Fundamental Models

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1 Operations on Outcomes

Proposition 1. *The sum of two outcomes is an outcome.*

Proof. Let $w, v \in \mathbb{Z}^{V_d}$ be outcomes. Let p be a pascal form. Then $p(w+v) = p(v) + p(w) = 0$ because $p(w) = 0$ and $p(v) = 0$. This proves that $w + v$ is an outcome. $\square$

Definition 2. *Let $w \in \mathbb{Z}^{V_d}$ be an outcome. Let $v \in \mathbb{Z}^{d+2}$ be some vector. We define $[v \mid w]$ to be the configuration obtained by left-appending v to w in the following way:*

$$\begin{array}{lllll}
v_{d+1} & & & & \\
v_d & w_{0,d} & & & \\
v_{d-1} & w_{0,d-1} & w_{1,d-1} & & \\
\vdots & \vdots & \vdots & \ddots & \\
v_0 & 0 & w_{1,0} & \dots & w_{d,0}
\end{array}$$

Bottom-appending $\left[\frac{w}{v}\right]$ is defined in the following way:

$$\begin{array}{lllll}
w_{0,d} & & & & \\
w_{0,d-1} & w_{1,d-1} & & & \\
\vdots & \vdots & \ddots & & \\
0 & w_{1,0} & \dots & w_{d,0} & \\
v_0 & v_1 & \dots & v_d & v_{d+1}
\end{array}$$

Proposition 3. *Left-appending the vector $v = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{Z}^{d+2}$ to a valid outcome $w \in \mathbb{Z}^{V_d}$ with $w_{00} = -1$ yields a valid outcome.*

Proof. We consider diagonal pascal forms in $\mathbb{Z}^{V_{d+2}}$ to show that $[v \mid w]$ is an outcome. Let p_k be the k-th diagonal pascal form. The case $k = 0$ is clear since $p_0([v \mid w]) = 0$ is easy to be seen. For $k = 1, \ldots, d+1$, we have

$$\begin{aligned} p_k([v \mid w]) &= -\binom{d+1}{k} + \binom{d}{k} + \sum_{(i,j)\in V_{d+1}, i>0} \binom{d+1-(i+j)}{k-i} w_{i-1,j} \\ &= -\binom{d+1}{k} + \binom{d}{k} + \sum_{(i,j)\in V_d} \binom{d-i-j}{k-(i+1)} w_{i,j}. \end{aligned}$$

By substituting $k = \tilde{k} + 1$ we obtain that

$$\sum_{(i,j)\in V_d} \binom{d-i-j}{\tilde{k}-i} w_{i,j} = \binom{d}{\tilde{k}} \quad \forall \tilde{k} = 0, \ldots, d$$

since w is an outcome. Plugging this expression back into $p_k([v \mid w])$ and using Pascal's rule yields $p_k([v \mid w]) = 0$. Hence, $[v \mid w]$ is an outcome.

Finally, $[v \mid w]$ is clearly valid because w is valid. $\square$

Corollary 4. *Left-appending the vector* $v = \begin{bmatrix} w_{00} & 1 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{Z}^{d+2}$ *to a valid non-initial outcome* $w \in \mathbb{Z}^{V_d}$ *yields a valid outcome.*

Proof. The proof is similar to the previous one except that we carry some coefficient w_{00} through the calculations. $\square$

Corollary 5. *Bottom-appending the vector* $v = \begin{bmatrix} w_{00} & 1 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{Z}^{d+2}$ *to a valid non-initial outcome* $w \in \mathbb{Z}^{V_d}$ *yields a valid outcome.*

Proof. We use symmetry. $\square$

2 Binomial Models are Fundamental

Fix some degree $d \in \mathbb{Z}_{\geq 0}$.

Definition 6. *Define* $T_d = \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid i + j = d\}$.

Lemma 7. *Let* $w \in \mathbb{Z}^{V_d}$ *be a nonzero valid outcome. If* $\operatorname{supp}^+(w) \subset T_d$, *then we have equality, i.e.* $\operatorname{supp}^+(w) = T_d$.

Proof. Fix some vertex $(i,j) \in T_d$. Define p to be the i-th diagonal pascal form. Since w is an outcome, we have that $p(w) = p_{0,0}w_{0,0} + p_{i,j}w_{i,j} = 0$. This is only the case if $w_{i,j} > 0$ because w is nonzero and valid. Hence, $T_d \subset \operatorname{supp}^+(w)$. $\square$

Proposition 8. *Binomial configurations are fundamental.*

Proof by Contradiction. Let $w \in \mathbb{Z}^{V_d}$ be a binomial configuration. Suppose w is not fundamental, i.e. $w = \alpha x + \beta y$ for valid outcomes x and y with $\mathrm{supp}^+(x), \mathrm{supp}^+(y) \subsetneq \mathrm{supp}^+(w) = T_d$ and positive $\alpha, \beta \in \mathbb{Q}_{>0}$. If $x = 0$, then $y = \beta^{-1} w$. Otherwise by the previous Lemma, we have $\mathrm{supp}^+(x) = T_d$. In either case, we have a contradiction to $\mathrm{supp}^+(x), \mathrm{supp}^+(y) \subsetneq T_d$. $\square$

Example 9. *Here is an example of a binomial configuration.*

```
 1
 *  3
 *  *  3
-1  *  *  1
```

3 Invariant Operations

Proposition 10. *Let $w \in \mathbb{Z}^{V_d}$ be a fundamental outcome. Then, λw is fundamental for all $\lambda \in \mathbb{Q}_{>0}$.*

Proof by Contraposition. Assume λw is not fundamental. Then, $\lambda w = \alpha x + \beta y$ for positive $\alpha, \beta \in \mathbb{Q}_{>0}$ and valid outcomes x, y with $\mathrm{supp}^+(x), \mathrm{supp}^+(y) \subsetneq \mathrm{supp}^+(\lambda w)$. Write $w = \frac{\alpha}{\lambda} x + \frac{\beta}{\lambda} y$, which shows that w is not fundamental. $\square$

Proposition 11. *Let $w \in \mathbb{Z}^{V_d}$ be a fundamental outcome, and let $v \in \mathbb{Z}^{V_d}$ be a valid outcome. If $\mathrm{supp}^+(w) = \mathrm{supp}^+(v)$, then $v = \lambda w$ for some $\lambda \in \mathbb{Q}_{>0}$.*

Proof. Assume that $\mathrm{supp}^+(w) = \mathrm{supp}^+(v)$. Then there exist fundamental statistical models $\mathcal{M}_1 = (w_v, i_v, j_v)_{v=0}^n$ and $\mathcal{M}_2 = (w'_v, i_v, j_v)_{v=0}^n$ that are associated to w and v respectively. Since $\mathcal{M}_1$ is fundamental, the values $(w_v)_{v=0}^n$ are uniquely determined given the values $(i_v, j_v)_{v=0}^n$. Thus, we conclude that $w'_v = w_v$ for all $v = 0, ..., n$. The rest follows from Proposition 4.5. $\square$

Corollary 12. *Let $w \in \mathbb{Z}^{V_d}$ be a fundamental outcome, and let $v \in \mathbb{Z}^{V_d}$ be a valid outcome. If $\mathrm{supp}^+(w) = \mathrm{supp}^+(v)$, then v is fundamental.*

Next, we show that certain unsplitting moves preserve the fundamental property.

Example 13. *Consider the fundamental outcome below.*

```
 1
 *  3
 *  *  3
-1  *  *  1
```

An unsplitting move at vertex $(0,2)$ *yields the outcome below.*

```
 *
 1  2
 *  *  3
-1  *  *  1
```

This outcome is fundamental as we will show in the next proposition.

Proposition 14. *Let* $w \in \mathbb{Z}^{V_d}$ *be a fundamental integral outcome. Let* u *be an unsplitting move at vertex* (i,j)*. If* $u(w)$ *is valid and* $|\mathrm{supp}^+(u(w))| = |\mathrm{supp}^+(w)|$*, then* $u(w)$ *is fundamental.*

Proof by Contradiction. Let $w \in \mathbb{Z}^{V_d}$ be fundamental. For the sake of contradiction, assume $u(w)$ is not fundamental, i.e. $u(w) = \alpha x + \beta y$ for valid outcomes x and y with $\mathrm{supp}^+(x), \mathrm{supp}^+(y) \subsetneq \mathrm{supp}^+(u(w))$. If we denote the splitting move at vertex (i,j) by s, then $w = s(u(w)) = s(\alpha x + \beta y) = \alpha s(x) + \beta y = \alpha x + \beta s(y)$.

Without loss of generality, we assume that $s(x)$ is valid, the reason being that $u(w)$ is valid, which implies $w_{i,j} \geq 1$, and in turn $x_{i,j} \geq 1$ or $y_{i,j} \geq 1$. Thus, we found a decomposition $w = \alpha s(x) + \beta y$ into valid outcomes.

Further examination of this decomposition reveals that $\mathrm{supp}^+(y) \subsetneq \mathrm{supp}^+(w)$ because $|\mathrm{supp}^+(u(w))| = |\mathrm{supp}^+(w)|$ leads to one of the following three cases: (A) we have $\mathrm{supp}^+(u(w)) = \mathrm{supp}^+(w)$, or (B) we have $w_{i,j} = 0$ and $w_{i,j+1} = 1$, which implies $(s(x))_{i,j}$, so $(s(x))_{i,j} = 1$ and therefore $y_{i,j} = 0$. □

Left-appending is also an invariant operation.

Proposition 15. *Left-appending the unit vector* $\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{Z}^{d+2}$ *to a fundamental outcome* $w \in \mathbb{Z}^{V_d}$ *yields a fundamental outcome.*

Proof. □