Homework 2

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MATH-693B Numerical Partial Differential Equations

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2.1.4

Use an argument similar to that used in (2.1.11) to show that the initial value problem for the equation $u_t = u_{xxx}$ is well-posed.

Solution

Definition 1.5.2 states that *The initial value problem for the first-order partial differential equation $Pu = 0$ is well-posed if for any time $T \geq 0$, there is a constant C_T such that any solution $u(t,x)$ satisfies*

$$\int_{-\infty}^{\infty} |u(t,x)|^2 \; dx \leq C_T \int_{-\infty}^{\infty} |u(0,x)|^2 \; dx$$

We can extend this to the third order partial differential equation given above. We first transform only the spatial variable, yielding:

$$\hat{u}_t = (i\omega)^3 \hat{u} = -i\omega^3 \hat{u}$$

Using the initial data, the solution must be:

$$\hat{u}(x,\omega) = e^{-i\omega^3 t} \hat{u}_0(\omega)$$

Like in Strikwerda's example, we use $|e^{-i\omega^3 t}| = 1$ and Parseval's relation to show:

$$\begin{aligned}\int_{-\infty}^{\infty} |u(t,x)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{u}(t,\omega)|^2 \, d\omega \\ &= \int_{-\infty}^{\infty} |e^{-i\omega^3 t}\hat{u}(0,\omega)|^2 \, d\omega \\ &= \int_{-\infty}^{\infty} |\hat{u}(0,\omega)|^2 \, d\omega \\ &= \int_{-\infty}^{\infty} |u(0,x)|^2 \, dx \qquad \longrightarrow \\ \int_{-\infty}^{\infty} |u(t,x)|^2 \, dx &\leq C_T \int_{-\infty}^{\infty} |u(0,x)|^2 \, dx \quad \forall\ C_T \geq 1\end{aligned}$$

Thus, the initial value problem is considered well-posed.

2.1.5

Use an argument similar to that used in (2.1.11) to show that the initial value problem for the equation $u_t + u_x + bu = 0$ is well-posed.

Solution

This solution follows from 2.1.4 in a similar fashion. First, we transform the spatial components:

$$\hat{u}_t = -u_x - bu = -i\omega\hat{u} - b\hat{u} = -(i\omega + b)\hat{u}$$

Using the initial data, the solution must be:

$$\hat{u}(x,\omega) = e^{-(i\omega+b)t}\hat{u}_0(\omega)$$

Now using $|e^{-(i\omega+b)t}| = 1$ and Parseval's relation gives us:

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(t,x)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{u}(t,\omega)|^2 \, d\omega \\
&= \int_{-\infty}^{\infty} |e^{-(i\omega+b)t}\hat{u}(0,\omega)|^2 \, d\omega \\
&= \int_{-\infty}^{\infty} |\hat{u}(0,\omega)|^2 \, d\omega \\
&= \int_{-\infty}^{\infty} |u(0,x)|^2 \, dx \qquad \longrightarrow \\
\int_{-\infty}^{\infty} |u(t,x)|^2 \, dx \leq C_T \int_{-\infty}^{\infty} |u(0,x)|^2 \, dx \quad \forall\ C_T \geq 1
\end{aligned}$$

Again, the initial value problem is considered well-posed.

2.2.1

Show that the backward-time central-space scheme (1.6.1) is consistent with equation (1.1.1) and is unconditionally stable.

Solution

The backward-time central-space scheme for $u_t + au_x = 0$ is given by:

$$\frac{u_m^{n+1} - u_m^n}{k} + a\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} = 0$$

Separating the n and $n+1$ terms, we get:

$$u_m^{n+1} + \frac{a\lambda}{2}u_{m+1}^{n+1} - \frac{a\lambda}{2}u_{m-1}^{n+1} = u_m^n$$

To show consistency and stability, we first use the Fourier transform on each of the terms $u_m^{n+1}, u_m^n, u_{m+1}^{n+1}, u_{m-1}^{n+1}$.

$$\hat{u}_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{u}^{n+1}(\xi)\ d\xi$$
$$\hat{u}_{m+1}^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} e^{ih\xi} \hat{u}^{n+1}(\xi)\ d\xi$$
$$\hat{u}_{m-1}^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} e^{-ih\xi} \hat{u}^{n+1}(\xi)\ d\xi$$
$$\hat{u}_m^{n} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{u}^{n}(\xi)\ d\xi$$

Plugging these values into the backward-time central-space scheme, we have:

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{u}^{n+1}(\xi)\ d\xi \ + \ \frac{a\lambda}{2\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} e^{ih\xi} \hat{u}^{n+1}(\xi)\ d\xi \ -$$
$$\frac{a\lambda}{2\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} e^{-ih\xi} \hat{u}^{n+1}(\xi)\ d\xi \ = \ \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{u}^{n}(\xi)\ d\xi$$

Combining like terms can simplify the equation:

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{u}^{n+1}(\xi) \left[1 + \frac{a\lambda}{2} e^{1h\xi} - \frac{a\lambda}{2} e^{-ih\xi}\right]\ d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{u}^{n}(\xi)\, d\xi$$

Now we can suggest that since the integrals are equal, their contents are equal as well. Simplifying further, we get:

$$\hat{u}^{n+1}(\xi) = \frac{1}{1 + \frac{a\lambda}{2} e^{ih\xi} - \frac{a\lambda}{2} e^{-ih\xi}} \hat{u}^{n}(\xi) = g(h\xi)\hat{u}^{n+1}(\xi)$$

In order to determine stability, we must set $|g(h\xi)| \leq 0$ and evaluate under what conditions this is true. It is first helpful to replace $h\xi$ with θ, and expand $g(\theta)$ into its sine and cosine parts. This leaves us with

$$\left|\frac{1}{1 + ia\lambda \sin\theta}\right| \leq 1 \quad \longrightarrow \quad 1 \leq |1 + ia\lambda \sin|^2 \quad \longrightarrow \quad 0 \leq a^2\lambda^2 \sin^2 \quad \longrightarrow |a\lambda \sin\theta| \geq 0$$

Since the inequality holds for all values of λ, the scheme is unconditionally stable.

2.2.4

Show that the box scheme

$$\frac{1}{2k}\left[\left(u_m^{n+1}+u_{m+1}^{n+1}\right)-\left(u_m^n+u_{m+1}^n\right)\right]+\frac{a}{2h}\left[\left(u_{m+1}^{n+1}-u_m^{n+1}\right)+\left(u_{m+1}^n-u_m^n\right)\right]=f_m^n$$

is consistent with the one-way wave equation $u_t + au_x = f$ and is stable for all values of λ.

Solution

Like the previous solution, we use the Fourier transform substitutes for values of u and plug them into the give scheme. After combining like terms and separating u^n from u^{n+1} values, we get:

$$u_m^{n+1}\left[\frac{1}{2}-\frac{a\lambda}{2}\right]+u_{m+1}^{n+1}\left[\frac{1}{2}+\frac{a\lambda}{2}\right]=f_{mn}+u_m^n\left[\frac{1}{2}+\frac{a\lambda}{2}\right]+u_{m+1}^n\left[\frac{1}{2}-\frac{a\lambda}{2}\right]$$

Now using Fourier transforms:

$$\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}e^{imh\xi}\hat{u}^{n+1}(\xi)\left[\frac{1}{2}-\frac{a\lambda}{2}+e^{-ih\xi}\left(\frac{1}{2}+\frac{a\lambda}{2}\right)\right]\,d\xi=$$

$$f_{mn}+\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}e^{imh\xi}\hat{u}^{n}(\xi)\left[\frac{1}{2}+\frac{a\lambda}{2}+e^{-ih\xi}\left(\frac{1}{2}-\frac{a\lambda}{2}\right)\right]\,d\xi$$

Neglecting the f_{mn} term and assuming the integrands are equal, we get:

$$\hat{u}^{n+1}(\xi)=\frac{1+a\lambda+e^{-ih\xi}\left(1-a\lambda\right)}{1-a\lambda+e^{ih\xi}\left(1+a\lambda\right)}\hat{u}^n(\xi)=g(h\xi)\hat{u}^n$$

For stability, we need $|g(h\xi)|\leq 1$. Replacing $h\xi$ with θ, we can form the inequality:

$$\left|1+a\lambda+e^{-\theta}\left(1-a\lambda\right)\right|^2\leq\left|1-a\lambda+e^{i\theta}\left(1+a\lambda\right)\right|^2$$

Squaring the real and non-real values, we get:

$$[(1+a\lambda)+(1-a\lambda)\cos\theta]^2+[-(1-a\lambda)\sin\theta]^2\leq[(1-a\lambda)+(1+a\lambda)\cos\theta]^2+[(1+a\lambda)\sin\theta]^2$$

After expanding, this simplifies quite nicely to

$$(1 - a\lambda)^2 \leq (1 + a\lambda)^2$$

Which is always true for all $a\lambda \geq 0$. Since $a\lambda$ is assumed to be non-negative, this scheme is stable for all λ.

3.2.1

Show that the (forward-backward) MacCormack scheme

$$\tilde{u}_m^{n+1} = u_m^n - a\lambda \left(u_{m+1}^n - u_m^n\right) + kf_m^n$$

$$u_m^{n+1} = \frac{1}{2}\left(u_m^n + \tilde{u}_m^{n+1} - a\lambda \left(\tilde{u}_m^{n+1} - \tilde{u}_{m-1}^{n+1}\right) + kf_m^{n+1}\right)$$

is a second-order accurate scheme for the one-way wave cquation (1.1.1). Show that for $f = 0$ it is identical to the Lax-Wendroff scheme (3.1.1)

Solution

To show that a scheme is accurate, we use definition 3.1.1., which states that *a scheme $P_{k,h}u = R_{k,h}f$ that is consistent with the differential equation $Pu = f$ is accurate of order p in time and order q in space if for any smooth function $\phi(t,x)$*

$$P_{k.h}\phi - R_{k,h}P\phi = \mathcal{O}\left(k^\rho\right) + \mathcal{O}\left(h^q\right)$$

We say that such a scheme is accurate of order (p,q).

For this case, we start with defining P and R by expanding and combining like terms, involving a fair amount of algebra when plugging the first step into the second.

$$\begin{aligned}
P_{kh}\phi &= \phi_m^{n+1} - \phi_m^n + \phi_{m+1}^n\left[\frac{1}{2}(a\lambda - a^2\lambda^2)\right] + \phi_{m-1}^n\left[\frac{1}{2}(a\lambda + a^2\lambda^2)\right] \\
&= f_m^{n+1}\left[\frac{1}{2}k\right] + f_m^n\left[\frac{1}{2}(k - ah)\right] + f_{m-1}^n\left[\frac{1}{2}ah\right] \\
&= R_{kh}f
\end{aligned}$$

Now we take a Taylor expansion for each term. The Taylor expansion for each term is give by:

$$\begin{aligned}
\phi_m^{n+1} &= \phi_m^n + k\phi_t + \mathcal{O}(k^2) \\
\phi_{m+1}^n &= \phi_m^n + h\phi_x + \mathcal{O}(h^2) \\
\phi_{m-1}^n &= \phi_m^n - h\phi_x + \mathcal{O}(h^2) \\
f_m^{n+1} &= f_m^n + kf_t + \mathcal{O}(k^2) \\
f_{m-1}^n &= f_m^n - hf_x + \ \mathcal{O}(h^2)
\end{aligned}$$

Now, we make a long substitution:

$$\begin{aligned}
P_{kh}\phi =& (\phi_m^n + k\phi_t) - \phi_m^n + (\phi_m^n + h\phi_x)\left[\frac{1}{2}(a\lambda - a^2\lambda^2)\right] + ... \\
& ... + (\phi_m^n - h\phi_x)\left[\frac{1}{2}(a\lambda + a^2\lambda^2)\right] + \mathcal{O}(k^2) + \mathcal{O}(h^2) \\
=& (f_m^n + kf_t)\left[\frac{1}{2}k\right] + f_m^n\left[\frac{1}{2}(k - ah)\right] + (f_m^n - hf_x)\left[\frac{1}{2}ah\right] + \mathcal{O}(k^2) + \mathcal{O}(h^2) \\
=& R_{kh}f
\end{aligned}$$

Now we have

$$P_{kh}\phi = k\phi_t + ak\phi_x + \mathcal{O}(k^2) + \mathcal{O}(h^2) \quad \longrightarrow \quad P_{kh}\phi = k\phi_t + ak\phi_x + \mathcal{O}(k^2) + \mathcal{O}(h^2)$$

and

$$R_{kh}f = kf + \frac{1}{2}k^2 f_t - \frac{1}{2}a\lambda khf_x + \mathcal{O}(k^2) + \mathcal{O}(h^2)$$

Now we can make the substitutions for the one-way wave equation

$$f = \phi_t + a\phi_x \qquad f_t = \phi_{tt} + a\phi_{xt} \qquad f_x = \phi_{tx} + a\phi_{xx}$$

This leads to

$$R_{kh}P\phi = k(\phi_t + a\phi_x) + \frac{1}{2}k^2(\phi_{tt} + a\phi_{xt}) - \frac{1}{2}ah^2(\phi_{tx} + a\phi_{xx}) + \mathcal{O}(k^2) + \mathcal{O}(h^2)$$

$$\longrightarrow R_{kh}P\phi = k\phi_t + ka\phi_x + \mathcal{O}(k^2) + \mathcal{O}(h^2)$$

This matches our definition. Since $P_{kh}\phi = R_{kh}O\phi$, the scheme is second order accurate.

If $f = 0$, then this scheme would look like:

$$u_m^{n+1} = \frac{1}{2}\left\{u_m^n + \left[u_m^n - a\lambda(u_{m+1}^n - u_m^n)\right](1 - a\lambda) - a\lambda\left[-(u_{m-1}^n - a\lambda(u_m^n - u_{m-1}^n))\right]\right\}$$

$$\longrightarrow u_m^{n+1} = u_m^n\left[1 - a^2\lambda^2\right] + u_{m+1}^n\left[\frac{1}{2}(a^2\lambda^2 - a\lambda)\right] + u_{m-1}^n\left[\frac{1}{2}(a\lambda + a^2\lambda^2)\right]$$

And compared with the Lax-Wendroff scheme, which is

$$\begin{aligned} u_m^{n+1} =& u_m^n - \frac{a\lambda}{2}\left(u_{m+1}^n - u_{m-1}^n\right) + \frac{a^2\lambda^2}{2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) \\ \longrightarrow u_m^{n+1} &= u_m^n\left[1 - a^2\lambda^2\right] + u_{m+1}^n\left[\frac{a^2\lambda^2}{2} - \frac{a\lambda}{2}\right] + u_{m-1}^n\left[\frac{a\lambda}{2} + \frac{a^2\lambda^2}{2}\right] \end{aligned}$$

which are equal in this instance.

3.2.3

Show that the box scheme

$$\frac{1}{2k}\left[\left(v_m^{n+1} + v_{m+1}^{n+1}\right) - \left(v_m^n + v_{m+1}^n\right)\right] \quad + \frac{a}{2h}\left[\left(v_{m+1}^{n+1} - v_m^{n+1}\right) + \left(v_{m+1}^n - v_m^n\right)\right] =$$

$$\frac{1}{4}\left(f_{m+1}^{n+1} + f_m^{n+1} + f_{m+1}^n + f_m^n\right)$$

is an approximation to the one-way wave equation $u_t + au_x = f$ that is accurate of order $(2, 2)$ and is stable for all values of λ

Solution

This is going to be a long one. First, I find it easiest to define $P_{kh}\phi$ and $R_{kh}f$ first:

$$\begin{aligned} P_{kh}\phi &= \phi_m^{n+1}[1 - a\lambda] + \phi_{m+1}^{n+1}[1 + a\lambda] + \phi_m^n[-1 - a\lambda] + \phi_{m+1}^n[-1 + a\lambda] \\ &= f_m^{n+1}\left[\frac{k}{2}\right] + f_{m+1}^{n+1}\left[\frac{k}{2}\right] + f_m^n\left[\frac{k}{2}\right] f_{m+1}^n\left[\frac{k}{2}\right] = R_{kh}f \end{aligned}$$

Now, for the Taylor expansion:

$$
\begin{aligned}
\phi_m^{n+1} &= \phi_m^n + k\phi_t + \mathcal{O}(k^2) \\
\phi_{m+1}^{n+1} &= \phi_m^n + k\phi_t + h\phi_x + \mathcal{O}(k^2) + \mathcal{O}(h^2) \\
\phi_{m+1}^{n} &= \phi_m^n + h\phi_x + \mathcal{O}(h^2) \\
f_m^{n+1} &= f_m^n + kf_t + \mathcal{O}(k^2) \\
f_{m+1}^{n+1} &= f_m^n + kf_t + +hf_x + \mathcal{O}(k^2) + \mathcal{O}(h^2) \\
f_{m+1}^{n} &= f_m^n + hf_x + \mathcal{O}(h^2)
\end{aligned}
$$

$$
\begin{aligned}
&(\phi_m^n + k\phi_t)[1 - a\lambda] + (\phi_m^n + k\phi_t + h\phi_x)[1 + a\lambda] + \phi_m^n[-1 - a\lambda] + (\phi_m^n + h\phi_x)[-1 + a\lambda] \\
&= (f_m^n + kf_t)\left[\frac{k}{2}\right] + (f_m^n + kf_t + hf_x)\left[\frac{k}{2}\right] + f_m^n\left[\frac{k}{2}\right](f_m^n + hf_x)\left[\frac{k}{2}\right] + \mathcal{O}(k^2) + \mathcal{O}(h^2)
\end{aligned}
$$

which simplifies to

$$(2k)\phi_t + (2ka)\phi_x = (2k)f + (k^2)f_t + (hk)f_x + \mathcal{O}(k^2) + \mathcal{O}(h^2)$$

Plugging this into $P_{kh}\phi - R_{kh}P\phi$ for the one-way wave equation yields:

$$(k^2)\phi_{tt} + (hka)\phi_{xx} + (hk)\phi_{tx} + (k^2a)\phi_{xt} = \mathcal{O}(k^2) + \mathcal{O}(h^2)$$

which we can see is solely comprised of higher order terms, this showing that this scheme has second order accuracy.

3.4.1 (Numerical)

Solve the initial-boundary value problem (1.2.1) with the leapfrog scheme and the following boundary conditions. Use $a = 1$. Only (d) should give good results. Why?

(a) At $x = 0$, specify $u(t, 0)$; at $x = 1$, use boundary condition (3.4.1b)
(b) At $x = 0$, specify $u(t, 0)$; at $x = 1$, specify $u(t, 1) = 0$
(c) At $x = 0$, use boundary condition (3.4.1b); at $x = 1$, use (3.4.1c)
(d) At $x = 0$, specify $u(t, 0)$; at $x = 1$, use boundary condition (3.4.1c)

Solution

Problem 1.2.1 states:

Consider system (1.2.2) on the interval [0, 1], with a equal to 0 and b equal to 1 and with the boundary conditions u^1 equal to 0 at the left and u^1 equal to 1 at the right boundary. Show that if the initial data are given by $u^1(0, x) = x$ and $u^2(0, x) = 1$, then the solution is $u^1(t, x) = x$ and $u^2(t, x) = 1 - t$ for all (t, x) with $0 \leq x \leq 1$ and $0 \leq t$

System (1.2.2) is given by:

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x = 0$$

And the leapfrog scheme is given by:

$$\frac{u_m^{n+1} - u_m^{n-1}}{2k} + a\frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0$$