Multistep Methods

Yu Geng

Outline

- Adams Methods
  - Adams-Bashforth methods (explicit)
  - Adams-Moulton methods (implicit)
- BDF Methods
- Numerical example (three-step AB method)

Introduction

- A well-known *two-step* method can be obtained by applying the centred finite difference

$$u_i^{CD} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}, \qquad 1 \le i \le n-1. \qquad (10.61)$$

- on

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I, \\ y(t_0) = y_0, & \end{cases} \qquad (11.1)$$

- which gives the *midpoint method,*

$$u_{n+1} = u_{n-1} + 2hf_n, \qquad n \ge 1 \qquad (11.43)$$

  - $u_0 = y_0$
  - u_1 is to be determined
  - $f_k = f(t_k, u_k)$

Introduction

- The Simpson method (implicit)

$$u_{n+1} = u_{n-1} + \frac{h}{3}[f_{n-1} + 4f_n + f_{n+1}], \qquad n \geq 1 \qquad (11.44)$$

  - $u_0 = y_0$
  - u_1 is to be determined
- obtained from

$$y(t) - y_0 = \int_{t_0}^{t} f(\tau, y(\tau))d\tau. \qquad (11.2)$$

- with $t_0 = t_{n-1}$ and $t = t_{n+1}$ and by using the Cavalieri-Simpson quadrature rule to approximate the integral of f.

Introduction

- A multistep method requires q initial values for "taking off".
- One way to assign the remaining values consists of resorting to explicit one-step methods of high order
  - Heun's method (Sec. 11.10)
  - Runge-Kutta methods (Sec. 11.8)
- The linear multistep methods

$$u_{n+1} = \sum_{j=0}^{p} a_j u_{n-j} + h \sum_{j=0}^{p} b_j f_{n-j} + h b_{-1} f_{n+1}, \quad n = p, p+1, \ldots \qquad (11.45)$$

  - which is called $(p+1)$-step methods.
  - When $p = 0$, we recover one-step methods.

Adams Methods

- Derived from the solution to (11.1)

$$y(t) - y_0 = \int_{t_0}^{t} f(\tau, y(\tau))d\tau. \qquad (11.2)$$

- through an approximate evaluation of the integral of f between t_n and t_{n+1}, assuming that the discretization nodes are equally spaced.

- The resulting schemes

$$u_{n+1} = u_n + h \sum_{j=-1}^{p} b_j f_{n-j}, \qquad n \geq p. \qquad (11.49)$$

- The interpolation nodes can be either
  - t_n, t_{n-1}, ..., t_{n-p} in which case $b_{-1} = 0$, which is called explicit;
  - t_{n+1}, t_n, ..., t_{n-p+1} in which case $b_{-1} \neq 0$, which is called implicit.

Adams-Bashforth methods (explicit)

- For $p = 1$, the two-step Adams-Bashforth method is

$$u_{n+1} = u_n + \frac{h}{2}\left[3f_n - f_{n-1}\right]. \tag{11.50}$$

- If $p = 2$, we find the three-step Adams-Bashforth method

$$u_{n+1} = u_n + \frac{h}{12}\left[23f_n - 16f_{n-1} + 5f_{n-2}\right],$$

- While for $p = 3$ we get the four-step Adams-Bashforth method

$$u_{n+1} = u_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right).$$

- q-step Adams-Bashforth methods have order q.

BDF Methods

- The so-called backward differentiation formulae

$$u_{n+1} = \sum_{j=0}^{p} a_j u_{n-j} + hb_{-1} f_{n+1}$$

- with $b_{-1} \neq 0$
- derived from directly approximating the value of the first derivative of y at node t_{n+1}

Numerical example

Three-step Adams-Bashforth method solving heat diffusion equation

```
In [6]: show_distribute(u_all[-1], 'Final solution')
```