

Discrete Geometry III

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These notes are not endorsed by the lecturer.

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1 Gröbner basis

1.1 Monomial order

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Definition 1.1 (Poset). (A, \preceq) is called a **poset** if \preceq is

- reflexive,
- anti-symmetric (i.e. $a \preceq b$ and $b \preceq a$ imply $a = b$), and
- transitive.

Definition 1.2 (Well-order). A poset is called a **well-order** if every nonempty subset $S \subset A$ has a minimum, i.e. there exists $s \in S$ such that $s \preceq s'$ for all $s' \in S$.

Remark 1.3 (well-order \implies total order). A well-order is a total order because for any $a, b \in A$ the set $\{a, b\}$ has a minimum; hence $a \preceq b$ or $b \preceq a$.

A total order need not be a well-order. As an example, take the integers \mathbb{Z} with the usual ordering \leq .

For a total order to be a well-order we need the **descending chain condition (DCC)**. DCC means that every descending chain $\cdots \preceq a_3 \preceq a_2 \preceq a_1$ stabilizes.

Proposition 1.4 (total order + DCC = well-order). Let \preceq be a total order. Then,

$$\preceq \text{ is well-order} \iff \preceq \text{ satisfies DCC.}$$

Equivalently, infinite strictly descending chains $\cdots \prec a_3 \prec a_2 \prec a_1$ do not exist.

Proof. \implies by contraposition. Assume there exists a non-terminating chain $a_1 \preceq a_2 \preceq a_3 \preceq \dots$. Then, $\{a_i\}_{i \in \mathbb{N}}$ has no minimum.

\impliedby by contraposition. Assume a nonempty subset has no minimum. Let a_0 be any element from the subset. Take a_i to be an element different from a_{i-1} such that $a_i \preceq a_{i-1}$ by axiom of choice. We found a non-terminating descending chain. \square

Definition 1.5 (Admissable order). A poset (\mathbb{N}^n, \preceq) is called **admissable** if for all $a, b, c \in \mathbb{N}^n$ we have $a \preceq b \implies a + c \preceq b + c$.

Definition 1.6 (Polynomial). Let k be a field. A polynomial $f \in k[x_1, \dots, x_n]$ is of the form $f = \sum_{\alpha \in A} c_\alpha x^\alpha$, where $A \subset \mathbb{N}^n$ is finite and $c_\alpha \in k$.

Definition 1.7 (Monomial order). A **monomial order** is an admissable, well-ordered and total order on \mathbb{N}^n .

It is called a monomial order because we define an ordering on monomials by

$$x^\alpha \prec x^\beta \iff \alpha \prec \beta.$$

Proposition 1.8 (Monomial order respects \leq). Let \preceq be a monomial order and $\alpha, \beta \in \mathbb{N}^n$. If $\alpha \leq \beta$, then $\alpha \preceq \beta$.

Proof. We have $\alpha + \gamma = \beta$ for some γ . Then, $0 \preceq x^\gamma \implies \alpha \preceq \alpha + \gamma = \beta$. \square

Checking if something is a well-order can be difficult. We have the following easy criterion for admissible total orderings. The proof is not trivial.

Proposition 1.9 (Easy criterion for well-order). Let \preceq be an admissible, total order on \mathbb{N}^n . It is a well-order if and only if $\mathbf{0} \preceq \mathbf{x}$ for all $\mathbf{x} \in \mathbb{N}^n$.

Proof.

\implies by contraposition. Assume $\mathbf{x} \prec \mathbf{0}$. Then we found an infinite strictly decreasing chain $\cdots \prec 3\mathbf{x} \prec 2\mathbf{x} \prec \mathbf{x} \prec \mathbf{0}$. Hence, DCC is not satisfied.

\Leftarrow by induction on n . For $n = 1$, \preceq must be the usual order \leq because we assume $0 \preceq 1$. (\mathbb{N}, \leq) is clearly a well-order, so $n = 1$ is done. For $n \rightsquigarrow n + 1$, we define an order \prec' on \mathbb{N}^n by

$$\mathbf{x} \preceq' \mathbf{y} \iff (\mathbf{x}, 0) \preceq (\mathbf{y}, 0) \quad \mathbf{x}, \mathbf{y} \in \mathbb{N}^n.$$

Let $S \subset \mathbb{N}^{n+1}$ be nonempty and define the projection $\pi : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ dropping the last coordinate. Set $S' = \text{im}(\pi) \subset \mathbb{N}^n$.

By induction hypothesis, S' has a minimum \mathbf{x}' with respect to \preceq' . Define $\mathbf{x} \in S$ to be the element that projects to \mathbf{x}' and is minimal in the sense that $\mathbf{x}_{n+1} \leq \mathbf{y}_{n+1}$ for all $\mathbf{y} \in \pi^{-1}(\mathbf{x}')$. Define $\ell = \mathbf{x}_{n+1}$. The following lemma shows that \mathbf{x} is minimal among all vectors in S that project to \mathbf{x}' .

Lemma. It holds: $\mathbf{x} \preceq \mathbf{y}$ for all $\mathbf{y} \in \pi^{-1}(\mathbf{x}')$.

Proof. We define $\delta = \mathbf{y}_{n+1} - \ell \geq 0$. By assumption, $\mathbf{0} \prec (\mathbf{0}, \delta)$. Adding (\mathbf{x}', ℓ) we get $(\mathbf{x}', \ell) \preceq (\mathbf{x}', \mathbf{y}_{n+1})$. Hence, $\mathbf{x} \preceq \mathbf{y}$. \square

Continuing the proof, we set $\mathbf{x}^{(i)} = \min_{\prec'} \{\mathbf{z} \in S' : (\mathbf{z}, i) \in S\}$ and $\mathbf{x}^{(i)} = (\mathbf{x}^{(i)}, i) \in S$ for $i \in \{1, \dots, \ell - 1\}$. Define

$$\mathbf{s} = \min \left\{ \mathbf{x}^{(0)}, \dots, \mathbf{x}^{(\ell-1)}, \mathbf{x} \right\}.$$

It is left to check that \mathbf{s} is the minimum of S . We use the following lemma.

Lemma. Assume $\mathbf{v}, \mathbf{w} \in \mathbb{N}^{n+1}$ with $\mathbf{v}_{n+1} = \mathbf{w}_{n+1}$. Then,

$$\mathbf{v} \preceq \mathbf{w} \iff (\mathbf{v}_1, \dots, \mathbf{v}_n, 0) \preceq (\mathbf{w}_1, \dots, \mathbf{w}_n, 0).$$

Proof. \implies by contraposition: Assume $(\mathbf{w}_1, \dots, \mathbf{w}_n, 0) \preceq (\mathbf{v}_1, \dots, \mathbf{v}_n, 0)$. Add $(\mathbf{0}, \mathbf{v}_{n+1})$ on both sides and we get $\mathbf{w} \preceq \mathbf{v}$. The converse direction \Leftarrow is similar. \square

With this lemma, it is easy to show that \mathbf{s} is the minimum of S . Assume $\mathbf{y} \in S$.

If $\mathbf{y}_{n+1} = \ell$, then the previous lemma implies $\mathbf{x} \preceq \mathbf{y}$.

If $\mathbf{y}_{n+1} < \ell$, then the previous lemma implies $\mathbf{x}^{(y_{n+1})} \preceq \mathbf{y}$.

If $\mathbf{y}_{n+1} > \ell$, then $\mathbf{x}' \preceq' \pi(\mathbf{y})$. So $(\mathbf{x}', \mathbf{y}_{n+1}) \preceq \mathbf{y}$. We know that $\mathbf{x} = (\mathbf{x}', \ell) \preceq (\mathbf{x}', \mathbf{y}_{n+1}) \preceq \mathbf{y}$ where the first inequality holds by the first lemma. \square

An alternative proof uses Dickson's lemma (Remark 1.11, Nonlinear Algebra by Michalek and Sturmfels).

Example 1.10. lex, grlex and grevlex are monomial orderings.

grevlex is defined as $\alpha \preceq \beta$ if either $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and the rightmost nonzero entry of $\beta - \alpha$ is negative.

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Definition 1.11 (Leading term, leading monomial). Let $f = \sum \lambda_\alpha x^\alpha \in k[x_1, \dots, x_n]$. We define the **leading term** $\text{LT}_\prec(f)$ as $\lambda_\alpha x^\alpha$ where x^α is the largest monomial in f with respect to \prec .

We call x^α the **leading monomial**, in symbol $\text{in}_\prec(f) = \text{LM}_\prec(f) = x^\alpha$.

Definition 1.12 (Multidegree). Fix a monomial order \prec in $k[x_1, \dots, x_n]$. Let $f = \sum \lambda_\alpha x^\alpha$. The **multidegree** of f is $\text{multideg}(f) = \max \{\alpha : \lambda_\alpha \neq 0\}$ with respect to \prec .

Remark 1.13 (Infinitely many distinct monomial orders exist). Let \prec be a monomial order and $w \in \mathbb{R}_{\geq 0}^n$. We define a monomial order $\alpha \prec_w \beta$ by

1. $w^T \alpha < w^T \beta$, or
2. $w^T \alpha = w^T \beta$ and $\alpha \prec \beta$.

For $n > 1$ this yields infinitely many monomial orders in \mathbb{N}^n by varying over w . To see this, we consider two monomial orders in \mathbb{N}^2 given by $w = (1, t)$ and $w' = (1, t')$. If we can show that for every $t \neq t' \in \mathbb{R}$ we have $\prec_w \neq \prec_{w'}$, then we have infinitely many distinct monomial orders.

Assume $t \neq t'$. Then there exists $t < \frac{r}{s} < t'$. Let $\alpha = (0, s)$ and $\beta = (r, 0)$. Then

$$w^T \alpha = ts < r = w^T \beta \quad \text{and} \quad w'^T \alpha = t's > r = w'^T \beta.$$

This shows that $\prec_w \neq \prec_{w'}$.

1.2 Polynomial division

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Remark 1.14. The polynomial ring $k[x]$ is a Euclidean domain. This means that for any polynomials f and g there exist q, r such that $f = qg + r$ and $r = 0$ or $\deg(r) < \deg(g)$.

Remark 1.15 (Euclidean domains are principal). A Euclidean domain is a principal ideal domain because every ideal in a Euclidean domain is generated by an element of minimal norm.

Remark 1.16. $k[x_1, \dots, x_n]$ is not a Euclidean domain because the ideal (x_1, x_2) is not principal.

Nevertheless, we want to do division with remainder in $k[x_1, \dots, x_n]$.

Proposition 1.17 (Multivariate division algorithm). Fix a monomial order \prec in $k[x_1, \dots, x_n]$. Let $F = (f_1, \dots, f_s) \subset k[x_1, \dots, x_n]$ be an s -tuple. For every $f \in k[x_1, \dots, x_n]$ there exist $q_i, r \in k[x_1, \dots, x_n]$ such that $f = q_1 f_1 + \dots + q_s f_s + r$ and either

- $r = 0$, or
- r is a k -linear combination of monomials, none of which is divisible by $\text{LT}(f_i)$.

More if $q_i f_i \neq 0$, then $\text{multideg}(f) \geq \text{multideg}(q_i f_i)$.

Multivariate division algorithm

Input: $F = (f_1, \dots, f_s)$ and f

Output: q_1, \dots, q_s and r

```

1:  $q_i \leftarrow 0, r \leftarrow 0$  and  $p \leftarrow f$ 
2: while  $p \neq 0$  do
3:    $i \leftarrow 0$ 
4:    $\text{div} \leftarrow \text{false}$ 
5:   while  $i \leq s$  and  $\text{div} = \text{false}$  do
6:     if  $\text{LT}(f_i)$  divides  $\text{LT}(p)$  then
7:        $q_i \leftarrow q_i + \frac{\text{LT}(p)}{\text{LT}(f_i)}$  and  $p \leftarrow p - \frac{\text{LT}(p)}{\text{LT}(f_i)} f_i$ 
8:        $\text{div} \leftarrow \text{true}$ 
9:     else
10:       $i \leftarrow i + 1$ 
11:    end if
12:  end while
13:  if  $\text{div} = \text{false}$  then
14:     $r \leftarrow r + \text{LT}(p)$  and  $p \leftarrow p - \text{LT}(p)$ 
15:  end if
16: end while
17: return  $q_1, \dots, q_s, r$ 

```

Proof. We prove the correctness of the algorithm and that the algorithm terminates.

1. We prove the invariant $f = \sum q_i f_i + p + r$
2. The algorithm stops when $p = 0$
3. The algorithm stops after finitely many steps (use that \prec is a well-ordering)

□

Definition 1.18 (Standard residue). We call r the **standard residue** \bar{f}^F .

Example 1.19 (Non-uniqueness of the residue when changing order). Divide $f = x^2y + xy^2 + y^2$ by $(xy - 1, y^2 - 1)$ and $(y^2 - 1, xy - 1)$.

- $x^2y + xy^2 + y^2 = (x + y)(xy - 1) + (y^2 - 1) + (x + y + 1)$
- $x^2y + xy^2 + y^2 = x(xy - 1) + (x + 1)(y^2 - 1) + (2x + 1)$

Remark 1.20 (Ideal membership problem). Given $f \in k[x_1, \dots, x_n]$, we want to find out if f is contained in an ideal $I \subset k[x_1, \dots, x_n]$. For $k[x]$ this is easy since every ideal is principal; we apply the division algorithm and check if $r = 0$.

Remark 1.21 (Vanishing residue is sufficient but not necessary). However, for $k[x_1, \dots, x_n]$ we have the following example: let $f = xy^2 - x$ and $f_1 = xy + 1$, $f_2 = y^2 - 1$. Then we compute f divided by (f_1, f_2) and (f_2, f_1) :

$$\begin{aligned} xy^2 - x &= y(xy + 1) + 0(y^2 - 1) + (-x - y) \\ &= 0(xy + 1) + x(y^2 - 1) + 0 \end{aligned}$$

So, $r = 0$ is sufficient but $r \neq 0$ does not imply $f \notin I$.

Here is another example.

Example 1.22 (Counter example: nonzero residue is not conclusive). Consider $f = x^2 + \frac{1}{2}y^2z - z - 1$ and $f_1 = x^2 + z^2 - 1$ and $f_2 = x^2 + y^2 + (z - 1)^2 - 4$. Computation yields that the standard residue is nonzero for (f_1, f_2) and (f_2, f_1) . Nevertheless, f is generated by f_1, f_2 .

1.3 Gröbner basis

Definition 1.23 (Monomial ideal). A **monomial ideal** I is an ideal generated by monomials x^α , i.e. $I = (x^\alpha : \alpha \in A)$ for some subset $A \subset \mathbb{N}^n$.

Lec04,
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Definition 1.24 (Initial ideal). Let I be a subset of $k[x_1, \dots, x_n]$. We define the **initial ideal** of I to be the ideal generated by leading terms of I , i.e.

$$\text{in}_<(I) = (\text{LT}_<(f) : f \in I \setminus \{0\}).$$

Clearly, an initial ideal is a monomial ideal.

Definition 1.25 (Gröbner basis). Let $I \subset k[x_1, \dots, x_n]$ be an ideal and fix a monomial order $<$. Let $G \subset I$ be a *finite subset* of I . We call G a **Gröbner basis**

if

$$\text{in}_{\prec}(I) = (\text{LT}_{\prec}(g) : g \in G).$$

Definition 1.26 (Minimal gröbner basis). A Gröbner basis G is called a *minimal Gröbner basis* if

1. every $p \in G$ has *leading coefficient* equal to 1
2. every p is *non-redundant*, i.e. if there exists some $p \in G$ such that the leading term of p can be obtained by $(\text{LT}(G \setminus \{p\}))$, we will remove p from G .

Definition 1.27 (Reduced gröbner basis). A minimal Gröbner basis G is called **reduced** if for all polynomials $p = \sum x^\alpha \in G$ no monomial x^α lies in $(\text{LT}(G \setminus \{p\}))$.

We will later see that a (reduced) Gröbner basis always exists.

Lemma 1.28 (We may remove redundant polynomials). Let G be a Gröbner basis for I . Removing a redundant $p \in G$ from G yields again a Gröbner basis for I .

Proof. We have $\text{LT}(G) = \text{LT}(G \setminus \{p\})$ because $\text{LT}(p) \in \text{LT}(G \setminus \{p\})$. Thus, $\text{LT}(I) = \text{LT}(G) = \text{LT}(G \setminus \{p\})$. \square

Lemma 1.29 (Minimal Gröbner basis respects fully reduced polynomials). Let \mathcal{G} be a minimal Gröbner basis of some ideal I . Assume that $p \in \mathcal{G}$ is fully reduced (that is if no monomial of p lies in $\text{LT}(G \setminus \{p\})$). If \mathcal{G}' is another minimal Gröbner basis containing p , then p stays fully reduced in \mathcal{G}' .

Proof. This follows from \mathcal{G} and \mathcal{G}' having the same leading coefficients. We know that no monomial of p lies in $\text{LT}(\mathcal{G} \setminus \{p\}) = \text{LT}(\mathcal{G}' \setminus \{p\})$. \square

Lemma 1.30 (Membership in monomial ideal). Let $I = (x^\alpha : \alpha \in A)$ be a monomial ideal. Then $x^\beta \in I$ if and only if x^α divides x^β for some $\alpha \in A$ if and only if $\alpha \leq \beta$.

Proof. Let $x^\beta = \sum f_\alpha x^\alpha$. Expanding the terms on the right-hand yields that $\alpha \leq \beta$ for some α . \square

Example 1.31.

- $I = (xy+1, y^2-1)$ and $G = \{xy+1, y^2-1\}$ is not a Gröbner basis since it cannot generate $x+y$. $G' = \{x+y, y^2-1\}$ is a *universal Gröbner basis*, i.e. it remains a Gröbner basis even if we change the monomial order.
- Let $I = (f)$ be principal. Then $G = (f)$ is a Gröbner basis because $\text{LT}(hf) = \text{LT}(h)\text{LT}(f)$ for any $h \in k[x_1, \dots, x_n]$.
- $I = 0$ has Gröbner basis \emptyset .

Proposition 1.32 (Gröbner basis is a basis). $I = (G)$ for any Gröbner basis G of an ideal $I \subset k[x_1, \dots, x_n]$.

Proof. Use the division algorithm and show that the residue must vanish. \square

We just proved that if f is in I , then $\bar{f}^G = 0$. This comes handy for the ideal membership problem.

Proposition 1.33 (Existence and uniqueness of the residue). Let G be a Gröbner basis of some ideal I . Then there exists a unique $r \in k[x_1, \dots, x_n]$ such that

- no term of r is divisible by any of $\text{LT}(g_1), \dots, \text{LT}(g_s)$, and
- $f = g + r$ for some $g \in I$

In particular, $r = \bar{f}^G$ no matter how the polynomials in G are listed.

Proof.

- Use the division algorithm to construct r and f .
- Assume we have $f = g + r = g' + r'$. Subtract and we see that $r - r'$ must be in I . So $r - r'$ must be zero.

\square

Remark 1.34. We earlier saw an example that it matters if we divide f by (f_1, f_2) or (f_2, f_1) because the residue changes. For a Gröbner basis the residue stays the same no matter how we list G .

Definition 1.35 (Normal form). Let G be a Gröbner basis. We call the residue $r = \bar{f}^G$ the **normal form** of f with respect to G .

Corollary 1.36 (Ideal membership problem). $f \in I$ if and only if $\bar{f}^G = 0$ if and only if the normal form of f with respect to G is zero.

Proof. Write $f = f + 0$. This is unique. \square

Proposition 1.37 (Existence of Gröbner bases (nonconstructive)). Let I be an ideal. Then there exists $G \subset I$ such that G is a Gröbner basis of I .

Proof. We have $\text{in}(I) = (\{\text{LT}(f) : f \in I\}) = (\{x^\alpha : \alpha \in A\})$. The latter is a monomial ideal. Use Hilbert's Basis Theorem to obtain a generating set $(x^{\beta_i})_{i=1, \dots, l}$. This generating set need not lie in I . However, each x^{β_i} must be divisible by some x^α because $(\{x^\alpha : \alpha \in A\}) = (x^{\beta_i})_{i=1, \dots, l}$. These x^α 's form our generating set. \square

Later, we will give a constructive proof, see Buchberger's algorithm 1.45.

Proposition 1.38 (Existence of reduced Gröbner basis). Let $I \neq 0$ be an ideal. Then I has a reduced Gröbner basis.

Proof. A minimal Gröbner basis of I exists. For any $g \in \mathcal{G}$, let $g' = \bar{g}^{G \setminus \{g\}}$ and set $G' = (G \setminus \{g\}) \cup \{g'\}$.

We see that $\text{LT}(G') = \text{LT}(G)$ and $G' \subset I$. So G' is Gröbner basis and also minimal (easy to check). By construction g' is fully reduced for G' by construction.

By repeating this process we obtain a reduced Gröbner basis. Note that the Gröbner basis changes with each iteration but elements that were fully reduced stay fully reduced. \square

Lemma 1.39. Let G, G' be minimal Gröbner basis for I . Then

$$\{\text{LT}(g) : g \in G\} = \{\text{LT}(g) : g \in G'\}.$$

Proof. Assume $x^\gamma \in \text{LT}(G)$. Hence, $x^\gamma \in \text{LT}(I)$. So x^γ is generated by a leading term of a polynomial in G' , say $x^\gamma = x^m x^\alpha$ for some $x^\alpha \in \text{LT}(G')$.

On the other hand, $x^\alpha = x^n x^\beta$ for some $x^\beta \in \text{LT}(G)$. Thus,

$$x^\gamma = x^m x^n x^\beta$$

which is only possible if $\beta = \gamma$ because G is minimal. Hence, $x^\alpha = x^\gamma$ which shows $x^\gamma \in \text{LT}(G')$. \square

Proposition 1.40 (Uniqueness of reduced Gröbner basis). The reduced Gröbner basis is unique.

Proof. Assume we have two reduced Gröbner bases G and G' for I . Since both are minimal, we apply the previous lemma to deduce $\text{LT}(G) = \text{LT}(G')$. Given $g \in G$, there is some $g' \in G'$ with the same leading term as g .

Now we show that $g = g'$. Consider $g - g' \in I$; hence $0 = \overline{g - g'}^G$. Note that $\text{LT}(g) = \text{LT}(g')$ gets cancelled in $g - g'$. So $g - g'$ consists of monomials none of which is divisible by $\text{LT}(G) = \text{LT}(G')$ since G and G' are reduced. Thus, $0 = \overline{g - g'}^G = g - g'$. \square

Remark 1.41 (Ideal equality problem). Question: Do $\{f_1, \dots, f_s\}$ and $\{g_1, \dots, g_t\}$ generate the same ideal?

Compute the unique reduced Gröbner basis and check if they are the same.

1.4 Buchberger's algorithm

Definition 1.42 (S -polynomial). Let $f, g \in k[x_1, \dots, x_n]$ be nonzero. We define the S -polynomial to be

$$S(f, g) = \frac{x^\gamma}{\text{LT}(f)} f - \frac{x^\gamma}{\text{LT}(g)} g$$

where x^γ is the least common multiple of the leading term of f and g .

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Cancellation of leading terms is a linear combination of S -polynomials.

Lemma 1.43 (Cancellation and S -polynomials). Let f_1, \dots, f_s be polynomials with the same multi-degree δ . If $g = \sum f_i$ has multi-degree less than δ , then g is a k -linear combination of S -polynomials $S(f_i, f_j)$.

Proof. Let λ_i be the leading coefficient of f_i . Since the leading term of g cancels, we easily see that $\sum_{i=1}^s \lambda_i = 0$. In particular, $-\lambda_s = \sum_{i=1}^{s-1} \lambda_i$.

Consider

$$\sum_{i=1}^{s-1} \lambda_i S(f_i, f_s) = \sum_{i=1}^{s-1} \lambda_i \left(\frac{1}{\lambda_i} f_i - \frac{1}{\lambda_s} f_s \right) = \sum_{i=1}^{s-1} f_i - \frac{\lambda_i}{\lambda_s} f_s = \sum_{i=1}^{s-1} f_i + \frac{\lambda_s}{\lambda_s} f_s = \sum_{i=1}^s f_i$$

□

With the following criterion we can check if a generating set is a Gröbner basis.

Proposition 1.44 (Buchberger's Criterion). Let $G = \{g_1, \dots, g_s\}$ be a generating set of I . G is a Gröbner basis of I if and only if $\overline{S(g_i, g_j)}^G = 0$ holds for all $i \neq j$.

Proof.

\Rightarrow : Note that $S(g_i, g_j) \in I$ and hence the residue vanishes.

\Leftarrow : Let $f \in I$. We want to show that $\text{LT}(f)$ is generated by $\text{LT}(g_1), \dots, \text{LT}(g_s)$.

Pick a representation $f = \sum h_i g_i$ such that $\delta = \max \{\text{multideg}(h_i g_i) : i = 1, \dots, s\}$ is minimal with respect to \prec ; we can do this because \prec is a well-order. It follows $\delta \geq \text{multideg}(f)$.

If $\delta = \text{multideg}(f)$, then $\text{LM}(f) = \text{LM}(h_i g_i)$. Hence, the leading term of f is generated by g_i .

Assume $\delta > \text{multideg}(f)$. We will see that this will lead to a contradiction of the minimality of δ . Write

$$f = \sum_{\text{multideg}(h_i g_i) = \delta} \text{LT}(h_i) g_i + \sum_{\text{multideg}(h_i g_i) < \delta} (h_i - \text{LT}(h_i)) g_i + \sum_{\text{multideg}(h_i g_i) < \delta} h_i g_i.$$

Since the second and third sum as well as f have multi-degree $< \delta$, the first sum is of multi-degree $< \delta$. Applying Lemma 1.43 on the first sum yields

$$\sum_{\text{multideg}(h_i g_i) = \delta} \text{LT}(h_i) g_i = \sum_{i,j} \tilde{\lambda}_{ij} S(\text{LT}(h_i) g_i, \text{LT}(h_j) g_j)$$

Note that

$$S(\text{LM}(h_i) g_i, \text{LM}(h_j) g_j) = \text{LM}(h_i) g_i - \text{LM}(h_j) g_j = \frac{x^\delta}{x^\gamma} S(g_i, g_j)$$

where x^γ is the least common multiple of the leading monomials of g_i and g_j . The last equality follows because $\text{LM}(h_i)\text{LM}(g_i) = \text{LM}(h_j)\text{LM}(g_j) = x^\delta$.

Therefore, the first sum is a k -linear combination of polynomials $x^{\delta-\gamma_{ij}}S(g_i, g_j)$, i.e.

$$\begin{aligned} \sum_{\text{multideg}(h_i g_i) = \delta} \text{LT}(h_i) g_i &= \sum_{i,j} \lambda_{ij} x^{\delta-\gamma_{ij}} S(g_i, g_j) = \sum_{i,j} \lambda_{ij} x^{\delta-\gamma_{ij}} \left(\sum_k q_k^{(ij)} g_k \right) \\ &= \sum_{i,j} \sum_k \underbrace{\lambda_{ij} x^{\delta-\gamma_{ij}} q_k^{(ij)} g_k}_{\text{multidegree} < \delta} \end{aligned}$$

where the second last equality follows because $S(g_i, g_j) \in (g_1, \dots, g_s)$ by assumption $\overline{S(g_i, g_j)}^G = 0$. To see why the multi-degree of the last expression is less than δ , recall that we obtained the q_k 's by applying the Multivariate division algorithm and hence $\text{multidegree}(q_k g_k) \leq \text{multidegree}(S(g_i, g_j))$. Then

$$\text{multidegree}(x^{\delta-\gamma_{ij}} q_k^{(ij)} g_k) \leq \text{multidegree}(x^{\delta-\gamma_{ij}} S(g_i, g_j)) < \delta.$$

In conclusion, we found a representation of f that has multi-degree $< \delta$ but δ was supposed to be minimal. Contradiction. \square

Proposition 1.45 (Buchberger's algorithm). Let $I = (f_1, \dots, f_s) \neq 0$. A Gröbner basis for I can be constructed in a finite number of steps by the following algorithm:

Buchberger's algorithm

Input: $F = (f_1, \dots, f_s)$

Output: Gröbner basis $G \supset F$

```

1:  $G \leftarrow F$ 
2: run  $\leftarrow$  true
3: while run = true do
4:    $G' \leftarrow G$ 
5:   for  $p, q \in G', p \neq q$  do
6:      $r \leftarrow \overline{S(p, q)}^{G'}$ 
7:     if  $r \neq 0$  then
8:        $G \leftarrow G \cup \{r\}$ 
9:     end if
10:  end for
11:  if  $G = G'$  then
12:    run  $\leftarrow$  false
13:  end if
14: end while
15: return  $G$ 
```

Proof. • Check: $G \subset I$

- Check: G is a generating set of I
- Check: G is a Gröbner basis
- Check that the algorithm terminates: Note that if $G' \subsetneq G$, then $\text{in}(G') \subsetneq \text{in}(G)$ by Corollary 1.55; this forms a strict ascending chain of ideals in $k[x_1, \dots, x_n]$. By the ACC, this chain must terminate. So, $\text{in}(G') = \text{indeed}(G)$ implies $G = G'$ by Corollary 1.55.

□

Example 1.46 (Gauss algorithm). Given an ideal generated by m linear forms

$$I = \left(\sum_{j=0}^n a_{ij}x_j : i = 1, \dots, m \right)$$

one can form the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

After applying the Gauss algorithm one obtains a matrix in row echelon form. The rows yield a *minimal Gröbner basis*. If the matrix is even in *reduced* row echelon form, we get a *reduced Gröbner basis*.

For example the system of linear equations

$$\begin{cases} 3x - 6y - 2z = 0 \\ 2x - 4y + 4w = 0 \\ x - 2y - z - w = 0 \end{cases}$$

yields the row echelon form

$$\begin{pmatrix} 1 & -2 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This admits the minimal Gröbner basis

$$G = \{x - 2y - z - w, z + 3w\}$$

for the ideal $I = (3x - 6y - 2z, 2x - 4y + 4w, x - 2y - z - w)$.

1.5 Macaulay's theorem

Definition 1.47 (Standard monomial). Let I be an ideal in $k[\mathbf{x}]$. Monomials x^α that are not contained in $\text{in}_\prec(I)$ are called **standard monomials**.

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Example 1.48. Consider the ideal $I = (xy^3 - x^2, x^3y^2 - y)$ in $k[x, y]$. Computing the standard monomials of I with respect to lex, grlex and grevlex yields a set of exactly 12 elements. It looks like that the number of standard monomials remains the same irrespective of the monomial ordering.

Proposition 1.49 (Every polynomial is represented mod I by standard monomials). Let I be an ideal in $k[\mathbf{x}]$. For any $f \in k[\mathbf{x}]$ there exists a unique $r \in k[\mathbf{x}]$ such that

$$f \equiv r \pmod{I}$$

and r is a k -linear combination of standard monomials.

Proof. Let $\mathcal{G} = \{g_1, \dots, g_s\}$ be the reduced Gröbner basis of I . Let $f \in k[\mathbf{x}]/I$. Compute

$$r := \bar{f}^{\mathcal{G}}.$$

So $f = g + r$ for some $g \in I$, and this r is unique. This shows $f \equiv r \pmod{I}$.

Note that no term of r is divisible by $\text{LT}(g_i)$ (by definition of the division algorithm), hence r is a k -linear combination of standard monomials. \square

Proposition 1.50 (Linear independence of standard monomials mod I). Standard monomials are linearly independent in $k[\mathbf{x}]/I$. That is, for all standard monomials x^α we have

$$\sum \lambda_\alpha x^\alpha \equiv 0 \pmod{I} \implies \lambda_\alpha = 0.$$

Proof. Assume $\sum_\alpha c_\alpha x^\alpha \equiv 0 \pmod{I}$. Then $f = \sum_\alpha c_\alpha x^\alpha \in I$. If we write $f = g + r$, then $g = 0$ and $r = \sum_\alpha c_\alpha x^\alpha$. Since r is unique, we must have $r = 0$. So $0 = \sum_\alpha c_\alpha x^\alpha$; this can only happen if $c_\alpha = 0$ for all α . \square

Proposition 1.51.

1. $\overline{f + g}^G = \bar{f}^G + \bar{g}^G$
2. $\overline{\lambda \cdot f}^G = \lambda \cdot \bar{f}^G$
3. $\overline{fg}^G = \bar{f}^G \cdot \bar{g}^G$

Proof. ... \square

Theorem 1.52 (Macaulay's Theorem). Let \prec be a monomial order. Let I be an ideal in $k[x_1, \dots, x_n]$. Then

$$k[x_1, \dots, x_n]/I \cong \text{span} \{x^\alpha : x^\alpha \text{ is a standard monomial with respect to } \prec\}$$

as a k -vector space.

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Proof.

- By Proposition 1.49, the map $f \mapsto \overline{f}^G$ is well-defined.
- It is injective because if f, f' have the same residue $r \bmod I$, then their difference $f - f' = g - g' \in I$. Hence, $f \equiv f' \bmod I$.
- It is clearly surjective.
- It is left to show that this map is a k -module homomorphism.
 - It respects addition: follows directly from $\overline{f+g}^G = \overline{f}^G + \overline{g}^G$.
 - It respects scalar multiplication: follows directly from $\overline{\lambda \cdot f}^G = \lambda \cdot \overline{f}^G$.

□

Example 1.53. Let $I = (x^2 + y^2 + x^2y, x^2 + xy + x^2y)$. A reduced Gröbner basis is given by $G = (x^3 + x^2 + y^2, xy - y^2, y^3 + x^2 + y^2)$. Thus, $\text{in}_{\text{grlex}} = (x^3, xy, y^3)$. We have $k[x, y]/I \cong \text{span}(1, x, y, x^2, y^2)$.

Corollary 1.54 (No proper inclusions for distinct monomial orders). Let I be an ideal in $k[x_1, \dots, x_n]$. Let \prec, \prec' be monomial orders. Then

$$\text{in}_{\prec}(I) \subset \text{in}_{\prec'}(I) \implies \text{in}_{\prec}(I) = \text{in}_{\prec'}(I).$$

Proof. We need to show $\text{in}_{\prec}(I) \supset \text{in}_{\prec'}(I)$. Assume $x^\gamma \in \text{in}_{\prec'}(I)$. Then $x^\gamma \equiv r \bmod I$ for some k -linear combination of standard monomials r with respect to \prec' , that is, there exists some $f \in I \setminus \{0\}$ (f must be nonzero, otherwise $x^\gamma = r$ which is a contradiction) such that

$$x^\gamma = f + r.$$

This shows $f = x^\gamma - r$. Then $\text{LT}_{\prec}(f) = \text{LT}_{\prec}(x^\gamma - r) \in \text{in}_{\prec}(I) \subset \text{in}_{\prec'}(I)$. Since r is not contained in $\text{in}_{\prec'}(I)$, we have $\text{LT}_{\prec}(f) = x^\gamma \in \text{in}_{\prec}(I)$. □

Corollary 1.55 (Proper inclusion are preserved under in_{\prec}). Let I be *strictly* contained in J . Then $\text{in}_{\prec}(I)$ is *strictly* contained in $\text{in}_{\prec}(J)$ for any monomial order \prec .

Proof. It is easy to see that $I \subset J$ implies $\text{in}_{\prec}(I) \subset \text{in}_{\prec}(J)$.

Let $f \in J \setminus I$. So the residue is nonzero and therefore $f \equiv r \bmod I$ where r is a nontrivial k -linear combination of standard monomials in I . Thus,

$$f = g + r, \quad g \in I \subset J.$$

Then $r = f - g \in J$. So $\text{in}_{\prec}(r) \in \text{in}_{\prec}(J)$ but $\text{in}_{\prec}(r) \notin \text{in}_{\prec}(I)$. □

Proposition 1.56 (Every ideal has only finitely many distinct initial ideals). For every ideal $I \subset k[x_1, \dots, x_n]$ it holds

$$|\{\text{in}_{\prec}(I) : \prec \text{ is a monomial order}\}| < \infty.$$

Proof. Let $I \neq 0$. Assume $\Sigma_0 = \{\text{in}_{\prec}(I) : \prec \text{ is a monomial order}\}$ is infinite. Choose some nonzero $f_1 = \sum \lambda_{\alpha} x^{\alpha} \in I$. Each ideal in Σ_0 contains some monomial of f_1 . Since Σ_0 is infinite, there exists a monomial x^{α_1} of f_1 such that $\Sigma_1 = \{J \in \Sigma_0 : x^{\alpha_1} \in J\}$ is infinite.

Note that $(x^{\alpha_1}) \subsetneq \text{in}_{\prec}(I)$ for some \prec since Σ_1 is infinite. Hence, the set of monomials outside x^{α_1} consists of standard monomials plus some monomials in $\text{in}_{\prec}(f)$. Hence, they are linearly dependent modulo I , which means that there is some $f_2 \in I$ whose monomials do not lie in (x^{α_1}) . Again, there is some monomial x^{α_2} of f_2 such that $\Sigma_2 = \{J \in \Sigma_1 : x^{\alpha_2} \in J\}$ is infinite.

Note that $(x^{\alpha_i}) \subsetneq (x^{\alpha_i} : i = 1, 2) \subsetneq \text{in}_{\prec}(I)$ for some \prec since Σ_2 is infinite. Thus, there is some $f_3 \in I$ whose monomials do not lie in $(x^{\alpha_i} : i = 1, 2)$. Choose a monomial x^{α_3} of f_3 ...

Repeating this process yields an infinite strictly increasing chain in $k[x]$

$$(x^{\alpha_1}) \subsetneq (x^{\alpha_1}, x^{\alpha_2}) \subsetneq (x^{\alpha_1}, x^{\alpha_2}, x^{\alpha_3}) \subsetneq \dots$$

This is not possible since $k[x]$ is Noetherian. \square

Definition 1.57 (Universal Gröbner basis). A finite subset $G \subset I$ is called a **universal Gröbner basis** of I if I is a Gröbner basis of I with respect to all monomial orders simultaneously.

Lemma 1.58. Let I be an ideal in $k[x_1, \dots, x_n]$. If $\text{in}_{\prec}(I) = \text{in}_{\prec'}(I)$ for two monomial orders \prec, \prec' , then the reduced Gröbner basis with respect to \prec and \prec' are identical.

Proof. Assume $\text{in}_{\prec}(I) = \text{in}_{\prec'}(I) = (x^{\alpha_1}, \dots, x^{\alpha_s})$ where each generator is minimal. Write $x^{\alpha_i} = g_i + r_i$ for $g_i \in I$ and terms of r_i not divisible by $x^{\alpha_1}, \dots, x^{\alpha_s}$. This is a unique representation: assume $x^{\alpha_i} = g_i + r_i = g'_i + r'_i$. Then $g_i - g'_i = r'_i - r_i \in I$. The leading term of $g_i - g'_i = r'_i - r_i$ is not divisible by $x^{\alpha_1}, \dots, x^{\alpha_s}$ which cannot happen since $g_i - g'_i \in I$. So $g_i - g'_i = r'_i - r_i = 0$.

Hence, we can take $G = \{g_1, \dots, g_s\}$ for both \prec and \prec' since the reduced Gröbner basis is unique. \square

Proposition 1.59 (Existence of a universal Gröbner basis). Every ideal in $k[\mathbf{x}]$ possesses a universal Gröbner basis.

Proof. Only finitely many distinct initial ideals exist. Hence, only finitely many distinct reduced Gröbner bases exist. \square

Example 1.60 (2×2 minors of a $2 \times m$ matrix). Given a polynomial ring k in $2m$ indeterminates

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \end{bmatrix}$$

we define $I = (D_{ij} : 1 \leq i < j \leq m)$ where $D_{ij} = x_{1i}x_{2j} - x_{1j}x_{2i}$. We claim that $G = \{D_{ij} : 1 \leq i < j \leq m\}$ is a universal Gröbner basis of I , that is

$$(\{\text{LT}_{\prec}(D_{ij}) : 1 \leq i < j \leq m\}) = \text{in}_{\prec}(I)$$

for all monomial orders \prec .

For $m = 3$ we have

$$D_{12} = x_{11}x_{22} - x_{12}x_{21},$$

$$D_{13} = x_{11}x_{23} - x_{13}x_{21},$$

$$D_{23} = x_{12}x_{23} - x_{13}x_{22}.$$

There exist 8 monomial orders such that the corresponding initial ideals $\text{in}_{\prec}(G)$ are distinct; here they are

1. $x_{11}x_{22} \succ x_{12}x_{21}, x_{11}x_{23} \succ x_{13}x_{21}, x_{12}x_{23} \succ x_{13}x_{22},$
2. $x_{11}x_{22} \succ x_{12}x_{21}, x_{11}x_{23} \succ x_{13}x_{21}, x_{12}x_{23} \prec x_{13}x_{22},$
3. $x_{11}x_{22} \succ x_{12}x_{21}, x_{11}x_{23} \prec x_{13}x_{21}, x_{12}x_{23} \succ x_{13}x_{22},$
4. $x_{11}x_{22} \succ x_{12}x_{21}, x_{11}x_{23} \prec x_{13}x_{21}, x_{12}x_{23} \prec x_{13}x_{22},$
5. $x_{11}x_{22} \prec x_{12}x_{21}, x_{11}x_{23} \succ x_{13}x_{21}, x_{12}x_{23} \succ x_{13}x_{22},$
6. $x_{11}x_{22} \prec x_{12}x_{21}, x_{11}x_{23} \succ x_{13}x_{21}, x_{12}x_{23} \prec x_{13}x_{22},$
7. $x_{11}x_{22} \prec x_{12}x_{21}, x_{11}x_{23} \prec x_{13}x_{21}, x_{12}x_{23} \succ x_{13}x_{22},$
8. $x_{11}x_{22} \prec x_{12}x_{21}, x_{11}x_{23} \prec x_{13}x_{21}, x_{12}x_{23} \prec x_{13}x_{22}.$

Note that case 3 cannot happen because multiplying the left-hand sides and right-hand sides of three given inequalities yields

$$(x_{11}x_{22}) \cdot (x_{13}x_{21}) \cdot (x_{12}x_{23}) \succ (x_{12}x_{21}) \cdot (x_{11}x_{23}) \cdot (x_{13}x_{22}),$$

but both sides are equal (just compare the variables on both sides). Similarly, we see that case 6 is impossible.

For the other cases it suffices to consider case 1 because the other cases can be obtained by permuting the columns of the matrix $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$. We want to check that G is indeed a Gröbner basis of I . This is not difficult to see; we just apply the S -pair criterion.

2 Gröbner fan

2.1 Representation of monomial orders by weight vectors

Definition 2.1 (Initial form and ideal). Let $f = \sum_{i=1}^k c_i x^{\alpha_i} \in k[x_1, \dots, x_n]$ and $\omega \in \mathbb{R}^n$. The **initial form** $\text{in}_\omega(f)$ is defined to be

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$$\text{in}_\omega(f) := \sum_{i \in M} c_i x^{\alpha_i} \quad \text{where } M = \{j : j = \text{argmax}_{l=1, \dots, k} (\langle \alpha_l, \omega \rangle)\}.$$

The **initial ideal** with respect to a weight vector is defined to be the ideal generated by all initial forms:

$$\text{in}_\omega(I) := (\text{in}_\omega(f) : f \in I).$$

Remark 2.2. The initial form and ideal need not be monomial. However, if we choose ω sufficiently generic, then $\text{in}_\omega(I)$ is a monomial ideal. We will prove this in this chapter.

Definition 2.3 (Weighted term order). For any monomial order \prec and $\omega \geq 0$ we define a new order \prec_ω by

$$a \prec_\omega b \iff \langle w, a \rangle < \langle w, b \rangle \text{ or } (\langle w, a \rangle = \langle w, b \rangle \text{ and } a \prec b)$$

Remark 2.4. If $\omega \in \mathbb{R}_{<0}^n$, then \prec_ω cannot be a monomial order. We know that for any order \prec we have $x \prec x^2$ because x divides x^2 . However, $x^2 \prec_\omega x$.

Example 2.5 (Initial ideal of a principal ideal). Define a principal ideal $I \subset k[x, y]$ generated by

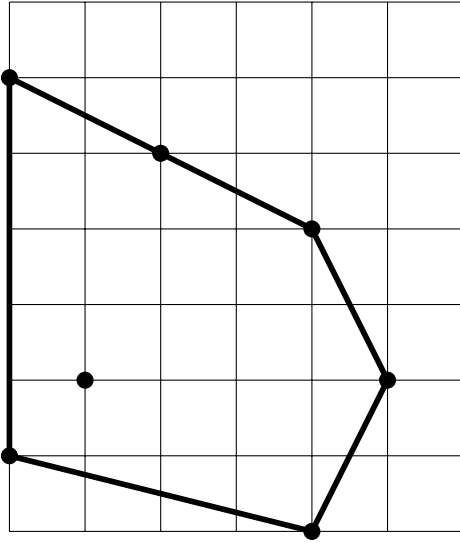
$$f(x, y) = x^5 y^2 + x^4 y^4 + x^4 + x^2 y^5 + x y^2 + y^6 + y.$$

Let $\omega = (1, 1)$. Then $\text{in}_\omega(I) = x^4 y^4$ is a *monomial ideal*.

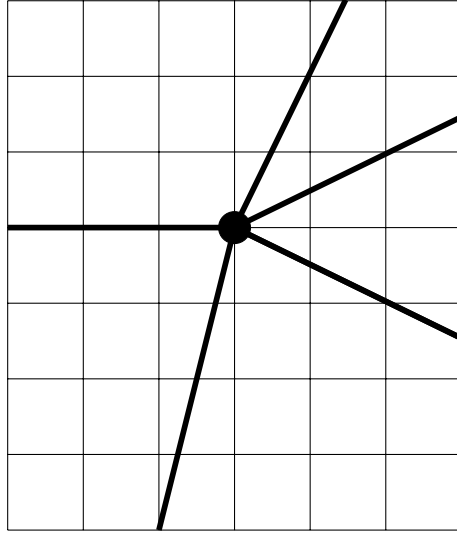
Let $\omega = (1, 2)$. Then $\text{in}_\omega(I) = (x^4 y^4 + x^2 y^5 + y^6)$ is *not* a monomial ideal.

Let $\omega = (0, 0)$. Then $\text{in}_\omega(I) = I$.

Below the convex hull of all monomials appearing in f are depicted on the left-hand side; this is also called the **Newton polytope** of f . Note that for any vertex v there exists a weight vector ω such that $\text{in}_\omega(I) = v$.



(a) Newton polytope of $f(x, y)$



(b) Normal fan of the Newton polytope

Proposition 2.6 (Useful formula). Let I be an ideal in $k[\mathbf{x}]$. Then,

$$\text{in}_{\prec_\omega}(I) = \text{in}_{\prec}(\text{in}_\omega(I)).$$

Proof. Reduce to the case $\text{in}_{\prec_\omega}(f) = \text{in}_{\prec}(\text{in}_\omega(f))$ for $f \in I$. Then, $\text{in}_{\prec_\omega}(I)$ and $\text{in}_{\prec}(\text{in}_\omega(I))$ contain the same monomials and hence are the same. \square

If the initial ideal $\text{in}_\omega(I)$ is a monomial ideal, say it is generated by monomials $\{f_1, \dots, f_s\}$, then we clearly found a Gröbner basis of $\text{in}_\omega(I)$ with respect to any monomial order \prec .

What if the initial ideal $\text{in}_\omega(I)$ is not a monomial ideal?

What is the Gröbner basis?

Assume $\text{in}_\omega(I) = (g_1, \dots, g_s)$ where g_1, \dots, g_s are not necessarily monomials. We want to find a Gröbner basis, i.e. find polynomials f_1, \dots, f_r such that

$$\text{in}_{\prec}(\text{in}_\omega(I)) = \text{in}_{\prec}(f_1, \dots, f_r).$$

Corollary 2.7 (Gröbner basis of initial ideals in_ω). Let $\omega \geq 0$. Let \mathcal{G} be a Gröbner basis of I with respect to \prec_ω . Then $\{\text{in}_\omega(g) : g \in \mathcal{G}\}$ is a Gröbner basis of $\text{in}_\omega(I)$ with respect to \prec , i.e.

$$\text{in}_{\prec_\omega}(I) = \text{in}_{\prec_\omega}(\mathcal{G}) \implies \text{in}_{\prec}(\text{in}_\omega(I)) = \text{in}_{\prec}(\text{in}_\omega(\mathcal{G})).$$

Corollary 2.8 (Initial ideal: weight = monomial order). Let $\omega \geq 0$. Then

$$\text{in}_\omega(I) \text{ is a monomial ideal} \implies \text{in}_\omega(I) = \text{in}_{\prec_\omega}(I).$$

Theorem 2.9 (Representation of initial ideals). For all ideals $I \subset k[\mathbf{x}]$ and monomial orders \prec there exists some $\omega \in \mathbb{N}^n$ such that

$$\text{in}_{\prec}(I) = \text{in}_{\omega}(I).$$

We say ω **represents** $\text{in}_{\prec}(I)$.

Proof. Let $\mathcal{G} = \{g_1, \dots, g_s\}$ be a reduced Gröbner basis of I with respect to \prec . We write for all $i = 1, \dots, s$:

$$g_i = c_{i0}\mathbf{x}^{\mathbf{a}_{i0}} + c_{i1}\mathbf{x}^{\mathbf{a}_{i1}} + \dots + c_{ij_i}\mathbf{x}^{\mathbf{a}_{ij_i}}, \quad \text{where } \mathbf{x}^{\mathbf{a}_{i0}} = \text{in}_{\prec}(g_i).$$

Next, we define

$$\begin{aligned} \mathcal{C} &= \{\omega \in \mathbb{R}_{\geq 0}^n : \text{in}_{\omega}(g_i) = \mathbf{x}^{\mathbf{a}_{i0}}, \forall i = 1, \dots, s\} \\ &= \{\omega \in \mathbb{R}_{\geq 0}^n : \langle \mathbf{a}_{i0} - \mathbf{a}_{ij}, \omega \rangle > 0, \forall i = 1, \dots, s; \forall j = 1, \dots, j_i\}. \end{aligned}$$

- Claim: any $\omega \in \mathcal{C}$ works. Let $\omega \in \mathcal{C}$. We know that

$$\text{in}_{\prec}(I) = \text{in}_{\prec}(g_i : i = 1, \dots, s) = (\mathbf{x}^{\mathbf{a}_{i0}} : i = 1, \dots, s) \subset \text{in}_{\omega}(I)$$

because $\text{in}_{\omega}(g_i) = \mathbf{x}^{\mathbf{a}_{i0}}$ and $g_i \in I$. Suppose for contradiction that $\text{in}_{\prec}(I) \subsetneq \text{in}_{\omega}(I)$. Applying in_{\prec} on both sides yields (see Corollary 1.55)

$$\text{in}_{\prec}(I) \subsetneq \text{in}_{\prec_{\omega}}(I)$$

which is not possible by Corollary 1.54.

- Claim: $\mathcal{C} \neq \emptyset$. Let's rewrite \mathcal{C} as a solution set of a linear system. Define the matrix \mathbf{A} with rows $(\mathbf{a}_{ij} - \mathbf{a}_{i0})$ for all i, j ; that is

$$\mathbf{A} := [-(\mathbf{a}_{ij} - \mathbf{a}_{i0})]_{i=1, \dots, s; j=1, \dots, j_i}.$$

Assume \mathcal{C} is empty. Then $\mathbf{A}\omega < \mathbf{0}$ has no solution for $\omega \geq \mathbf{0}$. Let $\mathbf{b} = [-1, \dots, -1]$. Then also $\mathbf{A}\omega \leq \mathbf{b}$ has no solution $\omega \geq \mathbf{0}$. By *Farkas Lemma*, there exists $\mathbf{c} \geq \mathbf{0}$ such that

$$\mathbf{c}^T \mathbf{A} \geq 0 \quad \text{and} \quad \mathbf{c}^T \mathbf{b} < 0.$$

Note that the vector \mathbf{c} has the form

$$\mathbf{c}^T = (\mathbf{c}_{0,0} \quad \dots \quad \mathbf{c}_{0,j_0} \quad \mathbf{c}_{1,0} \quad \dots \quad \mathbf{c}_{1,j_1} \quad \dots \quad \mathbf{c}_{s,0} \quad \dots \quad \mathbf{c}_{s,j_s});$$

thus the elements of \mathbf{c} are indexed by i, j . So the system $-\mathbf{A}^T \mathbf{c} \leq \mathbf{0}$ yields

$$\left[\begin{array}{c|c|c|c|c} | & | & | & | & | \\ \mathbf{a}_{10} - \mathbf{a}_{11} & \mathbf{a}_{10} - \mathbf{a}_{1j_1} & \dots & \mathbf{a}_{s0} - \mathbf{a}_{sj_s} & \\ | & | & | & | & | \end{array} \right] \mathbf{c} = \sum_{ij} \mathbf{c}_{ij} \left[\begin{array}{c|c} | & | \\ \mathbf{a}_{i0} - \mathbf{a}_{ij} & \\ | & | \end{array} \right] \leq \mathbf{0}.$$

Without loss of generality we can assume that $\mathbf{c}_{ij} \in \mathbb{N}$. Transposing again yields

$$\sum_{i,j} \mathbf{c}_{ij} (\mathbf{a}_{i0} - \mathbf{a}_{ij}) \leq 0.$$

This translates into

$$\prod_{i,j} (\mathbf{x}^{\mathbf{a}_{i0}})^{\mathbf{c}_{ij}} \preceq \prod_{i,j} (\mathbf{x}^{\mathbf{a}_{ij}})^{\mathbf{c}_{ij}}.$$

Note that not all \mathbf{c}_{ij} vanish, otherwise $\mathbf{c}^T \mathbf{b} = 0$. Thus, since we know that $\mathbf{x}^{\mathbf{a}_{i0}} \succ \mathbf{x}^{\mathbf{a}_{ij}}$ from $\text{in}_{\prec}(g_i) = \mathbf{x}^{\mathbf{a}_{i0}}$, we get

$$(\mathbf{x}^{\mathbf{a}_{i0}})^{\mathbf{c}_{ij}} \succ (\mathbf{x}^{\mathbf{a}_{ij}})^{\mathbf{c}_{ij}};$$

thus

$$\prod_{i,j} (\mathbf{x}^{\mathbf{a}_{i0}})^{\mathbf{c}_{ij}} \succ \prod_{i,j} (\mathbf{x}^{\mathbf{a}_{ij}})^{\mathbf{c}_{ij}}$$

This is a contradiction. So $\mathcal{C} \neq \emptyset$.

□

Fix an ideal $I \subset k[x_1, \dots, x_n]$. We define an **equivalence relation** on $\omega \in \mathbb{R}^n$ by $\omega \sim \omega'$ if $\text{in}_{\omega}(I) = \text{in}_{\omega'}(I)$. The **equivalence class** of ω is denoted

$$C_I[\omega] = \{\omega \in \mathbb{R}^n : \text{in}_{\omega}(I) = \text{in}_{\omega'}(I)\}.$$

We will see that $C_I[\omega]$ is in fact a polyhedral cone.

Definition 2.10 (Equivalent weight vector). We say two weight vectors $\omega, \omega' \in \mathbb{R}^n$ are **equivalent** (with respect to I) if $\omega \sim \omega'$.

Definition 2.11 (Gröbner region). The **Gröbner region** of an ideal I is defined as

$$\text{GR}(I) = \{\omega \in \mathbb{R}^n : \omega \sim \omega' \text{ for some } \omega' \in \mathbb{R}_{\geq 0}^n\}.$$

Remark 2.12. The Gröbner region always contains $\mathbb{R}_{\geq 0}^n$.

Remark 2.13 (Gröbner region = support of Gröbner fan). We will later see that closure of the Gröbner region $\overline{\text{GR}(I)}$ is the support of the Gröbner fan of I .

Example 2.14. Here is the Gröbner region of I . Note that any ω outside the Gröbner region cannot be monomial orders. For instance, if y were the leading term of I , then $y^6 \prec y$ but this is not possible because y divides y^6 .

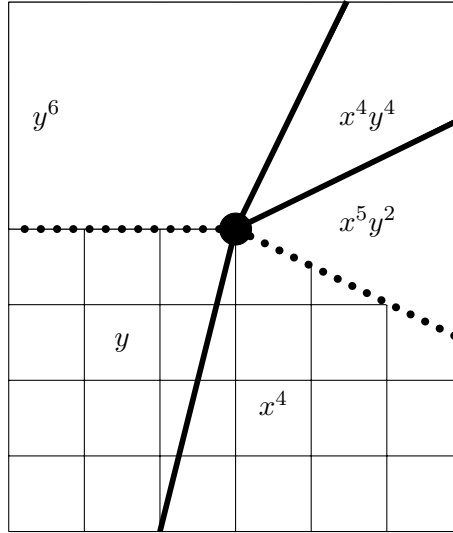


Figure 2: The white area is the Gröbner region of I .

2.2 Gröbner Fan

Our goal in this section is to prove the existence of the Gröbner fan.

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Recall that we defined two weight vectors to be equivalent with respect to I if $\text{in}_\omega(I) = \text{in}_{\omega'}(I)$. It turns out that if $\omega \in \mathbb{R}_{\geq 0}^n$ it suffices to check that they are equivalent with respect to the reduced Gröbner basis of I (for some monomial order \prec_ω).

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Remark 2.15. Observe that we have the following fact

$$\text{face}_\omega(\text{New}(f)) = \text{New}(\text{in}_\omega(f)).$$

Proof. Take a vertex $\alpha_j \in \text{face}_\omega(\text{New}(f))$. Then it is easy to see that $\alpha_j \in \text{New}(\text{in}_\omega(f))$. The converse direction is similar.

Proposition 2.16. $\text{New}(f \cdot g) = \text{New}(f) + \text{New}(g)$.

Proposition 2.17. Fix an ideal $I \subset k[x_1, \dots, x_n]$ and $\omega \in \mathbb{R}_{\geq 0}^n$. Let $\mathcal{G} = \{g_1, \dots, g_s\}$ be the reduced Gröbner basis of I with respect to \prec_ω for some monomial order \prec . Then, we have

$$C_I[\omega] = \{\omega' \in \mathbb{R}^n : \text{in}_{\omega'}(g_i) = \text{in}_\omega(g_i) \forall i = 1, \dots, s\}.$$

Proof.

- \supset : Let $\omega' \in \{\omega' \in \mathbb{R}^n : \text{in}_{\omega'}(g_i) = \text{in}_\omega(g_i) \forall i = 1, \dots, s\}$. Then

$$\text{in}_{\omega'}(I) \supset (\{\text{in}_{\omega'}(g_i) \forall i = 1, \dots, s\}) = (\{\text{in}_\omega(g_i) : \forall i = 1, \dots, s\}) = \text{in}_\omega(I)$$

where the last equality follows from Corollary 2.7. Assume for contradiction that the inclusion were strict, i.e. $\text{in}_{\omega'}(I) \supsetneq \text{in}_\omega(I)$. By Corollary 1.55, we have $\text{in}_{\prec_{\omega'}}(I) \supsetneq \text{in}_{\prec_\omega}(I)$. This contradicts Corollary 1.54. Hence, $\text{in}_{\omega'}(I) = \text{in}_\omega(I)$, and therefore $\omega' \in C_I[\omega]$.

- \subset : Let $\omega' \in C_I[\omega]$. By Corollary 2.7 we have $\{\text{in}_\omega(g_i) : \forall i = 1, \dots, s\}$ is a Gröbner basis of $\text{in}_\omega(I) = \text{in}_{\omega'}(I)$ with respect to \prec . We also have $\text{in}_{\omega'}(g_i) \in \text{in}_{\omega'}(I)$, and we just saw that $\text{in}_{\omega'}(I)$ has Gröbner basis $\{\text{in}_\omega(g_i) : \forall i = 1, \dots, s\}$. So,

$$\overline{\text{in}_{\omega'}(g_i)}^{\text{in}_\omega(\mathcal{G})} = 0.$$

Let $m = \text{in}_{\prec_\omega}(g_i)$. It is the only monomial of g_i divisible by any of $\text{in}_{\prec}(\text{in}_\omega(g_j))$, $j = 1, \dots, s$ since \mathcal{G} is reduced. Assume that m does not appear in $\text{in}_{\omega'}(g_i)$, so there is some other monomial of g_i whose residue vanishes with respect to $\text{in}_\omega(\mathcal{G})$. This means that this monomial is divisible by $\text{in}_{\prec}(\text{in}_\omega(g_j))$ which is not possible. Thus, m appears in $\text{in}_{\omega'}(g_i)$.

Write $\text{in}_\omega(g_i) = m + h \in \text{in}_\omega(I)$ and $\text{in}_{\omega'}(g_i) = m + h' \in \text{in}_{\omega'}(I)$ for some $h, h' \notin \text{in}_{\prec_\omega}(I)$ (if $h, h' \in \text{in}_{\prec_\omega}(I)$, then \mathcal{G} would not be reduced). Subtracting yields $h - h' \in \text{in}_\omega(I)$. Assume $h - h' \neq 0$. Then $\text{in}_{\prec}(h - h') \in \text{in}_{\prec_\omega}(I)$. This is a contradiction because no term of h, h' is divisible by $\text{in}_{\prec_\omega}(I)$.

We obtain $\text{in}_\omega(g_i) = m = \text{in}_{\omega'}(g_i)$.

□

Corollary 2.18. Let $\mathcal{N}_P(F)$ denote the normal cone of a polytope P with face $F \subset P$. The previous proposition shows that

$$C_I[\omega] = \mathcal{N}_{\text{New}(\prod g)}(\text{face}_\omega(\text{New}(\prod g))).$$

Proof. Take $\omega' \in \mathcal{N}(\text{face}_\omega(\text{New}(\prod g)))$. By definition of the normal cone \mathcal{N} , we have $\text{face}_{\omega'}(\text{New}(\prod g)) = \text{face}_\omega(\text{New}(\prod g))$. Thus, by Remark 2.15 we get

$$\text{New}(\text{in}_{\omega'}(\prod g)) = \text{New}(\text{in}_\omega(\prod g)).$$

Hence,

$$\sum \text{New}(\text{in}_{\omega'}(g)) = \text{New}(\prod \text{in}_{\omega'}(g)) = \text{New}(\prod \text{in}_\omega(g)) = \sum \text{New}(\text{in}_\omega(g)).$$

Hence, $\text{New}(\text{in}_{\omega'}(g)) = \text{New}(\text{in}_\omega(g))$ for all $g \in \mathcal{G}$. Therefore, the vertices of the newton polytopes coincide which means that $\text{in}_{\omega'}(g)$ and $\text{in}_\omega(g)$ coincide. Thus, $\omega' \in C_I[\omega]$. The reverse direction is similar. □

Corollary 2.19 (Polyhedral cone). Let I be an ideal in $k[x_1, \dots, x_n]$ and $\omega \in \mathbb{R}^n$. If $\omega \geq 0$, then $C_I[\omega]$ is a relatively open polyhedral cone in \mathbb{R}^n .

Proof. By the previous proposition we have

$$C_I[\omega] = \{\omega' \in \mathbb{R}^n : \text{in}_{\omega'}(g_i) = \text{in}_\omega(g_i) \forall i = 1, \dots, s\}.$$

Write $g_i = \sum \lambda_\alpha x^\alpha + \sum \lambda_\beta x^\beta$ such that $\text{in}_\omega(g_i) = \sum \lambda_\alpha x^\alpha$. Then $\text{in}_{\omega'}(g_i) = \text{in}_\omega(g_i)$ means that

$$\alpha \cdot \omega' = \alpha \cdot \omega' \quad \text{and} \quad \alpha \cdot \omega' > \beta \cdot \omega' \quad \forall \alpha, \beta.$$

Hence,

$$C_I[\omega] = \left\{ \omega' \in \mathbb{R}^n : \underbrace{\alpha \cdot \omega' = \alpha \cdot \omega'}_{\text{defines linear subspace}}, \underbrace{\alpha \cdot \omega' > \beta \cdot \omega'}_{\text{defines cone}} \forall i = 1, \dots, s, \forall \alpha, \beta \right\}.$$

□

Corollary 2.20 (Gröbner cone). Let I be an ideal in $k[x_1, \dots, x_n]$ and $\omega \in \mathbb{R}^n$. If $\omega \geq 0$, then

$$\overline{C_I[\omega]} = \{ \omega' \in \mathbb{R}^n : \text{in}_\omega(\text{in}_{\omega'}(g_i)) = \text{in}_\omega(g_i) \forall i = 1, \dots, s \}.$$

This closed polyhedral cone is called **Gröbner cone**.

Proof. Use

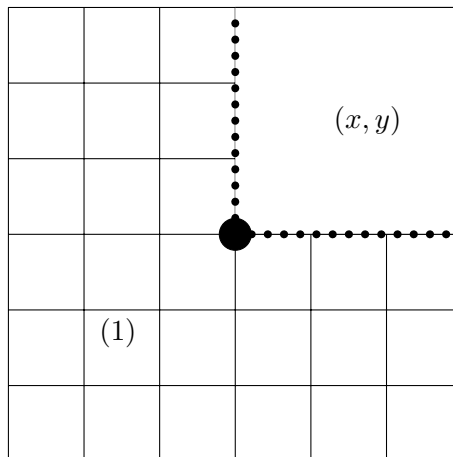
$$\overline{C_I[\omega]} = \{ \omega' \in \mathbb{R}^n : \alpha \cdot \omega' = \alpha \cdot \omega', \alpha \cdot \omega' \geq \beta \cdot \omega' \forall i = 1, \dots, s, \forall \alpha, \beta \}.$$

(Here is a dirty proof: If $\alpha\omega' = \alpha\omega'$ and $\alpha\omega' \geq \beta\omega'$, then after taking $\text{in}_\omega(\bullet)$ only the monomials with $\alpha\omega' > \beta\omega'$ survive. This proves \subset .

For \supset , take ω' from the right-hand side. Hence, $\text{in}_{\omega'}(g_i)$ contains monomials x^α . It may contain other monomials x^γ . Thus, $w'\alpha = w'\gamma$ and $w'\alpha > w'\beta'$. Hence, $\alpha\omega' = \alpha\omega'$ and $\alpha\omega' \geq \beta\omega'$ for $\beta \in \{\gamma, \beta'\}$.)

□

Example 2.21 ($\omega \geq 0$ is necessary). The assumption $\omega \geq 0$ is necessary. Let $I = (x-1, y-1) \subset \mathbb{R}[x, y]$. For $\omega_1 = (3, -1)$ and $\omega_2 = (-1, 3)$ we have $\text{in}_{\omega_1}(I) = \text{in}_{\omega_2}(I) = (-1) = (1)$. However, $\omega = \frac{\omega_1 + \omega_2}{2} = (2, 2)$ and hence $\text{in}_\omega(I) = (x, y)$. Thus, $\omega \notin C_I[\omega_1] = C_I[\omega_2]$ which shows that $C_I[\omega_1]$ is not a cone.



Definition 2.22 (Gröbner fan). Let $I \subset k[x_1, \dots, x_n]$ be an ideal. We define

$$\text{GFan}(I) = \left\{ \overline{C_I[\omega]} \mid \omega \in \mathbb{R}_{\geq 0}^n \right\}.$$

Remark 2.23. The closure $\overline{}$ is to be understood as the relative closure in $\text{GR}(I)$.

Lemma 2.24. Let $I \subset k[x_1, \dots, x_n]$ be an ideal and $\omega \in \mathbb{R}_{\geq 0}^n$. Then,

$$\omega' \in \overline{C_I[\omega]} \implies \overline{C_I[\omega']} \text{ is a face of } \overline{C_I[\omega]}.$$

Proof. Fix any monomial order \prec and let \mathcal{G} be the reduced Gröbner basis of I with respect to \prec_ω . Let $\omega' \in \overline{C_I[\omega]}$.

- We have $\text{in}_\omega(\text{in}_{\omega'}(g)) = \text{in}_\omega(g)$ for all $g \in \mathcal{G}$.
- Define $J = (\text{in}_{\omega'}(g) : g \in \mathcal{G} \subset I) \subset \text{in}_{\omega'}(I)$.
- By Corollary 2.7 we have $\text{in}_\omega(I) = (\text{in}_\omega(g) : g \in \mathcal{G}) \subset \text{in}_\omega(J)$ (use the first bullet point)
- Applying $\text{in}_\prec(\bullet)$ yields $\text{in}_{\prec_\omega}(I) \subset \text{in}_{\prec_\omega}(J) \subset \text{in}_{\prec_\omega}(\text{in}_{\omega'}(I)) = \text{in}_{(\prec_\omega)_{\omega'}}(I)$

By Macaulay's Corollary 1.54 all inclusions are equality, that is $\text{in}_{\prec_\omega}(I) = \text{in}_{(\prec_\omega)_{\omega'}}(I)$. Since \mathcal{G} is a Gröbner basis of I we have $\text{in}_{\prec_\omega}(\mathcal{G}) = \text{in}_{\prec_\omega}(I)$. Note that $\text{in}_{\prec_\omega}(\mathcal{G}) = \text{in}_{\prec}(\text{in}_\omega(\text{in}_{\omega'}(\mathcal{G}))) = \text{in}_{(\prec_\omega)_{\omega'}}(\mathcal{G})$. Hence,

$$\text{in}_{(\prec_\omega)_{\omega'}}(\mathcal{G}) = \text{in}_{\prec_\omega}(\mathcal{G}) = \text{in}_{\prec_\omega}(I) = \text{in}_{(\prec_\omega)_{\omega'}}(I)$$

which shows that \mathcal{G} is a reduced Gröbner basis of I with respect to $(\prec_\omega)_{\omega'}$. Applying Proposition 2.17 yields

$$\overline{C_I[\omega']} = \left\{ \omega'' \in \mathbb{R}^n : \text{in}_{\omega'}(\text{in}_{\omega''}(g)) = \text{in}_{\omega'}(g) \forall g \in \mathcal{G} \right\}.$$

Note that any $\omega'' \in \overline{C_I[\omega']}$ satisfies $\text{in}_\omega(\text{in}_{\omega''}(g)) = \text{in}_\omega(g)$ for all $g \in \mathcal{G}$. Hence, $\overline{C_I[\omega']} \subset \overline{C_I[\omega]}$. Moreover, transforming $\text{in}_{\omega'}(\text{in}_{\omega''}(g)) = \text{in}_{\omega'}(g)$ into a set of linear equalities and inequalities $\omega''\alpha = \omega''\alpha$ and $\omega''\alpha > \omega''\beta$ reveals that the set of β 's is a subset of the set of β 's of $\overline{C_I[\omega]}$ (the missing inequality becomes an equality in $\overline{C_I[\omega']}$). Hence, $\overline{C_I[\omega']}$ is a face of $\overline{C_I[\omega]}$. \square

Theorem 2.25 (Gröbner fan is a polyhedral fan). Let $I \subset k[x_1, \dots, x_n]$ be an ideal. Then $\text{GFan}(I)$ is a polyhedral fan.

Proof.

- All faces are in the fan: let F be a face of $\overline{C_I[\omega]}$. Take $\omega' \in \mathbb{R}_{\geq 0}^n$ in the relative interior and by the previous lemma we have $\overline{C_I[\omega']}$ is a face. Note that F is the only possible face of $\overline{C_I[\omega]}$ that can contain a relative interior ω' . Hence, $F = \overline{C_I[\omega']}$.
- Intersection of two cones in GRFan is a common face: let $P = \overline{C_I[\omega]} \cap \overline{C_I[\omega']}$. Take any $\omega'' \in P$. Then $\overline{C_I[\omega']}$ is a common face of $\overline{C_I[\omega]}$ and $\overline{C_I[\omega']}$. Hence, $P = \bigcup_{\omega'' \in P} \overline{C_I[\omega']}$. Since P is convex, the union is a singleton. So P is a common face.

□

Example 2.26. The Gröbner fan of $I = (f)$ where $f(x, y) = x^5y^2 + x^4y^4 + x^4 + x^2y^5 + xy^2 + y^6 + y$ is

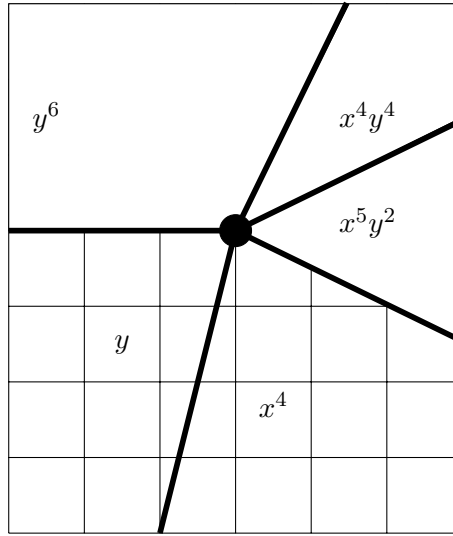
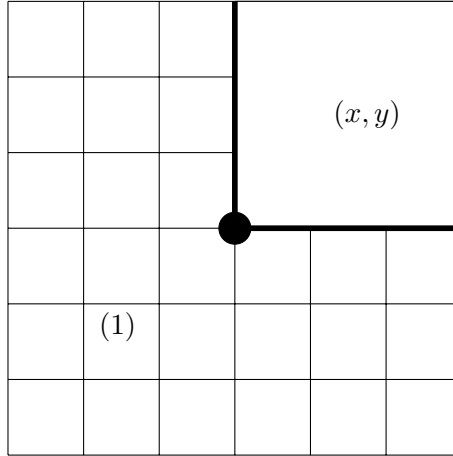


Figure 3: The Gröbner fan I .

Example 2.27. Let f be homogeneous. To draw the Gröbner fan of a principal ideal $I = (f)$ do the following: Draw the Newton polytope of f and then draw the normal fan of this Newton polytope. The Gröbner fan consists of cones intersecting the non-negative orthant.

Example 2.28. Let $I = (x - 1, y - 1)$. An universal Gröbner basis is given by $\mathcal{G} = \{x - 1, y - 1\}$ (check by computing the S -polynomial). There is a one-to-one correspondence between $\text{in}_\omega(I)$ where weights $\omega \geq 0$ represent a monomial ideal and $\text{in}_\prec(I)$. Thus, there is only one maximal cone in GFan since $\text{in}_\prec(I) = \text{in}_\prec(\mathcal{G}) = (x, y)$ for all monomial orders \prec (in other words, there exists exactly one initial ideal of I). In order to represent (x, y) , ω may only take non-negative values.



Remark 2.29 (How to draw the Gröbner fan by hand). Assume $I = (f_1, \dots, f_s)$. Compute a universal Gröbner basis \mathcal{G} . Then, compute all the possible initial ideals $\text{in}_{\prec}(\mathcal{G})$; for example, say we have $\text{in}_{\prec}(\mathcal{G}) = (x^{\alpha_1}, \dots, x^{\alpha_s})$. Write inequalities $\omega\alpha_i > \omega\beta_i$ where $f_i = \lambda_{\alpha_i}x^{\alpha_i} + \sum_{\beta_i} \lambda_{\beta_i}x^{\beta_i}$. Draw the cones $\omega \in \mathbb{R}^n$ defined by $\omega\alpha_i > \omega\beta_i$.

Repeat this process for every possible initial ideal $\text{in}_{\prec}(\mathcal{G})$ varying over \prec .

Example 2.30. Let $I = (y^2 + xy + 1, x^2) \subset k[x, y]$.

- Take a monomial order defined by $\omega = (1, 2)$. Then, $\text{in}_{\omega}(y^2 + xy + 1) = y^2$ and $\text{in}_{\omega}(x^2) = x^2$. We see that $\mathcal{G} = \{y^2 + xy + 1, x^2\}$ is a reduced Gröbner basis. Hence, $\text{in}_{\omega}(I) = (x^2, y^2)$. Then

$$\begin{aligned} C_I[(1, 2)] &= \{(\omega_1, \omega_2) \in \mathbb{R}^2 : 2\omega_2 > \omega_1 + \omega_2, 2\omega_2 > 0\} \\ &= \{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_2 > \omega_1, \omega_2 > 0\}. \end{aligned}$$

- Take grlex as a monomial order. We want to find $\omega \in \mathbb{R}_{\geq 0}^n$ representing grlex. A Gröbner basis is given by $\mathcal{G} = \{y^2 + xy + 1, x^2, y^3 - x + y\}$.
- Take lex as a monomial order.

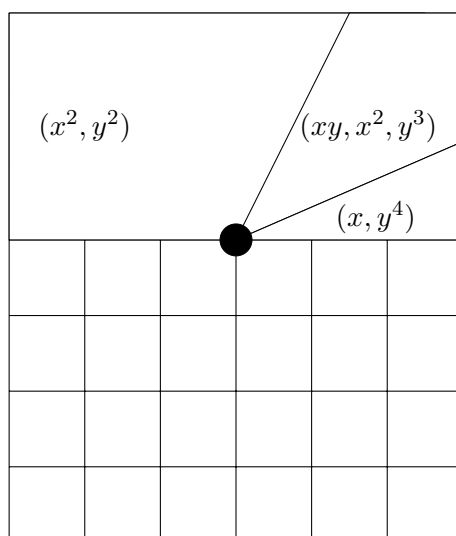


Figure 4: The Gröbner fan I .

2.3 Homogeneity

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Definition 2.31. Let $\omega \in \mathbb{R}^n$.

- A polynomial $f \in k[x_1, \dots, x_n]$ is ω -**homogeneous** if $\text{in}_\omega(f) = f$.
- An ideal is ω -homogeneous if it has an ω -**homogeneous** generating set.

Definition 2.32. An ideal is **homogeneous** if it is ω -homogeneous for some positive $\omega \in \mathbb{R}_{>0}^n$.

Definition 2.33. Let $f = \sum \lambda_\alpha x^\alpha$ be ω -homogeneous. Its ω -**degree** is $\omega \cdot \alpha$.

Example 2.34 (Total degree). The total degree is given by the $\mathbf{1} = (1, \dots, 1)$ -grading.

Example 2.35.

- Let $f = x^3 + y^2z + z^3$. f is $(1, 1, 1)$ -homogeneous with ω -degree 3.
- Let $f = x^2 - y$. f is $(1, 2)$ -homogeneous with ω -degree 2.

Example 2.36. Let $I = (x - y, x^2 - y + y^2 + y)$. The ideal I is homogeneous because it is generated by $(x - y, x^2 + y^2)$ and the generators are $(1, 1)$ -homogeneous polynomials.

Example 2.37. $I = (x - 1)$ is not homogeneous because $\text{in}_\omega(x - 1) = x$ for all $\omega > 0$.

Remark 2.38. Let I be an ideal and $\omega \in \mathbb{R}^n$. Then $\text{in}_\omega(I)$ is ω -homogeneous because $\text{in}_\omega(I) = \{\text{in}_\omega(f) : f \in I\}$ and $\text{in}_\omega(\text{in}_\omega(f)) = \text{in}_\omega(f)$.

Remark 2.39 (Unique decomposition into ω -homogeneous parts). Fix $\omega \in \mathbb{R}^n$. Let f be not ω -homogeneous. Then f can be *uniquely* decomposed into ω -homogeneous parts:

$$f = f_1 + \cdots + f_j \quad \text{where } f_j \text{ is } \omega\text{-homogeneous of different } \omega\text{-degree.}$$

Remark 2.40. Two observations:

- if f is ω -homogeneous, then $x^\alpha \cdot f$ is ω -homogeneous
- if f, g are ω -homogeneous of same ω -degree, then $f + g$ is ω -homogeneous.

Proposition 2.41 (Characterization of ω -homogeneous ideals). Let $I \subset k[x_1, \dots, x_n]$ be an ideal and $\omega \in \mathbb{R}_{>0}^n$. The following are equivalent

1. I is ω -homogeneous
2. $f \in I$ with $f = \sum f_j$ ω -homogeneous decomposition $\implies f_j \in I \forall i$
3. $\text{in}_\omega(I) = I$
4. the reduced Gröbner basis (with respect to any order \prec) of I consists of ω -homogeneous polynomials

Proof.

- 4. \implies (1) is clear
- 3. \implies (1): take $\text{in}_\omega(I)$ as a generating set of I . We saw that $\text{in}_\omega(I)$ is ω -homogeneous.
- 2. \implies (1): By Hilbert's Basis theorem, $I = (g_1, \dots, g_s)$. Write each g_i as a unique decomposition into ω -homogeneous parts: $g_i = \sum g_{ij}$. Then I is generated by ω -homogeneous g_{ij} .

- 1. \implies 4.: By assumption $I = (H)$ for some (infinite) generating set H that is ω -homogeneous. On the other hand, by Hilbert's basis theorem, there exists finite F that generates I . Hence, there exists finite $H' \subset H$ such that H' generates I .

Now that H' is finite, we apply Buchberger's algorithm to construct a Gröbner basis. We see that Buchberger's algorithm only appends ω -homogeneous polynomials.

- 4. \implies 3. We have $I = (G) = (\text{in}_\omega(G)) \subset \text{in}_\omega(I)$. This inclusion cannot be strict, if it were strict, we apply $\text{in}_\prec(\bullet)$ on both sides and by Macaulay's Corollary we obtain a contradiction.

- 4. \implies 2. Let $G = \{g_1, \dots, g_s\}$ be a reduced Gröbner basis that is ω -homogeneous. Let $f \in I$. Write $f = \sum h_i g_i$ and expand $h_i = \sum \lambda_{\alpha_i} x^{\alpha_i}$ to: $f = \sum_{i, \alpha_i} \lambda_{\alpha_i} x^{\alpha_i} g_i$. Note that $\lambda_{\alpha_i} x^{\alpha_i} g_i$ is ω -homogeneous. So any ω -homogeneous part of f is of the form $f_j = \sum_{i, \alpha_i} \lambda_{\alpha_i} x^{\alpha_i} g_i \in I$ because $g_i \in I$.

□

Lemma 2.42. Let I be a d -homogeneous ideal for some $d > 0$. Let $\omega \in \mathbb{R}^n$. Then $\text{in}_\omega(I) = \text{in}_{\omega+\lambda d}(I)$ for every $\lambda \in \mathbb{R}$.

Proof. If f is d -homogeneous, then $\text{in}_\omega(f) = \text{in}_{\omega+\lambda d}(f)$ for all $\lambda \in \mathbb{R}$ because

$$\omega\alpha > \omega\beta \iff (\omega + \lambda d)\alpha > (\omega + \lambda d)\beta$$

since $d\alpha = d\beta$ ($f = \text{in}_d(f) = \sum \lambda_\alpha x^\alpha$).

It suffices to only prove \subset because if we assume \subset is true, we can write $\omega' = \omega + \lambda d$ and thus

$$\text{in}_{\omega'}(I) \subset \text{in}_{\omega' - \lambda d}(I) = \text{in}_\omega(I).$$

Hence, we reduced the case \supset to \subset .

Let's prove \subset . Let $f \in I$ and write $f = \sum f_j$ d -homogeneous decomposition. We know that $f_j \in I$ for all j . Then

$$\text{in}_\omega(f) = \sum_i \text{in}_\omega(f_{j_i})$$

Since each f_{j_i} is d -homogeneous, we can write

$$\text{in}_\omega(f_{j_i}) = \text{in}_{\omega+\lambda d}(f_{j_i}) \in \text{in}_{\omega+\lambda d}(I)$$

Hence, $\text{in}_\omega(f) \in \text{in}_{\omega+\lambda d}(I)$.

□

Corollary 2.43. If $I \subset k[x_1, \dots, x_n]$ is a homogeneous ideal, then $\text{GR}(I) = \mathbb{R}^n$.

Proof. Assume I is d -homogeneous for $d > 0$. Then for any $\omega \in \mathbb{R}^n$ we find $\lambda > 0$ large enough such that $\omega + \lambda d > 0$. Use the previous lemma to show $\text{in}_\omega(I) = \text{in}_{\omega+\lambda d}(I)$. □

2.4 State polytope

The main theorem is: if an ideal is homogeneous, then a state polytope always exists.

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Definition 2.44 (State polytope). Let I be an ideal in $k[x_1, \dots, x_n]$. A polytope $P \subset \mathbb{R}^n$ is called a **state polytope** if its normal fan equals the Gröbner fan of I :

$$\mathcal{N}(\text{State}(I)) = \text{GFan}(I).$$

Remark 2.45. The state polytope is not unique. Two state polytopes are only strongly isomorphic (meaning that they have the same normal fan).

Example 2.46. ...

Theorem 2.47. If I is homogeneous, then a state polytope of I exists.

Proof sketch. Assume I is ω -homogeneous with $\omega > 0$. Define

$$\text{State}_d = \text{conv} \left\{ \sum (\text{in}_{\prec}(I))_d : \prec \text{ is any term order} \right\}.$$

$\sum (\text{in}_{\prec}(I))_d$ denotes the sum of $\mathbf{a} \in \mathbb{N}^n$ such that $\mathbf{x}^{\mathbf{a}} \in \text{in}_{\prec}(I)$ has ω -degree d . Let D be the largest degree of any element in a minimal universal Gröbner basis of I . We define

$$\text{State}(I) = \sum_{d=1}^D \text{State}_d(I).$$

We assume the lemma

Lemma 2.48. $\text{face}_{\omega}(\text{State}_d(I)) = \text{State}_d(\text{in}_{\omega}(I)) \quad \forall \omega \in \mathbb{R}^n$.

With this lemma we show that $\mathcal{N}(\text{State}(I)) = \text{GFan}(I)$. Note that polyhedral complexes are determined by their maximal faces. So it suffices to show that the open maximal cones of $\text{GFan}(I)$ coincide with the open maximal cones of $\mathcal{N}(\text{State}(I))$. This means that for any two *generic* vectors $\omega, \omega' \in \mathbb{R}^n$ we show that

$$\text{in}_{\omega}(I) = \text{in}_{\omega'}(I) \iff \text{face}_{\omega}(\text{State}(I)) = \text{face}_{\omega'}(\text{State}(I)).$$

- \Leftarrow : Assume $\text{in}_{\omega}(I) = \text{in}_{\omega'}(I)$. Applying $\text{State}_d(\bullet)$ and using the above lemma we get $\text{face}_{\omega}(\text{State}_d(I)) = \text{face}_{\omega'}(\text{State}_d(I))$. Now use

$$\text{face}_{\omega}(\text{State}(I)) = \sum \text{face}_{\omega}(\text{State}_d(I)) = \sum \text{face}_{\omega'}(\text{State}_d(I)) = \text{face}_{\omega'}(\text{State}(I)).$$

- \Rightarrow Assume $\text{face}_{\omega}(\text{State}(I)) = \text{face}_{\omega'}(\text{State}(I))$ where both faces are vertices since ω, ω' are chosen generically.

Observe the following general fact: let v be a vertex of $P + Q$, then there exists unique vertices $v = p + q$.

Hence,

$$\text{face}_{\omega}(\text{State}_d(I)) = \text{face}_{\omega'}(\text{State}_d(I))$$

for all $d = 1, \dots, D$ (the faces are vertices).

Also note that since ω, ω' are generic, we can replace the weights by term orders such that $\text{in}_{\omega}(I) = \text{in}_{\prec}(I)$ and $\text{in}_{\omega'}(I) = \text{in}_{\prec'}(I)$ (just take a Gröbner basis and since ω is generic, each $\text{in}_{\omega}(g)$ is monomial).

Apply the lemma again to obtain $\text{State}_d(\text{in}_{\prec}(I)) = \text{State}_d(\text{in}_{\prec'}(I))$ for all $d = 1, \dots, D$. Thus, $\sum (\text{in}_{\prec}(I))_d = \sum (\text{in}_{\prec'}(I))_d$. Hence, as a corollary of the previous lemma we get $\text{in}_{\prec}(I)_d = \text{in}_{\prec'}(I)_d$ for all $d = 1, \dots, D$. Then, $\text{in}_{\omega}(I) = \text{in}_{\omega'}(I)$.

□

We give two important examples of state polytopes.

Proposition 2.49 (Newton polytope is a state polytope). If f is homogeneous, then $\text{Newton}(f)$ is a state polytope of $I = (f)$.

Proof. Use Corollary 2.18.

□

Corollary 2.50. If G is a universal Gröbner basis of a homogeneous ideal I and G is reduced for every term order, then the state polytope of I is given by

$$\sum_{g \in G} \text{Newton}(g) = \text{Newton}\left(\prod_{g \in G} g\right)$$

Proof. Use Corollary 2.18.

□

3 Toric ideals

3.1 Basics of toric ideals

Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ be a lattice point configuration. We define a matrix $A \in \mathbb{Z}^{d \times n}$ by $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n]$. For an indeterminate \mathbf{t} we define $\mathbf{t}^{\mathbf{a}_i} = t_1^{\mathbf{a}_{1i}} t_2^{\mathbf{a}_{2i}} \dots t_d^{\mathbf{a}_{di}} \in K[\mathbf{t}^\pm]$. The ring $K[\mathbf{t}^\pm]$ is called **Laurent polynomial ring**. Lec15,
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Given \mathcal{A} we define a **monoid homomorphism** (this is a semigroup with an identity but no inverse) between $\mathbb{N}^n \rightarrow \mathbb{Z}^d$ by

$$\pi_A : u \mapsto Au.$$

The image of π_A is the monoid generated by A , denoted $\mathbb{N} \cdot A$.

The map π_A lifts to a **ring homomorphism**

$$\begin{aligned} \tilde{\pi}_A : k[x_1, \dots, x_n] &\rightarrow k[\mathbf{t}^\pm] \\ x_i &\mapsto \mathbf{t}^{\mathbf{a}_i} \end{aligned}$$

by the following diagram

$$\begin{array}{ccc} \mathbb{N}^n & \xrightarrow{\pi_A} & \mathbb{Z}^d \\ \downarrow & & \downarrow \\ k[x_1, \dots, x_n] & \xrightarrow{\tilde{\pi}_A} & k[\mathbf{t}^\pm] \end{array}$$

Definition 3.1 (Toric ideal). The **toric ideal** of A is defined to be $\ker(\tilde{\pi}_A) \subset k[x_1, \dots, x_n]$. It is denoted by I_A .

By definition, I_A is an ideal since an ideal is precisely the kernel of some ring homomorphism.

Remark 3.2 (Toric ideal is prime). The toric ideal is prime since the preimage of a prime ideal stays prime and (0) is prime in $k[\mathbf{t}^\pm]$.

Remark 3.3 (Toric ideal contains no monomial ideals). If $\mathbf{x}^\alpha \in I_A$, then $\mathbf{t}^{A\alpha} = 0$ which is impossible.

Example 3.4. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The map $\tilde{\pi}_A : k[x_1, x_2, x_3] \rightarrow k[s^\pm, t^\pm]$ maps

$$\begin{aligned} x_1 &\mapsto 1 \\ x_2 &\mapsto s \\ x_3 &\mapsto s \cdot t \end{aligned}$$

The kernel is $I_A = (x_1 - 1)$.

Example 3.5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. The map $\tilde{\pi}_A : k[x_1, x_2, x_3] \rightarrow k[s^\pm, t^\pm]$ maps

$$\begin{aligned} x_1 &\mapsto s \\ x_2 &\mapsto t \\ x_3 &\mapsto s \cdot t \end{aligned}$$

The kernel is $I_A = (x_1x_2 - x_3)$.

Here is an important characterization of the toric ideal I_A : it is a k -linear combination of binomials $x^\alpha - x^\beta$ where $A\alpha = A\beta$.

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Proposition 3.6 (Toric ideals are k -linear combinations of binomials). Let $A \in \mathbb{Z}^{d \times n}$. Then

$$I_A = \text{span}_k \left(\left\{ x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^n \text{ with } A\alpha = A\beta \right\} \right).$$

Proof. \supset is clear because $\tilde{\pi}_A(x^\alpha - x^\beta) = x^{A\alpha} - x^{A\beta} = 0$.

\subset : Let $S = \{f \in I_A : f \notin \text{span}_k(\{x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^n \text{ with } A\alpha = A\beta\})\}$ and assume for contradiction $S \neq \emptyset$. Fix some monomial order \prec and consider the initial ideal $\text{LT}_\prec(S)$. Since the initial ideal $\text{LT}_\prec(S)$ is nonempty and \prec is a well-order, there exists a minimal element $\lambda_\alpha x^\alpha \in \text{LT}_\prec(S)$. Let's say the monomial belongs to the polynomial $f \in S \subset I_A$, i.e. $\text{LT}_\prec(f) = \lambda_\alpha x^\alpha$. Since $f \in I_A$, we have $\tilde{\pi}_A(f) = 0$. Hence, $\tilde{\pi}_A(\lambda_\alpha x^\alpha)$ must cancel in $\tilde{\pi}_A(f)$. So there is some $\lambda_\beta x^\beta$ with $x^\beta \prec x^\alpha$ and $x^{A\alpha} = x^{A\beta}$. Therefore, $f' = f - \lambda_\alpha(x^\alpha - x^\beta)$ has a leading term smaller than $\lambda_\alpha x^\alpha$. Note that $f' \in I_A$. Assume that $f' \notin S$. Then $f \in \text{span}(\dots)$ which is not possible. So $f' \in S$ which is a contradiction because the leading term of f is not minimal anymore in S . \square

We can also take $k[x_1, \dots, x_n]$ -linear combinations of binomials.

Corollary 3.7. $I_A = (\{x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^n \text{ with } A\alpha = A\beta\})$

Proof. Clear. \square

Corollary 3.8 (Reduced Gröbner basis of a toric ideal). Let \prec be any monomial order. There exists a reduced Gröbner basis \mathcal{G} of I_A with respect to \prec where all elements of \mathcal{G} are of the form

$$x^\alpha - x^\beta \text{ for some } \alpha, \beta \in \mathbb{N}^n \text{ with } A\alpha = A\beta.$$

Proof. By Hilbert's basis theorem, we have a finite subset of binomials

$$\{x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^n \text{ with } A\alpha = A\beta\}$$

that generates I_A as an ideal. Apply Buchberger's algorithm and note that

- S -pairs of binomials is a binomial or 0:

$$\frac{lcm}{LT}(x^\alpha - x^\beta) - \frac{lcm}{LT}(x^{\alpha'} - x^{\beta'})$$

two of these terms cancel and it cannot be a monomial since I_A does not contain any monomials; so it is a binomial

- the residue of a binomial divided by another binomial is a binomial or 0: if division

$$(x^\alpha - x^\beta) : (x^{\alpha'} - x^{\beta'})$$

is possible, then the residue is again a binomial. If it is not possible, we add the leading term to the residue. We then obtain a monomial that we divide by a binomial

$$x^\alpha : (x^{\alpha'} - x^{\beta'})$$

This either yields a monomial as a remainder if division is possible (and we are in the base case again) or division is not possible, and we are done, so that the residue is a binomial.

- Reducing a basis of binomials stays a basis of binomials: reducing means computing the normal form, this is the above case.

□

Our next goal is to compute a finite set of generators of I_A . We know that I_A is generated by binomials of the form $\{x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^n \text{ with } A\alpha = A\beta\}$; however this set need not be finite. Hilbert's basis theorem assures the existence of a finite generating set of binomials but provides no way of computing it.

Remark 3.9. Assume $\alpha, \beta \in \mathbb{N}^n$. If $A\alpha = A\beta$, then $A(\alpha - \beta) = 0$. Hence, we have $\alpha - \beta \in \ker(A) \cap \mathbb{Z}^d$.

Remark 3.10 (Disjoint support). Let $x^\alpha - x^\beta \in I_A$. Without loss of generality, we can assume that α and β have disjoint support. Disjoint support means that if $\alpha_i \neq 0$ then $\beta_i = 0$ and the other way around.

To see this, note that if $x^\gamma(x^\alpha - x^\beta) \in I_A$, then $x^\alpha - x^\beta \in I_A$ because I_A is prime and I_A cannot contain monomials. Thus, if two vectors $\alpha, \beta \in \mathbb{N}^n$ have common support, say $\alpha_1 > \beta_1 > 0$, then we can factor $x^\alpha - x^\beta = x^{\beta_1}(x^{\alpha_1 - \beta_1, \dots} - x^{0, \dots})$. Repeating this process yields α and β with disjoint support such that $x^\alpha - x^\beta \in I_A$.

Definition 3.11 (Positive and negative part). Let $\alpha \in \mathbb{Z}^n$. We define α^+ to be the positive part and α^- to be the negative part of α .

Example 3.12. $(1, -2, 0, 4) = (1, 0, 0, 4) - (0, 2, 0, 0)$.

Corollary 3.13. We have

$$I_A = (x^{\alpha^+} - x^{\alpha^-} : \alpha \in \ker(A) \cap \mathbb{Z}^n).$$

Proof. \subset : Let $x^a - x^b \in I_A$. Without loss of generality, we can assume that a and b have disjoint support. Set $\alpha = a - b$. This easily implies \subset .

\supset : We have $0 = A\alpha = A\alpha^+ - A\alpha^-$. Hence, $A\alpha^+ = A\alpha^-$ with $\alpha^+, \alpha^- \in \mathbb{N}^n$. \square

Corollary 3.14. Let \prec be a monomial order. There exists a reduced Gröbner basis of I_A with respect to \prec with elements of the form $x^{\alpha^+} - x^{\alpha^-}$.

Here is an algorithm to compute the reduced Gröbner basis of a toric ideal.

Conti-Traverso algorithm

Input: $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$

Output: reduced Gröbner basis for I_A with respect to lex

- 1: Consider the polynomial ring $k[t_0, \dots, t_d, x_1, \dots, x_n]$ with lex such that $t_0 \succ t_1 \succ \dots \succ t_d \succ x_1 \succ \dots \succ x_n$
- 2: Compute the reduced Gröbner basis \mathcal{G} of $(t_0 t_1 \dots t_d - 1, x_i t^{\mathbf{a}_i^-} - t^{\mathbf{a}_i^+})$ with respect to lex
- 3: **return** $\mathcal{G} \cap k[x_1, \dots, x_n]$

Proof. Skipped. \square

Remark 3.15 (Special case). If A only takes non-negative entries, then it suffices to consider the polynomial ring $k[t_1, \dots, t_d, x_1, \dots, x_n]$ and the ideal $(x_i - t^{\mathbf{a}_i} : 1 \leq i \leq n)$.

Example 3.16. Let $A = \begin{bmatrix} 5 & 10 & 25 \end{bmatrix}$. Then, $\tilde{\pi}_A$ maps $x \mapsto t^5$, $y \mapsto t^{10}$ and $z \mapsto t^{25}$.

Consider the ideal $(x - t^5, y - t^{10}, z - t^{25}) \subset k[x, y, z, t]$ with respect to lex ($t \succ x \succ y \succ z$). Computing the Gröbner basis yields $\mathcal{G} = \{t^5 - x, x^2 - y, xy^2 - z, xz - y^3, y^5 - z^2\}$. Then,

$$I_A = \mathcal{G} \cap k[x, y, z] = (x^2 - y, xy^2 - z, xz - y^3, y^5 - z^2).$$

This expression is not very efficient. We also have the Gröbner basis $I_A = (x^2 - y, x^5 - z)$ which is much more efficient. Even $I_A = (x^2 - y, xy^2 - z)$ suffices. The universal Gröbner basis of I_A is $\{x^2 - y, xy^2 - z, xz - y^3, y^5 - z^2, x^5 - z\}$.

3.2 Graver basis and circuits

Definition 3.17 (Primitive binomial). A binomial $x^{\mathbf{a}^+} - x^{\mathbf{a}^-} \in I_A$ is called **primitive** if the following condition holds: if $x^{\beta^+} - x^{\beta^-} \in I_A$ with x^{β^+} dividing x^{α^+} and x^{β^-} dividing x^{α^-} , then $\beta^+ = \alpha^+$ and $\beta^- = \alpha^-$.

Remark 3.18. We will see next lecture that every binomial in a reduced Gröbner basis of I_A is primitive.

Definition 3.19. The **Graver basis** Gr_A of the toric I_A is the set of all primitive binomials $x^{\alpha^+} - x^{\alpha^-} \in I_A$.

Remark 3.20. The universal Gröbner basis can be smaller than the Graver basis.

Proposition 3.21. Every binomial in a universal Gröbner basis U_A of I_A is primitive.

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Proof. Let $x^{\alpha^+} - x^{\alpha^-} \in U_A$ correspond to some monomial order \prec (without loss of generality we assume $x^{\alpha^+} \succ x^{\alpha^-}$, otherwise multiply by -1). This means that x^{α^+} is one of the minimal generators of $\text{in}_{\prec}(I_A)$ and x^{α^-} is a standard monomial.

Let $x^{\beta^+} - x^{\beta^-} \in I_A$ with $x^{\beta^+} | x^{\alpha^+}$ and $x^{\beta^-} | x^{\alpha^-}$.

Claim: $x^{\beta^+} \succ x^{\beta^-}$. Otherwise, $x^{\beta^-} \in \text{in}_{\prec}(I_A)$ and therefore $x^{\alpha^-} \in \text{in}_{\prec}(I_A)$, but x^{α^-} is a standard monomial.

Thus, $x^{\beta^+} = x^{\alpha^+}$ because x^{α^+} is minimal. Therefore, $x^{\alpha^-} - x^{\beta^-} \in I_A$. Since both monomials x^{α^-} and x^{β^-} are standard monomials, their difference must be zero. Hence, $x^{\beta^-} = x^{\alpha^-}$. \square

Corollary 3.22. The Graver basis Gr_A is a universal Gröbner basis of I_A . Gr_A is a good approximation of U_A .

Definition 3.23 (Rational polyhedral cone). A **rational polyhedral cone** $C \subset \mathbb{R}^n$ is of the form $C = \text{cone}(v_1, \dots, v_k)$ with $v_i \in \mathbb{Z}^n$.

Definition 3.24 (Hilbert basis). A **Hilbert basis** H of a rational polyhedral cone C is a finite generating set of $\text{cone}(C) \cap \mathbb{Z}^n$. This means that every element $x \in \text{cone}(C) \cap \mathbb{Z}^n$ is obtained by repeatedly adding elements from H , i.e. $x = h_1 + \dots + h_s$, $h_i \in H$.

Remark 3.25 (Constructing a Hilbert basis). One way to obtain a Hilbert basis H of a cone $C = \text{cone}(v_1, \dots, v_k)$:

$$H = \{v_1, \dots, v_k\} \cup (\Pi \cap \mathbb{Z}^n)$$

where $\Pi = \{\sum \lambda_i v_i : \lambda_i \in [0, 1]\}$ is the fundamental parallelepiped.

Remark 3.26. It is a fact that if C is pointed, there is a unique minimal Hilbert basis. Pointed means: $x \in C$, $-x \in C$ then $x = 0$, or equivalently the cone contains no line.

Computing a Graver basisInput: $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$ Output: Graver basis Gr_A

- 1: for each sign pattern $\sigma \in \{+, -\}^n$ consider the closed orthant \mathbb{R}_σ^n
- 2: for each σ consider the pointed polyhedral cone $C_\sigma = \ker(A) \cap \mathbb{R}_\sigma^n$
- 3: Compute the minimal Hilbert basis of C_σ
- 4: **return** $\left\{ x^{\alpha^+} - x^{\alpha^-} : \alpha \in \bigcup_{\sigma \in \{+, -\}^n} H_\sigma \setminus \{0\} \right\}$

Remark 3.27. By symmetry, it is enough to go through half of the orthants.

Definition 3.28 (Circuit). An *irreducible* binomial in I_A is called a **circuit** if its support is minimal among all binomials in I_A . The **set of all circuits** is denoted C_A .

Note that $\text{supp}(x^u - x^v) = \text{supp}(x^u) \cup \text{supp}(x^v)$.

Theorem 3.29. $C_A \subset U_A \subset \text{Gr}_A$.

Proof. By Proposition 3.21, we have $U_A \subset \text{Gr}_A$.

Let $\mathbf{c} \in C_A$. We want to show that \mathbf{c} is contained in some reduced Gröbner basis \mathcal{G}_\prec of I_A . We define a term order \prec such that $x_i \notin \text{supp}(\mathbf{c}) \succ x_j \in \text{supp}(\mathbf{c})$ and $x^{\mathbf{c}^+} \succ x^{\mathbf{c}^-}$. Assume $x^{\mathbf{c}^+} - x^{\mathbf{c}^-}$ appears not in \mathcal{G}_\prec . Then there is some $\mathbf{v} \in I_A$ such that $x^{\mathbf{v}^+} \succ x^{\mathbf{v}^-}$ and $x^{\mathbf{v}^+}$ divides $x^{\mathbf{c}^+}$. By choice of the term order and since $\text{supp}(\mathbf{v}^+) \subset \text{supp}(\mathbf{c})$ we have $\text{supp}(\mathbf{v}^-) \subset \text{supp}(\mathbf{c})$, hence $\text{supp}(\mathbf{v}) \subset \text{supp}(\mathbf{c})$. Since \mathbf{c} is a circuit, \mathbf{v} must be a multiple of \mathbf{c} , but $x^{\mathbf{v}^+}$ divides $x^{\mathbf{c}^+}$. Hence $\mathbf{v} = \mathbf{c}$. \square

Example 3.30. Let $A = \begin{bmatrix} 5 & 10 & 25 \end{bmatrix}$. The kernel of A is given by

$$\{(x, y, z) : 5x + 10y + 25z = 0\}.$$

To compute the Graver basis, we just need to consider the cones $C_{+++}, C_{++-}, C_{+-+}, C_{-++}$, compute their Hilbert bases and join them.

- $C_{+++} = \{0\}$.
- ...

Example 3.31. Here is an example where C_A , U_A and the Graver basis coincide. Take

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \dots & & & & & \end{bmatrix}$$

Then $I_A = \left\{ 2 \times 2 \text{ minors of } \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \right\} = \{x_{11}x_{22} - x_{21}x_{12}, x_{11}x_{23} - x_{21}x_{13}, x_{13}x_{22} - x_{12}x_{23}\}.$

$$C_A = \left\{ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \dots \right\}$$

Lemma 3.32. Let $A \in \mathbb{Z}^{d \times n}$ of full row rank d . It holds:

1. For every circuit $\mathbf{c} \in C_A$ we have $|\text{supp}(\mathbf{c})| \leq d + 1$. In words, a circuit cannot have more than $d + 1$ nonzero entries.
2. Every vector $\mathbf{v} \in \ker(A) \cap \mathbb{Z}^n$ can be written as a non-negative rational linear combination of $n - d$ circuits

$$\mathbf{v} = \sum_{i=1}^{n-d} q_i \mathbf{c}_i, \quad q_i \in \mathbb{Q}_{\geq 0}, \mathbf{c}_i \in C_A,$$

such that

$$\text{supp}(\mathbf{c}_i^+) \subset \text{supp}(\mathbf{v}^+) \text{ and } \text{supp}(\mathbf{c}_i^-) \subset \text{supp}(\mathbf{v}^-) \quad \forall i = 1, \dots, n - d.$$

3. Define $D(A) = \max \{ |\det([a_{i_1} \dots a_{i_d}])| : 1 \leq i_1 < \dots < i_d \leq n \}$.
If $\mathbf{c} = (c_1, \dots, c_n) \in C_A$, then $|c_i| < D(A)$.

Proof.

1. By contraposition. Assume $\mathbf{c} \in I_A$ has r nonzero entries for some $r > d + 1$. Let $B \in \mathbb{Z}^{d \times r}$ be the submatrix of A given by the nonzero indices of \mathbf{c} . Then

$$\dim(\ker(B)) = r - \dim(\text{im}(B)) \geq r - d \geq 2.$$

So we can find $\mathbf{v} \in \ker(B) \setminus \{0\}$ with at least one zero coordinate (the kernel is a two-dimensional plane going through the origin, hence we can find such \mathbf{v}). Extend \mathbf{v} to $\mathbf{u} \in \ker(A) \cap \mathbb{Z}^n$ by adding zero to the other $n - r$ missing coordinates. Since $\mathbf{v} \in \mathbb{R}^r$, we found a vector \mathbf{u} whose support is contained in the support of \mathbf{c} . Hence, \mathbf{c} is not a circuit.

2. Skipped. Follows from Caratheodory's theorem.
3. Skipped. Use Cramer's rule. The claim states that all the coordinates v_i cannot be larger than the volume.

□

Theorem 3.33. Let $A \in \mathbb{Z}^{d \times n}$. Let $\mathbf{v} \in \text{Gr}_A$. Then the total degree of \mathbf{v} is bounded by $(d + 1)(n - d)D(A)$.

Proof. Recall that the total degree is the ω -degree for $\omega = (1, \dots, 1)$. Then the total degree of a binomial $x^{\alpha^+} - x^{\alpha^-}$ is $\max \{\|\alpha^+\|_1, \|\alpha^-\|_1\}$.

If $\mathbf{v} \in C_A$, then $|v_i| \leq D(A)$ by Lemma 3.32, (3). Hence, $\|\mathbf{v}\|_1 \leq (d+1)D(A)$, by Lemma 3.32, (1).

Assume \mathbf{v} is not a circuit. By Lemma 3.32, (2), we write

$$\mathbf{v} = q_1 \mathbf{c}_1 + \dots + q_{n-d} \mathbf{c}_{n-d}.$$

Hence, we can write

$$\begin{aligned} \mathbf{v}^+ &= q_1 \mathbf{c}_1^+ + \dots + q_{n-d} \mathbf{c}_{n-d}^+ \\ \mathbf{v}^- &= q_1 \mathbf{c}_1^- + \dots + q_{n-d} \mathbf{c}_{n-d}^- \end{aligned}$$

If $q_i \geq 1$ for some i , then $\mathbf{v}^+ \geq \mathbf{c}_i^+$. Since \mathbf{v} is primitive, we have $\mathbf{v} = \mathbf{c}_i$, which means that \mathbf{v} is a circuit. This is a contradiction. Thus, we assume

$$q_i < 1 \quad \forall i.$$

Without loss of generality we assume that the total degree $x^{\mathbf{v}^+} - x^{\mathbf{v}^-}$ is given by $\|\mathbf{v}^+\|_1$. Then,

$$\|\mathbf{v}^+\|_1 \leq \sum_{i=1}^{n-d} q_i \|\mathbf{c}_i^+\|_1 < (n-d) \cdot \max \{\|\mathbf{c}_i^+\|_1\} \leq (n-d)(d+1)D(A).$$

□

Remark 3.34. The degree bound of a primitive vector is at most single exponential in n . Sturmfels conjectures that one can remove the $(n-d)$ factor.

4 Triangulation

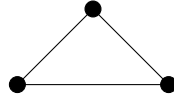
4.1 Simplicial complex, Stanley-Reisner complex and ideal

Definition 4.1 (Abstract simplicial complex). An abstract simplicial complex on $[n] = \{1, \dots, n\}$ is $\Delta \subset 2^{[n]}$ such that the following holds: if $\sigma \in \Delta$ and $F \subset \sigma$, then $F \in \Delta$.

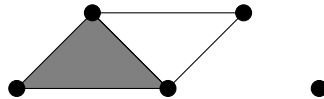
- Elements $\sigma \in \Delta$ are called **faces**.
- Maximal faces are called **facets**.
- If a face σ has $k+1$ elements, then it is called a k -face. We also say the **dimension of a face** σ is k .
- The **dimension** of Δ is defined as the maximal dimension of a face $\sigma \in \Delta$.

Example 4.2.

- $\Delta = \emptyset$
- $\Delta = 2^{[n]}$
- $\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ with facets $\{1, 2\}, \{1, 3\}, \{2, 3\}$.

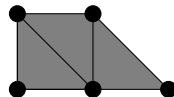


- Consider the following abstract simplicial complex



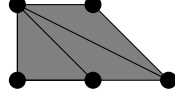
It has the following faces $\{1, 2, 4\}, \{4, 5\}, \{2, 5\}, \{3\}$.

- Consider the following triangulation

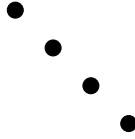


It has maximal cells $\{1, 2, 4\}, \{2, 4, 5\}, \{2, 3, 5\}$.

- Given a point configuration $A = (a_1, \dots, a_5)$ and $\omega = (1, 0, 0, 0, 1)$, we obtain a regular triangulation



- The twisted cube is given by $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$. Consider the point configuration given by A :



It has 4 regular triangulations:

$$\{\{1, 4\}\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}, \{\{1, 3\}, \{3, 4\}\}, \{\{1, 2\}, \{2, 4\}\}.$$

$\{\{1, 3\}, \{2, 4\}\}$ is not a *regular* triangulation.

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Definition 4.3 (Associated abstract simplicial complex). Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ be a point configuration. Let $\omega \in \mathbb{R}_{\geq 0}^n$. This induces a regular triangulation of A . To this regular triangulation we associate an abstract simplicial complex $\Delta_\omega(A)$ that consists of faces $\sigma \in 2^{[n]}$ that correspond to faces seen from below of the lifted point configuration. Algebraically, the **associated abstract simplicial complex** is defined as

$$\begin{aligned} \Delta_\omega(A) &= \left\{ \sigma \in 2^{[n]} : \exists \mathbf{c} \in \mathbb{R}^d \text{ with } \begin{bmatrix} \mathbf{c} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_i \\ w_i \end{bmatrix} = 0, \begin{bmatrix} \mathbf{c} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_j \\ w_j \end{bmatrix} < 0, \forall i \in \sigma, \forall j \notin \sigma \right\} \\ &= \left\{ \sigma \in 2^{[n]} : \exists \mathbf{c} \in \mathbb{R}^d \text{ with } \mathbf{c}\mathbf{a}_i = w_i, \mathbf{c}\mathbf{a}_j < w_j, \forall i \in \sigma, \forall j \notin \sigma \right\} \end{aligned}$$

Definition 4.4 (Initial complex). Let \prec be a monomial order. The **initial (abstract simplicial) complex** of a toric ideal I_A is defined as

$$\Delta_{\prec}(I_A) = \left\{ \sigma \in 2^{[n]} : \nexists f \in I_A \text{ with } \text{supp}(\text{in}_{\prec}(f)) = \sigma \right\}.$$

Proposition 4.5. $\Delta_{\prec}(I_A)$ is indeed an abstract simplicial complex.

Proof. We prove the claim by contraposition. Assume $\sigma \in \Delta_{\prec}(I_A)$ is a face and $F \subset \sigma$. Assume that there is some $f \in I_A$ such that $\text{supp}(\text{in}_{\prec}(f)) = F$. Define $\alpha = \sigma \setminus F$. Then $\text{supp}(\text{in}_{\prec}(x^\alpha \cdot f)) = \sigma$ and $x^\alpha \cdot f \in I_A$. \square

Our goal is to prove **Sturmfels Theorem** (1991): if ω represents $\text{in}_{\prec}(I_A)$, then

$$\Delta_{\prec}(I_A) = \Delta_{\omega}(A).$$

Definition 4.6 (Stanley-Reisner ideal). Let $\Delta \subset 2^{[n]}$ be an abstract simplicial complex. The **Stanley-Reisner ideal** $I(\Delta)$ of Δ is

$$I(\Delta) = \left(\prod_{i \in \sigma} x_i : \sigma \notin \Delta \right) \subset k[x_1, \dots, x_n].$$

The Stanley-Reisner ideal is a squarefree monomial ideal, i.e. it has squarefree monomial generators. A monomial x^α is called squarefree if $\alpha_i \in \{0, 1\}$ for all $i = 1, \dots, n$.

Remark 4.7. The Stanley-Reisner ideal is generated by the *minimal non-faces* of Δ .

Example 4.8.

- $I(2^{[n]}) = 0$
- $I(\emptyset) = (1)$
- ...

Remark 4.9 (Stanley-Reisner complex). We defined the following map:

$$\begin{aligned} \{\text{abstract simplicial complex in } [n]\} &\rightarrow \{\text{squarefree monomials in } k[x_1, \dots, x_n]\} \\ \Delta &\mapsto I(\Delta) \end{aligned}$$

This is even a bijection with an inverse given by

$$\Delta(I) = \left\{ \sigma \in 2^{[n]} : \prod_{i \in \sigma} x_i \notin I \right\}$$

for a squarefree ideal I . $\Delta(I)$ is called the **Stanley-Reisner complex**.

4.2 Sturmfels theorem

Note that the radical of a monomial ideal is squarefree. Square free ideals are radical.

Proposition 4.10. Let \prec be a monomial order and let I_A be a toric ideal. Then

$$\Delta_{\prec}(I_A) = \Delta(\sqrt{\text{in}_{\prec}(I_A)}).$$

Proof.

- \subset : Contraposition. Take $\sigma \notin \Delta(\sqrt{\text{in}_{\prec}(I_A)})$. This means that $\prod_{i \in \sigma} x_i^m \in \text{in}_{\prec}(I_A)$ for some $m \in \mathbb{N}$. So there exists $f \in I_A$ such that $\text{in}_{\prec}(f) = \prod_{i \in \sigma} x_i^m$. In particular, $\text{supp}(\text{in}_{\prec}(f)) = \sigma$. Hence, $\sigma \notin \Delta_{\prec}(I_A)$.

- \supset : Contraposition. Let $\sigma \notin \Delta_{\prec}(I_A)$. So there is some $f \in I_A$ such that $\text{supp}(\text{in}_{\prec}(f)) = \sigma$. Thus, $\text{in}_{\prec}(f) = \prod_{i \in \sigma} x_i^{\alpha_i}$. Then, $\prod_{i \in \sigma} x_i^m \in \text{in}_{\prec}(I_A)$ for $m = \max \alpha_i$.

□

Theorem 4.11 (Sturmfels 1991). If $\omega \in \mathbb{R}_{\geq 0}^n$ represents $\text{in}_{\prec}(I_A)$, then

$$\Delta_{\prec}(I_A) = \Delta_{\omega}(A).$$

Proof. It suffices to prove

$$I(\Delta_{\omega}(A)) = \sqrt{\text{in}_{\omega}(I_A)}$$

for generic ω . Applying $\Delta(\bullet)$ yields

$$\Delta_{\omega}(A) = \Delta(\sqrt{\text{in}_{\omega}(I_A)}) = \Delta(\sqrt{\text{in}_{\prec}(A)}) = \Delta_{\prec}(I_A).$$

Let's prove $I(\Delta_{\omega}(A)) = \sqrt{\text{in}_{\omega}(I_A)}$.

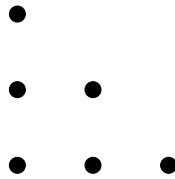
- \subset : Let $\sigma \notin \Delta_{\omega}(A)$, $k = |\sigma|$. This means that σ does not define a lower face of a regular triangulation induced by ω . So there is no $\mathbf{c} \in \mathbb{R}^d$ such that $\mathbf{c}\mathbf{a}_i = \omega_i$ and $\mathbf{c}\mathbf{a}_j < \omega_j$ for all $i \in \sigma$ and $j \notin \sigma$...
- \supset : carefully trace the proof back

□

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Example 4.12. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \in \mathbb{Z}^{3 \times 6}$. Here is the point configuration

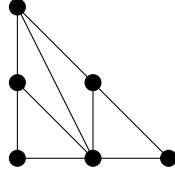
A cut at $x = 1$:



A toric ideal is given by

$$I_A = (ac - b^2, ae - bd, af - d^2, cd - be, ef - e^2, de - bf).$$

Here is regular triangulation induced by $\omega = (10, 1, 10, 5, 1, 1)$:

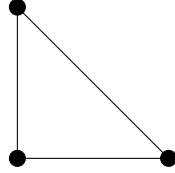


This triangulation is unimodular. We compute $\text{in}_\omega(I_A) = \sqrt{\text{in}_\omega(I_A)} = (ac, ae, af, cd, cf, de)$. We claim (without proof):

$$\text{in}_\omega(I_A) \text{ is squarefree} \iff \Delta_\omega \text{ is unimodular}$$

Here is a different triangulation: Given $\omega' = (1, 10, 1, 10, 10, 1)$ we obtain a non-unimodular triangulation. Moreover, we see that $\text{in}_{\omega'}(I_A) = (b^2, bd, d^2, be, e^2, de)$ is not squarefree.

Note that $\sqrt{\text{in}_{\omega'}(I_A)} = (b, d, e)$. The triangulation $\Delta_{\omega'}(A)$ has unique facet $\{a, c, f\}$.



The Stanley-Reisner ideal yields $I(\Delta_{\omega'}(A)) = (b, d, e)$. We see that the formula

$$I(\Delta_{\omega'}(A)) = \sqrt{\text{in}_{\omega'}(I_A)}$$

holds.

Example 4.13 (Twisted cube). Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$. Using a computer we compute the Gröbner fan of I_A . The Gröbner fan has 8 maximal cones that correspond to the following initial ideals

1. (bd, ad, ac)
2. (ad, bd, b^2)
3. (ad^2, bd, bc, b^2)
4. (bd, b^2, bc, c^3)
5. (ad, ac, c^2)
6. (a^2d, ac, c^2, bc)
7. (ac, c^2, bc, b^3)
8. (c^2, bc, b^2)

We can draw the Gröbner fan embedded in its 2-dimensional linearity space

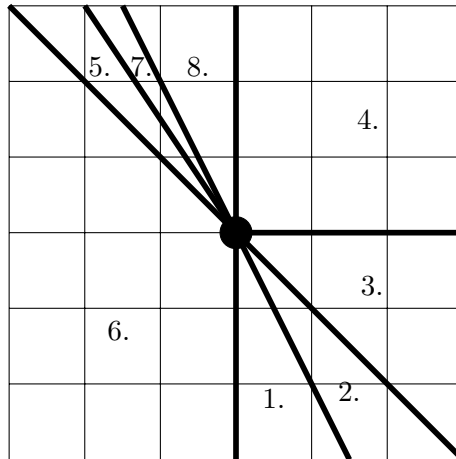
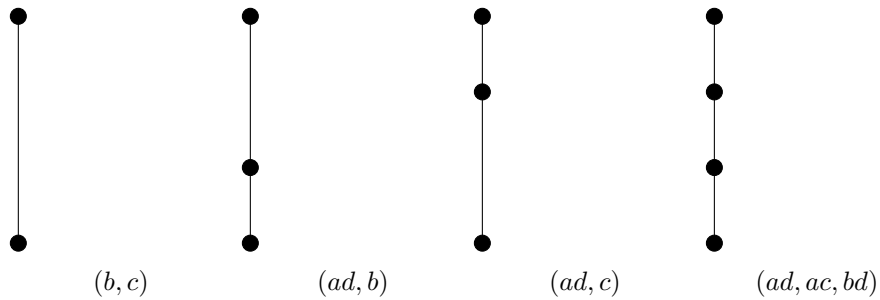


Figure 5: Gröbner fan embedded in its linearity space

If we take the radical ideals of the 8 distinct initial ideals, we obtain 4 distinct radical ideals: $(bd, ad, ac), (ad, b), (b, c), (c, ad)$. Since we have $I(\Delta_\omega(A)) = \sqrt{\text{in}_\omega(I_A)}$, there must exist 4 regular triangulations of A . Each regular triangulation corresponds to squarefree monomial ideals.



Let's group together the weights that induce the same regular triangulation of A .

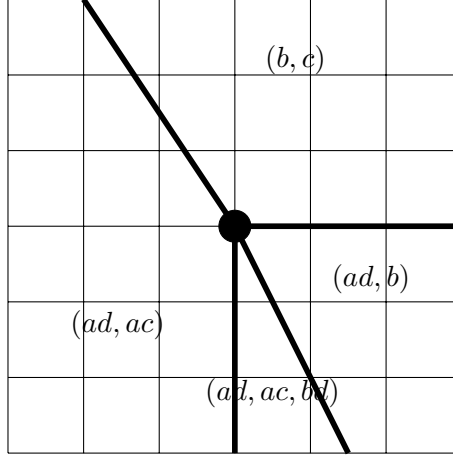


Figure 7: Secondary fan: Each maximal cone corresponds to a regular triangulation.

4.3 Secondary fan

Let's define the secondary fan. Given $A \in \mathbb{Z}^{d \times n}$, we define an equivalence relation

$$\omega \sim \omega' \iff \Delta_\omega(A) = \Delta_{\omega'}(A)$$

for $\omega, \omega' \in \mathbb{R}^n$. By Sturmfels theorem, the equivalence class $C_A[\omega]$ consists of a union of Gröbner cones $C_{I_A}[\omega] = \{\omega' \in \mathbb{R}^n : \text{in}_\omega(I_A) = \text{in}_{\omega'}(I_A)\}$ with same radical initial ideal $\sqrt{\text{in}_\omega(I_A)}$, i.e.

$$C_A[\omega] = \left\{ \omega' \in \mathbb{R}^n : \sqrt{\text{in}_\omega(I_A)} = \sqrt{\text{in}_{\omega'}(I_A)} \right\}.$$

There are adjacent cones in $\text{GFan}(I_A)$ because $C_A[\omega]$ is convex.

Proposition 4.14. $C_A[\omega]$ is convex.

Proof. Use the definition

$$\Delta_\omega(A) = \left\{ \sigma \in 2^{[n]} : \exists \mathbf{c} \in \mathbb{R}^d \text{ with } \mathbf{c}\mathbf{a}_i = w_i, \mathbf{c}\mathbf{a}_j < w_j, \forall i \in \sigma, \forall j \notin \sigma \right\}$$

and take two $\sigma, \sigma' \in C_A[\omega]$. Show that $\lambda\sigma + (1-\lambda)\sigma' \in C_A[\omega]$ which is easy to show. \square

Definition 4.15 (Secondary fan). The secondary fan of $A \in \mathbb{Z}^{d \times n}$ is the polyhedral fan S_A which has

$$\left\{ \overline{C_A[\omega]} : \omega \in \mathbb{R}^n \right\}$$

as maximal cones.

Remark 4.16. Clearly, $\omega \in \mathbb{R}^n$, $C_I[\omega] \subset C_A[\omega]$. Thus, the Gröbner fan of I_A is a refinement of S_A .

Remark 4.17. Gel'fand, Kapranov, Zelevinsky proved in 1994 that there exists a polytope $\sum(A)$ such that $\mathcal{N}(\sum(A)) = S_A$ if $(1, \dots, 1) \in \text{rowspan}(A)$ (this condition makes I_A homogeneous). This is called the **secondary polytope** of A .

Remark 4.18. A is called unimodular if all triangulations of A are unimodular.

If A is unimodular, then $\text{GFan}(I) = S_A$ and $\text{State}(I_A) = \sum(A)$.

If A is unimodular, then $C_A = \text{Gr}(A)$.

5 Tropical Geometry

See last two lecture notes uploaded on ISIS.

6 Review

- What is a monomial order?
- How to characterize a well-order?
- When is a total order a well-order?
- Give examples of monomial orders
- State the multivariate division algorithm and prove the correctness
- What is Gröbner basis? What is a minimal Gröbner basis? A reduced Gröbner basis?
- Does a Gröbner basis always exist?
- Is the Gröbner basis really a basis?
- State the main theorem about Gröbner basis
- Does a reduced Gröbner basis always exist?
- Prove the uniqueness of the reduced Gröbner basis
- State Buchberger's criterion and prove it
- State Buchberger's algorithm
- What are standard monomials?
- Prove Macaulay's Theorem
- Prove the two corollaries of Macaulay's Theorem
- Prove that every ideal has only finitely many distinct initial ideals
- Does a universal Gröbner basis always exist?
- $\text{in}_{\prec_{\omega}}(I) = \text{in}_{\prec}(\text{in}_{\omega}(I))$
- Give a Gröbner basis of $\text{in}_{\omega}(I)$.
- Representation of initial ideals by weight vectors
- Explain the relationship between $\text{New}()$, in , face
- $C_I[\omega]$ is a polyhedral cone
- Explain the formula $C_I[\omega] = N_P(\text{face}_{\omega}(P))$ with $P = \text{New}(\prod_{g \in \mathcal{G}} g) = \sum \text{New}(g)$
- Is the Gröbner fan really a fan?

- How to compute the Gröbner fan of a principal ideal?
- Can there be infinitely many Gröbner cones $\overline{C_I[\omega]}$?
- How to draw the Gröbner fan?
- Characterization of ω -homogeneous ideals
- Gröbner region of homogeneous ideals
- Newton polytope and state polytope
- Existence of state polytope
- Give a formula for the state polytope
- What is a toric ideal?
- Is it prime?
- Can a toric ideal contain a monomial?
- Explain the relationship between toric ideals and linear combinations of binomials
- How does the reduced Gröbner basis of a toric ideal look like?
- How to compute the Gröbner basis of a toric ideal?
- What is a Graver basis?
- What is the relationship between a Graver basis and a Gröbner basis?
- What is a rational polyhedral cone?
- What is a circuit?
- Relationship between circuits, Gröbner basis and primitive binomials
- Can we say something about the total degree of a primitive binomial?
- Let's move on to triangulations: What is an abstract simplicial complex?
- What is the associated abstract simplicial complex?
- What is the initial complex?
- What is the Stanley-Reisner complex / ideal?
- What is Sturmfels theorem?

A Appendix: Rings and ideals

We assume that all rings are commutative and have 1.

Definition A.1 (Ideal). Let R be a ring. An **ideal** $I \subset R$ is an additive subgroup that also satisfies $ab \in I, \forall b \in R, \forall a \in I$.

Definition A.2 (Generating system). The ideal generated by $A \subset R$ is defined as

$$(A) = \{Ra_1 + \cdots + Ra_n : a_i \in A, n \in \mathbb{N}\}$$

It is the smallest ideal containing A . A **principal ideal** I is an ideal generated by one element. We say an ideal is **finitely generated** if $I = (A)$ for some finite A . We call such A a **basis** of I .

Remark A.3. The ideal (\emptyset) generated by the empty set is the zero ideal.

Remark A.4.

1. $I + J = \{a + b : a \in I, b \in J\}$ is an ideal
2. $I \cap J$ is an ideal
3. Products of ideal need not be ideals, so we define IJ as the ideal generated by $\{ab : a \in I, b \in J\}$. It is always true that $IJ \subset I \cap J$. The converse fails for $I = J = (2) \subset \mathbb{Z}$.

Remark A.5. An ideal $I \subset R$ always contains 0.

Remark A.6. An ideal $I \subset R$ contains 1 if and only if $I = R$.

Theorem A.7 (Hilbert's Basis Theorem). Every ideal $I \subset k[x_1, \dots, x_n]$ is finitely generated.

B Appendix: Farkas Lemma

Proposition B.1. Given $A \in \mathbb{R}^{d \times n}$ and $b \in \mathbb{R}^d$, exactly one of the following holds

- there exists $x \in \mathbb{R}_{\geq 0}^n$ such that $Ax = b$ or
- there exists $c \in \mathbb{R}^d$ such that $c^T A \geq 0$ and $c^T b < 0$.

Proposition B.2. Given $A \in \mathbb{R}^{d \times n}$ and $b \in \mathbb{R}^d$, exactly one of the following holds

- there exists $x \in \mathbb{R}_{\geq 0}^n$ such that $Ax \leq b$ or
- there exists $c \in \mathbb{R}_{\geq 0}^d$ such that $c^T A \geq 0$ and $c^T b < 0$.

C Appendix: Polyhedra, cones and fans

We quickly summarize all important facts regarding polyhedra, cones and fans.

Definition C.1 (Convex cone). A set $C \subset \mathbb{R}^n$ is called a **convex cone** if for every $x, y \in C$ and $\lambda > 0$ we have $x + y \in C$ and $\lambda x \in C$.

Definition C.2 (Conic hull). Given vectors $x_1, \dots, x_k \in \mathbb{R}^n$ the **conic hull** is defined to be the set that contains all vectors of the form

$$\alpha x_1 + \dots + \alpha_k x_k$$

where $\alpha \geq 0$.

Definition C.3 (Face of a cone). A face of a cone C is of the form $\text{face}_\omega(C) = \{x \in C : \omega x \geq \omega y, \forall y \in C\}$.

Definition C.4 (Polyhedral cone). A cone is **polyhedral** if $C = \text{cone}(v_1, \dots, v_k)$. Equivalently by Minkowski-Weyl, a cone C is polyhedral if C is a polyhedron, that is, it is the intersection of a finite number of half-spaces, each containing 0 on their boundary.

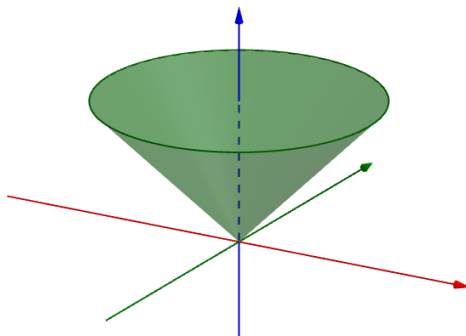


Figure 8: A cone that is not polyhedral

Definition C.5 (Polyhedral fan). A polyhedral fan is a collection $\mathcal{F} = \{C_1, \dots, C_n\}$ of polyhedral cones such that

1. every face of a cone in \mathcal{F} is also a cone in \mathcal{F} , and
2. the intersection of any two cones in \mathcal{F} is a common face.

The **support of a fan** is $\bigcup C_i$. If the support of a fan is \mathbb{R}^n then we say that \mathcal{F} is **complete**.

Example C.6 (Normal fan). Let $P \subset \mathbb{R}^n$ be a polytope. Its normal fan \mathcal{N}_P consists of normal cones $\{\mathcal{N}_P(F) : F \neq \emptyset \text{ is a face of } P\}$ where $\mathcal{N}_P(F) = \{\omega \in \mathbb{R}^n : \text{face}_\omega(P) = F\}$.

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