On the Hairer-Caravenna-Zambotti Reconstruction

by

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Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Berlin, den

Zusammenfassung in deutscher Sprache

Gegeben sei eine Familie von Distributionen $(F_x)_{x\in\mathbb{R}^d}$. Gesucht ist eine Distribution, die für jeden Punkt $x \in \mathbb{R}^d$ durch F_x lokal gut approximiert wird. Wir stellen das *Reconstruction Theorem* vor, welche die Existenz solch einer Distribution sichert und die Eindeutigkeit in bestimmten Fällen.

Das Problem ist von großer Bedeutung für die Behandlung stochastischer Differentialgleichungen; genauer ist es das zentrale Theorem der Theorie der *Regularity Structures* von Martin Hairer, welche auch in diesem Zusammenhang zum ersten Mal bewiesen wurde. Wir geben einen alternativen Zugang zum Reconstruction Theorem, die ohne Regularity Structures auskommt. Stattdessen betten wir das Reconstruction Theorem in die Theorie der Distributionen ein. Mit elementaren Techniken beweisen wir das Theorem. Wir erhalten am Ende ein mächtiges Theorem der Analysis, welche nicht nur für die stochastische Analysis von Interesse ist.

Als Anwendung des Reconstruction Theorems führen wir negative Hölderräume ein und beweisen das Sewing Lemma, ein wichtiges Hilfsmittel in der Theorie der rauen Pfade. Das Sewing Lemma wird oft als das eindimensionale Analogon des Reconstruction Theorems betrachtet. Wir überprüfen diese Aussage.

Abstract

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We give a self-contained and elementary proof of Hairer's Reconstruction Theorem using only the theory of distributions. We introduce an optimal condition for the Reconstruction Theorem called coherence

Using the Reconstruction Theorem, we characterize negative Hölder

Contents

Chapter 1

Introduction

The Reconstruction Theorem allows to construct a distribution f from a family of distributions $(F_x)_{x\in\mathbb{R}^d}$ such that f is locally well-approximated by F_x around $x \in \mathbb{R}^d$. It may be viewed as a converse to Taylor's Theorem if F_x is a Taylor polynomial. However, this view fails to capture the true importance. The Reconstruction Theorem is the most fundamental theorem in the theory of regularity structures — a novel theory proposed by Hairer [7] that provides a robust solution theory to many ill-posed stochastic partial differential equations. In fact, the theory of regularity structures was so groundbreaking that Hairer was awarded the Fields medal for his *"creation of regularity structures"* in 2014 [15]. It is the Reconstruction Theorem, a purely deterministic tool from analysis, that enables the theory of regularity structures.

The original proof of the Reconstruction Theorem relied heavily on wavelet analysis; since then numerous proofs have been published [8, 12, 6, 11, 14]. All of these proofs required profound knowledge of regularity structures, rough path theory or paracontrolled distributions. A concise treatment of the Reconstruction Theorem for a broader audience was not available until Caravenna and Zambotti gave a proof in 2020 based on elementary distribution theory [3].

The aim of this thesis is to give a self-contained proof of the Reconstruction Theorem. Hence, we mimic the proof by Caravenna and Zambotti, which requires no prior knowledge. We hope that this thesis introduces the Reconstruction Theorem to a broader audience. Since the Reconstruction Theorem is a purely analytical tool, it may find applications outside regularity structures or stochastic partial differential equations.

The thesis is structured in the following way: Chapter 1.1 introduces notation. Chapter 1.2 gives a concise overview of the theory of distributions. In Chapter 2 we state the Reconstruction Theorem with its assumptions. This leads to the central notion of *coherence*, an optimal assumption coined by Caravenna and Zambotti. In Chapter 3, 4 and 5 we prove the Reconstruction Theorem. We split the proof into three parts because we consider two cases: $\gamma > 0$ and $\gamma \leq 0$, where γ is a parameter occuring in the Reconstruction Theorem. A major part of the proof holds for all $\gamma \in \mathbb{R}$ and is presented in Chapter 3. We continue the proof for $\gamma > 0$ in Chapter 4, and the proof for $\gamma \leq 0$ in Chapter 5. In Chapter 6 we give an application of the Reconstruction Theorem, where we extend Hölder spaces to negative exponents. In Chapter 7, we end the thesis with a discussion of the relationship between the Reconstruction Theorem and the Sewing Lemma, another analytical tool from Rough Path theory that is often considered as the one-dimensional analogue of the Reconstruction Theorem.

1.1 Notation

Throughout this thesis, the symbol $\mathbb{R}^d$ denotes the d-dimensional Euclidean space with the Euclidean norm $|\cdot|$ for some $d \in \mathbb{N}$. The *open ball* $B(x_0, r)$ centered around $x_0 \in \mathbb{R}^d$ with radius $r > 0$ is defined as $B(x_0, r) = \{x \in \mathbb{R}^d : |x - x_0| \leq r\}$. We write $\bar{A}_\epsilon$ to enlarge a set $A \subset \mathbb{R}^d$ by some $\epsilon \in \mathbb{R}$:

$$\bar{A}_\epsilon \coloneqq A + B(0, \epsilon) \coloneqq \{x \in \mathbb{R}^d : |x - a| \leq \epsilon \text{ for some } a \in A\}.$$

The *multi-index notation* makes many theorems for functions in multiple variables appear as if there is only one variable. A *multi-index* $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ is a d-tuple of non-negative integers. The *length* of k is defined as $|k| = k_1 + \dots + k_d$. We define for all $x, k, l \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$

$$x^k = x_1^{k_1} \cdot \dots \cdot x_d^{k_d}, \qquad k! = k_1! k_2! \cdots k_d!, \qquad \binom{k}{l} = \binom{l_1}{k_1} \cdots \binom{l_d}{k_d} = \frac{k!}{l!(k-l)!}.$$

A polynomial $f(x)$ with real coefficients $\alpha_k \in \mathbb{R}$ of degree $m \in \mathbb{N}_0$ in d variables can be written as

$$f(x) = \sum_{|k| \leq m} \alpha_k x^k \quad \text{and} \quad \partial^k f(x) = \partial_1^{k_1} \cdots \partial_d^{k_d} f(x).$$

We say

- $f \in C$ or $f \in C^0$ if f is continuous,
- $f \in C^k$ if f is k-times continuously differentiable for $k \in \mathbb{N}$, and
- f is smooth if $f \in C^\infty$.

We define the C^k-*norm* as $\|f\|_{C^k} = \max_{|i| \leq k} \|\partial^i f\|_\infty$ where $\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$. Next, we state classical results from analysis without their proofs; the proofs can be found in any standard analysis book, for example [13].

Theorem 1.1 (Taylor's Theorem). *Let $f \in C^k(B(x_0, r))$ and $k \in \mathbb{N}_0^d$. Then, we have $f(x) = \sum_{|j| \leq k} \partial^j f(x_0) \frac{(x-x_0)^j}{j!} + R(x)$ with $\frac{R(x)}{|x-x_0|^k} \to 0$ as $x \to x_0$ for all $x \in B(x_0, r)$.*

Theorem 1.2 (Leibniz Rule)**.** *The Leibniz rule is a generalization of the product rule, that is*

$$\partial^\alpha(fg) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta f)(\partial^{\alpha-\beta}g).$$

Theorem 1.3 (Mean Value Inequality)**.** *Let $f : G \to \mathbb{R}$ be differentiable, where G is an open convex subset of $\mathbb{R}^n$. Let $a, b \in G$. Then, $|f(b) - f(a)| \le \sup_{x\in\overline{ab}} |f'(x)|\,|b-a|$, where $f'(x) = \begin{pmatrix}\partial_{x_1} f & \cdots & \partial_{x_d} f\end{pmatrix}$ is the gradient of f.*

Later, we will consider functions that are said to *annihilate monomials*; they play an essential role in the proof of the Reconstruction Theorem, and we will spend a considerate amount of time constructing such functions. The definition reads as follows.

Definition 1.4 (Annihilation of Monomials)**.** A function $g : \mathbb{R}^d \to \mathbb{R}$ *annihilates monomials* of degree $j \in \mathbb{N}$ if for all $n \in \mathbb{N}_0^d$ with $|n| = j$ we have

$$\int_{\mathbb{R}^d} y^n g(y)\,\mathrm{d}y = 0.$$

For later applications of the Reconstruction Theorem, we introduce the space of *locally α-Hölder functions* denoted by $\mathcal{C}^\alpha$ for positive exponents $\alpha > 0$. It will be our task to extend this space to non-positive exponents $\alpha \le 0$.

Definition 1.5 (Locally α-Hölder Functions)**.** Let $\alpha > 0$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$. We say $\varphi \in \mathcal{C}^\alpha$ if

- $\varphi \in C^r$ for $r = \max\{n \in \mathbb{N}_0 : n < \alpha\}$, and
- there exists a constant $C < \infty$ such that $|\varphi(y) - F_x(y)| \le C|y-x|^\alpha$ uniformly for all x, y in compact sets, where F_x is the Taylor polynomial of φ of order r at x.

For non-positive exponents α, the space $\mathcal{C}^\alpha$ is no longer a space of continuously differentiable functions but *a space of distributions*. It is now time to introduce distributions along with test functions.

1.2 Theory of Distributions

In distribution theory one is interested in

The beauty of [3], on which this bachelor thesis is based, lies within the fact that the Reconstruction Theorem can be stated in terms of elementary distribution theory without the need of regularity structures.

The first concept we will encounter is that of a *support* of a function $\varphi : \mathbb{R}^d \to \mathbb{R}$, which is defined as $\operatorname{supp}(\varphi) = \overline{\{x \in \mathbb{R}^d : \varphi(x) \neq 0\}}$.

Definition 1.6 (Test Function)**.** *Test functions* $\varphi : \mathbb{R}^d \to \mathbb{R}$ are smooth functions that have compact support. The *space of test functions* $\mathcal{D}$ is the set that contains all test functions:

$$\begin{aligned} \mathcal{D} = \mathcal{D}(\mathbb{R}^d) &= \left\{\varphi \in C^\infty(\mathbb{R}^d) : \operatorname{supp}(\varphi) \text{ is compact}\right\}, \\ \mathcal{D}(A) &= \{\varphi \in \mathcal{D} : \operatorname{supp}(\varphi) \subset A\} \qquad \text{for any subset } A \subset \mathbb{R}^d. \end{aligned}$$

A standard example for a test function is the *bump function*:

$$\mathcal{B}(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right), & |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the bump function has compact support in $B(0,1)$. The proof that it is smooth can be found in any standard analysis book, e.g. see (22.2) in [4].

Distributions are the key objects in Distribution Theory.

Definition 1.7 (Distribution)**.** A functional $u : \mathcal{D} \to \mathbb{R}$ is called a *distribution* if u is linear, and if for every compact set $K \subset \mathbb{R}^d$ there exist $r \in \mathbb{N}_0$ and $C < \infty$ such that

$$|u(\varphi)| \leq C\|\varphi\|_{C^r}, \quad \forall \varphi \in \mathcal{D}(K).$$

The *space of all distributions* is denoted $\mathcal{D}' = \{u : \mathcal{D} \to \mathbb{R} \,|\, u \text{ is a distribution}\}$.

Next, we give *convergence in* $\mathcal{D}$ a meaning.

Definition 1.8 (Convergence)**.** Let (φ_j) be a sequence in $\mathcal{D}$ and $\varphi \in \mathcal{D}$. We say $\varphi_j \to \varphi$ in $\mathcal{D}$ if

(i) there exists a compact set $K \in \mathbb{R}^d$ such that $\operatorname{supp}(\varphi)$ and $\operatorname{supp}(\varphi_j)$ are contained in K for all j, and

(ii) $\|\varphi_j - \varphi\|_{C^r} \to 0$ as $j \to \infty$ for all $r \in \mathbb{N}_0$.

This allows us to give an alternative characterization of distributions: a distribution is a linear functional that is *continuous*.

Lemma 1.9. *Let $u : \mathcal{D} \to \mathbb{R}$ be a linear functional. Then, u is a distribution if and only if $\varphi_j \to \varphi$ in $\mathcal{D}$ implies $u(\varphi_j) \to u(\varphi)$ for all test functions φ_j and φ.*

Proof. Let u be a distribution and $\varphi_j \to \varphi$ in $\mathcal{D}$. Then, there exist C and r such that $|u(\varphi_j - \varphi)| \leq C\|\varphi_j - \varphi\|_{C^r} \to 0$ as $j \to \infty$. By linearity, it follows $u(\varphi_j) \to u(\varphi)$.

For the converse direction, we argue by contradiction. Let $\varphi_j \to \varphi$ in $\mathcal{D}$ imply $u(\varphi_j) \to u(\varphi)$ for all test functions φ_j and φ. Assume, there is a compact set $K \subset \mathbb{R}^d$ such that for all $r \in \mathbb{N}_0$ and $C < \infty$ the inequality $|u(\varphi)| \leq C\|\varphi\|_{C^r}$ is violated for some $\varphi \in \mathcal{D}$. Then, we can find φ_n with $|u(\varphi_n)| > n\|\varphi_n\|_{C^n}$ for all $n \in \mathbb{N}$. Next, we define $\phi_n := \frac{\varphi_n}{n\|\varphi_n\|_{C^n}}$. So, we get $u(\phi_n) > 1$. However, $\phi_n \to 0$ in $\mathcal{D}$ as $n \to \infty$ because $\|\phi_n\|_{C^r} \leq \frac{1}{n}$ for all $n \geq r$. $\square$

Quite often, we are given a test function φ, and we would like to construct a sequence (φ_j) such that $\varphi_j \to \varphi$ in $\mathcal{D}$. *Mollifiers* are an indispensable tool for constructing such sequences, which we will use throughout the paper. First, we need to scale and translate arbitrary functions $\varphi : \mathbb{R}^d \to \mathbb{R}$:

$$\varphi_y^\epsilon(x) \coloneqq \frac{1}{\epsilon^d}\,\varphi\left(\frac{x-y}{\epsilon}\right), \quad \varphi^\epsilon(x) \coloneqq \varphi_0^\epsilon(x), \quad \varphi_y(x) \coloneqq \varphi_y^1(x).$$

Given a compactly supported function $\rho : \mathbb{R}^d \to \mathbb{R}$ that integrates to one, a family of scaled and translated versions of ρ is called a *mollifier*.

Definition 1.10 (Mollifier)**.** Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a function with compact support and $\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1$. Then, the family of scaled functions $(\rho^\epsilon)_{\epsilon>0}$ is called a *mollifier*.

Constructing a sequence (φ_j) such that $\varphi_j \to \varphi$ in $\mathcal{D}$ becomes an easy task with the help of mollifiers and *convolutions*. The *convolution* of two functions $f, g \in L^1(\mathbb{R}^d)$ is defined as $(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y$.

Lemma 1.11. *Let $f, g \in L^1(\mathbb{R}^d)$. Then,*

1. *$f * g$ is well-defined almost everywhere, and*
2. *$f * g \in L^1(\mathbb{R}^d)$.*

Proof. We can safely assume f and g to be representatives of the equivalence classes, because we treat $\int f(x)\,\mathrm{d}x$ as Lebesgue integrals and these integrals are independent of the chosen representatives.

First, we check that $(x, y) \mapsto f(x-y)g(y) \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ in order to apply Fubini. From Tonelli's theorem and the translation invariance of the Lebesgue integral we obtain

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |f(x-y)g(y)|\,\mathrm{d}(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\,\mathrm{d}x\,\mathrm{d}y = \|f\|_1\|g\|_1 < \infty.$$

Then, by Fubini's theorem we obtain that $y \mapsto f(x-y)g(y)$ is integrable for almost every $x \in \mathbb{R}^d$. Thus, $(f*g)(x) = \int f(x-y)g(y)\,\mathrm{d}y$ is well-defined for almost every $x \in \mathbb{R}^d$. Also by Fubini, $f * g$ is integrable (if we assign zero in the points of the null set where it is not defined). $\square$

Additionally, the proof shows that

$$\|f * g\|_1 \le \|f\|_1\|g\|_1. \tag{1.1}$$

When we convolute a test function $\varphi \in \mathcal{D}$ against a mollifier (ρ^ϵ), we obtain a sequence $(\varphi * \rho^\epsilon) \subset \mathcal{D}$ that converges to φ in $\mathcal{D}$ as $\epsilon \to 0$.

Lemma 1.12. *Let (ρ^ϵ) be a mollifier. For all test functions $\varphi \in \mathcal{D}$, we have*

1. $\varphi * \rho^\epsilon \in \mathcal{D}$ *for all* $\epsilon > 0$*, and*
2. $\varphi * \rho^\epsilon \to \varphi$ *in* $\mathcal{D}$ *as* $\epsilon \to 0$.

Proof. We first show $\frac{\partial(\varphi * \rho^\epsilon)}{\partial x_j} = \left(\frac{\partial \varphi}{\partial x_j}\right) * \rho^\epsilon$. Applying that rule inductively implies $\varphi * \rho^\epsilon \in C^\infty$. Let e_j denote the j-th unit vector. Consider the difference quotient

$$\frac{(\varphi * \rho^\epsilon)(x + te_j) - (\varphi * \rho^\epsilon)(x)}{t} = \int \frac{\varphi(x + te_j - y) - \varphi(x - y)}{t} \rho^\epsilon(y) \mathrm{d}y.$$

By the mean value inequality (Theorem 1.3), we can bound

$$\frac{\varphi(x + te_j - y) - \varphi(x - y)}{t} \leq \max_{\xi \in [0,t]} \frac{\partial}{\partial x_j} \varphi(x - y + \xi e_j) \leq C$$

for some $C > 0$ since φ is continuously differentiable. Thus, we found an integrable function that dominates $\frac{\varphi(x+te_j-y)-\varphi(x-y)}{t}\rho^\epsilon(y)$, and we can apply Lebesgue's dominated convergence theorem to obtain $\frac{\partial(\varphi * \rho^\epsilon)}{\partial x_j} = \left(\frac{\partial \varphi}{\partial x_j}\right) * \rho^\epsilon$.

The convolution $\varphi * \rho^\epsilon$ has compact support because φ and ρ^ϵ have also compact support. Therefore, we conclude $\varphi * \rho^\epsilon \in \mathcal{D}$.

Now, we show $\varphi * \rho^\epsilon \to \varphi$ in $\mathcal{D}$ as $\epsilon \to 0$. First, there exists a compact set K that contains the support of $\varphi * \rho^\epsilon$ for all $\epsilon \in (0, 1)$ because φ and ρ have compact support. Let the support of ρ be contained in $B(0, r)$ for some $r > 0$. For any multi-index k, $\epsilon \in (0, 1)$ and $x \in K$, we have

$$\partial^k(\varphi * \rho^\epsilon)(x) - \partial^k \varphi(x) = \int \left(\partial^k \varphi(x - y) - \partial^k \varphi(x)\right) \rho^\epsilon(y) \mathrm{d}y$$

because $\int \rho^\epsilon(y)\, \mathrm{d}y = \int \rho(y)\, \mathrm{d}y = 1$. Hence,

$$\begin{aligned} |\partial^k(\varphi * \rho^\epsilon)(x) - \partial^k \varphi(x)| &\leq \int |\partial^k \varphi(x - y) - \partial^k \varphi(x)|\, |\rho^\epsilon(y)| \mathrm{d}y \\ &\Downarrow \text{Mean value theorem} \\ &\leq \max_{z \in K_r} |\partial^{k+1} \varphi(z)| \int |y|\, |\rho^\epsilon(y)| \mathrm{d}y \\ &\Downarrow \text{Substitution: } y = \epsilon \tilde{y} \\ &= \max_{z \in K_r} |\partial^{k+1} \varphi(z)|\, \epsilon \underbrace{\int |\tilde{y}|\, |\rho(\tilde{y})| \mathrm{d}\tilde{y}}_{<\infty}. \end{aligned}$$

As $\epsilon \to 0$, we see that $\sup_{x \in K} |\partial^k(\varphi * \rho^\epsilon)(x) - \partial^k \varphi(x)| \to 0$. □

Next, we can study the effect of applying a distribution F on $\varphi * \rho^\epsilon$, where ρ is a mollifier. We know that $\varphi * \rho^\epsilon \to \varphi$ in $\mathcal{D}$ as $\epsilon \to 0$. Then, $F(\varphi * \rho^\epsilon) \to F(\varphi)$ because F is a distribution. $F(\varphi * \rho^\epsilon)$ is also known as a *mollified distribution*. We have the following result for mollified distributions.

Lemma 1.13. *Let $F \in \mathcal{D}'$. Let $\varphi \in \mathcal{D}$ and g be locally integrable and compactly supported. Then,*

$$F(\varphi * g) = \int_{\mathbb{R}^d} F(\varphi_y) g(y)\,\mathrm{d}y. \tag{1.2}$$

Proof. This follows from linearity of F and Riemann sum approximation; see Lemma 4.12 in [10]. □

If we let $g = \rho$, we obtain the crucial relationship $\int F(\varphi_y)\rho^\epsilon(y)\,\mathrm{d}y \to F(\varphi)$ as $\epsilon \to 0$; or equivalently

$$\int F(\rho_y^\epsilon)\varphi(y)\,\mathrm{d}y \to F(\varphi) \quad \text{as } \epsilon \to 0. \tag{1.3}$$

This is *key* to proving the Reconstruction Theorem.

For future reference, we state the following corollary.

Corollary 1.14. *Let $F \in \mathcal{D}'$ and $g, h, \psi \in \mathcal{D}$. Then,*

$$\int_{\mathbb{R}^d} F((g * h)_z)\psi(z)\,\mathrm{d}z = \iint_{\mathbb{R}^{d\times d}} F(g_y)h(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z.$$

Proof. Note that $(g * h)_z(x) = (g*h)(x-z) = \int g(y)h(x-z-y)\,\mathrm{d}y = (g*h_z)(x)$. Using (1.2) we get $F((g * h)_z) = F(g * h_z) = \int F(g_y)h_z(y)\,\mathrm{d}y$. This proves the corollary. □

Chapter 2

Coherence, Homogeneity and the Reconstruction Theorem

The Reconstruction Theorem was originally stated in the context of regularity structures by Hairer [7]. Later, it was revisited by Caravenna and Zambotti [3], where the Reconstruction Theorem was framed in the theory of distributions. In this chapter, we will closely follow the spirit of Caravenna and Zambotti with the advantage being that it allows for an easily accessible and self-contained treatment of the Reconstruction Theorem.

2.1 A First Peek at the Reconstruction Theorem

Problem: Given a family of distributions $(F_x)_{x\in\mathbb{R}^d}$ we would like to find a distribution $f \in \mathcal{D}'$ that is locally well approximated by F_x around x for every $x \in \mathbb{R}^d$.

We can think of $(F_x)_{x\in\mathbb{R}^d}$ as a family of local candidate approximations for an unknown distribution f. The *Reconstruction Theorem* reconstructs a function f that is well approximated by F_x at any point $x \in \mathbb{R}^d$. Our objective in this section is to find the conditions under which finding a reconstruction f is possible.

First, we only consider a measurable family of distributions $(F_x)_{x\in\mathbb{R}^d}$ which we call *germs* — a notion first introduced in [3].

Definition 2.1 (Germ)**.** A family of distributions $(F_x)_{x\in\mathbb{R}^d}$ is called a *germ* if for all test functions $\psi \in \mathcal{D}$ the map $x \mapsto F_x(\psi)$ is measurable.

Next, being *locally well approximated* by a germ $(F_x)_{x\in\mathbb{R}^d}$ means that there exists a test function $\psi \in \mathcal{D}$, $\int \psi(x)\,\mathrm{d}x \neq 0$ such that for all compact sets $K \subset \mathbb{R}^d$ we have

$$\lim_{\epsilon\to 0} |(f - F_x)(\psi_x^\epsilon)| = 0 \quad \text{uniformly for } x \in K\ . \tag{2.1}$$

The reconstruction theorem states that *under some condition* we can find a reconstruction $f \in \mathcal{D}'$ that satisfies (2.1).

Conjecture 2.2. *Let $(F_x)_{x\in\mathbb{R}^d}$ be a germ that satisfies some yet unknown condition* ???. *Then, there exists a reconstruction $f \in \mathcal{D}'$ and $\gamma > 0$ such that for every test function $\psi \in \mathcal{D}$ there exists $C < \infty$ with*

$$|(f - F_x)(\psi_x^\epsilon)| \leq C\epsilon^\gamma \tag{2.2}$$

uniformly for x in compact sets and $\epsilon \in (0, 1]$.

Note that the distribution f is indeed a reconstruction of the germ $(F_x)_{x\in\mathbb{R}^d}$ in the sense of (2.1) because $|(f - F_x)(\psi_x^\epsilon)| \leq C\epsilon^\gamma \to 0$ as $\epsilon \to 0$.

We would like to find a condition ??? that leads to the above conjecture. Let $(F_x)_{x\in\mathbb{R}^d}$ be a germ. Equation (1.3) will be our starting point for our search of ???. For any $x \in \mathbb{R}^d$ and any test function ψ, the distribution F_x evaluated for ψ can be approximated by the mollified distribution $F_x(\psi * \rho^\epsilon)$ for some mollifier ρ, i.e.

$$\lim_{\epsilon\to 0} F_x(\psi * \rho^\epsilon) \overset{(1.2)}{=} \lim_{\epsilon\to 0} \int F_x(\rho_y^\epsilon)\psi(y)\,\mathrm{d}y \overset{(1.3)}{=} F_x(\psi).$$

This observation inspires us to replace F_x under the integral by F_y so that we obtain the map $f_\epsilon : \psi \mapsto \int F_y(\rho_y^\epsilon)\psi(y)\,\mathrm{d}y$. The motivation for f_ϵ is that we hope for

$$\lim_{\epsilon\to 0} f_\epsilon = \mathcal{R}f \quad \text{where } \mathcal{R}f \text{ is approximated by } F_x \text{ around any } x \in \mathbb{R}^d.$$

Definition 2.3 (Approximating distributions)**.** Let $(F_x)_{x\in\mathbb{R}^d}$ be a germ and $\epsilon_n = 2^{-n}$ for $n \in \mathbb{N}$. The *approximating distribution* $f_n \in \mathcal{D}'$ is defined as

$$f_n : \psi \mapsto \int_{\mathbb{R}^d} F_y(\rho_y^{\epsilon_n})\psi(y)\,\mathrm{d}y$$

for some mollifier ρ.

It is now our task to find ??? such that (H1) the limit $\lim_{n\to\infty} f_n$ exists, and (H2) that this limit satisfies (2.2). This is an easier task since now we are only required to find a promising condition ??? such that (H1) and (H2) hold. We further simplify this problem by ignoring (H2) for the beginning. So, the question becomes: *Under which condition does f_n converge?*

To discuss this question in more depth, we write f_n as a telescopic sum $f_n = f_1 + \sum_{k=1}^{n-1} g_k$ with $g_k = f_{k+1} - f_k$. So, the limit $\lim_{n\to\infty} f_n$ exists if and only if $\sum_{k=1}^{\infty} g_k < \infty$. By definition of f_k, the term g_k can be written as

$$g_k(\psi) = \int_{\mathbb{R}^d} F_y(\rho_y^{\epsilon_{k+1}} - \rho_y^{\epsilon_k})\psi(y)\,\mathrm{d}y.$$

Here, we encounter our very first obstacle. What is an appropriate choice for our mollifier ρ? It turns out that if we can write the *difference* of two mollifiers as a *convolution* of two nice test functions $\hat{\varphi}$ and $\check{\varphi}$, i.e. $\rho_y^{\epsilon_{k+1}} - \rho_y^{\epsilon_k} = (\hat{\varphi}^{\epsilon_k} * \check{\varphi}^{\epsilon_k})_y$, we can write with the help of Corollary 1.14

$$g_k(\psi) = \int_{\mathbb{R}^d} F_z((\hat{\varphi}^{\epsilon_k} * \check{\varphi}^{\epsilon_k})_z)\psi(z)\,\mathrm{d}z = \iint_{\mathbb{R}^{d\times d}} F_z(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z.$$

We are intentionally vague about what *nice test functions* are in this context; we will discuss them in depth in Chapter 3.2, where these nice test functions are called *tweaked test functions*.

Taking a closer look at $F_z(\hat{\varphi}_y^{\epsilon_k})$, we recognize a problem when we let k approach zero: the support of $\hat{\varphi}_y^{\epsilon_k}$ shrinks to some small compact set around y, but F_z is a local candidate approximation around z; so F_z cannot capture the behavior around the point of interest y. We circumvent this problem with the triangle inequality: $F_z(\hat{\varphi}_y^{\epsilon_k}) = F_y(\hat{\varphi}_y^{\epsilon_k}) + \big(F_z(\hat{\varphi}_y^{\epsilon_k}) - F_y(\hat{\varphi}_y^{\epsilon_k})\big)$. Therefore, we can write g_k as

$$g_k(\psi) = \iint F_y(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z + \iint (F_z - F_y)(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z.$$

Remember that we are interested in finding a condition ??? for the germ $(F_x)_{x\in\mathbb{R}^d}$ such that $\sum_{k=1}^{\infty} g_k < \infty$.

To find ??? we let us guide by a closely related problem in another branch of mathematics: rough differential equations. There, one would like to make sense of an integral I_t of the form $I_t = \int_0^t X_s\,\mathrm{d}Y_s$ where X_s and Y_s are paths of low regularity. For instance, let $G \in \mathcal{V}^p$ and $F \in \mathcal{V}^q$ with $\frac{1}{p} + \frac{1}{q} > 1$, where $\mathcal{V}^j$ is the space of all functions with finite j-variation for $j \in \{p, q\}$. Then, there exists a canonical integration theory for this setting (the so called *Young* regime [16]) such that the integral $I_t = \int_0^t G\,\mathrm{d}F$ is defined. The idea is that for very small $|t-s|$ we approximate

$$\int_s^t G\,\mathrm{d}F \approx G(s)(F(t) - F(s)) =: A_{s,t}.$$

We use this approximation to give a meaning to the integral $\int_0^t G\,\mathrm{d}F$. The *Sewing Lemma* [5], an analytical tool, which let integrals of low regularity to be defined in a meaningful sense, allows us to sew the approximations $A_{s,t}$ together to obtain an integral as a Riemann-type sum

$$I_t = \int_0^t G\,\mathrm{d}F \coloneqq \lim_{|\pi|\to 0} \sum_{i=0}^{\#\pi-1} A_{t_i,t_{i+1}} \tag{2.3}$$

for arbitrary partitions[1] π of $[0,t]$ with $|\pi| \to 0$ as $n \to 0$.

[1] Here, a partition of $[0,t]$ is an ordered set $\pi = \{0 = t_0 < t_1 < \cdots < t_k = t\}$, $\#\pi = k$ and $|\pi| = \max\limits_{i=0,\ldots,\#\pi-1} |t_{i+1} - t_i|$.

Lemma 2.4 (Sewing Lemma [2]). *Let $\gamma > 1$ and $\Delta = \{(s,t) : 0 \leq s \leq t \leq T\}$ for some fixed $T > 0$. Let $A : \Delta \to \mathbb{R}$ be a continuous function such that there exists $C < \infty$ with*

$$\delta A_{s,u,t} \coloneqq |A_{s,t} - A_{s,u} - A_{u,t}| \leq C(\max\{|u-s|, |t-u|\})^\gamma \tag{2.4}$$
$$\textit{uniformly for } 0 \leq s \leq u \leq t \leq T.$$

Then, there exists a unique function $I : [0,T] \to \mathbb{R}$ and $\tilde{C} < \infty$ such that $I_0 = 0$ and

$$|I_t - I_s - A_{s,t}| \leq \tilde{C}|t-s|^\gamma$$
$$\textit{uniformly over } 0 \leq s \leq t \leq T.$$

Furthermore, I is the limit of Riemann-type sums as in (2.3).

The connection to the Reconstruction Theorem can be seen in the following way: From a distributional viewpoint, we approximate the integral I_t by F_x:

$$G(x)\int_0^t \psi \,\mathrm{d}F \eqqcolon F_x(\psi) \rightsquigarrow I_t(\psi) \coloneqq \int_0^t G\psi \,\mathrm{d}F$$

If we let $\psi = 1_{[s,t]}$, we get

$$F_s(1_{[s,t]}) = G(s)\int_s^t \mathrm{d}F = G(s)(F(t) - F(s)) = A_{s,t}.$$

If we further assume that $F_s(1_{[s,t]})$ satisfies (2.4), we have by the Sewing Lemma

$$\begin{aligned}(F_x - F_u)((1_{[0,1]})_u^{y-u}) = \frac{(F_x - F_u)(1_{[u,y]})}{y-u} &= \frac{(G(x) - G(u))(F(y) - F(u))}{y-u} \\ &= \frac{\delta A_{x,u,y}}{y-u} \\ &\leq C\frac{(|u-x| + |y-u|)^\gamma}{y-u}.\end{aligned}$$

Hence the germ $(F_x)_{x \in \mathbb{R}^d}$ satisfies

$$(F_x - F_u)((1_{[0,1]})_u^\epsilon) \leq C\epsilon^{-1}(|u-x| + \epsilon)^\gamma$$

for $\epsilon = y - u$ as long as $A_{s,t} = F_s(1_{[s,t]})$ satisfies the Sewing Lemma condition (2.4). This inspires us to define a property coined *coherence* — an optimal condition for the Reconstruction Theorem that was found by Caravenna and Zambotta in [3]. Coherence states that a germ satisfies

$$|(F_z - F_y)(\varphi_y^\epsilon)| \leq C\epsilon^a(|z-y| + \epsilon)^{c-a} \tag{2.5}$$
$$\text{uniformly for } z, y \text{ in compact sets and } \epsilon \in (0,1].$$

for some test function φ, constants c and a. The precise definition will occur in Chapter 2.2. In our previous example $A_{s,t} = F_s(1_{[s,t]})$, we have $a = -1$ and $c = \gamma - 1$ for our coherence condition.

Returning to our problem of finding a bound for g_k (recall that we aim to show that $\sum_{k=1}^{\infty} g_k < \infty$):

$$g_k(\psi) = \iint F_y(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z + \iint (F_z - F_y)(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z,$$

we now have the condition of *coherence* (2.5) to control g_k. It is not difficult to show that the second term in g_k can be bounded with coherence. As we let $k \to \infty$, we have $\epsilon_k \to 0$. Moreover by coherence, $(F_z - F_y)(\hat{\varphi}_y^{\epsilon_k}) \leq C\epsilon_k^a(|z-y| + \epsilon_)k^{c-a}$. If $|z-y| < \epsilon_k$, then $(F_z - F_y)(\hat{\varphi}_y^{\epsilon_k}) \leq C2^{c-y}\epsilon_k^c \to 0$ as $k \to \infty$; this is great news because the remaining part of the second term $\check{\varphi}^{\epsilon_k}(y-z)\psi(z)$ is easily bounded.

Regarding the first term, we want $F_y(\hat{\varphi}_y^{\epsilon}) \to 0$ as $\epsilon \to 0$. One way to achieve this is by imposing a condition which we will call *homogeneity*: if $F_y(\hat{\varphi}_y^{\epsilon}) \leq B\epsilon^{\beta}$ for some constant $B < \infty$, we will say that the germ $(F_x)_{x\in\mathbb{R}^d}$ has homogeneity bound β. If $\beta > 0$, then $F_y(\hat{\varphi}_y^{\epsilon}) \leq B\epsilon^{\beta} \to 0$ as $\epsilon \to 0$. Hence, the first term in g_k can be controlled thanks to homogeneity; this condition in turn with coherence will allow us to show $\sum_{k=1}^{\infty} g_k < \infty$.

It seems that we need a germ to satisfy the coherence and homogeneity condition. Fortunately, we get the homogeneity for free if a germ is coherent[2]. So, requiring a germ to be coherent is all we need to get the Reconstruction Theorem going! It gets even better: so far we showed with the help of coherence that a limiting sequence f_n exists that converges to some f which we *might* call our reconstruction. However, it is not clear if f is a reconstruction in the sense of (2.2). We will see that coherence suffices to show that f is indeed a reconstruction.

2.2 Coherence and Homogeneity

In this chapter we will rigorously introduce the notion of *coherence* and *homogeneity*. We will later see that coherence is sufficient and even necessary for the Reconstruction Theorem. Moreover, homogeneity follows from coherence.

We gave a heuristic motivation for the coherence condition in Chapter 2.1, where we started with the Sewing Lemma and ended up with the following definition for a germ to be *coherent*.

Definition 2.5 (γ-coherent germs). Let $\gamma \in \mathbb{R}$. A germ $(F_x)_{x\in\mathbb{R}^d}$ is called γ-*coherent* if there exists a test function $\varphi \in \mathcal{D}$ with $\int \varphi(x)\,\mathrm{d}x \neq 0$ such that for every compact set $K \subset \mathbb{R}^d$

[2] A germ is said to be coherent if it satisfies the coherence condition in (2.5). The precise definition will be given in Chapter 2.2.

there exists a non-positive real number $\alpha_K \leq \min\{0, \gamma\}$ and a constant $C < \infty$ with

$$|(F_z - F_y)(\varphi_y^\lambda)| \leq C\lambda^\alpha(|z-y| + \lambda)^{\gamma-\alpha} \tag{2.6}$$
uniformly for $z, y \in K$, $|y-z| \leq 2$ and $\lambda \in (0,1]$.

Remark 2.6. Note that we require $|y-z| \leq 2$, which appears rather arbitrary. Indeed, one could also define coherence with $|y-z| \leq R$ for any $R \in \mathbb{R}$ instead; they are both equivalent. Furthermore, we can even drop the constraint $|y-z| \leq 2$ entirely, see Proposition 2.12. In the end, we choose $|y-z| \leq 2$ because it is convenient for our purpose of proving the Reconstruction Theorem.

We say that $(F_x)_{x\in\mathbb{R}^d}$ is $(\boldsymbol{\alpha}, \gamma)$-coherent if $\boldsymbol{\alpha} = (\alpha_K)$ and α_K is the exponent required for the coherence condition (2.6) to hold for the compact set K.

Fix $K, \varphi, \alpha, \gamma$. The *semi-norm* $|||\cdot|||^{\text{coh}}_{K,\varphi,\alpha,\gamma}$ is the smallest constant $C \in \mathbb{R} \cup \{\infty\}$ such that the coherence condition (2.6) holds for $K, \varphi, \alpha, \gamma$. Concretely, we define

$$|||F|||^{\text{coh}}_{K,\varphi,\alpha,\gamma} = \sup\left\{\frac{(F_z - F_y)(\varphi_y^\lambda)}{\lambda^\alpha(|z-y| + \epsilon)^{\gamma-\alpha}} : y, z \in K, |z-y| \leq 2, \lambda \in (0,1]\right\}.$$

We briefly discuss the meaning of coherence. For some constant $C' < \infty$ we rewrite the inequality (2.6) in the coherence assumption as

$$|(F_z - F_y)(\varphi_y^\epsilon)| \leq C' \begin{cases} \epsilon^\gamma & \text{if } |z-y| \leq \epsilon \\ \epsilon^\alpha |z-y|^{\gamma-\alpha} & \text{otherwise} \end{cases}. \tag{2.7}$$

- First, note that $\epsilon^\gamma \leq \epsilon^\alpha$ because of $\epsilon \in (0,1]$ and $\gamma \geq \alpha$. As $|z-y|$ decreases to ϵ, the difference between the two distributions F_z and F_y (evaluated at φ_y^ϵ) changes from magnitude ϵ^α to ϵ^γ. This change becomes very dramatic when $\gamma > 0$ and $\alpha < 0$. Then, ϵ^α diverges while ϵ^γ vanishes as $\epsilon \to 0$.

- Second, observe that the right hand side of (2.7) shrinks as $\alpha \nearrow 0$ for fixed γ, y and z. In other words, the larger α (remember that $\alpha < 0$), the better the estimate gets. Hence, without loss of generality we assume that the map $K \mapsto \alpha_K$ is *monotone*, i.e.

$$K \subset K' \implies \alpha_K \geq \alpha_{K'}. \tag{2.8}$$

  This is achieved by choosing the exponents α_K in the following way: for balls $K = B(0,n)$ of radius $n \in \mathbb{N}$ choose $\alpha_K = \min\left\{\alpha_{B(0,i)} : 1 \leq i \leq n\right\}$; otherwise for general compact sets K choose $\alpha_K = \min\left\{\alpha_{B(0,i)} : 1 \leq i \leq n\right\}$ with $n \in \mathbb{N}$ such that $B(0,n) \supset K$. This ensures that the family of exponents (α_K) is monotone. It will play an important role in the proof of the Reconstruction Theorem in case $\gamma < 0$, see Chapter 5.2.

To conclude, for fixed $y \in \mathbb{R}^d$ and distribution F_y

1. we know more about distributions F_z if $|z-y| \leq \epsilon$ because then $|(F_z - F_y)(\varphi_y^\epsilon)|$ is smaller than for $|z-y| > \epsilon$, and
2. in case of $\gamma > 0$ and $\alpha < 0$ we obtain even more information about F_z where $|z-y| \leq \epsilon$ as ϵ decreases.

Next, we discuss how we need to utilize the coherence condition (2.6) to get the most out of it. Remember, we want to control the $(**)-$part of g_k which is $\iint (F_z - F_y)(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z$. As we found out, the coherence property gives us the most information when $|z-y| \leq \epsilon_k$. This means that we need to carefully select $\check{\varphi}$ such that its support $\mathrm{supp}(\check{\varphi})$ has diameter smaller or equal ϵ_k as this implies $|z-y| \leq \epsilon_k$. When we then apply the coherence condition (2.6), we can bound

$$
\begin{aligned}
&\iint (F_z - F_y)(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z \\
&\quad \leq \sup_{|y-z|\leq\epsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\epsilon_k})| \iint \check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z \\
&\quad \Downarrow \text{coherence condition} \\
&\quad \leq C'\epsilon_k^\gamma \cdot \{\text{constant}\}\,.
\end{aligned}
$$

As we sum $\sum_{k=1}^\infty g_k$, we want $\sum_{k=1}^\infty C'\epsilon_k^\gamma \cdot \{\text{constant}\}$, which is a geometric sum (since $\epsilon_k := 2^{-k}$), to be finite. That is the case when $\gamma > 0$. Thus, we can bound one part of $\sum_{k=1}^\infty g_k$, and the coherence condition (2.6) helped us successfully to show that the approximating distributions f_n do converge.

To control the $(*)$-part of g_k, we have introduced the notion of *homogeneity bound*. We could demand that the germ $(F_x)_{x\in\mathbb{R}^d}$ needs to satisfy the homogeneity bound on top of the coherence condition (2.6), but luckily we get it for free when the germ $(F_x)_{x\in\mathbb{R}^d}$ is γ-coherent. The following lemma is definition and lemma at the same time.

Lemma 2.7 (Homogeneity bound)**.** *Let $(F_x)_{x\in\mathbb{R}^d}$ be a γ-coherent germ. Then, for every compact set $K \subset \mathbb{R}^d$ there exists a real number $\beta_K < \gamma$ and a constant $B < \infty$ such that the* homogeneity bound *holds, i.e.*

$$|F_y(\varphi_y^\epsilon)| \leq B\epsilon^{\beta_K} \quad \textit{uniformly for } y \in K \textit{ and } \epsilon \in (0,1].$$

We say the germ $(F_x)_{x\in\mathbb{R}^d}$ has local homogeneity bound $\boldsymbol{\beta} = (\beta_K)$. *We say the germ $(F_x)_{x\in\mathbb{R}^d}$ has* global homogeneity bound $\beta \in \mathbb{R}$ *if $\beta_K = \beta$ for all compact sets $K \subset \mathbb{R}^d$.*

Proof. We know how to bound $|(F_y - F_z)(\varphi_y^\epsilon)|$ by the coherence condition (2.6). If we can bound $|F_z(\varphi_y^\epsilon)|$, then we can easily obtain

$$|F_y(\varphi_y^\epsilon)| \leq |(F_y - F_z)(\varphi_y^\epsilon) + F_z(\varphi_y^\epsilon)| \leq \{\text{constant}\} \cdot \epsilon^\beta$$

for some β.

Fix any compact set $K \subset \mathbb{R}^d$ and $z \in K$. We use the coherence condition to estimate $|(F_y - F_z)(\varphi_y^\epsilon)| \leq C\epsilon^\alpha(|z-y|+\epsilon)^{\gamma-\alpha} \leq \{C(\operatorname{diam}(K)+1)^{\gamma-\alpha}\} \cdot \epsilon^\alpha$ uniformly for $y \in K$ and $\epsilon \in (0,1]$ (where $\operatorname{diam}(K) := \sup_{y,z\in K}|y-z|$).

To estimate $|F_z(\varphi_y^\epsilon)|$, we know there exist $\tilde{C} < \infty$ and $r \in \mathbb{N}_0$ such that $|F_z(\varphi_y^\epsilon)| \leq \tilde{C}\|\varphi_y^\epsilon\|_{C^r}$ for all $y \in K$ and $\epsilon \in (0,1]$ because F_z is a distribution. Also, we have $\|\partial^k\varphi_y^\epsilon\|_\infty \leq \epsilon^{-|k|-d}\|\partial^k\varphi\|_\infty \leq \epsilon^{-r-d}\|\varphi\|_{C^r}$. Thus, $\|\varphi_y^\epsilon\|_{C^r} \leq \epsilon^{-r-d}\|\varphi\|_{C^r}$ follows. In the end, we obtain $|F_z(\varphi_y^\epsilon)| \leq \{\tilde{C}\|\varphi\|_{C^r}\} \cdot \epsilon^{-r-d}$.

We choose $B = C(\operatorname{diam}(K)+1)^{\gamma-\alpha} + \tilde{C}\|\varphi\|_{C^r}$ and $\beta \leq \min\{\alpha, -r-d, \gamma\}$. We can further decrease β to ensure $\beta < \gamma$. □

Similar to (α_K), the family (β_K) is *monotone* in the sense that

$$K \subset K' \implies \beta_K \geq \beta_{K'}. \tag{2.9}$$

We also introduce a semi-norm that quantifies the homogeneity of a coherent germ F

$$\vert\kern-0.25ex\vert\kern-0.25ex\vert F \vert\kern-0.25ex\vert\kern-0.25ex\vert^{\text{hom}}_{K,\varphi,\beta} = \sup_{\substack{x\in K\\ \epsilon\in(0,1]}} \frac{|F_x(\varphi_x^\epsilon)|}{\epsilon^\beta} \qquad \text{compact set } K \subset \mathbb{R}^d. \tag{2.10}$$

We are ready to state a preliminary version of the Reconstruction Theorem.

Theorem 2.8 (Preliminary Reconstruction Theorem)**.** *Let $\gamma \in \mathbb{R}$. Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a γ-coherent germ. Then, there exists a distribution $f \in \mathcal{D}'$ such that for every test function $\psi \in \mathcal{D}$ and compact set $K \subset \mathbb{R}^d$ there exists a constant $C < \infty$ with*

$$|(f - F_x)(\psi_x^\epsilon)| \leq C \begin{cases} \epsilon^\gamma & \text{if } \gamma \neq 0 \\ 1 + |\log\epsilon| & \text{if } \gamma = 0 \end{cases}$$

uniformly for $x \in K$ and $\epsilon \in (0,1]$.

If $\gamma > 0$, the distribution f is unique, and we say $f = \mathcal{R}F$ (in words: f is the reconstruction *of F).*

2.3 The Reconstruction Theorem in Detail

We state the Reconstruction Theorem of Hairer [7] in the language of distribution theory.

Theorem 2.9 (Reconstruction Theorem)**.** *Let $\gamma \in \mathbb{R}$ and $(F_x)_{x\in\mathbb{R}^d}$ be a $(\boldsymbol{\alpha}, \gamma)$-coherent germ with local homogeneity bounds $\boldsymbol{\beta}$. Then, there exists a distribution $f \in \mathcal{D}'$ such that for every compact set $K \subset \mathbb{R}^d$ and all $r \in \mathbb{N}$ with $r > \max\{-\alpha_{\bar{K}_2}, -\beta_{\bar{K}_2}\}$ we have*

$$|(f - F_x)(\psi_x^\lambda)| \leq \{\text{constant}\} \cdot \vert\kern-0.25ex\vert\kern-0.25ex\vert F \vert\kern-0.25ex\vert\kern-0.25ex\vert^{\text{coh}}_{\bar{K}_2,\varphi,\alpha,\gamma} \cdot \begin{cases} \lambda^\gamma & \text{if } \gamma \neq 0 \\ 1 + |\log\lambda| & \text{if } \gamma = 0 \end{cases} \tag{2.11}$$

uniformly for $\psi \in \mathcal{B}_r$, $x \in K$, $\lambda \in (0,1]$.

The multiplicative constant may only depend on α, γ, r, d and φ. It is explicitly computed in (4.4), (5.4) *and* (5.5) *for the cases $\gamma > 0$, $\gamma < 0$ and $\gamma = 0$ respectively.*

Remark 2.10. We gather some remarks about the reconstruction theorem.

1. As usual, φ denotes the test function in the coherence condition (2.6).

2. For $\gamma > 0$, $f = \mathcal{R}F$ is unique and we call it the *reconstruction* of $F = (F_x)_{x\in\mathbb{R}^d}$. Moreover, the map $F \mapsto \mathcal{R}F$ is linear.

3. For $\gamma \leq 0$, the distribution f need not be unique, but for any fixed $\alpha \leq \min\{0, \gamma\}$, we can choose f in such a way that the map $F \mapsto \mathcal{R}F$ is linear on the vector space of (α, γ)-coherent germs with global homogeneity bound β.

4. The preliminary reconstruction theorem found in the previous section is a special case of the reconstruction theorem stated here. Proof. TO-DO

Remarkably, coherence is not only *sufficient*, that is coherence implies the Reconstruction Theorem. Coherence is also necessary. We say that coherence is an *optimal* condition because it is necessary and sufficient.

Theorem 2.11 (Coherence is necessary)**.** *Fix any $\gamma \in \mathbb{R}$. Let $(F_x)_{x\in\mathbb{R}^d}$ be a germ. Let $f \in \mathcal{D}'$ be a distribution such that for every compact set $K \subset \mathbb{R}^d$ there exists $C < \infty$ and $r \in \mathbb{N}$ with*

$$|(f - F_y)(\psi_y^\lambda)| \leq C\lambda^\gamma \tag{2.12}$$
$$\textit{for all } y \in K, \lambda \in (0, 1] \textit{ and } \psi \in \mathcal{B}_r.$$

Then, $(F_x)_{x\in\mathbb{R}^d}$ is γ-coherent.

Proof. To show that $(F_x)_{x\in\mathbb{R}^d}$ is γ-coherent, we prove that there exists $\alpha \leq \min\{0, \gamma\}$ and a constant $C < \infty$ such that

$$|(F_z - F_y)(\varphi_y^\lambda)| \leq C\lambda^\alpha(|z - y| + \lambda)^{\gamma-\alpha}$$
$$\text{uniformly for } z, y \in K,\ |y - z| \leq \frac{1}{2} \text{ and } \lambda \in (0, 1].$$

Note that we require $|y - z| \leq \frac{1}{2}$ but this is equivalent to Definition 2.5, see Remark 2.6.

Fix a compact set $K \subset \mathbb{R}^d$. Let $x, y \in K$ with $|x - y| \leq \frac{1}{2}$, $\lambda \in (0, \frac{1}{2}]$ and $\psi \in \mathcal{B}_r$. Then,

$$|(F_x - F_y)(\psi_y^\lambda)| \leq |(F_x - f)(\psi_y^\lambda)| + |(f - F_y)(\psi_y^\lambda)| \overset{(2.12)}{\leq} |(F_x - f)(\psi_y^\lambda)| + C\lambda^\gamma.$$

Next, estimating $|(F_x - f)(\psi_y^\lambda)|$ is nontrivial because ψ_y^λ is centered around y and not x. We overcome this obstacle by substituting $\psi_y^\lambda \rightsquigarrow \xi_x^{\lambda_1}$, where

$$\xi := \psi_w^{\lambda_2}, \quad w := \frac{y-x}{|x-y|+\lambda},$$
$$\lambda_1 := |x-y|+\lambda, \text{ and } \lambda_2 := \frac{\lambda}{|x-y|+\lambda}.$$

We quickly verify the correctness of the substitution

$$\xi_x^{\lambda_1} = \frac{\psi\left(\lambda_2^{-1}\left(\frac{\cdot - x}{|x-y|+\lambda} - w\right)\right)}{\left((|x-y|+\lambda)\frac{\lambda}{|x-y|+\lambda}\right)^d} = \frac{\psi\left(\frac{\cdot - x - (y-x)}{\lambda}\right)}{\lambda^d} = \lambda^{-d}\psi\left(\frac{\cdot - y}{\lambda}\right) = \psi_y^\lambda.$$

Hence,

$$|(F_x - f)(\psi_y^\lambda)| = |(F_x - f)\left(\xi_x^{\lambda_1}\|\xi\|_{C^r}^{-1}\right)| \cdot \|\xi\|_{C^r} \overset{(2.12)}{\leq} C\lambda_1^\gamma \|\xi\|_{C^r}. \tag{2.13}$$

To justify (2.12), observe that

- $\lambda_1 \in (0,1]$, and
- $\xi_x^{\lambda_1}\|\xi\|_{C^r}^{-1} \in \mathcal{B}_r$ because $\lambda_2 + |w| = 1$ and $\operatorname{supp}(\psi) \subset B(0,1)$; both imply that $\xi = \psi_w^{\lambda_2}$ is supported in $B(0,1)$, and the scaling factor $\|\xi\|_{C^r}^{-1}$ ensures that the C^r-norm is one.

Additionally, $\|\xi\|_{C^r} = \max_{k\leq r}\|\partial^k \psi_w^{\lambda_2}\|_\infty = \max_{k\leq r}\lambda_2^{-d-k}\|\partial^k\psi\|_\infty \leq \lambda_2^{-d-r}$. So,

$$\begin{aligned} |(F_x - f)(\psi_y^\lambda)| \leq C\lambda_1^\gamma\lambda_2^{-d-r} &= C(|x-y|+\lambda)^\gamma\left(\frac{\lambda}{|x-y|+\lambda}\right)^{-d-r} \\ &\leq C(|x-y|+\lambda)^{\gamma-\alpha}\lambda^\alpha, \end{aligned}$$

where we define $\alpha = \min\{-d-r, \gamma\}$. Then,

$$\begin{aligned} |(F_x - F_y)(\psi_y^\lambda)| &\leq C(|x-y|+\lambda)^{\gamma-\alpha}\lambda^\alpha + C\lambda^\gamma \\ &\Downarrow \text{ where } \lambda^\gamma = \lambda^{\gamma-\alpha}\lambda^\alpha \leq (|x-y|+\lambda)^{\gamma-\alpha}\lambda^\alpha \\ &\leq 2C(|x-y|+\lambda)^{\gamma-\alpha}\lambda^\alpha. \end{aligned}$$

□

We slightly modify the previous proof to prove that the constraint $|z-y| \leq 2$ in the coherence condition (see Definition 2.5) can be dropped, i.e. if (2.6) holds uniformly for any $y, z \in K$ with $|z-y| \leq 2$, then it also holds for any $\tilde{y}, \tilde{z} \in K$ with $|\tilde{z} - \tilde{y}| > 2$ (possibly with a different multiplicative constant C). Hence, we could also define coherence as

$$|(F_z - F_y)(\varphi_y^\lambda)| \leq C\lambda^\alpha(|z-y|+\lambda)^{\gamma-\alpha} \tag{2.14}$$
$$\text{uniformly for } z, y \in K \text{ and } \lambda \in (0,1].$$

Proposition 2.12. *Let F be a γ-coherent germ as in Definition 2.5. Then, it satisfies* (2.14) *for any compact set K provided the multiplicative constant C is adjusted.*

Proof. Let F be a γ-coherent germ and φ be as in Definition 2.5. Fix a compact set $K \subset \mathbb{R}^d$. Assume $y, z \in K$ with $|y-z| > 2$. Let A be a finite family of points in $\mathbb{R}^d$ such that K is covered by A and for each point $x \in K$ there exists $a_x \in A$ with $|x-a| < 2$. Such A exists because K is compact. Then, we have $|(F_z - F_y)(\varphi_z^\lambda)| \leq |(F_z - F_{a_z})(\varphi_z^\lambda)| + |(F_{a_z} - F_y)(\varphi_z^\lambda)|$. The first summand is bounded by (2.6). Bounding the second summand is nontrivial since φ is centered around z and not a_z or y. This is the same situation as in the proof of Theorem 2.11. So, we write

$$|(F_{a_z} - F_y)(\varphi_z^\lambda)| \leq |(F_{a_z} - f)(\varphi_z^\lambda)| + |(f - F_y)(\varphi_z^\lambda)|,$$

where f is the reconstruction of the germ F; note that f exists by the Reconstruction Theorem and the Reconstruction Theorem only requires F to be coherent in the sense of Definition 2.5. Next, we use the same substitution as in (2.13) to obtain an upper bound for both summands. These upper bounds only depend on $|a_z - z|$, $|y-z|$ and λ, which ends the proof. □

Next, we show uniqueness of the reconstruction.

Theorem 2.13 (Uniqueness)**.** *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a germ and $\varphi \in \mathcal{D}$ be a test function with $\int \varphi(x)\mathrm{d}x \neq 0$. Let $K \subset \mathbb{R}^d$ be a compact set, and let $f, g \in \mathcal{D}'$ be any two distributions such that*

$$\lim_{\lambda\to 0} |(f - F_x)(\varphi_x^\lambda)| = 0 \quad \textit{uniformly for } x \textit{ in } K$$
$$\lim_{\lambda\to 0} |(g - F_x)(\varphi_x^\lambda)| = 0 \quad \textit{uniformly for } x \textit{ in } K$$

Then, $f(\psi) = g(\psi)$ for all test functions $\psi \in \mathcal{D}(K)$.

Proof. Define F, φ, K, f and g as in the theorem. Next, we define $T := f - g$, fix $\psi \in \mathcal{D}(K)$ and show $T(\psi) = 0$.

We assume that $\int \varphi(x)\mathrm{d}x = 1$ (otherwise we replace φ by $(\int \varphi(x)\mathrm{d}x)^{-1}\varphi$). Then, the family $(\varphi^\lambda)_{\lambda\in(0,1]}$ is a mollifier, and thus $T(\psi) = \lim_{\lambda\to 0} T(\psi * \varphi^\lambda)$. This allows us to estimate

$$|T(\psi * \varphi^\lambda)| = \left| \int T(\varphi_x^\lambda)\psi(x)\mathrm{d}x \right| \leq \|\psi\|_{L^1} \sup_{x\in K} |T(\varphi_x^\lambda)|,$$

where for the last inequality we recall that ψ has compact support in K. Using the triangle inequality, we bound

$$|T(\varphi_x^\lambda)| = |(f-g)(\varphi_x^\lambda)| \leq |(f - F_x)(\varphi_x^\lambda)| + |(g - F_x)(\varphi_x^\lambda)|.$$

Taking the limit $\lambda \to 0$ proves the uniqueness theorem. □

Chapter 3

Proof of the Reconstruction Theorem

From chapter 1 TO-DO that given a germ $(F_x)_{x\in\mathbb{R}^d}$ we introduced the *approximating distributions*

$$f_n(\psi) \coloneqq \int_{\mathbb{R}^d} F_y(\rho_y^{\epsilon_n})\psi(y)\,\mathrm{d}y, \qquad \text{(Approximating distributions)}$$

which was motivated by $F_x(\psi*\rho^\epsilon) \to F_x(\psi)$. We *guessed* that if the *coherence condition* (2.6) holds, i.e. the germ $(F_x)_{x\in\mathbb{R}^d}$ is $(\boldsymbol{\alpha},\gamma)$-coherent, then

(H1) $\lim_{n\to\infty} f_n$ exists, and

(H2) $\lim_{n\to\infty} f_n$ satisfies the inequality (2.11) of the reconstruction theorem

Then, we *decomposed* the approximating distributions f_n using a telescopic into

$$f_n = f_1 + \sum_{k=1}^{n-1} g_k, \quad g_k(\psi) \coloneqq f_{k+1}(\psi) - f_k(\psi) = \int_{\mathbb{R}^d} F_y(\rho_y^{\epsilon_{k+1}} - \rho_y^{\epsilon_k})\psi(y)\,\mathrm{d}y.$$

For (H1) to hold, the series must therefore be finite. When proving the reconstruction theorem, we will largely be concerned with showing (H1) and (H2). However, there is one caveat that we need to be aware of: the limit $\lim_{n\to\infty} f_n$ of claim (H1) need not exist for $\gamma \le 0$, and we need to divide the proof into two cases $\gamma > 0$ and $\gamma \le 0$. Fortunately, claim (H1) for $\gamma \le 0$ can be fixed in a simple way (TO-DO), and both cases share a large part of the proof.

In the following sections, we prove the reconstruction theorem in multiple steps. As mentioned, a large part of the proof for the case $\gamma > 0$ holds for the case $\gamma \le 0$, too; so the steps we will be presenting now *hold for both cases*; only in the latest steps we divide the proof. We will specifically mention in later steps when we need $\gamma > 0$ or $\gamma \le 0$, but for now we do not worry about it.

3.1 Step 0: Setup

First, we lay the foundation of the proof. We have already given an informal ansatz how we wanted to approach the proof. Specifically, the proof is centered around showing that the limit of the approximating distributions $\lim_{n\to\infty} f_n$ do exist and satisfy the inequality (2.11) of the reconstruction theorem.

Let $\gamma \in \mathbb{R}$ and $F = (F_x)_{x\in\mathbb{R}^d}$ be a $(\boldsymbol{\alpha}, \gamma)$-coherent germ with local homogeneity bounds $\boldsymbol{\beta}$ and test function φ. Without loss of generality, we assume that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are monotone. Let $K \subset \mathbb{R}^d$ be a compact set. Define

$$\alpha \coloneqq \alpha_{\bar{K}_{3/2}} \quad \text{and} \quad \beta \coloneqq \beta_{\bar{K}_{3/2}}$$

such that the coherence condition (2.6) with homogeneity bound (2.7) holds for the $\frac{3}{2}$-enlargement $\bar{K}_{3/2}$, i.e. there exist constants $C, B < \infty$ such that

$$|(F_z - F_y)(\varphi_y^\epsilon)| \leq C\,\epsilon^\alpha(|z-y|+\epsilon)^{\gamma-\alpha} \quad \text{and} \quad |F_y(\varphi_y^\epsilon)| \leq B\epsilon^\beta \tag{3.1}$$
$$\text{for all } y, z \in \bar{K}_{3/2} \text{ with } |z-y| \leq 2, \epsilon \in (0,1]$$

We define a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ by $\epsilon_k = 2^{-k}$. Next define $r \in \mathbb{N}$ such that

$$r > \max\{-\alpha, -\beta\}. \tag{R}$$

This particular choice will allow us to bound $\sum_{k=0}^\infty g_k$.

3.2 Step 1: Tweaking

We briefly discuss the motivation and concept of tweaking: in previous chapters we wrote $f_n = f_1 + \sum_{k=1}^{n-1} g_k$ where

$$g_k(\psi) = f_{k+1}(\psi) - f_k(\psi) = \int_{\mathbb{R}^d} F_y(\rho_y^{\epsilon_{k+1}} - \rho_y^{\epsilon_k})\psi(y)\,\mathrm{d}y$$

for some mollifier ρ. Finding a suitable ρ is the task that we confront ourselves with in this step, which we shall call *tweaking*.

We will see that it turns out to be quite useful if we can write $\rho_y^{\epsilon_{k+1}} - \rho_y^{\epsilon_k}$ as a difference of two test functions $\hat{\varphi}$ and $\check{\varphi}$:

$$\rho_y^{\epsilon_{k+1}} - \rho_y^{\epsilon_k} = (\hat{\varphi}^{\epsilon_k} * \check{\varphi}^{\epsilon_k})_y. \tag{MOL}$$

Additionally, we want $\hat{\varphi}$ and $\check{\varphi}$ to possess some advantageous properties that are listed in the following table.

$\operatorname{supp}(\hat\varphi) = B(0, \frac{1}{2})$	$\operatorname{supp}(\check\varphi) = B(0, 1)$
$\int \hat\varphi(x)\,\mathrm{d}x = 1$	$\int \check\varphi(x)\,\mathrm{d}x = 0$
$\hat\varphi$ annihilates monomials of degree from 1 to $r-1$	$\check\varphi$ annihilates monomials of degree from 0 to $r-1$
$\hat\varphi$ satisfies the coherence condition (3.1)	

Table 3.1: Properties of $\hat\varphi$ and $\check\varphi$

Definition 3.1. A function $g \in \mathcal{D}$ is said to *annihilate monomials* of degree $j \in \mathbb{N}$ if for all $n \in \mathbb{N}_0^d$ with $|n| = j$ we have

$$\int_{\mathbb{R}^d} y^n g(y)\,\mathrm{d}y = 0.$$

Tweaking φ allows us to construct such nice test functions $\hat\varphi$ and $\check\varphi$.

Lemma 3.2 (Tweaking)**.** *Let $r \in \mathbb{N}$, and let $\lambda_0, \dots, \lambda_{r-1} \in \mathbb{R}_{>0}$ be pairwise distinct. Define*

$$c_0 = 1 \quad \textit{and} \quad c_i = \prod_{k\in\{0,\dots,r-1\}\setminus\{i\}} \frac{\lambda_k}{\lambda_k - \lambda_i}, \quad i > 0.$$

Then, for every measurable and compactly supported function $\varphi : \mathbb{R}^d \to \mathbb{R}$ and every $a \in \mathbb{R}$ the tweaked function

$$\mathcal{T}_\varphi : x \mapsto a \sum_{i=0}^{r-1} c_i \varphi^{\lambda_i}(x)$$

has integral equal to $a \int \varphi(x)\,\mathrm{d}x$ and annihilates monomials of degree from 1 to $r-1$.

Proof. The case for $r = 1$ is simple: $\int \mathcal{T}_\varphi(x)\mathrm{d}x = a \int \varphi^{\lambda_0}(x)\mathrm{d}x = a \int \varphi(x)\mathrm{d}x$.

Let $r \geq 2$. Given all λ_i's we solve for the variables c_i's such that the desired properties hold. Luckily for us, this is a simple system of linear equations. Write

$$\int y^k \mathcal{T}_\varphi(y)\,\mathrm{d}y = a \sum_{i=0}^{r-1} c_i \int y^k \varphi^{\lambda_i}(y)\,\mathrm{d}y = a \sum_{i=0}^{r-1} c_i \lambda_i^{|k|} \int x^k \varphi(x)\,\mathrm{d}x, \quad \forall k \in \mathbb{N}^d$$

where we substituted $y \rightsquigarrow \lambda_i x$. Now observe that for $k = 0$ we get

$$\int \mathcal{T}_\varphi(y)\,\mathrm{d}y = a \sum_{i=0}^{r-1} c_i \lambda_i^{|k|} \int \varphi(x)\,\mathrm{d}x.$$

Thus, if we find c_i's such that the constraint $\sum_{i=0}^{r-1} c_i\lambda_i = 1$ holds, the tweaked function $\mathcal{T}_\varphi$ has integral equal to $a\int \varphi(x)\mathrm{d}x$. Next, if we let $1 \le |k| \le r-1$, we want $\int y^k \mathcal{T}_\varphi(y) = 0$; so the constraint $\sum_{i=0}^{r-1} c_i\lambda_i^{|k|} = 0$ needs to be satisfied.

In the language of linear algebra, we try to solve

$$\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_0 & \cdots & \lambda_{r-1} \\ \lambda_0^2 & \cdots & \lambda_{r-1}^2 \\ & & \\ \lambda_0^{r-1} & \cdots & \lambda_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \\ c_{r-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \\ 0 \end{pmatrix}.$$

The matrix on the left is a *Vandermonde matrix* for which it is easy to compute the determinant: $\det = \prod_{1\le i<j\le r-1} \lambda_j - \lambda_i$. Therefore, a solution c exists if and only if the determinant does not vanish if and only if all λ_i's are distinct. If we let A denote the left hand side matrix, the inverse of A can be explicitly stated

$$(A^{-1})_{i=0,\dots,r-1}^{j=0,\dots,r-1} = (-1)^j \frac{\sum\limits_{\substack{U\subset\{0,\dots,r-1\}\setminus\{i\} \\ |U|=r-1-j}} \prod\limits_{u\in U} \lambda_u}{\prod\limits_{v\in\{0,\dots,r-1\}\setminus\{i\}} (\lambda_v - \lambda_i)},$$

see equation (7) in [9] for more details. We left-multiply the linear system with this inverse

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \\ c_{r-1} \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \\ 0 \end{pmatrix},$$

and we finally confirm that the vector c is a solution if and only if

$$c_i = \frac{\prod\limits_{u\in\{0,\dots,r-1\}\setminus\{i\}} \lambda_u}{\prod\limits_{v\in\{0,\dots,r-1\}\setminus\{i\}} (\lambda_v - \lambda_i)} = \prod_{k\in\{0,\dots,r-1\}\setminus\{i\}} \frac{\lambda_k}{\lambda_k - \lambda_i}.$$

□

We define $\hat{\varphi}$ as

$$\varphi = \mathcal{T}_\varphi \tag{3.2}$$

for $a = \frac{1}{\int \varphi(x)\,\mathrm{d}x}$ and $\lambda_i = \frac{2^{-(i+1)}}{1+R_\varphi}$ for all $i = 0, \dots, r-1$.

Lemma 3.3. *$\hat{\varphi}$ satisfies the properties in Table 3.1.*

Proof. By the tweaking lemma, $\hat{\varphi}$ integrates to one and annihilates all monomials from degree 1 to $r-1$. We also have $\operatorname{supp}(\hat{\varphi}) = B(0, \frac{1}{2})$ because the support of $\hat{\varphi}$ depends on the largest λ_i, and $\lambda_i = \frac{1}{2^{i+1}(1+R_\varphi)} \leq \frac{1}{2R_\varphi}$.

It remains to show that the coherence inequality (3.1) holds if φ is replaced by $\hat{\varphi}$ (possibly with different constants C and B). By definition of $\hat{\varphi} = \mathcal{T}_\varphi$ we have $|(F_z - F_y)(\hat{\varphi}_y^\epsilon)| = a\sum_{i=0}^{r-1} |c_i|\,|(F_z - F_y)(\varphi_y^{\lambda_i\epsilon})|$. Next, we bound

$$
\begin{aligned}
|(F_z - F_y)(\varphi_y^{\epsilon\lambda_i})| &\overset{(3.1)}{\leq} C(\epsilon\lambda_i)^\alpha(|z-y| + \epsilon\lambda_i)^{\gamma-\alpha} \\
&\Downarrow \text{because } \alpha < 0 \text{ and } \lambda_i > \frac{2^{-(r+1)}}{1+R_\varphi} \\
&\leq \left(\frac{2^{-(r+1)}}{1+R_\varphi}\right)^\alpha C\epsilon^\alpha(|z-y| + \epsilon\lambda_i)^{\gamma-\alpha} \\
&\Downarrow \text{because } \gamma - \alpha \geq 0 \text{ and } \lambda_i \leq 1 \\
&\leq \left(\frac{2^{-(r+1)}}{1+R_\varphi}\right)^\alpha C\epsilon^\alpha(|z-y| + \epsilon)^{\gamma-\alpha}.
\end{aligned}
$$

To estimate the constants c_i, we use $|c_i| \leq e^2$ — this fact will be proved in the next lemma (see equation (3.5)), but we can already use it here. Altogether, we have

$$
|(F_z - F_y)(\hat{\varphi}_y^\epsilon)| \leq \frac{e^2 r}{|\int \varphi(x)\,\mathrm{d}x|}\left(\frac{2^{-(r+1)}}{1+R_\varphi}\right)^\alpha C\,\epsilon^\alpha(|z-y| + \epsilon)^{\gamma-\alpha}.
$$

Analogously,

$$
|F_y(\hat{\varphi}_y^\epsilon)| \leq \frac{e^2 r}{|\int \varphi(x)\,\mathrm{d}x|} B\epsilon^\beta\lambda_i^\beta \leq \frac{e^2 r}{|\int \varphi(x)\,\mathrm{d}x|}\left(\frac{2^{-(r+1)}}{1+R_\varphi}\right)^{\min\{\beta,0\}} B\epsilon^\beta
$$

Therefore, the coherence and homogeneity condition still hold if we do the following replacements:

$$
\begin{aligned}
\varphi &= \hat{\varphi}, \\
\hat{C} &= C\frac{e^2 r}{|\int \varphi(x)\,\mathrm{d}x|}\left(\frac{2^{-(r+1)}}{1+R_\varphi}\right)^\alpha \\
\hat{B} &= B\frac{e^2 r}{\int \varphi(x)\,\mathrm{d}x}\left(\frac{2^{-(r+1)}}{1+R_\varphi}\right)^{\min\{\beta,0\}}.
\end{aligned}
\tag{3.3}
$$

□

What we have just proven is of elementary importance for the next steps; in other words the existence of the tweaked function $\hat{\varphi}$ does a lot of heavylifting for us. The main takeaway is the following fact: if φ satisfies the coherence condition, so does $\hat{\varphi}$:

$$|(F_z - F_y)(\hat{\varphi}_y^\epsilon)| \leq \hat{C}\epsilon^\alpha(|z-y|+\epsilon)^{\gamma-\alpha} \quad \text{and} \quad |F_y(\hat{\varphi}_y^\epsilon)| \leq \hat{B}\epsilon^\beta \qquad (\widehat{\texttt{COH}})$$
$$\text{for all } y, z \in \bar{K}_{3/2} \text{ with } |z-y| \leq 2, \epsilon \in (0,1].$$

Next, we define $\check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2$, and quickly verify the properties in Table 3.1.

- First, $\operatorname{supp}(\check{\varphi}) \subset B(0,1)$ because $\operatorname{supp}(\hat{\varphi}) \subset B(0, \frac{1}{2})$.
- Second, $\int \check{\varphi}(x)\,\mathrm{d}x = 0$ because $\int \hat{\varphi}^{\frac{1}{2}}(x)\,\mathrm{d}x = \int \hat{\varphi}^2(x)\,\mathrm{d}x$.
- Third, $\check{\varphi}$ annihilates monomials of degree 1 to $r-1$ because $\hat{\varphi}$ annihilates monomials of degree 1 to $r-1$.

Finally, we set $\rho = \hat{\varphi}^2 * \hat{\varphi}$. This is a mollifier because $\int \hat{\varphi}^2(x)\,\mathrm{d}x = \int \hat{\varphi}(x)\,\mathrm{d}x = 1$. Then, $\rho^{\frac{1}{2}} - \rho = (\hat{\varphi}^2 * \hat{\varphi})^{\frac{1}{2}} - (\hat{\varphi}^2 * \hat{\varphi}) = (\hat{\varphi} * \hat{\varphi}^{\frac{1}{2}}) - (\hat{\varphi}^2 * \hat{\varphi}) = \hat{\varphi} * (\hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2) = \hat{\varphi} * \check{\varphi}$. Hence, we get $\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k} = (\rho^{\frac{1}{2}} - \rho)^{\epsilon_k} = (\hat{\varphi} * \check{\varphi})^{\epsilon_k} = \hat{\varphi}^{\epsilon_k} * \check{\varphi}^{\epsilon_k}$ — this is exactly what we want: *we found a mollifier whose difference is the convolution of two tweaked test functions.*

The remaining chapter is devoted to three technical lemmas that involve estimating the tweaked test functions $\hat{\varphi}$ and $\check{\varphi}$. They are only used in some minor parts of the proofs that are about to come. You can either skip to Chapter 3.3: Step 2 right away and come back when the lemmas are actually needed.

Many upcoming proofs require estimating the L^1-norm of $\hat{\varphi}$. We even used an estimate in Lemma 3.3.

Lemma 3.4. *We estimate*

$$\|\hat{\varphi}\|_{L^1} \leq \frac{e^2 r}{|\int \varphi(x)\mathrm{d}x|}\|\varphi\|_{L^1}. \tag{3.4}$$

Proof. We start with

$$\|\hat{\varphi}\|_{L^1} = \int |\hat{\varphi}(x)|\mathrm{d}x = \int |\mathcal{T}_\varphi(x)|\,\mathrm{d}x \leq |a| \sum_{i=0}^{r-1} |c_i| \int |\varphi^{\lambda_i}(x)|\mathrm{d}x$$

where $c_i = \prod_{k\in\{0,\dots,r-1\}\setminus\{i\}} \frac{\lambda_k}{\lambda_k - \lambda_i}$ and $\lambda_k = \frac{2^{-(k+1)}}{1+R_\varphi}$. So,

$$|c_i| = \left| \prod_{k\in\{0,\dots,r-1\}\setminus\{i\}} \frac{2^{-(k+1)}}{2^{-(k+1)} - 2^{-(i+1)}} \right| = \prod_{k\in\{0,\dots,r-1\}\setminus\{i\}} \frac{1}{|1-2^{k-i}|}.$$

Since $|1-2^{k-i}| \geq 1$ for all $k > i$, we have

$$|c_i| = \prod_{k\in\{0,\dots,r-1\}\setminus\{i\}} \frac{1}{|1-2^{k-i}|} \leq \prod_{k=1}^{\infty} \frac{1}{1-2^{-m}}$$

Note that $(1-x)^{-1} \leq 1+2x \leq e^{2x}$ for $x \in [0, \frac{1}{2}]$. So, by substituting $2^{-m} \rightsquigarrow x$, we get

$$|c_i| \leq \prod_{m=1}^{\infty} 1 + 2^{-m} \leq e^2 \tag{3.5}$$

Then, using $a = \frac{1}{\int \varphi(x)\mathrm{d}x}$ and $\|\varphi^{\lambda_i}\|_{L^1} = \|\varphi\|_{L^1}$ we end up with

$$\|\hat{\varphi}\|_{L^1} \leq |a| \sum_{i=0}^{r-1} |c_i| \int |\varphi^{\lambda_i}(x)|\mathrm{d}x \leq \frac{1}{|\int \varphi(x)\mathrm{d}x|} e^2 r \|\varphi\|_{L^1}.$$

□

Estimating the convolution of the tweaked test function $\check{\varphi}$ with some test function g is also of interest for us (especially in Step 3).

Lemma 3.5. *Let $K \subset \mathbb{R}^d$ be a compact set, and let $g \in \mathcal{D}(K)$. For any $\epsilon > 0$, the function $\check{\varphi}^\epsilon * g$ is supported in $\bar{H}_\epsilon$, and we estimate*

$$\|\check{\varphi}^\epsilon * g\|_{L^1} \leq \mathrm{Vol}(\bar{H}_\epsilon) \|\check{\varphi}\|_{L^1} \|g\|_{C^r} \epsilon^r.$$

Proof. The core idea is to utilize the annihilation property of $\check{\varphi}$. Let $\epsilon > 0$, and let $T_y g$ be the Taylor polynomial of g of order $(r-1)$ at $y \in \mathbb{R}^d$, i.e. $T_y g(x) = \sum\limits_{|k| \leq r-1} \frac{1}{k!} \partial^k g(y)(x-y)^k$. We estimate the error term of the Taylor polynomial to be bounded by

$$|g(x) - T_y g(x)| \leq \|g\|_{C^r} |x-y|^r.$$

This follows from the fact that the error term can be explicitly given by $\frac{1}{r!}\partial^r g(\xi)(x-y)^r$ for some ξ between x and y. Next, by the annihilation property of $\check{\varphi}$ we have

$$\int \check{\varphi}^\epsilon(x-y) T_x g(y)\, \mathrm{d}y = 0.$$

Hence, we write

$$\begin{aligned} |\check{\varphi}^\epsilon * g(x)| = \left| \int \check{\varphi}^\epsilon(x-y)(g(y) - T_x g(y))\, \mathrm{d}y \right| &\leq \int |\check{\varphi}^\epsilon(x-y)| \|g\|_{C^r} |x-y|^r \, \mathrm{d}y \\ &\Downarrow \mathrm{supp}(\check{\varphi}^\epsilon) \subset B(0,\epsilon) \\ &\leq \|\check{\varphi}\|_{L^1} \|g\|_{C^r} \epsilon^r. \end{aligned}$$

Since g has compact support in H and $\check{\varphi}$ in $B(0,1)$, $\check{\varphi}^\epsilon * g$ is supported in $\bar{H}_\epsilon$. So, we integrate $|\check{\varphi}^\epsilon * g|$ over $\bar{H}_\epsilon$, and we obtain the claim. □

For the last lemma, consider the problem of finding an upper bound for the convolution of any test function ψ with $\hat{\varphi}$ or $\check{\varphi}$ against some arbitrary function G.

Lemma 3.6. *Let $\lambda, \epsilon \in (0,1]$, $K \subset \mathbb{R}^d$ be compact and $G : \mathbb{R}^d \to \mathbb{R}$ be measurable. Then, for any $x \in K$ and $\psi \in \mathcal{B}_r$ we have*

$$\Big| \int_{\mathbb{R}^d} G(y) \left(\hat{\varphi}^{2\epsilon} * \psi_x^\lambda\right)(y)\,\mathrm{d}y \Big| \leq 2^d \|\hat{\varphi}\|_{L^1} \sup_{B(x,\lambda+\epsilon)} |G|, \tag{3.6}$$

$$\Big| \int_{\mathbb{R}^d} G(y) \left(\check{\varphi}^{\epsilon} * \psi_x^\lambda\right)(y)\,\mathrm{d}y \Big| \leq 4^d \|\check{\varphi}\|_{L^1} \min\left\{ \left(\frac{\epsilon}{\lambda}\right)^r, 1\right\} \sup_{B(x,\lambda+\epsilon)} |G| \tag{3.7}$$

Proof. Regarding the first inequality, note that $\hat{\varphi}^{2\epsilon} * \psi_x^\lambda$ has support in $B(x, \lambda+\epsilon)$ because ψ is supported in $B(0,1)$ (due to $\psi \in \mathcal{B}_r$) and $\hat{\varphi} \in B(0, \frac{1}{2})$. So,

$$\Big| \int_{\mathbb{R}^d} G(y) \left(\hat{\varphi}^{2\epsilon} * \psi_x^\lambda\right)(y)\,\mathrm{d}y \Big| \leq \|\hat{\varphi}^{2\epsilon} * \psi_x^\lambda\|_{L^1} \sup_{B(x,\lambda+\epsilon)} |G| \overset{(1.1)}{\leq} \|\hat{\varphi}^{2\epsilon}\|_{L^1} \|\psi_x^\lambda\|_{L^1} \sup_{B(x,\lambda+\epsilon)} |G|.$$

$\|\psi\|_{L^1}$ is bounded by the volume of the unit ball in $\mathbb{R}^d$ because $\|\psi\|_\infty \leq 1$ and ψ has support in $B(0,1)$. Hence,

$$\|\psi\|_{L^1} \leq 2^d. \tag{3.8}$$

With $\|\hat{\varphi}^{2\epsilon}\|_{L^1} = \|\hat{\varphi}\|_{L^1}$, we obtain the first inequality.

For the second inequality, we use the exact same argument as for the first inequality to obtain $\left| \int_{\mathbb{R}^d} G(y) \left(\check{\varphi}^{\epsilon} * \psi_x^\lambda\right)(y)\,\mathrm{d}y \right| \leq 2^d \|\check{\varphi}\|_{L^1} \sup_{B(x,\lambda+\epsilon)} |G|$. This yields the case $\lambda \leq \epsilon$.

If $\epsilon < \lambda$, we get an even sharper bound. We use Lemma 3.5 to get

$$\|\check{\varphi}^{\epsilon} * \psi_x^\lambda\|_{L^1} \leq \mathrm{Vol}(B(x, \lambda+\epsilon)) \|\psi_x^\lambda\|_{C^r} \epsilon^r \|\check{\varphi}\|_{L^1}.$$

The ball $B(x, \lambda+\epsilon)$ has diameter $2(\lambda+\epsilon)$, so its volume is smaller than $(2(\lambda+\epsilon))^d$. Since $\epsilon < \lambda$, we get $\mathrm{Vol}(B(x,\lambda+\epsilon)) \leq 4^d \lambda^d$. The C^r-norm of ψ_x^λ can be computed by

$$\|\psi_x^\lambda\|_{C^r} = \max_{|k| \leq r} \|\partial^k \psi_x^\lambda\|_\infty = \max_{|k|\leq r} \frac{1}{\lambda^{d+|k|}} \|\psi\|_\infty \leq \frac{1}{\lambda^{d+r}}.$$

This completes the proof of the second inequality. □

3.3 Step 2: Decomposition

Let $\psi \in \mathcal{D}$ be any test function. With the right mollifier in our toolkit, we can decompose the approximating distribution $f_n(\psi) = \int_{\mathbb{R}^d} F_z(\rho_z^{\epsilon_n}) \psi(z)\,\mathrm{d}z$ into $f_n(\psi) = f_1(\psi) + \sum_{k=1}^{n-1} g_k(\psi)$, where

$$
\begin{aligned}
g_k(\psi) \coloneqq f_{k+1}(\psi) - f_k(\psi) &= \int_{\mathbb{R}^d} F_z(\rho_z^{\epsilon_{k+1}} - \rho_z^{\epsilon_k})\psi(z)\,\mathrm{d}z \\
&\overset{\texttt{(MOL)}}{=} \int_{\mathbb{R}^d} F_z((\hat{\varphi}^{\epsilon_k} * \check{\varphi}^{\epsilon_k})_z)\psi(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_z(\hat{\varphi}^{\epsilon_k}_{y+z})\check{\varphi}^{\epsilon_k}(y)\psi(z)\,\mathrm{d}y\,\mathrm{d}z \\
&\Downarrow \text{substitute } y+z \text{ by a new variable} \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_z(\hat{\varphi}^{\epsilon_k}_{y})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}
$$

Note that $\hat{\varphi}$ is centered around y and not z. So, we use the triangle inequality and write

$$
\begin{aligned}
g_k(\psi) &= \iint F_z(\hat{\varphi}^{\epsilon_k}_{y})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z \\
&= \iint F_y(\hat{\varphi}^{\epsilon_k}_{y})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z + \iint (F_z - F_y)(\hat{\varphi}^{\epsilon_k}_{y})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}
$$

Thus, we have decomposed g_k into $g_k = g'_k + g''_k$ where

$$
\begin{aligned}
g'_k(\psi) &\coloneqq \iint F_y(\hat{\varphi}^{\epsilon_k}_{y})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z \\
g''_k(\psi) &\coloneqq \iint (F_z - F_y)(\hat{\varphi}^{\epsilon_k}_{y})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}
$$

This allows us to use the coherence condition for g''_k and the homogeneity bound for g'_k. Finally, we write

$$
f_n(\psi) = f_1(\psi) + \sum_{k=1}^{n-1} g_k(\psi) = f_1 + \sum_{k=1}^{n-1} g'_k(\psi) + \sum_{k=1}^{n-1} g''_k(\psi). \tag{3.9}
$$

Our goal is to show that $\lim_{n\to\infty} f_n(\psi)$ exists for all $\psi \in \mathcal{D}$. Hence, our goal for the subsequent chapters is to prove that the series $\sum g'_k(\psi)$ and $\sum g''_k(\psi)$ converge.

Remark 3.7. The choice of our mollifier ρ depends on K since $\rho \coloneqq \hat{\varphi}^2 * \hat{\varphi}$, and $\hat{\varphi} \coloneqq$. Thus, the approximating distributions depend on TO-DO

We will give a brief overview over the next steps because now we must be careful about γ. Remember that we have fixed a compact set K. Furthermore, notice that the approximating distributions f_n depend on K since our mollifier ρ depends on $\hat{\varphi}$, which depends on r, and the value of r depends on $\alpha = \alpha_{\bar{K}_{3/2}}$ and $\beta = \beta_{\bar{K}_{3/2}}$.

- In step 3, we show that for every $\gamma \in \mathbb{R}$ the series $\sum_{k=1}^{\infty} g'_k(\psi)$ converges for all test functions $\psi \in \mathcal{D}(\bar{K}_1)$.
- In step 4, we show that for $\gamma > 0$ the series $\sum_{k=1}^{\infty} g''_k(\psi)$ converges for all test functions $\psi \in \mathcal{D}(\bar{K}_1)$. Thus, $\lim_{n\to\infty} f_n(\psi)$ is well defined for $\gamma > 0$, and we define

$$f^K := \lim_{n\to\infty} f_n.$$

- In step 5, we show that for all $\gamma > 0$ the function f^K is a distribution on $\bar{K}_1$ which satisfies (2.11)

$$|(f^K - F_x)(\psi_x^\epsilon)| \leq C\epsilon^\gamma$$
$$\text{uniformly for } \psi \in \mathcal{B}_r,\ x \in K,\ \epsilon \in (0,1],$$

  where $C := \text{const}(\alpha, \gamma, r, d, \varphi) \cdot |||F|||^{\text{coh}}_{\bar{K}_{3/2},\varphi,\alpha,\gamma}$.

- In step 6, we show that for all $\gamma > 0$, the distributions f^K are consistent, i.e.

$$\text{for } K \subset K' : f^K(\psi) = f^{K'}(\psi) \quad \forall \psi \in \mathcal{D}(\bar{K}_1).$$

  This will allow us to build a global distribution $f \in \mathcal{D}'$.

We see that we need to divide the proof in two cases $\gamma > 0$ and $\gamma \leq 0$ when we reach step 4. Fortunately, step 4 can be fixed for $\gamma \leq 0$ and thus the proof for $\gamma \leq 0$ resembles that of $\gamma > 0$.

3.4 Step 3: $\sum |g'_k|$ converges for all $\gamma \in \mathbb{R}$

In this step, we want to show that $\sum |g'_k(\psi)|$ converges for all test functions $\psi \in \mathcal{D}(\bar{K}_1)$. To show this, we estimate $g'_k(\psi) := \iint F_y(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z$. We make use of the following three facts:

1. a *compact support* argument for $\check{\varphi}^{\epsilon_k}$ and ψ,
2. a *local homogeneity bound* argument ($\widehat{\texttt{COH}}$), and
3. an *annihilation* argument due to $\check{\varphi}$.

Let, $\psi \in \mathcal{D}(\bar{K}_1)$. First, we rewrite g'_k as

$$g'_k(\psi) = \iint F_y(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z = \int F_y(\hat{\varphi}_y^{\epsilon_k})\,(\check{\varphi}^{\epsilon_k} * \psi)(y)\,\mathrm{d}y. \tag{3.10}$$

The compact support argument for $\check{\varphi}^{\epsilon_k}$ and ψ works as follows:

- the tweaked test function $\check{\varphi}^{\epsilon_k}$ has a compact support in $B(0,\epsilon_k)$ because $\check{\varphi}$ has a compact support in the unit ball $B(0,1)$; since $\epsilon_k = 2^{-k} \le \frac{1}{2}$ for all $k \in \mathbb{N}$, we have $\operatorname{supp}(\check{\varphi}^{\epsilon_k}) \subset B(0,\frac{1}{2})$;
- by assumption $\psi \in \mathcal{D}(\bar{K}_1)$, the compact support of ψ lies in $\bar{K}_1$.

Thus by Lemma 3.5, we have $\operatorname{supp}(\check{\varphi}^{\epsilon_k} * \psi) \subset \bar{K}_{3/2}$. We therefore obtain

$$|g_k'(\psi)| = \left| \int F_y(\hat{\varphi}_y^{\epsilon_k})\,(\check{\varphi}^{\epsilon_k} * \psi)(y)\,\mathrm{d}y \right| \le \sup_{y\in\overline{K}_{3/2}} \left(|F_y(\hat{\varphi}_y^{\epsilon_k})|\right) \|\check{\varphi}^{\epsilon_k} * \psi\|_{L^1}.$$

Next, by local homogeneity bound argument we have $\sup_{y\in\overline{K}_{3/2}} |F_y(\hat{\varphi}_y^{\epsilon_k})| \overset{\widehat{(\mathtt{COH})}}{\le} \hat{B}\epsilon_k^{\beta}$. So, we have

$$|g_k'(\psi)| \le \hat{B}\epsilon_k^{\beta}\,\|\check{\varphi}^{\epsilon_k} * \psi\|_{L^1}$$

With Lemma 3.5, we finalize step 3 as follows

$$\begin{aligned} |g_k'(\psi)| \le \hat{B}\epsilon_k^{\beta}\,\|\check{\varphi}^{\epsilon_k} * \psi\|_{L^1} &\le \hat{B}\epsilon_k^{\beta}\,\mathrm{Vol}(\overline{K}_{3/2})\|\check{\varphi}\|_{L^1}\|\psi\|_{C^r}\epsilon_k^r \\ &= \left(\hat{B}\,\mathrm{Vol}(\overline{K}_{3/2})\|\check{\varphi}\|_{L^1}\|\psi\|_{C^r}\right)\epsilon_k^{\beta+r}. \end{aligned} \tag{3.11}$$

In *Step 0: Setup*, we chose r such that $\beta + r > 0$; hence $\sum_{k=1}^{\infty} \epsilon_k^{\beta+r} < \infty$. Thus, $\sum_{k=1}^{\infty} g_k'(\psi)$ is finite for all $\gamma \in \mathbb{R}$ and all $\psi \in \mathcal{D}(\bar{K}_1)$.

Remark 3.8. One might wonder why we even needed Lemma 3.5 if we can just estimate $\|\check{\varphi}^{\epsilon_k} * \psi\|_{L^1} \le \|\check{\varphi}^{\epsilon_k}\|_{L^1}\|\psi\|_{L^1}$. However, this estimate does not make use of the scaling of $\check{\varphi}^{\epsilon_k}$. If we had used that estimate, we would have ended up with $|g_k'(\psi)| \le \text{constant} \cdot \epsilon_k^{\beta}$. Since β could be negative, the series need not converge.

It is worth noting that Step 3 works for $\gamma > 0$ and $\gamma \le 0$. This is not the case for the subsequent steps.

Chapter 4

Proof Continued for $\gamma > 0$

4.1 Step 4: $\sum |g''_k|$ converges for $\gamma > 0$

In Step 3, we showed that the first sum $\sum_{k=1}^{\infty} g'_k$ in $f_n = f_1 + \sum_{k=1}^{\infty} g'_k + \sum_{k=1}^{\infty} g''_k$ is finite, but what about the other sum? The answer is simple: it is indeed finite *if* $\gamma > 0$. The proof is short, and uses a *compact support* argument and the *coherence condition* ($\widehat{\texttt{COH}}$).

Let $\psi \in \mathcal{D}(\overline{K}_1)$. We start with $g''_k(\psi) := \iint (F_z - F_y)(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi(z)\,\mathrm{d}y\,\mathrm{d}z$. The tweaked test function $\check{\varphi}^{\epsilon_k}$ has compact support in $B(0, \epsilon_k)$, and ψ has compact support in $\overline{K}_1$. So, we have

$$
\begin{aligned}
|g''_k(\psi)| &\leq \sup_{\substack{z \in \overline{K}_1 \\ |y-z| \leq \epsilon_k}} \left(|(F_z - F_y)(\hat{\varphi}_y^{\epsilon_k})| \right) \|\check{\varphi}^{\epsilon_k} * \psi\|_{L^1} \\
&\leq \sup_{\substack{z \in \overline{K}_1 \\ |y-z| \leq \epsilon_k}} \left(|(F_z - F_y)(\hat{\varphi}_y^{\epsilon_k})| \right) \|\check{\varphi}^{\epsilon_k}\|_{L^1} \|\psi\|_{L^1}.
\end{aligned}
$$

Using the coherence condition ($\widehat{\texttt{COH}}$), we get

$$
\begin{aligned}
|g''_k(\psi)| \leq \hat{C}\epsilon_k^{\alpha}(|y-z| + \epsilon_k)^{\gamma-\alpha} \|\check{\varphi}^{\epsilon_k}\|_{L^1} \|\psi\|_{L^1} &\leq \hat{C}\epsilon_k^{\alpha}(2\epsilon_k)^{\gamma-\alpha} \|\check{\varphi}^{\epsilon_k}\|_{L^1} \|\psi\|_{L^1} \\
&= \left(\hat{C} 2^{\gamma-\alpha} \|\check{\varphi}\|_{L^1} \|\psi\|_{L^1} \right) \epsilon_k^{\gamma}. \qquad (4.1)
\end{aligned}
$$

Thus, $\sum |g''_k|$ converges if $\gamma > 0$.

4.2 Step 5: f^K is a local reconstruction

In Step 3 and Step 4 we showed that $\lim\limits_{n \to \infty} f_n(\psi)$ exists for all $\psi \in \mathcal{D}(\overline{K}_1)$. Hence, we define

$$
f^K := \lim_{n \to \infty} f_n \overset{(3.9)}{=} f_1 + \sum_{k=1}^{n-1} g'_k + \sum_{k=1}^{n-1} g''_k.
$$

The notation f^K explicitly emphasizes that $\lim_{n\to\infty} f_n$ depends on the compact set K that we fixed in *Step 0: Setup*; recall that the construction of the mollifier ρ depends on K (see Remark 3.7).

In this step, we prove for $\gamma > 0$ that f^K is a reconstruction in the following sense: (1) f^K is a distribution, and (2) there exists a constant $C < \infty$ such that

$$|(f^K - F_x)(\psi_x^\lambda)| \leq C\lambda^\gamma \tag{2.11}$$
$$\text{uniformly for } \psi \in \mathcal{B}_r,\ x \in K,\ \lambda \in (0,1].$$

The constant C will be explicitly computed in (4.4).

f^K is a distribution

We want to show that $f^K \in \mathcal{D}'(\bar{K}_1)$. f_1 is a distribution. So, we find a constant such that $|f_1(\psi)| \leq \text{constant} \cdot \|\psi\|_{C^r}$ for all $\psi \in \mathcal{D}(\bar{K}_1)$. Then, we use the upper bounds for $\sum |g_k'|$ and $\sum |g_k''|$ established in Step 3 and Step 4 (see (3.11) and (4.1)) to find a constant such that

$$|f^K(\psi)| \leq \text{constant} \cdot \{\|\psi\|_{L^1} + \|\psi\|_{C^r}\} \leq \text{constant} \cdot \left\{\text{Vol}(\bar{K}_{3/2}) + 1\right\} \|\psi\|_{C^r}.$$

This proves that f^K is a distribution.

Setup

We fix $\psi \in \mathcal{B}_r, x \in K$ and $\lambda \in (0,1]$. We define a function $\tilde{f}$ that measures the error of the reconstruction f^K to F_x:

$$\begin{aligned} \tilde{f}(\phi) &= f^K(\phi) - F_x(\phi), && \phi \in \mathcal{D}(\overline{K}_1) \\ \tilde{f}_n(\phi) &= f_n(\phi) - F_x(\phi * \rho^{\epsilon_n}), && n \in \mathbb{N}. \end{aligned}$$

By Lemma 1.12, $\tilde{f}_n(\phi)$ converges to $\tilde{f}(\phi)$ as $n \to \infty$.

Let $N \in \mathbb{N}$ be the smallest index such that $\epsilon_N \leq \lambda$, i.e. $N = \min\{k \in \mathbb{N} : \epsilon_k \leq \lambda\}$. Using the triangle inequality, we then write

$$|\tilde{f}(\psi_x^\lambda)| \leq |\tilde{f}_N(\psi_x^\lambda)| + |(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)|.$$

Bounding $|\tilde{f}_N(\psi_x^\lambda)|$

We start with

$$\begin{aligned}
|\tilde{f}_N(\psi_x^\lambda)| = |f_N(\psi_x^\lambda) - F_x(\psi_x^\lambda * \rho^{\epsilon_N})| &= |\int_{\mathbb{R}^d} F_y(\rho_y^{\epsilon_N})\psi_x^\lambda(y)\,\mathrm{dy} - F_x(\psi_x^\lambda * \rho^{\epsilon_N})| \\
&\Downarrow \text{Lemma 1.2} \\
&= |\int_{\mathbb{R}^d} F_y(\rho_y^{\epsilon_N})\psi_x^\lambda(y)\,\mathrm{dy} - \int_{\mathbb{R}^d} F_x(\rho_y^{\epsilon_N})\psi_x^\lambda(y)\,\mathrm{dy}| \\
&= |\int_{\mathbb{R}^d} (F_y - F_x)((\hat{\varphi}^{2\epsilon_N} * \hat{\varphi}^{\epsilon_N})(\cdot - y))\psi_x^\lambda(y)\,\mathrm{dy}| \\
&\Downarrow \text{Lemma 1.2} \\
&= |\iint (F_y - F_x)(\hat{\varphi}_z^{\epsilon_N})\hat{\varphi}^{2\epsilon_N}(z-y)\psi_x^\lambda(y)\,\mathrm{d}z\,\mathrm{dy}| \\
&\le \sup_{\substack{y \in \overline{K}_1, \\ |z-y| \le \epsilon_N}} |(F_y - F_x)(\hat{\varphi}_z^{\epsilon_N})| \cdot \|\hat{\varphi}^{2\epsilon_N}\|_{L^1}\,\|\psi_x^\lambda\|_{L^1}.
\end{aligned}$$

In the last line, we used a *compact support* argument as in step 4. Note that we have the term $\sup |(F_y - F_x)(\hat{\varphi}_z^{\epsilon_N})|$, which subtly indicates that we need to use ($\widehat{\texttt{COH}}$). To do so, we write for all $y \in B(x, \epsilon), |z - y| \le \epsilon_N$

$$\begin{aligned}
|(F_y - F_x)(\hat{\varphi}_z^{\epsilon_N})| &\le |(F_y - F_z)(\hat{\varphi}_z^{\epsilon_N})| + |(F_z - F_x)(\hat{\varphi}_z^{\epsilon_N})| \\
&\le \hat{C}\epsilon_N^\alpha(|z-y| + \epsilon_N)^{\gamma-\alpha} + \hat{C}\epsilon_N^\alpha(|z-x| + \epsilon_N)^{\gamma-\alpha} \\
&\le \hat{C}\epsilon_N^\alpha(2\epsilon_N)^{\gamma-\alpha} + \hat{C}\epsilon_N^\alpha(|z-y| + |y-x| + \epsilon_N)^{\gamma-\alpha} \\
&\le \hat{C}2^{\gamma-\alpha}\lambda^\gamma + \hat{C}\lambda^\alpha(3\lambda)^{\gamma-\alpha} \\
&= 2\hat{C}3^{\gamma-\alpha}\lambda^\gamma.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
|\tilde{f}_N(\psi_x^\lambda)| &\le 2\hat{C}3^{\gamma-\alpha}\lambda^\gamma\|\hat{\varphi}^{2\epsilon_N}\|_{L^1}\,\|\psi_x^\lambda\|_{L^1} \\
&= 2\hat{C}3^{\gamma-\alpha}\lambda^\gamma\|\hat{\varphi}\|_{L^1}\,\|\psi_x^\lambda\|_{L^1} \\
&\Downarrow \|\psi_x^\lambda\|_{L^1} \le 2^d, \text{ see (3.8)} \\
&\le \left\{3^{\gamma-\alpha}2^{d+1}\hat{C}\|\hat{\varphi}\|_{L^1}\right\}\lambda^\gamma.
\end{aligned}$$

Bounding $|(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)|$

We begin with

$$|(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)| = |\lim_{n\to\infty}(\tilde{f}_n - \tilde{f}_N)(\psi_x^\lambda)| \le \sum_{k \ge N} |(\tilde{f}_{k+1} - \tilde{f}_k)(\psi_x^\lambda)|.$$

By definition $\tilde{f}_k(\psi) = f_k(\psi) - F_x(\psi * \rho^{\epsilon_k})$, we then obtain

$$\begin{aligned}
|(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)| &\leq \sum_{k \geq N} |(f_{k+1} - f_k)(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))| \\
&\Downarrow \text{by definition } f_k = f_1 + \sum_{j=1}^{k-1} g_j' + \sum_{j=1}^{k-1} g_j'' \\
&\leq \sum_{k \geq N} |(g_k' + g_k'')(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))| \\
&\leq \sum_{k \geq N} \underbrace{|g_k''(\psi_x^\lambda)|}_{\text{(A)}} + \underbrace{|g_k'(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))|}_{\text{(B)}}.
\end{aligned}$$

We are going to bound (A) and (B) separately. Our short-term goal is to find a constant such that $|(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)| \leq \{\text{constant}\} \cdot \lambda^\gamma$.

- (A): We start with (A) that follows almost immediately from Step 4. In Chapter 4.1: Step 4 we showed that $|g_k''(\psi_x^\lambda)| \leq \left(\hat{C} 2^{\gamma-\alpha} \|\check{\varphi}\|_{L^1} \|\psi_x^\lambda\|_{L^1}\right) \epsilon_k^\gamma$. We also know that $\|\psi_x^\lambda\|_{L^1} = \|\psi\|_{L^1} \leq 2^d$ by (3.8). Then, we have

$$\sum_{k \geq N} |g_k''(\psi_x^\lambda)| \leq \left(\hat{C} 2^{\gamma-\alpha} \|\check{\varphi}\|_{L^1} \|\psi_x^\lambda\|_{L^1}\right) \sum_{k \geq N} \epsilon_k^\gamma \leq \left(\hat{C} 2^{\gamma-\alpha+d} \|\check{\varphi}\|_{L^1}\right) \sum_{k \geq N} \epsilon_k^\gamma.$$

  The geometric series $\sum_{k \geq N} \epsilon_k^\gamma$ converges because $\gamma > 0$, and therefore

$$\sum_{k \geq N} |g_k''(\psi_x^\lambda)| \leq \frac{\hat{C} 2^{\gamma-\alpha+d} \|\check{\varphi}\|_{L^1}}{1 - 2^{-\gamma}} \epsilon_N^\gamma \leq \left\{ \frac{\hat{C} 2^{\gamma-\alpha+d} \|\check{\varphi}\|_{L^1}}{1 - 2^{-\gamma}} \right\} \lambda^\gamma.$$

- (B): Next, we bound $|g_k'(\psi_x^\epsilon) - F_x(\psi_x^\epsilon * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))|$. For that we need a technical lemma that also turns out to be useful in the case $\gamma \leq 0$. Hence, we state it as a lemma here, so we can refer to it in later chapters.

  Lemma 4.1. *For any $\gamma \in \mathbb{R}$ we have*

$$|g_k'(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))| \leq 4^{d+\gamma-\alpha} \hat{C} \|\check{\varphi}\|_{L^1} \begin{cases} \lambda^{\gamma-\alpha-r} \epsilon_k^{\alpha+r} & \text{if } \epsilon_k < \lambda \\ \epsilon_k^\gamma & \text{if } \epsilon_k \geq \lambda \end{cases}$$

  and

$$\sum_{k \geq N} |g_k'(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))| \leq \left\{ \frac{\hat{C} 4^{\gamma-\alpha+d} \|\check{\varphi}\|_{L^1}}{1 - 2^{-\alpha-r}} \right\} \lambda^\gamma.$$

  Recall that $\epsilon_k = 2^{-k}$ and $N = \min\{k \in \mathbb{N} : \epsilon_k \leq \epsilon\}$.

Proof. By (MOL) and Corollary 1.14, we obtain

$$\begin{aligned} F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k})) &= \iint F_x(\hat{\varphi}_y^{\epsilon_k})\check{\varphi}^{\epsilon_k}(y-z)\psi_x^\lambda(z)\,\mathrm{d}y\mathrm{d}z \\ &= \int F_x(\hat{\varphi}_y^{\epsilon_k})(\check{\varphi}^{\epsilon_k} * \psi_x^\lambda)(y)\,\mathrm{d}y. \end{aligned}$$

By definition of g_k' (see (3.10)), we have

$$g_k'(\psi_x^\lambda) = \int F_y(\hat{\varphi}_y^{\epsilon_k})(\check{\varphi}^{\epsilon_k} * \psi_x^\lambda)(y)\,\mathrm{d}y.$$

Hence, we get

$$|g_k'(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))| = \int (F_y - F_x)(\hat{\varphi}_y^{\epsilon_k})(\check{\varphi}^{\epsilon_k} * \psi_x^\lambda)(y)\,\mathrm{d}y, \tag{4.2}$$

for which we find an upper bound using the second inequality of Lemma 3.6

$$(4.2) \le 4^d \|\check{\varphi}\|_{L^1} \min\left\{\frac{\epsilon_k}{\lambda}, 1\right\}^r \sup_{y\in B(x,\lambda+\epsilon_k)} |(F_y - F_x)(\hat{\varphi}_y^{\epsilon_k})|.$$

The supremum of $|F_y - F_x|$ is estimated with the coherence condition

$$\begin{aligned} \sup_{y\in B(x,\lambda+\epsilon_k)} |(F_y - F_x)(\hat{\varphi}_y^{\epsilon_k})| &\overset{(\widehat{\mathtt{COH}})}{\le} \hat{C}\epsilon_k^\alpha \sup_{y\in B(x,\lambda+\epsilon_k)} (|x-y| + \epsilon_k)^{\gamma-\alpha} \\ &\le \hat{C}\epsilon_k^\alpha(\lambda + 2\epsilon_k)^{\gamma-\alpha} \\ &\le \hat{C}\epsilon_k^\alpha 3^{\gamma-\alpha} \begin{cases} \epsilon_k^{\gamma-\alpha} & \text{if } \epsilon_k > \lambda \\ \lambda^{\gamma-\alpha} & \text{if } \epsilon_k \le \lambda \end{cases} \end{aligned}$$

This proves the first claim of the lemma.

To prove the last line, note that $\sum_{k\ge N} \epsilon_k^{\alpha+r} = \frac{\epsilon_N^{\alpha+r}}{1-2^{-\alpha-r}} \le \frac{\lambda^{\alpha+r}}{1-2^{-\alpha-r}}$ (regarding the last inequality, we chose $r \in \mathbb{N}$ such that $\alpha + r > 0$). Since $k \ge N$, we have $\epsilon_k < \lambda$. So, we use the first claim of the lemma together with $\sum_{k\ge N} \epsilon_k^{\alpha+r} \le \frac{\lambda^{\alpha+r}}{1-2^{-\alpha-r}}$ to obtain

$$\sum_{k\ge N} |g_k'(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))| \le \left\{\frac{4^{d+\gamma-\alpha}}{1-2^{-\alpha-r}}\|\check{\varphi}\|_{L^1}\hat{C}\right\}\lambda^\gamma.$$

$\square$

With this lemma proved, we use the second line to find an upper bound for (B).

Finally, we use (A) and (B) to estimate

$$\begin{aligned}
|(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)| &\leq \left\{ \frac{\hat{C} 4^{\gamma-\alpha+d} \|\check{\varphi}\|_{L^1}}{1 - 2^{-\alpha-r}} \right\} \lambda^\gamma + \left\{ \frac{\hat{C} 2^{\gamma-\alpha+d} \|\check{\varphi}\|_{L^1}}{1 - 2^{-\gamma}} \right\} \lambda^\gamma \\
&\leq \left\{ 2 \frac{\hat{C} 4^{\gamma-\alpha+d} \|\check{\varphi}\|_{L^1}}{1 - 2^{-\min\{\alpha+r,\gamma\}}} \right\} \cdot \lambda^\gamma \\
&\Downarrow \|\check{\varphi}\|_{L^1} \leq 2\|\hat{\varphi}\|_{L^1} \text{ by definition of } \check{\varphi} \\
&\leq \left\{ \frac{\hat{C} 4^{\gamma-\alpha+d+1} \|\hat{\varphi}\|_{L^1}}{1 - 2^{-\min\{\alpha+r,\gamma\}}} \right\} \cdot \lambda^\gamma.
\end{aligned}$$

Finish

Now, we are finally able to prove that f^K is a local reconstruction. We show that f^K satisfies (2.11) uniformly for all $\psi \in \mathcal{B}_r, x \in K$ and $\lambda \in (0, 1]$. We have

$$\begin{aligned}
|f^K(\psi_x^\lambda) - F_x(\psi_x^\lambda)| = |\tilde{f}(\psi_x^\lambda)| &\leq |\tilde{f}_N(\psi_x^\lambda)| + |(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)| \\
&\leq \left\{ 3^{\gamma-\alpha} 2^{d+1} \hat{C} \|\hat{\varphi}\|_{L^1} \right\} \lambda^\gamma + \left\{ \frac{4^{\gamma-\alpha+d+1} \hat{C} \|\hat{\varphi}\|_{L^1}}{1 - 2^{-\min\{\alpha+r,\gamma\}}} \right\} \cdot \lambda^\gamma \\
&\leq \left\{ 2 \frac{4^{\gamma-\alpha+d+1} \hat{C} \|\hat{\varphi}\|_{L^1}}{1 - 2^{-\min\{\alpha+r,\gamma\}}} \right\} \lambda^\gamma.
\end{aligned}$$

This proves that f^K is a local reconstruction of $(F_x)_{x \in \mathbb{R}^d}$.

For completeness, we estimate $\hat{C}\|\hat{\varphi}\|_{L^1}$. Lemma 3.4 gives us $\|\hat{\varphi}\|_{L^1} \leq \frac{e^2 r \|\varphi\|_{L^1}}{|\int \varphi(x) \mathrm{d}x|}$. The bound for $\hat{C}$ in (3.3) then yields

$$\hat{C}\|\hat{\varphi}\|_{L^1} \leq \left(C \frac{e^2 r}{|\int \varphi(x)\, \mathrm{d}x|} \left(\frac{2^{-(r+1)}}{1 + R_\varphi} \right)^\alpha \right) \left(\frac{e^2 r}{|\int \varphi(x) \mathrm{d}x|} \|\varphi\|_{L^1} \right). \tag{4.3}$$

So, the constant factor in front of λ^γ reads

$$\left\{ 2 \frac{4^{\gamma-\alpha+d+1}}{1 - 2^{-\min\{\alpha+r,\gamma\}}} \left(C \frac{e^2 r}{|\int \varphi(x)\, \mathrm{d}x|} \left(\frac{2^{-(r+1)}}{1 + R_\varphi} \right)^\alpha \right) \left(\frac{e^2 r}{|\int \varphi(x) \mathrm{d}x|} \|\varphi\|_{L^1} \right) \right\}.$$

Finally, we bound $e^2 \leq 4$ to obtain the constant

$$\left\{ C \frac{4^{\gamma-\alpha+d+6} r^2}{1 - 2^{-\min\{\alpha+r,\gamma\}}} \frac{\|\varphi\|_{L^1}}{2^{\alpha(r+1)} (1 + R_\varphi)^\alpha |\int \varphi(x) \mathrm{d}x|^2} \right\}, \tag{4.4}$$

where R_φ is the radius of the ball such that $\mathrm{supp}(\varphi) \subset B(0, R_\varphi)$.

4.3 Step 6: (f^K) are consistent

In this final step, we construct a global reconstruction $f = \mathcal{R}F$ of $(F_x)_{x\in\mathbb{R}^d}$ using the local reconstructions f^K. First, we prove that the family of local reconstructions $(f^K)_{K\subset\mathbb{R}^d, K \text{ compact}}$ is consistent in the sense of

$$\forall \text{compact sets } K \text{ and } H \text{ with } K \subset H : \psi \in \mathcal{D}(\bar{K}_1) \implies f^K(\psi) = f^H(\psi).$$

To prove this claim, we use the Uniqueness Theorem 2.13. We have $(f^K - F_x)(\varphi_x^\lambda) \to 0$ and $(g^K - F_x)(\varphi_x^\lambda) \to 0$ as $\lambda \to 0$ uniformly for $x \in \bar{K}_1$ because of $\gamma > 0$ and (2.11). Hence, we apply the Uniqueness Theorem and conclude that $f^K(\psi) = f^H(\psi)$ for all $\psi \in \mathcal{D}(\bar{K}_1)$.

With the consistency property being shown, we move on to construct a global reconstruction. Let $\psi \in \mathcal{D}$. Then, we define $\mathcal{R}F : \psi \mapsto f^K(\psi)$ where $K \subset \mathbb{R}^d$ is a compact set large enough such that ψ is supported in $\bar{K}_1$. The map $\mathcal{R}F$ is well-defined because we showed the consistency property. Moreover, $\mathcal{R}F$ is a reconstruction in the sense of (2.11) because f^K is a local reconstruction. This ends the proof of the Reconstruction Theorem in the case $\gamma > 0$.

Chapter 5

Proof Continued for $\gamma \leq 0$

The main idea of the proof for $\gamma > 0$ was that f_n converges to some reconstruction $f = \mathcal{R}f$ if $\gamma > 0$, where $f_n = f_1 + \sum_{k=1}^{n-1} g'_k + \sum_{k=1}^{n-1} g''_k$, see (3.9). If however $\gamma \leq 0$, the series $\sum g''_k(\psi)$ need not converge. We fix this by ignoring $\sum g''_k(\psi)$; the approximating distribution then reads $f_1 + \sum_{k=1}^{n-1} g'_k(\psi)$. We set

$$f^K(\psi) = f_1(\psi) + \lim_{n\to\infty} \sum_{k=1}^{n-1} g'_k(\psi),$$

which is well-defined because $\sum g'_k(\psi)$ converges for all $\gamma \in \mathbb{R}$ (see Chapter 3.4). Next, f^K is a distribution on $\bar{K}_1$ because TO-DO

In the next steps, we will show that f^K satisfies (2.11), i.e.

$$|(f^K - F_x)(\psi_x^\lambda)| \leq \mathfrak{C} |||F|||^{\mathrm{coh}}_{\bar{K}_2,\varphi,\alpha,\gamma} \begin{cases} \lambda^\gamma & \text{if } \gamma < 0 \\ 1 + |\log(\lambda)| & \text{if } \gamma = 0 \end{cases}$$
$$\text{uniformly for } \psi \in \mathcal{B}_r,\ x \in K \text{ and } \lambda \in (0,1].$$

where the constant $\mathfrak{C}$ is given by TO-DO and TO-Do. We will then spend another chapter to build a *global* distribution $f \in \mathcal{D}'$ out of the local distributions $f^K \in \mathcal{D}(\bar{K}_1)$ such that f satisfies (2.11), as well.

5.1 Step 5: f^K is a local reconstruction

We have the same setup as in Chapter 3.1: Step 0. Let $K \subset \mathbb{R}^d$ be a compact set, $x \in K$, $\psi \in \mathcal{B}_r$ and $\lambda \in (0,1]$. Then, we have

$$\begin{aligned}
|(f^K - F_x)(\psi_x^\lambda)| &= |((f_1 + \lim_{n\to\infty}\sum_{k=1}^{n-1} g_k') - F_x)(\psi_x^\lambda)| \\
&= |(f_1 + \lim_{n\to\infty}\sum_{k=1}^{n-1} g_k')(\psi_x^\lambda) - \lim_{n\to\infty} F_x(\psi_x^\lambda * \rho^{\epsilon_n})| \\
&= \lim_{n\to\infty} |\underbrace{f_1(\psi_x^\lambda) + \left\{\sum_{k=1}^{n-1} g_k'(\psi_x^\lambda)\right\} - F_x(\psi_x^\lambda * \rho^{\epsilon_n})}_{:=\bar{f}_n(\psi_x^\lambda)}|.
\end{aligned}$$

Next, we write the above expression as a telescopic sum

$$|(f^K - F_x)(\psi_x^\lambda)| = \lim_{n\to\infty} |\bar{f}_n(\psi_x^\lambda)| \leq |\left(\lim_{n\to\infty} \bar{f}_n(\psi_x^\lambda)\right) - \bar{f}_N(\psi_x^\lambda)| + |\bar{f}_N(\psi_x^\lambda)|$$

where N is chosen such that $\epsilon_N \leq \lambda < \epsilon_{N-1}$. The first summand is estimated by Lemma 4.1

$$\begin{aligned}
|\left(\lim_{n\to\infty} \bar{f}_n(\psi_x^\lambda)\right) - \bar{f}_N(\psi_x^\lambda)| &\leq \sum_{k\geq N} |(\bar{f}_{k+1} - \bar{f}_k)(\psi_x^\lambda)| \\
&= \sum_{k\geq N} |g_k'(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))| \\
&\Downarrow \text{Lemma 4.1} \\
&\leq \left\{\frac{\hat{C}4^{\gamma-\alpha+d}\|\check{\varphi}\|_{L^1}}{1-2^{-\alpha-r}}\right\}\lambda^\gamma.
\end{aligned}$$

The second summand $|\bar{f}_N(\psi_x^\lambda)|$ is also bounded by Lemma 4.1

$$\begin{aligned}
|\bar{f}_N(\psi_x^\lambda)| &\leq |\bar{f}_1(\psi_x^\lambda)| + \sum_{k=1}^{N-1} |(\bar{f}_{k+1} - \bar{f}_k)(\psi_x^\lambda)| \\
&= |\bar{f}_1(\psi_x^\lambda)| + \sum_{k=1}^{N-1} |g_k'(\psi_x^\lambda) - F_x(\psi_x^\lambda * (\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k}))| \\
&\Downarrow \text{Lemma 4.1 (use the case } \epsilon_k \geq \epsilon_{N-1} > \lambda) \\
&\leq |\bar{f}_1(\psi_x^\lambda)| + \sum_{k=1}^{N-1} 4^{d+\gamma-\alpha}\hat{C}\|\check{\varphi}\|_{L^1}\epsilon_k^\gamma.
\end{aligned}$$

Next, we observe

$$
\begin{aligned}
|\bar{f}_1(\psi_x^\lambda)| &= |f_1(\psi_x^\lambda) - F_x(\psi_x^\lambda * \rho^{\epsilon_1})| \\
&\Downarrow \text{use (1.2)} \\
&= \left| \int_{\mathbb{R}^d} F_z(\rho_z^{\epsilon_1}) \psi_x^\lambda(z) \mathrm{d}z - \int F_x(\rho_z^{\epsilon_1}) \psi_x^\lambda(z) \mathrm{d}z \right| \\
&= \left| \int_{\mathbb{R}^d} (F_z - F_x)(\rho_z^{\epsilon_1}) \psi_x^\lambda(z) \mathrm{d}z \right| \\
&\Downarrow \text{Recall } \rho = \hat{\varphi}^2 * \hat{\varphi} \text{ and use (1.2)} \\
&= \left| \iint (F_z - F_x)(\hat{\varphi}_y^{\epsilon_1}) \hat{\varphi}^{2\epsilon_1}(y-z) \psi_x^\lambda(z) \, \mathrm{d}y \, \mathrm{d}z \right|.
\end{aligned}
$$

The tweaked test function $\hat{\varphi}$ has a compact support in $B(0, \frac{1}{2})$; hence $\hat{\varphi}^{2\epsilon}$ is supported in $B(0, \epsilon_1)$. Thus, the integral is nonzero if $|y - z| \leq \epsilon_1 = \frac{1}{2}$. Additionally, we have $|x - z| \leq \lambda$ because $\psi_x^\lambda(z)$. So, we estimate

$$
|\bar{f}_1(\psi_x^\lambda)| \leq \sup_{\substack{z \in B(x,\lambda) \\ |y-z| \leq \frac{1}{2}}} \left| (F_z - F_x)(\hat{\varphi}_y^{\epsilon_1}) \right| \cdot \|\hat{\varphi}^{2\epsilon_1}\|_{L^1} \|\psi_x^\lambda\|_{L^1}.
$$

Moreover, $z \in \bar{K}_1$ (recall that $x \in K$ and $\lambda \in (0,1]$), $y \in \bar{K}_{\frac{3}{2}}$ and $|x-y| \leq |x-z|+|z-y| \leq \frac{3}{2}$. Hence, we use the triangle inequality and the coherence condition to obtain

$$
\begin{aligned}
\sup_{\substack{z \in B(x,\lambda) \\ |y-z| \leq \frac{1}{2}}} \left| (F_z - F_x)(\hat{\varphi}_y^{\epsilon_1}) \right| &\leq \sup_{\substack{y,z \in \bar{K}_{3/2} \\ |y-z| \leq \frac{1}{2}}} \left| (F_z - F_y)(\hat{\varphi}_y^{\epsilon_1}) \right| + \sup_{\substack{x,y \in \bar{K}_{3/2} \\ |x-y| \leq \frac{3}{2}}} \left| (F_y - F_x)(\hat{\varphi}_y^{\epsilon_1}) \right| \\
&\overset{(\widehat{\texttt{COH}})}{\leq} \hat{C} \epsilon_1^\alpha (|z-y| + \epsilon_1)^{\gamma-\alpha} + \hat{C} \epsilon_1^\alpha (|y-x| + \epsilon_1)^{\gamma-\alpha} \\
&\leq \hat{C} \left(\frac{3}{2} \right)^{\gamma-\alpha} + \hat{C} \left(\frac{5}{2} \right)^{\gamma-\alpha} \\
&\leq 2\hat{C} \cdot 3^{\gamma-\alpha}.
\end{aligned}
$$

We bound

$$
\begin{aligned}
|\bar{f}_1(\psi_x^\lambda)| \leq \left\{ 2\hat{C} \cdot 3^{\gamma-\alpha} \right\} \|\hat{\varphi}^{2\epsilon_1}\|_{L^1} \|\psi_x^\lambda\|_{L^1} &= \left\{ 2\hat{C} \cdot 3^{\gamma-\alpha} \right\} \|\hat{\varphi}\|_{L^1} \|\psi\|_{L^1} \\
&\leq \left\{ 2\hat{C} \cdot 3^{\gamma-\alpha} \right\} \|\hat{\varphi}\|_{L^1} \cdot \sup_{\psi \in \mathcal{B}_r} \|\psi\|_{L^1} \\
&\leq \left\{ 2\hat{C} \cdot 3^{\gamma-\alpha} \right\} \|\hat{\varphi}\|_{L^1} \cdot 2^d \cdot \underbrace{\sup_{\psi \in \mathcal{B}_r} \|\psi\|_\infty}_{\leq 1} \\
&\leq 2^{d+1} \hat{C} \cdot 3^{\gamma-\alpha} \|\hat{\varphi}\|_{L^1}.
\end{aligned}
$$

Then, we have $|\bar{f}_N(\psi_x^\lambda)| \leq 2^{d+1}\hat{C} \cdot 3^{\gamma-\alpha}\|\hat{\varphi}\|_{L^1} + \sum_{k=1}^{N-1} 4^{d+\gamma-\alpha}\hat{C}\|\check{\varphi}\|_{L^1}\epsilon_k^\gamma$. Also observe that $\|\check{\varphi}\|_{L^1} = \int |\check{\varphi}(x)|\mathrm{d}x \leq \int |\hat{\varphi}^{\frac{1}{2}}(x)| + |\hat{\varphi}^2(x)|\mathrm{d}x = 2\int |\hat{\varphi}(x)|\mathrm{d}x = 2\|\hat{\varphi}\|_{L^1}$. So, we get

$$|\bar{f}_N(\psi_x^\lambda)| \leq 4^{d+\gamma-\alpha+1}\hat{C}\|\hat{\varphi}\|_{L^1}\sum_{k=0}^{N-1}\epsilon_k^\gamma.$$

Note that $\sum_{k=0}^{N-1}\epsilon_k^\gamma$ is a geometric sum which we explicitly compute

$$\sum_{k=0}^{N-1}\epsilon_k^\gamma = \sum_{k=0}^{N-1} 2^{-\gamma k} \leq \begin{cases} \dfrac{\lambda^\gamma}{1-2^\gamma} & \text{if } \gamma < 0 \\ \dfrac{\log(\frac{2}{\lambda})}{\log 2} & \text{if } \gamma = 0 \end{cases}.$$

Finally,

$$|(f^K - F_x)(\psi_x^\lambda)| \leq \frac{\hat{C}4^{\gamma-\alpha+d}\|\check{\varphi}\|_{L^1}}{1-2^{-\alpha-r}}\lambda^\gamma + 4^{d+\gamma-\alpha+1}\hat{C}\|\hat{\varphi}\|_{L^1}\begin{cases} \dfrac{\lambda^\gamma}{1-2^\gamma} & \text{if } \gamma < 0 \\ \dfrac{\log(\frac{2}{\lambda})}{\log 2} & \text{if } \gamma = 0 \end{cases}.$$

If $\gamma < 0$, then

$$\begin{aligned} |(f^K - F_x)(\psi_x^\lambda)| &\leq \left\{\hat{C}\|\hat{\varphi}\|_{L^1}\frac{4^{\gamma-\alpha+d+1}}{1-2^{-\min\{\alpha+r,-\gamma\}}}\right\}\lambda^\gamma \\ &\overset{(4.3)}{\leq} \left\{\frac{r^2 2^{\alpha(-r-1)}4^{d+\gamma-\alpha+6}\|\varphi\|_{L^1}}{1-2^{-\min\{\alpha+r,|\gamma|\}}(1+R_\varphi)^\alpha|\int\varphi(x)\,\mathrm{d}x|^2}|||F|||^{\mathrm{coh}}_{\bar{K}_{3/2},\varphi,\alpha,\gamma}.\right\}\lambda^\gamma. \end{aligned} \tag{5.1}$$

Otherwise if $\gamma = 0$, then we know $\log(\frac{2}{\lambda})(\log 2)^{-1} \leq 2(1+|\log\lambda|)$. Thus

$$|(f^K - F_x)(\psi_x^\lambda)| \leq \left\{\frac{r^2 2^{\alpha(-r-1)}4^{d-\alpha+6}\|\varphi\|_{L^1}}{1-2^{-\min\{\alpha+r,|\gamma|\}}(1+R_\varphi)^\alpha|\int\varphi(x)\,\mathrm{d}x|^2}|||F|||^{\mathrm{coh}}_{\bar{K}_{3/2},\varphi,\alpha,\gamma}\right\}(1+|\log\lambda|).$$

This shows that f^K is a local reconstruction.

5.2 Step 6: Localization

Similar to the case $\gamma > 0$, we need to build a global distribution f from the local reconstructions f^K. For that, we make use of a localization argument. First, we construct a partition of unity. Fix some test function $\eta \in \mathcal{D}(B(0,\frac{1}{4}))$ such that $\eta \geq 0$ on $B(0,\frac{1}{4})$ and $\eta \geq 1$ on $B(0,\frac{1}{8})$. Define

$$\xi(x) = \frac{\eta(x)}{\sum_{z\in E}\eta_z(x)} \qquad \text{where } E = \frac{1}{4\sqrt{d}}\mathbb{Z}^d.$$

Note that $\xi_y \in \mathcal{D}(B(y, \frac{1}{4}))$ for every $y \in \mathbb{R}^d$ and $\sum_{y \in E} \xi_y \equiv 1$. We call $(\xi_y)_{y \in E}$ a *partition of unity subordinated to the cover* $B(y, \frac{1}{4})_{y \in E}$. Define $B_y = B(y, \frac{1}{4})$. Note that B_y has diameter $\frac{1}{2}$. The global reconstruction f is then defined as

$$f(\psi) = \sum_{y \in E} f^{B_y}(\xi_y \psi), \quad \forall \psi \in \mathcal{D}.$$

Now, we show that f satisfies the Reconstruction Theorem. Fix a compact set $K \subset \mathbb{R}^d$, and define

$$\alpha = \alpha_{\bar{K}_2}, \quad \beta = \beta_{\bar{K}_2}, \quad r > \max\{-\alpha_{\bar{K}_2}, -\beta_{\bar{K}_2}\}. \tag{5.2}$$

Let $\psi \in \mathcal{B}_r$, $x \in K$ and $\lambda \in (0, 1]$. For $\gamma < 0$ we have

$$|(f - F_x)(\varphi_x^\lambda)| = |\sum_{y \in E}(f^{B_y} - F_x)(\xi_y \varphi_x^\lambda)| \leq \sum_{y \in E} |(f^{B_y} - F_x)(\xi_y \psi_x^\lambda)|. \tag{5.3}$$

To justify the first equality, note that $F_x(\varphi_x^\lambda) = F_x(\sum_{y \in E} \xi_y \varphi_x^\lambda) = \sum_{y \in E} F_x(\xi_y \varphi_x^\lambda)$ because $\sum_{y \in E} \xi_y \equiv 1$.

Next, we make sure that we only sum over a finite number of $y \in E$. Note that ξ_y has compact support in $B(y, \frac{1}{4})$ and ψ_x^λ has compact support in $B(x, \lambda)$. So, $\xi_y \psi_x^\lambda \not\equiv 0$ only if $|y - x| \leq |y - z| + |z - x| \leq \frac{1}{4} + \lambda \leq \frac{5}{4}$. There are at most $(2 \cdot \frac{5}{4} \cdot 4\sqrt{d} + 1)^d \leq (11\sqrt{d})^d$ many points $y \in E$ that satisfy this.

Then, we write

$$|(f^{B_y} - F_x)(\xi_y \psi_x^\lambda)| = |(f^{B_y} - F_x)(\zeta_x^\lambda)|$$

for $\zeta(z) = \xi_y(x + \lambda z)\psi(z)$. We would like to apply (5.1) for the compact set B_y and $\zeta / \|\zeta\|_{C^r} \in \mathcal{B}_r$. Here, we need to be careful about α, β and r because we must check that (5.1) still holds if we choose α, β and r as in (5.2). Let $\Gamma = \{y_1, \ldots, y_n\} \subset \mathbb{R}^d$ such that $y_i \cap K \neq \emptyset$ for all $1 \leq i \leq n$. We have $\bigcup_{y \in \Gamma} B_y \subset \bar{K}_{\frac{1}{2}}$ because each ball B_y has diameter $\frac{1}{2}$. So, the $\frac{3}{2}$-enlargement of $\bigcup_{y \in \Gamma} B_y$ is contained in $\bar{K}_2$. By (2.8) and (2.9), we know that the maps $K \mapsto \alpha_K$ and $K \mapsto \beta_K$ are monotone. So, we have $\alpha_{\bar{K}_2} \leq \alpha_{\overline{(\bigcup_{y \in \Gamma} B_y)}_{3/2}}$ and $\beta_{\bar{K}_2} \leq \beta_{\overline{(\bigcup_{y \in \Gamma} B_y)}_{3/2}}$. This, together with Step 5 ($\gamma > 0$), shows that (5.1) remains true for α, β and r. Applying (5.1) then yields

$$\begin{aligned}
|(f^{B_y} - F_x)(\zeta_x^\lambda)| = |(f^{B_y} - F_x)(\zeta_x^\lambda / \|\zeta\|_{C^r})| \|\zeta\|_{C^r} &\leq \{\text{constant}\} \cdot \|\zeta\|_{C^r} |\!|\!| F |\!|\!|^{\text{coh}}_{\bar{K}_2, \varphi, \alpha, \gamma} \lambda^\gamma \\
&\Downarrow \text{Leibniz Rule and } \sum_{k=0}^{r} \binom{r}{k} = 2^r \\
&\leq 2^r \{\text{constant}\} \|\xi\|_{C^r} \|\psi\|_{C^r} |\!|\!| F |\!|\!|^{\text{coh}}_{\bar{K}_2, \varphi, \alpha, \gamma} \lambda^\gamma.
\end{aligned}$$

Continuing the estimate (5.3)

$$\begin{aligned}|(f - F_x)(\varphi_x^\lambda)| &\overset{(5.3)}{\leq} \sum_{y \in E} |(f^{B_y} - F_x)(\xi_y \psi_x^\lambda)| \\ &\leq \left\{(11\sqrt{d})^d \,\{\text{constant}\}\, \|\xi\|_{C^r} \|\psi\|_{C^r} |\!|\!| F |\!|\!|^{\text{coh}}_{\bar{K}_2,\varphi,\alpha,\gamma}\right\} \lambda^\gamma .\end{aligned}$$

The constant before λ^γ then reads

$$\left\{2^r \|\xi\|_{C^r} (11\sqrt{d})^d \frac{r^2 2^{-(r+1)\alpha} 4^{d+\gamma-\alpha+6}}{1 - 2^{-\min\{\alpha+r,-\gamma\}} |\int \varphi(x)\,\mathrm{d}x|^2 (1+R_\varphi)^\alpha} \|\varphi\|_{L^1}\right\} \quad \text{in the case } \gamma < 0 \;. \tag{5.4}$$

The proof for the case $\gamma = 0$ is done similarly and gives the constant

$$\left\{2^r \|\xi\|_{C^r} (11\sqrt{d})^d \frac{r^2 2^{-(r+1)\alpha} 4^{d-\alpha+6}}{1 - 2^{-\alpha-r} |\int \varphi(x)\,\mathrm{d}x|^2 (1+R_\varphi)^\alpha} \|\varphi\|_{L^1}\right\} \quad \text{in the case } \gamma = 0 \;. \tag{5.5}$$

This ends the proof of the Reconstruction Theorem for $\gamma \leq 0$.

Chapter 6

Applications

6.1 Negative Hölder Spaces

In Chapter 1.1 we introduced the space of locally α-Hölder functions $\mathcal{C}^\alpha$ for positive exponents $\alpha > 0$. We drop the adjective *locally* and simply call it the space of α-Hölder functions. We also say that f is α-Hölder continuous if $f \in \mathcal{C}^\alpha$.

Our goal is to extend $\mathcal{C}^\alpha$ to non-positive exponents $\alpha \leq 0$. In this case, $\mathcal{C}^\alpha$ is no longer a space of continuously differentiable functions but a space of *distributions*. Then, we will see that the reconstruction $\mathcal{R}F$ of a γ-coherent F germ with negative $\gamma < 0$ lies in $\mathcal{C}^\gamma$.

Hölder spaces $\mathcal{C}^\alpha$ play an important role in stochastic processes. The Hölder exponent α encodes the "regularity" or "roughness" of a process. That is the smaller α gets, the more irregular the process becomes. Take the *Brownian Motion*, which is almost surely locally α-Hölder continuous for $\alpha \in (0, \frac{1}{2})$. In case of $\alpha = 1$, we obtain Lipschitz continuity. Hence, Hölder continuity is a generalization of Lipschitz continuity. For $\alpha > 1$, the only α-Hölder continuous functions in $\mathbb{R}$ are constant functions. Clearly, constant functions are the most regular functions we can think of.

Let us extend $\mathcal{C}^\alpha$ to negative exponents $\alpha < 0$. It then becomes a space of *distributions*.

Definition 6.1 (Negative Hölder Space)**.** Let $\alpha \leq 0$ and let $r_\alpha = \min\{n \in \mathbb{N} : r > -\alpha\}$. We define the Hölder space C^α as the space of all distributions T such that for any compact set $K \subset \mathbb{R}^d$ there exists $C < \infty$ with

$$|T(\psi_x^\epsilon)| \leq C\epsilon^\alpha \tag{6.1}$$
$$\text{for all } x \in K, \epsilon \in (0,1] \text{ and } \psi \in \mathcal{B}_{r_\alpha}.$$

The semi-norm $\|\cdot\|_{\mathcal{C}^\alpha(K)}$ is defined as

$$\|T\|_{\mathcal{C}^\alpha(K)} = \sup_{\substack{x \in K, \\ \lambda \in (0,1], \\ \psi \in \mathcal{B}_{r_\alpha}}} \frac{|T(\psi_x^\lambda)|}{\lambda^\alpha} \qquad \forall T \in \mathcal{D}'. \tag{6.2}$$

Clearly, $T \in \mathcal{C}^\alpha \iff \|T\|_{\mathcal{C}^\alpha(K)} < \infty$ for all compact sets $K \subset \mathbb{R}^d$.

We present our main result. A distribution $T \in \mathcal{D}'$ lies in $\mathcal{C}^\alpha$ if inequality (6.1) holds for a *single* arbitrary test function $\varphi \in \mathcal{D}$ rather for all $\psi \in \mathcal{B}_{r_\alpha}$. This characterization of negative Hölder spaces is obtained from the following theorem, which we prove in a similar fashion as in the proof of the Reconstruction Theorem.

Theorem 6.2. *Let $T \in \mathcal{D}'$. If there exists a set $K \subset \mathbb{R}^d$ and a test function $\varphi \in \mathcal{D}, \int \varphi(x)\,\mathrm{d}x \neq 0$ such that*

$$\begin{aligned}&\forall x \in \bar{K}_2, \epsilon \in \left\{2^k\right\}_{k\in\mathbb{N}}: \quad |T(\varphi_x^\epsilon)| \leq \epsilon^\alpha f(\epsilon, x) \\ &\textit{for some } \alpha \leq 0 \textit{ and } f:(0,1]\times\bar{K}_2 \to [0,\infty),\end{aligned} \tag{6.3}$$

then the above inequality (6.3) *also holds for* every *test function $\psi \in \mathcal{B}_r$ and integer $r > -\alpha$ in the following sense: for every $\psi \in \mathcal{B}_r$ and $r > -\alpha$ there exists a constant such that*

$$\forall x \in K, \epsilon \in (0,1]: \quad |T(\psi_x^\epsilon)| \leq \{\text{constant}\}\cdot \epsilon^\alpha \sup_{\substack{\epsilon'\in(0,\epsilon] \\ x'\in B(x,2\epsilon)}} f(\epsilon', x').$$

Proof. Let T, φ, K, α and f be as above. Fix an integer $r > -\alpha$. As in the second step of the proof of the Reconstruction Theorem (see (3.2)), we define the tweaked test function $\hat{\varphi} = T_\varphi$ for $a = \frac{1}{\int \varphi(x)\,\mathrm{d}x}$ and $\lambda_i = \frac{2^{-(i+1)}}{1+R_\varphi}$, $i = 0, \ldots, r-1$.

We claim that $\hat{\varphi}$ satisfies a slightly modified inequality (6.3), i.e. for all $x \in \bar{K}_2, \epsilon = 2^{-k}$:

$$|T(\hat{\varphi}_x^\epsilon)| \leq C\epsilon^\alpha \sup_{\epsilon'\in(0,\epsilon]} f(\epsilon', x) \quad \text{with} \quad C = \frac{e^2 r}{\int \varphi(x)\,\mathrm{d}x}\left(\frac{2^{-r-1}}{1+R_\varphi}\right)^\alpha. \tag{6.4}$$

This part is also very similar to the second step of the proof of the Reconstruction Theorem (see Lemma 3.3). Let $\epsilon \in \left\{2^{-k}\right\}_{k\in\mathbb{N}}$ and $x \in \bar{K}_2$. By definition of $\hat{\varphi}$ we have

$$\begin{aligned}|T(\hat{\varphi}_x^\epsilon)| \leq \frac{1}{\int \varphi(x)\,\mathrm{d}x}\sum_{i=0}^{r-1}|c_i||T(\varphi_x^{\epsilon\lambda_i})| &\overset{(6.3)}{\leq} \frac{1}{\int \varphi(x)\,\mathrm{d}x}\sum_{i=0}^{r-1}|c_i|(\epsilon\lambda_i)^\alpha f(\epsilon, x) \\ &\Downarrow \text{Note that } \frac{2^{-r-1}}{1+R_\varphi} \leq \lambda_i \text{ and } |c_i| \leq e^2 \text{ see (3.5)} \\ &\leq \frac{e^2 r}{\int \varphi(x)\,\mathrm{d}x}\left(\frac{2^{-r-1}}{1+R_\varphi}\right)^\alpha \epsilon^\alpha f(\epsilon, x)\end{aligned}$$

This proves the claim (6.4).

Next, we define our usual mollifier $\rho = \hat{\varphi}^2 * \hat{\varphi}$ and $\epsilon_k = 2^{-k}$. Recall from Step 2: Tweaking that we have the crucial property $\rho^{\epsilon_{k+1}} - \rho^{\epsilon_k} = \hat{\varphi}^{\epsilon_k} * \check{\varphi}^{\epsilon_k}$ with $\check{\varphi} = \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2$. This allows us to write $T(\psi_x^\lambda) = \lim_{n\to\infty} T(\rho^{\epsilon_n} * \psi_x^\lambda)$. Furthermore, we define $N = \min\{k \in \mathbb{N} : \epsilon_k \leq \lambda\}$. Hence, $\frac{1}{2}\lambda < \epsilon_N \leq \lambda$. Fix $x \in K, \lambda \in (0,1]$ and $\psi \in \mathcal{B}_r$. Then, we write

$$T(\psi_x^\lambda) = \underbrace{T(\rho^{\epsilon_N} * \psi_x^\lambda)}_{=A} + \underbrace{\left(T(\psi_x^\lambda) - T(\rho^{\epsilon_N} * \psi_x^\lambda)\right)}_{=B}.$$

- We estimate A. We begin with $A = T(\rho^{\epsilon_N} * \psi_x^\lambda) = \iint T(\hat{\varphi}_y^{\epsilon_N})\hat{\varphi}^{2\epsilon_N}(y-z)\psi_x^\lambda(z)\,\mathrm{d}y\,\mathrm{d}z$, where for the last equality we used Corollary 1.14 twice. The last expression can also be written as a convolution $\int T(\hat{\varphi}_y^{\epsilon_N})(\hat{\varphi}^{2\epsilon_N} * \psi_x^\lambda)(y)\,\mathrm{d}y$. We apply the first inequality of Lemma 3.6 and obtain $|A| \le 2^d \|\hat{\varphi}\|_{L^1} \sup_{y\in B(x,\lambda+\epsilon_N)} |T(\hat{\varphi}_y^{\epsilon_N})|$. Then, we use our claim (6.4) to get (recall that $\alpha \le 0, \epsilon_N \le \lambda, \lambda + 2\epsilon_N \le 2\lambda$ and $\epsilon_N \ge \frac{\lambda}{2}$):

$$|A| \le 2^d \|\hat{\varphi}\|_{L^1} C\epsilon_N^\alpha \sup_{\substack{x'\in B(x,2\lambda)\\ \epsilon'\in(0,\lambda]}} f(\epsilon',x') \le \left\{2^{d-\alpha}C\|\hat{\varphi}\|_{L^1}\right\}\lambda^\alpha \sup_{\substack{x'\in B(x,2\lambda)\\ \epsilon'\in(0,\lambda]}} f(\epsilon',x').$$

- We estimate B. First, we define a sequence $(B_k)_{k\ge N}$ such that $B = \sum_{k=N}^\infty B_k$. We set $B_k = T(\rho^{\epsilon_{k+1}} * \psi_x^\lambda) - T(\rho^{\epsilon_k} * \psi_x^\lambda) \overset{(1.2)}{=} \iint T(\hat{\varphi}_y^{\epsilon_k})(\check{\varphi}^{\epsilon_k}(y-z)\psi_x^\lambda(z))\,\mathrm{d}y\,\mathrm{d}z$. Again, the last expression can be written as a convolution $\int T(\hat{\varphi}_y^{\epsilon_k})(\check{\varphi}^{\epsilon_k} * \psi_x^\lambda)(y)\,\mathrm{d}y$ so that we can use Lemma 3.6. Applying the second inequality of Lemma 3.6 yields

$$|B_k| \le 4^d \|\check{\varphi}\|_{L^1}\epsilon_k^r\lambda^{-r} \sup_{y\in B(x,\lambda+\epsilon_k)} |T(\hat{\varphi}_y^{\epsilon_k})| \overset{(6.4)}{\le} 4^d\|\check{\varphi}\|_{L^1}\epsilon_k^{\alpha+r}\lambda^{-r}C \sup_{\substack{x'\in B(x,2\lambda)\\ \epsilon'\in(0,\lambda]}} f(\epsilon',x').$$

We see that $\sum_{k=N}^\infty |B_k| < \infty$ because $\sum_{k=N}^\infty \epsilon_k^{\alpha+r} = \frac{\epsilon_N^{\alpha+r}}{1-2^{-\alpha-r}}$ for $\alpha + r > 0$. So,

$$|B| \le \sum_{k=N}^\infty |b_k| \le \frac{C4^d\|\check{\varphi}\|_{L^1}}{1-2^{-\alpha-r}}\lambda^{-r}\epsilon_N^{\alpha+r} \sup_{\substack{x'\in B(x,2\lambda)\\ \epsilon'\in(0,\lambda]}} f(\epsilon',x').$$

We are ready to estimate $T(\psi_x^\lambda)$. Recall that $\|\check{\varphi}\|_{L^1} \le 2\|\hat{\varphi}\|_{L^1}$. Then, we have

$$|T(\psi_x^\lambda)| \le \left\{\frac{4^{d-\alpha+1}}{1-2^{-\alpha-r}}\|\hat{\varphi}\|_{L^1}C\right\}\lambda^\alpha \sup_{\substack{x'\in B(x,2\lambda)\\ \epsilon'\in(0,\lambda]}} f(\epsilon',x').$$

Using (6.4) and (3.4) we explicitly state the constant:

$$\left\{\frac{4^{d-\alpha+1}}{1-2^{-\alpha-r}}\left(\frac{e^2 r}{|\int\varphi(x)\,\mathrm{d}x|}\right)^2\left(\frac{2^{-r-1}}{1+R_\varphi}\right)^\alpha\|\varphi\|_{L^1}\right\}. \tag{6.5}$$

□

As a corollary, we get the following characterization of negative Hölder spaces, where testing for a *single* test function $\varphi \in \mathcal{D}$ suffices to show that a distribution lies in a negative Hölder space.

Corollary 6.3 (Characterization of Negative Hölder Spaces)**.** *Let $\alpha \le 0$ and $T \in \mathcal{D}'$. Then, the following conditions are equivalent*

1. *$T \in \mathcal{C}^\alpha$;*
2. *There exists an integer $r > -\alpha$ such that* (6.1) *holds for all test functions $\psi \in \mathcal{B}_r$;*
3. *There exists $\varphi \in \mathcal{D}$ with $\int \varphi(x)\,\mathrm{d}x \neq 0$ such that for any compact set $K \subset \mathbb{R}^d$ there exists a constant $\tilde{C} < \infty$ with*

$$|T(\varphi_x^\epsilon)| \leq \tilde{C}\epsilon^\alpha$$
$$\textit{for all } x \in K \textit{ and } \epsilon \in \left\{2^{-k}\right\}_{k\in\mathbb{N}}.$$

Proof.

- 1. $\implies$ 2. This holds because $\mathcal{B}_r \subset \mathcal{B}_{r_\alpha}$ for $r \geq r_\alpha$.
- 2. $\implies$ 3. Choose any $\varphi \in \mathcal{B}_r$ with $\int \varphi(x)\,\mathrm{d}x$.
- 3. $\implies$ 1. Apply Theorem 6.2 with $f \equiv \tilde{C}$.

□

We further exploit that negative Hölder-continuity is based on a single test function $\varphi \in \mathcal{D}$. We estimate the $\|\cdot\|_{\mathcal{C}^\alpha(K)}$-norm using φ.

Corollary 6.4. *Let $\alpha \leq 0$ and $T \in \mathcal{D}'$. Let $\varphi \in \mathcal{D}$ be the test function as in Corollary 6.3. Then, we have for any compact set $K \subset \mathbb{R}^d$*

$$\|T\|_{\mathcal{C}^\alpha(K)} \leq \{\text{constant}\} \cdot \sup_{\substack{x\in\bar{K}_2\\ \epsilon\in(0,1]}} \frac{|T(\varphi_x^\epsilon)|}{\epsilon^\alpha},$$

where the constant is given as in (6.5).

Proof. By Corollary 6.3 there exists a test function $\varphi \in \mathcal{D}$ such that $|T(\varphi_x^\epsilon)| \leq \tilde{C}\epsilon^\alpha$ uniformly for x in compact sets and $\epsilon \in \left\{2^{-k}\right\}_{k\in\mathbb{N}}$ with some constant $\tilde{C}$. Thus, we apply Theorem 6.2 with $\tilde{C} = \sup_{\substack{x\in\bar{K}_2\\ \epsilon\in(0,1]}} \frac{|T(\varphi_x^\epsilon)|}{\epsilon^\alpha}$ and we obtain

$$\|T\|_{\mathcal{C}^\alpha(K)} = \sup_{\substack{x\in K,\\ \epsilon\in(0,1],\\ \psi\in\mathcal{B}_{r_\alpha}}} \frac{|T(\psi_x^\epsilon)|}{\epsilon^\alpha} \leq \frac{\{\text{constant}\}\cdot\epsilon^\alpha}{\epsilon^\alpha} \sup_{\substack{\epsilon'\in(0,\epsilon]\\ x'\in B(x,2\epsilon)}} f(\epsilon', x') = \{\text{constant}\} \cdot \sup_{\substack{x\in\bar{K}_2\\ \epsilon\in(0,1]}} \frac{|T(\varphi_x^\epsilon)|}{\epsilon^\alpha}.$$

□

To the end of this chapter, we prove that $\mathcal{R}F$ lies in a negative Hölder-space if the germ F has *global* homogeneity bound. Local homogeneity bound, that we get for free by the coherence condition (Lemma 2.7), does not suffice. This fact is later important to see why the Sewing Lemma is slightly more general than the Reconstruction Theorem.

Theorem 6.5 (Reconstruction Theorem and Hölder Spaces)**.** *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be (α,γ)-coherent germ with global homogeneity bound $\beta < \gamma$. If $\beta > 0$, then $\mathcal{R}F = 0$. If $\beta \leq 0$, then $\mathcal{R}F \in \mathcal{C}^\beta$. Additionally in case of $\beta \leq 0$, the reconstruction operator $\mathcal{R}$ which maps coherent germs F to their reconstruction $\mathcal{R}F$ is continuous in the following sense: there exists a constant such that for every compact set $K \subset \mathbb{R}^d$ the operator $\mathcal{R}$ satisfies*

$$\|\mathcal{R}F\|_{\mathcal{C}^\beta(K)} \leq \{\text{constant}\} \cdot \left(|||F|||^{\text{coh}}_{\bar{K}_4,\varphi,\alpha,\gamma} + |||F|||^{\text{hom}}_{\bar{K}_2,\varphi,\beta}\right) \quad \textit{for all germs } F = (F_x)_{x\in\mathbb{R}^d}. \tag{6.6}$$

Proof. Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a (α,γ)-coherent germ with homogeneity bound $\beta > 0$. Then, $f \equiv 0$ satisfies $\lim_{\lambda\to 0} |(f - F_x)(\varphi_x^\lambda)| = 0$ uniformly for x in compact sets. Theorem 2.13 guarantees the uniqueness of a reconstruction $\mathcal{R}F$. Hence, $\mathcal{R}F = 0$.

Let $\beta \leq 0$. Fix a compact set $K \subset \mathbb{R}^d$. To show that $\mathcal{R}F$ lies in $\mathcal{C}^\beta$, it suffices to show (6.6) because $\mathcal{R}F \in \mathcal{C}^\beta \iff \|\mathcal{R}F\|_{\mathcal{C}^\beta(K)} < \infty$. So, we compute the $\mathcal{C}^\beta$-norm by Corollary 6.4

$$\|\mathcal{R}F\|_{\mathcal{C}^\beta(K)} \leq \{\text{constant}\} \cdot \sup_{\substack{x\in\bar{K}_2\\ \lambda\in(0,1]}} \frac{|\mathcal{R}F(\varphi_x^\lambda)|}{\lambda^\beta}.$$

Let $f = \mathcal{R}F$ and φ be the test function in the coherence condition. We claim that

$$\sup_{\substack{x\in\bar{K}_2\\ \lambda\in(0,1]}} \frac{|\mathcal{R}F(\varphi_x^\lambda)|}{\lambda^\beta} \leq \{\text{constant}\} \left(|||F|||^{\text{coh}}_{\bar{K}_4,\varphi,\alpha,\gamma} + |||F|||^{\text{hom}}_{\bar{K}_2,\varphi,\beta}\right).$$

We choose $\bar{r} = \min\{r \in \mathbb{N} : r > \max\{-\alpha, -\beta\}\}$. We apply the Reconstruction Theorem (Theorem 2.9) for $r = \bar{r}$ and $K = \bar{K}_2$. Let $x \in \bar{K}_2$ and $\lambda \in (0,1]$. Note that $\varphi \in \mathcal{D}$ does not necessarily lie in $\mathcal{B}_{\bar{r}}$. However, it is easy to find parameters c, z and η such that the test function $\xi : x \mapsto c\varphi^\eta(x - z)$ lies in $\mathcal{B}_{\bar{r}}$; the parameter c scales the $C^{\bar{r}}$-norm to one, and η and z shift the compact support of φ to $B(0,1)$. Using the Reconstruction Theorem, we obtain

$$\begin{aligned}
|(f - F_x)(\varphi_x^\lambda)| \leq \frac{1}{c}|(f - F_x)(\xi_{x-z}^{\eta^{-1}\lambda})| &\leq \frac{1}{c}(|(f - F_{x-z})(\xi_{x-z}^{\eta^{-1}\lambda})| + |(F_{x-z} - F_x)(\xi_{x-z}^{\eta^{-1}\lambda})|) \\
&\leq \frac{1}{c}\{\text{constant}\}\, |||F|||^{\text{coh}}_{\bar{K}_4,\varphi,\alpha,\gamma} \begin{cases} (\eta^{-1}\lambda)^\gamma & \text{if } \gamma \neq 0 \\ 1 + |\log(\eta^{-1}\lambda)| & \text{if } \gamma = 0 \end{cases} \\
&\quad + \frac{1}{c}|(F_{x-z} - F_x)(\xi_{x-z}^{\eta^{-1}\lambda})|
\end{aligned}$$

Since η, z and c only depend on φ, we encode η, z and c in the multiplicative constant factor. The second summand is estimated with the coherence condition. Then, we get

$$|(f - F_x)(\varphi_x^\lambda)| \leq \{\text{constant}\} \cdot |||F|||^{\text{coh}}_{\bar{K}_4,\varphi,\alpha,\gamma} \begin{cases} \lambda^\gamma & \text{if } \gamma \neq 0 \\ 1 + |\log(\lambda)| & \text{if } \gamma = 0 \end{cases}.$$

Next, observe that $\lambda^\gamma < \lambda^\beta$ because $\gamma > \beta$. Moreover, a lengthy computations shows $1+|\log(\lambda)| \le \left\{-\beta^{-1}e^{-(1+\beta)}\right\}\lambda^\beta$ for all $\lambda \in (0,1]$ and $\beta \le 0$, see Theorem 12.7 in [3]. Hence,

$$|(f - F_x)(\varphi_x^\lambda)| \le \{\text{constant}\} \cdot (1 + \left\{-\beta^{-1}e^{-(1+\beta)}\right\}) \cdot |||F|||^{\text{coh}}_{\bar{K}_4,\varphi,\alpha,\gamma}\lambda^\beta.$$

Finally, we use the above estimate and the homogeneity semi-norm (2.10) to bound

$$\begin{aligned}\sup_{\substack{x\in\bar{K}_2\\ \lambda\in(0,1]}} \frac{|\mathcal{R}F(\varphi_x^\lambda)|}{\lambda^\beta} &\le \sup_{\substack{x\in\bar{K}_2\\ \lambda\in(0,1]}} \frac{|(f - F_x)(\varphi_x^\lambda)| + |F_x(\varphi_x^\lambda)|}{\lambda^\beta} \\ &\le \{\text{constant}\}\, |||F|||^{\text{coh}}_{\bar{K}_4,\varphi,\alpha,\gamma} + |||F|||^{\text{hom}}_{\bar{K}_2,\varphi,\beta}.\end{aligned}$$

We proved the theorem. □

It follows $\mathcal{R}F \in \mathcal{C}^\gamma$ because $\mathcal{C}^\beta \subset \mathcal{C}^\gamma$ for $\beta < \gamma$ (to see this note that $\lambda^\beta < \lambda^\gamma$ for $\lambda \in (0,1]$). Moreover, $\mathcal{R}F$ is unique up to an element of $\mathcal{C}^\gamma$ as we show next.

Corollary 6.6 (Non uniqueness). *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a γ-coherent germ for some $\gamma < 0$. Then, the reconstruction $\mathcal{R}F$ is unique up to an element of $\mathcal{C}^\gamma$.*

Proof. Let f and g be two reconstructions of a γ-coherent germ F with negative $\gamma < 0$. Let $K \subset \mathbb{R}^d$ be a compact set. Then,

$$|(f-g)(\psi_x^\lambda)| \le |(f-F)(\psi_x^\lambda)| + |(F-g)(\psi_x^\lambda)| \le \{\text{constant}\}\,\lambda^\gamma$$

holds uniformly for all test functions $\psi \in \mathcal{B}_r, x \in K$ and $\lambda \in (0,1]$. Thus, $f - g \in \mathcal{C}^\gamma$ by Corollary 6.3.

On the other hand let $g \in \mathcal{C}^\gamma$. Then, $\mathcal{R}F + g$ is a reconstruction because

$$|(\{\mathcal{R}F + g\} - F_x)(\psi_x^\lambda)| \le |(\mathcal{R}F - F_x)(\psi_x^\lambda)| + |g(\psi_x^\lambda)| \le \{\text{constant}\}\,\lambda^\gamma.$$

□

To conclude, we defined negative Hölder spaces. We also showed that $\mathcal{R}F$ lives in a negative Hölder space and is not unique.

6.2 Enhanced Coherence and Homogeneity

We return to the coherence condition and homogeneity bound. In the previous chapter, we saw that it suffices to test for a single test function $\varphi \in \mathcal{D}$ if we want some distribution to lie in a negative Hölder-space. We will prove something similar for coherence and homogeneity. If a test function $\varphi \in \mathcal{D}$ satisfies the inequality (2.6) in the coherence condition, then so do all test functions in $\mathcal{B}_r$. Hence, our definition of coherence is quite powerful; testing a germ against a single test function φ allows us to inspect the germ's behavior for the whole class of test functions in $\mathcal{B}_r$.

Theorem 6.7 (Enhanced Coherence)**.** *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be an $(\boldsymbol{\alpha}, \gamma)$-coherent germ. For any compact set $K \subset \mathbb{R}^d$ and $r > -\alpha_{\bar{K}_2}$ there exists a constant $C < \infty$ such that*

$$|(F_z - F_y)(\psi_y^\lambda)| \leq C\lambda^{\alpha_{\bar{K}_2}}(|z-y| + \lambda)^{\gamma-\alpha_{\bar{K}_2}}$$

uniformly for $z, y \in K, \lambda \in (0, 1]$ and $\psi \in \mathcal{B}_r$.

We stress that our definition of coherence requires only a single test function $\varphi \in \mathcal{D}$ with $\int \varphi(x)\,\mathrm{d}x \neq 0$ to be tested.

Proof. The proof is straighforward and uses Theorem 6.2.

Let $K \subset \mathbb{R}^d$ be a compact set and $z, y \in K$. We choose a third point $x \in \bar{K}_2$ just for the purpose of applying Theorem 6.2. Using the triangle inequality, we have for all $\lambda \in (0, 1]$ that

$$\begin{aligned}|(F_z - F_y)(\varphi_x^\lambda)| &\leq |(F_z - F_x)(\varphi_x^\lambda)| + |(F_x - F_y)(\varphi_x^\lambda)| \\ &\Downarrow \text{Apply coherence for } K = \bar{K}_2 \\ &\leq \{\text{constant}\}\, \lambda^{\alpha_{\bar{K}_2}}(|z-x| + |y-x| + \lambda)^{\gamma-\alpha_{\bar{K}_2}}.\end{aligned}$$

The constant only depends on K and φ; this fact is important because the inequality must hold uniformly for points in compact sets. Next, we apply Theorem 6.2 for $T = F_z - F_y$ and $f(\lambda, x) = (|z-x| + |y-x| + \lambda)^{\gamma-\alpha_{\bar{K}_2}}$, which yields

$$|(F_z - F_y)(\psi_x^\lambda)| \leq \{\text{constant}\}\, \lambda^{\alpha_{\bar{K}_2}}(|z-x| + |x-y| + 5\lambda)^{\gamma-\alpha_{\bar{K}_2}}$$

$$\text{for any } r > -\alpha_{\bar{K}_2}, \lambda \in (0, 1] \text{ and } \psi \in \mathcal{B}_r.$$

Note that the multiplicative constant still only depends on K and φ; hence the above inequality holds uniformly for $x, y \in K, \lambda \in (0, 1]$ and $\psi \in \mathcal{B}_r$. We set $x = y$, which ends the proof. □

In the proof we replaced α_K by $\alpha_{\bar{K}_2}$ to prove an enhanced coherence condition. If we replace β_K by $\beta_{\bar{K}_2}$, we obtain an enhanced homogeneity bound.

Theorem 6.8 (Enhanced Local Homogeneity)**.** *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be an $(\boldsymbol{\alpha}, \gamma)$-coherent germ with local homogeneity bound $\boldsymbol{\beta}$. For any compact set $K \subset \mathbb{R}^d$ and $r > -\max\{\alpha_{\bar{K}_2}, -\beta_{\bar{K}_2}\}$ there exists a constant $C < \infty$ such that*

$$|F_x(\psi_x^\lambda)| \leq C\lambda^{\beta_{\bar{K}_2}}$$

uniformly for $x \in K, \lambda \in (0, 1]$ and $\psi \in \mathcal{B}_r$.

Proof. Let $K \subset \mathbb{R}^d$ be a compact set and $r > -\max\{\alpha_{\bar{K}_2}, -\beta_{\bar{K}_2}\}$. Using the triangle inequality, we have $|F_x(\psi_x^\lambda)| \leq |(F_x - \mathcal{R}F)(\psi_x^\lambda)| + |\mathcal{R}F(\psi_x^\lambda)|$. The first summand is estimated with the Reconstruction Theorem, which yields $|(F_x - \mathcal{R}F)(\psi_x^\lambda)| \leq \{\text{constant}\}\, \lambda^\gamma \leq \{\text{constant}\}\, \lambda^\beta$ uniformly for $x \in K, \lambda \in (0, 1], \psi \in \mathcal{B}_r$. For the second summand, we use that $\mathcal{R}F \in \mathcal{C}^\beta$, see (6.1). Hence, $|\mathcal{R}F(\psi_x^\lambda)| \leq \{\text{constant}\}\, \lambda^\beta$ uniformly for $x \in K, \lambda \in (0, 1], \psi \in \mathcal{B}_r$. This ends the proof. □

Clearly, enhanced coherence implies coherence, and enhanced homogeneity implies homogeneity. At first sight, the converse seems false. However, we proved that our initial definition of coherence and homogeneity offers as much as information as the enhanced versions. The proof was simple and relied on the Reconstruction Theorem.

6.3 Sewing Lemma

The Sewing Lemma is closely related to the Reconstruction Theorem; often it is called the 1-dimensional analogue of the Reconstruction Theorem. Originally, introduced by ... HISTORY

Our goal is to prove the Sewing Lemma with the Reconstruction Theorem. We closely follow the proof of [2, Chapter 5]. We will see that the Sewing Lemma is slightly more general than the Reconstruction Theorem in the 1-dimensional setting $\mathbb{R}$. That is, it is no hurdle to show that Sewing implies the Reconstruction Theorem. Proving the converse is nontrivial without additional assumptions. We need to assume a *global homogeneity bound*. To date, a proof that the Sewing Lemma implies the Reconstruction Theorem without any additional assumption is still missing.

Yet, the Sewing Lemma has a drawback. It is not applicable in the multi-dimensional setting $\mathbb{R}^d$ in contrast to the Reconstruction Theorem. Here, the distributional view begins to shine.

We state the Sewing Lemma. Recall that in Chapter 2.1, we have already introduced the Sewing Lemma by Lemma 2.4. We were trying to find a suitable assumption for the Reconstruction Theorem, which ultimately led to coherence.

Lemma 6.9 (Sewing Lemma for $\gamma > 1$, [2])**.** *Let $\gamma > 1$. Define $\Delta = \{(s,t) : 0 \leq s \leq t \leq T\}$ for some fixed $T > 0$. Let $A : \Delta \to \mathbb{R}$ be a continuous function such that there exists $C < \infty$ with*

$$\delta A_{s,u,t} := |A_{s,t} - A_{s,u} - A_{u,t}| \leq C|t-s|^\gamma$$
$$\text{uniformly for } 0 \leq s \leq u \leq t \leq T.$$

Then, there exists a unique function $I : [0,T] \to \mathbb{R}$ and $\tilde{C} < \infty$ such that $I_0 = 0$ and

$$|I_t - I_s - A_{s,t}| \leq \tilde{C}|t-s|^\gamma$$
$$\text{uniformly for } 0 \leq s \leq t \leq T.$$

Furthermore, I is the limit of Riemann-type sums. That is

$$I_t = \lim_{|\pi|\to 0} \sum_{i=0}^{\#\pi - 1} A_{t_i,t_{i+1}}$$

where the limit is taken over partitions π of $[0,T]$.

This Sewing Lemma holds for $\gamma > 1$. Recently, the Sewing Lemma was extended to $\gamma \in (0, 1]$ by Broux and Zambotti [2].

Lemma 6.10 (Sewing Lemma for $0 < \gamma \leq 1$, [2])**.** *Let $0 < \gamma \leq 1$. Let $A : \Delta \to \mathbb{R}$ be a continuous function that satisfies*

$$\delta A_{s,u,t} \leq C|t-s|^{\gamma}$$
$$\textit{uniformly for } 0 \leq s \leq u \leq t \leq T.$$

Then, there exists a (non-unique) function $I : [0, T] \to \mathbb{R}$ and $\tilde{C} < \infty$ such that $I_0 = 0$ and

$$|I_t - I_s - A_{s,t}| \leq \tilde{C} \begin{cases} |t-s|^{\gamma} & \textit{if } 0 < \gamma < 1, \\ |t-s|(1 + |\log(|t-s|)|) & \textit{if } \gamma = 1, \end{cases}$$
$$\textit{uniformly for } 0 \leq s \leq t \leq T.$$

We are going to prove both Sewing Lemmas using the Reconstruction Theorem. Let's briefly discuss the idea of the proof.

- Given a family of $(A_{s,t})$ we construct a coherent germ $F = (F_x)_{x \in \mathbb{R}^d}$.
- We apply the Reconstruction Theorem on F and obtain $\mathcal{R}F$.
- We find a primitive I of $\mathcal{R}F$. That is we find a function I such that $I' = \mathcal{R}F$. Then, I is our wanted function.

Finding a primitive I is nontrivial if we only assume $A_{s,t}$ to satisfy the sewing condition

$$\delta A_{s,u,t} \leq C|t-s|^{\gamma} \tag{6.7}$$
$$\text{uniformly for } 0 \leq s \leq u \leq t \leq T.$$

However, if we further assume that $A_{s,t}$ satisfies a *global* homogeneity bound β, that is

$$|A_{s,t}| \leq \{\text{constant}\} \cdot |t-s|^{\beta},$$

then the constructed germ F also has a *global* homogeneity bound $\beta - 1$. By Theorem 6.5, the reconstruction $\mathcal{R}F$ is in $\mathcal{C}^{\beta-1}$. Then, there exists $I \in \mathcal{C}^{\beta}$ such that $I_0 = 0$ and $I' = \mathcal{R}F$. This is a well-known fact, see [1, Lemma 3.10]. In other words, the global homogeneity bound gives us enough information about the regularity of $\mathcal{R}F$; then knowing the regularity allows us to claim the existence of a primitive of $\mathcal{R}F$.

We emphasize that the constructed germ must have a *global* homogeneity bound. It trivially has a *local* homogeneity bound by Lemma 2.7 but not necessarily a global bound.

We begin with a naive proof of the Sewing Lemma for $\gamma > 1$ and $0 < \gamma \leq 1$. Assume $A \in C([0,1]\times[0,1])$ is a continuous function satisfying the sewing condition (6.7). Let $\gamma > 0$. First, we extend A to the entire domain $\mathbb{R}^2$. We set

$$\begin{aligned} p(s) &= \max\{0, \min\{s, 1\}\} \\ A_{s,t} &= A_{p(s),p(t)}. \end{aligned}$$

Next, we define a germ $F = (F_s)_{s\in\mathbb{R}^d}$ by differentiating $A_{\cdot,\cdot}$ with respect to its second variable; here we mean the distributional derivative. So,

$$F_s(\varphi) = -\int_{\mathbb{R}^d} A_{s,t}\varphi'(t)\,\mathrm{d}t \qquad \forall\varphi\in\mathcal{D}(\mathbb{R}^d).$$

We claim that $F = (F_s)_{s\in\mathbb{R}^d}$ is a $(-1, \gamma-1)$-coherent germ.

Proof. Fix any test function $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Let $s, u, t \in \mathbb{R}$ and $\lambda \in (0, 1]$. Then,

$$\begin{aligned} |(F_t - F_s)(\varphi_s^\lambda)| &= \left|-\lambda^{-1}\int_{\mathbb{R}^d} A_{p(t),p(x)}\varphi_s'^\lambda(x)\,\mathrm{d}x + \lambda^{-1}\int_{\mathbb{R}^d} A_{p(s),p(x)}\varphi_s'^\lambda(x)\,\mathrm{d}x\right| \\ &= \left|\lambda^{-1}\int_{\mathbb{R}^d}(A_{p(t),p(x)} - A_{p(s),p(x)})\varphi_s'^\lambda(x)\,\mathrm{d}x\right| \\ &\Downarrow \text{Substitution } v = \frac{x-s}{\lambda} \\ &= \lambda^{-1}\left|\int_{\mathbb{R}^d}\delta A_{p(t),p(s),p(s+\lambda v)}\varphi'(v)\,\mathrm{d}v\right|. \end{aligned}$$

For the last equation we also used that $\int_{\mathbb{R}^d} A_{p(t),p(s)}\varphi'(v)\,\mathrm{d}v = 0$; this follows from integration by parts. Since $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the integral $|\int \varphi'(v)\,\mathrm{d}v|$ is bounded from above by some constant. So, we use the sewing condition (2.4) to estimate

$$\begin{aligned} |(F_t - F_s)(\varphi_s^\lambda)| &\leq \{\text{constant}\}\,\lambda^{-1}\left(\max\{|p(s+\lambda v) - p(s)|, |p(s) - p(t)|\}\right)^\gamma \\ &\Downarrow p \text{ is Lipschitz-continuous} \\ &\leq \{\text{constant}\}\,\lambda^{-1}(\lambda + |t-s|)^\gamma. \end{aligned}$$

This proves that F is a $(-1, \gamma-1)$-coherent germ. □

We apply the Reconstruction Theorem. Then, there exists $\mathcal{R}F$ such that for any compact set $K \subset \mathbb{R}$ we have

$$|(F_s - \mathcal{R}F)(\psi_s^\lambda)| \leq \{\text{constant}\}\,\lambda^{\gamma-1} \tag{6.8}$$

uniformly for $s \in K, \lambda \in (0, 1]$ and $\psi \in \mathcal{B}_{r_K}$.

Naively, we assume that there exists a primitive I of $\mathcal{R}F$. Then, $\mathcal{R}F(1_{[s,s+\lambda]}) = -I_{s,s+\lambda}$. Also, we have $F_s(1_{[s,s+\lambda]}) = -A_{s,s+\lambda}$. Then, we write $1_{[s,s+\lambda]} = \lambda(1_{[0,1]})_s^\lambda$ so that we can apply the Reconstruction Theorem:

$$|(\mathcal{R}F - F_s)(1_{[s,s+\lambda]})| = |(I-A)_{s,s+\lambda}| = |I_t - I_s - A_{s,t}| \overset{(6.8)}{\leq} \{\text{constant}\}\, \lambda^\gamma.$$

This *would* prove the Sewing Lemma *if* $1_{(0,1)}$ were a test function (it is clearly not smooth), and if we knew that a primitive I exists in the first place. We fix both issues.

Approximating an Indicator Function

The indicator function $1_{(0,1)}$ is not a test function. However, we find smooth functions φ_n and ψ_n that approximate $1_{(0,1)}$.

Lemma 6.11 (Dyadic Approximation of Indicator Functions)**.** *There exist $\varphi_n, \psi_n \in \mathcal{D}$ for $n \in \mathbb{N}_0$ such that*

- $\sup(\varphi_n) \subset [\frac{1}{16}2^{-n}, \frac{15}{16}2^{-n}]$ *and* $\sup(\psi_n) \subset [1 - \frac{15}{16}2^{-n}, 1 - \frac{1}{16}2^{-n}]$ *for all* $n \in \mathbb{N}_0$,
- $\sup\limits_{n\in\mathbb{N}_0} \sup\limits_{0\leq k\leq r, k\in\mathbb{N}} \frac{\|\partial^k\varphi_n\|_\infty + \|\partial^k\psi_n\|_\infty}{2^{kn}} < \infty$, *and*
- $1_{0,1} = \sum\limits_{n\geq 0} \varphi_n + \psi_n$.

Proof. We skip the proof to turn our attention to the interesting part; instead we refer to [2, Lemma 5.2]. □

We approximate $1_{0,1}$ with the following theorem.

Theorem 6.12. *Assume that I is a primitive of $\mathcal{R}F$ in the sense of distributions, and $I_0 = 0$. Define for $n \in \mathbb{N}, s \in [0,T]$ and $\lambda > 0$*

$$\Delta_{s,\lambda}^N = \sum_{n=0}^N (\mathcal{R}F - F_s)(\lambda(\varphi_n + \psi_n)_s^\lambda).$$

Then, we have for all $s \in \mathbb{R}$ and $\lambda > 0$

$$\lim_{N\to\infty} \Delta_{s,\lambda}^N = (I-A)_{s,s+\lambda}.$$

Additionally, for any compact set $K \subset \mathbb{R}$ there exists a constant $C < \infty$ such that

$$|\Delta_{s,\lambda}^N| \leq C\lambda^\gamma$$

uniformly for $s \in K, N \in \mathbb{N}$ and $\lambda \in (0,1]$.

We give a proof but omit minor technical details to keep this section concise; again we refer to [2] for the full proof.

Sketch of proof. Let $N \in \mathbb{N}_0$. Then,

$$\Delta^N_{s,\lambda} = \sum_{n=0}^{N} (\mathcal{R}F - F_s)(\lambda(\varphi_n + \psi_n)^\lambda_s) = -\int_{\mathbb{R}} (I_u - A_{s,u}) \sum_{n=0}^{N} ((\varphi_n + \psi_n)')^\lambda_s (u)\,\mathrm{d}u.$$

We define for $N \in \mathbb{N}$ and $u \in \mathbb{R}$

$$\begin{aligned}\eta_N(u) &= 2^{-N} \sum_{n=0}^{N} (\varphi_n + \psi_n)'(2^{-N}u) 1_{[0,T]}(u)\\ \tilde{\eta}_N(u) &= -2^{-N} \sum_{n=0}^{N} (\varphi_n + \psi_n)'(1 + 2^{-N}u) 1_{[-1,0]}(u).\end{aligned}$$

Then, by substitution and rearranging the terms we get

$$\sum_{n=0}^{N} (\varphi_n + \psi_n)' = (\eta_N)^{\frac{1}{2^N}} - (\tilde{\eta}_N)^{\frac{1}{2^N}}_1.$$

After a substitution $\frac{u-s}{\lambda} \rightsquigarrow u$, we write

$$\begin{aligned}\Delta^N_{s,\lambda} = &-\int I_{s+\lambda 2^{-N}u}\eta_N(u)\,\mathrm{d}u + \int A_{s,s+\lambda 2^{-N}u}\eta_N(u)\,\mathrm{d}u\\ &+\int I_{s+\lambda+\lambda 2^{-N}u}\tilde{\eta}_N(u)\,\mathrm{d}u - \int A_{s,s+\lambda+\lambda 2^{-N}u}\tilde{\eta}_N(u)\,\mathrm{d}u\end{aligned}$$

We show that the first summand converges to $-I_s$. We have

$$\int I_{s+\lambda 2^{-N}u}\eta_N(u)\,\mathrm{d}u = I_s + \int (I_{s+\lambda 2^{-N}u} - I_s)\eta_N(u)\,\mathrm{d}u.$$

By assumption, I is continuous. Hence, the integrand $(I_{s+\lambda 2^{-N}\cdot} - I_s)\eta_N(\cdot) \to 0$ converges pointwise to 0 as $N \to \infty$ if η_N is continuous. If we further assume that $\eta_N \in \mathcal{D}(B(0,1))$, then the integrand is bounded by a constant. We can then apply the dominated convergence theorem, which shows that the first summand converges to $-I_s$. It remains to prove that $\eta_N \in \mathcal{D}(B(0,1))$, for which we use the properties of the dyadic approximation. It is not difficult to show but we omit this part.

Similarly, we show that the other three summands converge to $I_{s+\lambda}$ or $A_{s,s+\lambda}$ or they vanishes. We then obtain

$$\Delta^N_{s,\lambda} = I_{s+\lambda} - I_s - A_{s,s+\lambda} + o_{N\to\infty}(1).$$

This proves our first claim.

Next, we want to bound $|\Delta^N_{s,\lambda}|$. This is the step where we use the Reconstruction Theorem. We define

$$\eta_n(x) = \varphi_n(2^{-n}x) \quad \text{and} \quad \tilde{\eta}_n(x) = \psi_n(2^{-n}+1).$$

Thus, we have $(\varphi_n)^\lambda_s = 2^{-n}(\eta_n)^{2^{-n}\lambda}_s$ and $(\psi_n)^\lambda_s = 2^{-n}(\tilde{\eta}_n)^{2^{-n}\lambda}_{s+\lambda}$. Writing

$$\Delta^N_{s,\lambda} = \sum_{n=0}^{N}(\mathcal{R}F - F_s)(\lambda 2^{-n}(\eta_n)^{\lambda^{2^{-n}}}_s) + \sum_{n=0}^{N}(\mathcal{R}F - F_{s+\lambda})(\lambda 2^{-n}(\tilde{\eta}_n)^{\lambda^{2^{-n}}}_{s+\lambda})$$
$$+ \sum_{n=0}^{N}(F_{s+\lambda} - F_s)(\lambda 2^{-n}(\tilde{\eta}_n)^{\lambda^{2^{-n}}}_{s+\lambda}),$$

we apply the Reconstruction Theorem to estimate the first two summands. The first summand is then bounded from above by $\leq \{\text{constant}\} \sum_{n=0}^N \lambda 2^{-n}(\lambda 2^{-n})^{\gamma-1} \leq \{\text{constant}\}\, \lambda^\gamma$. Similarly for the second summand we obtain a bound $\leq \{\text{constant}\}\, \lambda^\gamma$.

Finally, it remains to bound the third summand. By definition of F_s we write

$$\sum_{n=0}^{N}(F_{s+\lambda} - F_s)(\lambda 2^{-n}(\tilde{\eta}_n)^{\lambda^{2^{-n}}}_{s+\lambda}) = -\lambda \int_{\mathbb{R}} \delta A_{s+\lambda,s,u} \sum_{n=0}^{N} 2^{-n} \left((\tilde{\eta}_n)^{\lambda 2^{-n}}_{s+\lambda}\right)'(u)\, \mathrm{d}u,$$

where we again used the fact that $\int A_{s+\lambda,s}\, \tilde{\eta}'(u)\, \mathrm{d}u$ vanishes by integration by parts. Since $\tilde{\eta}_n \in \mathcal{D}(B(0,1))$, we have $\operatorname{supp}((\tilde{\eta})_n)^{\lambda 2^{-n}}_{s+\lambda} \subset [s+\lambda, s+\lambda+2^{-n}\lambda] \subset [s+\lambda, s+2\lambda]$. Hence,

$$|\sum_{n=0}^{N}(F_{s+\lambda} - F_s)(\lambda 2^{-n}(\tilde{\eta}_n)^{\lambda^{2^{-n}}}_{s+\lambda})| \leq \left(\sup_{u\in[s+\lambda,s+2\lambda]} |\delta A_{s+\lambda,s,u}|\right) \int_{\mathbb{R}} |\sum_{n=0}^{N} \lambda 2^{-n}((\tilde{\eta}_n)^{\lambda 2^{-n}}_{s+\lambda})'(u)|\, \mathrm{d}u.$$

We use the Sewing condition to estimate $\delta A_{s+\lambda,s,u} \leq (|u-s|+|s-(s+\lambda)|)^\gamma \leq \{\text{constant}\}\, \lambda^\gamma$.

To find a constant that bounds the integral $\int_{\mathbb{R}} |\sum_{n=0}^N \lambda 2^{-n}((\tilde{\eta}_n)^{\lambda 2^{-n}}_{s+\lambda})'(u)|\, \mathrm{d}u$ we use the properties of the dyadic approximation. The details are found in [2, Proposition 5.3]. This finishes the proof of $|\Delta^N_{s,\lambda}| \leq \{\text{constant}\}\, \lambda^\gamma$. □

This theorem establishes our wanted function I in the Sewing Lemma. Hence, we prove the Sewing Lemma with the Reconstruction Theorem *as long as* there exists a primitive I of $\mathcal{R}F$.

Existence of a Primitive

We assume there exists $0 < \beta < \min\{1, \gamma\}$ such that there exists a constant $C < \infty$ with

$$|A_{s,t}| \leq C|t-s|^\beta$$

uniformly for $s, t \in [0,1]$. We claim that the constructed germ F has a *global* homogeneity bound $\beta - 1$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be any test function. Let $s, t \in \mathbb{R}$ and $\lambda \in (0,1]$. Then,

$$|F_s(\varphi_s^\lambda)| = \lambda^{-1} \int_{\mathbb{R}} |A_{p(s),p(s+\lambda v)} \varphi'(v)\, \mathrm{d}v| \leq \{\text{constant}\}\, \lambda^{\beta-1},$$

where we again used the Lipschitz-continuity of p; see the proof that F is a $(-1, \gamma-1)$-coherent germ at the beginning of this chapter. □

Hence, $\mathcal{R}F \in \mathcal{C}^{\beta-1}$. This proves the existence of a primitive I; that is $I_0 = 0$ and $I' = \mathcal{R}F$. We summarize our insights in a theorem.

Theorem 6.13 (Sewing via the Reconstruction Theorem)**.** *Let $\beta, \gamma > 0$ with $\beta < 1$. Let $A \in C([0,1] \times [0,1])$ be a function that satisfies*

$$\begin{aligned} |\delta A_{s,u,t}| &\leq \{\text{constant}\}\, (\max\{|t-u|, |u-s|\})^\gamma, \\ |A_{s,t}| &\leq \{\text{constant}\}|t-s|^\beta. \end{aligned}$$

uniformly for $s, u, t \in [0,T]$. Then, there exists $I \in \mathcal{C}^\beta$ such that

$$|I_t - I_s - A_{s,t}| \leq \{\text{constant}\} \begin{cases} |t-s|^\gamma & \text{if } \gamma \neq 1 \\ |t-s|(1 + |\log(|t-s|)|) & \text{if } \gamma = 1 \end{cases}$$

uniformly for $s, t \in [0,T]$.

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