Event ticket pricing with capacity constraints and price restrictions

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Motivated by a ticket pricing problem facing a manager of live events, we study how the manager must maximize revenues by simultaneously determining prices for multiple categories of tickets (because their demands are interdependent) while incorporating contextual restrictions on capacities and prices. Focusing on the context of a single live event, we define the manager's decision problem as the *Event Ticket Pricing* (ETP) problem and formulate it as a constrained non-linear optimization model. Different from existing literature and recognizing contextual details, the ETP problem incorporates constraints based on capacity considerations and pricing restrictions. Our formulation for the ETP problem is robust and can embed many demand functions. Importantly, our analysis of the optimal solution to the ETP problem reveals vital insights for practitioners. To better understand the impact of the nature of product differentiation on event revenue, we solve the ETP problem considering two types of demand stemming from vertically- and horizontally-differentiated ticket categories respectively. To isolate the impact of individual constraints, for each demand type, we apply convex optimization techniques to analyze a series of optimization problems resulting from progressively adding constraint classes. In each case, we identify structural properties of the problem that help us either fully characterize the optimal solution or develop an efficient solution algorithm. When the products are vertically-differentiated, we prove that the optimal solution has a *sold-out threshold structure*, where only products with quality higher than a certain threshold, determined by the partial sum of capacities in the higher quality categories, are sold out. We find that the partially sold products have zero sales when including only the capacity constraint but could have positive sales when adding price restrictions. The latter is different in structure from the unconstrained optimal solution. When the products are horizontally-differentiated, the sold-out threshold structure remains optimal. Importantly, this structure is in contrast to the well-known "equal markup" result that applies in the unconstrained context, confirming the importance of incorporating practically relevant considerations. With the addition of constraints, rather than an equal markup, we find that the prices are non-increasing in quality. Finally, to understand the value of flexibility in seat capacities, we show how the optimal sales, prices, and revenue are affected when possible flexible changes are made to the ticket categories offered in the collection. Among other applications, by highlighting the impact of realistic considerations on optimal prices, our results can provide useful insights for live entertainment promotion planners facing capacity constraints and price restrictions.

Key words: live entertainment events; multi-product pricing; vertical and horizontal product differentiation; constrained optimization

1. Introduction

Live entertainment events that include musical performances, theatrical presentations, and sports have been part of an influential industry with an enormously large market. In 2023, the global revenue for the concert and live music industry alone reached over $33 billion; moreover, the market is seeing promising growth and the revenue figure is predicted to surpass $50 billion in 2030 (Statista 2024). Since the majority of these revenues are derived from ticket sales, pricing event tickets is a key consideration in the live entertainment business. Recognizing the importance of optimizing ticket revenues, managers of live events are looking to data and analytics to enhance their pricing decisions; indeed, industry experts have observed a typical revenue uplift of 5%-15% for firms through better pricing[1].

Our work on event ticket pricing is motivated by a problem facing the box office of a live music venue in a large metropolitan area. The box office's approach to ticket pricing (with the objective of maximizing revenues) is guided by an understanding of the demand. Specifically, the box office recognizes that customers' preferences depend on not only the price, but also non-price characteristics such as the visual and auditory experience and access to concessions and restrooms. Consequently, the quality of the experience, and in turn, demand, varies by seat location. Recognizing these differences in value, the box office groups seats with more or less the same viewing and participation experience into a single *ticket category* and offers all the seats within a category at the same price. It is important to note, however, that individual customers may *perceive* the value of a seat in a ticket category differently, which suggests a diversity of opinions on "ideal" seat locations (see, e.g., Kawase 2013). For instance, in terms of the viewing experience, customers may uniformly prefer a location with a direct viewing angle. In contrast, customers may have idiosyncratic preferences for seats as in the case where rear seats are favored by connoisseurs in audio concerts. Thus, the box office's approach to pricing must explicitly incorporate customer preferences and anticipate their impact on prices and revenue. Finally, since the ticket categories are natural substitutes, the price of a seat in one category has an impact on the demand not only for that category but also for other categories that have comparable viewing and participation experiences. As such, the box office must jointly determine the price of all ticket categories.

In addition, the box office is also concerned with several practical considerations that are vital in this context. Ticket prices of live concerts are keenly monitored by fans and, if perceived as unaffordable, can cause outrage and create a backlash for both the artist and the venue (e.g., Clark

[1] Source: `https://tinyurl.com/betterpricing`

2021, della Cava 2022, Mendez 2022). The performing artists themselves are unhappy about overly-high ticket prices and are keen to ensure access and not antagonize committed fans (Courty 2015). Thus, the need to ensure affordable access is a key consideration for the box office. Audiences and journalists typically use measures such as the average ticket price and the price of the cheapest ticket to gauge the affordability of events[2]. Thus, the box office manager must ensure that the lowest ticket price and the average ticket price across all categories do not exceed pre-determined upper limits. The box office that motivated our work faced such affordable access mandates as well. Finally, the box office is attuned to the impact that prices have on space considerations. Because the allocation of seats to categories precedes the sales of tickets, the prices cannot be set too low lest demand should exceed capacity. Together, these considerations define the box office's decision problem as one of maximizing revenues by choosing a price for each ticket category while satisfying capacity constraints and price restrictions, and incorporating the impact that both prices and non-price factors have on the demand. We refer to this problem as the *Event Ticket Pricing* (ETP) problem.

Although the literature on revenue management is extensive, to the best of our knowledge, we are not aware of any work that incorporates the practical constraints of the ETP problem in a systematic manner. As such, while the literature offers analytical results and insights for unconstrained or lightly constrained revenue optimization problems, there is a need to develop a rigorous approach to understand the impact that incorporating these practical aspects may have on optimal prices and revenue. Our work attempts to fill this gap by modeling and solving the ETP problem. Specifically, we model the ETP problem as a static, constrained, multi-product price optimization to maximize revenue. Recognizing the need to incorporate customer behavior appropriately, our model can embed demand functions stemming from different consumer choice models. We focus on two models – vertical and horizontal differentiation – that are commonly considered in revenue management literature with similar application context (e.g., Budish and Bhave 2023, Arslan et al. 2022). When the ticket categories are vertically differentiated, customers' purchasing behavior is solely steered by their sensitivity to quality; when they are horizontally differentiated, however, the customer valuation is heterogeneous and reflects idiosyncratic preferences. Analytically characterizing the optimal prices, even with a specific demand function, is challenging due to the presence of constraints. Therefore, we adopt a step-by-step approach by adding the constraints to the unconstrained price optimization one at a time. Progressively tightening the model in this manner allows us to develop insights into the solution structure that, in turn, helps us solve the ETP problem.

[2] See, for example, SeatGeek: `https://seatgeek.com/live-tickets`

Another advantage of this approach is that it can isolate the impact of each constraint, enabling managers to understand the trade-offs clearly. We summarize our main findings below.

First, we consider the case where ticket categories are vertically-differentiated. We show that the ETP is a convex optimization problem *if and only if* the random sensitivity to quality is *uniformly distributed* among all customers. While this condition is only sufficient for the unconstrained problem, it becomes necessary as well with capacity constraints, which are not linear in price, illustrating the critical role of the constraints in optimal pricing. When considering only the capacity constraints, we find that a ticket category starts to generate sales *only* when all the categories with higher quality are sold out. Hence, the optimal solution exhibits a *sold-out threshold* structure, where ticket categories with quality higher than a threshold level are all sold out. Moreover, the threshold depends on the aggregate capacity limits of the sold-out categories. In contrast to the unconstrained optimization solution that sells tickets only of the highest quality category, the capacitated problem solution prioritizes the sales of higher quality classes, which may leave lower quality seats unsold, which could be detrimental from the standpoint of ensuring affordable access. Adding the average price restriction to the capacity-constrained problem, we show that the sold-out threshold structure is still optimal, and the threshold is no less than that in the previous scenario (with only the capacity constraints). Different from the previous structure, we find that *all* ticket categories may generate sales now, and the optimal sales of the partially sold categories are all the same. Besides, compared to the unconstrained model and the one with only the capacity constraints, more of the potential market is covered. Effectively, the average price constraint restricts the box office's ability to set high prices at higher quality levels, forcing the box office to offer tickets at lower quality categories. The tighter this restriction, the greater the accessibility and affordability provided to the customers. Finally, we include the upper bound constraint for the price of the lowest quality ticket category to the model augmented by the capacity and average price restrictions. By recursively reducing the current problem to a certain form of the previous problem where the upper bound constraint is absent, we design an iterative polynomial-time algorithm that solves for the optimal prices. In this case, since the ceiling on the lowest price also restricts the prices of adjacent categories, lower quality categories achieve higher sales and may also be sold out. Hence, the optimal solution features two sold-out thresholds, such that tickets of the highest categories and the lowest categories are all sold out.

Next, we consider the case with horizontally-differentiated ticket categories. Because customers are idiosyncratic in their evaluation of ticket categories, one category does not uniformly dominate another in terms of its utility to consumers. Such consumer choice patterns are well-modeled by the

MNL model (e.g., Li and Huh 2011, Arslan et al. 2022). In contrast to the classic result of "equal-markup" optimal pricing (see Dong et al. 2009), with constraints on the prices, we find that the optimal prices (price equals to markup in our context because there is no production cost) are no longer the same across ticket categories. First, with only the capacity constraints, we show that the sold-out threshold structure is optimal and the prices of the sold-out ticket categories are decreasing in quality. However, for the remaining categories that have leftover supplies, their prices are still the same; i.e., this subset of ticket categories still has an equal markup. When also considering the average price restriction, we first show that this constraint is quasi-convex and derive sufficient and necessary optimality conditions. Then, we characterize optimal sales as having a sold-out threshold structure. Moreover, different from the capacitated problem, all ticket categories in this case, regardless of their sales, have different prices that decrease in quality. Lastly, we also include the upper bound constraint for the lowest quality ticket category. Although this constraint has an equivalent convex form and thus the optimality conditions derived for the previous problem still hold, it is analytically challenging to derive simple structural properties and useful insights in this case. Hence, we study the relaxed problem where the average price restriction is assumed to be non-binding. In this case, we again show the optimality of the sold-out threshold structure and the monotonicity of the sold-out categories' prices. Additionally, we find that the price of the lowest quality category is lower than the (identical) price of all other partially sold categories. Overall, our results showcase the significant impact the practical constraints may have on the structure of the optimal solution to the ETP problem, providing vital insights for practicing revenue managers. Indeed, the constraints will have an adverse impact on revenues, but more importantly, they change the solution structure in a manner that deviates from conventional wisdom, which largely stems from unconstrained optimization.

While the above findings characterize the optimal prices for a given collection of ticket categories and capacities, the box office might be able to adjust the capacities of ticket categories in practice. As a first step towards understanding the value of this flexibility, we examine two scenarios. First, we consider a scenario where the box office manager could either *close* or *add* a ticket category. We define *closing* as the decision to make the lowest quality category unavailable, whereas *adding* refers to introducing a new category with higher quality than all existing ones. We characterize the impact of these changes on optimal outcomes. In the vertical ticket categories case, closing has no or negligible impact on the solution, but adding increases the total revenue and changes the optimal sold-out threshold by at most one category. For the horizontal ticket categories case, assuming the optimal threshold remains the same, we show that closing can increase the revenue of each category while adding will cause the partially sold categories to have even lower sales. Second, we consider *capacity reallocation flexibility*, where the box office manager could convert some capacity of one ticket category to another (either higher or lower quality). Moreover, we assume that the reallocated amount is small and the optimal sold-out threshold is unchanged. In our setting, the segmentation of the physically adjacent seats in the venue emerged as a potential decision variable. Results show that, although the sales and prices may change differently for the vertical and horizontal cases, the total revenue always increases. Collectively, our model and analytical results show that, to optimize revenues, the box office manager must thoroughly understand the different impacts these additional considerations may have on the optimal solutions.

The rest of the paper is organized as follows. In the next section, we review the extant literature and position our work. Section 3 formulates the problem and lays out the model fundamentals. We investigate the ETP problem for the vertical case in Section 4 and for the horizontal case in Section 5. Then, Section 6 discusses the impact of additional considerations, and Section 7 concludes the paper. All proofs are in the Appendix.

2. Literature Review

Our paper is primarily related to the literature on multi-product price optimization. We discuss the relevant work in this area, classifying it on the basis of how the demand is modeled.

Since the demand model we consider in the ETP problem can stem from customers' heterogeneous valuation towards non-price attributes of the ticket categories, our paper is built upon prior research on pricing with quality differentiated products. Deriving from the classical framework of vertical product differentiation (Tirole 1988), researchers in the areas of economics (Choi and Shin 1992), marketing (Moorthy 1988), and information systems (Bhargava and Choudhary 2001) have commonly adopted this demand model. In the revenue management literature, previous papers have used the vertically-differentiated demand model to investigate problems such as dynamic pricing (Akçay et al. 2010, Stamatopoulos and Tzamos 2019, Caldentey and Liu 2017), bundling (Banciu et al. 2010, Honhon and Pan 2017), and assortment planning (Honhon et al. 2012, Pan and Honhon 2012, Chen and Yang 2019). More related to our setting, Budish and Bhave (2023) consider vertically-differentiated customers in an event pricing context faced by Ticketmaster and analyze the company's auction design.

Two of the above papers are particularly related to our work. First, Akçay et al. (2010) study the multi-product dynamic pricing problem with the consumer choice model derived from either vertical or horizontal differentiation. Hence, in terms of underscoring the critical role played by the nature of product differentiation in firms' optimal pricing, our paper is closely related to theirs. However, unlike their work, we consider a *constrained* multi-product optimization problem. Second, Banciu et al. (2010) study a seller who owns two capacity-constrained products and makes pricing decisions for the two products as well as a bundle of the two. While capacity constraints are also our focus, we consider a context with more than two products. In addition, we also consider price restrictions in our model. Thus, our work differs from prior work by incorporating additional practical considerations in revenue optimization. Moreover, our analysis of the constrained optimization problems and comparing the structure of optimal solutions with prior work highlights the importance of incorporating these considerations.

Our paper is also related to the stream of literature on choice-based pricing problems that capture horizontal product differentiation. Researchers have employed the multinomial logit (MNL) model and its extensions, such as nested logit (NL), paired combinatorial logit (PCL), and mixed logit models to describe the demand and characterize optimal prices (Dong et al. 2009, Li and Huh 2011, Li and Webster 2017, Li et al. 2019). We focus on the MNL model in an event ticket selling setting where the quality of ticket categories plays an important role. In this regard, a related work is Li et al. (2020). They study a product-line design problem in which the price and quality have an interactive effect on the customers' utility and propose a single-decision variable approach for the joint quality-price optimization. While there is a vast literature on choice-based price optimization, all the above mentioned papers only consider unconstrained problems. Capacity constraints, specifically the cardinality or space constraints, have been studied by papers on assortment optimization (e.g., Désir et al. 2022, Wang 2012, Rusmevichientong et al. 2010, Gallego and Topaloglu 2014). However, differently, we study a seat capacity limit for every ticket category.

Thus, from both theoretical and practical perspectives, there is a need to explore and understand constrained price optimization. Three prior works are relevant in this regard. Keller (2013) studies the demand and price-constrained optimization problems under different logit demand models and provides some sufficient conditions for concavity. However, they do not characterize the optimal prices in closed form and instead solve it via approximation. Shao and Kleywegt (2020) examine a multi-attribute optimization problem with both resource and upper/lower bounds constraints. They reformulate the problem as a convex conic program and solve it numerically. Arslan et al. (2022) conduct an empirical study on price optimization in the sports industry with multiple sales channels and heterogeneous customers. They also consider several constraints in a real business setting, but their research does not analytically investigate optimality or solution structure.

To the best of our knowledge, our study on the ETP problem is the first to analytically solve a constrained price optimization for multiple vertically/horizontally-differentiated products. The characterization sheds light on how capacity and price constraints influence optimal prices, providing insights for managers and paving the way to understanding optimal pricing with important practical considerations.

3. Problem Formulation

Consider a box office manager that offers multiple ticket categories that are substitutable products[3] but vary in utility. The manager's goal is to maximize revenues by setting appropriate prices for each category. Formally, suppose there are N ticket categories with differentiated quality levels represented by the totality of all non-price attributes for the event, denoted by $q_1 > q_2 > \cdots > q_N > 0$. The box office manager decides prices $\mathbf{p} = (p_1, p_2, \cdots, p_N)$ for each category. Here, we consider static, rather than dynamic, pricing for two reasons. First, as previously mentioned, event ticket prices are keenly monitored by fans, and the price changes brought by dynamic pricing may be viewed as unfair and akin to price gouging (Mohammed 2022). Second, for live shows performed by popular artists, high volumes of tickets are usually sold out in seconds, barely leaving time to adjust the ticket price. Ticketmaster, a leading online seller of event tickets, reports that, on average, just 12% of tickets for a live event were dynamically priced while the rest were sold at fixed prices (della Cava 2022). Therefore, there are many instances where static pricing is appropriate and our model applies in these contexts. Finally, even the static optimization problem with constraints is difficult to solve. Thus, given our intent to understand and derive insights for practicing managers, we did not consider dynamic pricing and hope that our work can serve as a foundation to address the more complex problem of dynamic pricing.

The underlying consumer choice model stems from a linear random utility framework; specifically, we assume that a consumer derives the following utility from purchasing product j:

$$u_j = \theta q_j - p_j + \mu \zeta_j. \tag{1}$$

There are two random parts in the above utility function. Corresponding to the idea of vertical product differentiation, each consumer has a random component θ that represents the sensitivity to product quality. Similarly, to model horizontal product differentiation, the random term ζ_j captures each consumer's idiosyncratic utility when purchasing product j, and the parameter $\mu \geq 0$ measures the strength of the idiosyncrasy. This consumer choice model is commonly known as the

[3] Hereafter, we will use "ticket category" and "product" interchangeably.

mixed logit model (McFadden and Train 2000). With this setup, the demands of each category $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is simply the probability of a representative consumer choosing product j. Despite its fitness for a wide range of practical problems, the mixed logit model's analytical intractability prevents us from deriving any useful insights. To circumvent this difficulty and, more importantly, to understand the role of product differentiation in the ETP problem, we take a similar approach followed by Akçay et al. (2010) and examine two specific cases of the mixed logit model, namely, the vertical and horizontal product differentiation cases. We elaborate on these cases next.

Vertically-Differentiated Demand. In this case, we focus on the random quality sensitivity θ and assume that $\mu = 0$. Let $F(\cdot)$ be the distribution function of θ with support $[0,1]$; furthermore, we focus on the case where F is at least twice differentiable. Then, due to the monotonicity of the quality levels, the demand for product j, $\alpha_j^V(\mathbf{p})$, can be written as

$$\alpha_j^V(\mathbf{p}) = \begin{cases} 1 - F\left(\frac{p_1 - p_2}{q_1 - q_2}\right), \ j = 1; \\ F\left(\frac{p_{j-1} - p_j}{q_{j-1} - q_j}\right) - F\left(\frac{p_j - p_{j+1}}{q_j - q_{j+1}}\right), \ \forall\, 1 < j < N; \\ F\left(\frac{p_{N-1} - p_N}{q_{N-1} - q_N}\right) - F\left(\frac{p_N}{q_N}\right), \ j = N. \end{cases} \tag{2}$$

Note that the demand function (2) depends critically on the distribution of θ. Moreover, the cross-price elasticity only exists for products whose quality levels are adjacent to each other.

Horizontally-Differentiated Demand. Suppose that θ is a positive constant and ζ_j follows the standard Gumbel distribution. Furthermore, we normalize $\mu = 1$ for simplicity. Then, the consumer choice model reduces to the well-known multinomial logit (MNL) model. Consequently, we have the following demand function in the horizontal differentiation case:

$$\alpha_j^H(\mathbf{p}) = \frac{e^{(\theta q_j - p_j)}}{1 + \sum_{i=1}^N e^{(\theta q_i - p_i)}}, \ \forall\, 1 \le j \le N. \tag{3}$$

In contrast to the vertically-differentiated model, here the demand function of product j is related to the prices of all products. Although the products are still quality differentiated, the utilities derived from them cannot be ordered universally.

With this demand setup ((2) and (3)), we can formulate the price optimization problem and incorporate constraints. First, since defining ticket categories precedes the pricing decision, each category j has limited capacity. Let β_j be the scaled capacity limit for product j (e.g., $\beta_j = C_j/\Lambda$ where Λ is the total market size and C_j is the total number of tickets in category j). In the analysis, we assume that the capacity limits satisfy $0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_N$, which is consistent with practices in the live entertainment industry[4]. Thus, it ensures analytical tractability without significantly

[4] Appendix B shows typical venue charts that highlight the non-decreasing trend in seat category capacity that is modeled in our assumption. Conversations with the box office manager of our motivating application also supported this assumption.

sacrificing practical relevance or generalizability. Second, we consider the average price constraint which requires that the average price[5] of all products does not exceed a pre-determined threshold, which we denote as $\bar{P}$. Third, the price of the lowest quality product, p_N, is bounded above by an exogenous ceiling price U_N. A lower U_N forces prices to be low, permitting a greater level of access, whereas a larger U_N may allow high prices for the least preferred seats in the venue. In practice, managers determine the above thresholds using their judgment to ensure that it does not represent an unpalatable deviation from historical prices. We name the three constraints as (**Cap**), (**AveP**), and (**CeilN**), respectively. Then, the ETP problem is:

$$
\begin{aligned}
\textbf{[ETP]} \quad \max_{\mathbf{p}} \quad & R(\mathbf{p}) = \sum_j \alpha_j(\mathbf{p}) p_j \\
\text{s.t.} \quad & \alpha_j \le \beta_j, \ \forall\, 1 \le j \le N; \quad (\textbf{Cap}) \\
& \frac{1}{N} \sum_j p_j \le \bar{P}; \quad (\textbf{AveP}) \\
& p_N \le U_N. \quad (\textbf{CeilN})
\end{aligned}
$$

4. Pricing under Vertically-Differentiated Demand

Consider the ETP problem for a vertically-differentiated set of ticket categories. This setup has been used to study event ticket pricing by other researchers (e.g., Budish and Bhave 2023, Akçay et al. 2010). In this case, the demand function is given by (2). From the problem formulation, we can see that the structure of the problem hinges on the distribution of the quality sensitivity parameter θ. While the distribution can be general, we would ideally want the resulting problem to be a convex program for tractability. For the unconstrained problem, the convexity can be achieved under mild conditions on the distribution function F. However, for the constrained problem, the convexity is guaranteed only with specific conditions. The following lemma characterizes these conditions.

LEMMA 1. *Consider the vertical product differentiation setting where the demand function is given by $\alpha_j^V(\mathbf{p})$. The revenue function $R(\mathbf{p})$ is concave in $\mathbf{p}$ if the following condition holds:*

$$xF^{''}(x) + 2F^{'}(x) \ge 0, \ \forall x \in [0,1]. \tag{4}$$

Furthermore, for product $j = 1,2,...,N$, the demand function $\alpha_j^V(\mathbf{p})$ is convex if and only if $F(x) = x$, i.e., the quality sensitivity θ is uniformly distributed over $[0,1]$.

Note that condition (4) stated in Lemma 1 is a sufficient condition, and clearly the uniform distribution satisfies it. Hence, assuming a uniform distribution for θ yields a convex program

[5] Alternatively, one may look at the weighted average price restriction (e.g., using β_j as the weight of p_j). In this case, similar analysis can be carried through and our results continue to hold; see proofs in Appendix A for details.

just as in the unconstrained problem (see, e.g., Choi and Shin 1992, Akçay et al. 2010). Here, for the constrained problem to be convex, the constraint functions also need to be convex (Boyd and Vandenberghe 2004). The price restrictions are already linear, and Lemma 1 implies that $\alpha_j^V(\mathbf{p}) - \beta_j$ is convex if and only if $F(x) = x$. Therefore, consistent with the literature (e.g., Akçay et al. 2010), we hereafter assume uniform distribution for θ.

To analytically investigate the **[ETP]** problem in the vertical products setting, we sequentially add constraints to build progressively constrained models. We define **[ETP-v0]** as the unconstrained pricing problem for vertical products; similarly, we define problems **[ETP-v1]**, **[ETP-v2]**, and **[ETP-v3]** as the price optimization problems with (**Cap**), with (**Cap**) and (**AveP**), and with (**Cap**), (**AveP**), and (**CeilN**), respectively. In this section, we write $\boldsymbol{\alpha}$ instead of $\boldsymbol{\alpha}^V$ for brevity. Additionally, let $\alpha_{N+1} := 1 - \sum \alpha_j$ be the no-purchase probability, i.e., the size of the uncovered market. Now, we sequentially investigate problems **[ETP-vx]**, $x = 0, 1, 2, 3$.

4.1. Unconstrained Optimization of Vertically-Differentiated Categories ([ETP-v0])

A direct corollary following Theorem 3 in Akçay et al. (2010) is that the optimal prices are given by $p_j^* = \frac{q_j}{2}$, $1 \le j \le N$. Moreover, the optimal sales are $\alpha_1^* = \frac{1}{2}$ and $\alpha_j^* = 0$ for $j = 2, \cdots, N$; and therefore the no purchase probability $\alpha_{N+1}^* = \frac{1}{2}$. As such, the optimal strategy is to extract the value from the highest quality product alone. This approach is feasible because there are unlimited such units available for sale, and the lower quality products are not as profitable as product 1. Hence, the box office will price in a way such that half of the market is served with the best seats whereas the other half is left uncovered. In practice, however, this solution is not implementable because of the capacity constraint (**Cap**) on ticket categories. Moreover, pricing for the highest quality category in this manner may not satisfy the accessibility mandate facing the event manager. The constrained problems we consider next aim to address these drawbacks.

4.2. Optimization with Capacity Constraints ([ETP-v1])

Let us consider the following constrained price optimization:

$$[\textbf{ETP-v1}] \quad \max_{\mathbf{p}} \left\{ \sum_j \alpha_j(\mathbf{p}) p_j \,|\, \alpha_j(\mathbf{p}) \le \beta_j, \forall 1 \le j \le N \right\}.$$

To solve problem **[ETP-v1]**, we adopt the idea of variable transformation. Note that the demand function (2) with $F(x) = x$ is a system of linear equations $\boldsymbol{\alpha}^T = M\mathbf{p}^T + (1, 0, \cdots, 0)^T$, where M is an $N \times N$ matrix. Since we can show that M is invertible, a one-to-one correspondence between the price vector and the demand vector is guaranteed. Specifically, we can write the price vector as a function of the demand vector and the no-purchase probability; i.e., $p_j = p_j(\boldsymbol{\alpha}, \alpha_{N+1})$, $j =$

$1,2,\cdots,N$. e., $p_j = p_j(\boldsymbol{\alpha}, \alpha_{N+1})$, $j = 1,2,\cdots,N$. Note that the one-to-one mapping still holds here due to the relationship $\alpha_{N+1} = 1-(\alpha_1+\alpha_2+\cdots+\alpha_N)$. Therefore, by treating the demand vector $\boldsymbol{\alpha}$ as the decision variable, we can obtain a convex program with linear constraints, which is relatively easier to solve.

Our analysis of the KKT conditions for this case (see Proposition 1) has two important implications. First, no more than half of the market will be covered at optimality. As in **[ETP-v0]**, this is due to the potential revenue dilution if more tickets are sold at lower prices. Second, the optimal sales have a *threshold* structure that is determined by the partial sum of the capacity limits, defined as

$$K(s) = \sum_{j=1}^{s} \beta_j, s = 1,2,\cdots,N.$$

PROPOSITION 1. *Consider problem **[ETP-v1]**. The optimal sales $\boldsymbol{\alpha}^*$ can be obtained in the following three cases.*

*(1) If $K(1) > \frac{1}{2}$, then **(Cap)** is redundant and the problem reduces to **[ETP-v0]**.*

(2) If $K(N) \le \frac{1}{2}$, then $\alpha_j^* = \beta_j$ for all $j = 1,2...,N$.

(3) If $K(1) \le \frac{1}{2} < K(N)$, then there exists a threshold

$$s_1 = \min\left\{K(s+1) \ge \frac{1}{2} \mid s = 1,2,\cdots,N-1\right\},$$

such that $\alpha_j^* = \beta_j$ for $j \le s_1$, $\alpha_{s_1+1}^* = \frac{1}{2} - K(s_1)$, and all remaining products (if any) have zero sales.

For a price p, we define a product as unsold if $\alpha_j(p) = 0$, as sold-out if $\alpha_j(p) = \beta_j$, and as partially sold if $\alpha_j(p) < \beta_j$. As such, Proposition 1 characterizes conditions that help us identify the three classes of products. In the first case, the box office has enough capacity for the highest quality product to serve half of the market, just as in the unconstrained problem **[ETP-v0]**, and the second case occurs when the capacities of all the categories are so limited that together they cannot serve even half of the market, resulting in all products being sold out. Interestingly, in the third case in which there exists a threshold s_1 such that the ticket categories $1,\cdots,s_1$ are sold out. We refer to this as the *sold-out threshold structure* of the optimal solution. Moreover, the next product s_1+1 is partially sold such that exactly half of the market is covered, leaving all other products unsold. It is worth mentioning that our result is in line with Akçay et al. (2010), where the aggregated inventories determine the optimal prices, and only the part of the inventories that equals the potential demand has a positive value.

As for the optimal prices, it is straightforward to find the optimal price vector for the first two cases. For the third case, by noting that $\alpha_{N+1} = \frac{1}{2}$, we can obtain the optimal price $p_j^* = q_j/2$ for the partially sold and unsold products, i.e., $j = s_1 + 1, \cdots, N$. For the sold-out products, i.e., the first s_1 products, we can recursively find their prices, which can be written as

$$p_j^* = \frac{q_{s_1+1}}{2} + \sum_{n=j}^{s_1} (1 - K(n)) Q_n, \quad j = 1, \cdots, s_1. \tag{5}$$

In the above, $Q_j := q_j - q_{j+1}$ for $j = 1, \cdots, N-1$ and $Q_N := q_N$. It is interesting to note that for every sold-out product, its price is affected by the capacity limits of all higher quality products and not by those of the lower quality products. However, all the partially sold products' prices are independent of any capacity information.

From the above proposition, we can see that the partial sum (from the highest quality product and in decreasing order) of the capacity limits plays a vital role in the optimal solution. While the box office still wants to extract most of the value from higher quality products, that goal is constrained by the sub-aggregate capacity, $K(s)$, at quality level $s \le s_1$. As a result, the box office can only extract as much value from high quality products as their capacity limits will allow.

4.3. Optimization with Capacity Constraints and Average Price Restriction ([ETP-v2])

Next, we augment **[ETP-v1]** with average price restriction (**AveP**) to obtain the following price optimization problem:

$$\textbf{[ETP-v2]} \quad \max_{\mathbf{p}} \left\{ \sum_{j=1}^{N} \alpha_j(\mathbf{p}) p_j \,|\, \alpha_j(\mathbf{p}) \le \beta_j, \forall 1 \le j \le N; \sum_{j=1}^{N} p_j \le N\bar{P} \right\}.$$

The extra restriction in the form of a linear inequality in price vector $\mathbf{p}$ causes the optimization problem to have a complex form in terms of demand vector $\boldsymbol{\alpha}$, which makes this problem more difficult than **[ETP-v1]**. Nevertheless, the next lemma still uncovers useful properties of the optimal solution to **[ETP-v2]**.

LEMMA 2. *Suppose that there exists a feasible solution to the problem **[ETP-v2]**. Then, at optimality, there exists an s ($1 \le s \le N$) such that only products $1, 2, \cdots, s$ are sold out. Moreover, if $s \le N-1$, then the optimal prices of the sold-out products can be expressed as a function of the optimal price of product $s+1$ in the following way:*

$$p_j^* = p_{s+1}^* + \sum_{n=j}^{s} (1 - K(n)) Q_n, \quad 1 \le j \le s. \tag{6}$$

Lemma 2 offers an important insight into the optimal solution to problem **[ETP-v2]**; that is, there exists a threshold structure in terms of product sales. Therefore, we may leverage our understanding of such a structure to simplify the N-dimensional optimization problem **[ETP-v2]** and recast it as a one-dimensional search to find the optimal threshold. In turn, as we will see next, this threshold allows us to fully specify the optimal solution. In particular, if the threshold s turns out to be N, then all products are sold out and we can easily obtain the optimal prices based on the one-to-one mapping of the demand and price. If the threshold $s<N$, then we can reduce the problem to a lower-dimensional sub-problem with fewer constraints. To be specific, since the sales of the sold-out products are fully characterized by their capacity limits ($\alpha_j=\beta_j$ for $j=1,\cdots,s$), and their prices can be derived from p^*_{s+1}, we can focus on the remaining $N-s$ products. Because these are partially sold or unsold products, their capacity constraints become redundant. In addition, the average price restriction can be rewritten by substituting (6) into (**AveP**) s times. Then, we can formulate a sub-problem where only products $s+1,\cdots,N$ exist in the offered collection:

$$[\textbf{ETP-v2}(s)] \qquad \max_{p_{s+1},\cdots,p_N}\left\{\sum_{j=s+1}^{N}\tilde{\alpha}_j p_j \,|\, sp_{s+1}+\sum_{j=s+1}^{N}p_j\le N\bar{P}'\right\}.$$

Here, the demand function $\tilde{\alpha}_j$ is simply the $(N-s)$-product equivalence of the N-product demand function (2); i.e., product $s+1$ has the highest quality and product N has the lowest quality. Moreover, the price constraint in the above sub-problem is derived from the original constraint (**AveP**) and the equation (6); therefore, we have $\bar{P}'=\bar{P}-\sum_{j=1}^{s}\sum_{n=j}^{s}(1-K(n))Q_n/N$.

Compared to the original problem [**ETP-v2**], the sub-problem [**ETP-v2(s)**] has fewer decision variables and only one constraint. Moreover, we can readily verify that problem [**ETP-v2(s)**] is a convex program. With these insights, we can develop an approach to solve [**ETP-v2**]. Let $\lambda\ge 0$ be the Lagrange multiplier associated with the price constraint in [**ETP-v2(s)**]. Then, the KKT conditions ensure the following complementary slackness requirement:

$$\lambda\left(2N\bar{P}'-\sum_{n=s+1}^{N}Q_n(n-n^2\lambda)\right)=0.$$

If the above constraint is non-binding, then (**AveP**) in problem [**ETP-v2**] is redundant, which reduces [**ETP-v2**] to [**ETP-v1**]. Otherwise, we can solve the sub-problem [**ETP-v2(s)**] based on the multiplier λ, which is given by

$$\lambda=\frac{\sum_{n=s+1}^{N}nQ_n-2N\bar{P}'}{\sum_{n=s+1}^{N}n^2Q_n}. \tag{7}$$

Lemma 3. *Given an $s=1,\cdots,N-1$, let λ be defined by (7) above.*

*(1) If $\sum_{n=s+1}^{N}nQ_n\le 2N\bar{P}'$, then (**AveP**) is redundant and [**ETP-v2**] becomes [**ETP-v1**].*

*(2) If $\sum_{n=s+1}^{N}nQ_n> 2N\bar{P}'$, then sub-problem [**ETP-v2(s)**] can be solved as follows. The optimal demand of product $s+1$ is $\alpha^*_{s+1}=\frac{1}{2}-K(s)+\frac{s+1}{2}\lambda$, and the optimal demand of each of the remaining product (if there is any) is $\frac{\lambda}{2}$. The optimal prices are given by*

$$p^*_j=\frac{q_j}{2}-\frac{\lambda}{2}\sum_{n=j}^{N}nQ_n, \quad j=s+1,\cdots,N.$$

When $\lambda > 0$, the average price constraint is binding. In this case, Lemma 3 contrasts with Proposition 1. That is, the lower quality products ($j > s$) have positive demand and, since the no-purchase probability $\alpha_{N+1} = (1 - N\lambda)/2 < 1/2$, the market coverage increases to more than half. This illustrates the impact of the average price constraint (**AveP**) on the optimum market coverage. Recall that without (**AveP**), the box office manager could charge a premium in the high quality ticket categories and only half of the customers are served; by contrast, a binding (**AveP**) (meaning that the manager's pricing power is limited) would help more customers get access to the tickets. Thus, constraint (**AveP**) addresses one of the drawbacks of the unconstrained solution by ensuring affordability and helping the box office manage public perceptions regarding the cost of access in some cases, and align with mission-driven accessibility goals in others.

To solve problem [**ETP-v2**], based on Lemma 3, we only need to find the threshold of the sold-out products. Our approach is as follows. First, because Lemma 2 assures the threshold structure in terms of product sales, we begin with the all-sold-out scenario and evaluate the box office's revenue. Then, we construct a threshold-type solution that is feasible and yields a higher revenue. We proceed in this manner until we reach a situation where the next solution does not satisfy the average price constraint. Recall that we can write the price as a function of the demand vector and the no-purchase probability; i.e., $p_j = p_j(\boldsymbol{\alpha}, \alpha_{N+1})$, $j = 1, 2, \cdots, N$. Then, we define the s-sold-out demand vector as $\boldsymbol{\alpha}^{(s)} := (\beta_1, \cdots, \beta_s, 0, \cdots, 0)$, and then the resulting no-purchase probability is $\alpha_{N+1}^{(s)} = 1 - K(s)$. We focus on the total price given the s-sold-out demand vector,

$$T(s) := \sum_{j=1}^{N} p_j(\boldsymbol{\alpha}^{(s)}, \alpha_{N+1}^{(s)}).$$

Clearly, the total price $T(s)$ is a decreasing function in s; i.e., $T(1) \geq T(2) \geq \cdots \geq T(N)$. Hence, whenever feasible, we can traverse all possible s to find the smallest one at which $T(s)$ meets the average price restriction. In addition, recall that s_1 is the critical value for problem [**ETP-v1**]. Hence, being the optimum of a less constrained problem, s_1 serves as a lower bound for the optimal threshold of problem [**ETP-v2**]. Using that threshold, we could solve the [**ETP-v2**] by reducing it to a smaller sub-problem.

Proposition 2. *Problem [**ETP-v2**] can be solved as follows:*

*(1) If $T(N) > N\bar{P}$, then (**AveP**) is always violated and there is no feasible solution.*

(2) If $T(N) < N\bar{P} < T(N-1)$, all products are sold out except N, i.e., $\alpha_j^* = \beta_j$ for $j = 1, \cdots, N-1$; otherwise, if $T(N) = N\bar{P}$, then all products are sold out.

(3) If $T(N-1) \le N\bar{P}$, then there exists a threshold

$$s_2 = \max\left\{s_1, \min\left\{T(s+1) \le N\bar{P} \mid s = 1, 2, \cdots, N-1\right\}\right\},$$

*such that $\alpha_j^* = \beta_j$ for $j = 1, 2..., s_2$; moreover, the remaining demand vector $(\alpha_{s_2+1}^*, \cdots, \alpha_N^*)$ can be obtained by solving the sub-problem **[ETP-v2(s_2)]** as prescribed by Lemma 3.*

Proposition 2 considers three cases. The first case leads to an infeasible problem. In particular, $T(N)$ represents the total price when all products are sold out, implying that prices are low. Hence, if $T(N) > N\bar{P}$, then there is no way to lower prices further, and therefore problem [**ETP-v2**] does not have a feasible solution. In the second case, all products are sold out at optimality if $T(N) = N\bar{P}$; but if $T(N) < N\bar{P} < T(N-1)$, the revenue could be enhanced by setting a slightly higher price for product N, resulting in all but the last product being sold out.

In the third case, we find the threshold $s_2 < N$ and invoke sub-problem [**ETP-v2(s_2)**] to solve for the optimal prices. First, we have $s_1 \le s_2$. That is, adding an average price restriction may help the box office sell out more ticket categories. While this may not be profitable for the box office, it is beneficial to the consumers, as more seats are sold and a greater number of tickets are offered at lower prices. Second, from the solution of the sub-problem [**ETP-v2(s_2)**], we can see that the optimal prices of *all* products are affected by the capacity limits of the sold-out ones. This result is in contrast with problem [**ETP-v1**], where the capacities of the sold-out products impact just their own prices. Thus, we find that the average price restriction accentuates and spreads the impact of the capacity constraints of the sold-out products to the pricing decisions of all products.

Together, Proposition 2 uses Lemma 2 and Lemma 3 to simplify a constrained, multi-dimensional optimization problem into a one-dimensional search, permitting us to understand the nature of the optimal solution.

4.4. Optimization with All Three Constraint Classes ([ETP-v3])

Augmenting [**ETP-v2**] with the constraint (**CeilN**) define

$$[\textbf{ETP-v3}] \quad \max_{\mathbf{p}} \left\{ \sum_j \alpha_j(\mathbf{p}) p_j \mid \alpha_j(\mathbf{p}) \le \beta_j, \forall 1 \le j \le N; \sum_j p_j \le N\bar{P}; p_N \le U_N \right\}.$$

The last constraint (**CeilN**), a restriction on the upper bound of the price of the lowest quality product, reflects the practical concern of providing a minimum level of affordability to the event. Despite its simple structure, (**CeilN**) can cause a considerable structural change to the optimal solution of the previous problem, [**ETP-v2**]. To tackle this difficulty, we recursively construct sub-problems that have fewer decision variables and take the form of the [**ETP-v2**]. Specifically, given an integer $s = N-1, N-2, \cdots, 1$, we define a facilitating s-dimensional sub-problem:

$$[\textbf{ETP-v3}(s)] \quad \max_{p_1,\cdots,p_s} \left\{ \sum_{j=1}^{s} \bar{\alpha}_j(\mathbf{p}) p_j \,|\, \bar{\alpha}_j(\mathbf{p}) \leq \beta_j, \forall 1 \leq j \leq s; \sum_{j=1}^{s} p_j \leq N\bar{P} - (s+1) \sum_{j=s+1}^{N} U_j \right\}.$$

Note that problem [**ETP-v3**(s)] is essentially problem [**ETP-v2**] with s products. However, the demand function and the average price restriction are modified. In particular, consider products $1, 2, \cdots, s$; the quality of product j is modified to be $\bar{q}_j = q_j - q_{s+1}$ ($j = 1, 2, \cdots, s$). Additionally, the demand function $\bar{\alpha}_j$ is modified from (2) by using quality $\bar{q}_j$ and letting product s to be the lowest quality product. Lastly, the average price restriction is transformed by adding an extra term that is a function of the sequence $(U_N, U_{N-1}, \cdots, U_{s+1})$ where U_j is a function of $(\beta_j, \cdots, \beta_N, U_N)$, which can be calculated by the iterative formula below:

$$U_j = Q_j(U_{j+1}/Q_{j+1} + \beta_{j+1}), \quad s \leq j \leq N-1. \tag{8}$$

Hence, the upper bound on the price of the lowest quality product has a ripple effect across the collection of all products, because the impact of price may pass along adjacent products in the vertically-differentiated demand function (2).

Based on this insight, we develop an easily implemented algorithm to solve problem [**ETP-v3**].

- *Step 1: Initialization.* Given the quality levels $q_1, \cdots, q_N$ and the capacity limits $\beta_1, \cdots, \beta_N$, solve the corresponding [**ETP-v2**] to obtain the optimal price vector $(p_1^*, \cdots, p_N^*)$.
  (a) If $p_N^* \leq U_N$, then return $\mathbf{p}^*$ and STOP.
  (b) Else, let $p_N^* = U_N$ and go to Step 2.
- *Step 2: Iteration.* For $s = N-1, N-2, \cdots, 1$, do the following:
  (a) Update the quality levels $q_1, \cdots, q_s$ such that $q_j \to q_j - q_{s+1}$.
  (b) Generate the $(U_N, \cdots, U_{s+1})$ sequence based on (8).
  (c) Use these parameters to construct a sub-problem [**ETP-v3**(s)].
  (d) Solve [**ETP-v3**(s)] to obtain $(p_1^*, \cdots, p_s^*)$.
    - If $p_s^* \leq U_s$, then update $p_j^* \to p_j^* + \sum_{n=s+1}^{N} U_n$, ($j = 1, \cdots, s$) and go to Step 3.
    - Else, let $p_s^* = U_s$ and proceed to the next iteration.
- *Step 3: Conclusion.* Concatenate the prices obtained in the above procedure to form the price vector $\mathbf{p}^*$. Return it as the optimal solution.

Note that, in the above algorithm, we assume that all problems are feasible. Otherwise, if any of the above problems do not have a feasible solution, then neither does problem [**ETP-v3**]. Intuitively, we recursively transfer the upper bound constraint (**CeilN**) to the next sub-problem by removing the incumbent lowest-quality product. Thus, we may reduce the current problem to [**ETP-v2**], whose structural results and solution are already known. Note that the iteration is one-dimensional, and as such the algorithm is computationally efficient. The next proposition formally establishes the validity of the above algorithm.

Proposition 3. *Consider problem [**ETP-v3**]. The above algorithm produces an optimal price vector* $\mathbf{p}^*$*. Moreover, if the algorithm ever enters into Step 2 and stops at* $s < N-1$*, there exist two thresholds* $1 \le s_3 \le s_3' \le N$ *such that* $\alpha_j^* = \beta_j$ *for* $1 \le j \le s_3$ *or* $s_3' \le j \le N$*.*

Compared to the optimal solution of problem [**ETP-v2**], we see two important effects of (**CeilN**) on the ETP problem. First, under the vertical structure, even though the restriction applies directly only to the lowest quality product, its effect propagates through all of the ticket categories in an increasing order in quality. Hence, every product effectively faces an implicit price upper bound. Second, there exists a new sold-out threshold s_3' where the sales of the last $N - s_3' + 1$ products reach their capacity limits. Indeed, when U_N is small enough, all p_j $(j \ge s_3')$ is determined by relatively low ceiling prices U_j $(j \ge s_3')$. As a result, the ticket categories with the lowest appeal attract enough demand to sell out. Despite seemingly restricting the price of the lowest quality product, (**CeilN**) has a broader impact and can meaningfully address concerns of affordability and access to tickets, giving the box office a powerful tool to structure their approach to price setting.

Additionally, as with [**ETP-v2**], the optimal prices in [**ETP-v3**] are affected by the capacities of the sold-out products. Applying Proposition 2 to sub-problem [**ETP-v3(**s**)**] with $s = s_3' - 1$, we see that the prices of products $j < s_3'$ are determined by the capacities of sold-out products $j \le s_3$ and the modified average price, which depends on the capacities of sold-out products $j \ge s_3'$. Thus, the prices of the high quality sold-out products are determined by the capacity limits of *all* sold-out products. For the low quality sold-out products $j \ge s_3'$, however, their optimal prices are independent of the capacities of the sold-out products at the higher end and are only affected by their own capacities.

5. Pricing under Horizontally-Differentiated Demand

Idiosyncratic preferences often manifest in purchasing behavior, and in turn, researchers have adopted the well-known MNL model to analyze such contexts. Accordingly, we study the price optimization problem where products are horizontally-differentiated. Particularly, we look at a special case of the general mixed logit utility (1), where θ is fixed to be a positive constant, ζ_j $(j = 1, \cdots, N)$ is assumed to be i.i.d. with the standard Gumbel distribution, and μ is normalized to 1. That is, all customers have the same sensitivity to the vertical attributes of the products but derive random idiosyncratic utility when purchasing a certain ticket category. The choice model then reduces to the well-known MNL model. The demand function of each ticket category is given by equation (3); and we write $\boldsymbol{\alpha}$ instead of $\boldsymbol{\alpha}^H$ in this section for brevity.

Different from the vertically-differentiated collection, the pricing of each ticket category is now intertwined with that of all others, which adds complexity to the analysis. Note that all customers are assumed to have an identical price sensitivity (equalling one) in the utility function. This is a common assumption in literature and, more importantly, it can protect the concavity of the revenue function, which may otherwise be lost with heterogeneous price sensitivity (see, e.g., Gallego and Wang 2014, Li et al. 2020). Next, as in Section 4, we progressively add the three constraints sequentially to construct and study problems **[ETP-h**x], $x = 0, 1, 2, 3$.

5.1. Unconstrained Optimization of Horizontally-Differentiated Categories ([ETP-h0])

As a benchmark, we solve the unconstrained problem **[ETP-h0]** using the price-demand variable transformation, which is a well-developed method in the literature (see, e.g., Bitran and Caldentey 2003). In accordance with this technique, our analysis is based on the one-to-one correspondence between the price vector $\mathbf{p}$ and the demand vector $\boldsymbol{\alpha}$:

$$p_j(\boldsymbol{\alpha}) = \theta q_j + \log(1 - \sum_{i=1}^{N} \alpha_i) - \log \alpha_j, \quad j = 1, 2, \cdots, N. \tag{9}$$

When writing the revenue as a function of the demand vector, the objective function becomes concave, and the maximization problem can be solved using the first-order conditions. To facilitate exposition, we define the Lambert W function, $W(z)$, as the real solution w to the equation $z = we^w$ for any given $z \geq 0$. Then, the maximized revenue, R^*, and the optimal prices, p_j^*, are given by

$$R^* = W(\sum_{j=1}^{N} \exp(\theta q_j - 1)) \quad \text{and} \quad p_j^* = R^* + 1, \quad j = 1, 2, \cdots, N.$$

Observe that the prices for all ticket categories at optimality are equal. Researchers (e.g., Anderson and De Palma 1992, Dong et al. 2009) have identified this structure as the "equal markup" property. Note that the production cost is assumed to be zero in our model; with positive cost, the above prices would be replaced with unit profit margins (i.e., the markup). Building on the MNL model, we investigate the impact of capacity constraints and price restrictions on the optimality structure of the pricing problem.

5.2. Optimization with the Capacity Constraints ([ETP-h1])

Consider the price optimization **[ETP-h0]** augmented with the capacity constraints (**Cap**). Using the one-to-one mapping (9), we rewrite the resulting optimization problem using the demand vector $\boldsymbol{\alpha}$ as the decision variable:

$$\textbf{[ETP-h1]} \quad \max_{\boldsymbol{\alpha}} \left\{ \sum_j \alpha_j p_j(\boldsymbol{\alpha}) \,|\, \alpha_j \leq \beta_j, \forall 1 \leq j \leq N \right\}.$$

With concave objective function and linear constraints, the problem is convex and we can use the KKT conditions to compute the globally optimal solution. From the KKT conditions, we deduce an interesting threshold structure as characterized by the following lemma.

LEMMA 4. *In the solution $\boldsymbol{\alpha}^*$ to the KKT conditions of problem [**ETP-h1**], if product j is sold out, then product i with $i \le j$ must also be sold out.*

Therefore, the optimal solution to problem [**ETP-h1**] is well-structured. In fact, the optimal demands feature a certain threshold such that all ticket categories with quality above that threshold are sold out at optimality. As a result, we can narrow the search for the optimal solution down to the sold-out threshold type. Thus, we only need to characterize the optimal demands given a specific threshold. In other words, for any threshold, we want to compute the corresponding optimal demand vector and maximal revenue. Recall the partial sum of capacity limits $K(s) = \sum_{j=1}^{s} \beta_j$ $(s = 1, 2, \cdots, N)$. Then, given an $s = 0, 1, \cdots, N$, the following algorithm will produce a solution $\boldsymbol{\alpha}(s)$ with the maximized revenue $R(s)$:

(1) If $s = 0$, then $\boldsymbol{\alpha}(s)$ equals to the optimal solution to problem [**ETP-h0**].

(2) If $s \ge 1$, then $\alpha_j(s) = \beta_j$ for $j = 1, \cdots, s$. Furthermore, if $s < N$, then for $j = s+1, \cdots, N$,

$$\alpha_j(s) = \frac{W(Q)(1-K(s))^2}{1 + W(Q)(1-K(s))} \frac{e^{\theta(q_j - q_N)}}{\sum_{i=s+1}^{N} e^{\theta(q_i - q_N)}}, \tag{10}$$

where $Q = \sum_{i=s+1}^{N} e^{\theta q_i} f(1-K(s))$ and $f(x) = x^{-1} e^{-x^{-1}}$.

(3) Lastly, if $s < N$ and $\alpha_{s+1}(s) > \beta_{s+1}$, then $R(s) = 0$; otherwise, substituting $\boldsymbol{\alpha}(s)$ into the objective function, we obtain the total revenue:

$$R(s) = \sum_{j=1}^{N} \alpha_j(s) p_j(\boldsymbol{\alpha}(s)). \tag{11}$$

Interestingly, every product $j = s+1, \cdots, N$ is partially sold as its demand α_j is not zero, which is a contrast to the vertical products case (see Proposition 1). Rather, its demand is correlated with the aggregate capacity of all the sold-out products $K(s)$. Furthermore, since the demand given by (10) can be shown to decrease in K (keeping s unchanged), a larger capacity of sold-out products will negatively affect the demand for every partially sold product. In addition, the optimal sales of the partially sold products also depend on their quality levels $q_{s+1}, \cdots, q_N$. In particular, we have $\alpha_j > \alpha_{j+1}$, indicating that the product with higher quality has larger sales. Therefore, product quality has a considerable impact on product substitutions and in turn, influences product sales. Note that the condition $\alpha_{s+1}(s) \le \beta_{s+1}$ guarantees that the capacity constraints are met for all products $s+1, \cdots, N$, because $\alpha_j > \alpha_{j+1}$ and $\beta_j \le \beta_{j+1}$.

Now, once the optimal total revenue is expressed as a function of the threshold s, the [**ETP-h1**] can be solved via a one-dimensional search. Moreover, we can uncover interesting properties of the optimal prices based on the KKT conditions.

PROPOSITION 4. *Consider problem* ***[ETP-h1]****. The optimal sold-out threshold* s_1[6] *is given by*

$$s_1 = \max\{R(s) \mid s = 0, 1, \cdots, N\},$$

where $R(s)$ *is the total revenue function* (11)*. Hence, the optimal demand is* $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}(s_1)$*. Moreover, the optimal prices have the following property:*

(1) If $s_1 = 0$*, then* $p_1^* = \cdots = p_N^*$*.*

(2) If $s_1 = N$*, then* $p_1^* > \cdots > p_N^*$*.*

(3) If $0 < s_1 < N$*, then* $p_1^* > \cdots > p_{s_1}^* > p_{s_1+1}^* = \cdots = p_N^*$*.*

First, unlike Proposition 1 in the vertical product case, Proposition 4 reveals that the sold-out threshold s_1 cannot be pre-determined. Rather, it can only be searched by comparing the resulting optimal revenues. Nevertheless, such a one-dimensional search is computationally efficient. Second, in contrast to problem [**ETP-h0**], the equal-markup result does not hold for problem [**ETP-h1**]. Instead, the optimal prices have the monotonicity property, which is aligned with practice. Particularly, the prices of the sold-out products can be ordered based on their quality levels – higher quality products are priced higher. On the other hand, the partially sold products, although having non-zero sales that increase in quality, are still valued identically at the same price. Hence, the capacity constraints (**Cap**), which are commonly seen in practice, have a strong impact on the structure of the optimal solution, and do so in a manner that contrasts with insights that stem from unconstrained models.

5.3. Optimization with Capacity Constraints and Average Price Restriction ([ETP-h2])

Problem [**ETP-h2**] includes the average price restriction (**AveP**) in addition to the capacity constraints. To solve this model, we treat the demand vector as the decision variable and write the problem via the relationship described in (9).

$$\textbf{[ETP-h2]} \quad \max_{\boldsymbol{\alpha}} \left\{ \sum_j \alpha_j p_j(\boldsymbol{\alpha}) \mid \alpha_j \le \beta_j, \forall 1 \le j \le N; \sum_j p_j(\boldsymbol{\alpha}) \le N\bar{P} \right\}.$$

Note that the main difficulty introduced by the constraint (**AveP**) is that the price function $p_j(\boldsymbol{\alpha})$ is neither convex nor concave in $\boldsymbol{\alpha}$. However, we can show that the sum of all prices is quasi-convex in the demand $\boldsymbol{\alpha}$.[7] As such, both the feasible region and objective function of problem [**ETP-h2**] are convex, and we can solve for optimality via the KKT conditions. In fact, as Lemma 5 shows, the sold-out threshold structure holds in this case as well.

[6] In this section, we continue to use s_x $(x = 1, 2, 3)$ to denote the threshold for problem [**ETP-h**x], with the understanding that it now pertains to the case of horizontal products.

[7] Note that, even considering weighted average price, we can still show the same result; see proof in the Appendix.

LEMMA 5. *The KKT conditions are necessary and sufficient for the optimality of problem **[ETP-h2]**. Moreover, the optimal demand vector has a sold-out threshold structure; i.e., if product j is sold out, then product i with $i \leq j$ must also be sold out.*

The threshold-type optimal solution described in Lemma 5 not only facilitates the algorithm for the optimality search, but also helps us characterize the optimal prices and demands of the products. Although we cannot specify the threshold or present the optimal solution in closed form, our analysis uncovers several useful insights concerning the optimality structure.

PROPOSITION 5. *Let s_2 $(0 \leq s_2 \leq N)$ be the optimal sold-out threshold for problem **[ETP-h2]**. Then, $s_2 \geq s_1$. Moreover, unless the problem is equivalent to the unconstrained case, the optimal prices have the strict monotonicity property $p_1^* > p_2^* > \cdots > p_N^*$; and for the optimal demands, $\alpha_j^* = \beta_j$ for the sold-out products $j \leq s_2$ and, if $s_2 < N$, $\alpha_{s_2+1}^* > \cdots > \alpha_N^*$ for the remaining products.*

Proposition 5 reveals important structural results about the optimal solution to problem [**ETP-h2**]. First, similar to the vertical differentiation case, the thresholds for [**ETP-h1**] and [**ETP-h2**] are comparable and we have $s_1 \leq s_2$. Suppose that we search for the sold-out threshold in decreasing order from the all-sold-out scenario. Then, a lower sold-out threshold indicates higher prices for certain products and they may violate the average price restriction, at which time we terminate the search. If (**AveP**) is not violated even when the threshold is down to s_1, then the average price restriction is redundant and s_1 is the optimal threshold.

Second, unless the problem is equivalent to its unconstrained counterpart, there exists a monotonicity property such that higher quality products charge higher prices. Therefore, we show that the average price restriction can differentiate the pricing of the partially sold products as well, further breaking down the equal-markup structure observed in the unconstrained scenario. Additionally, while the sales of the sold-out products are known, the demands of the remaining products can be ordered such that the higher quality products attract more customers. Such a partial order among products' sales implies that the manager must prioritize and monitor the high quality partially sold products, as they can generate higher revenue. Although the sales of the sold-out products decrease in quality and the sales of partially sold products increase in quality, the two monotonic chains are not comparable, because either β_{s_2} or $\alpha_{s_2+1}^*$ could be larger. As such, in our setting, customers' substitution behavior is affected by both the product quality and the constraints. Our results therefore provide vital insights for the manager to better understand how the two factors jointly steer the demand share among ticket categories.

5.4. Optimization with All Three Constraint Classes ([ETP-h3])

Next, we consider the optimization problem with all of the constraints. As in Section 5.3, we write the problem using $\boldsymbol{\alpha}$ as the decision variable based on the one-to-one mapping (9):

$$\textbf{[ETP-h3]} \quad \max_{\boldsymbol{\alpha}} \left\{ \sum_j \alpha_j p_j(\boldsymbol{\alpha}) \,|\, \alpha_j \le \beta_j, \forall 1 \le j \le N; \sum_j p_j(\boldsymbol{\alpha}) \le N\bar{P}; p_N(\boldsymbol{\alpha}) \le U_N \right\}.$$

For the constraint (**CeilN**), we can write its equivalence as a linear inequality in $\boldsymbol{\alpha}$. Hence, just as in Lemma 5, the KKT conditions are still necessary and sufficient for problem [**ETP-h3**]. However, the specific KKT conditions in this case are not amenable to analysis, and we are unable to generate useful structural results. Therefore, we will focus on a special case where the constraint (**AveP**) is not binding. Consider the problem

$$\textbf{[ETP-h3']} \quad \max_{\boldsymbol{\alpha}} \left\{ \sum_j \alpha_j p_j(\boldsymbol{\alpha}) \,|\, \alpha_j \le \beta_j, \forall 1 \le j \le N; p_N(\boldsymbol{\alpha}) \le U_N \right\}.$$

Solving the above problem analytically may help us better understand the solution to [**ETP-h3**]. Without the average price restriction, problem [**ETP-h3'**] is a convex optimization problem. We can therefore solve it by the KKT condition and the solution actually has a closed form. Moreover, similar to the previous scenarios, the optimal sales of all ticket categories exhibit a sold-out threshold structure. Furthermore, when the constraint (**CeilN**) is binding, we only need to conduct a one-dimensional search for the optimal threshold from 0 to $N-1$. For a given threshold s, we give the optimal solution in the lemma below.

LEMMA 6. *Consider problem [**ETP-h3'**] and assume that the constraint (**CeilN**) is binding (otherwise the problem is reduced to [**ETP-h1**]). Then, its optimal solution has a sold-out threshold structure. Given the threshold $s = 0, 1, \cdots, N-1$, the sales/prices are as follows:*

(1) If $s = N-1$, then $\alpha_j^* = \beta_j$ for $j = 1, \cdots, N-1$ and $\alpha_N^* = \frac{1-\sum_{i=1}^{N-1}\beta_i}{1+e^{U_N-\theta q_N}}$.

(2) If $s = 1, \cdots, N-2$, then the optimal demand $\alpha_j^* = \beta_j$ for $j = 1, \cdots, s$; $\alpha_j^* = \alpha_N^* \exp(\omega + \theta(q_j - q_N))$ for $j = s+1, \cdots, N-1$; and $\alpha_N^* = \left[e^{(U_N-\theta q_N)}(U_n + \omega) + \omega\right]^{-1}$, where $\omega > 0$ is uniquely determined by the equation

$$(1-K(s))(e^{(U_N-\theta q_N)}(U_N+\omega)+\omega) = 1 + \sum_{j=s+1}^{N-1} e^{\theta(q_j-q_N)-\omega} + e^{(U_N-\theta q_N)}.$$

(3) If $s = 0$, then the optimal price $p_j^* = U_N + \omega$ for $j = 1, \cdots, N-1$ and $p_N^* = U_N$, where ω is determined by the above equation with $K(0) = 0$.

The above lemma gives the sales/prices of the products for any sold-out threshold s. However, we only focus on the feasible thresholds, which are the ones that correspond to feasible sales $\boldsymbol{\alpha}$.

Moreover, although we only provide either the sales or the prices in the lemma, note that we may easily derive the full optimal solution using the optimality condition as well as the one-to-one mapping (9) between the price vector and demand vector. For example, in case (2) above, since (**CeilN**) is binding, we have $p_N^* = U_N$. Then, for the sold-out products, subtracting (9) for product j and N, we have $p_j^* = U_N + \theta(q_j - q_N) + \log\alpha_N^* - \log\beta_j$ for $j = 1, \cdots, s$. For the remaining products (if any) $j = s, \cdots, N-1$, we can use optimality conditions to obtain $p_j^* = U_N + \omega$. In fact, we can perform similar operations in cases (1) and (3) to compute the full optimal solution. It is worth noting that the optimal prices of all products are influenced by the upper bound U_N; however, for sales, only the demands of the partially sold products depend on U_N. Hence, the ceiling constraint on the price of the lowest quality ticket category propagates across all the other ticket categories.

From part (2) of Lemma 6, an important factor in deciding the optimal solution to **[ETP-h3']** is ω, which is the multiplier of the constraint (**CeilN**) in the KKT conditions; i.e., $\omega(U_N - p_N) = 0$. As a result, all the partially sold products have the same demand structure in the optimal solution. We can express this optimal demand by the optimal demand of product N, the quality gap, and the multiplier ω. Lastly, we compare the optimal ticket price across categories in the next proposition.

Proposition 6. *Suppose that (**CeilN**) is binding. Problem **[ETP-h3']** can be optimally solved by searching the feasible threshold $s_3 \in \{0, 1, \cdots, N-1\}$. The optimal solution is given by Lemma 6. Moreover, we have $p_1^* > \cdots p_{s_3}^* > p_{s_3+1}^* = \cdots = p_{N-1}^* > p_N^*$.*

Although the impact of U_N ripples through the optimal prices of all products, its impact on the equal-markup property manifests only for the last product. Furthermore, the optimal prices have the monotonicity property, which is similar to what we observed for **[ETP-h1]** (see Proposition 4). Specifically, the prices first decrease with quality levels and then stay the same, but the lowest quality product has an even lower price. Additionally, we have $\alpha_{s_3+1} > \cdots > \alpha_{N-1}$, meaning that the lower quality products are generally sold less than the higher quality ones. Furthermore, this statement holds true even for the last product, i.e., $\alpha_{N-1} > \alpha_N$, if $\omega < \theta(q_{N-1} - q_N)$ at optimality.

6. Additional Considerations

The main structural theme in the price optimization in Sections 4 and 5 is that the optimal solution has a sold-out threshold structure. Although our analysis is based on the assumption that the number of ticket categories and the capacity in each category are fixed, the box office may have the flexibility to mildly adjust the capacity available to various ticket categories. Next, we examine these considerations and see how they could influence optimal prices. To systematically analyze the flexibility, we consider two scenarios. First, we consider the case where the box office either drops the lowest quality product or introduces a new highest quality product. Second, we study a capacity reallocation scenario, where the box office moves a small amount of capacity from a lower quality product to add to the capacity of a higher product. These flexible adjustments to the collection of ticket categories, although minor, may have a considerable impact on revenue. Since the two additional considerations are focused on capacity planning, we limit our analysis of these scenarios to the problems [**ETP-v1**] and [**ETP-h1**] that incorporate (**Cap**) alone, and examine how the additional considerations affect their respective optimal solutions.

6.1. Closing and Adding Ticket Categories

In the setting that motivated our work, the box office manager was able to make small changes to the collection of ticket categories to improve profitability. One scenario, called *closing*, considers the option of eliminating the lowest quality ticket category. Alternatively, a new ticket category with quality higher than all the existing ones could be created; we refer to this as *adding*. Thus, in case of closing, the ticket categories change from $j = 1, \cdots, N$ to $j = 1, \cdots, N-1$, and in case of adding, the ticket categories become $j = 0, 1, \cdots, N$, where product 0 is a new product[8] with quality $q_0 > q_1$ and capacity $\beta_0 < \beta_1$. Below, we examine the impact of these changes on problem [**ETP-v1**] and problem [**ETP-h1**].

Impact on the optimal solution of [ETP-v1]. When products are vertically differentiated, because the lower quality products may have no demand (see Proposition 1), *closing* the last product is likely to have no impact. However, adding a new product with the highest quality will, intuitively, impact the optimal solution. Proposition 7 below details our findings.

PROPOSITION 7. *Let s_1 be the sold-out threshold index before the change.*

(1) Closing: If $s_1 < N-1$ or $s_1 = N-1$ with $K(N-1) = \frac{1}{2}$, then closing has no effect on the optimal solution. If $s_1 > N-1$ or $s_1 = N-1$ with $K(N-1) < \frac{1}{2}$, then all products are sold out; moreover, their optimal prices and the maximized revenue will decrease after closing.

(2) Adding: When adding product 0 to the ticket categories, the optimal revenue will increase and the index of the optimal sold-out threshold will either stay the same or decrease by 1; if it stays the same, then the optimal prices of all existing products will (weakly) decrease.

Consistent with our intuition, Proposition 7 reveals that closing indeed has little impact on the pricing decision because product N almost always has no sales anyway. The exception occurs when there is limited capacity across all products and the sold-out threshold is high enough, in which case

[8] Note that using index 0 for the new product allows us to keep the same indexes for all the existing products.

closing product N would decrease revenue. In the case of adding product 0 to the ticket categories, we find that the new highest quality product will attract sales from the lower quality products and force their prices to decrease. As a result, product 0 becomes the main revenue generator across categories. Moreover, despite lower prices of the existing products, the total revenue strictly increases. Finally, depending on q_0, the quality of product 0, the previous sold-out product (s_1) may either still be sold out or have seats left, which could change the new threshold to $s_1 - 1$. It is worth noting that, in the expanded collection of ticket categories, the threshold index $s_1 - 1$ implies that the number of sold-out products is actually unchanged. That is, adding a new product with the highest quality may cause a previously sold-out product to have leftover capacity. Moreover, no product that was previously unsold would be impacted by adding.

Impact on the optimal solution of [ETP-h1]. When products are horizontally differentiated, the demand function is more involved, and comprehensive analytical results are difficult to obtain. One of the major challenges is that the optimal sold-out threshold may change arbitrarily. Therefore, the optimal prices and demands are not directly comparable. This is especially the case when adding a new product to the ticket categories. Hence, to obtain useful insights, we focus on the case where the threshold remains the same after the adjustments.

PROPOSITION 8. *Consider ticket category adjustments for problem **[ETP-h1]**. Suppose the sold-out threshold index, $s_1 < N$, remains the same before and after the change.*

(i) Closing: When closing product N, for the sold-out products, their optimal prices increase; but for the partially sold products, their optimal prices decrease and optimal demands increase. Moreover, the optimal revenue associated with each product $j = 1, \cdots, N-1$ becomes higher.

(ii) Adding: When adding product 0 to the ticket categories, the optimal demands for each of the partially sold products ($j = s_1 + 1, \cdots, N$) will decrease.

In the horizontal differentiation case, the customers who were supposed to purchase product N will switch to other products after closing, which is in contrast to the vertical case. Such a substitution effect explains the findings in Proposition 8(i). In particular, with fewer ticket categories, one would expect the high-quality products to be priced higher (given that they are already sold out) and the low-quality products to be priced lower to generate more sales. Moreover, due to substitution, the revenue for each product will increase. However, because there is one less ticket category, the total revenue may either increase or decrease. The case of adding a new product is much more complicated analytically. The revenue and product prices are difficult to compare before and after adding. Nevertheless, we have the following result regarding the optimal demands of the products that are not sold out. According to the closed-form solution (10), the demand of product $j = s_1 + 1, \cdots, N$ decreases in K, the total capacity of all sold-out products. Therefore, with the sold-out threshold unchanged and the increase of total capacity by β_0, there will be fewer sales for the partially sold products. This result shares a similarity with the analogous result for the vertical case – the new product would take sales away from the low-quality products.

6.2. Reallocating Capacities Between Ticket Categories

Although capacity planning is a relatively long-term decision, the box office, when selling tickets, may have the flexibility to convert some seats from one ticket category into another, seeking to expand the capacity of the most profitable tickets. In practice, for instance, the seat section in a stadium may be rearranged between physically adjacent areas. Since the highest quality product potentially generates the most revenue, we focus on the capacity reallocation from product $r > 1$ to product 1. Moreover, given the sold-out threshold of the optimal solution, we consider two kinds of reallocation, namely, reallocating the excessive supply from a partially sold category, or converting tickets from a sold-out category. Furthermore, we assume that the amount of reallocated capacity is relatively small and, as a result, the sold-out threshold always stays the same. In the following, we will again separately study problem **[ETP-v1]** and problem **[ETP-h1]**.

Impact on the optimal solution of [ETP-v1]. For the vertical differentiation case, we consider the capacity reallocation of amount $\epsilon > 0$ from product r to product 1. Our objective is to examine how the reallocation influences the optimal sales, prices, and total revenue. The next proposition summarizes our findings.

PROPOSITION 9. *Consider problem **[ETP-v1]** and assume that the capacity of $\epsilon > 0$ is removed from product $r > 1$ and reallocated to product 1. Furthermore, assume that the optimal sold-out threshold s_1 is unchanged and satisfies $1 \le s_1 \le N-1$. Then we have the following:*

(i) Suppose $r > s_1$. All optimal demands remain the same except product 1, whose demand increases by ϵ. All optimal prices remain the same except products $j = 1, \cdots, s_1$, whose prices decrease. The optimal total revenue increases.

(ii) Suppose $1 < r \le s_1$. The optimal demand of product 1 increases by ϵ and that of product r decreases by ϵ; all others remain the same. All optimal prices remain the same except products $j = 1, \cdots, r-1$, whose prices decrease. The optimal total revenue increases.

Since the optimal sold-out threshold is unchanged and the capacity reallocation is relatively small, the optimal sales are affected in a straightforward way. That is, all products have the same sales except the affected products, 1 and r. The impact on the optimal prices is more interesting.

Specifically, all the sold-out products or the high-quality products ($j < r$) have a lower price, while the prices of the rest are the same. This result is due to the fact that the price of a sold-out product is negatively correlated with the aggregate capacity of products that have higher quality. For this reason, we can see that, in Proposition 9(ii), even though products $j = r, \cdots, s_1$ are sold-out, their prices will remain the same as the corresponding aggregate capacity is the same. Finally, regardless of the source of the capacity reallocation, whether from the excessive supply or from the category that could be sold out, the optimal total revenue will increase. In fact, we observe that all products, except product 1, have revenue that is lower or unaffected. Product 1's revenue, in contrast, increases. Therefore, the highest quality product is really at the core of the ticket categories and its revenue-generating power prevails across all ticket categories.

Impact on the optimal solution of [ETP-h1]. For the horizontal differentiation case, we conduct a similar study on the impact of capacity reallocation. The results are different from the vertical case. However, as the following proposition shows, adding more capacity to the highest quality category is indeed beneficial to the total revenue, although it impacts the demands and prices of all products.

PROPOSITION 10. *Consider problem **[ETP-h1]** and assume that the capacity of $\epsilon > 0$ is removed from product $r > 1$ and reallocated to product 1. Furthermore, assume that the optimal sold-out threshold s_1 is unchanged and satisfies $1 \le s_1 \le N - 1$. Then we have the following:*

(i) Suppose $r > s_1$. The optimal demand of product 1 increases by ϵ, the optimal demand of product $j = s_1 + 1, \cdots, N$ decreases, and the no-purchase probability α^*_{N+1} increases; all other demands are the same. The optimal price of the product 1 decreases whereas those of all other products increase. The optimal total revenue increases.

(ii) Suppose $1 < r \le s_1$. The optimal demand of product 1 increases by ϵ and that of product r decreases by ϵ. The optimal price of product 1 decreases and that of product r increases. The demands and prices of all other products are the same. The optimal total revenue increases.

In Proposition 10(i), some excessive supply from the low-quality partially sold product is reallocated to product 1. While this increases the sales of product 1, it decreases the sales of all the partially sold products. Moreover, the total sales reduction of the partially sold products exceeds ϵ because the no-purchase probability is larger and the market coverage becomes smaller. As for the prices, the increased demand for product 1 can lead to a price reduction for itself. All remaining products have higher optimal prices. As a result, the optimal total revenue becomes larger, which is consistent with intuition.

Proposition 10(ii) depicts the case where the ϵ capacity of a category that could be sold out is now reallocated to product 1. Such a *conversion* between sold-out products can lead to a simple characterization of the impact on the optimality outcomes. Specifically, only products 1 and r are affected by the capacity reallocation. Product 1 will have higher sales but a lower price, whereas product r will have lower sales but a higher price. The net effect in terms of revenue, however, favors the higher-quality product. As can be seen from the proof, the revenue increase, in this case, is approximately $\epsilon\theta(q_1 - q_r) + \log\frac{\beta_r}{\beta_1}$ when ϵ is small. Therefore, the revenue increase is linear in the reallocated amount of the capacity, and the increase is higher if the quality gap is larger. We note that this result can be generalized to the case where a small amount of product i is reallocated to product j, where $j < i \leq s_1$.

7. Concluding Remarks

Motivated by an application in the live entertainment industry, we study a multi-product pricing problem under two types of product differentiation: vertical and horizontal. Different from existing research and stemming from vital practical requirements in this context, we consider capacity constraints and price restrictions in our problem. The capacity constraints reflect the fact that the supply (e.g., seats in a venue) is limited. Price restrictions set a ceiling on the average price across all products and the price of the product with the lowest quality. Such price restrictions can enhance customer satisfaction and further ensure affordable access for customers on a limited budget. Incorporating these practical constraints in a revenue optimization problem is one of our major contributions. Not only are these considerations vital from a practical standpoint, but they also lead to optimal solutions and structures that are significantly different from their unconstrained counterparts. As such, well-known insights from the unconstrained price optimization context do not readily apply in this setting. Our analysis reveals exact characterizations of optimal policies under vertical and horizontal product differentiation, permitting managers to understand the impact of each constraint under different demand models.

In the vertical products case, we prove that the constraints give rise to a convex region if and only if the consumers' valuation towards quality forms a uniform distribution. Under the uniform distribution assumption, we find that the optimal solution has a sold-out threshold structure under the capacity constraints and the average price restriction. At optimality, only some high quality products are sold out and the optimal sales for the majority of the partially sold products are identical. With the price upper bound on the lowest quality product, we find that, in addition to the sold-out structure of the highest quality products, there exists another threshold where products with lower quality are also sold out. In the horizontally-differentiated products case, we prove that the sold-out threshold structure is still optimal. One important finding is that the equal-markup result, which is well-known in literature, does not hold in the constrained problem. In fact, with capacity constraints only, the prices of the sold-out products are decreasing in their quality while the remaining products have the same price. When the average price constraint is considered, the prices of all products are decreasing in quality. As such, our findings show the significant impact of the practical constraints on the price optimization problem.

Overall, our paper makes the following two major contributions. From the modeling perspective, we build a robust model to handle different consumer choice contexts, which are commonly seen in many practical settings beyond the live entertainment industry. In doing so, we construct an architecture for researchers to study and optimize constrained revenue management problems. Analytically, we are able to characterize the optimal solution despite the complexity of the optimization problem. The uncovered structural properties isolate the impact of the practical considerations individually and collectively to better inform managerial decision-making.

To conclude, we discuss three interesting future directions that may address the limitations of our model. First, since our constrained optimization stops at finding the prices for a given collection of product types, we can extend the optimization over the collection/assortment. In this regard, our analysis in Section 6, which looks at the box office's flexible management of capacity across ticket categories, might be helpful. Second, considering possible stock-out substitution will add significant analytical complexity to the ETP problem, but is nevertheless worthy of study. We hope future works along this research direction may benefit from the results in this paper. Third, although our demand model in the horizontal differentiation case can support empirical research that uses MNL model (e.g., Arslan et al. 2022), future research could address more general logit models while studying the impact of the practical constraints.

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Appendix A: Proofs of Statements

For convenience, we use α to represent α^V in the paper in the following proofs for the vertical case.

Proof of Lemma 1. To prove the objective function is concave and the demand function is convex in $\mathbf{p}$, it suffices to prove the Hessian matrix of revenue function $R(\mathbf{p})$ is negative semi-definite and the Hessian matrix of demand function $\alpha_j(\mathbf{p})$ $(j=1,2,\cdots,N)$ is positive semi-definite.

Therefore, first, consider the Hessian matrix of revenue function $R(\mathbf{p})$, which is a symmetric diagonal matrix as follows:

$$\Delta R(\mathbf{p}) = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & 0 & \cdots & 0 \\ 0 & b_2 & a_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & \cdots & 0 & b_{n-1} & a_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -b_n \end{bmatrix} + \sum_{i=1}^{n-1} \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -b_i & b_i & \cdots & 0 \\ 0 & \cdots & b_i & -b_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

where we define a, b and Z as

$$\begin{cases} a_j := \frac{\partial^2 R}{\partial p_j^2} = -\left(\frac{F''(Z_{j-1})Z_{j-1}}{q_{j-1}-q_j} + \frac{F''(Z_j)Z_j}{q_j-q_{j+1}} + \frac{2F'(Z_{j-1})}{q_{j-1}-q_j} + \frac{2F'(Z_j)}{q_j-q_{j+1}}\right),\ j=2,\cdots,N-1; \\ a_1 := -\left(\frac{F''(Z_1)Z_1}{q_1-q_2} + \frac{2F'(Z_1)}{q_1-q_2}\right),\ a_N := -\left(\frac{F''(Z_{N-1})Z_{N-1}}{q_{N-1}-q_N} + \frac{F''(Z_N)Z_N}{q_N} + \frac{2F'(Z_{N-1})}{q_{N-1}-q_N} + \frac{2F'(Z_N)}{q_N}\right); \\ b_j := \frac{\partial^2 R}{\partial p_j \partial p_{j+1}} = \frac{F''(Z_j)Z_j}{q_j-q_{j+1}} + \frac{2F'(Z_j)}{q_j-q_{j+1}},\ j=1,2,\cdots,N-1; \\ b_N := \frac{F''(Z_N)Z_N}{q_N} + \frac{2F'(Z_N)}{q_N}; \\ z_j := \frac{p_j-p_{j+1}}{q_j-q_{j+1}},\ j=1,2,\cdots,N-1;\ z_N := \frac{p_N}{q_N}. \end{cases}$$

The second equation is because $a_1 = -b_1, a_j = -(b_j + b_{j+1})$, $j = 2, ..., N$. For any $X \in R^{1\times N}$, if the equation (4) satisfies, then we get b_j $(j=1,2,\cdots,N)$ are non-negative, which implies $XRX^T = -\sum_{j=1}^{N-1} b_j(x_j - x_{j+1})^2 - b_N x_N^2 \leq 0$ will always hold. Thus, $R(\mathbf{p})$ is concave in $\mathbf{p}$.

Second, we prove that for any $j=1,\cdots,N$, the Hessian matrix of $\alpha_j(\mathbf{p})$ is semi-definite by contradiction. For notation simplicity, we set $Q_j := q_j - q_{j+1}$ for $j=1,\cdots,N-1$ and $Q_N := q_N$. Then, we have the Hessian matrix as follows:

$$\Delta\alpha_j(\mathbf{p}) = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & A_j & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix},\ \text{where } A_j = \begin{bmatrix} 0 & -F''(z_{j-1})Q_{j-1}^2 & 0 \\ -F''(z_{j-1})Q_{j-1}^2 & F''(z_{j-1})Q_{j-1}^2 - F''(z_j)Q_j^2 & F''(z_j)Q_j^2 \\ 0 & F''(z_j)Q_j^2 & 0 \end{bmatrix} \quad \forall j > 1.$$

Based on the above equation, it suffices to prove A_j is the positive semi-definite. Let $X \in R^n$ and check the quadratic form

$$g_j := X\Delta\alpha_j(\mathbf{p})X^T = x_{j-1}x_{j+1}\left[F''(z_{j-1})Q_{j-1}^2\left(\frac{x_j^2}{x_{j-1}x_{j+1}} - 2\frac{x_j}{x_{j+1}}\right) - F''(z_j)Q_j^2\left(\frac{x_j^2}{x_{j-1}x_{j+1}} - 2\frac{x_j}{x_{j-1}}\right)\right], \forall j > 1.$$

We claim that only if the $F''(z_j)Q_j^2 = 0$ for all $j = 1, 2..., N$, $g_j \geq 0$ will always hold. Otherwise, we can always find the counterexamples of x as follows. Assume that there exists a j such that $F''(z_j)Q_j^2 \neq 0$:

- if $F''(z_j)Q_j^2 < F''(z_{j-1})Q_{j-1}^2$, let $x_{j-1} = x_{j+1} = x_j \neq 0$, then, we have $g_j = x_{j-1}x_{j+1}(-F''(z_{j-1})Q_{j-1}^2 + F''(z_j)Q_j^2) < 0$;
- if $F''(z_j)Q_j^2 > F''(z_{j-1})Q_{j-1}^2$, let $x_{j-1} = x_{j+1} = -x_j \neq 0$, then, we have $g_j = 3x_{j-1}x_{j+1}(F''(z_{j-1})Q_{j-1}^2 - F''(z_j)Q_j^2) < 0$;

- if $F''(z_j)Q_j^2 = F''(z_{j-1})Q_{j-1}^2$, let $0 < x_{j+1} < x_{j-1} = x_j$, then, we have $g_j = 2x_{j-1}x_{j+1}F''(z_{j-1})Q_{j-1}^2(\frac{x_j}{x_{j-1}} - \frac{x_j}{x_{j+1}}) < 0$.

Therefore, $F''(z_j)Q_j^2 = 0$ for all $j = 1, 2..., N$ when the demand function $\alpha_j(\mathbf{p})$ is convex, which means the quality sensitivity θ is uniformly distributed over $[0,1]$. Q.E.D.

Proof of Proposition 1. Based on the proposition structure, we prove the optimal outcomes in three cases one by one.

Case 1: Consider the $K(1) \geq \frac{1}{2}$, which is equivalent to $\beta_1 \geq \frac{1}{2}$. It is trivial that the optimal solution will be the same as the unconstrained one.

Next, to prove the main result in Case 3, the key idea is to exclude all the other possible solutions since they will cause contradiction.

Case 3: If $K(1) < \frac{1}{2} < K(N)$, we have $\beta_1 < \frac{1}{2} < \sum_{j=1}^{N} \beta_j$. We denote the optimal solution $\mathbf{p}^* = (p_1^*, p_2^*, ..., p_N^*)$, $\boldsymbol{\alpha}^* = (\alpha_1^*, \alpha_2^*, ..., \alpha_N^*)$. The proof will be finished in two steps: first, we prove that $\alpha_1^* = \beta_1$ by contradiction method; after that, we can transform the original problem to a new $N-1$ class ticket pricing problem. By repeating the first step s times, we will find the sold-out threshold and determine α_{s+1}^*. For the first step, assume that $\alpha_1^* < \beta_1$, there are two possible sub-cases: $\alpha_2^* > 0$ or $\alpha_2^* = 0$. By contradictions, we can prove that neither sub-case holds.

Sub-case 1: Consider $\alpha_2^* > 0$, we let $\epsilon_1 = \min\{\beta_1 - \alpha_1^*, \alpha_2^*\}$ and define new solution p' as following:

$$p_1' = p_1^* - \epsilon_1(q_1 - q_2),\ p_j' = p_j^*,\ j = 2, 3, \cdots, N. \tag{12}$$

It is easy to prove that

$$\begin{aligned}
p_1' =& p_1^* - \epsilon_1(q_1 - q_2) > p_1^* - \alpha_2^*(q_1 - q_2) \\
=& p_1^* - (\frac{p_1^* - p_2^*}{q_1 - q_2} - \frac{p_2^* - p_3^*}{q_2 - q_3})(q_1 - q_2) \\
=& p_2^* + \frac{p_2^* - p_3^*}{q_2 - q_3}(q_1 - q_2) > 0
\end{aligned}$$

and

$$0 \leq 1 - \frac{p_1' - p_2^*}{q_1 - q_2} = 1 - \frac{p_1^* - p_2^*}{q_1 - q_2} + \epsilon_1 \leq \beta_1,$$

which shows that the new solution p' is well-defined. Check the corresponding demand α', we have

$$\begin{cases}
\alpha_1' = \alpha_1^* + \epsilon_1 \leq \beta_1, \\
\alpha_2' = \frac{p_1^* - p_2^*}{q_1 - q_2} - \epsilon_1 - \frac{p_2^* - p_3^*}{q_2 - q_3} = \alpha_2^* - \epsilon_1 \geq 0, \\
\alpha_j' = \frac{p_{j-1}^* - p_j^*}{q_{j-1} - q_j} - \frac{p_j^* - p_{j+1}^*}{q_j - q_{j+1}} = \alpha_j^*,\ 2 < j \leq N.
\end{cases}$$

Then, we can prove that the revenue under the new solution p' is larger than the p's:

$$\begin{aligned}
\sum_{j=1}^{N} \alpha_j' p_j' - \sum_{j=1}^{N} \alpha_j^* p_j^* =& \alpha_1' p_1' + \alpha_2' p_2' - \alpha_1^* p_1^* - \alpha_2^* p_2^* \\
=& (\alpha_1^* + \epsilon_1)(p_1^* - \epsilon_1(q_1 - q_2)) + (\alpha_2^* - \epsilon_1)p_2^* - \alpha_1^* p_1^* - \alpha_2^* p_2^* \\
=& (p_1^* - p_2^*)\epsilon_1 - \epsilon_1(q_1 - q_2)(\alpha_1^* + \epsilon_1) \\
=& \epsilon_1(1 - \alpha_1^*)(q_1 - q_2) - \epsilon_1(q_1 - q_2)(\alpha_1^* + \epsilon_1^*)
\end{aligned}$$

$$
\begin{aligned}
&=\epsilon_1(1-2\alpha_1^*-\epsilon_1)(q_1-q_2)\\
&>\epsilon_1(1-\alpha_1^*-\beta_1)(q_1-q_2).
\end{aligned}
$$

The last inequality is because $1-\epsilon_1>1-\beta_1+\alpha_1^*$. Since $\alpha_1<\beta_1<\frac{1}{2}$, we have $1-\beta_1+\alpha_1^*>0$ so that $\sum_{j=1}^N\alpha_j'p_j'-\sum_{j=1}^N\alpha_j^*p_j^*>0$, which contradicts with optimality assumption. Therefore, Sub-case 1 can be excluded.

Sub-case 2: First, we prove that α^* is not optimal when $\alpha_2^*=0$. Note that if $\alpha_3^*=0$, we can move out product 3 from the set since it has 0 sales, and product 4 can be treated as new product "3". If we repeat this method and can't find a positive sale, we have all left products $\alpha_j^*=0$, $\alpha_1^*=\beta_1<\frac{1}{2}$, where the optimal revenue is $R^*=\beta_1(1-\beta_1)q_1$, respectively. However, we can prove this case doesn't exist by a counterexample: let $\alpha_1=\beta_1$, $\alpha_2=\min\{\frac{1}{2}-\beta_1,\beta_2\}$ and $\alpha_j=0$ for all $j=3,\cdots,N$, we have the revenue function as

$$
R=\begin{cases}\beta_1(1-\beta_1)q_1+\frac{q_2}{2}(\frac{1}{2}-\beta_1),\ \beta_1+\beta_2\geq\frac{1}{2};\\ \beta_1(1-\beta_1)q_1+\beta_2\left(\frac{1}{2}-\beta_1+(\frac{1}{2}-\beta_1-\beta_2)\right)q_2,\ \beta_1+\beta_2\leq\frac{1}{2}.\end{cases}
$$

Because $R>R^*=\beta_1(1-\beta_1)q_1$, the new solution generates higher revenue, which is in contradiction to the optimality. Therefore, we assume a new product n $(n>2)$ with positive sales exists in this logic. By $\alpha_j^*=0$ $(j=2,3,\cdots,n-1)$, we have

$$
\frac{p_1^*-p_2^*}{q_1-q_2}=\frac{p_2^*-p_3^*}{q_2-q_3}=\cdots=\frac{p_{n-1}^*-p_n^*}{q_{n-1}-q_n}\Longrightarrow\frac{p_1^*-p_2^*}{q_1-q_2}=\frac{\sum_{j=1}^{n-1}(p_j^*-p_{j+1}^*)}{\sum_{j=1}^{n-1}(q_j-q_{j+1})}=\frac{p_1^*-p_n^*}{q_1-q_n}.
$$

Note that these products j $(1<j<n)$ will not generate any revenue due to $\alpha_j^*=0$. Similarly, we can also move them out from the ticket category without changing the demand structure and value of the optimal revenue. So the optimal demand and price for the $N-n+1$ products will be $(\alpha_1^*,\alpha_n^*,\cdots,\alpha_N^*)$ and $(p_1^*,p_n^*,\cdots,p_N^*)$, where $\alpha_1^*=1-\frac{p_1^*-p_n^*}{q_1-q_n}<\beta_1$, $\alpha_n^*=\frac{p_1^*-p_n^*}{q_1-q_n}-\frac{p_n^*-p_{n+1}^*}{q_n-q_{n+1}}>0$. By the Sub-case 1, we know $(\alpha_1^*,\alpha_n^*,\cdots,\alpha_N^*)$ is not optimal for $N-n+1$ product cases, consequently, $(\alpha_1^*,0,\cdots,0,\alpha_n^*,\cdots,\alpha_N^*)$ will not be the optimal demands under N product cases as well.

In summary, we prove that $\alpha_1=\beta_1$ when $K(1)<\frac{1}{2}<K(N)$.

Next, to finish the second step, we do the following transformation on the objective function:

$$
\begin{aligned}
\sum_{j=1}^N\alpha_jp_j&=\beta_1p_1+\alpha_2p_2+\sum_{j=3}^N\alpha_jp_j\\
&=\beta_1p_1+(1-\beta_1-\frac{p_2-p_3}{q_2-q_3})p_2+\sum_{j=3}^N\alpha_jp_j\\
&=\beta_1(p_1-p_2)+(1-\frac{p_2-p_3}{q_2-q_3})p_2+\sum_{j=3}^N\alpha_jp_j\\
&=\beta_1(1-\beta_1)(q_1-q_2)+(1-\frac{p_2-p_3}{q_2-q_3})p_2+\sum_{j=3}^N\alpha_jp_j.
\end{aligned}
$$

If we consider the maximum revenue R_{N-1} of the class set $\{2,3,...,N\}$, the demand of product 2 will be $1-\frac{p_2-p_3}{q_2-q_3}$, which is exactly the vertical demand function when the highest quality is q_2. However, note that to keep the equivalence with the original problem, the capacity limit for new product '2' will be $\beta_2'=\beta_1+\beta_2$. Under the new capacity constraints, the constrained maximization problem of N class tickets is equivalent to the constrained maximization problem of $N-1$ class tickets with new capacity limits. Now consider the $N-1$ product pricing problem, we get the two possible scenarios: $\beta_2' = \beta_1 + \beta_2 \geq \frac{1}{2}$ and $\beta_2' = \beta_1 + \beta_2 < \frac{1}{2}$, which can either be solved by Case 1 (unconstrained optimization problem) or the first step in the Proof of Case 3, respectively. Then, by repeating this procedure until finding an s such that $\beta_{s+1}' = \sum_{j=1}^{s+1} \beta_i \geq \frac{1}{2}$. Same with the Case 1, we apply the F.O.C of $R = \sum_{j=1}^{N} \alpha_j p_j$ in terms of p_j, $j = s, s+1, \cdots, N$:

$$1 - \frac{2(p_s^* - p_{s+1}^*)}{q_s - q_{s+1}} = 0 \Longrightarrow \frac{p_s^* - p_{s+1}^*}{q_s - q_{s+1}} = \frac{1}{2},$$

and

$$\frac{p_j^* - p_{j+1}^*}{q_j - q_{j+1}} = \frac{p_N^*}{q_N} = \frac{1}{2}, \; j = s+1, s+2, ...N-1,$$

which equivalently shows that $\alpha_i^* = 0$ for $s+1 < j \leq N$, $\alpha_{s+1}^* = \frac{1}{2} - \sum_{j=1}^{s} \beta_j$ and $\alpha_{N+1}^* = \frac{1}{2}$. In summary, we have $\alpha_j^* = \beta_j$ for $j = 1, 2, \cdots s$, $\alpha_{s+1}^* = \frac{1}{2} - \sum_{j=1}^{s} \beta_j$ and $\alpha_{N+1} = \frac{1}{2}$ in Case 3. Notice that the threshold s_1 we defined in the proposition is equivalent to $s-1$ here.

Case 2: If $K(s) < \frac{1}{2}$, we can't find such j that $\beta_j' > \frac{1}{2}$ to implement the procedure in Case 3, so we have $\alpha_j = \beta_j$, $j = 1, 2 \cdots, N$. Q.E.D.

Proof of Lemma 2. First, we will prove the threshold structure at optimality. The idea is to adopt the proof in Proposition 1. We assume $\mathbf{p}$ is the optimal solution for **[ETP-v2]**, then, we will show that the feasible solution (12) in **[ETP-v1]** will also be feasible in **[ETP-v2]**. The feasibility of (12) is trivial since

$$p_1' = p_1 - \epsilon_1(q_1 - q_2) < p_1, \; p_j' = p_j, j = 2, 3, \cdots, N,$$

which suggests that p' is still feasible under (**AveP**). Therefore, all the logic in the proof of Prop 1 still holds in **[ETP-v2]**. So we conclude that the optimal structure of threshold will be the same with what has been proved in Prop 1.

Now, we get the optimal demands $\alpha_j^* = \beta_j$ for $j \leq s$ in **[ETP-v2]**. So we have

$$\beta_1 = 1 - \frac{p_1^* - p_2^*}{q_1 - q_2}, \; \beta_j = \frac{p_{j-1}^* - p_j^*}{q_{j-1} - q_j} - \frac{p_j^* - p_{j+1}^*}{q_j - q_{j+1}}, j = 2, 3, \cdots, N.$$

Here we set $p_{N+1}^* = q_{N+1} = 0$ for convenience. Iterating the equations, for any $j \leq s$,

$$1 - \sum_{i=1}^{j} \beta_i = \frac{p_j^* - p_{j+1}^*}{q_j - q_{j+1}} \Longrightarrow p_j^* = p_{s+1}^* + \sum_{n=j}^{s} (1 - \sum_{i=1}^{n} \beta_i) Q_n.$$

Substitute $K(n) = \sum_{i=1}^{n} \beta_i$, we have

$$p_j^* = p_{s+1}^* + \sum_{n=j}^{s} (1 - K(n)) Q_n, \quad j = 1, \cdots, s. \quad Q.E.D.$$

Proof of Lemma 3. Consider $0 \leq s \leq N-1$, since the **[ETP-v2(s_2)]** is convex, we can apply the KKT condition. Let

$$L = \sum_{n=1}^{N} \alpha_n(p) p_n - \lambda(s p_{s+1} + \sum_{n=s+1}^{N} p_n - \bar{P}').$$

We have the following conditions:

$$\frac{\partial L}{\partial p_j} = 0 = \begin{cases} 1 - 2\frac{p_j - p_{j+1}}{q_j - q_{j+1}} - j\lambda, \; j = s+1, \\ 2\frac{p_{j-1} - p_j}{q_{j-1} - q_j} - 2\frac{p_j - p_{j+1}}{q_j - q_{j+1}} - \lambda, \; s+2 \le j \le N, \end{cases}$$

where we set $p_{N+1} = q_{N+1} = 0$ for convenience. Then, we have

$$\frac{p_j - p_{j+1}}{q_j - q_{j+1}} = \frac{1 - j\lambda}{2}, \; s+1 \le j \le N.$$

Substitute the above equations into the price constraint:

$$\begin{aligned} sp_{s+1} + \sum_{n=s+1}^{N} p_n &= \sum_{n=s+1}^{N} n(p_n - p_{n+1}) \\ &= \sum_{n=s+1}^{N} n(q_n - q_{n+1}) \frac{p_n - p_{n+1}}{q_n - q_{n+1}} \\ &= \sum_{n=s+1}^{N} n(q_n - q_{n+1}) \frac{1 - n\lambda}{2} \\ &= \sum_{n=s+1}^{N} Q_n \frac{n - n^2\lambda}{2} = N\bar{P}'. \end{aligned}$$

Therefore, if $\frac{\sum_{n=s+1}^{N} nQ_n - 2N\bar{P}'}{\sum_{n=s+1}^{N} n^2 Q_n} > 0$ (or $\sum_{n=s+1}^{N} nQ_n - 2N\bar{P}' > 0$, equivalently), we have the Lagrange multiplier

$$\lambda = \frac{\sum_{n=s+1}^{N} nQ_n - 2N\bar{P}'}{\sum_{n=s+1}^{N} n^2 Q_n} > 0.$$

Otherwise, $\lambda = 0$ and the average price constraint is redundant. Now we substitute the λ into p_j^*

$$\begin{aligned} p_j^* &= \sum_{n=j}^{N} Q_n \frac{p_n^* - p_{n+1}^*}{Q_n} \\ &= \sum_{n=j}^{N} Q_n \frac{1 - n\lambda}{2} \\ &= \frac{q_j}{2} - \frac{\lambda}{2} \sum_{n=j}^{N} nQ_n, \; j = s+1, \cdots, N. \end{aligned}$$

Given $\alpha_j^* = \beta_j, j = 1, 2, \cdots, s$, we substitute the optimal prices into the demand functions:

$$\begin{cases} \alpha_j^* = \frac{p_j - p_{j+1}}{q_j - q_{j+1}} - \frac{p_j - p_{j+1}}{q_j - q_{j+1}} = \frac{\lambda}{2}, j = s+2, \cdots, N-1; \\ \alpha_{s+1}^* = 1 - \sum_{j=s+2}^{N+1} \alpha_j^* - \sum_{j=1}^{s} \beta_j = \frac{1}{2} - K(s) + \frac{(s+1)\lambda}{2}. \end{cases}$$

Note that if we change the simple average constraint (**AveP**) into the weighted average, Proposition 1 and Lemmas 1&2 still hold; i.e., the structural results do not change. Moreover, the problem **[ETP-v2]** continues to be convex and the above proof can be carried through without any difficulty. As a result, the above solution α_j^* is only mildly affected; specifically, assuming the weights are w_j $(j = 1, \cdots, N)$, we get similar optimal outcomes ($\hat{\lambda}$, $\hat{\mathbf{p}}^*$, $\hat{\boldsymbol{\alpha}}^*$) as below:

$$\hat{\lambda} = \frac{\sum_{n=s+1}^{N} Q_n \sum_{i=1}^{n} w_i - 2\hat{\bar{P}}'}{\sum_{n=s+1}^{N} (\sum_{i=1}^{n} w_i)^2 Q_n}, \; \hat{\bar{P}}' = \bar{P} - \sum_{j=1}^{s} w_j \sum_{n=j}^{s} (1 - K(s)) Q_n$$

$$\hat{p}_j^* = \frac{q_j}{2} - \frac{\hat{\lambda}}{2} \sum_{n=j}^{N} Q_n \sum_{i=1}^{n} w_i, \; j = s+1, \cdots, N.$$

and

$$\begin{cases} \hat{\alpha}_j^* = \frac{w_i \lambda_w}{2}, j = s+2, \cdots, N-1; \\ \hat{\alpha}_{s+1}^* = \frac{1}{2} - K(s) + \frac{\sum_{i=1}^{s+1} w_i \hat{\lambda}}{2}. \quad Q.E.D. \end{cases}$$

Proof of Proposition 2. Here, to better understand how the average price constraint affects the optimal solution under capacity constraints, we change the decision variables to $\boldsymbol{\alpha}$ instead of $\mathbf{p}$. Reminding that $q_N = Q_N$, $Q_n = q_n - q_{n+1}, 1 \le n < N$, it is easy to get

$$\begin{aligned} p_N(\boldsymbol{\alpha}) =& (1 - \sum_{n=1}^{N} \alpha_n) Q_N = \alpha_{N+1} Q_N, \\ p_j(\boldsymbol{\alpha}) =& (1 - \sum_{n=1}^{j} \alpha_n) Q_j + p_{j+1} = \sum_{n=j}^{N} Q_n \sum_{k=n+1}^{N+1} \alpha_k. \end{aligned}$$

Furthermore, we have

$$\sum_{j=1}^{N} p_j(\alpha) = \sum_{j=1}^{N} \left((1 - \sum_{n=1}^{j} \alpha_n) Q_j + p_{j+1} \right) = \sum_{j=1}^{N} \sum_{n=j}^{N} Q_n \sum_{k=n+1}^{N+1} \alpha_k = \sum_{j=2}^{N+1} \omega_j \alpha_j,$$

where $\omega_{i+1} = \sum_{j=1}^{i} \sum_{n=j}^{N} Q_n$, $i = 1, 2, \cdots, N$. We see that the summation of price is a linear combination of $\boldsymbol{\alpha}$ with the weights ω determined by $Q_j, j = 1, 2, ..., N$. It is easy to find that the weight w of each $(\alpha_2, \cdots, \alpha_{N+1})$ is in increasing order: $w_2 \le w_3 \le \cdots \le w_{N+1}$. Now, we can redeem the summation of the prices as the weighted average of $(\omega_2, \omega_3, \cdots, \omega_{N+1})$. In such a sense, we can clearly see how the demand $\boldsymbol{\alpha}$ affects the price $\mathbf{p}$.

The first case is trivial since $T(N)$ is the total price of the all sold out solution $\boldsymbol{\alpha}^{(N)} = (\beta_1, \beta_2, \cdots, \beta_N)$, which is also the minimum. Therefore, if $T(N) > N\bar{P}$, for any other solution $\boldsymbol{\alpha}'$, we know that $\alpha'_{N+1} > 1 - \sum_{j=1}^{N} \beta_j$, so the

$$\sum_{j=1}^{N} p_j(\boldsymbol{\alpha}') > T(N) > N\bar{P},$$

which means there won't be a feasible solution.

For the second case, when $T(N-1) > N\bar{P} \ge T(N)$, we see that at least $N-1$ products are sold out. For product N, if $T(N) = N\bar{P}$, we know that $\alpha_j^* = \beta_j$ for all $j = 1, 2, \cdots, N$ by the definition of $T(N)$. But if $T(N-1) > N\bar{P}$, product N is not sold out and we can adopt Lemma 3 to solve the optimization problem: the threshold $s+1 = N$ so that

$$\lambda = \frac{Q_N - 2\bar{P}'}{NQ_N}, \ \alpha_N^* = 1 - K(N) - \frac{\bar{P}'}{Q_N}.$$

For the last case, by Lemma 2, we know that the threshold structure still holds but can be different with the threshold s_1 in [**ETP-v1**]. To figure out the real threshold in [**ETP-v2**], since $T(N-1) \le N\bar{P}$, we claim that there exists an s such that

$$T(s+1) \le N\bar{P} < T(s).$$

Note we can represent the s by $\min\left\{T(s+1) \le N\bar{P} \mid s = 1, 2, \cdots, N-1\right\}$. Since $T(s)$ is a decreasing function, if $s_1 \ge s+1$, we get $T(s_1) \le T(s+1) \le N\bar{P}$, which implies the optimal solution given by threshold s_1 under [**ETP-v1**] is also feasible under [**ETP-v2**], furthermore, it is also optimal for [**ETP-v2**]; Otherwise, if $s_1 < s+1$, we show that in such case, the (**AveP**) must be binding and the threshold in [**ETP-v2**] is s, respectively. If (**AveP**) is not binding, we can solve **[ETP-v2]** by Proposition 1 and obtain the threshold s_1. But, by $s_1 < s+1$, we get $T(s_1) \geq T(s) > N\bar{P}$ so that the (**AveP**) is violated, which is in contradiction.

In summary, we have the real threshold $s_2 = \max\left\{s_1, \min\left\{T(s+1) \leq N\bar{P} \mid s = 1, 2, \cdots, N-1\right\}\right\}$. Q.E.D.

Proof of Proposition 3. To prove the proposition, first, we consider that at least (**CeilN**) is binding so that the algorithm will enter Step 2. Otherwise, if (**AveP**) is binding but (**CeilN**) not, the **[ETP-v3]** can be solved by **[ETP-v2]**; if the (**AveP**) and (**CeilN**) are not binding, the problem will degenerate to **[ETP-v1]**. Therefore, we show that the algorithm can find the optimal solution when $p_N^* = U_N$ and stops at $s < N-1$. First, we prove the equivalence between **[ETP-v3]** and **[ETP-v3(s)]**. Define new price and quality vectors $(\mathbf{p}^1, \mathbf{q}^1)$ as below:

$$\begin{aligned}
&p_N^1 = U_N,\, p_j^1 = p_j - U_N,\, 1 \leq j \leq N-1,\\
&q_N^1 = q_N,\, q_j^1 = q_j - q_N,\, 1 \leq j \leq N-1.
\end{aligned}$$

Consequently, we have

$$\begin{cases}
\alpha_N^1 = \frac{p_{N-1}^1}{q_{N-1}^1} - \frac{U_N}{q_N};\\
\alpha_{N-1}^1 = \frac{p_{N-2}^1 - p_{N-1}^1}{q_{N-2}^1 - q_{N-1}^1} - \frac{p_{N-1}^1}{q_{N-1}^1} = \alpha_{N-1};\\
\alpha_j^1 = \frac{p_{j-1}^1 - p_j^1}{q_{j-1}^1 - q_j^1} - \frac{p_j^1 - p_{i+1}^1}{q_j^1 - q_{j+1}^1} = \alpha_j,\, 2 \leq j \leq N-2;\\
\alpha_1^1 = 1 - \frac{p_1^1 - p_2^1}{q_1^1 - q_2^1} = \alpha_1.
\end{cases}$$

In this way, we can equivalently solve the **[ETP-v3]** under new variables and parameters:

$$\begin{aligned}
\max_{\mathbf{p}^1} \quad & \sum_{j=1}^{N} \alpha_j p_j = \sum_{j=1}^{N-1} \alpha_j^1 p_j^1 + U_N\left(1 - \frac{U_N}{q_N^1}\right)\\
\text{s.t.} \quad & \frac{1}{N}\sum_{j=1}^{N} p_j^1 \leq \bar{P} - \frac{N-1}{N} U_N\\
& \alpha_j^1 \leq \beta_j,\, \forall\, 1 \leq j \leq N-1\\
& \alpha_N^1 = \frac{p_{N-1}^1}{q_{N-1}^1} - \frac{U_N}{q_N} \leq \beta_N
\end{aligned}$$

In fact, the above problem can be seen as an $N-1$ product problem with the constraint of no purchase probability α_N^1. Since p_{N-1}^1 and α_N^1 have a one-to-one mapping, to make the new $N-1$ problem consistent with the form of **[ETP-v3]**, we let $U_{N-1} = q_{N-1}^1(\beta_N + \frac{U_N}{q_N})$, which transfers the capacity constraint $\alpha_N^1 \leq \beta_N$ to ceiling price constraint $p_{N-1}^1 \leq U_{N-1}$. Rearrange the problem, we get the same format with **[ETP-v3]** for $N-1$ class ticketing problem:

$$\begin{aligned}
\max_{\mathbf{p}^1} \quad & \sum_{j=1}^{N} \alpha_j p_j = \sum_{j=1}^{N-1} \alpha_j^1 p_j^1 + U_N\left(1 - \frac{U_N}{q_N^1}\right)\\
\text{s.t.} \quad & \sum_{j=1}^{N-1} p_j^1 \leq N(\bar{P} - U_N)\\
& \alpha_j^1 \leq \beta_j,\, \forall\, 1 \leq j \leq N-1\\
& p_{N-1}^1 \leq U_{N-1}
\end{aligned}$$

Following this logic, we consider the price constraint $p_{N-1}^1 = p_{N-1} - U_N \le U_{N-1}$. If it is binding, we can do the same procedure until it stops at product s, which indicates the left s products are not restricted by the ceiling constraints. By the iteration, we have $p_j^{N-j} = U_j$ and $\{U_j\}$ is determined by

$$\alpha_j^{N-j} = \frac{p_{j-1}^{N-j+1}}{q_{j-1}^{N-j+1}} - \frac{U_j}{q_j^{N-j}} = \beta_j \Longrightarrow \frac{U_{j-1}}{q_{j-1}^{N-j+1}} = \frac{U_j}{q_j^{N-j}} + \beta_j \Longrightarrow U_{j-1} = Q_{j-1}(U_j/Q_j + \beta_j),$$

where $q_n^{N-j} = q_n - q_{j+1}$, $s \le j \le N-1$, $1 \le n < j$. Hence, the new price and quality vectors ($\mathbf{p}^{N-s}$, $\mathbf{q}^{N-s}$) for the left products are given by

$$p_j^{N-s} = p_j - \sum_{n=s+1}^{N} U_n,\ q_j^{N-s} = q_j - q_{s+1},\ 1 \le j \le s.$$

Under the new $\mathbf{p}^{N-s}$ and $(U_{s+1}, \cdots, U_N)$, we rewrite the **[ETP-v3]**, especially, the (**AveP**):

$$\sum_{j=1}^{N} p_j = \sum_{j=1}^{s} (p_j^{N-s} + \sum_{n=s+1}^{N} U_n) + \sum_{j=s+1}^{N} U_j \le N\bar{P} \Longrightarrow \sum_{j=1}^{s} p_j^{N-s} \le N\bar{P} - (s+1) \sum_{j=s+1}^{N} U_j.$$

and

$$\sum_{j=1}^{N} \alpha_j p_j = \sum_{j=1}^{N-1} \alpha_j^1 p_j^1 + U_N(1 - \frac{U_N}{q_N^1}) = \sum_{j=1}^{s} \alpha_j^{N-s} p_j^{N-s} + \sum_{n=s+1}^{N} U_n (1 - \frac{U_n}{q_n^{N-n+1}}).$$

Now, we have the equivalent problem as follows:

$$\begin{aligned} \max_{\mathbf{p}^s} \quad & \sum_{j=1}^{N} \alpha_j p_j = \sum_{j=1}^{s} \alpha_j^{N-s} p_j^{N-s} + \sum_{n=s+1}^{N} U_n (1 - \frac{U_n}{q_n^{N-n+1}}) \\ \text{s.t.} \quad & \sum_{j=1}^{s} p_j^{N-s} \le N\bar{P} - (s+1) \sum_{j=s+1}^{N} U_j \\ & \alpha_j^{N-s} \le \beta_j,\ \forall\, 1 \le j \le s \end{aligned}$$

which is same with the **[ETP-v3(s)]** defined in Prop 3. Once we get the optimal solution of **[ETP-v3(s)]**, adopting

$$p_j = \begin{cases} p_j^{N-s} + \sum_{n=s+1}^{N} U_n, 1 \le j \le s; \\ U_j, s+1 \le j \le N, \end{cases}$$

we get the optimal solution of **[ETP-v3]**. Q.E.D.

From Lemma 4, all results are about the horizontal demand model, and for convenience, we use α to represent α^H in the paper. And all notations in the following proof will follow the same definitions we defined before.

Proof of Lemma 4. Before proving this lemma, we need some preparations first. Let

$$L = \sum_{i=1}^{N} \alpha_i p_i + \sum_{i=1}^{N} \lambda_i (\beta_i - \alpha_i).$$

We can apply the KKT condition,

$$\begin{aligned} \frac{\partial L}{\partial p_j} =& \alpha_j - p_j \alpha_j (1 - \alpha_j) + \sum_{i \ne j} p_i \alpha_j \alpha_i - \sum_{i=1}^{N} \lambda_i \frac{\partial \alpha_i}{\partial p_j} \\ =& -\alpha_j \left(p_j - 1 - \lambda_j - \sum_{i=1}^{N} (p_i - \lambda_i) \alpha_i \right) = 0, \end{aligned}$$

$$\lambda_j(\alpha_j - \beta_j) = 0,\ \lambda_j \geq 0,\ \alpha_j \leq \beta_j,\ j = 1, 2..., N,$$

where

$$\frac{\partial \alpha_j}{\partial p_j} = -\alpha_j(1-\alpha_j),\ \frac{\partial \alpha_j}{\partial p_i} = \alpha_j \alpha_i.$$

By F.O.C, there exists $\hat{R}$ that $\hat{R} = p_j - 1 - \lambda_j = \sum_{i=1}^{N}(p_i - \lambda_i)\alpha_i$. If all λ_i, $i = 1, 2, \cdots N$, are 0, the capacity constraints are not binding and the problem can be solved by the unconstrained case. Otherwise, assume there exists j such that $\lambda_j > 0$. So the product j is sold out. Consider $i < j$ and there will be two possible scenarios:

$$\alpha_i^* < \beta_i, \alpha_j^* = \beta_j \textbf{ or } \alpha_i^* = \beta_i, \alpha_j^* = \beta_j.$$

When $\alpha_i^* < \beta_i$, $\alpha_j^* = \beta_j$, by KKT condition we know that

$$\lambda_i = 0,\ \lambda_j > 0,\ p_i^* = p_j^* - \lambda_j < p_j^*,$$

then, we can prove that

$$p_i^* < p_j^* \Longrightarrow \theta q_i - p_i^* > \theta q_j - p_j^* \Longrightarrow \beta_i > \alpha_i^* > \alpha_j^* = \beta_j,$$

which contradicts the monotone assumption on β. Therefore, for any $i < j$, we have $\alpha_i^* = \beta_i^*$. Q.E.D.

Proof of Proposition 4. The proof of the proposition will be finished in two parts: first, we prove some properties when $0 < s_1 < N$; then, we find the optimal solution in close form in terms of s_1. Assume the optimal solution of **[ETP-h1]** is $(\boldsymbol{\alpha}^*, \mathbf{p}^*)$. By the Lemma 4, we know that for the product $j > s_1$, we have $\lambda_j = \lambda_N = 0$. Applying F.O.C $\frac{\partial L}{\partial p_j} = \frac{\partial L}{\partial p_N} = 0$, we have $p_j^* = p_N^*$. In addition, for the sold-out products $i \leq j \leq s_1$, since we have $q_i > q_j$, $\beta_i \leq \beta_j$, it is easy to derive that

$$p_i^* = \theta q_i + \log \alpha_0^* - \log \beta_i > \theta q_j + \log \alpha_0^* - \log \beta_j = p_j^* \Longrightarrow p_1^* > \cdots > p_{s_1}^*.$$

In conclusion, we have $p_1^* > \cdots > p_{s_1}^*$ and $p_{s_1+1}^* = \cdots = p_N^*$. About $p_{s_1}^*$ and $p_{s_1+1}^*$, by F.O.C, we get

$$\hat{R} = p_{s_1}^* - \lambda_{s_1} - 1 = p_{s_1+1}^* - 1,\ \lambda_{s-1} > 0 \Longrightarrow p_{s_1}^* > p_{s_1+1}^*.$$

Next, we will prove the close-form solution. Applying the above results $p_j^* = p_N^*$ for $j > s_1$ and the definition of the demand function, we have

$$\alpha_j^* = \begin{cases} \beta_j, 1 \leq j \leq s_1; \\ e^{\theta(q_j - q_N)}\alpha_N^*, s_1 < j \leq N. \end{cases}$$

Substitute them into optimal price functions $p_j^* = (\theta q_j + \log(1 - \sum_{i=1}^N \alpha_i^*) - \log \alpha_j^*)$:

$$p_j^* = \begin{cases} \theta q_j + \log\left(1 - \sum_{i=1}^{s_1}\beta_i - \alpha_N^* \sum_{i=s_1+1}^N e^{\theta(q_i - q_N)}\right) - \log \beta_j, 1 \leq j \leq s_1; \\ \theta q_N + \log\left(1 - \sum_{i=1}^{s_1}\beta_i - \alpha_N^* \sum_{i=s_1+1}^N e^{\theta(q_i - q_N)}\right) - \log \alpha_N, s_1 < j \leq N. \end{cases}$$

Next, we substitute the optimal prices and demands into the objective function:

$$\begin{aligned} R = \sum_{j=1}^N \alpha_j^* p_j^* = &\sum_{j=1}^{s_1} \beta_j \left(\theta q_j + \log\left(1 - \sum_{i=1}^{s_1}\beta_i - \alpha_N^* \sum_{i=s_1+1}^N e^{\theta(q_i - q_N)}\right) - \log \beta_j\right) \\ &+ \sum_{j=s_1+1}^N e^{\theta(q_j - q_N)}\alpha_N^* \left(\theta q_N + \log\left(1 - \sum_{i=1}^{s_1}\beta_i - \alpha_N^* \sum_{i=s_1+1}^N e^{\theta(q_i - q_N)}\right) - \log \alpha_N^*\right). \end{aligned}$$

So we have R as a function of single decision variable α_N^* and the first order derivative can be calculated as follows:

$$\frac{dR}{d\alpha_N^*} = \sum_{j=s_1+1}^{N} e^{\theta(q_j-q_N)}\left(\theta q_N - \log\alpha_N^* - \frac{1}{1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}} + \log\left(1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}\right)\right).$$

Apply F.O.C:

$$\theta q_N - \log\alpha_N^* = \frac{1}{1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}} - \log\left(1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}\right). \tag{13}$$

Then, taking the exponential of both sides, we get

$$e^{\theta q_N}/\alpha_N^* = \frac{1}{1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}} e^{\frac{1}{1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}}. \tag{14}$$

Besides, combining equation (9) and (13), it is also straightforward that

$$p_N^* = \frac{1}{1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}} = \frac{1}{\alpha_0^*}. \tag{15}$$

Define $v = \frac{\alpha_N^*}{1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)})} = \frac{\alpha_N^*}{\alpha_0^*}$, equivalently, we have

$$\alpha_N^* = \frac{v(1-\sum_{i=1}^{s_1}\beta_i)}{1+v\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}. \tag{16}$$

Applying the definition of v into equation (14):

$$\frac{e^{\theta q_N}}{\alpha_N^*} = \frac{1}{1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}} e^{\frac{1}{1-\sum_{i=1}^{s_1}\beta_i - \alpha_N^*\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}} \Longrightarrow$$
$$e^{\theta q_N} = v e^{\frac{1+v\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}{1-\sum_{i=1}^{s_1}\beta_i}} \Longrightarrow \frac{\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}{1-\sum_{i=1}^{s_1}\beta_i} e^{\theta q_N - \frac{1}{1-\sum_{i=1}^{s_1}\beta_i}} = \frac{v\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}{1-\sum_{i=1}^{s_1}\beta_i} e^{\frac{v\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}{1-\sum_{i=1}^{s_1}\beta_i}},$$

then, using Lambert W function to the above equation, we get

$$v = W\left(\frac{\sum_{i=s_1+1}^{N} e^{(\theta q_i - \frac{1}{1-\sum_{i=1}^{s_1}\beta_i})}}{1-\sum_{i=1}^{s_1}\beta_i}\right)\frac{1-\sum_{i=1}^{s_1}\beta_i}{\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}. \tag{17}$$

Let $Q = \frac{\sum_{i=s_1+1}^{N} e^{(\theta q_i - \frac{1}{1-\sum_{i=1}^{s_1}\beta_i})}}{1-\sum_{i=1}^{s_1}\beta_i} = \sum_{i=s_1+1}^{N} e^{\theta q_i}\frac{e^{-\frac{1}{1-K(s_1)}}}{1-K(s_1)}$, which is a decreasing function in terms of $K(s)$ given s_1. By equation (16) and the definition of v, we get the close-form solutions of α_N^* and α_0^*:

$$\alpha_N^* = \frac{v(1-\sum_{i=1}^{s_1}\beta_i)}{1+v\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}} = \frac{W(Q)(1-K(s_1))^2}{1+W(Q)(1-K(s_1))}\frac{1}{\sum_{i=s_1+1}^{N} e^{\theta(q_i-q_N)}}, \tag{18}$$

$$\alpha_0^* = \frac{\alpha_N^*}{v} = \frac{1-K(s_1)}{1+W(Q)(1-K(s_1))}. \tag{19}$$

Since α_N^* is a function of s_1 and the value of threshold s_1 is finite, we calculate the objective function $R^*(\alpha_N(s_1))$ with all threshold values $s_1 = 1, 2, ..., N$. The solution corresponding to the largest $R^*(\alpha_N(s_1))$ is optimal.

For the other two cases, if $s_1 = 0$, the property of the optimal prices is trivial known as the 'equal-markup' property; if $s_1 = N$, we get the property by the same method for $0 < s_1 < N$ case: since $q_1 > \cdots > q_N$ and $\beta_1 \le \beta_2 \le \cdots \beta_N$, by equation (9), it is easy to show the monotonicity of optimal prices. Q.E.D.

Proof of Lemma 5. Since the objective function is concave in α, it suffices to prove the average price constraint is quasi-convex. Consider the average price $g(\boldsymbol{\alpha})$ as a function of $\boldsymbol{\alpha}$ ($\alpha_0 := 1 - \sum_{j=1}^{N} \alpha_j$):

$$g(\boldsymbol{\alpha}) = \sum_{i=1}^{N} (\theta q_i + \log \alpha_0 - \log \alpha_i).$$

For any feasible vector x and y, we need to prove that $\nabla g(y)^T(x-y) \le 0$ whenever $g(x) \le g(y)$. By the assumption of $g(x) \le g(y)$, we have

$$\sum_{i=1}^{N} (\theta q_i + \log x_0 - \log x_i) \le \sum_{i=1}^{N} (\theta q_i + \log y_0 - \log y_i) \Leftrightarrow N \log \frac{x_0}{y_0} \le \sum_{i=1}^{N} \log \frac{x_i}{y_i} \Leftrightarrow \frac{x_0}{y_0} \le (\prod_{i=1}^{N} \frac{x_i}{y_i})^{1/N}.$$

Note that

$$\partial g(\boldsymbol{\alpha})/\partial \alpha_i = -(\frac{N}{\alpha_0} + \frac{1}{\alpha_i}), i = 1, 2, \cdots, N.$$

Therefore

$$\begin{aligned}
\nabla g(y)^T(x-y) &= -\sum_{i=1}^{N} (\frac{N}{y_0} + \frac{1}{y_i})(x_i - y_i) \\
&= -N \sum_{i=1}^{N} (\frac{N(1-x_0)}{y_0} + \frac{x_i}{y_i}) - (\sum_{i=1}^{N} \frac{N x_i}{y_0} + N) \\
&= N \frac{x_0}{y_0} - \sum_{i=1}^{N} \frac{x_i}{y_i}.
\end{aligned}$$

Since the geometric means is no larger than the arithmetic means, we have:

$$N \frac{x_0}{y_0} \le N (\prod_{i=1}^{N} \frac{x_i}{y_i})^{1/N} \le \sum_{i=1}^{N} \frac{x_i}{y_i}.$$

Therefore, $g(\boldsymbol{\alpha})$ is quasi-convex and the KKT condition is necessary and sufficient.

It is worth noting that the above proof continues to carry through if it is the weighted average price that we consider in the constraint. Define the weighted average price $h(\boldsymbol{\alpha})$ as follows:

$$h(\boldsymbol{\alpha}) = \sum_{i=1}^{N} w_i (\theta q_i + \log \alpha_0 - \log \alpha_i), \ \sum_{i=1}^{N} w_i = 1.$$

Then, we use the same argument as above, and then it boils down to simply showing

$$\prod_{i=1}^{N} (\frac{x_i}{y_i})^{w_i} \le \sum_{i=1}^{N} w_i \frac{x_i}{y_i}.$$

Applying logarithm on both sides, the desired result then follows directly from Jensen's inequality. Therefore, in the weighted average price case, the constraint is still quasi-convex and the KKT condition is necessary and sufficient. One example of a weighted average price is to consider the capacity limits of each ticket category, i.e., the weights are given by $w_j = \beta_j / \sum_k \beta_k$.

Second, we follow the same procedures in the proof of Lemma 4 to show the threshold structure of the optimal solution here. We prove it by contradiction. Assume product j is sold out, i.e., $\lambda_j > 0$, but product i $(i < j)$ is not sold out, i.e, $\lambda_i = 0$. Consider the KKT condition:

$$\frac{\partial L}{\partial p_j} = -\alpha_j \left(p_j - 1 - \lambda_j - \sum_{i=1}^{N} (p_i - \lambda_i)\alpha_i + \frac{\phi}{\alpha_j} \right) = 0,$$
$$N\bar{P} \geq \sum_{i=1}^{N} p_i, \phi \geq 0,$$
$$\lambda_j(\alpha_j - \beta_j) = 0,\ \lambda_j \geq 0,\ \alpha_j \leq \beta_j,\ j = 1, 2..., N.$$

By KKT conditions, there exists $\hat{R}$ that

$$\hat{R} = p_j - 1 - \lambda_j + \frac{\phi}{\alpha_j} = \sum_{i=1}^{N} (p_i - \lambda_i)\alpha_i.$$

For products i and j, we have

$$\lambda_i = 0 < \lambda_j \Longrightarrow p_i^* + \frac{\phi}{\alpha_i^*} - \hat{R} - 1 < p_j^* + \frac{\phi}{\alpha_j^*} - \hat{R} - 1 \Longrightarrow p_i^* + \frac{\phi}{\alpha_i^*} < p_j^* + \frac{\phi}{\alpha_j^*}.$$

Since $\alpha_i^* < \beta_i \leq \beta_j = \alpha_j^*$, above inequality deduces $p_i^* < p_j^*$. Hence, $\theta q_i - p_i^* > \theta q_j - p_j^*$, which by the definition of demand function leads to $\alpha_i^* > \alpha_j^*$, a contradiction. Q.E.D.

Proof of Proposition 5. First, we prove all the results of $s_2 > 0$. Here we assume the average price constraint is binding, i.e., $N\bar{P} = \sum_{i=1}^{N} p_i$ and $\phi > 0$. Otherwise, the problem is simply **[ETP-h1]** which can be solved by Proposition 4. Now, we prove the monotonous properties of the optimal outcome by two different cases. In case (1), consider the KKT for $j > i > s_2$:

$$p_i^* + \frac{\phi}{\alpha_i^*} = \theta q_i - \log \alpha_i^* + \log \alpha_0^* + \frac{\phi}{\alpha_i^*} = \theta q_j - \log \alpha_j^* + \log \alpha_0^* + \frac{\phi}{\alpha_j^*} = p_j^* + \frac{\phi}{\alpha_j^*},$$

since $\theta q_i > \theta q_j$ and $-\log \alpha_i^* + \log \alpha_0^* + \frac{\phi}{\alpha_i^*}$ is decreasing in α_i^*, to maintain the above equation, $\alpha_i^* > \alpha_j^*$ must hold. Therefore, we get $\alpha_{s_2+1}^* > \cdots > \alpha_N^*$. To prove the monotonous properties of optimal prices, go back to the above equation when $j > i > s_2$:

$$p_i^* + \frac{\phi}{\alpha_i^*} = p_j^* + \frac{\phi}{\alpha_j^*}, \alpha_i^* \geq \alpha_j^* \Longrightarrow p_i^* \geq p_j^*.$$

In case (2), when $i < j < s_2$, since $\beta_i \leq \beta_j$, $q_i > q_j$, we have

$$p_i^* = \theta q_i + \log \alpha_0^* - \log \beta_i > \theta q_j + \log \alpha_0^* - \log \beta_j = p_j^*.$$

The last and most important, we prove that $p_{s_2}^* > p_{s_2+1}^*$ by contradiction. Assume that $p_{s_2}^* \leq p_{s_2+1}^*$, then, by KKT condition, we have

$$p_{s_2}^* + \frac{\phi}{\alpha_{s_2}^*} - \lambda_{s_2} = p_{s_2+1}^* + \frac{\phi}{\alpha_{s_2+1}^*} \Longrightarrow \frac{\phi}{\alpha_{s_2}^*} > \frac{\phi}{\alpha_{s_2+1}^*} \Longrightarrow \alpha_{s_2}^* < \alpha_{s_2+1}^*$$

However, if $\alpha_{s_2}^* = \beta_{s_2} < \alpha_{s_2+1}^*$, substitute it into equation (9), we see

$$p_{s_2}^* = \theta q_{s_2} + \log \alpha_0^* - \log \beta_{s_2} > \theta q_{s_2+1} + \log \alpha_0^* - \log \alpha_{s_2+1}^* = p_{s_2+1}^*,$$

which is a contradiction.

Second, we show the optimal properties under $s_2 = 0$. If $\lambda_j = 0$ for all $1 \le j \le N$:

$$\hat{R} = p_j - 1 + \frac{\phi}{\alpha_j} = \sum_{i=1}^{N} p_i \alpha_i, \ j = 1, 2, \cdots, N.$$

In the same way, if $p_i \le p_j$ for some $1 \le i < j \le N$, we find a contradiction between equation (9) and $p_i^* + \frac{\phi}{\alpha_i^*} = p_j^* + \frac{\phi}{\alpha_j^*}$: we can derive that the $\frac{\phi}{\alpha_i^*} \ge \frac{\phi}{\alpha_j^*}$, $\alpha_i^* \le \alpha_j^*$ based on $\frac{\phi}{\alpha_i^*} \ge \frac{\phi}{\alpha_j^*}$; however, by the demand function (9), if $p_i \le p_j$, we have

$$\theta q_i - \log \alpha_i^* \le \theta q_j - \log \alpha_j^*, q_i > q_j \Longrightarrow \alpha_i^* > \alpha_j^*.$$

Therefore, $p_i > p_j$ for all $1 \le i < j \le N$, which suggests optimal prices are monotonously decreasing. Q.E.D.

Proof of Lemma 6 and Proposition 6. Since their results directly follow the procedure for solving [**ETP-h3'**], we prove them together. Assume that the (**CeilN**) is binding, applying the KKT condition to (**Ph-3'**):

$$\begin{cases} \dfrac{\partial L}{\partial p_j} = -\alpha_j \left(p_j - 1 - \lambda_j - \displaystyle\sum_{i=1}^{N} (p_i - \lambda_i)\alpha_i \right) = 0, j = 1, 2, \cdots, N-1, \\ \dfrac{\partial L}{\partial p_N} = -\alpha_N \left(p_N - 1 - \lambda_N - \displaystyle\sum_{i=1}^{N} (p_i - \lambda_i)\alpha_i + \omega \right) = 0, \\ \lambda_j(\alpha_j - \beta_j) = 0, \ \lambda_j \ge 0, \ \alpha_j \le \beta_j, \ j = 1, 2, \cdots, N, \\ \omega > 0, U_N - p_N = 0. \end{cases}$$

Following the proof in Proposition 4, we can get the threshold structure by the same method. First, let's consider the case (1) in Lemma 6 where all products are sold out except class N. It is easy to calculate the p_j^* by $\alpha_j^* = \beta_j$ $(j \le N-1)$ and $p_N^* = U_N$ with equation (9):

$$p_N^* = U_N = \theta q_N + \log(1 - K(N-1) - \alpha_N^*) - \log \alpha_N^* \Longrightarrow \alpha_N^* = \frac{1 - K(N-1)}{1 + e^{(U_N - \theta q_N)}}.$$

Next, consider general case $0 < s < N$, we get the following optimal prices by the KKT condition based on the threshold structure:

$$p_j^* = \begin{cases} U_N, & j = N, \\ U_N + \omega, & s+1 \le j < N, \\ \theta q_j + \log \left(1 - K(s) - \alpha_N^* (1 + \displaystyle\sum_{i=s+1}^{N-1} e^{\theta(q_i - q_N) - \omega}) \right) - \log \beta_j, & j < s+1. \end{cases}$$

Substitute p_j^* into the revenue function:

$$R = \sum_{j=1}^{N} \alpha_j^* p_j^* = \sum_{j=1}^{s} \beta_j \left(U_N + \theta(q_j - q_N) + \log \alpha_N^* - \log \beta_j \right) + \sum_{j=s+1}^{N-1} e^{\theta(q_j - q_N) - \omega} \alpha_N^* (U_N + \omega) + \alpha_N^* U_N.$$

Consider the F.O.C of R in term of α_N^*:

$$\frac{\partial R}{\partial \alpha_N^*} = \sum_{j=1}^{s} \frac{\beta_j}{\alpha_N^*} + \left(\sum_{i=s+1}^{N-1} e^{\theta(q_i - q_N) - \omega} - \alpha_N^* \sum_{i=s+1}^{N-1} e^{\theta(q_i - q_N) - \omega} \frac{\partial \omega}{\partial \alpha_N^*} \right) (U_N + \omega)$$

$$+\frac{\partial\omega}{\partial\alpha_N^*}\sum_{j=s+1}^{N-1}e^{\theta(q_j-q_N)-\omega}\alpha_N^*+U_N.$$

Since $\alpha_0^*=1-K(s)-\alpha_N^*(1+\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega})$, we have

$$\frac{\partial\alpha_0^*}{\partial\alpha_N^*}=-1-\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}+\alpha_N^*\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}\frac{\partial\omega}{\partial\alpha_N^*},$$

In another way, we can also represent α_0^* by $\alpha_0^*=\alpha_N^*e^{(U_N-\theta q_N)}$. Therefore, we have

$$\frac{\partial\alpha_0^*}{\partial\alpha_N^*}=e^{(U_N-\theta q_N)}=\frac{\alpha_0^*}{\alpha_N^*}.$$

Combining the above two equations, we have

$$\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}-\alpha_N^*\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}\frac{\partial\omega}{\partial\alpha_N^*}=-\frac{\alpha_0^*}{\alpha_N^*}-1,$$

and

$$\frac{\partial\omega}{\partial\alpha_N^*}\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}=\frac{\alpha_0^*}{\alpha_N^*}+1+\alpha_N^*\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}.$$

Substitute the above two equations into the F.O.C of R:

$$\begin{aligned}\frac{\partial R}{\partial\alpha_N^*}&=\sum_{j=1}^{s}\frac{\beta_j}{\alpha_N^*}+\left(\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}-\alpha_N^*\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}\frac{\partial\omega}{\partial\alpha_N^*}\right)(U_N+\omega)\\&\quad+\frac{\partial\omega}{\partial\alpha_N^*}\sum_{j=s+1}^{N-1}e^{\theta(q_j-q_N)-\omega}\alpha_N^*+U_N\\&=\sum_{j=1}^{s}\frac{\beta_j}{\alpha_N^*}+(-\frac{\alpha_0^*}{\alpha_N^*}-1)(U_n+\omega)+(\frac{\alpha_0^*}{\alpha_N^*}+1+\sum_{j=s+1}^{N-1}e^{\theta(q_j-q_N)-\omega})+U_N\\&=\frac{1}{\alpha_N^*}-\frac{\alpha_0^*}{\alpha_N^*}(U_n+\omega)-\omega=\frac{1}{\alpha_N^*}-e^{(U_N-\theta q_N)}(U_n+\omega)-\omega=0.\end{aligned}$$

Hence, we get

$$\alpha_N^*=\frac{1}{e^{(U_N-\theta q_N)}(U_N+\omega)+\omega}.$$

At the last step, substitute the above equation of α_N^* into $\frac{\alpha_0^*}{\alpha_N^*}=\frac{1-K(s)-\alpha_N^*(1+\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega})}{\alpha_N^*}=e^{(U_N-\theta q_N)}$, we have the implicit function of ω as follows

$$e^{(U_N-\theta q_N)}=(1-K(s))\left(e^{(U_N-\theta q_N)}(U_n+\omega)+\omega\right)-(1+\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-\omega}).\tag{20}$$

Define $f(x)=(1-K(s))(e^{(U_N-\theta q_N)}(U_n+x)+x)$, $g(x)=(1+\sum_{i=s+1}^{N-1}e^{\theta(q_i-q_N)-x})+e^{(U_N-\theta q_N)}$, we will show that f is monotonously increasing and g is monotonously decreasing. Let $x=0$, we have

$$\begin{aligned}f(0)-g(0)=&(1-K(s))e^{(U_N-\theta q_N)}U_n-(1+\sum_{j=s+1}^{N}e^{\theta(q_j-q_N)})-e^{(U_N-\theta q_N)}\\=&\frac{1}{\alpha_N^*}\left((1-K(s))\alpha_0^*U_n-\alpha_N^*(1+\sum_{j=s+1}^{N}e^{\theta(q_j-q_N)})-\alpha_0^*\right)\\=&\frac{1}{\alpha_N^*}\left((1-K(s))\alpha_0^*U_n-(1-K(s)-\alpha_0^*)-\alpha_0^*\right)\\=&\frac{1}{\alpha_N^*}(1-K(s))(U_N\alpha_0^*-1)\end{aligned}$$

Again, since

$$\frac{1}{\alpha_N^*} - \frac{\alpha_0^*}{\alpha_N^*}(U_N+\omega) - \omega = 0 \Longrightarrow \frac{1-U_N\alpha_0^*}{\alpha_N^*} - \omega(1+\frac{\alpha_0^*}{\alpha_N^*}) = 0, \omega > 0 \Longrightarrow 1 - U_N\alpha_0^* > 0,$$

we get $f(0)-g(0)<0$. Next, consider $\omega \to \infty$, it is clear that $f \to \infty$ and $g \to 1+e^{(U_N-\theta q_N)}$. Hence, $f(M) > g(M)$ for some M. By the Intermediate Value Theorem, there exists a unique solution of equation (20). At last, when $s=0$, we have $\lambda_j = 0$ for all $j=1,\cdots,N$. Substitute them into the KKT conditions:

$$\begin{cases} p_j - 1 - \sum_{i=1}^{N} p_i\alpha_i = 0, j = 1,2,...,N-1, \\ p_N - 1 - \sum_{i=1}^{N} p_i\alpha_i + \omega = 0, \\ \omega > 0, p_N = U_N. \end{cases}$$

So, we have the following solution:

$$p_N^* = U_N,\ \omega > 0,\ p_j^* = U_N + \omega,\ j = 1,2,\cdots,N-1.$$

The ω is determined by the same method above with modification:

$$e^{(U_N-\theta q_N)} = e^{(U_N-\theta q_N)}(U_n+\omega) + \omega - (1 + \sum_{i=1}^{N-1} e^{\theta(q_i-q_N)-\omega}).$$

For the monotonicity of optimal prices and demands, it can be proved by the same way adopted in the proof of Proposition (4). Q.E.D.

To differentiate the threshold, we use s_1^{N-1}, s_1^{N+1} and s_1^{ϵ} to represent the thresholds under the different strategies: s_1^{N-1}, s_1^{N+1} are the new thresholds after closing and adding decisions; s_1^{ϵ} is the new threshold after capacity reallocation. Correspondingly, (p^X, α^X) $(X = N-1, N+1, \epsilon)$ will be the optimal solutions after different operations. The following is about the flexible capacity management problems based on [**ETP-v1**] and [**ETP-h1**].

Proof of Proposition 7. First, for the closing decision, we consider the three different cases separately: if $s_1 < N-1$ or $s_1 = N-1$, $K(N-1) = \frac{1}{2}$, then closing does not affect the optimality outcome since product N has 0 demand. However, all the remaining products will still follow the optimal solution in [**ETP-v1**] so that this decision does not affect the optimal outcome; if $s_1 > N-1$, we know that

$$p_j^* - p_{j+1}^* = (1-K(j))Q_j \Longrightarrow p_j^* = \sum_{i=j}^{N}(1-K(i))Q_i,\ j = 1,2,\cdots,N.$$

From the above equations, closing section N will lead to a decrease in the optimal prices of all products by $(1-K(N))Q_N$. Furthermore, it will also decrease the maximized revenue since all optimal prices decrease and demands are non-increasing. In the end, if $s_1 = N-1$, $K(N-1) < \frac{1}{2}$, then the optimal sale α_N^* is less than β_N and

$$p_N^* = \frac{Q_N}{2}, p_j^* = \frac{Q_N}{2} + \sum_{i=j}^{N-1}(1-K(i))Q_i,\ j = 1,2,\cdots,N-1.$$

Similarly, the new optimal prices p_j^{N-1} $(j=1,2,\cdots,N-1)$ will decrease by $\frac{Q_N}{2}$ and decrease the maximized revenue as well.

Second, for the new product adding decision, we show that the index of the optimal sold-out threshold will either stay the same or decrease by 1. When $\beta_0 \le \frac{1}{2} - K(s_1)$, we have the

$$\frac{1}{2} - \sum_{i=1}^{s_1} \beta_i - \beta_0 \ge 0,$$

and

$$\frac{1}{2} - \sum_{i=1}^{s_1+1} \beta_i - \beta_0 < \frac{1}{2} - \sum_{i=1}^{s_1+1} \beta_i < 0.$$

By definition, we have $s_1^{N+1} = s_1$. In other case, when $\beta_0 > \frac{1}{2} - K(s_1)$, we have the

$$\frac{1}{2} - \sum_{i=1}^{s_1} \beta_i - \beta_0 < 0,$$

and

$$\frac{1}{2} - \sum_{i=1}^{s_1-1} \beta_i - \beta_0 > \frac{1}{2} - \sum_{i=1}^{s_1-1} \beta_i - \beta_{s_1} > 0.$$

Therefore, $s_1^{N+1} = s_1 - 1$. In summary, when the new product capacity is small enough ($\beta_0 \le \frac{1}{2} - K(s_1)$), the index of the threshold will not change; when the capacity is bigger than $\frac{1}{2} - K(s_1)$, the demand of s_1 will be attracted to the product 0 so that the threshold decreases by 1:

$$s_1^{N+1} = \begin{cases} s_1, & \beta_0 \le \dfrac{1}{2} - K(s_1), \\ s_1 - 1, & \beta_0 > \dfrac{1}{2} - K(s_1). \end{cases}$$

By the optimal prices in **[ETP-v1]**,

$$p_j^* = \begin{cases} \dfrac{q_j}{2}, & s_1 + 1 \le j \le N, \\ \dfrac{q_{s_1+1}}{2} + \displaystyle\sum_{i=j}^{s_1} (1 - K(i)) Q_i, & 1 \le j \le s_1, \end{cases}$$

we can see how the optimal prices change under adding decision: first, if $s_1^{N+1} = s_1$, we see that for the sold-out product, optimal prices p_j^{N+1} decrease by $\sum_{i=j}^{s_1^{N+1}} \beta_0 Q_i$ but the partially sold ones remain the same by above equation. Q.E.D.

Proof of Proposition 8. Unlike the vertical case, we can't compare the value of $R^*(s_1 - 1)$ and $R^*(s_1^{N-1} - 1)$ analytically. But, it is easy to show that $\alpha_0^* = \frac{1-K(s_1)}{1+W(Q)(1-K(s_1))}$ decrease in N. Therefore, we can characterize the changes in the optimal sales and prices for each ticket category. For the closing decision, assume that $s_1^{N-1} = s_1$, we have

$$\alpha_0^{N-1} > \alpha_0^*.$$

For the sold-out product $j \le s_1^{N-1}$, we have the demand $\alpha_j^{N-1} = \beta_j$ will not change. Then, by $\alpha_0^{N-1} > \alpha_0^*$, we have the optimal price

$$p_j^* = \theta q_j + \log \alpha_0^* - \log \beta_j < \theta q_j + \log \alpha_0^{N-1} - \log \beta_j = p_j^{N-1},\ 1 \le j \le s_1^{N-1}.$$

For the partially sold products, by equation $p_{N-1}^{N-1} = \frac{1}{\alpha_0^{N-1}}$, we have

$$p_{N-1}^{N-1} = \frac{1}{\alpha_0^{N-1}} < \frac{1}{\alpha_0^*} = p_N^*.$$

Furthermore, since the optimal prices p_j^* for $s_1^{N-1}+1 \le j \le N$ are identical, we have $p_j^{N-1} < p_j^*$. Next, it is easy to show that optimal sales of all partially sold products increase:

$$\alpha_j^{N-1} = \alpha_0^{N-1} e^{\theta q_j - p_j^{N-1}} > \alpha_0^N e^{\theta q_j - p_j^*} = \alpha_j^*.$$

At last, we prove the close decision will increase the revenue for each ticket category:

(1) for $j \ge s_1^{N-1}$, by $p_j^{N-1} = p_{N-1}^{N-1} = \frac{1}{\alpha_0^{N-1}}$:

$$p_j^{N-1}\alpha_j^{N-1} = \frac{\alpha_j^{N-1}}{\alpha_0^{N-1}} = e^{\theta q_j - p_j^{N-1}} = e^{\theta q_j - p_{N-1}^{N-1}} > e^{\theta q_j - p_N^*} = e^{\theta q_j - p_j^*} = \frac{\alpha_j^*}{\alpha_0^*} = p_j^*\alpha_j^*;$$

(2) for $j < s_1^{N-1}$, by $\alpha_0^{N-1} > \alpha_0^*$:

$$p_j^{N-1}\beta_j = (\theta q_j + \log\alpha_0^{N-1} - \log\beta_j)\beta_j > (\theta q_j + \log\alpha_0^* - \log\beta_j)\beta_j = p_j^*\beta_j.$$

For the adding decision, by equation (10), we have

$$\alpha_j = \left(1 - K(s_1^{N+1}) - \frac{(1-K(s_1^{N+1}))}{1+W(Q)(1-K(s_1^{N+1}))}\right)\frac{e^{\theta(q_j-q_N)}}{\sum_{i=s_1^{N+1}}^N e^{\theta(q_i-q_N)}},\ j > s_1^{N+1}.$$

Assume that we add product 0 such that $s_1^{N+1} = s_1$, since

$$\partial\alpha_j/\partial(1-K(s_1^{N+1})) = \left(1 - \frac{1}{(1+W(Q)(1-K(s_1^{N+1})))^2}\right)\frac{e^{\theta(q_j-q_N)}}{\sum_{i=s_1^{N+1}}^N e^{\theta(q_i-q_N)}} > 0,\ j > s_1^{N+1}.$$

we have α_j is also a decreasing function of $K(s)$. By $K(s_1) < K(s_1^{N+1})$, hence, the optimal demands $\alpha_j^{N+1} < \alpha_j^*$ for all partially sold products. Q.E.D.

Proof of Proposition 9. Assuming the reallocation has been done, the new capacity for product 1 is $\beta_1 + \epsilon$ and r is $\beta_r - \epsilon$. There are two different cases based on the capacity reduced category r.

(1) $r > s_1^\epsilon$. For the optimal solution of **[ETP-v1]**, we have the optimal prices for partially sold products:

$$p_j^\epsilon = \begin{cases} \dfrac{q_j}{2}, & s_1^\epsilon + 1 \le j \le N, \\ \dfrac{q_{s_1^\epsilon+1}}{2} + \displaystyle\sum_{i=j}^{s_1^\epsilon}(1-K(i)-\epsilon)Q_i, & 1 \le j \le s_1^\epsilon. \end{cases}$$

Therefore, there is no change in the optimal solution for the partially sold products. For sold-out product j, from the equation we see the optimal prices decrease by $\sum_{i=j}^{s_1^\epsilon-1}\epsilon Q_i$. In the end, since the optimal solution without capacity reallocation is still feasible with the new capacity setting, we conclude that the optimal total revenue increases.

(2) $r \le s_1^\epsilon$. In this case, the optimal price structure will change as follows:

$$p_j^\epsilon = \begin{cases} \dfrac{q_j}{2}, & s_1^\epsilon + 1 \le j \le N, \\ \dfrac{q_{s_1^\epsilon+1}}{2} + \displaystyle\sum_{i=j}^{s_1^\epsilon}(1-K(i))Q_i, & r \le j \le s_1^\epsilon, \\ \dfrac{q_{s_1^\epsilon+1}}{2} + \displaystyle\sum_{i=j}^{s_1^\epsilon}(1-K(i)-\epsilon)Q_i, & 1 \le j \le r-1. \end{cases}$$

It is clear that if the product $j \geq r$, there is no change in the optimal solution on a partially sold product. Similarly, for the left products $j < r$, from the equation we see the optimal prices decrease by $\sum_{i=j}^{s_1^\epsilon - 1} \epsilon Q_i$ and total revenue increases. Q.E.D.

Proof of Proposition 10. Still, assume that the reallocation doesn't have an impact on the threshold.

(1) Suppose that $r > s_1^\epsilon$. First, we check the α_0^ϵ.

$$\alpha_0^\epsilon = \frac{1 - K(s_1^\epsilon)}{1 + W(Q)(1 - K(s_1^\epsilon))}, \tag{21}$$

Recall $Q = \sum_{j=s_1^\epsilon}^N e^{\theta q_j} f(1 - K(s))$ and $f(x) = x^{-1} e^{-x^{-1}}$. Let $x = \frac{1}{1-K(s_1^\epsilon)} \geq 1$ and check the F.O.C of the $\alpha_0^\epsilon(x)$:

$$\begin{aligned}
\frac{d\alpha_0^\epsilon(x)}{dx} =& \frac{d\alpha_0^\epsilon(x)}{dW} \frac{dW}{dQ} \frac{dQ}{dx} \\
=& \frac{-1}{(x + W(Q))^2} \frac{W(Q)}{(1 + W(Q))Q} (1 - x) e^{-x} \sum_{j=s_1^\epsilon}^N e^{\theta q_j} > 0, \; x > 1.
\end{aligned}$$

Therefore, optimal no-purchase probability α_0^ϵ is an increasing function in x, thus, α_0^ϵ increases once reallocating. Furthermore, consider the optimal demands $j > s_1^\epsilon$, since the sold-out threshold remains the same and α_0^ϵ increases, the total optimal demands of partially sold products will decrease. Due to the equal-markup, it shows that the demand for each partially sold product will decrease as well. Then, consider the optimal prices, for the sold-out products $1 < j \leq s_1^\epsilon$, it is trivial that optimal prices will increase by equation (9), except product 1:

$$p_1^\epsilon = \theta q_1 + \log \alpha_0^\epsilon - \log \beta_1.$$

By the definition of x, we have the F.O.C:

$$\frac{dp_1^\epsilon(x)}{dx} = \frac{-1}{x + W(Q)} \frac{W(Q)}{Q(W(Q) + 1)} (1 - x) e^{-x} \sum_{j=s_1^\epsilon}^N e^{\theta q_j} - \frac{1}{x^2 \beta_1} < 0.$$

So the optimal price of product 1 decreases. For $j > s_1^\epsilon$, since α_0^ϵ increases and α_j^ϵ decreases, by equation (9), the optimal prices p_j^ϵ will also increase. In the end, since the optimal solution before the reallocation is still feasible under new capacity limits, the new optimal revenue is greater than the optimal revenue before reallocation.

(2) Now, we will prove a more general case comparing with $r \leq s_1^\epsilon$. Consider that we transfer capacity ϵ from β_i to β_j where $1 \leq j < i \leq s_1^\epsilon$, since $1 - K(s_1^\epsilon)$ doesn't change, the optimal solutions will not change except p_j^ϵ decreases and p_i^ϵ increases by the equation (9). The change in the optimal revenue can also be calculated by

$$p_i^\epsilon \alpha_i^\epsilon + p_j^\epsilon \alpha_j^\epsilon - \alpha_i p_i - \alpha_j p_j = \epsilon\theta(q_j - q_i) - (\beta_j + \epsilon)\log(\beta_j + \epsilon) + \beta_j \log \beta_j - (\beta_i - \epsilon)\log(\beta_i - \epsilon) + \beta_i \log \beta_i.$$

Assume that ϵ is small and we use the Taylor expansion:

$$(\beta_j + \epsilon)\log(\beta_j + \epsilon) = (\beta_j + \epsilon)\left(\log \beta_j + \frac{\epsilon}{\beta_j} + o(\epsilon)\right)$$

$$=\beta_j \log \beta_j + \epsilon \log \beta_j + \epsilon + o(\epsilon),$$

and

$$\begin{aligned}(\beta_i - \epsilon)\log(\beta_i - \epsilon) =& (\beta_i - \epsilon)\left(\log \beta_i - \frac{\epsilon}{\beta_i} + o(\epsilon)\right)\\ =& \beta_i \log \beta_i - \epsilon \log \beta_i - \epsilon + o(\epsilon).\end{aligned}$$

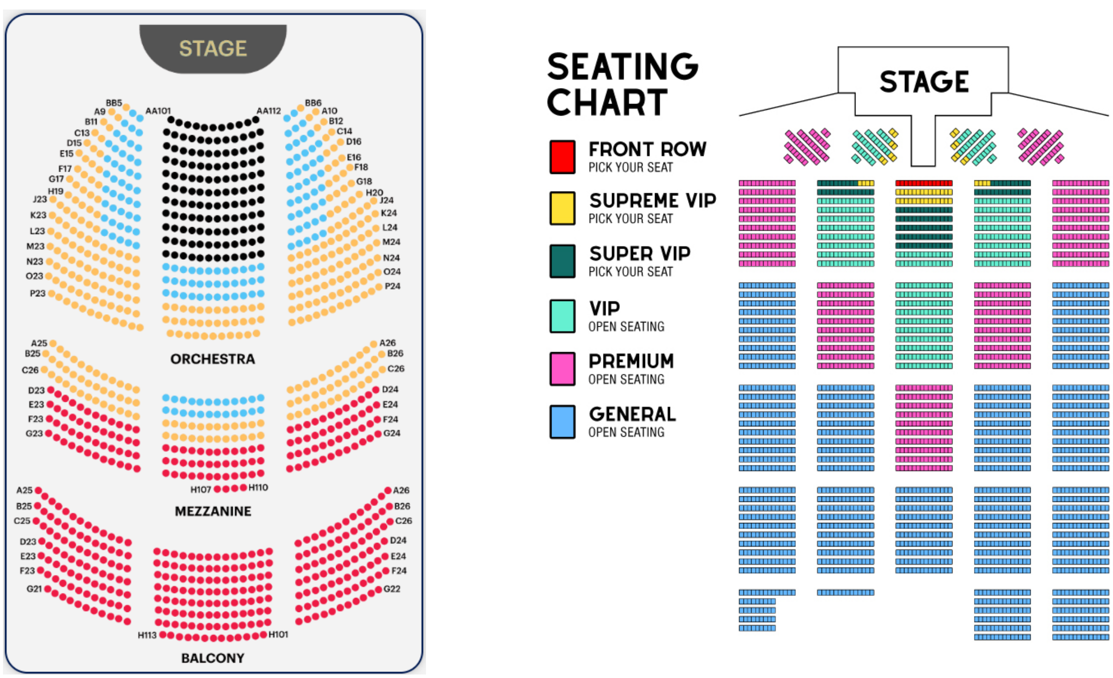
Therefore, substituting the above Taylor expansion equations into the revenue difference, we get

$$p_i^\epsilon \alpha_i^\epsilon + p_j^\epsilon \alpha_j^\epsilon - \alpha_i p_i - \alpha_j p_j = \epsilon\theta(q_j - q_i) + \epsilon \log(\frac{\beta_i}{\beta_j}). \quad Q.E.D.$$

Appendix B: Typical Venue Charts Showing Seating Maps

We provide two typical venue charts of live events showing the segmentation of seats with different characteristics. These seat maps serve as supporting evidence for our assumption of the order of capacities for different ticket categories. In particular, one may observe from the charts that higher quality seats are overall less than in number than lower quality ones.

Figure 1 Two Typical Venue Charts that Support Our Assumption.



(a) Broadway Theater Seating Charts (b) Seating Map of The BTC Show 2022

Sources: (a) `https://broadwayshow.tickets/at/value-city-arena-at-the-schottenstein-center/`; (b) `https://www.tix.com/ticket-sales/behindthechair/5232/event/1269736`