A Fast Quasi-Linear Heuristic for the Close-Enough Traveling Salesman Problem

1 Introduction

The *Close-Enough Traveling Salesman Problem* (CETSP) is a continuous generalization of the classical Traveling Salesman Problem (TSP), where each target is associated with a circular neighborhood that the salesman must intersect rather than a precise point. The CETSP thus combines the discrete optimization of the visiting sequence with the continuous optimization of visiting positions, and it naturally models path-planning tasks such as radio meter reading, drone-based inspection, and robotic welding (Lei and Hao 2024; Di Placido, Archetti, and Cerrone 2022).

Despite recent progress, most high-performing CETSP algorithms rely on population-based or local-improvement metaheuristics that achieve strong solution quality at the expense of high computational cost. In this work, we present a new heuristic for the CETSP that is inspired by the quasi-linear *pair-center algorithm* for the Euclidean TSP proposed by Formella (2024). While originally designed for point-based TSP instances, the pair-center concept extends naturally to the CETSP once disk geometry and intersection logic are handled properly. Our resulting method runs in expected $O(n \operatorname{polylog} n)$ time, making it markedly faster and more scalable than current state-of-the-art CETSP solvers. Although it does not seek to outperform metaheuristic approaches in final tour length, its runtime efficiency allows application to extremely large or real-time instances that remain intractable for existing techniques.

2 Related Work

3 Algorithm

Our approach follows the overall structure of the quasi-linear *pair-center algorithm* introduced by Formella (2024) for the Euclidean TSP, but extends it to handle circular neighborhoods as in the Close-Enough Traveling Salesman Problem. Conceptually, the method retains the same two-phase organization: a *clustering phase* that builds a hierarchical representation of the instance, and a *construction phase* that incrementally expands a feasible tour from this hierarchy.

In the clustering phase, the algorithm recursively merges the closest pair of geometric objects into a new composite "proxy" object until a single hierarchy remains. In the CETSP setting, these objects are circles rather than points as in Formella (2024). The result is a binary clustering tree whose internal nodes represent proxy circles.

The construction phase then traverses this tree to build a closed tour. As in the original pair-center approach, the tour is dynamically maintained so that it remains feasible at every step. However, the continuous nature of the CETSP introduces two additional challenges: first, multiple circles may correspond to a single effective tour point when their feasible regions overlap; second, each tour point admits continuous local optimization within its circle. To address these, the algorithm performs lightweight dynamic updates—reinserting or locally re-optimizing tour points—while preserving near-linear expected runtime.

Overall, the method retains the speed and structural simplicity of the pair-center algorithm while incorporating the geometric flexibility required for close-enough constraints. Detailed definitions, data structures, and optimization steps are presented in the following sections.

3.1 Clustering Phase

The clustering phase closely follows the structure of the pair-center algorithm described by Formella (2024), with appropriate modifications to handle circular regions instead of point targets. As in the original approach, the goal is to construct a hierarchical binary tree over all targets by iteratively merging the closest pair of geometric objects, replacing them with a single representative (a *proxy circle*). The leaves of the resulting tree correspond to the original input disks, and each internal node stores its proxy circle and the associated merge distance.

Data Structures. We make extensive use of geometric search structures to maintain and query spatial relationships efficiently. In particular, we employ an R*-tree to store bounding boxes of the current set of circles, enabling logarithmic-time expected queries for nearest neighbors, insertions, and deletions. The resulting operations have expected $O(n \operatorname{polylog} n)$ runtime; see Formella (2024) for a detailed review and analysis.

Preprocessing. Before clustering, we remove redundant circles—specifically, any circle that is completely contained within another. This is performed by approximating each circle with its axis-aligned bounding box, sorting all boxes by decreasing radius, and using the R*-tree to query for boxes that fully cover the current one. To mitigate alignment bias from axis-aligned boxes, we randomize the instance by applying a random global rotation to all circle centers. This randomization slightly improves expected merging quality while maintaining near-linear preprocessing cost.

Merge Step. Let c_1 and c_2 denote the two circles chosen for merging. Their *effective distance* is defined as

$$d(c_1, c_2) = \|p_1 - p_2\| - r_1 - r_2,$$

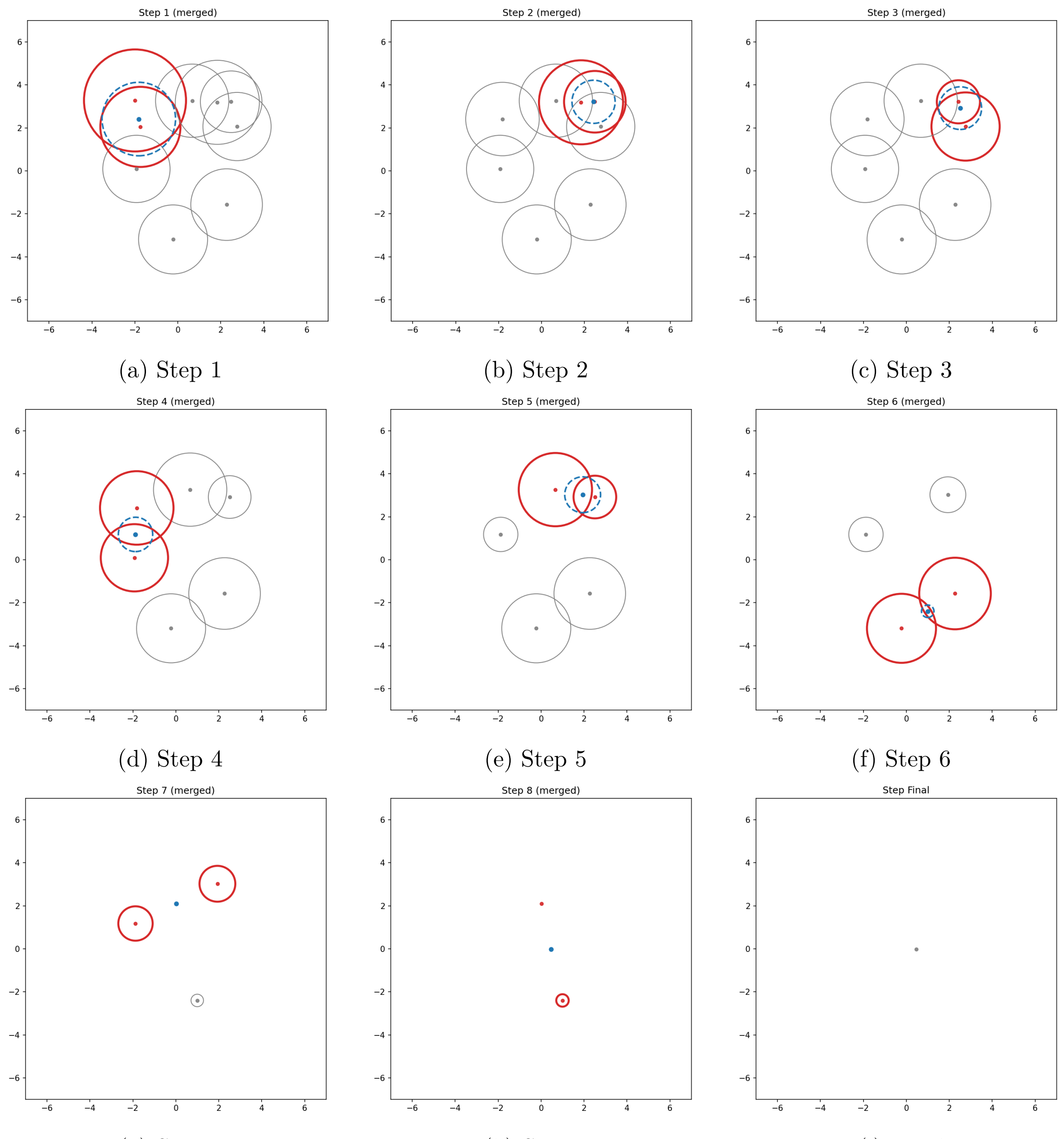
that is, the Euclidean gap between their boundaries. Intuitively, smaller values indicate strong overlap or proximity, which makes the pair more representative for early merging.

The merge procedure operates as follows:

1. Select the pair (c_1, c_2) with the minimal $d(c_1, c_2)$ value.
2. Remove c_1 and c_2 from the R*-tree and from the corresponding neighbor structures.
3. Compute their proxy circle c', defined as the minimal enclosing circle covering both c_1 and c_2. This can be computed in constant time using geometric primitives.
4. Insert c' into the R*-tree and update all relevant neighbor lists.
5. Insert c' as an internal node in the binary tree, with c_1 and c_2 as its children.

To maintain efficiency, we adopt the same general mechanism as Formella (2024). Each active circle keeps a list of its current nearest neighbors, and a global min-heap stores tuples of the form $(d(c_i, c_j), i, j)$. When a pair is merged, only the affected neighbors of c_i and c_j are updated—an operation bounded by a constant factor related to the planar kissing number (six in $\mathbb{R}^2$).

Approximate Neighbor Search. Because the effective distance $d(c_1, c_2) = \|p_1 - p_2\| - r_1 - r_2$ depends on both position and radius, exact nearest-neighbor search under this metric is impractical. Instead, we perform approximate search by querying the k closest bounding boxes (from a certain center p_1) and evaluating $d(c_1, c_2)$ explicitly for those candidates. This is inexact in two ways: boxes do not perfectly encapsulate the circles, and the distance any value of $\|p_1 - p_2\| - r_2$ not greater than 0 are treated the same. However, in practice, small constant values of k suffice to preserve both runtime efficiency and clustering quality.



(a) Step 1 (b) Step 2 (c) Step 3

(d) Step 4 (e) Step 5 (f) Step 6

(g) Step 7 (h) Step 8 (i) Final

Figure 1: Visualization of the clustering phase for a sample instance with $n = 9$ disks. Each subfigure shows one merge step, where the two closest circles (highlighted) are replaced by their proxy circle (dashed blue outline). The final frame (bottom right) shows the single remaining cluster.

Proxy Circle Computation. When merging two circles $c_1 = (p_1, r_1)$ and $c_2 = (p_2, r_2)$, we define their *proxy circle* $c' = (p', r')$ as a compact approximation of the lens-shaped region covered by their union. In contrast to simply using the midpoint between centers, our implementation constructs c' geometrically along the line connecting p_1 and p_2.

Specifically, let $d = \|p_1 - p_2\|$ denote the center distance. If a circle covers another circle, we simply take the smaller circle. If the circles do not intersect at all ($d \geq r_1 + r_2$), we take the midpoint between the circle boundaries and radius 0 (i.e. a single point).

Otherwise, we construct a proxy circle. We determine the two points where the line connecting p_1 and p_2 intersects the boundaries of the two circles, and place the new center p' at the midpoint between these boundary points. The resulting circle approximates the geometric "lens" formed by the intersection. We compute both the overlap depth

$$\delta = \tfrac{1}{2}(r_1 + r_2 - d)$$

and the half-chord length

$$h = \sqrt{r_1^2 - a^2}, \quad a = \frac{r_1^2 - r_2^2 + d^2}{2d},$$

which jointly define a plausible range for the new radius. To introduce mild stochastic diversity, the final radius r' is drawn uniformly from the interval $[\delta, h]$. This randomization helps avoid degenerate clustering patterns and allows multiple independent runs to explore slightly different hierarchical structures.

Although this proxy construction is heuristic, it consistently yields stable and geometrically balanced merges in practice, while maintaining constant-time computation and preserving the algorithm's overall quasi-linear runtime.

References

Di Placido, Andrea, Claudia Archetti, and Carmine Cerrone (Sept. 2022). "A genetic algorithm for the close-enough traveling salesman problem with application to solar panels diagnostic reconnaissance". In: *Computers & Operations Research* 145, p. 105831. ISSN: 03050548. DOI: `10.1016/j.cor.2022.105831`. URL: `https://linkinghub.elsevier.com/retrieve/pii/S0305054822001095` (visited on 06/05/2025).

Formella, Arno (Oct. 2024). "Quasi-linear time heuristic to solve the Euclidean traveling salesman problem with low gap". In: *Journal of Computational Science* 82, p. 102424. ISSN: 18777503. DOI: `10.1016/j.jocs.2024.102424`. URL: `https://linkinghub.elsevier.com/retrieve/pii/S1877750324002175` (visited on 10/13/2025).

Lei, Zhenyu and Jin-Kao Hao (Mar. 2024). "An effective memetic algorithm for the close-enough traveling salesman problem". In: *Applied Soft Computing* 153, p. 111266. ISSN: 15684946. DOI: `10.1016/j.asoc.2024.111266`. URL: `https://linkinghub.elsevier.com/retrieve/pii/S1568494624000401` (visited on 06/05/2025).