Homework 1

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MATH-693B Numerical Partial DIfferential Equations

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1.1.1

Consider the initial value problem for the equation

$$u_t + au_x = f(t,x)$$

$$\text{with } u(0,x) = 0 \text{ and } \quad f(t,x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume that a is positive. Show that the solution is given by

$$u(t,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/a & \text{if } x \geq 0 \text{ and } x - at \leq 0 \\ t & \text{if } x \geq 0 \text{ and } x - at \geq 0 \end{cases}$$

Solution

When $f(t,x) = 0$, we have the unique solution $u(t,x) = u_0(x - at)$. This gives the answer $u_0(x-at) = u(0, x-at) = 0$, which is the indicated solution for all $x < 0$.

For $f(t,x) = 1$ and $x-at < 0$, we can show that the solution $u(t,x) = x/a$ is valid. Observe:

$$u(t,x) = \frac{x}{a} \quad \longrightarrow \quad u_t = 0 \quad \text{and} \quad u_x = \frac{1}{a}$$

Plugging this into our problem, we see that the result is $0 + a(1/a) = 1$, which is true. So this solution is correct.

For $f(t,x) = 1$ and $x - at \geq 0$, let's change variables so that

$$\tau = t \qquad \text{and} \qquad \xi = x - at \quad \longrightarrow \quad x = \xi + a\tau.$$

Now we have $\tilde{u}(\tau, \xi) = u(t, x)$, and it follows that

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial \tau} &= \frac{\partial t}{\partial \tau} u_t + \frac{\partial x}{\partial \tau} u_x \\
&= u_t + a u_x = f(\tau, \xi + a\tau).
\end{aligned}$$

With $\frac{\partial \tilde{u}}{\partial \tau} = f(\tau, \xi + a\tau)$, we can solve this as an ordinary differential equation, which has the following solution:

$$\begin{aligned}
\tilde{u}(\tau, \xi) &= u_0(\xi) + \int_0^\tau f(\sigma, \xi + a\sigma) d\sigma \quad \longrightarrow \\
u(t, x) &= u_0(x - at) + \int_0^t f(s, x - a(t - s)) ds \\
&= 0 + \int_0^t ds = \left. s \right|_0^t = t
\end{aligned}$$

And we see that this is indeed the solution we were seeking.

1.3.1

For values of x in the interval $[-1, 3]$ and t in $[0, 2.4]$, solve the one-way wave equation $u_t + u_x = 0$ with the initial data

$$u(0, x) = \begin{cases} \cos^2 \pi x & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and the boundary data $u(t, -1) = 0$. Use the following four schemes for $h = 1/10,\ 1/20$, and $1/40$:

(a) Forward-time backward-space scheme (1.3.2) with $\lambda = 0.8$

(b) Forward-time central-space scheme (1.3.3) with $\lambda = 0.8$

(c) Lax-Friedrichs scheme (1.3.5) with $\lambda = 0.8$ and 1.6

(d) Leapfrog scheme (1.3.4) with $\lambda = 0.8$.

For schemes (b). (c), and (d), at the right boundary use the condition $v_M^{n+1} = v_{M-1}^{n+1}$, where $x_M = 3$. For scheme (d) use scheme (b) to compute the solution at $n = 1$ For each scheme determine whether the scheme is a useful or useless scheme. For the purposes of this exercise only, a scheme will be useless if $|v_m^n|$ is greater than 5 for any value of m and n. It will be regarded as a useful scheme if the solution looks like a reasonable approximation to the solution of the differential equations. Graph or plot several solutions at the last time they were computed. What do you notice about the "blow-up time" for the useless schemes as the mesh size decreases? Is there a pattern to these solutions? For the useful cases, how does the error decrease as the mesh decreases; i.e., as h decreases by one-half, by how much does the error decrease?

Solution

1.4.2

Solution

1.5.1

Solution

MATLAB Command Comments