Homework 1

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MATH-693B Numerical Partial DIfferential Equations

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1.1.1

Consider the initial value problem for the equation

$$u_t + au_x = f(t,x)$$

$$\text{with } u(0,x) = 0 \text{ and } \quad f(t,x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume that a is positive. Show that the solution is given by

$$u(t,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/a & \text{if } x \geq 0 \text{ and } x - at \leq 0 \\ t & \text{if } x \geq 0 \text{ and } x - at \geq 0 \end{cases}$$

Solution

When $f(t,x) = 0$, we have the unique solution $u(t,x) = u_0(x - at)$. This gives the answer $u_0(x-at) = u(0,x-at) = 0$, which is the indicated solution for all $x < 0$.

For $f(t,x) = 1$ and $x-at < 0$, we can show that the solution $u(t,x) = x/a$ is valid. Observe:

$$u(t,x) = \frac{x}{a} \quad \longrightarrow \quad u_t = 0 \quad \text{and} \quad u_x = \frac{1}{a}$$

Plugging this into our problem, we see that the result is $0 + a(1/a) = 1$, which is true. So this solution is correct.

For $f(t,x) = 1$ and $x - at \geq 0$, let's change variables so that

$$\tau = t \qquad \text{and} \qquad \xi = x - at \quad \longrightarrow \quad x = \xi + a\tau.$$

Now we have $\tilde{u}(\tau, \xi) = u(t,x)$, and it follows that

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial \tau} &= \frac{\partial t}{\partial \tau} u_t + \frac{\partial x}{\partial \tau} u_x \\ &= u_t + a u_x = f(\tau, \xi + a\tau). \end{aligned}$$

With $\frac{\partial \tilde{u}}{\partial \tau} = f(\tau, \xi + a\tau)$, we can solve this as an ordinary differential equation, which has the following solution:

$$\begin{aligned} \tilde{u}(\tau, \xi) &= u_0(\xi) + \int_0^\tau f(\sigma, \xi + a\sigma) d\sigma \quad \longrightarrow \\ u(t,x) &= u_0(x - at) + \int_0^t f(s, x - a(t-s)) ds \\ &= 0 + \int_0^t ds = \left. s \right|_0^t = t \end{aligned}$$

And we see that this is indeed the solution we were seeking.

1.3.1

For values of x in the interval $[-1, 3]$ and t in $[0, 2.4]$, solve the one-way wave equation $u_t + u_x = 0$ with the initial data

$$u(0,x) = \begin{cases} \cos^2 \pi x & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and the boundary data $u(t,-1) = 0$. Use the following four schemes for $h = 1/10,\ 1/20$, and $1/40$:

1. (a) Forward-time backward-space scheme (1.3.2) with $\lambda = 0.8$
   (b) Forward-time central-space scheme (1.3.3) with $\lambda = 0.8$
   (c) Lax-Friedrichs scheme (1.3.5) with $\lambda = 0.8$ and 1.6
   (d) Leapfrog scheme (1.3.4) with $\lambda = 0.8$.

Solution

(a) Forward Time, Backward Space

Figure (1) shows this scheme. Notice that the approximate solution dampens to zero (see video, emailed). For $h = 1/10$, it hits zero by about 0.72 seconds. For $h = 1/20$, it hits zero by about 0.36 seconds. For $h = 1/40$, it hits zero by about 0.18 seconds. As the density of the mesh doubles, it converges twice as quickly. Error?

(b) Forward Time, Central Space

Figure (2) shows this scheme. Notice that the approximate solution explodes to infinity (see video, emailed). For $h = 1/10$, the solution exceeds a value of 5 after about 1.76 seconds. For $h = 1/20$, the solution exceeds 5 after about 1.2 seconds. For $h = 1/40$, the solution passes 5 after about .72 seconds. As the density of the mesh doubles, the solution becomes useless more quickly.

(c) Lax-Friedrichs

Figure (3) shows this scheme. Notice that the approximate solution explodes to infinity (see video, emailed). For $h = 1/10$, the solution exceeds a value of 5 after about 1.6 seconds. For $h = 1/20$, the solution exceeds 5 after about 1.12 seconds. For $h = 1/40$, the solution passes 5 after about .72 seconds. As the density of the mesh doubles, the solution becomes useless more quickly, in a similar scale to the forward time/central space scheme.

(d) Leapfrog

Figure (4) shows this scheme. Notice that the approximate solution dampens to zero (see video, emailed). For $h = 1/10$, it hits zero by about 0.72 seconds, like the forward time, backward space scheme. For $h = 1/20$, it hits zero by about 0.36 seconds. For $h = 1/40$, it hits zero by about 0.18 seconds. Just like (a), as the mesh density doubles, the time to convergence halves. Error?

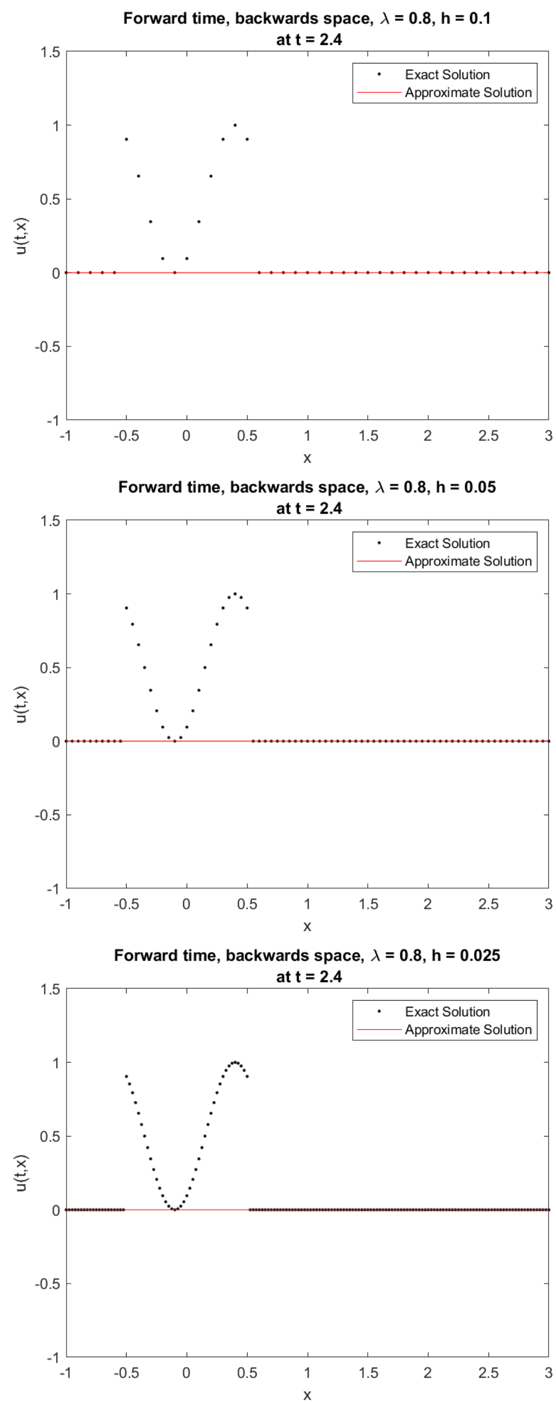

Figure 1: Forward time, backward space scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

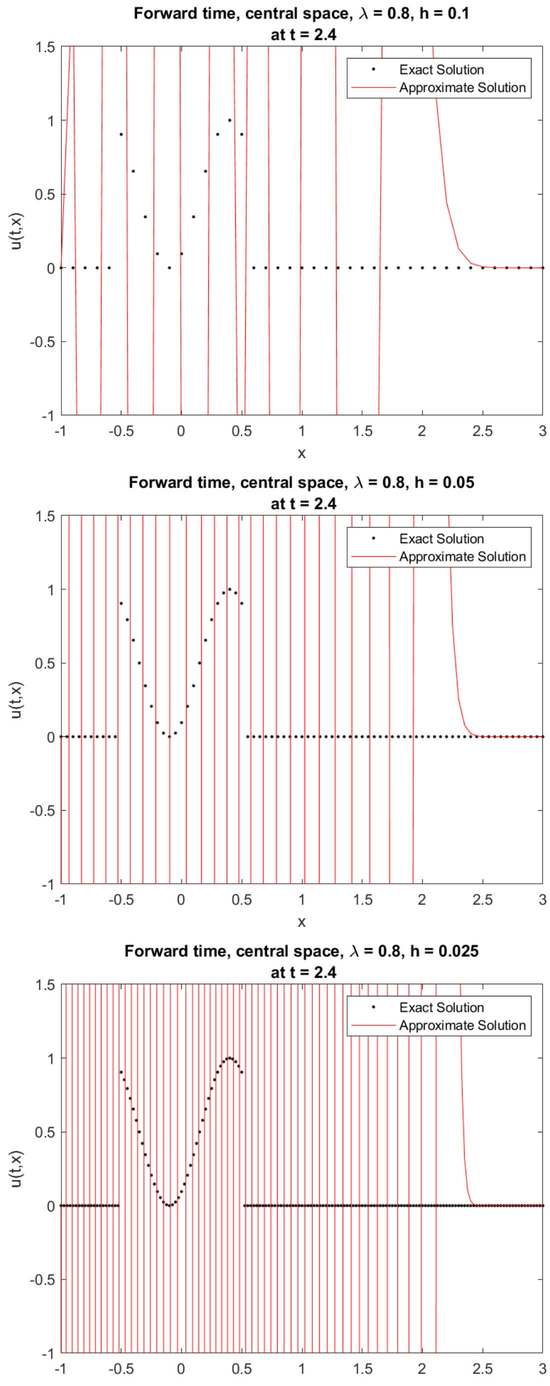

Figure 2: Forward time, central space scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

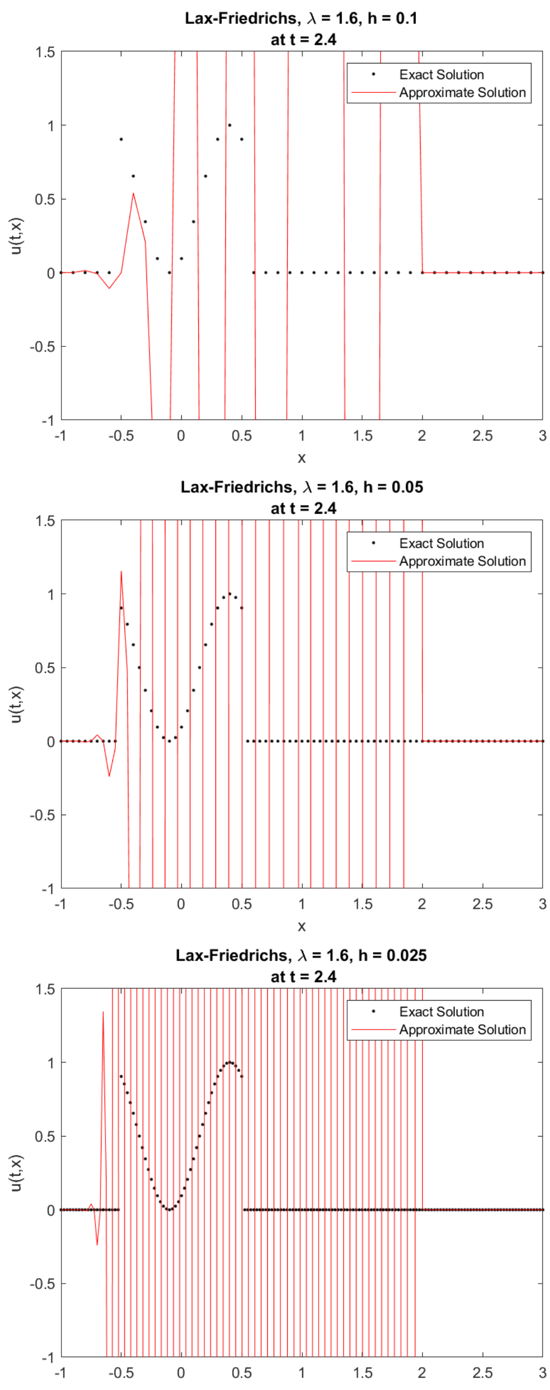

Figure 3: Lax-Friedrichs at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

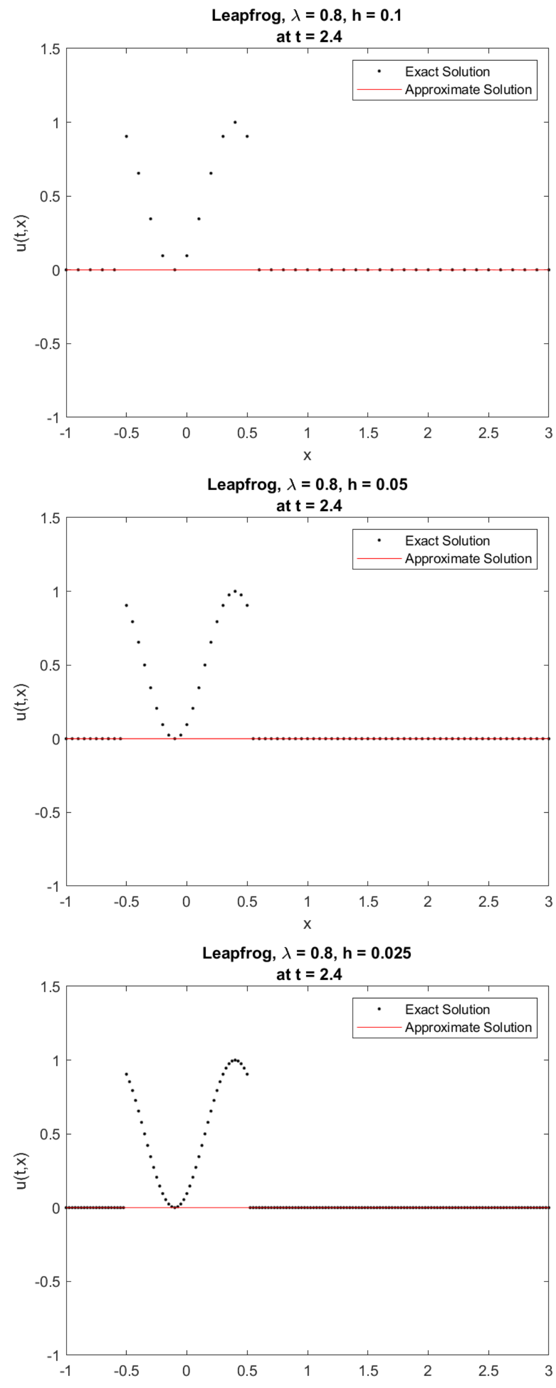

Figure 4: Forward time, backward space scheme at $t = 2.4$ for $h - 1/10, h = 1/20$, and $h = 1/40$, respectively

1.4.2

Show that the leapfrog scheme is consistent with the one-way wave equation.

Solution

1.5.1

Show that schemes of the form

$$u_m^{n+1} = \alpha u_{m+1}^n + \beta u_{m-1}^n$$

are stable if $|\alpha| + |\beta|$ is less than or equal to 1. Conclude that the Lax-Friedrichs scheme is stable if $|a\lambda|$ is less than or equal to 1 .

Solution

MATLAB Command Comments