels depend exclusively on the externality term γ, which we have used to represent patent breadth.

$$z^*_{1,a}(\gamma)\ ;\ z^*_{1,b}(\gamma)\ ;\ z^*_{2,a}(\gamma)\ ;\ z^*_{2,b}(\gamma)\ ;\ z^*_{2,c}(\gamma)\ ;\ z^*_{3,a}(\gamma)\ ;\ z^*_{3,b}(\gamma)$$

In the output files available in the GitHub repository, we have plugged in the optimal investments at different levels of γ to observe some structural differences valid for every firm. We have found some predictable yet reassuring patterns. For $\gamma = 0$, the investment size is equal for all technologies within a certain firm. This trend happens because the investment is fully protected, so there is no need to diversify investments since it is more efficient to profit with convex costs by investing a balanced amount in both technologies. The linear and nonlinear effects on profits here would be the same since the cost reduction is equal for all patented improvements within a certain firm. Contrarily, by increasing γ, an increasingly extensive amount of R&D goes to the technologies that are less connected with the other firms.

This trend is valid for all of our cases except when, for example, a firm shares the technology for which they have patented improvements only with one firm that does not compete at the demand level with the firm in question but is connected at the demand level with a competitor of the firm in question. In that case, a larger level of investment will be devoted to the shared technology the more γ increases because by capturing part of the cost reduction, the other firm will incentivize the competitor to reduce its quantity. This event does happen in network D for firm 1. Since Firm 2 is connected to both technologies, for all $\gamma > 0$, Firm 1 invests increasingly more in the technology shared with Firm 3, which is a competitor of Firm 2 but not a competitor of Firm 1. Instead, Firm 3, which is connected to technology a to both firms but on technology b with no one, chooses to invest more in technology b to avoid helping Firm 2.

3.3 Optimal Patent Breadth

Now we plug back the optimal investment values into the profit functions. We want profits to be only a function of γ.

Since quantities $q_n(z_{n,t_n}, \gamma)$ were a function of γ and the investments of all firms z_{n,t_n}, we had to find the investment levels that depended only on γ. Solving the system of 7 equations for the optimal values of investment returns the amount of investment that depends only on γ and not on the other investments, in a similar way that optimal quantities do not depend on other quantities.

Since each value represents the value that maximizes profits for the firm that controls the investments' decision, if every firm takes the values that maximize profits of other firms as given and chooses the investment values in such a way as to maximize profits, all the values of investment will be given by the maximum point reached in the concave profit functions. The only unknowns will be the n-th order values interacting with γ.

Since investments depend only on patent breadth, with the new optimal values $z^*_{n,t_n}(\gamma)$ plugged inside the optimal quantities, profits are now composed of optimal quantities and optimal investments, thus profits too depend only on γ. Aggregate and single firms' profits can be expressed and plotted for all the values of γ in the continuum between 0 and 1. The plots for aggregate profits are available in Appendix II.

Regarding utility, we need to come back to the utility functions for the consumers:

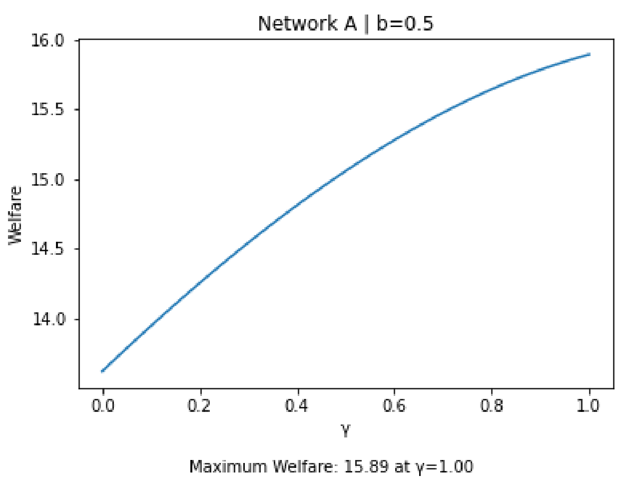
$$U = \alpha \sum_n q_n - \sum_n \frac{q_n^2}{2} - \sum_{n \neq \eta} b_{n,\eta} q_n q_\eta - \sum p_n q_n \tag{16}$$

The first derivative of the utility function with respect to unknown quantity gave us the prices. Now we have to plug back in the utility function the optimal quantities and prices. Since we have all the prices, quantities, and investment levels expressed in terms of γ, the utility can be plotted for all the values of γ in the continuum between 0 and 1. The plots for consumers' utility are available in Appendix III.

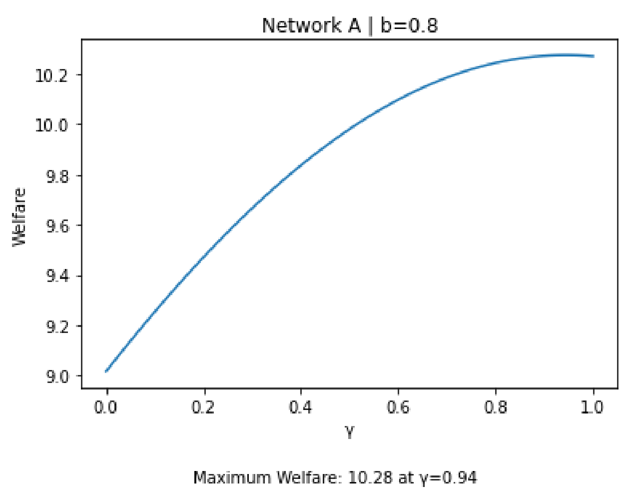
By adding utility and profits together, we have the Welfare function. We have plotted it for all the values of γ in the continuum 0 and 1.

We will see for all 5 Networks structures, with both intermediate and high levels of substitution, what are the welfare values at every level of γ, and thus what is the optimal externality term γ which maximizes welfare for every network. These plots are shown in the following subsection. We will comment on the results in the Discussion section.

3.4 Comparative statics

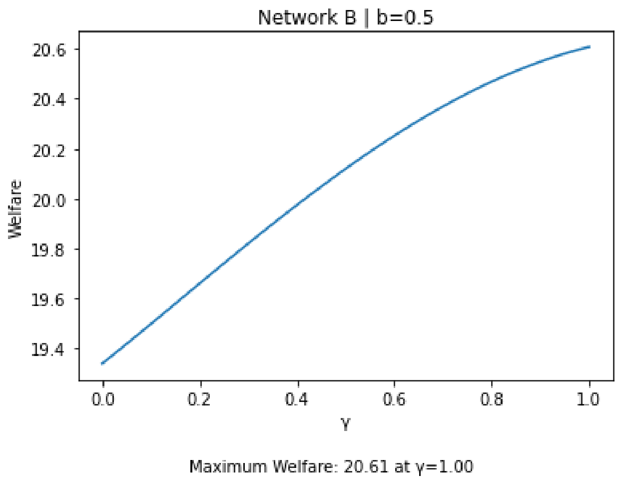


(a) Intermediate Substitution

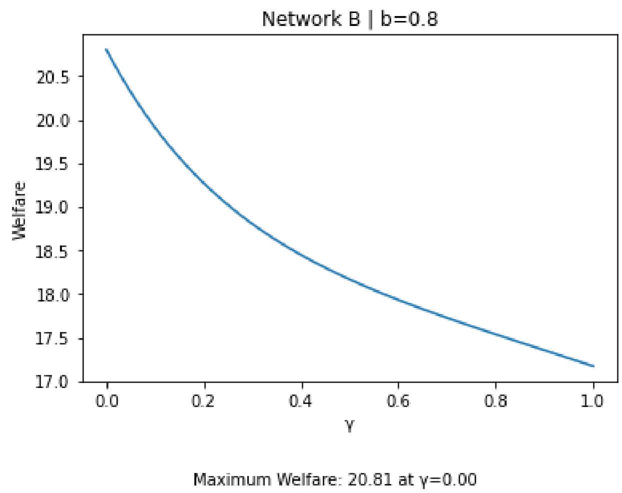


(b) High Substitution

Figure 6: Network A

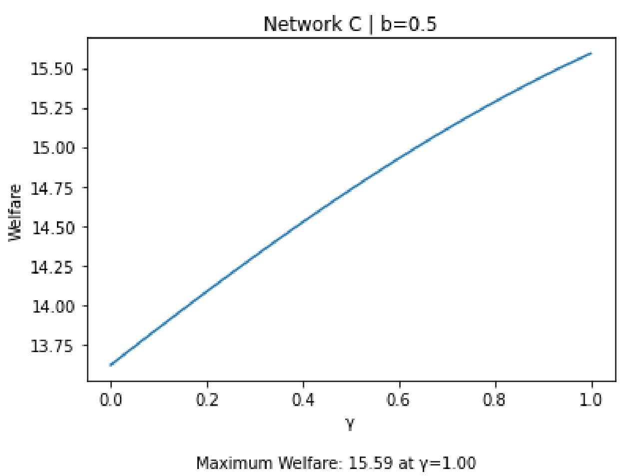


(a) Intermediate Substitution

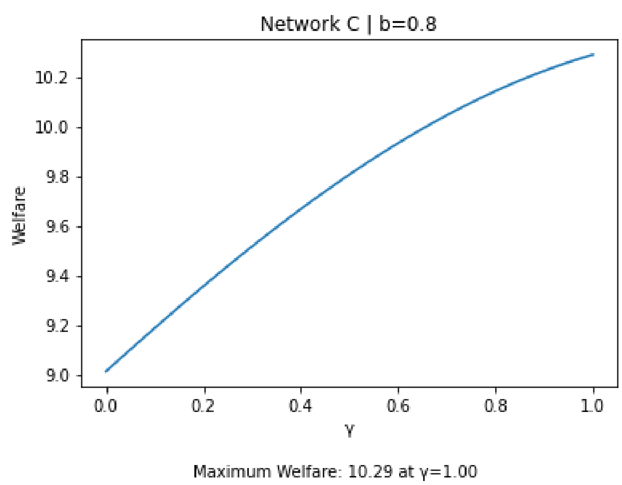


(b) High Substitution

Figure 7: Network B

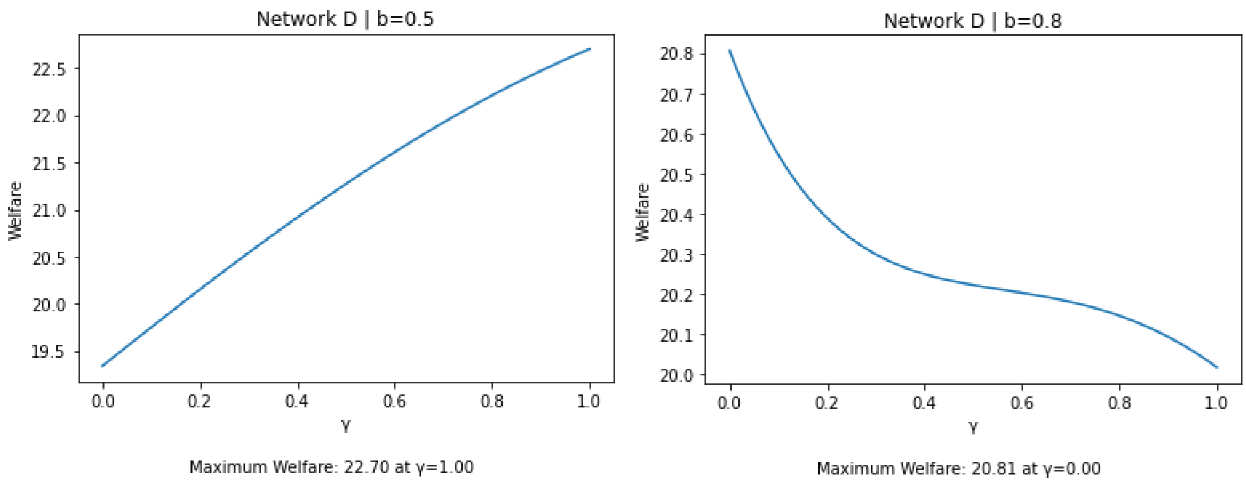


(a) Intermediate Substitution



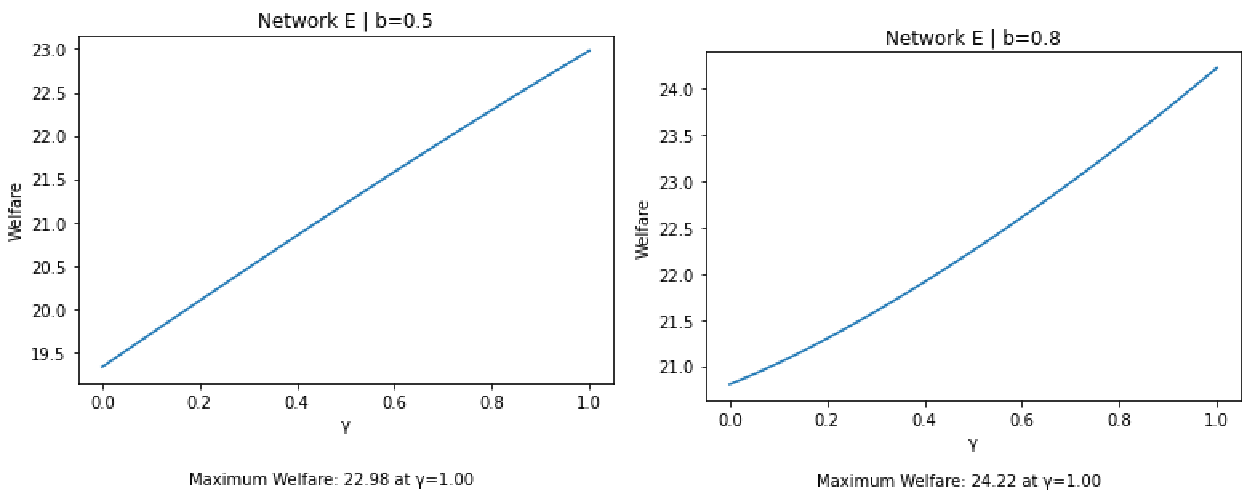
(b) High Substitution

Figure 8: Network C



(a) Intermediate Substitution (b) High Substitution

Figure 9: Network D



(a) Intermediate Substitution (b) High Substitution

Figure 10: Network E

4 Discussion

All the networks with intermediate substitution maximize welfare with $\gamma^* = 1$. Regarding high substitution, two networks out of 5 maximize welfare with $\gamma^* = 0$. One maximizes it with $\gamma^* = 0.94$ and the other two with $\gamma^* = 1$. This result means that in 8 Networks out of 10, the welfare maximization is obtained with full externality (or almost full externality in one case).

Regarding the firms which, in aggregate, invest more at equilibrium, we have found that firms 2 and 3 are the ones that spend more in absolute terms in R&D. Firm 2 invests more in four networks, but in network C with intermediate substitution, we can see that it is almost at the same level of Firm 3. Firm 3 invests more in 6 networks. In two particular networks (the only ones with $\gamma^* = 0$) it invests at the same level as A.

The reader could find it puzzling that the firm with more patents (Firm 2) does not invest much in the cases with $\gamma = 0$. However, in those settings, Firm 2 does not invest much with all levels of γ because is competing with two firms that do not compete with each other, thus it is strongly disadvantaged in a quantity competition setting. We can see this from the fact that they are the only networks when Firm 2 is in the middle of the line. This means that good 2 can be substituted both for good 1 and 3, but good 1 and 3 are not threatened by each other. Indeed Firm 1 behaves similarly when it is in the middle of the line for the only case of Network E.

We can see the sum of investments for all the patented improvements of each firm plotted in the following graphs:

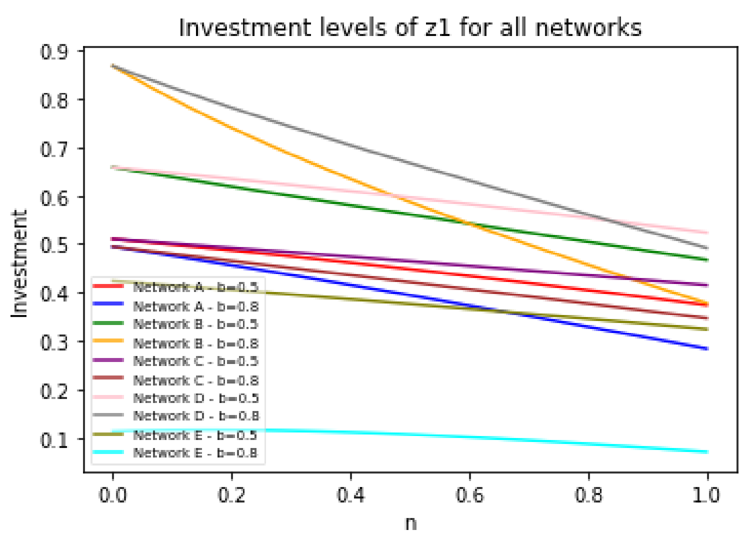


Figure 11: Firm 1 Total Investments

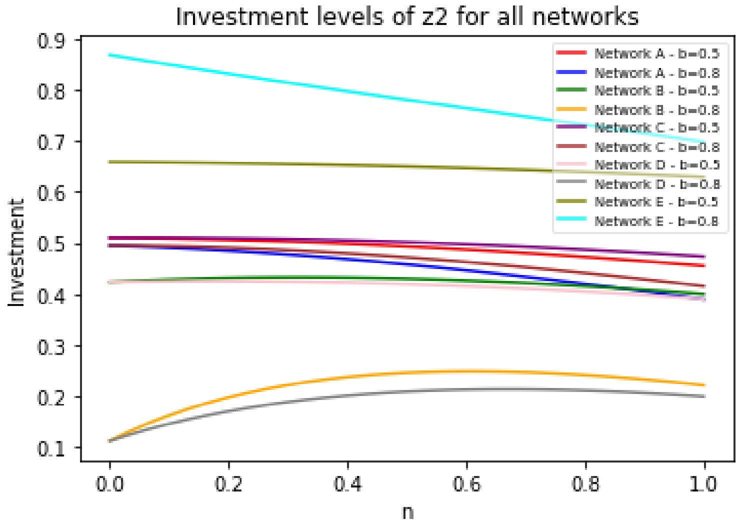


Figure 12: Firm 2 Total Investments

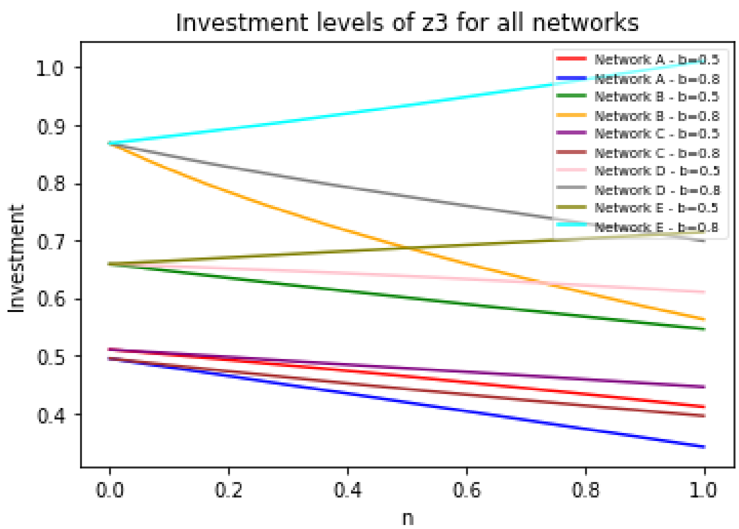


Figure 13: Firm 3 Total Investments

The relative change from $\gamma^* = 0$ to $\gamma^* = 1$ can reach up to 50% change. Considering that the marginal cost c is 5, an oscillation of the aggregate optimal investments of around 0.5 has a significant absolute and relative effect on output. The firm with fewer connections (Firm 3) invests more in most cases because it can retain further returns from the investments compared to the other firms. Firm 3 also relies less on the other firms compared to the most technology-connected ones (Firms 1 and 2).

By plotting each firm profit for all networks, we see that they behave correctly and there are no negative profits. The plots are available in the disaggregated profits branch of the GitHub repository. In both cases where the optimal externality is $\gamma = 0$, we observe a contrast in the profits trend, namely Firm 2 maximizes profits with $\gamma = 1$ and the other 2 firms (and consumers too) gain more with $\gamma = 0$. As we have already said before, Firm 2 in this case is threatened at the demand level by both firms, which do not threaten each other. Therefore the market available for Firm 2 is smaller, while Firm 1 and 3 have their own market where they can invest without competitors. Therefore for Firm 2, it would be more convenient to profit from the connection with these two firms instead of investing more to compete for a market accessible by three firms.

Innovation stealing is not the only effect that plays a role in the investments' optimal decisions. The concept of externality requires an effect duality, in the sense that the benefits of innovation can be given or received. The common idea around patents focuses extensively on the effect that competitors can have by capturing one firm's improvements. However, even with a small number of connected firms, we have already seen how other interactions are also playing a role. When another firm receives the cost improvement, it is not necessarily detrimental if the receiving firm is not demand-connected to the innovating firm. And even if it is demand-connected it is still important to take into account the effect that improving a competitor firm's production function has on all the other competitors. In this setting costs are convex, thus it is always convenient to invest at least a small part of profits in investments. Therefore, since all the connected firms always invest, reducing γ diminishes the sure rent of technology connection from all the other firms involved. Regarding the interaction between two firms' investment levels and the level of γ, it becomes difficult to say if an external firm improvement reduces the need for the other firm to invest in its technology or if either it improves its position on the market at the level that investing becomes more convenient. We will see in the following points what general conclusions we can derive from all these conflicting effects.

4.1 Optimal Patent Breadth for all substitution levels

We compared five networks with two substitution levels each, to conceive how a full externality equilibrium could be robust, even with significant substitutability between products. However, we still do not know where the transition from optimal full externality to no externality happens and whether it is sudden and steep. To investigate this mechanism, we plot the optimal level of patent breadth γ^* with respect to all the substitution levels. We can see the plot for all the networks of the different optimal externality levels γ^* for every substitution level $b_{g,j}$:

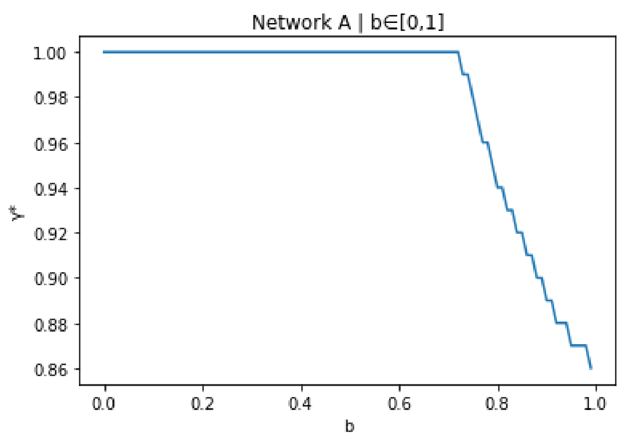


Figure 14: Value of γ^* for all the substitution levels - Network A

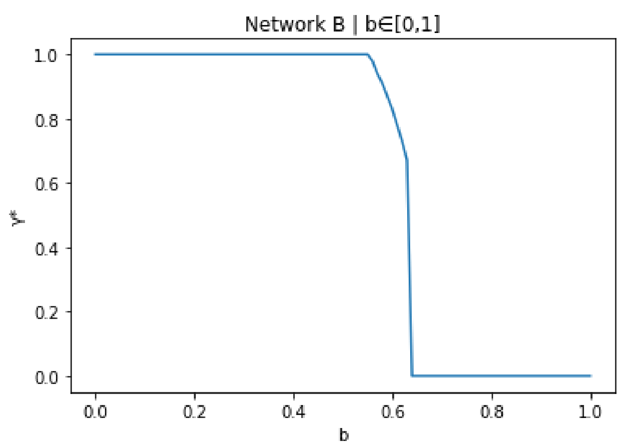


Figure 15: Value of γ^* for all the substitution levels - Network B

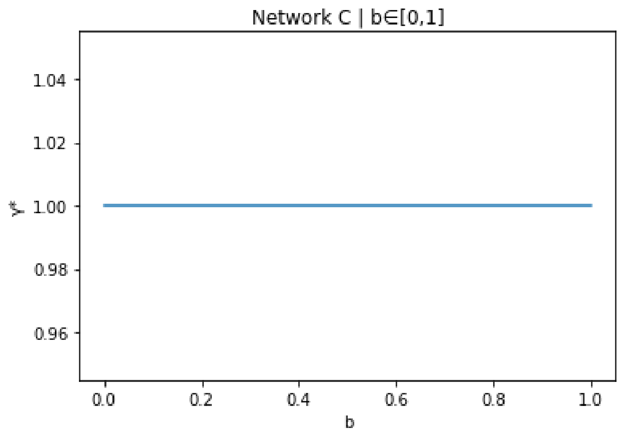


Figure 16: Value of γ^* for all the substitution levels - Network C

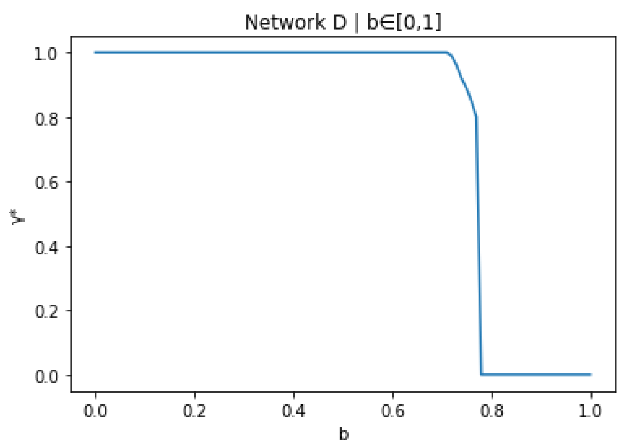


Figure 17: Value of γ^* for all the substitution levels - Network D

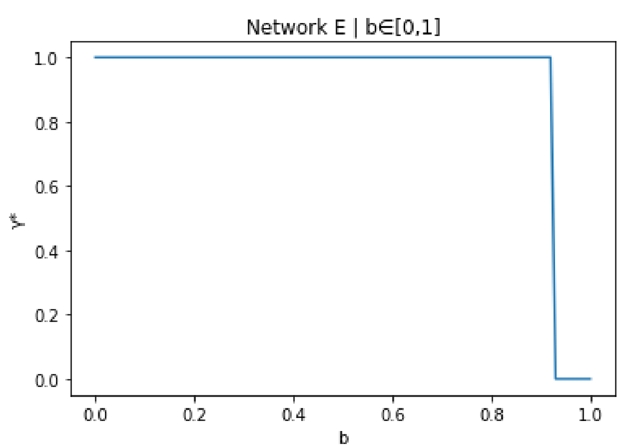


Figure 18: Value of γ^* for all the substitution levels - Network E

Networks B and D are the only ones where both firms 1 and 3 are demand connected to Firm 2 without being demand connected between each other. Therefore we have that the two firms can set a higher level of investment to force Firm 2 to produce a lower quantity while making no threat to each other. For this reason, a certain level of protection will be necessary for Firms 1 and 3 to invest more and reduce the quantity of Firm 2. Otherwise, Firm 2 would capture the improvements and increase its profits. Firms 1 and 3 will see their profit decrease when the exogenous externality γ increases, while the opposite happens for Firm 2. Firms 1 and 3 can flood the market with their product while increasing consumer utility. Firm 2 cannot compete and so will reduce its quantity, and firms 1 and 3 can increase their presence in the common market while profiting from the absence of demand competition with the other good. Network B and D are predictably similar in their behavior. Network B goes quicker to the optimal equilibrium with no externality because the link between firms 1 and 3 is suppressed, making the loss of externality smaller compared to the complete technology network.

Network C maximizes welfare with full externality even with perfect substitution between goods. Looking back at the investments graphs, we see that in Network C, the level of investments for all firms monotonically declines, but the magnitude of this change is not substantial. Since firms 1 and 3 are competitors between each other, firm 2 does not reduce investments significantly when substitution increases. Firms are disciplined by the complete demand network, in the sense that the competitive damage is felt simultaneously by both competitors. Reducing the costs for one absorbing firm is a direct competitive damage for the investing firm, however it entails the indirect competitive advantage of helping the rival of the absorbing firm, which is also a rival of the investing firm. This balancing effect makes Firm 2 decrease its investment by an amount that still makes it convenient for the general welfare to preserve absolute externality. Network A has a complete network both at the demand and technology levels. In this case, the optimal externality level remains higher even at $b = 1$ with $\gamma^* = 0.86$. However, it is smaller compared to Network C. This difference could derive from the fact that a more connected network increases open innovation externality but at the same time reduces the incentive to innovate. The discipline from indirect competitive advantage is still present since each patented improvement is experienced by all the remaining firms, which are also competitors between each other. However, at some point, all the firms find it optimal to reduce investment in Network A, thus making necessary a certain level of protection. While in Network C firms 1 and 3 remain protected by the negative effect of externalities on each other. Therefore both firms remain around the same investment proportions for all the substitution levels and in response firm 2 also maintains similar investment proportions.

Network E experiences a sudden jump from optimal full externality to no externality at around 95% of substitutability. We can see again from the investment plots that firm 3 increases its investment with γ while firm 2 decreases it. Protection usually comes when the imbalance between investments is considerable, due to the opportunity cost of losing investments of one firm with convex costs. For all the networks observed, the magnitude of investments is crucial. Namely, when all firms do not experience significant changes in investment with respect to γ, a full externality is preferable. Predictably, substitutability makes investment incentives more fragile. Therefore, for large levels of $b_{g,j}$, patent protection becomes welfare maximizing, in line with the usual mechanism of investments' incentive. However, the comparison also showed that network structures make the difference, at every level of $b_{g,j}$, between a full externality optimal equilibrium and a zero externality one.

Once more, we have observed the monotonic behavior of γ^* on welfare. Throughout the network, we have seen that the aggregate welfare was monotonically increasing or decreasing depending on the substitution level. Investment levels respected this trend too. As we see in Appendix II and III, the disaggregation of general welfare between Utility and Profits also respects this trend. The only nonmonotonic decrease happens for the Profits in Network D for high substitution. However, by plotting separately the profits of firms 1,2, and 3, we can see that this difference comes from the fact that firm 2 increases surplus monotonically with γ while firms 1 and 3 decrease it monotonically. Even if high-order effects of γ can arrive at the 32nd order, the aggregate effect remains monotonic for almost all substitution levels.

In this sense, it will be interesting to see what will be the predicted effects with real data, since monotonicity could be an exclusive characteristic of the 3-firms example. For now, we can say that monotonicity shows us the importance of the network structure, which can change the total welfare from monotonically increasing to decreasing for a given substitution parameter.

In some rare cases, we have observed nonmonotonic effects of γ^* on welfare. We do not find nonmonotonic effects in Networks C and E, while for B and D the nonmonotonic interval covers around 2% of all the possible values of $b_{g,j}$. In Network A, we see nonmonotonicity for around 25% of the $b_{g,j}$ values. In this case, by disaggregating profits, we notice that firm 2 profits experience a decline of profits for a smaller level of γ, compared to the other firms. Due to convex costs, every investment imbalance increases the need for patent protection. However, compared to Networks B and D where this effect does not exist, firms still find it convenient to help the rivals of the rivals for all the levels of $b_{g,j}$. Therefore, negligible intermediate patent protection reaches the objective of maximizing welfare for Network A. Indeed we see that the minimum level of welfare maximizing externality permitted in Network A is still 0.86.

4.2 Uniform Investment Comparison

In the uniform investment context, the results are the same, apart from Network D, which has an optimal $\gamma^* = 0.92$ instead of $\gamma^* = 0$ in the high substitution case. This only switch could derive from the fact that, with homogeneous investment, there is a pointwise lower absolute investment level compared to the heterogeneous case. From this characteristic, we can imagine that, since consumers have the same preferences in both cases, the level of investment at any γ is small enough to prefer a high transmission level between firms. Furthermore, firms cannot switch to non-shared technologies. Therefore they will be bounded to invest as a whole in a technology that will be transmitted to other firms, making the parameter γ more welfare-increasing compared to the non-homogeneous case. Remember that, in Network D, the transmission is between all three firms (complete network of technologies). Instead, network B with symmetric lines of demand and technologies still maximizes welfare at $\gamma = 0$ even in the homogeneous case. Therefore, we can say that the bounding investment effect exists. However, due to its small size with respect to γ, it is observable only in the case of Network D which, as we have seen in the general plots before, has a sudden switch from $\gamma^* = 1$ to $\gamma^* = 0$ in a homogeneous value of $b_{g,j}$ close to 0.8.

The different cost structure could have also contributed since all the three technologies of Firm 2 are multiplied by ϕ_{2,t_2} compared to the other two firms' technologies which are multiplied by ϕ_{1,t_1} and ϕ_{3,t_3} for both technologies. While in the uniform case, only one parameter is multiplied by its ϕ_{n,t_n}, and considering that Firm 2 has three patents, we can say that in the uniform investment case, the cost borne by the firms is also more uniform, making more convenient for Firm 2 to invest and thus making the high externality case $\gamma^* = 0.92$ welfare maximizing for Network D with high substitution, compared to the zero externality $\gamma^* = 0$ in the non-uniform investment case. For our master case of non-uniform investment, we have run the same plots with uniform cost structures and similar non-uniform cost structures without finding significant differences. It makes sense because the uniform case is, for the most part, a different cost structure that yields similar results with only one flipping. Therefore the issue of costs is relevant, but if they remain around the proportion between costs and gains that we have set, the model should remain consistent. The plots for all networks in the uniform investment case are available in Appendix IV.

4.3 Binary Outcome in Optimal Patent Breadth

Another interesting phenomenon is the substantial difference between optimal values of γ. The outcome is almost binary. This tendency increases the hope of applicability for two reasons. First, at the legislative level is simpler to convert a 0-1 result in patent law instead of understanding the applied difference for the intermediate values of γ. Because each improvement would have different transmission drivers, allowing full transmission or protecting it in an absolute way from imitation is easier compared to assigning a value of intermediate externality to technological improvement with unique transmission drivers. Binary variables are also easier to use in a Machine Learning environment. For example, if we generalize the model and still observe binary levels of optimal patent breadth, we can apply gradient boosting to observe more patterns that we cannot notice in the 3-Firms scale.

4.4 Balanced Innovation

We have seen that the optimal patent breadth can be drastically different from one environment to another after the network structures in the interaction between demand and technologies have been taken into account.

When the optimal patent breadth $\gamma^* = 0$ prohibits externality sharing, we can imagine that the usual benefit of investment protection dominates. Therefore, even by shutting the externality between firms we still increase aggregate welfare because this protection increases investment levels for certain firms up to a point where they more than compensate for the absence of technology externality.

However, the explanation for the opposite behavior, open-source innovation with optimal $\gamma = 1$, is sometimes overlooked in the theoretical modeling literature. The reported benefits of open innovation have been already mentioned in the literature review. In this step, we need to analyze the interplay between networks that give birth to this $\gamma = 1$ equilibrium. We see that in Networks A, C, and E, the full externality equilibrium does not change when the substitution parameter goes up from the intermediate to the high level. However, the aggregate welfare contracts because firms retrieve lower profits while consumers acquire a smaller amount of more similar goods. The balancing effect between the different substitution levels comes from the fact that with more product homogeneity there is also a lower need for investments from the consumers' side. For this reason, the quantity demanded will be lower due to its ease of substitutability between products. In this case, an expansion of patent breadth will have a limited impact on the investment level since the market profitability is already reduced due to the high substitutability. Therefore in all three cases of high substitution environment with optimal $\gamma^* = 1$, the reduction in investment caused by the absence of patent protection yielded an aggregate welfare that was still above the opposite case. With $\gamma = 0$, the aggregate technological investments would have been higher. However, considering consumers' demand and the resulting market profitability, reaching that higher level of non-shared investment won't be optimal for aggregate welfare.

Therefore we can summarize the Balanced Innovation as the idea that firms will invest ex-ante more for a less homogeneous market which yields higher profit and requires no protection due to low products' similarity while investing ex-ante less for a high homogeneous market which also does not require absolute protection due to its smaller level of investment used to satisfy the elastic demand.

If this trend dominates throughout all substitution levels for different network structures, we would expect a nonlinear performance for the optimal γ^* with respect to all substitution levels. The relation between patent breadth and similarity should start with free externality since investments would be protected by low substitution, rising to complete patent protection for an intermediate substitution level, and finally relaxing intellectual protection by reducing patent breadth when similarity converges to homogeneity, with substitution values around 1, where the optimal investment will be lower due to extensive products' similarity.

To derive this trend with simple structures we have looked at the difference between 3-Firms networks, justifying the different $b_{g,j}$ turning points, where the switch from complete externality to full protection happened, with a nonlinear relation between patent breadth and product similarity. However, a clearer picture of this tendency would require observing this nonlinear trend within a single network. Since we are now in a phase with balanced innovation without nonlinear behavior, we can only remark on the different turning points and justify them by comparing networks' characteristics without seeing a single network interaction as complex as the five networks all merged. Consequently, we design more complex structures to find this nonlinear relation within every network.

5 Balance and Nonlinearity

We now extend the model to eight firms to capture the interaction between other network shapes. We will also represent eight firms in the multi-product example developed in the next section, making it easier to compare the two extensions.

We will connect circles and core-periphery network shapes. We have 5 cases from combining the two structures in the demand and technology space.

Network F represents the interaction of a core-periphery demand with a core-periphery technology structure. The core periphery demand results in the following price structure:

$$\begin{aligned}
P_1 &= \alpha - q_1 - b_{1,5}q_5 \\
P_2 &= \alpha - q_2 - b_{2,6}q_6 \\
P_3 &= \alpha - q_3 - b_{3,7}q_7 \\
P_4 &= \alpha - q_4 - b_{4,8}q_8 \\
P_5 &= \alpha - q_5 - b_{5,1}q_1 - b_{5,6}q_6 - b_{5,7}q_7 - b_{5,8}q_8 \\
P_6 &= \alpha - q_6 - b_{6,2}q_2 - b_{6,5}q_5 - b_{6,7}q_7 - b_{6,8}q_8 \\
P_7 &= \alpha - q_7 - b_{7,3}q_3 - b_{7,5}q_5 - b_{7,6}q_6 - b_{7,8}q_8 \\
P_8 &= \alpha - q_8 - b_{8,4}q_4 - b_{8,5}q_5 - b_{8,6}q_6 - b_{8,7}q_7
\end{aligned} \tag{17}$$

We expect this network to be the one that reaches quicker the no externality optimal value for an intermediate value of $b_{g,j}$. The core is composed of firms that own the majority of technologies and are also more exposed to the abating effects that an intense connection at the demand level has on profits. Followers firms instead own a lesser amount of technologies and are also shielded by competition effects since only one firm in the core produces a good with an undirected positive substitution value.

In Network G we connect an inverse core-periphery demand and a core-periphery technology network. In the inverse core-periphery, the peripheral nodes become the cohesive core sub-graph and vice versa. The inverse core-periphery demand results in the following price structure:

$$
\begin{aligned}
P_1 &= \alpha - q_1 - b_{1,5}q_5 - b_{1,2}q_2 - b_{1,3}q_3 - b_{1,4}q_4 \\
P_2 &= \alpha - q_2 - b_{2,6}q_6 - b_{2,1}q_1 - b_{2,3}q_3 - b_{2,4}q_4 \\
P_3 &= \alpha - q_3 - b_{3,7}q_7 - b_{3,1}q_1 - b_{3,2}q_2 - b_{3,4}q_4 \\
P_4 &= \alpha - q_4 - b_{4,8}q_8 - b_{4,1}q_1 - b_{4,2}q_2 - b_{4,3}q_3 \\
P_5 &= \alpha - q_5 - b_{5,1}q_1 \\
P_6 &= \alpha - q_6 - b_{6,2}q_2 \\
P_7 &= \alpha - q_7 - b_{7,3}q_3 \\
P_8 &= \alpha - q_8 - b_{8,4}q_4
\end{aligned}
\tag{18}
$$

Network H represents the interaction of a circle demand with a core-periphery technology structure. The circle demand results in the following price structure:

$$
\begin{aligned}
P_1 &= \alpha - q_1 - b_{1,5}q_2 - b_{1,2}q_8 \\
P_2 &= \alpha - q_2 - b_{2,6}q_3 - b_{2,1}q_1 \\
P_3 &= \alpha - q_3 - b_{3,7}q_4 - b_{3,1}q_2 \\
P_4 &= \alpha - q_4 - b_{4,8}q_5 - b_{4,1}q_3 \\
P_5 &= \alpha - q_5 - b_{5,1}q_6 - b_{4,1}q_4 \\
P_6 &= \alpha - q_6 - b_{6,2}q_7 - b_{4,1}q_5 \\
P_7 &= \alpha - q_7 - b_{7,3}q_8 - b_{4,1}q_6 \\
P_8 &= \alpha - q_8 - b_{8,4}q_1 - b_{4,1}q_7
\end{aligned}
\tag{19}
$$

Network H could also reflect more accurately the trends of an innovative sector since we can observe a core of established firms that invests in more technologies which are all used by the core firms while the followers are attached only in one technology in which they are specialized.

The core periphery technology results in the following cost structure:

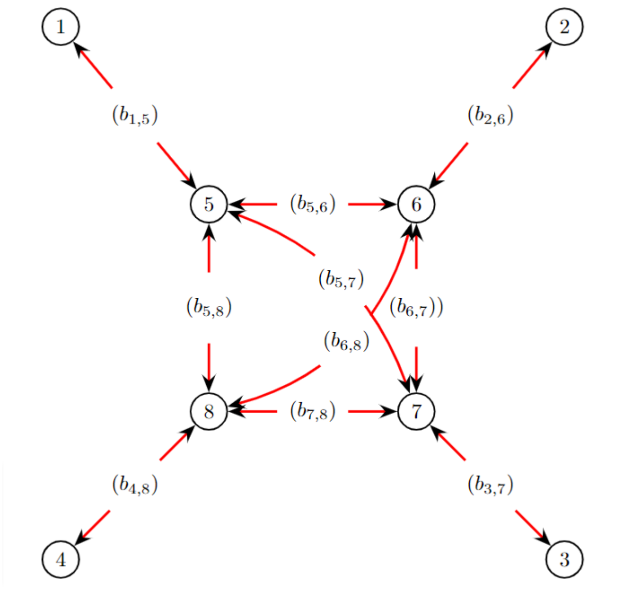
$$\begin{aligned}
&C_1 = c - z_{1,a} - \gamma z_{5,a} \\
&C_2 = c - z_{2,b} - \gamma z_{6,b} \\
&C_3 = c - z_{3,c} - \gamma z_{7,c} \\
&C_4 = c - z_{4,d} - \gamma z_{8,d} \\
&C_5 = c - z_{5,e} - z_{5,a} - \gamma z_{1,a} - z_{5,f} - z_{5,g} - z_{5,h} - \gamma z_{6,e} - \gamma z_{6,f} - \gamma z_{6,g} - \gamma z_{6,h} - \\
&\gamma z_{7,e} - \gamma z_{7,f} - \gamma z_{7,g} - \gamma z_{7,h} - \gamma z_{8,e} - \gamma z_{8,f} - \gamma z_{8,g} - \gamma z_{8,h} \\
&C_6 = c - z_{6,f} - z_{6,b} - \gamma z_{2,b} - z_{6,e} - z_{6,g} - z_{6,h} - \gamma z_{5,e} - \gamma z_{5,f} - \gamma z_{5,g} - \gamma z_{5,h} - \\
&\gamma z_{7,e} - \gamma z_{7,f} - \gamma z_{7,g} - \gamma z_{7,h} - \gamma z_{8,e} - \gamma z_{8,f} - \gamma z_{8,g} - \gamma z_{8,h} \\
&C_7 = c - z_{7,g} - z_{7,c} - \gamma z_{3,c} - z_{7,e} - z_{7,f} - z_{7,h} - \gamma z_{6,e} - \gamma z_{6,f} - \gamma z_{6,g} - \gamma z_{6,h} - \\
&\gamma z_{5,e} - \gamma z_{5,f} - \gamma z_{5,g} - \gamma z_{5,h} - \gamma z_{8,e} - \gamma z_{8,f} - \gamma z_{8,g} - \gamma z_{8,h} \\
&C_8 = c - z_{8,h} - z_{8,d} - \gamma z_{4,h} - z_{8,e} - z_{8,f} - z_{8,g} - \gamma z_{6,e} - \gamma z_{6,f} - \gamma z_{6,g} - \gamma z_{6,h} - \\
&\gamma z_{7,e} - \gamma z_{7,f} - \gamma z_{7,g} - \gamma z_{7,h} - \gamma z_{5,e} - \gamma z_{5,f} - \gamma z_{5,g} - \gamma z_{5,h}
\end{aligned} \tag{20}$$

The core-periphery technology system is the most computationally demanding structure that we had until now since we had to find 24 optimal investment values with a system of 24 equations that maximized profits for all the firms with respect to all the investment decisions stored in the set $\mathbb{Z}_n = \{z_{n,a}, z_{n,b}, \ldots, z_{\eta,t}\}$ for all $n = 8$ firms.

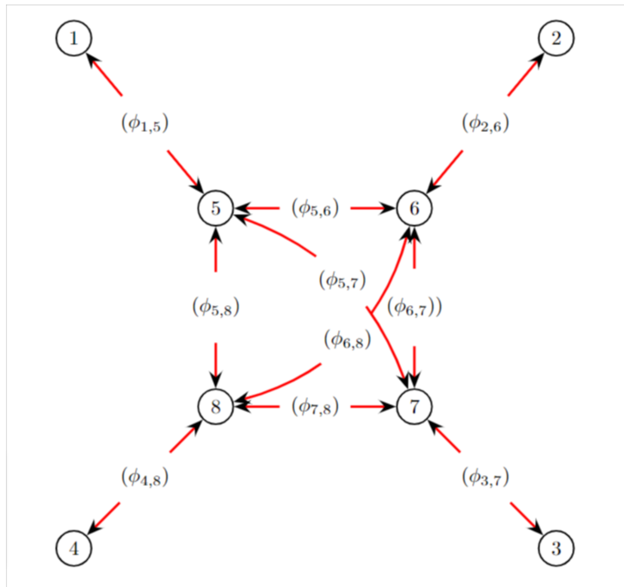
Network I represents the interaction of a circle demand with a circle technology structure. The circle technology results in the following cost structure:

$$\begin{aligned}
C_1 &= c - z_{1,a} - z_{1,h} - \gamma z_{2,a} - \gamma z_{8,h} \\
C_2 &= c - z_{2,b} - z_{2,a} - \gamma z_{3,b} - \gamma z_{1,a} \\
C_3 &= c - z_{3,c} - z_{3,b} - \gamma z_{4,c} - \gamma z_{2,b} \\
C_4 &= c - z_{4,d} - z_{4,c} - \gamma z_{5,d} - \gamma z_{3,c} \\
C_5 &= c - z_{5,e} - z_{5,d} - \gamma z_{6,e} - \gamma z_{4,d} \\
C_6 &= c - z_{6,f} - z_{6,e} - \gamma z_{7,f} - \gamma z_{5,e} \\
C_7 &= c - z_{7,g} - z_{7,f} - \gamma z_{8,g} - -\gamma z_{6,f} \\
C_8 &= c - z_{8,h} - z_{8,g} - \gamma z_{1,h} - \gamma z_{7,g}
\end{aligned} \tag{21}$$

Network J represents the interaction of a core-periphery demand with a circle technology.

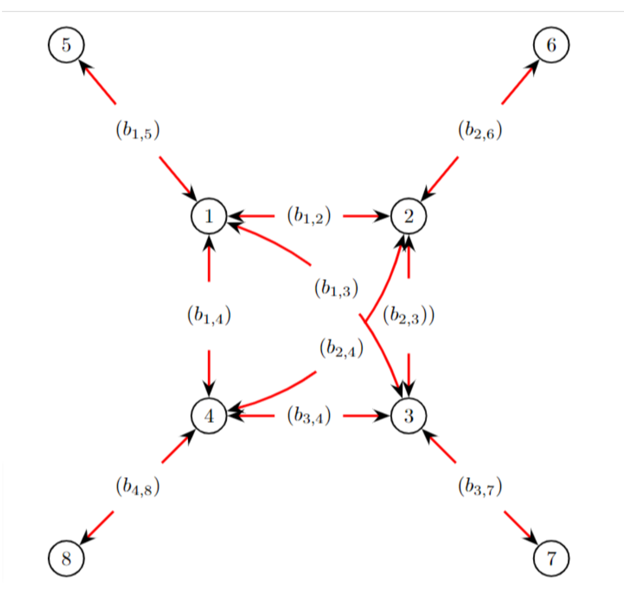


(a) Demand Network

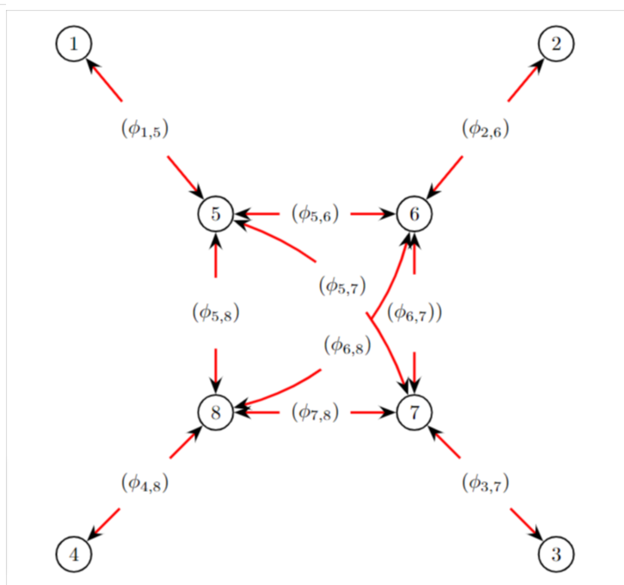


(b) Technology Network

Figure 19: Network F

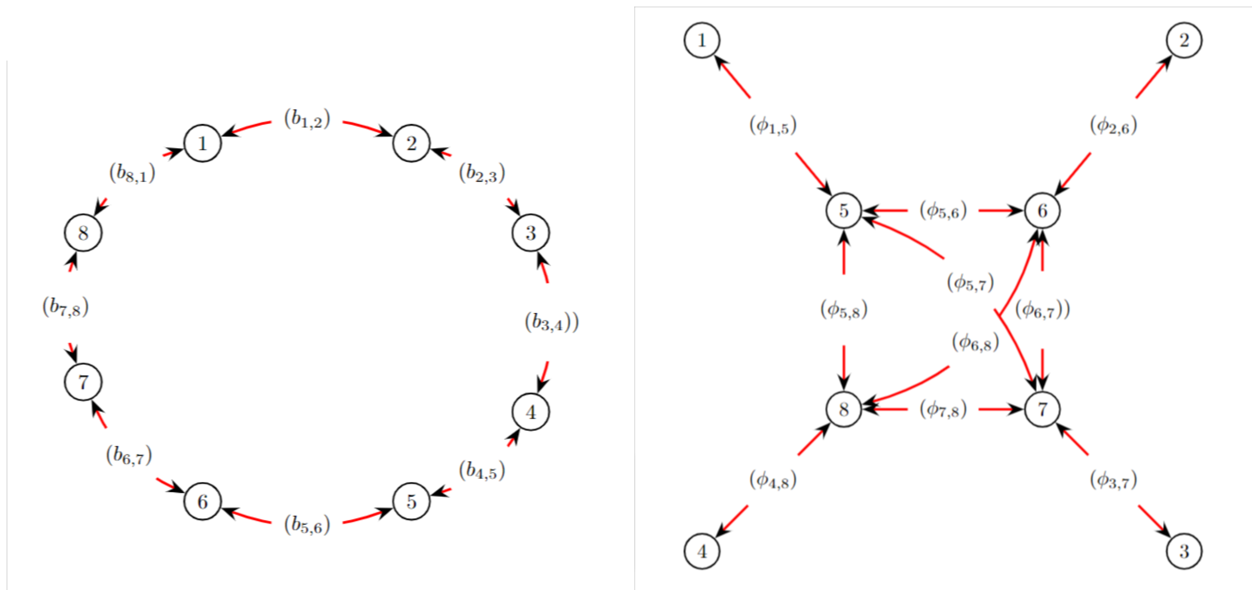


(a) Demand Network



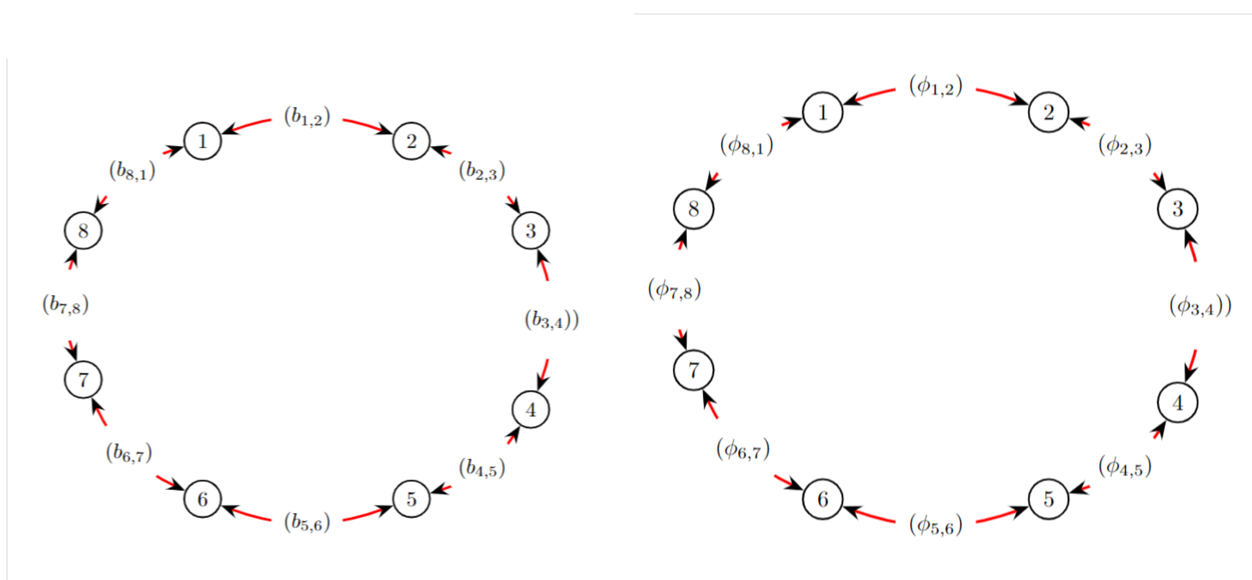
(b) Technology Network

Figure 20: Network G



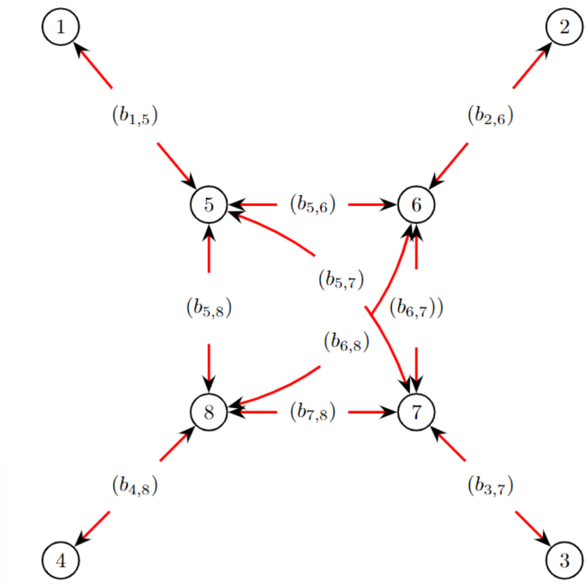
(a) Demand Network (b) Technology Network

Figure 21: Network H

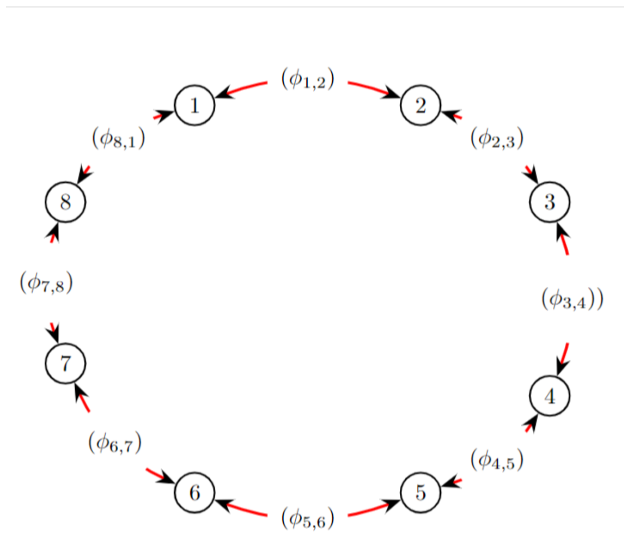


(a) Demand Network (b) Technology Network

Figure 22: Network I



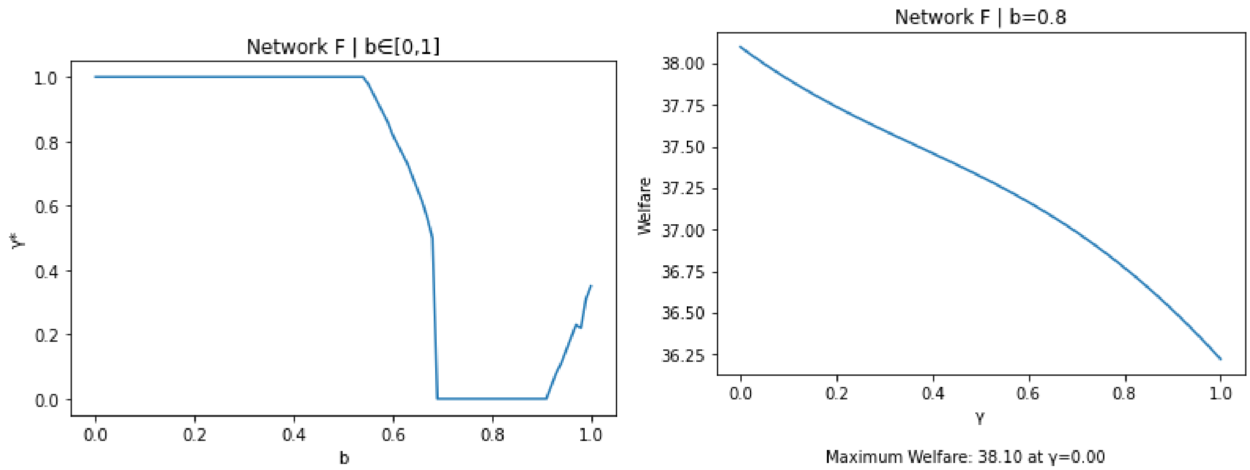
(a) Demand Network



(b) Technology Network

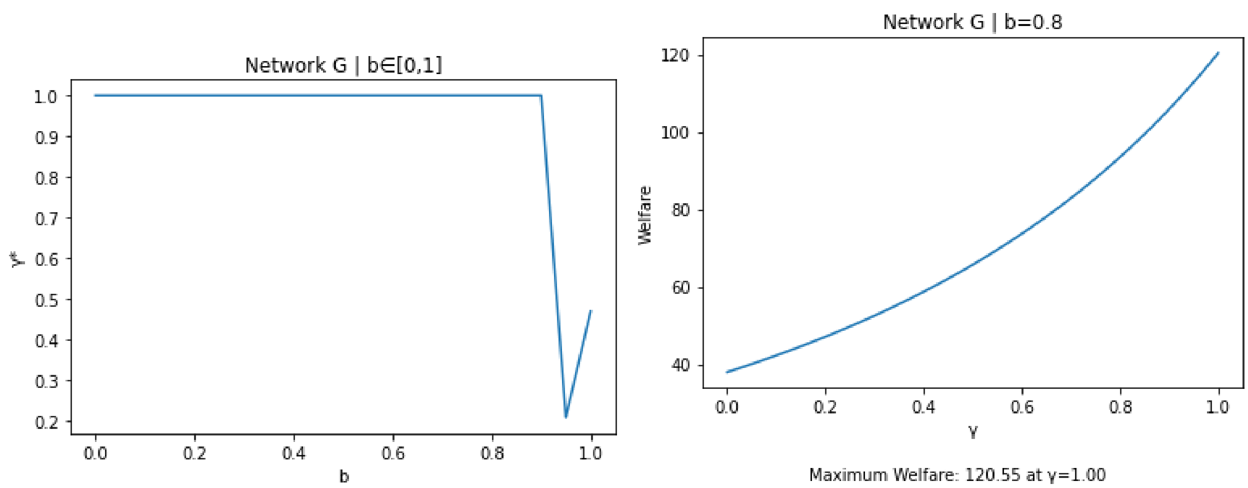
Figure 23: Network J

Here we show the optimal externality level γ^* for every substitution level $b_{g,j}$ for all the networks together with the high substitution welfare maximization.



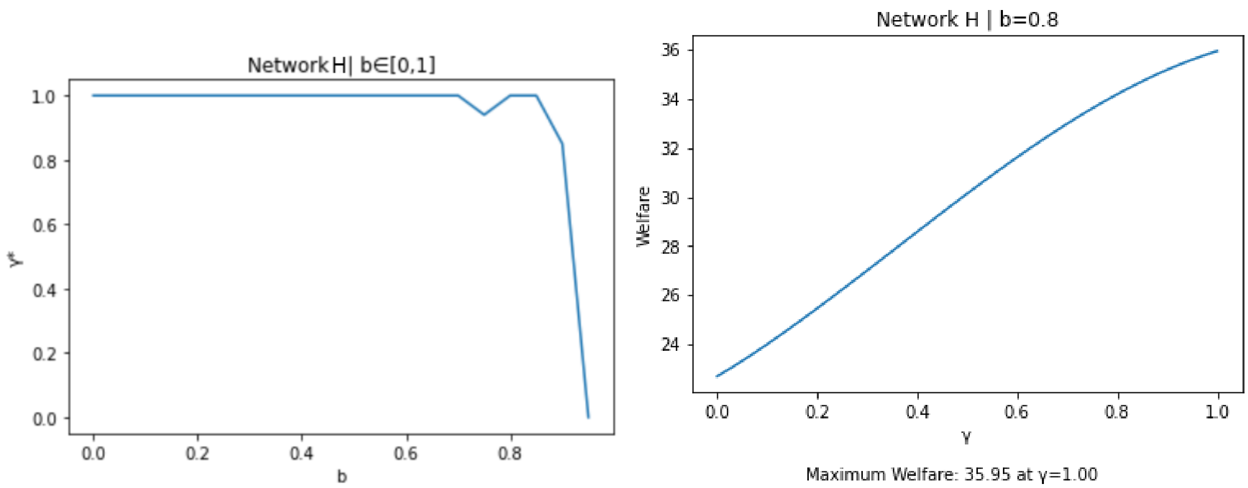
(a) Optimal γ^* for all $b_{g,j}$ (b) High Substitution ($b = 0.8$)

Figure 24: Network F



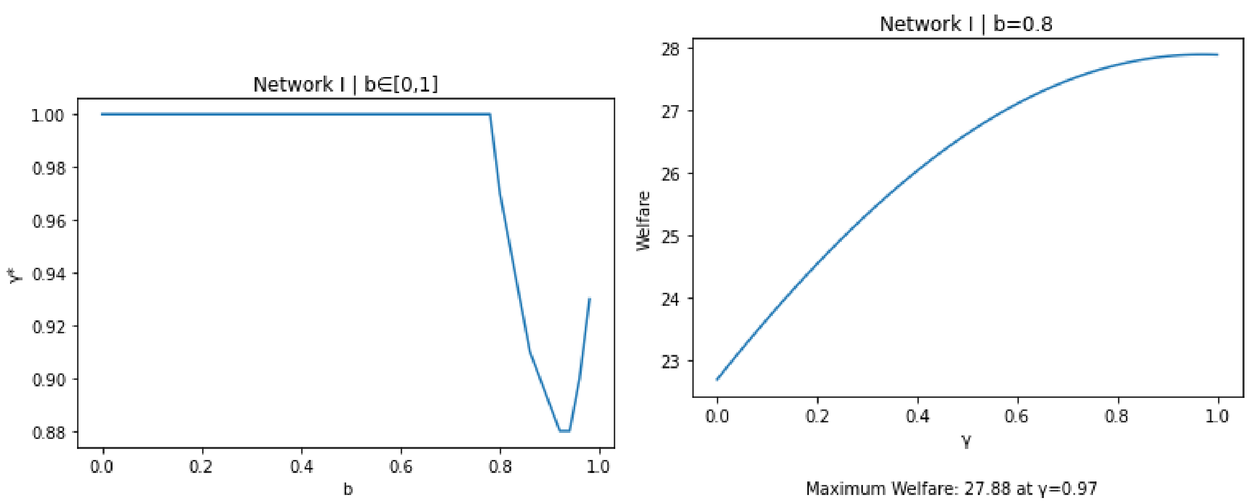
(a) Optimal γ^* for all $b_{g,j}$ (b) High Substitution ($b = 0.8$)

Figure 25: Network G



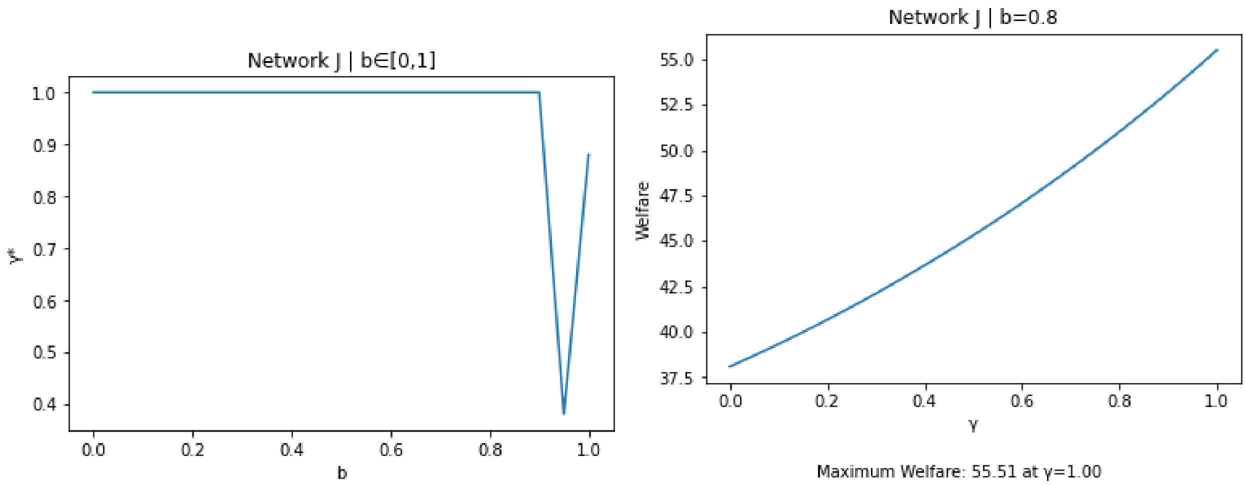
(a) Optimal γ^* for all $b_{g,j}$ (b) High Substitution ($b = 0.8$)

Figure 26: Network H



(a) Optimal γ^* for all $b_{g,j}$ (b) High Substitution ($b = 0.8$)

Figure 27: Network I



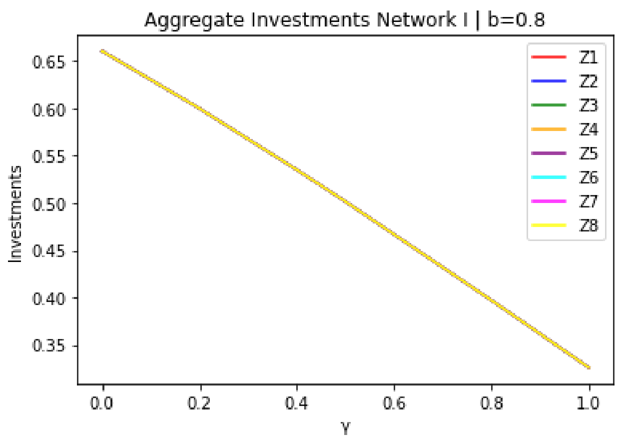
(a) Optimal γ^* for all $b_{g,j}$ (b) High Substitution ($b = 0.8$)

Figure 28: Network J

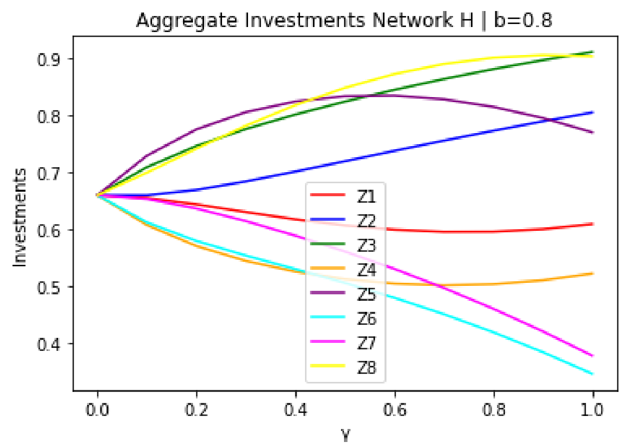
As we can see from most binary values of the optimal γ^* for all $b_{g,j}$, welfare maximization with respect to γ^* remains binomial for most substitution parameters $b_{g,j}$ between 0 and 1. Most importantly, the balance trend discussed for the 3-Firms case is still valid, with the difference that for a larger number of firm we can see it within each network without the need for comparing different structures. In 4 out of 5 networks, including Network H, we see a drop in the optimal γ^* with a final spike. When the final spike looks extremely steep is not due to graphic reasons, it is due to a sharp monotonic relation between $b_{g,j}$ and γ in the extreme product similarity values. Indeed even by using 0.005 γ plot frequency instead of the standard 0.01 interval in the x-axis, the final spike remains the same.. Our explanation for this nonlinear trend remains the same. For minor substitution levels, firms are protected by the extensive difference between their goods. While for high substitution levels, the need for protection is lower for two reasons. First, on the profit side, firms' optimal quantity is limited for any externality level due to consumers' market power which leads to lower profit for any patent breadth setting. Secondly, on the utility side, consumers are less harmed by a quantity reduction due to the high substitution between goods. As we will see in the final section, the future empirical study based on this theoretical paper will test demand and technology indexes that should behave nonlinearly on microdata.

This balanced pattern requires attention because it is the main finding of this theoretical study, namely that intellectual protection could behave nonlinearly in its welfare maximization with respect to patent breadth.

Finally, we can show an interesting property of the overlapping circles' networks, such as the one in Network I. Since the structure is symmetric for both demand and technology networks, we can see that all the firms invest the same quantity for all the levels of γ. While for the non-symmetric or non-overlapping networks, we see that the investments' levels diverge. We can see this feature in the following plots, where the investments in all technologies available by each firm are summed and plotted for all the levels of γ from 0 to 1:



(a) Aggregate Investments for Network I



(b) Aggregate Investment for Network H

Figure 29: Aggregate Profit-Maximizing Investments for two Networks

6 Theoretical Extensions

6.1 Multiple Products Setting

As shown in Maraghy et al. 2013, numerous product variants are introduced in the market each year. The automotive sector perfectly exemplifies this variety. Therefore, it is interesting for this article to conclude with an extension with firms that can produce more than one good. Specifically, we have three core firms that produce five goods each, while five follower firms produce only one of these goods.

The vector of goods is:

$$\mathbb{G} = \{1, 2, 3, 4, 5\}$$

Core Firms 1,2,3 produce all goods:

$$||f_1 \Leftrightarrow 1, 2, 3, 4, 5||f_2 \Leftrightarrow 1, 2, 3, 4, 5||f_3 \Leftrightarrow 1, 2, 3, 4, 5||$$

Follower Firms 4,5,6,7,8 produce one good each:

$$||f_4 \Leftrightarrow 1||f_5 \Leftrightarrow 2||f_6 \Leftrightarrow 3||f_7 \Leftrightarrow 4||f_8 \Leftrightarrow 5||$$

The prices for each good are the same as the 3-firm example, with the addition of one follower firm connected to every core firm. For example, good 1 has the Network A structure with a complete network with the addition of one follower firm fully connected both on demand and technology. For that good 1, we have the following prices and marginal costs:

$$\begin{aligned}
P_{1,1} &= \alpha - q_{1,1} - b_{1,2}q_{2,1} - b_{1,3}q_{3,1} - b_{1,4}q_4 \\
P_{2,1} &= \alpha - q_{2,1} - b_{2,1}q_{1,1} - b_{2,3}q_{3,1} - b_{2,4}q_4 \\
P_{3,1} &= \alpha - q_{3,1} - b_{3,1}q_{1,1} - b_{3,2}q_{2,1} - b_{3,4}q_4 \\
P_4 &= \alpha - q_4 - b_{4,1}q_{1,1} - b_{4,2}q_{2,1} - b_{4,3}q_{3,1}
\end{aligned} \tag{22}$$

$$\begin{aligned}
C_{1,1} &= c - z_{1,a} - z_{1,b} - \gamma z_{2,a} - \gamma z_{2,b} - \gamma z_{3,a} - \gamma z_{4,a} - \gamma z_{4,b} \\
C_{2,1} &= c - z_{2,a} - z_{2,b} - z_{2,c} - \gamma z_{1,a} - \gamma z_{1,b} - \gamma z_{3,a} - \gamma z_{4,a} - \gamma z_{4,b} - \gamma z_{4,c} \\
C_{3,1} &= c - z_{3,a} - z_{3,b} - \gamma z_{1,a} - \gamma z_{2,a} - \gamma z_{4,a} \\
C_4 &== c - z_{4,a} - z_{4,b} - z_{4,c} - \gamma z_{1,a} - \gamma z_{1,b} - \gamma z_{3,a} - \gamma z_{2,a} - \gamma z_{2,b} - \gamma z_{2,c}
\end{aligned} \tag{23}$$

Remember that the substitution levels are the same between every good, as in every other setting treated until now. It is possible to relax this assumption to have a specific value of $b_{g,j}$ for every relation within and even between all goods, in case we want to allow substitution between varieties. Every $b_{n,n'}$ would thus be a set that contains every substitution link between all the goods produced by the two firms mentioned, so that we can retrieve the substitution value for each specific good for which the price is computed. We compute the optimal quantities and investments as in the 3-Firms case. The difference for the Multiple Products case is that core firms have total profits which are the sum of the profits derived from all five goods. Good 1 has demand and technology connections in line with network A, same logic for the remaining goods until Good 5 which has technology and demand connections of Network E. The follower firm is fully connected in all goods, even when the good's network is not complete at the demand or technology level. The follower firms obtain profit from the only good that they sell.

The vector of technologies in our Multiple Products setting is:

$$
\begin{aligned}
\mathbb{T}_1 &= \{a, b\} \\
\mathbb{T}_2 &= \{a, b, c\} \\
\mathbb{T}_3 &= \{a, b\} \\
\mathbb{T}_4 &= \{a, b, c\} \\
\mathbb{T}_5 &= \{a, b, c\} \\
\mathbb{T}_6 &= \{a, b, c\} \\
\mathbb{T}_7 &= \{a, b, c\} \\
\mathbb{T}_8 &= \{a, b, c\}
\end{aligned} \tag{24}
$$

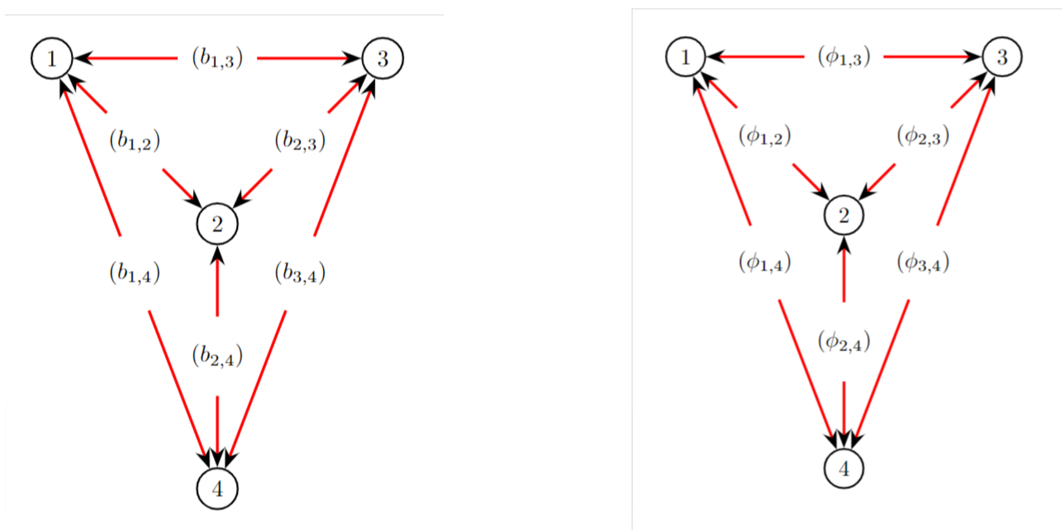
The vector of patent sets in our Multiple Products setting is:

$$
\begin{aligned}
\mathbb{P}_a = \{[1, 2, 3, 4, 5, 6, 7, 8]\} \\
\mathbb{P}_b = \{[1, 2, 4, 5, 6, 7, 8]\} \ ; \ \mathbb{P}'_b = \{[3]\} \\
\mathbb{P}_c = \{[2, 4, 5, 6, 7, 8]\}
\end{aligned} \tag{25}
$$

Convex costs incentivize firms to produce a minimum amount of all goods. Core firms use the same technologies to produce five goods that are not substitutable between themselves. As we said before, we could consider substitution between goods even in the case of within-firm substitution. In that case, the network structure would not be composed of five separate components. If all goods are substitutable with each other, at least between two firms or within the same firm, we would have a lattice formed by a single component. For the technology case, we could follow the same logic. Due to the differences between networks, the linear cost reduction transmission is already different from one good to another. However, we could have externalities of cost improvements for one product shared in another production function. It becomes a question of how we want to characterize the network. The Python code can easily handle a more heterogeneous characterization if, by observing the microdata, we notice that a specific feature needs to be integrated into the theoretical model. For now we keep the simple model with homogeneous substitution within but not between goods. We would thus have networks of five components both for demand and technology.

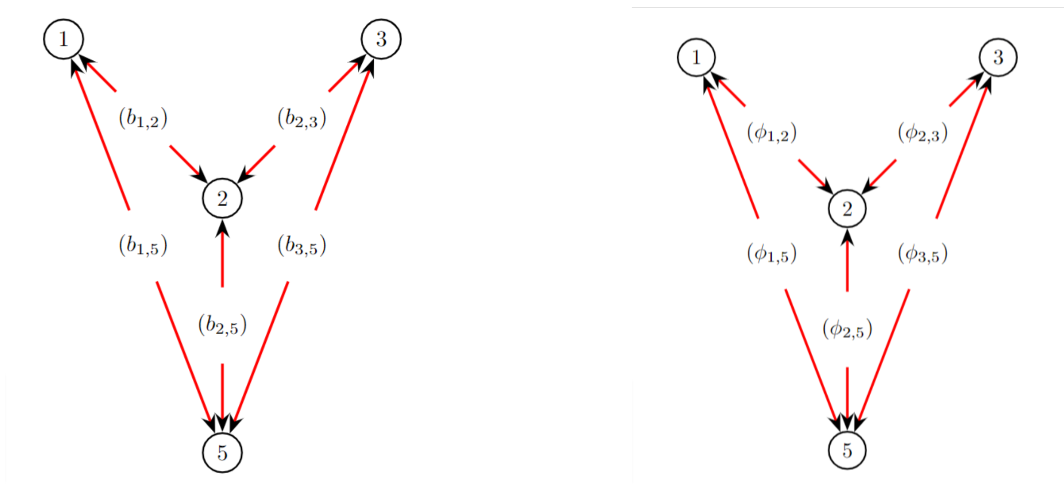
For the sake of clarity, we now show the graphical representation of the Multi-Product Network. In this case, we won't have five different networks to compare, but rather the connections at good and technology levels for all five goods, which are part of the same production network. The core firms 1-2-3 will produce all five goods, while one of the five follower firms will connect to all the core firms for each good, both at demand and technology level,. We decided to assign this behavior to the follower firms with the idea that by coming into the market at a later stage compared to the core firms, the follower firm produces its single good by using all the technologies already available. The final product is also substitutable with all the products already in the market. The profits of the core firms are the sum of all the profits coming from the five goods. The follower firms' profits come from the production of their single product.

The plots for the five components of the Multi-Product Network are:



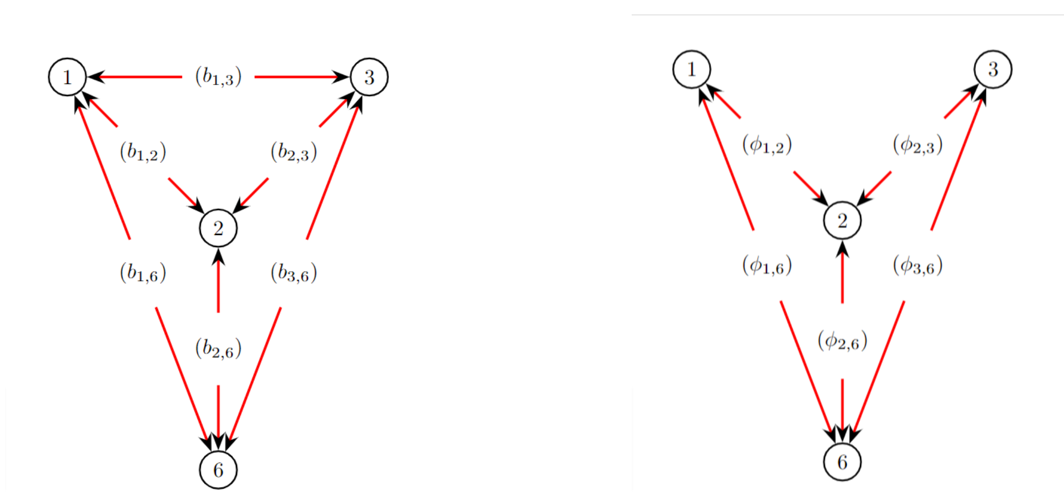
(a) Demand (b) Technology

Figure 30: Good 1 Connections



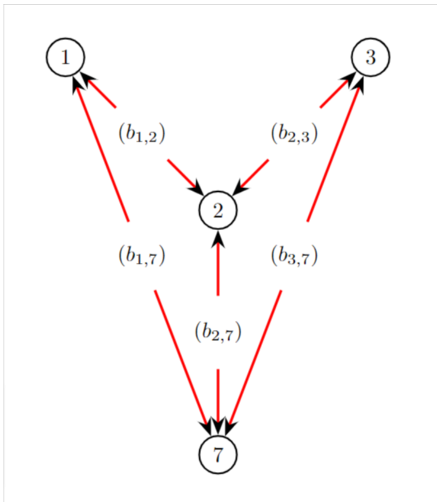
(a) Demand (b) Technology

Figure 31: Good 2 Connections

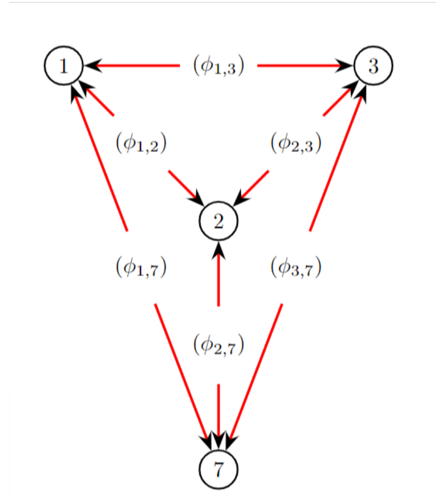


(a) Demand (b) Technology

Figure 32: Good 3 Connections

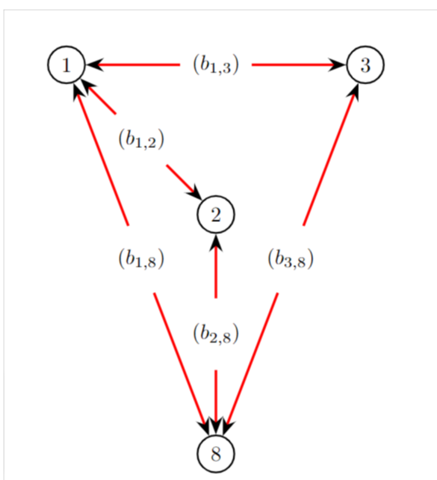


(a) Demand

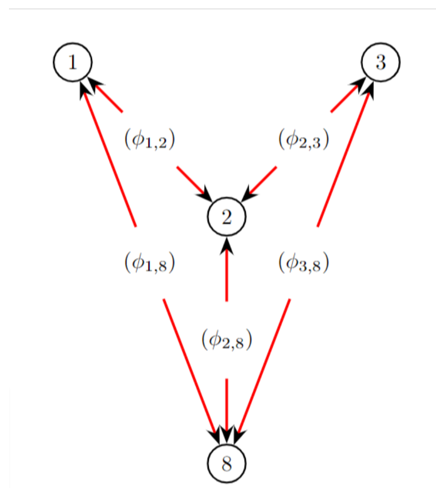


(b) Technology

Figure 33: Good 4 Connections



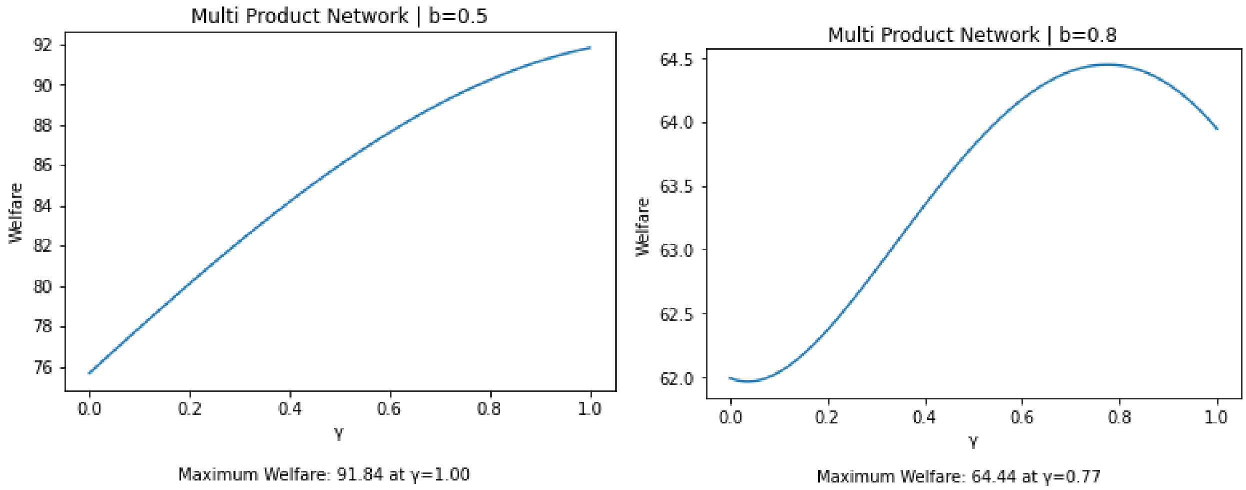
(a) Demand



(b) Technology

Figure 34: Good 5 Connections

We can see here the optimal level of γ^* for both intermediate and high substitution levels.



(a) Optimal γ^* - Intermediate Substitution

(b) Optimal γ^* - High Substitution

Figure 35: Multiple Products Optimal γ^*

Core firms experience linear returns on all the products from the same technology. The only change occurring between one Network and another is the externality linkages between technologies. Given the investments' convex costs, core firms are incentivized to invest a positive amount in every technology available. In a self-reinforcing cycle, the investment level chosen by each core firm linearly decreases costs by the same amount throughout all goods, core firms have a strong incentive to produce, after investing any positive amount, to experience the benefit of investment in all five different goods.

Follower firms can tap into this larger investment pool. Due to the different network connections between products, core firms produce one good that yields higher profits compared to the others. Since the same technologies are present in all production functions, every follower firm can experience an externality rent from the core firms' investment in their preferred product.

We now see the optimal γ^* plotted for all substitution levels:

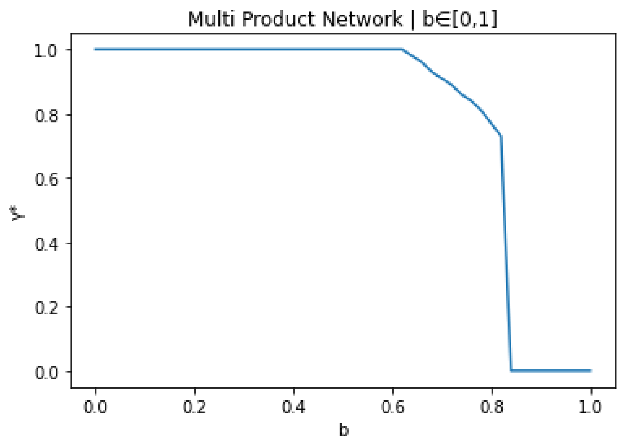


Figure 36: Value of γ^* for all the substitution levels - Multiple Products Network

From these values, we can see an effect of γ on welfare which follows a similar trend to the most exposed networks in the 3-Firms example, like Network D, with the difference that we have a larger interval of intermediate values. By saying exposed, we mean the ones for which the firms' position at the demand level requires a no externality equilibrium for a lower level of $b_{g,j}$, such as in Network B and D for the 3-Firms case.

We don't see a spike for the extreme values of $b_{g,j}$ here, meaning that the trend is again monotonically decreasing for γ with respect to the substitution parameter. The interaction between complex product structures for firms producing more than one good could help to clarify the conditions for a nonlinear effect to predominate. We have already seen that more complex structures create a nonlinear effect of γ on welfare. The Multiple Products network by itself does not enforce the monotonicity of patent breadth welfare effect, but rather it mostly takes into account the proportion of the goods structures in the total production, especially when goods are not substitutable between each other.

In the initial case, we examined five simple structures with a maximum of four firms connected for the same good. The results were in line with the 3-firms example, with a larger interval of intermediate values coming from the interaction of the five networks. Considering the same eight firms but with the five circle and core-periphery networks, we would have results more consistent with the nonlinearity trend, especially for the case without between-good substitution. The Multiple Products setting mostly amplifies the centrality of a specific complex structure depending on its presence in the whole network.

The reason for this multiple product characterization was to show that the model is flexible enough to take into account it. We will leave the verification of the interesting patterns that we have found to future structural econometric insights. The reason for this decision comes from the fact that the choice of what goods connect in between and what structures represent becomes increasingly arbitrary the more the structures become complex. Therefore we need empirical reasons to represent particular links at the demand or technological level, for us to have an idea of what would be a realistic sectoral optimal patent breadth and whether the nonlinearity of patent breadth on welfare is widespread as it seems from our complex structures example.

6.2 Possible Theoretical Extensions

A possible extension of this model could be the inclusion of predatory pricing. In the previous cases, companies are heterogeneous but they do not take into account the possibility of predatory pricing. In the sense that they know the effect that they have on each other, but they do not take into account the fact that they could drive out of the market another firm. We would need to extend our model to a multi production period setting, which is necessary to design this representation because firm need to evaluate the cost of immediate sub-optimal prices in exchange for a subsequent production period without one or more rivals. We could add the possibility for the firms to choose to drive out a competitor by setting lower prices. The predatory price setting would make us add entry costs which is a feature that this model does not have. It could be interesting to observe it because, with the convex costs of technology, there is always the possibility to enter with a small amount of research. However, significant entry costs could make production unsustainable even for a small amount of investment. Even in this case, we would need to estimate the size of entry costs before being taken into account by a model. Otherwise, without data, the value could be seen as an arbitrary cost increase.

Regarding the cumulative value of investment instead, an extensive literature on the cumulative value of innovation (Chang 1995) addresses this phenomenon. The cumulative value of investment could be added to the utility or profit functions. In this case as well, it is necessary to estimate the empirical cumulative innovation effect. Otherwise, it would just be an arbitrary addition of a positive value to the welfare function.

An extensive literature (Axtell 2001 and Gabaix 2011) covers also the Pareto distribution of firms dimension. With a general model, we could differentiate between firms' resources, to see what is the behavior of each firm in a monopolistic competition throughout different periods with a small number of important firms. I would expect contrasting effects, because large firms rely for innovations on small ones, which need a certain level of externality and accessible entry to imitate and innovate for a certain period with positive profits. However, crucial firms also need a level of protection from the few other important firms. Other factors such as multimarket competition would enter the game, making the model even more complex.

7 Future Structural Econometrics

The general case of this setting should keep the same logic regarding the connection for demand and technology networks, while being effective for an arbitrarily large number of firms and technologies. This generalization could be done by creating a Python code for a network-generating model of firms competing on quantity, with the same logic as in the 3-Firms example. The network should adequately represent a realistic example of 1000 firms interacting, making the task more adequate to be dealt with real microdata. The network-generating model could become more sophisticated with some degree of correlation between connected firms at both levels. However, some conclusions will be determined ex-ante by the characterization of this model. Therefore, it would be more honest to test the following indexes with real microdata.

I would expect to see the same patterns that we have seen in the previous examples. However, to be sure that our explanation is not spurious we would need to develop indices to test the predictive power of our theory against optimal patent breadth.

This index could start by taking into account other patterns that we have already seen in the previous examples, such as the technological link with the competitor of a competitor. We would use the characteristic function χ_L where the set L is the intersection of firms with the following properties:

1. $f_{n'} \in L$ if: $b_{n,n'} = 0$ with $f_{n'} \neq f_n$ (f_n is not connected at the demand level with $f_{n'}$).

2. There exists a set $\mathbb{P}_t$ that contains both f_n and the other firm $f_{n'}$ (f_n and $f_{n'}$ are connected at the supply level).

3. There exists a firm f_η that is connected at the demand level with both f_n and $f_{n'}$, namely $b_{n,\eta} > 0$; $b_{n',\eta} > 0$ (we find that in the utility function, the good η produced by f_η can be substituted with the good produced by the other two firms f_n and $f_{n'}$ which from point 1 cannot be substituted with each other and from point 2 are connected at the supply level)

Then, the set L can be expressed as:

$$
\begin{aligned}
L = \left[f_n \in \mathcal{F} : f_n, f_{n'} : b_{n,n'} = 0\right], \\
\exists f_n \in P_t, f_{n'} \in P_t, \\
\exists f_\eta \in \mathcal{F} : b_{n,\eta} > 0, b_{n',\eta} > 0.
\end{aligned}
\tag{26}
$$

χ_L is also called the indicator function. It is the function defined as equal to one on L and zero elsewhere. The index would just be the number of ones in the indicator function, namely:

$$
\sum_{f_n \in \chi_L} \chi_L(f_n)
$$

We would expect that the networks with larger values of this index would also have a greater value of γ^*, because technological sharing should have a relatively larger negative average demand impact on competitors and an average smaller demand impact on the quantity of the firm itself.

Balanced Innovation happens when firms invest more ex-ante for a less homogeneous market which yields higher profit and requires no protection due to low substitutability while investing ex-ante less for a high homogeneous market which also does not require protection due to its smaller level of investment at any point of the market. This phenomenon does not need an index because it is already captured by the utility functions, which represent the characteristics of the available demand at the substitution level and willingness to buy. A simple ratio between α and average substitution would be enough: $\frac{\alpha}{\mu(b)}$. I would also expect optimal externality to increase with this index because there would be either lower substitution or a larger willingness to buy which would incentivize investments.

We could observe another general interaction between available demand and technology to derive a ratio between the degree of connection that would be tested to predict the optimal level of patent breadth γ^*. The index would have at the numerator the number of firms connected on technology and at the denominator the number of firms connected at the demand level.

$$
\frac{\sum_{t_\eta}^{T_{\phi_{n,\eta}}} \left(\frac{\sum_{t_\eta \neq t_n}^{T_{\phi_{n,\eta}}} z_{n,t_n}}{\sum_{t_\eta \neq t_n}^{T_{\phi_{n,\eta}}} z_{n,t_n}} \right)}{\sum_{n=1}^{N} \left(\frac{\sum_{n \neq \eta}^{N} b_{n,\eta} q_n q_\eta}{\sum_{n \neq \eta}^{N} b_{n,\eta} q_n q_\eta} \right)}
$$

This measure should be interpreted as a simple general-purpose technology index which would be a proxy for the number of weak competitors with access to a defined firm's improvements. A network with more general-purpose technology would require slighter protection because there is a further possibility for firms to profit from the technological improvements of other firms while impacting less on the incentives to innovate.

Testing for the validity of these indices would be possible after observing the n-firms network and computing the optimal externality level γ. Therefore, the indices would represent the conjectured drivers of the different optimal values γ^*. The predictive power of each index would verify the adequacy between the explanations for the differences in the optimal levels of patent breadth.

I would expect a nonlinear prediction performance for this index, where small values indicate a large substitution level with respect to technological connectedness (and so pointwise lower optimal investment), and large values indicate a small substitution level with respect to connectedness (and so investment protected by low substitution or high amount of general-purpose technologies). Therefore the two extremes would require a smaller patent protection. However, for intermediate values, I would expect a demand and technology environment balanced enough to require a larger level of patent protection to satisfy that market.

In a following paper we will try to compute these indices and test them with patent data and costs estimations.

Conclusion

In this study, we examined the interaction between demand and technology networks in a 3-Firms setting to determine the optimal patent breadth for all relevant structures. Our analysis considered the technological connections between firms based on the intersection of their patent sets and the demand connections based on the substitutability of their goods. We weighted each edge on the demand network by its corresponding substitution parameter and on the technological network depending on the number of technologies in common. The three firms competed à la Cournot and we computed the quantities and investments that maximized their profits. Firms were able to invest in all technologies available in their patent sets at a convex cost, resulting in an average linear cost reduction that was transmitted to other connected firms depending on the weight of their technology links and the externality level γ set by the regulator. In our setting, this externality level coincided with patent breadth. We computed optimal investments so that they could depend only on γ. By plugging these investments back into the optimal quantities of the corresponding network, we were able to compute the optimal externality level γ^* that maximizes the general welfare for each network. General welfare was calculated as the sum of consumer utility and firms' profits. Our results indicate that all networks with an intermediate substitution level have a welfare-maximizing $\gamma^* = 1$, meaning that no protection with the full transmission is optimal in these cases. 2 out of 5 networks with high substitution have an optimal transmission $\gamma^* = 1$ and similarly, Network A with high substitution has an optimal $\gamma^* = 0.94$. However, the remaining two networks B and D with high substitution have an optimal $\gamma^* = 0$.

We can summarize the Balanced Innovation as the idea that firms will invest ex-ante more for a less homogeneous market which yields higher profit and requires no protection due to low products' similarity while investing ex-ante less for a high homogeneous market which also does not require absolute protection due to its smaller level of investment used to satisfy the elastic demand.

If this trend dominates throughout all substitution levels for different network structures, we would expect a nonlinear performance for the optimal γ^* with respect to all substitution levels. The relation between patent breadth and product similarity should start with free externality since investments would be protected by low substitution, rising to complete patent protection for an intermediate substitution level, and finally relaxing intellectual protection by reducing patent breadth when similarity converges to homogeneity, with substitution values around 1, where the optimal investment will be lower due to extensive products' similarity.

To derive this trend with simple structures we have looked at the difference between 3-Firms networks, justifying the different $b_{g,j}$ turning points, where the switch from complete externality to full protection happened, with a nonlinear relation between patent breadth and product similarity.

However, a clearer picture of this tendency would require observing this nonlinear trend within a single network. Consequently, we also designed more complex structures to find this nonlinear relation within every network. We observed five demand and technology networks of cores and circles with eight firms.

The balance trend discussed for the 3-Firms case is still valid, with the difference that for a larger number of firms we can see it within each network without the need for comparing different structures. In 4 out of 5 networks, we see a drop in the optimal γ^* with a spike in the end. Our explanation for this nonlinear trend remains the same. For minor substitution levels, firms are protected by the extensive difference between their goods. While for high substitution levels, the need for protection is lower for two reasons. First, on the profit side, firms' optimal quantity is limited for any externality level due to consumers' market power which leads to lower profit for any patent breadth setting. Secondly, on the utility side, consumers are less harmed by a quantity reduction due to the high substitution between goods.

In the last two sections, we analyze possible extensions such as multiple product networks to understand what characteristic we should take into account in a more complex network, and we also develop indices to test our theory in a general model. The general case of this setting should keep the same logic regarding the connection for demand and technology networks, while being effective for an arbitrarily large number of firms and technologies. This generalization could be done by creating a Python code for a network-generating model of firms competing on quantity, with the same logic as in the 3-Firms example. However, some conclusions will be determined ex-ante by the characterization of this model. Therefore, it would be more honest to test the already described indexes with real microdata. This will be the objective of a future project.

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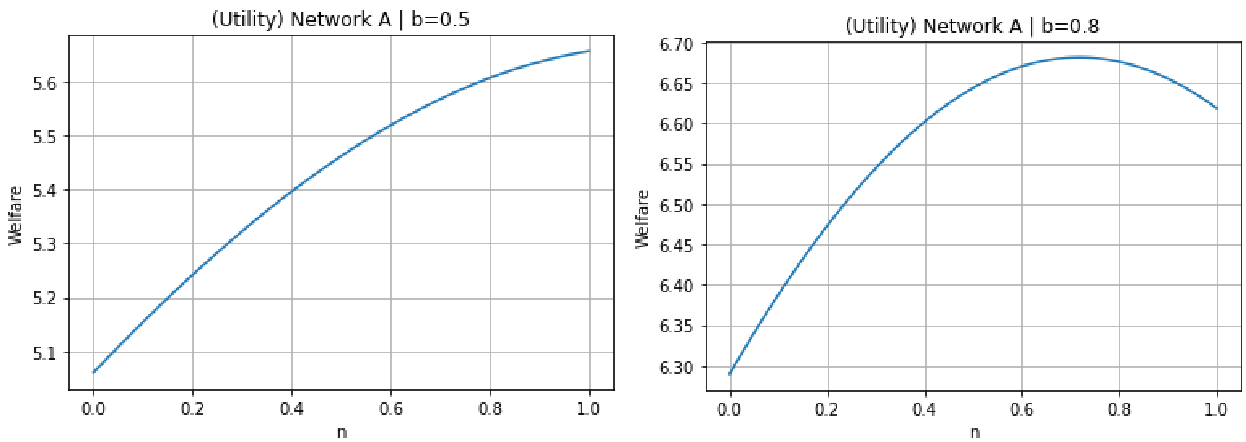
Appendix I

Find codes, plots and computations in Dario Marino GitHub

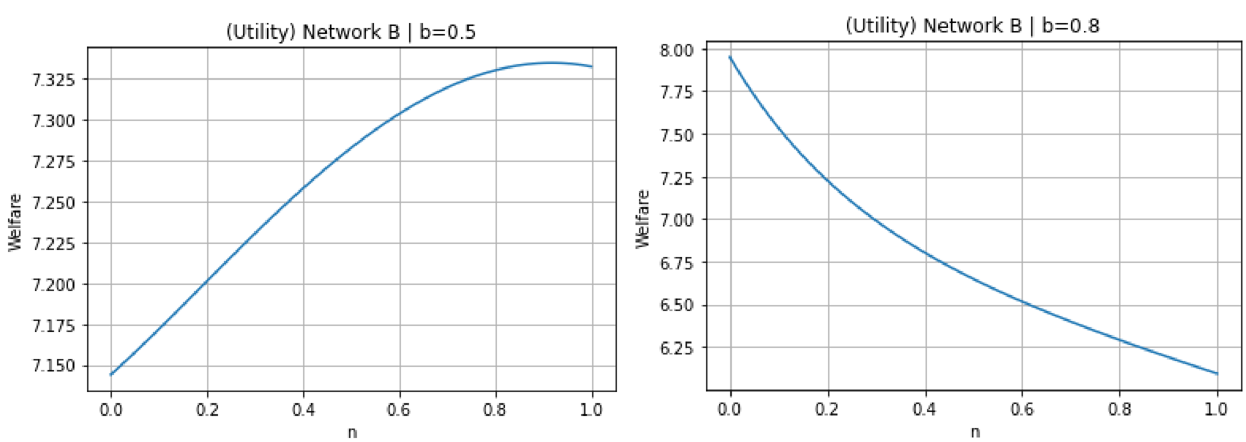
Appendix II



(a) Intermediate Substitution

(b) Strong Substitution

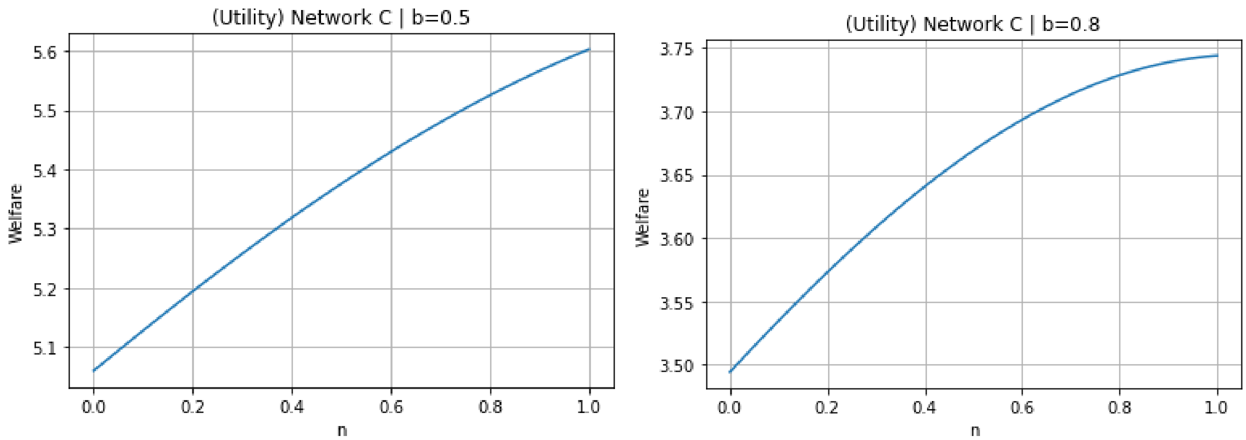
Figure 37: (Utility) Network A



(a) Intermediate Substitution

(b) Strong Substitution

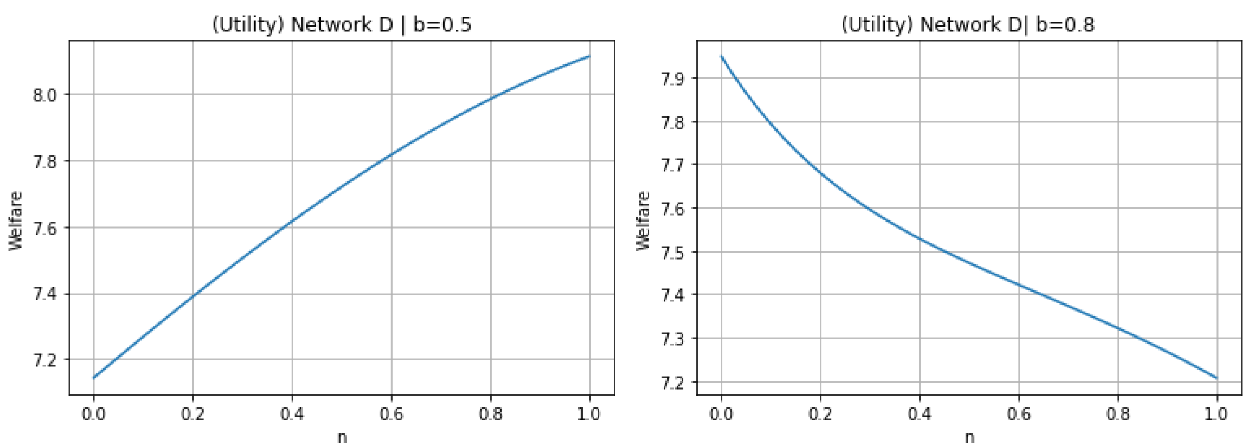
Figure 38: (Utility) Network B



(a) Intermediate Substitution

(b) Strong Substitution

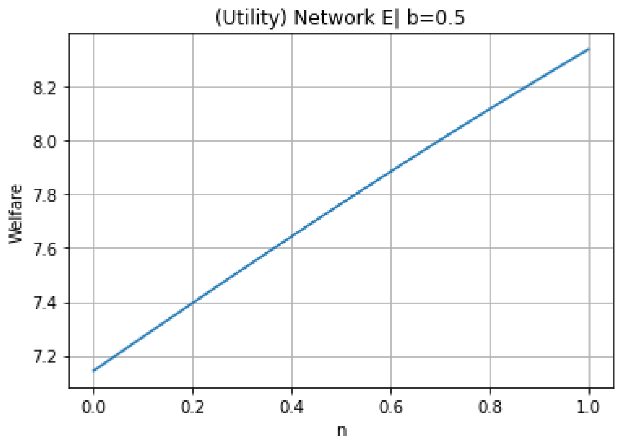
Figure 39: (Utility) Network C



(a) Intermediate Substitution

(b) Strong Substitution

Figure 40: (Utility) Network D



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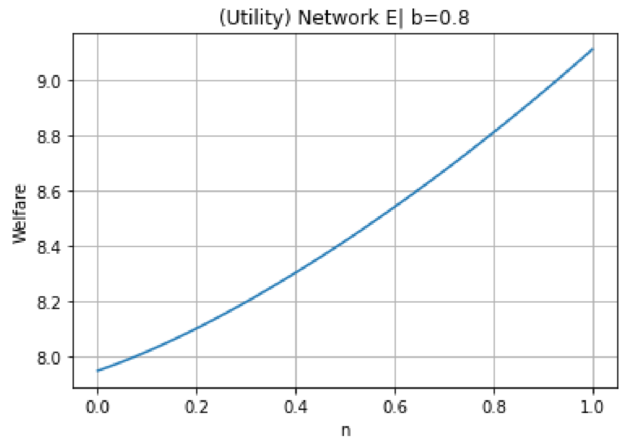
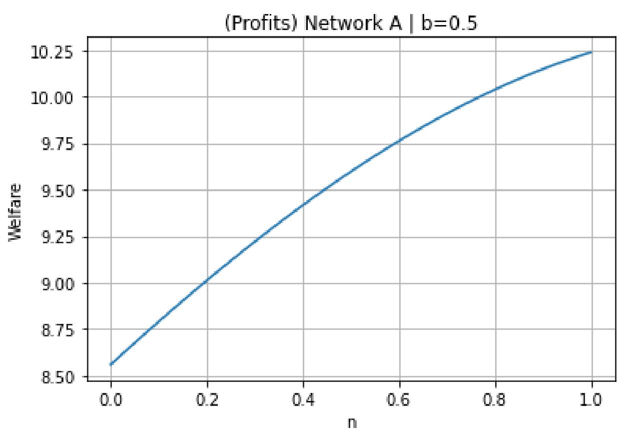


(b) Strong Substitution

Figure 41: (Utility) Network E

Appendix III



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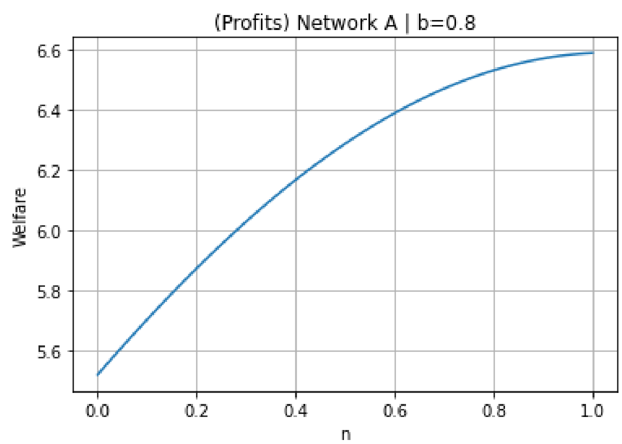
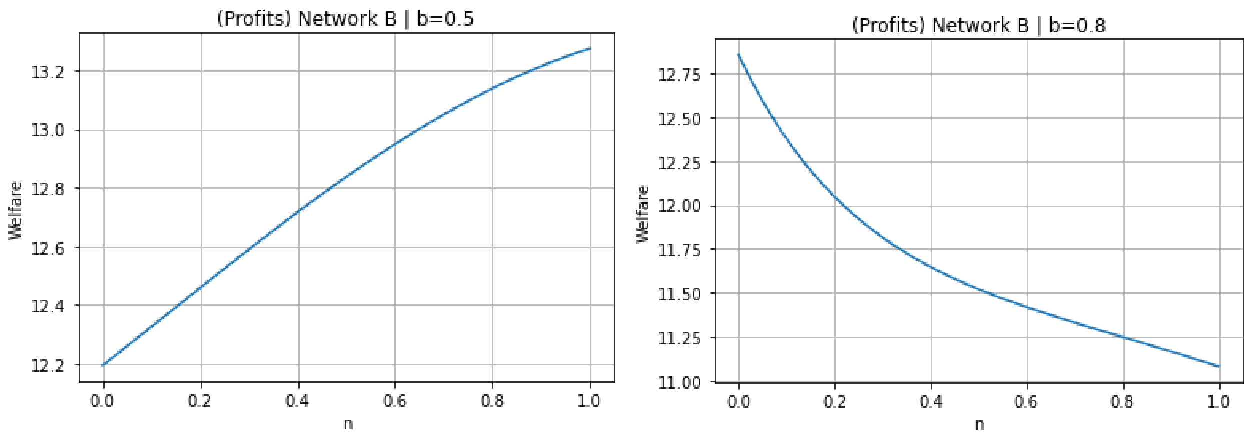


(b) Strong Substitution

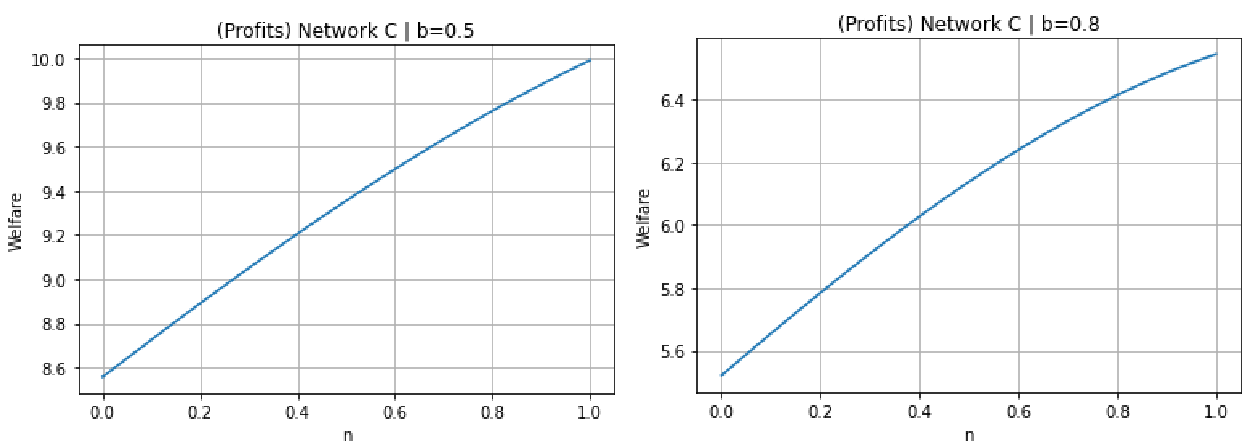
Figure 42: (Aggregate Profits) Network A



(a) Intermediate Substitution

(b) Strong Substitution

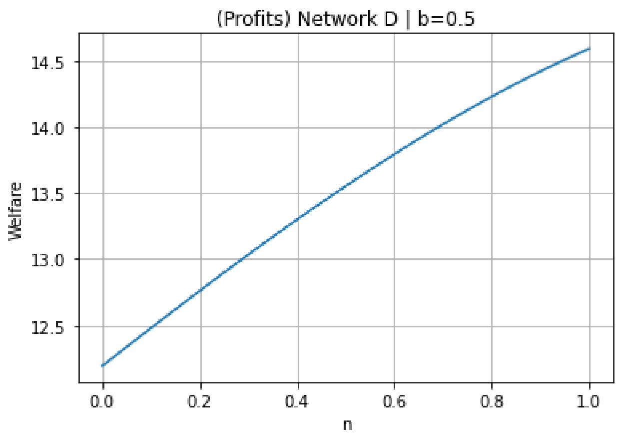
Figure 43: (Aggregate Profits) Network B



(a) Intermediate Substitution

(b) Strong Substitution

Figure 44: (Aggregate Profits) Network C



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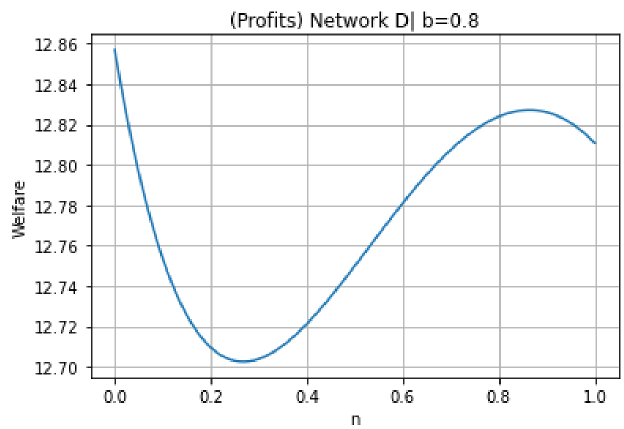
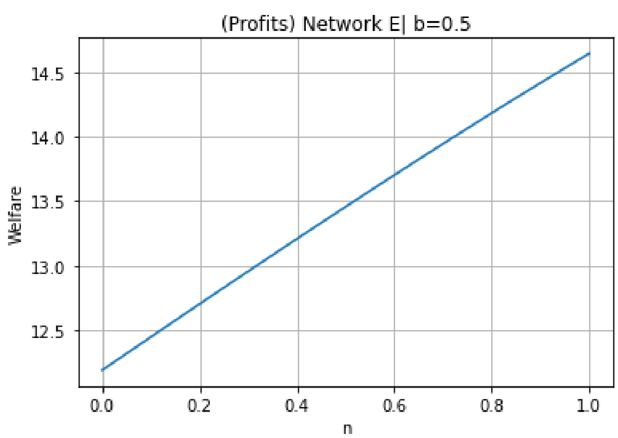


(b) Strong Substitution

Figure 45: (Aggregate Profits) Network D



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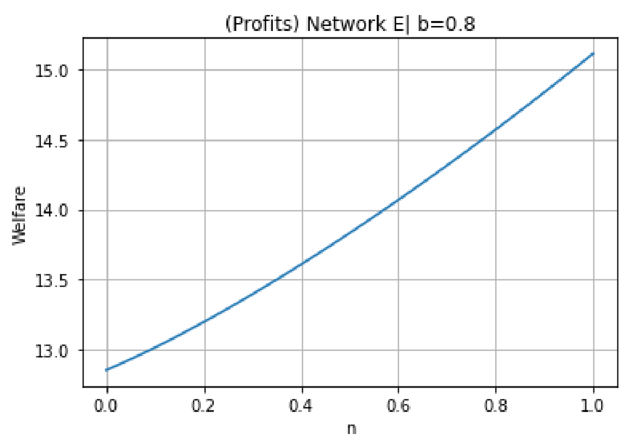
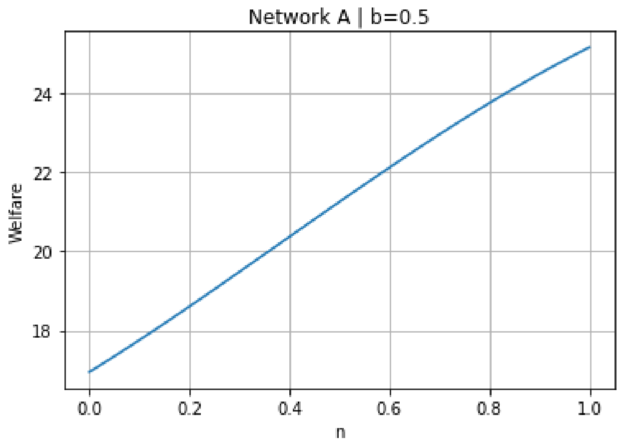


(b) Strong Substitution

Figure 46: (Aggregate Profits) Network E

Appendix IV



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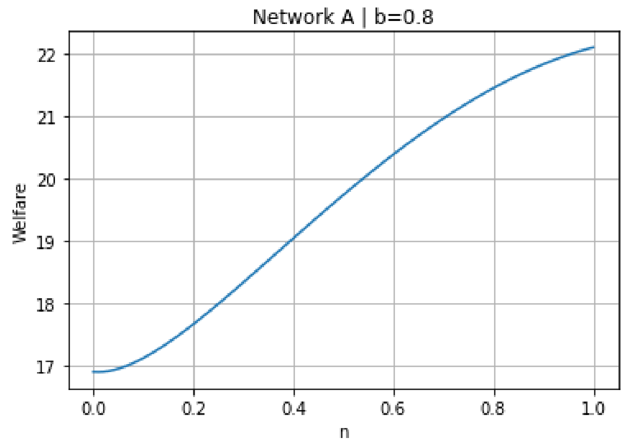
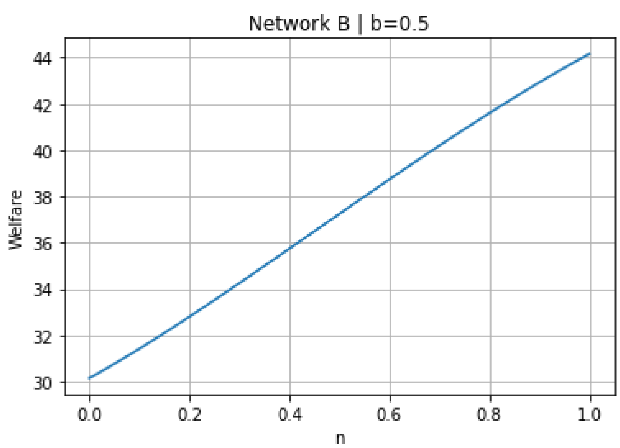
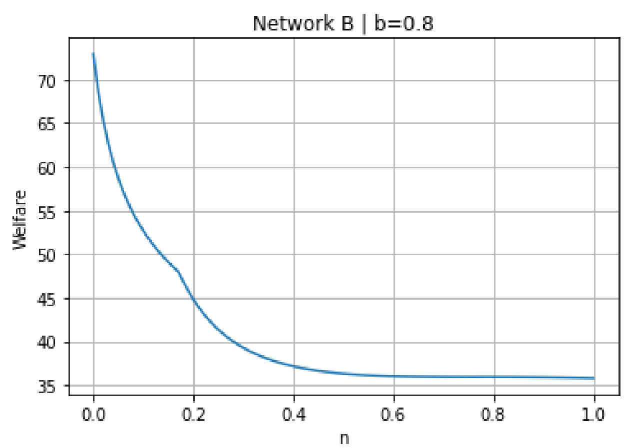


(b) Strong Substitution

Figure 47: (Uniform) Network A

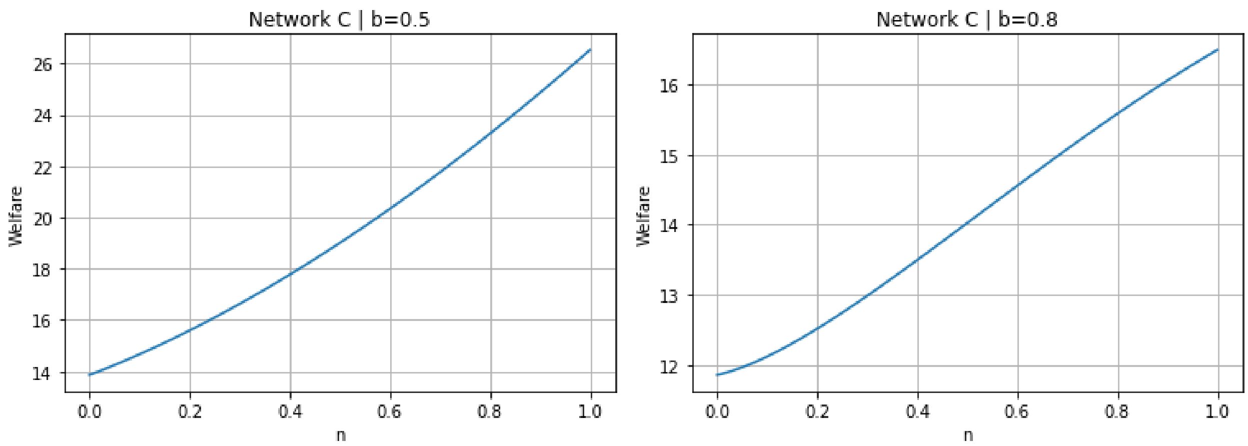


(a) Intermediate Substitution



(b) Strong Substitution

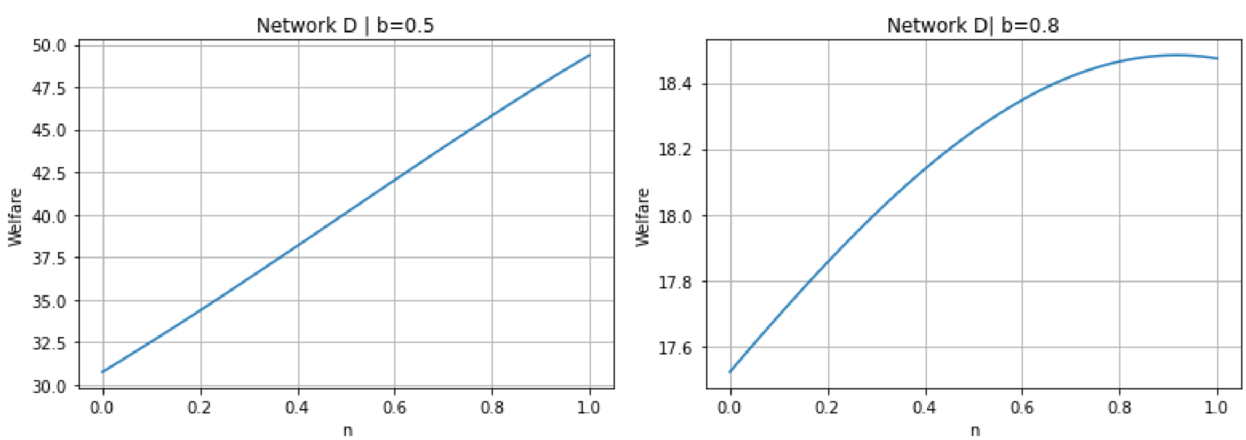
Figure 48: (Uniform) Network B



(a) Intermediate Substitution

(b) Strong Substitution

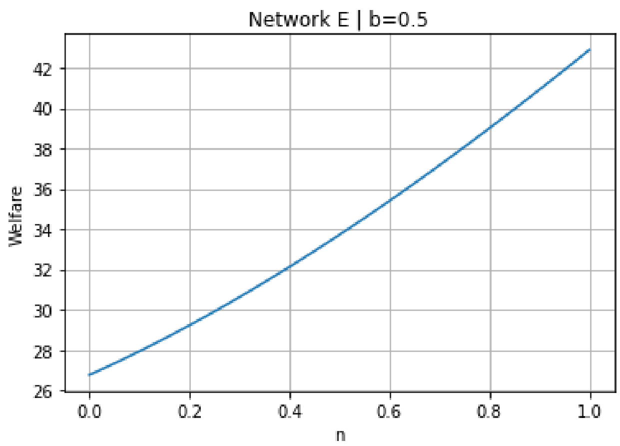
Figure 49: (Uniform) Network C



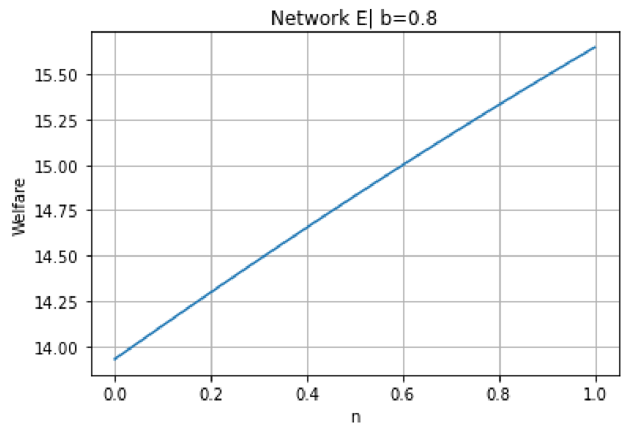
(a) Intermediate Substitution

(b) Strong Substitution

Figure 50: (Uniform) Network D



(a) Intermediate Substitution



(b) Strong Substitution

Figure 51: (Uniform) Network E

Appendix V

In the uniform case we have that the optimal quantities with the following fixed parameters are:

$$[\alpha = 10, c = 5, \phi_{1,1} = \phi_{3,3} = 2, \phi_{2,2} = 3]$$

1. Network A ($\phi_{1,2} = 2, \phi_{1,3} = 1, \phi_{2,3} = 1, b_{i,j} = 0.5$):

$$q_1 = \frac{566 + z_1(466 - 165\gamma) + z_2(421\gamma - 141) + z_3(186\gamma - 90)}{420}$$

$$q_2 = \frac{1234 - z_1(106 - 434\gamma) + z_2(681 - 121\gamma) + 6z_3(29\gamma - 5)}{420}$$

$$q_3 = \frac{600 + z_1(210\gamma - 90) + z_2(135\gamma - 135) + z_3(450 - 90\gamma)}{420}$$

$$Q = \frac{2565 + z_1(270 + 479\gamma) + z_2(405 + 435\gamma) + z_3(330 + 270\gamma)}{420}$$

2. Network A ($\phi_{1,2} = 2, \phi_{1,3} = 1, \phi_{2,3} = 1, b_{i,j} = 0.8$):

$$q_1 = \frac{25 + z_1(23 - 10\gamma) - z_2(10 - 20\gamma) - z_3(7 - 8\gamma)}{18}$$

$$q_2 = \frac{25 - z_1(6 - 20\gamma) + z_2(35 - 10\gamma) - z_3(6 - 9\gamma)}{18}$$

$$q_3 = \frac{25 - z_1(7 - 5\gamma) - z_2(10 - 5\gamma) + z_3(23 - 7\gamma)}{18}$$

$$Q = \frac{(75 + z_1(10 + 15\gamma) + z_2(15 + 15\gamma) + z_3(10 + 10\gamma)}{18}$$

3. Network B ($\phi_{1,2} = 2, \phi_{2,3} = 1, b_{i,j} = 0.5$):

$$q_1 = \frac{1575 + 15z_1(60 - n) + z_2(930\gamma - 360) + z_3(60 - 120\gamma)}{840}$$

$$q_2 = \frac{21 + z_1(15\gamma - 4) + z_2(24 - 6\gamma) + z_3(8\gamma - 4)}{14}$$

$$q_3 = \frac{119 + z_1(4 - 15\gamma) + z_2(34\gamma - 24) + z_3(60 - 8\gamma)}{56}$$

$$Q = \frac{154 + z_1(24 + 22\gamma) + z_2(24 + 36n) + z_3(24 + 8\gamma)}{28}$$

4. Network B ($\phi_{1,2} = 2, \phi_{2,3} = 1, b_{i,j} = 0.8$):

$$q_1 = \frac{75 + z_1(42 - 20\gamma) - z_2(30 - 46\gamma) + z_3(8 - 10\gamma)}{34}$$

$$q_2 = \frac{55 - z_1(50 - 125\gamma) + z_2(188 - 75\gamma) - z_3(42 - 53\gamma)}{85}$$

$$q_3 = \frac{1875 + z_1(200 - 500\gamma) + z_2(725\gamma - 750) + z_3(1050 - 250\gamma)}{850}$$

$$Q = \frac{860 + z_1(150 + 50\gamma) + z_2(72 + 225\gamma) + z_3(166 + 6n)}{170}$$

5. Network C ($\phi_{1,2} = 2, \phi_{2,3} = 1, b_{i,j} = 0.5$):

$$q_1 = \frac{9210 + z_1(8164 - 1224\gamma) - z_2(1836 - 5522\gamma) - z_3(1228 - 204\gamma)}{5523}$$

$$q_2 = \frac{3120 - z_1(3120 - 2080\gamma) + z_2(3120 - 624\gamma) - z_3(416 + 1040\gamma)}{1841}$$

$$q_3 = \frac{1290 - z_1(172 + 192\gamma) - z_2(288 - 268\gamma) + z_3(880 + 96\gamma)}{789}$$

$$Q = \frac{9200 + z_1(-800 + 1244\gamma) + z_2(1836 + 1842\gamma) + z_3(1228 - 748\gamma)}{1841}$$

6. Network C ($\phi_{1,2} = 2, \phi_{2,3} = 1, b_{i,j} = 0.8$):

$$q_1 = \frac{375 + z_1(350 - 100\gamma) - z_2(150 - 300\gamma) - z_3(100 + 50\gamma)}{382}$$

$$q_2 = \frac{862 - z_1(150 - 725\gamma) + z_2(1087 - 265\gamma) - z_3(230 - 362\gamma)}{573}$$

$$q_3 = \frac{8625 - z_2(3450 - 2125\gamma) - z_1(1500 + 2300\gamma) + z_3(7250 - 1150\gamma)}{5730}$$

$$Q = \frac{4574 + z_1(450 + 690\gamma) + z_2(1034 + 795\gamma) + z_3(690 + 344\gamma)}{1146}$$

7. Network D ($\phi_{1,2} = 2, \phi_{1,3} = 1, \phi_{2,3} = 1, b_{i,j} = 0.5$):

$$q_1 = \frac{15 + z_1(7.5 - 1.75\gamma) - z_2(3 - 8\gamma) + 3.25cn)}{7}$$
$$q_2 = \frac{10 - z_1(2 - 7\gamma) + z_2(12 - 3\gamma) + cn)}{7}$$
$$q_3 = \frac{15 + z_1(0.5 + 1.75\gamma) + -z_2(3 - 4\gamma) + z_3(7 - 0.25\gamma)}{7}$$

$$Q = \frac{40 + z_1(6 + n) + z_2(6 + 9\gamma) + z_3(7 + 4\gamma)}{7}$$

8. Network D ($\phi_{1,2} = 2, \phi_{1,3} = 1, \phi_{2,3} = 1, b_{i,j} = 0.8$):

$$q_1 = \frac{75 + z_1(42 - 16\gamma) - z_2(30 - 46\gamma) + z_3(1.6 + 16\gamma)}{34}$$
$$q_2 = \frac{25 - z_1(20 - 40\gamma) + z_2(75 - 30\gamma) - z_3(4 - 3\gamma)}{34}$$
$$q_3 = \frac{75 + z_1(8 - n) - z_2(30 - 29\gamma) + z_3(35 - \gamma)}{34}$$

$$Q = \frac{175 + z_1(30 + 23\gamma) + z_2(15 + 45\gamma) + z_3(33 + 18\gamma)}{34}$$

9. Network E ($\phi_{1,2} = 2, \phi_{2,3} = 1, b_{i,j} = 0.5$):

$$q_1 = \frac{10 + z_1(8 - 2\gamma) - z_2(3 - 7\gamma) - z_3(2 + n)}{7}$$
$$q_2 = \frac{15 - z_1(2 - 7.5\gamma) + z_2(11.25 - 1.75\gamma) + z_3(0.5 + 3.75\gamma)}{7}$$
$$q_3 = \frac{15 - z_1(4 - 0.5\gamma) + z_2(0.75 + 1.75\gamma) + z_3(7.5 + 0.25\gamma)}{7}$$

$$Q = \frac{40 + z_1(2 + 6n) + z_2(9 + 7\gamma) + z_3(6 + 3\gamma)}{7}$$

10. Network E ($\phi_{1,2} = 2, \phi_{2,3} = 1, b_{i,j} = 0.8$):

$$q_1 = \frac{25 + z_1(50 - 20\gamma) - z_2(30 - 40\gamma) - z_3(20 + 10\gamma)}{34}$$
$$q_2 = \frac{75 - z_1(20 - 42\gamma) + z_2(63 - 16\gamma) + z_3(8 + 21\gamma)}{34}$$
$$q_3 = \frac{75 - z_1(20 - 8\gamma) + z_2(12 + n) + z_3(42 + 4\gamma)}{34}$$

$$Q = \frac{175 + z_1(10 + 22\gamma) + z_2(45 + 25\gamma) + z_3(30 + 15\gamma)}{34}$$