Homework 1

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MATH-693B Numerical Partial DIfferential Equations

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1.1.1

Consider the initial value problem for the equation

$$u_t + au_x = f(t,x)$$

$$\text{with } u(0,x) = 0 \text{ and } \quad f(t,x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume that a is positive. Show that the solution is given by

$$u(t,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/a & \text{if } x \geq 0 \text{ and } x - at \leq 0 \\ t & \text{if } x \geq 0 \text{ and } x - at \geq 0 \end{cases}$$

Solution

When $f(t,x) = 0$, we have the unique solution $u(t,x) = u_0(x - at)$. This gives the answer $u_0(x-at) = u(0, x-at) = 0$, which is the indicated solution for all $x < 0$.

For $f(t,x) = 1$ and $x - at < 0$, we can show that the solution $u(t,x) = x/a$ is valid. Observe:

$$u(t,x) = \frac{x}{a} \quad \longrightarrow \quad u_t = 0 \quad \text{and} \quad u_x = \frac{1}{a}$$

Plugging this into our problem, we see that the result is $0 + a(1/a) = 1$, which is true. So this solution is correct.

For $f(t,x) = 1$ and $x - at \geq 0$, let's change variables so that

$$\tau = t \qquad \text{and} \qquad \xi = x - at \quad \longrightarrow \quad x = \xi + a\tau.$$

Now we have $\tilde{u}(\tau,\xi) = u(t,x)$, and it follows that

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial \tau} &= \frac{\partial t}{\partial \tau} u_t + \frac{\partial x}{\partial \tau} u_x \\ &= u_t + a u_x = f(\tau, \xi + a\tau). \end{aligned}$$

With $\frac{\partial \tilde{u}}{\partial \tau} = f(\tau, \xi + a\tau)$, we can solve this as an ordinary differential equation, which has the following solution:

$$\begin{aligned} \tilde{u}(\tau,\xi) &= u_0(\xi) + \int_0^\tau f(\sigma, \xi + a\sigma)d\sigma \quad \longrightarrow \\ u(t,x) &= u_0(x - at) + \int_0^t f(s, x - a(t-s))ds \\ &= 0 + \int_0^t ds = \left. s \right|_0^t = t \end{aligned}$$

And we see that this is indeed the solution we were seeking.

1.3.1

For values of x in the interval $[-1, 3]$ and t in $[0, 2.4]$, solve the one-way wave equation $u_t + u_x = 0$ with the initial data

$$u(0,x) = \begin{cases} \cos^2 \pi x & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and the boundary data $u(t,-1) = 0$. Use the following four schemes for $h = 1/10,\ 1/20$, and $1/40$:

1. (a) Forward-time backward-space scheme (1.3.2) with $\lambda = 0.8$
   (b) Forward-time central-space scheme (1.3.3) with $\lambda = 0.8$
   (c) Lax-Friedrichs scheme (1.3.5) with $\lambda = 0.8$ and 1.6
   (d) Leapfrog scheme (1.3.4) with $\lambda = 0.8$.

For schemes (b). (c), and (d), at the right boundary use the condition $v_M^{n+1} = v_{M-1}^{n+1}$, where $x_M = 3$. For scheme (d) use scheme (b) to compute the solution at $n = 1$ For each scheme determine whether the scheme is a useful or useless scheme. For the purposes of this exercise only, a scheme will be useless if $|v_m^n|$ is greater than 5 for any value of m and n. It will be regarded as a useful scheme if the solution looks like a reasonable approximation to the solution of the differential equations. Graph or plot several solutions at the last time they were computed. What do you notice about the "blow-up time" for the useless schemes as the mesh size decreases? Is there a pattern to these solutions? For the useful cases, how does the error decrease as the mesh decreases; i.e., as h decreases by one-half, by how much does the error decrease?

Solution

(a) Forward Time, Backward Space

Figure (1) shows this scheme. Notice that the approximate solution dampens to zero (see video, emailed). For $h = 1/10$, it hits zero by about 0.72 seconds. For $h = 1/20$, it hits zero by about 0.36 seconds. For $h = 1/40$, it hits zero by about 0.18 seconds. As the density of the mesh doubles, it converges twice as quickly. Error?

(b) Forward Time, Central Space

Figure (2) shows this scheme. Notice that the approximate solution explodes to infinity (see video, emailed). For $h = 1/10$, the solution exceeds a value of 5 after about 1.76 seconds. For $h = 1/20$, the solution exceeds 5 after about 1.2 seconds. For $h = 1/40$, the solution passes 5 after about .72 seconds. As the density of the mesh doubles, the solution becomes useless more quickly.

(c) Lax-Friedrichs

Figure (3) shows this scheme. Notice that the approximate solution explodes to infinity (see video, emailed). For $h = 1/10$, the solution exceeds a value of 5 after about 1.6 seconds. For $h = 1/20$, the solution exceeds 5 after about 1.12 seconds. For $h = 1/40$, the solution passes 5 after about .72 seconds. As the density of the mesh doubles, the solution becomes useless more quickly, in a similar scale to the forward time/central space scheme.

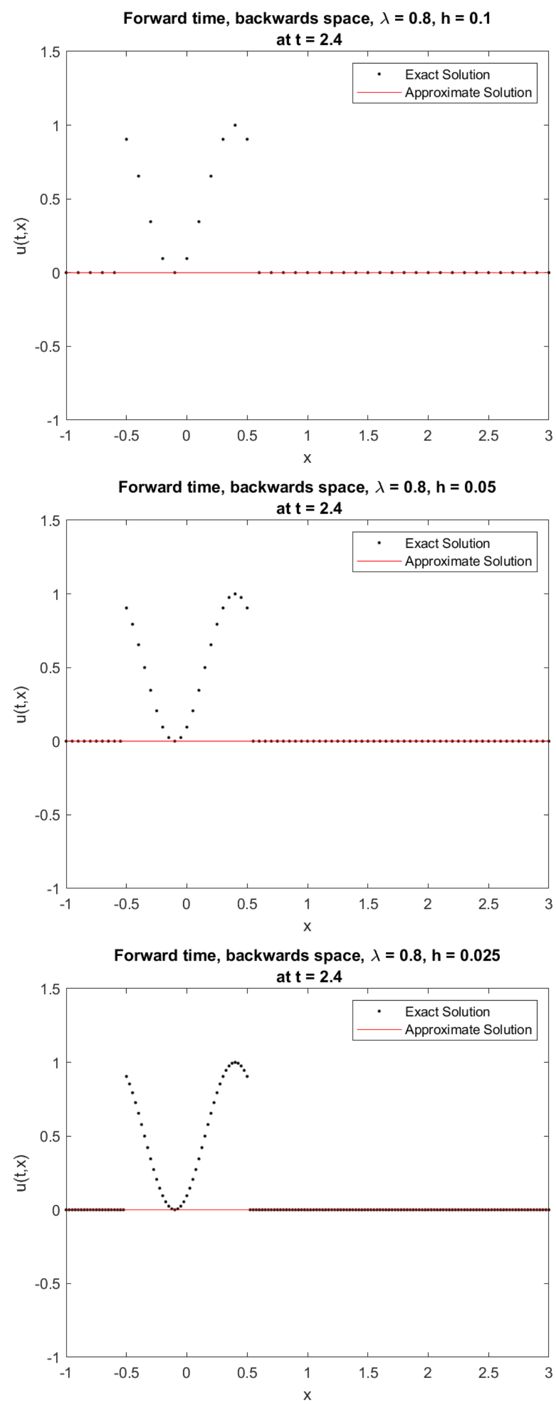

Figure 1: Forward time, backward space scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

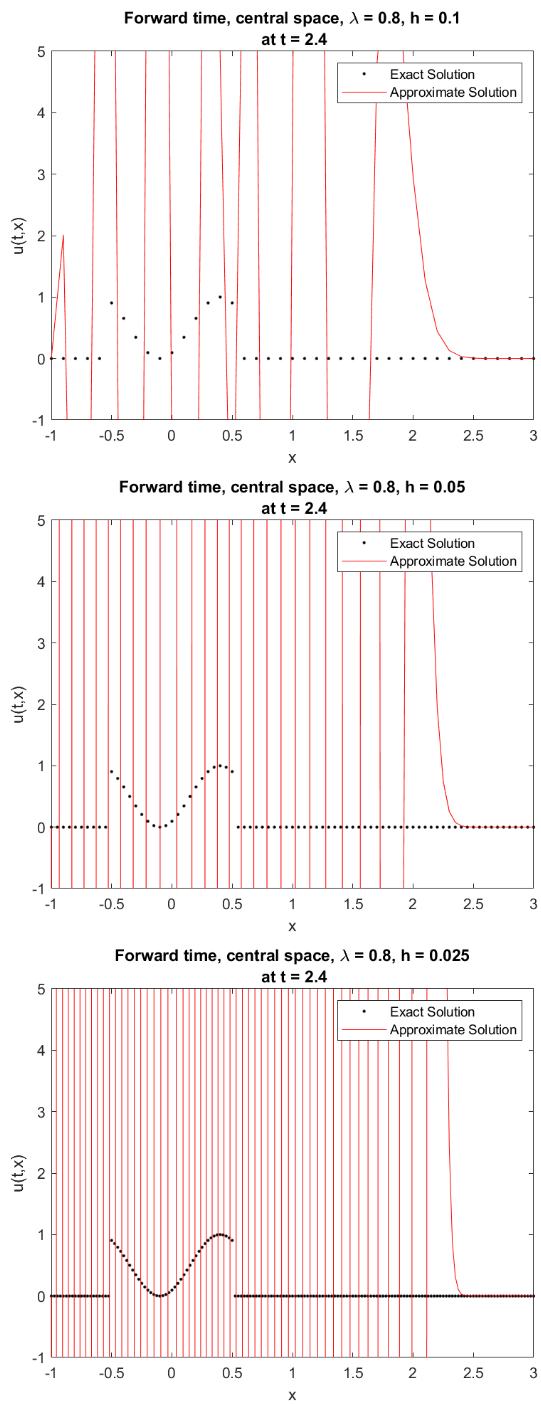

Figure 2: Forward time, central space scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

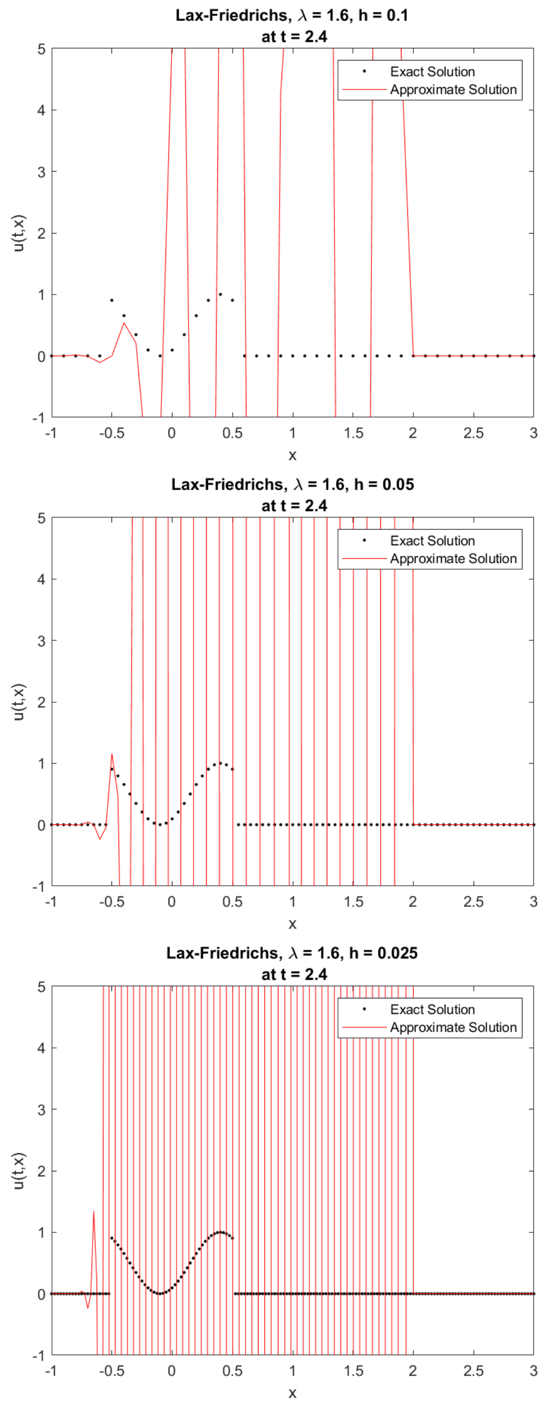

Figure 3: Lax-Friedrichs at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

(d) Leapfrog

Figure (4) shows this scheme. Notice that the approximate solution dampens to zero (see video, emailed). For $h = 1/10$, it hits zero by about 0.72 seconds, like the forward time, backward space scheme. For $h = 1/20$, it hits zero by about 0.36 seconds. For $h = 1/40$, it hits zero by about 0.18 seconds. Just like (a), as the mesh density doubles, the time to convergence halves. Error?

1.4.2

Show that the leapfrog scheme is consistent with the one-way wave equation.

Solution

1.5.1

Show that schemes of the form

$$u_m^{n+1} = \alpha u_{m+1}^n + \beta u_{m-1}^n$$

are stable if $|\alpha| + |\beta|$ is less than or equal to 1. Conclude that the Lax-Friedrichs scheme is stable if $|a\lambda|$ is less than or equal to 1 .

Solution

MATLAB Command Comments

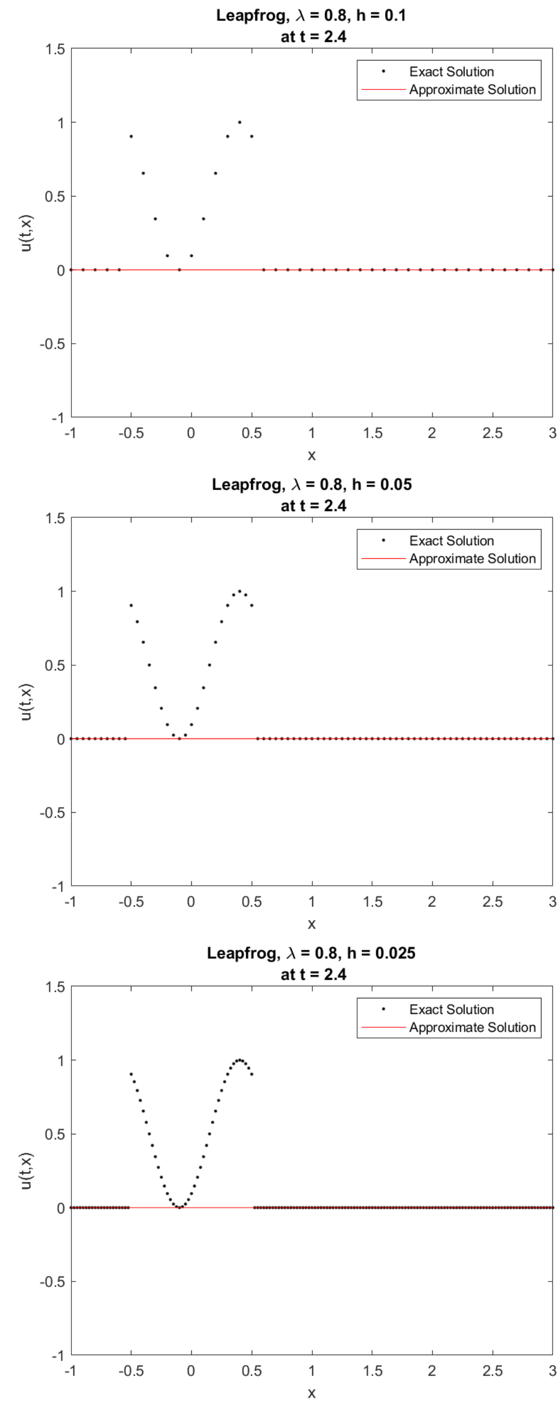

Figure 4: Forward time, backward space scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively