

## 7. Structural Stability

In the previous two sections, we described how to get a rough picture of the trajectories of a non-linear system by linearizing at each of its critical points. The basic assumption of the method is that the linearized system will be a good approximation to the original non-linear system if you stay near the critical point.

The method only works however if the linearized system turns out to be a node, saddle, or spiral. What is it about these geometric types that allows the method to work, and why won't it work if the linearized system turns out to be one of the other possibilities (dismissed as "borderline types" in the previous section)?

Briefly, the answer is that nodes, saddles, and spirals are *structurally stable*, while the other possibilities are not. We call a system

$$(1) \quad \begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

**structurally stable** if small changes in its parameters (i.e., the constants that enter into the functions on the right hand side) do not change the geometric type and stability of its critical points (or its limit cycles, if there are any — see the next section, and don't worry about them for now.)

**Theorem.** *The  $2 \times 2$  autonomous linear system*

$$(2) \quad \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

*is structurally stable if it is a spiral, saddle, or node (but not a degenerate or star node).*

**Proof.** The characteristic equation is

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0,$$

and its roots (the eigenvalues) are

$$(3) \quad \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$

Let's look at the cases one-by-one; assume first that the roots  $\lambda_1$  and  $\lambda_2$  are real and distinct. The possibilities in the theorem are given by the following (note that since the roots are distinct, the node will not be degenerate or a star node):

$$\begin{aligned} \lambda_1 > 0, \lambda_2 > 0 &\text{ } \textit{unstable node} \\ \lambda_1 < 0, \lambda_2 < 0 &\text{ } \textit{asymptotically stable node} \\ \lambda_1 > 0, \lambda_2 < 0 &\text{ } \textit{unstable saddle}. \end{aligned}$$

The quadratic formula (3) shows that the roots depend continuously on the coefficients  $a, b, c, d$ . Thus if the coefficients are changed a little, the roots  $\lambda_1$  and  $\lambda_2$  will also be changed a little to  $\lambda'_1$  and  $\lambda'_2$  respectively; the new roots will still be real, and will have the

same sign if the change is small enough. Thus the changed system will still have the same geometric type and stability.  $\square$

If the roots of the characteristic equation are complex, the reasoning is similar. Let us denote the complex roots by  $r \pm si$ ; we use the root  $\lambda = r + si$ ,  $s > 0$ ; then the possibilities to be considered for structural stability are

$$\begin{aligned} r > 0, s > 0 &\quad \text{unstable spiral} \\ r < 0, s > 0 &\quad \text{asymptotically stable spiral.} \end{aligned}$$

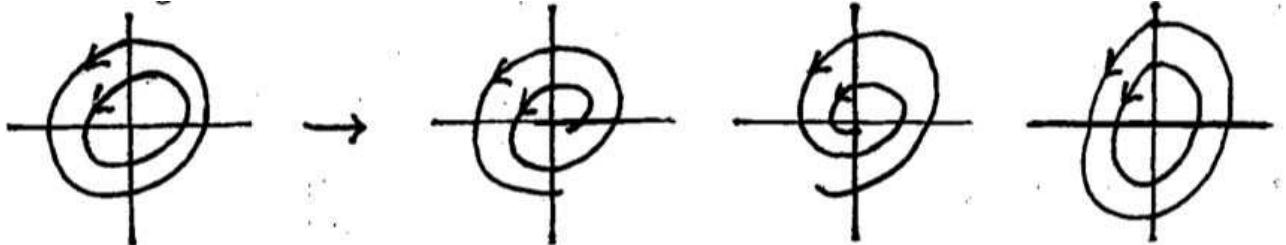
If  $a, b, c, d$  are changed a little, the root is changed to  $\lambda' = r' + s'i$ , where  $r'$  and  $s'$  are close to  $r$  and  $s$  respectively, since the quadratic formula (3) shows  $r$  and  $s$  depend continuously on the coefficients. If the change is small enough,  $r'$  will have the same sign as  $r$  and  $s'$  will still be positive, so the geometric type of the changed system will still be a spiral, with the same stability type.  $\square\square$

**8. The borderline geometric types** All the other possibilities for the linear system (2) we call borderline types. We will show now that none of them is structurally stable; we begin with the center.

**Eigenvalues pure imaginary.** Once again we use the eigenvalue with the positive imaginary part:  $\lambda = 0 + si$ ,  $s > 0$ . It corresponds to a *center*: the trajectories are a family of concentric ellipses, centered at the origin. If the coefficients  $a, b, c, d$  are changed a little, the eigenvalue  $0 + si$  changes a little to  $r' + s'i$ , where  $r' \approx 0$ ,  $s' \approx s$ , and there are three possibilities for the new eigenvalue:

$$\begin{array}{llll} 0 + si \rightarrow r' + s'i : & r' > 0 & r' < 0 & r' = 0 \\ s > 0 & s' > 0 & s' > 0 & s' > 0 \\ \text{center} & \text{source spiral} & \text{sink spiral} & \text{center} \end{array}$$

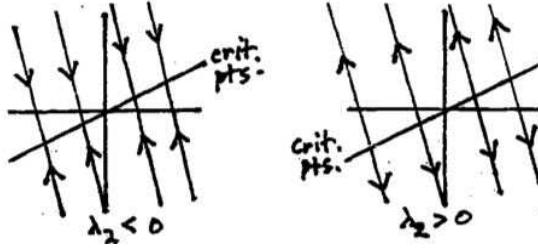
Correspondingly, there are three possibilities for how the geometric picture of the trajectories can change:



**Eigenvalues real; one eigenvalue zero.** Here  $\lambda_1 = 0$ , and  $\lambda_2 > 0$  or  $\lambda_2 < 0$ . The general solution to the system has the form ( $\alpha_1, \alpha_2$  are the eigenvectors)

$$\mathbf{x} = c_1 \alpha_1 + c_2 \alpha_2 e^{\lambda_2 t}.$$

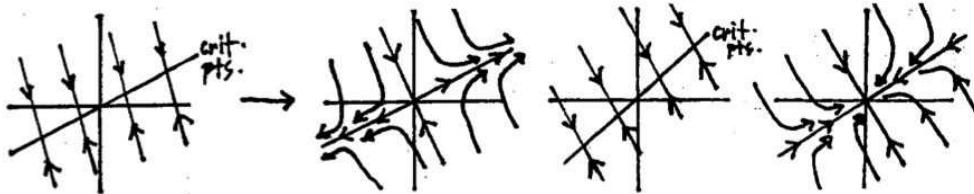
If  $\lambda_2 < 0$ , the geometric picture of its trajectories shows a line of critical points (constant solutions, corresponding to  $c_2 = 0$ ), with all other trajectories being parallel lines ending up (for  $t = \infty$ ) at one of the critical points, as shown below.



We continue to assume  $\lambda_2 < 0$ . As the coefficients of the system change a little, the two eigenvalues change a little also; there are three possibilities, since the 0 eigenvalue can become positive, negative, or stay zero:

$$\begin{array}{lll} \lambda_1 = 0 \rightarrow \lambda'_1 : & \lambda'_1 > 0 & \lambda'_1 = 0 \\ \lambda_2 < 0 \rightarrow \lambda'_2 : & \lambda'_2 < 0 & \lambda'_2 < 0 \\ \text{critical line} & \text{unstable saddle} & \text{critical line} \\ & & \text{sink node} \end{array}$$

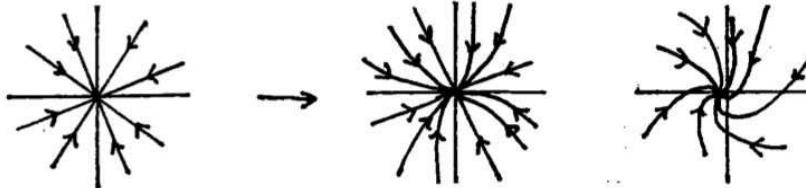
Here are the corresponding pictures. (The pictures would look the same if we assumed  $\lambda_2 > 0$ , but the arrows on the trajectories would be reversed.)



**One repeated real eigenvalue.** Finally, we consider the case where  $\lambda_1 = \lambda_2$ . Here there are a number of possibilities, depending on whether  $\lambda_1$  is positive or negative, and whether the repeated eigenvalue is complete (i.e., has two independent eigenvectors), or defective (i.e., incomplete: only one eigenvector). Let us assume that  $\lambda_1 < 0$ . We vary the coefficients of the system a little. By the same reasoning as before, the eigenvalues change a little, and by the same reasoning as before, we get as the main possibilities (omitting this time the one where the changed eigenvalue is still repeated):

$$\begin{array}{lll} \lambda_1 < 0 & \rightarrow & \lambda'_1 < 0 & r + si \\ \lambda_2 < 0 & \rightarrow & \lambda'_2 < 0 & r - si \\ \lambda_1 = \lambda_2 & & \lambda'_1 \neq \lambda'_2 & r \approx \lambda_1, s \approx 0, \\ \text{sink node} & & \text{sink node} & \text{sink spiral} \end{array}$$

Typical corresponding pictures for the complete case and the defective (incomplete) case are (the last one is left for you to experiment with on the computer screen)



complete: star node

incomplete: defective node

**Remarks.** Each of these three cases—one eigenvalue zero, pure imaginary eigenvalues, repeated real eigenvalue—has to be looked on as a borderline linear system: altering the coefficients slightly can give it an entirely different geometric type, and in the first two cases, possibly alter its stability as well.

### Application to non-linear systems.

All the preceding analysis discussed the structural stability of a linear system. How does it apply to non-linear systems?

Suppose our non-linear system has a critical point at  $P$ , and we want to study its trajectories near  $P$  by linearizing the system at  $P$ .

This linearization is only an approximation to the original system, so if it turns out to be a borderline case, i.e., one sensitive to the exact value of the coefficients, *the trajectories near  $P$  of the original system can look like any of the types obtainable by slightly changing the coefficients of the linearization.*

It could also look like a combination of types. For instance, if the linearized system had a critical line (i.e., one eigenvalue zero), the original system could have a sink node on one half of the critical line, and an unstable saddle on the other half. (This actually occurs.)

In other words, the method of linearization we used in Sections 6 and 7 to analyze a non-linear system near a critical point doesn't fail entirely, but we don't end up with a definite picture of the non-linear system near  $P$ ; we only get a list of possibilities. In general one has to rely on computation or more powerful analytic tools to get a clearer answer. The first thing to try is a computer picture of the non-linear system, which often will give the answer.

# **M.I.T. 18.03 Ordinary Differential Equations**

## **18.03 Notes and Exercises**

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