Numerical Solutions to PDEs

Lecture Notes #3

— Hyperbolic Partial Differential Equations —

Convergence, Consistency, and Stability; the CFL Condition

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Outline

Previously...

Classification of PDEs: — Hyperbolic, Parabolic, and Elliptic.

Exact solutions of the hyperbolic One-Way Wave Equation

$$u_t + a(t,x)u_x + b(t,x)u = f(t,x)$$

constant/variable coefficient, lower-order term, forcing.

Systems of hyperbolic PDEs: propagation along characteristics, initial and boundary values.

Introduction to Finite Difference Schemes: gridding; forward, backward, and central differences; notation $v_m^{n+1} = (1 + a\lambda)v_m^n - a\lambda v_{m+1}^n$, $\lambda = k/h = \Delta t/\Delta x$; Numerical example — the leapfrog scheme (dependence on λ).

Convergence and Consistency — Introduction

The most basic property a finite difference scheme must have in order to be useful is that:

Property (Convergence)

The approximate solution to the corresponding PDE must improve as the grid spacing $(k, h) \to 0$.

For now we consider linear PDEs in the form

$$P(\partial_t, \partial_x)u = f(t, x),$$

which are first order in the t-derivative. Also, we assume that specifying the initial condition $u(0, x) = u_0(x)$, $\forall x \in \mathbb{R}$ determines a unique solution. (Infinite domain, no boundaries).

Convergence — Definition

Definition (Convergent Scheme)

A one-step finite difference scheme approximating a PDE is a convergent scheme if for any solution to the PDE, $u(t,x)$, and solutions to the finite difference scheme, v_m^n, such that v_m^0 converges to the initial condition $u_0(x)$ as $m \cdot h$ converges to x, then v_m^n converges to $u(t,x)$ as $(n \cdot k, m \cdot h)$ converges to (t,x) as k, h converge to 0.

This definition is formally not quite complete until we clarify the convergence of v_m^n (on the discrete grid) to $u(t,x)$, which is continuously varying.

If this was an "analysis for PDEs" course, we would go to the trouble of filling all theoretical gaps; but, alas, we take a more practical view.

Two Schemes: Leapfrog and Lax-Friedrichs

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \qquad \text{Leapfrog}$$

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \qquad \text{Lax-Friedrichs}$$

Note that the Lax-Friedrichs scheme is a one-step scheme, whereas the leapfrog scheme is a 2-step scheme.

For n-step schemes, the definition must be changed to allow for initialization of the first n time-levels. — Before we can apply an n-step scheme we must define $v_m^0, \ldots, v_m^{n-1}$, $\forall m$.

Grid + Initialization Suitable for 1-Step Schemes

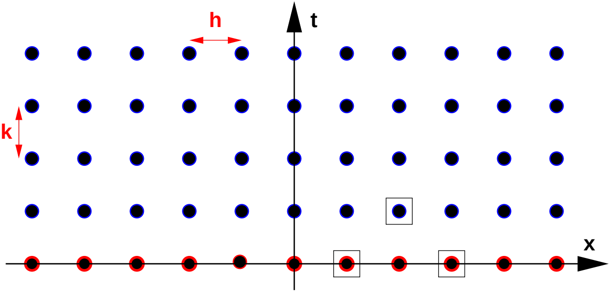

Figure: Once the first level v_m^0 is defined using the initial conditions $v_m^0 = u_0(x_m)$ on the red grid-points, a one-step scheme, such as Lax-Friedrichs

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

can be applied to compute the values v_m^n at the remaining grid-points, one level at a time.

Grid + Initialization Suitable for 2-Step Schemes

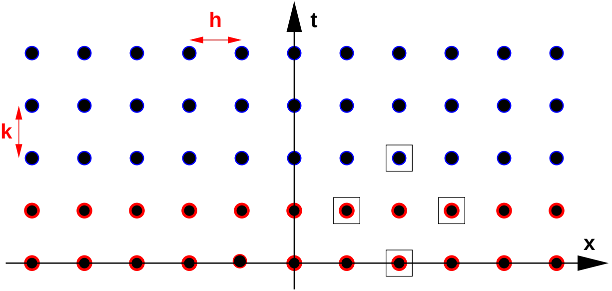

Figure: Once the first two levels v_m^0, and v_m^1 are defined using the initial conditions $v_m^0 = u_0(x_m)$ and some other clever initialization scheme for v_m^1, on the red grid-points, a two-step scheme, such as the Leapfrog scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

can be applied to compute the values v_m^n at the remaining grid-points, one level at a time.

The Lax-Friedrichs Solution of the One-Way Wave Equation

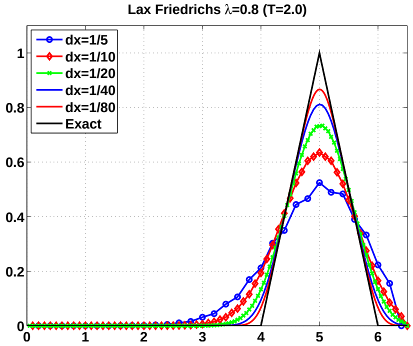

Figure: Five numerical solutions of the one-way wave equation

$$u_t + u_x = 0, \quad u_0 = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, |x| \geq 1, \end{cases}$$

as $k, h \to 0$, the solution seems to approach the exact solution.

The Road To Convergence

Directly proving that a given scheme is convergent is not easy in general. However, we can get there by checking two other properties: **consistency**, and **stability**.

Definition (Consistent Scheme)

Given a PDE, $P(\partial_t, \partial_x)u = f$, and a finite difference scheme $P_{k,h}v = f$, we say that the finite difference scheme is **consistent** with the PDE if for any smooth function $\Phi(t, x)$

$$P(\partial_t, \partial_x)\Phi - P_{k,h}\Phi \to 0, \quad \text{as} \quad k, h \to 0,$$

the convergence being *pointwise convergence* at each point (t, x).

Consistency: Notes

- For some schemes, we may have to restrict **how** $k, h \to 0$ in order to be consistent.
- A "smooth function" is a function which is differentiable (at least) as many times as required for the expression to make sense.
- The difference operator $P_{k,h}$ when applied to a function of (t, x) does not need to be restricted to grid-points. A forward-space difference operator applied on Φ at (t, x) gives

$$\frac{\Phi(t, x+h) - \Phi(t, x)}{h}.$$

Checking Consistency: Lax-Friedrichs

The Lax-Friedrichs difference operator is given by

$$P_{k,h}^{\mathrm{LF}}\Phi = \frac{\Phi_m^{n+1} - \frac{1}{2}(\Phi_{m+1}^n + \Phi_{m-1}^n)}{k} + a\frac{\Phi_{m+1}^n - \Phi_{m-1}^n}{2h}.$$

We use **Taylor expansion** around the point (t_n, x_m)

$$\begin{cases} \Phi_{m\pm1}^n &= \Phi_m^n \pm h\Phi_x + \frac{1}{2}h^2\Phi_{xx} \pm \frac{1}{6}h^3\Phi_{xxx} + \mathcal{O}\left(h^4\right) \\ \Phi_m^{n+1} &= \Phi_m^n + k\Phi_t + \frac{1}{2}k^2\Phi_{tt} + \frac{1}{6}k^3\Phi_{ttt} + \mathcal{O}\left(k^4\right) \end{cases}$$

Noting that

$$\begin{cases} \frac{1}{2}(\Phi_{m+1}^n + \Phi_{m-1}^n) &= \Phi_m^n + \frac{1}{2}h^2\Phi_{xx} + \mathcal{O}\left(h^4\right) \\ \dfrac{\Phi_{m+1}^n - \Phi_{m-1}^n}{2h} &= \Phi_x + \frac{1}{6}h^2\Phi_{xxx} + \mathcal{O}\left(h^4\right) \end{cases}$$

Checking Consistency: Lax-Friedrichs

We can now write

$$P_{k,h}^{\mathrm{LF}}\Phi = \Big[\Phi_t + a\Phi_x\Big] + \frac{1}{2}k\Phi_{tt} - \frac{1}{2}\frac{h^2}{k}\Phi_{xx} + \frac{1}{6}ah^2\Phi_{xxx} + \mathcal{O}\left(h^4 + \frac{h^4}{k} + k^2\right)$$

Hence,

$$P\Phi - P_{k,h}^{\mathrm{LF}}\Phi = -\frac{1}{2}k\Phi_{tt} + \frac{1}{2}\frac{h^2}{k}\Phi_{xx} - \frac{1}{6}ah^2\Phi_{xxx} + \mathcal{O}\left(h^4 + \frac{h^4}{k} + k^2\right)$$

As long as $k, h \to 0$ in such a way that also $\frac{h^2}{k} \to 0$, we have $P\Phi - P_{k,h}^{\mathrm{LF}}\Phi \to 0$, *i.e.* the Lax-Friedrichs scheme is consistent. $\square$

Consistency $\not\Rightarrow$ Convergence

- Consistency implies that the solution of the PDE, *if it is smooth*, is an approximate solution of the finite difference scheme (FDS).
- Convergence means that a solution of the FDS approximates a solution of the PDE.
- It turns out that consistency is **necessary**, but **not sufficient** for a FDS to be convergent.

We illustrate this with an example.

Example: Consistency $\not\Rightarrow$ Convergence

We consider the one-way wave equation with constant $a = 1$ propagation speed, and apply the forward-space-forward-time scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{v_{m+1}^n - v_m^n}{h} = 0.$$

A quick Taylor expansion shows that this indeed is consistent with the PDE, with an error term

$$P\Phi - P_{k,h}\Phi \sim k\Phi_{tt} + h\Phi_{xx} + \mathcal{O}\left(k^2 + h^2\right).$$

We rewrite the scheme using $\lambda = k/h$:

$$v_m^{n+1} = v_m^n - \frac{k}{h}\left(v_{m+1}^n - v_m^n\right) = (1+\lambda)v_m^n - \lambda v_{m+1}^n.$$

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Example: Consistency $\not\Rightarrow$ Convergence

Let the initial condition be given by

$$u_0(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0, \\ 0 & \text{elsewhere} \end{cases}$$

Hence the exact solution is $u(t,x) = u_0(x-t)$, *i.e.* a "hump" of height and width one, traveling to the right with speed one.

The initial data for the FDS are given by

$$v_m^0 = \begin{cases} 1 & \text{if } -1 \leq m \cdot h \leq 0, \\ 0 & \text{elsewhere} \end{cases}$$

Example: Consistency $\not\Rightarrow$ Convergence

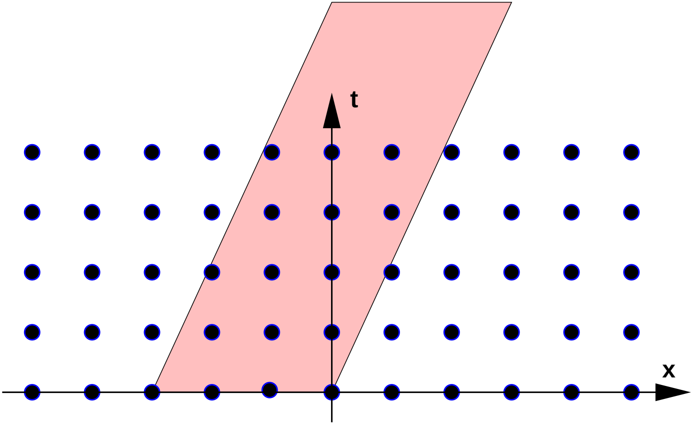


Figure: Illustration of how the exact solution propagates; it is one in the band, and zero outside the band. The initial condition for the FDS are zeros everywhere, except in the four points v_0^0, v_{-1}^0, v_{-2}^0, and v_{-3}^0, where it is one.

Example: Consistency $\not\Rightarrow$ Convergence

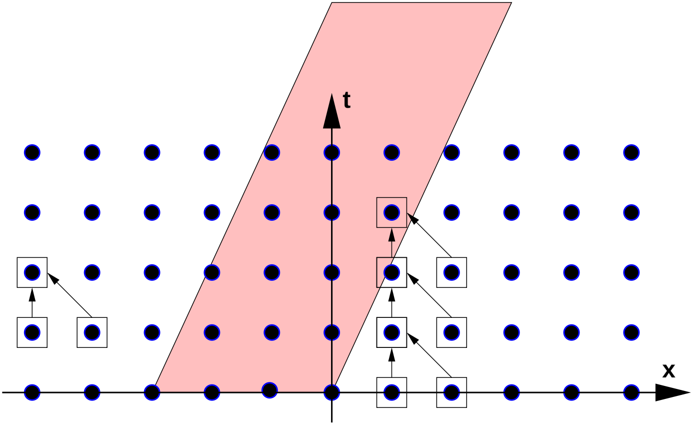
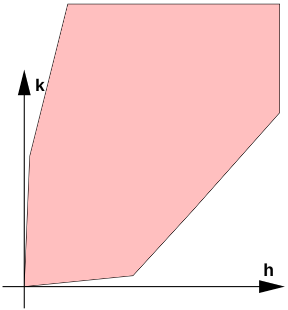


Figure: Illustration of how the FDS solution propagates. In particular we have that $v_m^n \equiv 0, \; \forall m > 0, \; n \geq 0$. Hence, $v_m^n \not\to u(t_n, x_m)$ for (t_n, x_m) in the part of the band strictly in the right half plane — u is one there, but v_m^n is zero, no matter how much we refine the grid.

Stability — The "Missing" Property

If a scheme is convergent, then as v_m^n converges to $u(t,x)$, then v_m^n is bounded in some sense; this is the essence of stability.

For almost all schemes there are restrictions on the way h and k can be chosen so that the particular scheme is stable. A **stability region** is any bounded non-empty region of the first quadrant of $\mathbb{R}^2$ that has the origin as an accumulation point:



Stability — Definition

Definition (Stable Scheme)

A finite difference scheme $P_{k,h}v_m^n = 0$ for a first-order equation is **stable** in a stability region Λ if there is an integer J such that for any positive time T, there is a constant C_T such that

$$h \sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq C_T h \sum_{j=0}^{J} \sum_{m=-\infty}^{\infty} \left|v_m^j\right|^2$$

for $0 \leq nk \leq T$, with $(k, h) \in \Lambda$.

Stability — Notation

The quantity

$$\|w\|_h = \left[h \sum_{m=-\infty}^{\infty} |w_m|^2 \right]^{1/2}$$

is the L^2 norm of the grid function w, and is a measure of the size (energy) of the solution. — The multiplication by h is needed so that the norm is not sensitive to grid refinements (the number of points increase as $h \to 0$).

With this notation, the inequality in the definition can be written

$$\|v^n\|_h \leq \left[C_T \sum_{j=0}^{J} \|v^j\|_h^2 \right]^{1/2} \quad \Leftrightarrow \quad \|v^n\|_h \leq C_T^* \sum_{j=0}^{J} \|v^j\|_h$$

The inequality expresses a limit (in terms of energy) of how much the solution can grow. Typically $J = (n-1)$ for an n-step scheme.

Checking for Stability... Using the Definition

Checking whether $\|v^n\|_h \leq C_T^* \sum_{j=0}^{J} \|v^j\|_h$ holds for a particular scheme directly from the definition can be a formidable task.

Example-1.5.1 in Strikwerda performs this test for the forward-time-forward-space scheme; the analysis takes up a good page of algebraic manipulations... We will return to this issue very soon with better tools in hand.

We note that there is a strong relation between the Stability of Finite Difference Schemes, and the Well-Posedness of PDEs (IVPs).

Well-Posedness for the IVP — Definition

Definition (Well-Posed IVP)

The initial value problem for the first-order partial differential equation $Pu = 0$ is well-posed if for any time $T \geq 0$, there exists a constant C_T such that any solution $u(t, x)$ satisfies

$$\int_{-\infty}^{\infty} |u(t, x)|^2 \, dx \leq C_T \int_{-\infty}^{\infty} |u(0, x)|^2 \, dx$$

for $0 \leq t \leq T$.

All these concepts, consistency, well-posedness, stability, and convergence come together in the **Lax-Richtmyer equivalence theorem.**

The Lax-Richtmyer Equivalence Theorem

Theorem (The Lax-Richtmyer Equivalence Theorem)

A consistent finite difference scheme for a partial differential equation for which the initial value problem is well-posed is convergent if and only if it is stable.

This theorem is extremely useful:

- Checking consistency is straight-forward (Taylor expansions).
- We are going to introduce tools (based on Fourier transforms) which make checking stability quite easy.
- Thus, if the problem is well-posed, we get the more difficult (and desirable) result — **Convergence** — by checking two (relatively) easy things — consistency and stability.
- The relationship is one-to-one, hence only consistent and stable schemes need to be considered.

Condition for Stability

We now turn our attention to the key stability criterion for hyperbolic PDEs.

In last lecture we saw some numerical evidence of the leapfrog scheme (applied to $u_t + au_x = 0$, $a = 1$) breaking down when $\lambda > 1$.

The condition $|a\lambda| < 1$ is necessary for stability of many explicit FDS.

An explicit scheme is a scheme that can be written as

$$v_m^{n+1} = \sum_{n' \leq n}^{\text{finite}} v_{m'}^{n'}$$

Implicit schemes, where the sum may contain terms with $n' = n + 1$, will be discussed soon.

The Courant-Friedrichs-Lewy Condition

The following result covers all one-step schemes we have seen so far:

Theorem (The CFL Condition)

For an explicit scheme for the hyperbolic equation

$$u_t + a u_x = 0$$

of the form

$$v_m^{n+1} = \alpha v_{m+1}^n + \beta v_m^n + \gamma v_{m-1}^n$$

with $\lambda = k/h$ held constant, a necessary condition for stability is the **Courant-Friedrichs-Lewy (CFL) condition**,

$$|a\lambda| \leq 1.$$

For systems of equations for which $\bar{\mathbf{v}}$ is a vector and α, β, and γ are matrices, we must have $|a_i \lambda| \leq 1$ for all eigenvalues a_i of the matrix A.

The CFL Condition: "Proof By Picture"

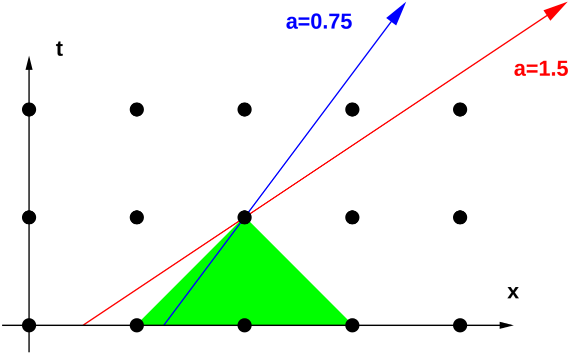


Figure: Illustration of the CFL condition, with $\lambda = 1$ held fixed. The **green** triangle shows the region of dependence, *i.e.* what region influences v_m^{n+1} (actually only the three points at the base of the triangle contribute). The **blue** arrow corresponds to a characteristic with speed $a = 0.75$, which carries information **inside** the region of dependence; the **red** arrow, corresponding to a characteristic speed of $a = 1.5$ carries information from **outside** the region of dependence — this information cannot be captured by the scheme.

Another Theorem

Courant, Friedrichs and Lewy also proved the following theorem:

Theorem

There are no explicit, unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations.

One way of thinking about these theorems is to define the **numerical speed of propagation** as $\lambda^{-1} = h/k$, and note that a necessary condition for the stability of a scheme is

$$\lambda^{-1} \geq |a|.$$

This guarantees that the FDS can propagate information (energy) at least as fast as the PDE.

λ^{-1} is the "speed limit" on the grid; which explains why (with $a = 1$), we saw the breakdown of the leapfrog scheme when $\lambda > 1 \Leftrightarrow \lambda^{-1} < 1$.

Homework #1 — Due 2/13/2020, in class

- **Strikwerda-1.1.1** — Theoretical
- **Strikwerda-1.3.1** — Numerical
- **Strikwerda-1.4.2** — Theoretical; but use software for Taylor expansions*.
- **Strikwerda-1.5.1** — Theoretical; but use software for Taylor expansions.

* In matlab, try:

```
>> syms f(t,x) k h
>> taylor( f(t+k,x+h), [h,k], 'ExpansionPoint', [0,0], 'Order', 5)
```

... and ponder what the output means. In the long run you will save a lot of time if you can parse that output.