Homework 1

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MATH-693B Numerical Partial DIfferential Equations

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1.1.1

Consider the initial value problem for the equation

$$u_t + au_x = f(t,x)$$

$$\text{with } u(0,x) = 0 \text{ and } \quad f(t,x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume that a is positive. Show that the solution is given by

$$u(t,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/a & \text{if } x \geq 0 \text{ and } x - at \leq 0 \\ t & \text{if } x \geq 0 \text{ and } x - at \geq 0 \end{cases}$$

Solution

When $f(t,x) = 0$, we have the unique solution $u(t,x) = u_0(x - at)$. This gives the answer $u_0(x-at) = u(0,x-at) = 0$, which is the indicated solution for all $x < 0$.

For $f(t,x) = 1$ and $x-at < 0$, we can show that the solution $u(t,x) = x/a$ is valid. Observe:

$$u(t,x) = \frac{x}{a} \quad \longrightarrow \quad u_t = 0 \quad \text{and} \quad u_x = \frac{1}{a}$$

Plugging this into our problem, we see that the result is $0 + a(1/a) = 1$, which is true. So this solution is correct.

For $f(t,x) = 1$ and $x - at \geq 0$, let's change variables so that

$$\tau = t \quad \text{and} \quad \xi = x - at \quad \longrightarrow \quad x = \xi + a\tau.$$

Now we have $\tilde{u}(\tau,\xi) = u(t,x)$, and it follows that

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial \tau} &= \frac{\partial t}{\partial \tau} u_t + \frac{\partial x}{\partial \tau} u_x \\ &= u_t + a u_x = f(\tau, \xi + a\tau).\end{aligned}$$

With $\frac{\partial \tilde{u}}{\partial \tau} = f(\tau, \xi + a\tau)$, we can solve this as an ordinary differential equation, which has the following solution:

$$\begin{aligned}\tilde{u}(\tau,\xi) &= u_0(\xi) + \int_0^\tau f(\sigma, \xi + a\sigma) d\sigma \quad \longrightarrow \\ u(t,x) &= u_0(x - at) + \int_0^t f(s, x - a(t-s)) ds \\ &= 0 + \int_0^t ds = \left. s \right|_0^t = t\end{aligned}$$

And we see that this is indeed the solution we were seeking.

1.3.1

For values of x in the interval $[-1,3]$ and t in $[0,2.4]$, solve the one-way wave equation $u_t + u_x = 0$ with the initial data

$$u(0,x) = \begin{cases} \cos^2 \pi x & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and the boundary data $u(t,-1) = 0$. Use the following four schemes for $h = 1/10,\ 1/20$, and $1/40$:

1. (a) Forward-time backward-space scheme (1.3.2) with $\lambda = 0.8$
   (b) Forward-time central-space scheme (1.3.3) with $\lambda = 0.8$
   (c) Lax-Friedrichs scheme (1.3.5) with $\lambda = 0.8$ and 1.6
   (d) Leapfrog scheme (1.3.4) with $\lambda = 0.8$.

(a) Forward Time, Backward Space

Figure (1) shows this scheme. Notice that the approximate solution has a significant error at first, but is still "useful" (see video, emailed). For $h = 1/10$, the largest error is 0.3094. For $h = 1/20$, the error is 0.1888. For $h = 1/40$, the error is 0.1055. As the density of the mesh doubles, the error reduces by roughly half.

(b) Forward Time, Central Space

Figure (2) shows this scheme. Notice that the approximate solution explodes to infinity (see video in code). For $h = 1/10$, the solution exceeds a value of 5 after about 1.76 seconds. For $h = 1/20$, the solution exceeds 5 after about 1.2 seconds. For $h = 1/40$, the solution passes 5 after about .72 seconds. As the density of the mesh doubles, the solution becomes useless more quickly.

(c_1) Lax-Friedrichs, $\lambda = 0.8$

Figure (3) shows this scheme. Notice that the approximate solution has a significant error at first, but is still "useful" (see video, emailed). For $h = 1/10$, the largest error is 0.4705. For $h = 1/20$, the error is 0.3315. For $h = 1/40$, the error is 0.2064. As the density of the mesh doubles, the error decreases by about 1/3.

(c_2) Lax-Friedrichs, $\lambda = 1.6$

Figure (4) shows this scheme. Notice that the approximate solution explodes to infinity (see video in code). For $h = 1/10$, the solution exceeds a value of 5 after about 1.44 seconds. For $h = 1/20$, the solution exceeds 5 after about 1.12 seconds. For $h = 1/40$, the solution passes 5 after about 0.72 seconds. As the density of the mesh doubles, the solution becomes useless about 30% more quickly.

(d) Leapfrog

Figure (5) shows this scheme. Notice that the approximate solution has a significant error at first, but is ultimately very useful (see video, emailed). For $h = 1/10$, the error is 0.0755. For $h = 1/20$, the error is 0.0151. For $h = 1/40$, the error is 0.0030. As the density of the mesh doubles, the error decreases by about 3/4. This is clearly the most successful scheme.

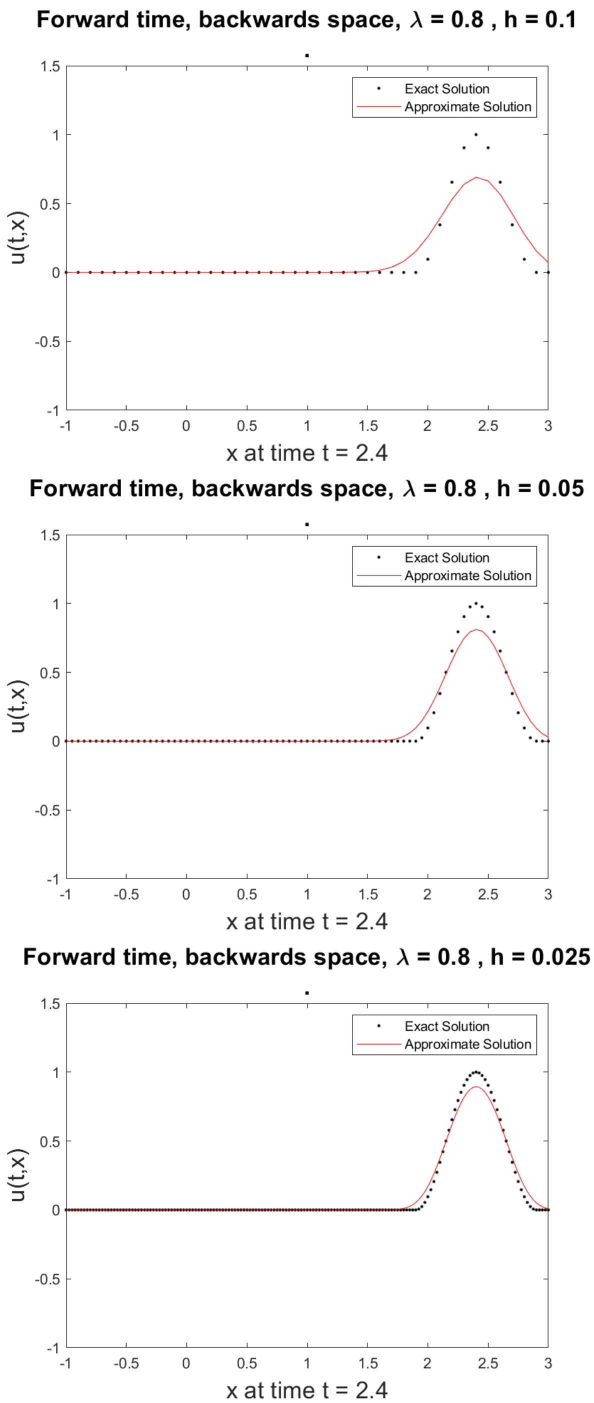

Figure 1: Forward time, backward space scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

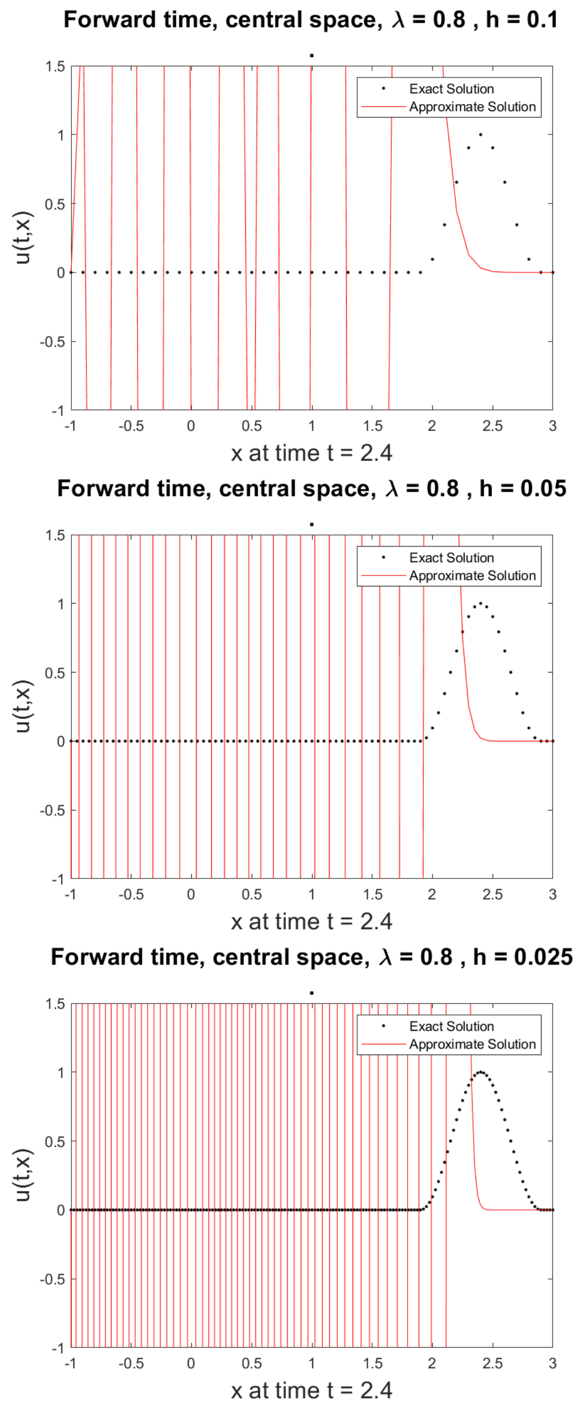

Figure 2: Forward time, central space scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

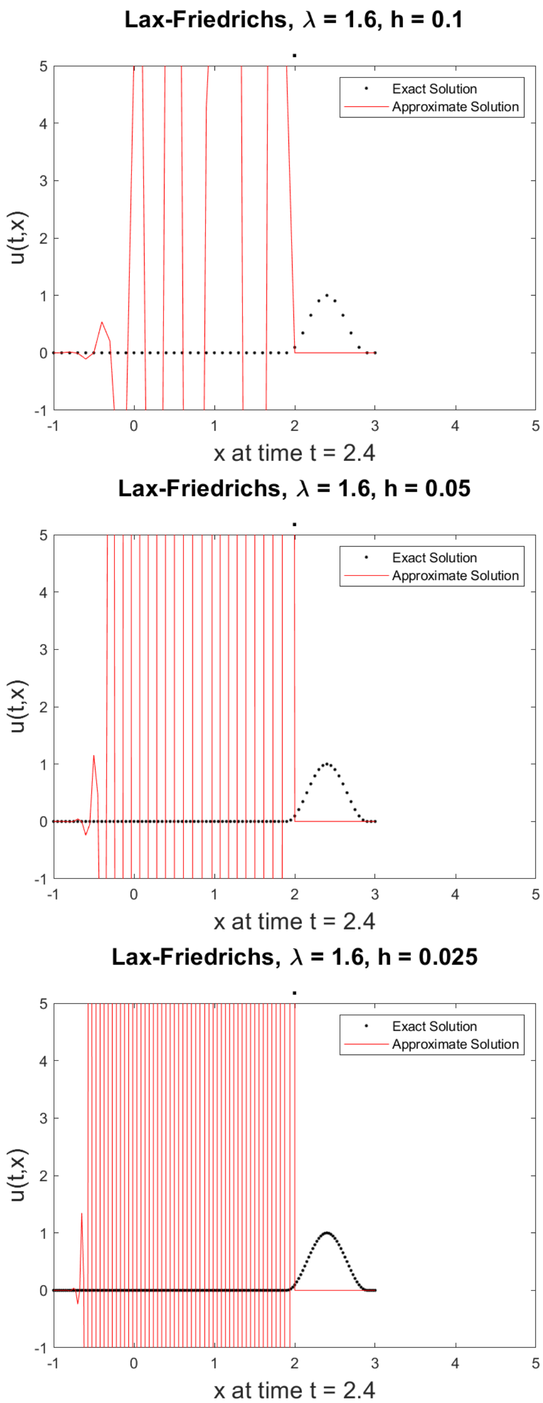

Figure 3: Lax-Friedrichs at $\lambda = 0.8$ and $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

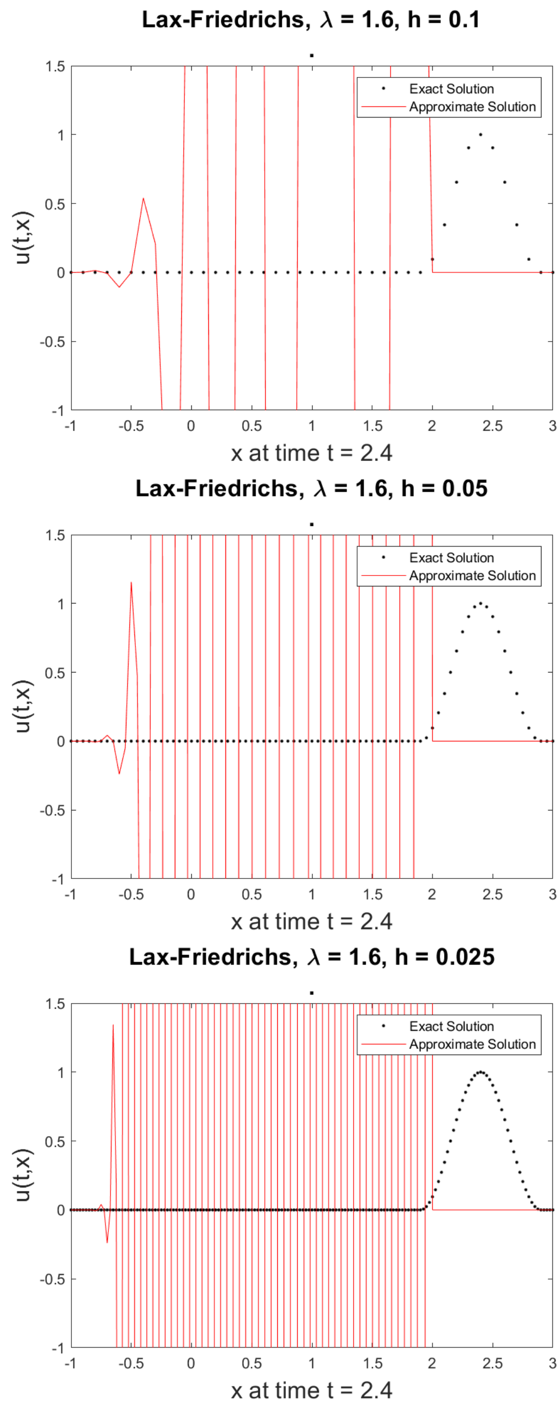

Figure 4: Lax-Friedrichs at $\lambda = 1.6$ and $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

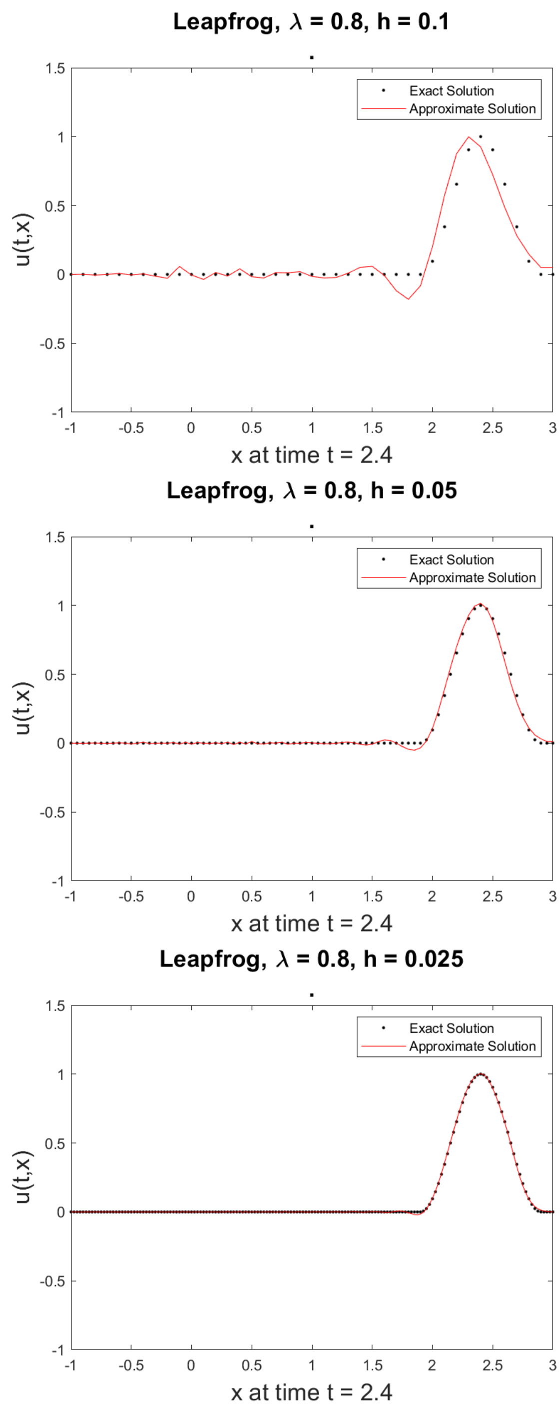

Figure 5: Leapfrog scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

1.4.2

Show that the leapfrog scheme is consistent with the one-way wave equation.

Solution

First, let's establish the equations we are working with.

$$\begin{array}{ll} \text{one-way wave eq.} & P(u) = u_t + u_x = 0 \\ \text{leapfrog scheme} & P_{k,h}(u)\dfrac{u_m^{n+1} - u_m^{n-1}}{2k} + a\dfrac{u_{m+1}^n - u_{m-1}^n}{2h} \end{array}$$

Now we write out the Taylor expansion for each term of the leapfrog scheme. The respective MATLAB commands used to quickly find the Taylor expansions for u_m^{n+1}, u_m^{n-1}, u_{m+1}^n, u_{m-1}^n were:

```
taylor(f(t+k, x), [h,k], 'ExpansionPoint', [0,0], 'Order', 3)
taylor(f(t-k, x), [h,k], 'ExpansionPoint', [0,0], 'Order', 3)
taylor(f(t, x+h), [h,k], 'ExpansionPoint', [0,0], 'Order', 3)
taylor(f(t, x-h), [h,k], 'ExpansionPoint', [0,0], 'Order', 3)
```

The Taylor expansion for these terms gives us:

$$\begin{aligned} u_m^{n+1} &= u_m^n + ku_t + \frac{1}{2}k^2u_{tt} + \mathcal{O}(k^3) \\ u_m^{n-1} &= u_m^n - ku_t + \frac{1}{2}k^2u_{tt} + \mathcal{O}(k^3) \\ u_{m+1}^n &= u_m^n + hu_x + \frac{1}{2}h^2u_{xx} + \mathcal{O}(h^3) \\ u_{m-1}^n &= u_m^n - hu_x + \frac{1}{2}h^2u_{xx} + \mathcal{O}(h^3) \end{aligned}$$

Now we substitute the expansion into the leapfrog scheme, yielding:

$$\begin{aligned} &\frac{(u_m^n + ku_t + \frac{1}{2}k^2u_{tt} + \mathcal{O}(k^3)) - (u_m^n - ku_t + \frac{1}{2}k^2u_{tt} + \mathcal{O}(k^3))}{2k} - \\ &a\frac{(u_m^n + hu_x + \frac{1}{2}h^2u_{xx} + \mathcal{O}(h^3)) - (u_m^n - hu_x + \frac{1}{2}h^2u_{xx} + \mathcal{O}(h^3))}{2h} \\ &= u_t + au_x \end{aligned}$$

Next, it is necessary to subtract the expanded scheme from the original wave equation, and since they are identical, we get zero:

$$P(u) - P_{k,h}(u) = (u_t + au_x) - (u_t + au_x) = 0$$

Therefore, the leapfrog scheme is consistent with the one-way wave equation.

1.5.1

Show that schemes of the form

$$u_m^{n+1} = \alpha u_{m+1}^n + \beta u_{m-1}^n$$

are stable if $|\alpha| + |\beta|$ is less than or equal to 1. Conclude that the Lax-Friedrichs scheme is stable if $|a\lambda|$ is less than or equal to 1 .

Solution

By definition, a first order equation scheme is stable if there is an integer J such that for any positive time T, there is a constant C_T such that

$$h \sum_{m=-\infty}^{\infty} |u_m^n|^2 \le C_T h \sum_{j=0}^{J} \sum_{m=-\infty}^{\infty} |u_m^j|^2$$

$$\text{for } 0 \le nk \le T, \text{ with } (k,h) \in \Lambda$$

For a scheme of the form $u_m^{n+1} = \alpha u_{m+1}^n + \beta u_{m-1}^n$, we have:

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} \left|u_{i_m}^{n+1}\right|^2 &= \sum_{m=-\infty}^{\infty} \left|\alpha u_n^n + \beta u_{m+1}^n\right|^2 \\
&\le \sum_{m=-\infty}^{\infty} |\alpha|^2 \left|u_m^n\right|^2 + 2|\alpha| \left|\beta \left\|u_m^n\right\| u_{m+1}^n\right| + |\beta|^2 \left|u_{m+1}^n\right|^2 \\
&\le \sum_{m=-\infty}^{\infty} |\alpha|^2 \left|u_m^n\right|^2 + |\alpha||\beta| \left(\left|u_m^n\right|^2 + \left|u_{m+1}^n\right|^2\right) + |\beta|^2 \left|u_{m+1}^n\right|^2 \\
&= \sum_{m=-\infty}^{\infty} |\alpha|^2 \left|u_m^n\right|^2 + |\alpha\|\beta\|u_m^n|^2 + \sum_{m=-\infty}^{\infty} |\alpha||\beta| \left|u_{m+1}^n\right|^2 + |\beta|^2 \left|u_{m+1}^n\right|^2
\end{aligned}$$

$$= \sum_{m=-\infty}^{\infty} |\alpha|^2 |u_m^n|^2 + |\alpha| |\beta\|u_m^n|^2 + \sum_{m=-\infty}^{\infty} |\alpha||\beta| |u_m^n|^2 + |\beta|^2 |u_m^n|^2$$

$$= \sum_{m=-\infty}^{\infty} \left(|\alpha|^2 + 2|\alpha||\beta| + |\beta|^2\right) |u_m^n|^2$$

$$= (|\alpha| + |\beta|)^2 \sum_{m=-\infty}^{\infty} |u_m^n|^2$$

$$\longrightarrow \quad \sum_{m=-\infty}^{\infty} \left|v_m^{n+1}\right|^2 \leq (|\alpha| + |\beta|)^2 \sum_{m=-\infty}^{\infty} |v_m^n|^2$$

$$\longrightarrow \quad \sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq (|\alpha| + |\beta|)^{2n} \sum_{m=-\infty}^{\infty} \left|v_m^0\right|^2$$

This derivation also appears in Strikwerda pages 30-31. We can see that this can always be true for any n, so long as the term $(|\alpha| + |\beta|)^{2n}$ does not approach infinity. This would be possible only if $|\alpha| + |\beta| \leq 1$.

The Lax-Friedrichs scheme is similar to this form, and the stability derivation is performed in nearly the same way:

$$\frac{u_m^{n+1} - \frac{1}{2}\left(u_{m+1}^n + u_{m-1}^n\right)}{k} + a\frac{u_{m+1}^n - u_{m-1}^n}{2h}$$

$$\longrightarrow \quad u_m^{n+1} = (1/2 - a/2\lambda)u_{m+1}^n + (1/2 + a/2\lambda)u_{m-1}^n$$

Following the previous form, we have $\alpha = 1/2 - a/2\lambda$ and $\beta = 1/2 + a/2\lambda$. Therefore,

$$|\alpha| + |\beta| = |1/2 - a/2\lambda| + |1/2 + a/2\lambda| = \frac{1}{2}(|1 - a\lambda| + |1 + a\lambda|) \leq 1 \quad \longrightarrow$$

$$2|1 - a\lambda| \leq |1 - a\lambda| + |1 + a\lambda| \leq 2 \quad \longrightarrow |1 - a\lambda| \leq 1$$

For this equivalence to be true, we need $|a\lambda| \leq 1$, which is our required term for stability.