Chipsplitting Games: A Combinatorial Approach to Classifying One-Dimensional Discrete Statistical Models with Rational Maximum Likelihood Estimator

by

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Sofern generative KI-Tools verwendet wurden, habe ich Produktnamen, Hersteller, die jeweils verwendete Softwareversion und die jeweiligen Einsatzzwecke (z.B. sprachliche Überprüfung und Verbesserung der Texte, systematische Recherche) benannt. Ich verantworte die Auswahl, die Übernahme und sämtliche Ergebnisse des von mir verwendeten KI-generierten Outputs vollumfänglich selbst.

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Ich erkläre weiterhin, dass ich die Arbeit in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegt habe.

Berlin, den

Zusammenfassung in deutscher Sprache

Die vorliegende Ausarbeitung setzt die Forschung von Arthur Bik und Orlando Marigliano zur Klassifizierung ein-dimensionaler diskreter statistischer Modelle mit rationalen Maximum Likelihood Schätzern unter Verwendung fundamentaler Modelle fort. Wir erzielen bedeutende Fortschritte beim Beweis zur endlichen Anzahl der fundamentalen Modelle im Wahrscheinlichkeitssimplex Δ_5. Zudem bestimmen wir die Anzahl der fundamentalen Modelle im Simplex Δ_6 mit einem maximalen Grad von 11.

Abstract

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This paper continues the research of Arthur Bik and Orlando Marigliano on the classification of one-dimensional discrete statistical models with rational maximum likelihood estimators using fundamental models. We present a missing proof of an algorithm from their work. Furthermore, we make significant progress in proving the finite number of fundamental models in the probability simplex Δ_5. We also determine the number of fundamental models in the simplex Δ_6 with a maximum degree of 11.

Contents

Chapter 1

Introduction

In statistics we come across various collections of probability distributions, such as the normal distribution, Poisson distribution, and binomial distribution. These distributions are used to model random variables in applications, and are referred to as *statistical models*. Precisely, a statistical model is just a set of probability distributions. If the set contains only discrete distributions, we call it a *discrete statistical model*. In this case, discrete statistical models are just subsets of the probability simplex $\Delta_n := \{p \in \mathbb{R}^{n+1} \mid \sum p_i = 1\}$.

A discrete distribution $p \in \mathcal{M} \subset \Delta_n$ from a discrete statistical model encapsulates the probabilities of observing the states $0, \dots, n$, i.e. if $X \in \{0, \dots, n\}$ is a discrete random variable, then the state $X = i$ occurs with probability p_i for all $i = 0, \dots, n$. Say we have a binomial random variable X with $n + 1$ states, then $p_i = \binom{n}{i}\theta^i(1-\theta)^{n-i}$ computes the probability of observing i successes in n trials with success probability $\theta \in [0, 1]$. The set $\mathcal{M}$ of all probability distributions of that form, i.e. $\mathcal{M} = \left\{(\binom{n}{i}\theta^i(1-\theta)^{n-i})_{i=0}^n \mid \theta \in [0,1]\right\}$, is our first example of a discrete statistical model, and is known as the *binomial model*.

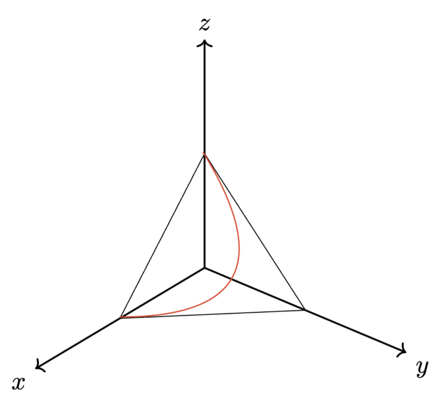

Figure 1.1: This figure shows the probability simplex Δ_2 with the binomial model (red curve). Every point on the curve is a binomial distribution.

Given a statistical model $\mathcal{M} \subset \Delta_n$ and data $u \in \mathbb{N}^{n+1}$, a typical problem in statistics is to find a distribution from a statistical model that best describes the data. "Best" can mean a lot of things, but in *maximum likelihood estimation* it means finding the distribution that maximizes the probability of observing the data; the map $\Phi : \Delta_n \to \mathcal{M}, u \mapsto \hat{p}$ that assigns the data u to a distribution $\hat{p} \in \mathcal{M}$ from the statistical model is called the *maximum likelihood estimator (MLE)*. This map is characterized by the property that $\hat{p}$ maximizes the log-likelihood function $\ell(p) = \sum u_i \log p_i$ for all $p \in \mathcal{M}$.

We focus on **one-dimensional discrete statistical models with rational MLE**. These are models $\mathcal{M}$ satisfying

- $\mathcal{M} = \text{image}(p)$ for some rational map $p = (p_0, \dots, p_n) : I \to \Delta_n$ where p_i is rational, $I \subset \mathbb{R}$ is a union of closed intervals and $p(\partial I) \subset \partial \Delta_n$,
- all the $n+1$ coordinates of the maximum likelihood estimator Φ are rational functions in the data u.

There are two intriguing questions to ask about statistical models with rational MLE: the first one is about which *form* do the maximum likelihood estimators take; the second one is more concerned with the *classification* of the statistical models, i.e. can we divide these models into easier to understand classes? An answer to the first question was given by June Huh. He showed that if Φ is rational, then each of its coordinates is an alternating product of linear forms with numerator and denominator of the same degree, see [5, 3]. For the second question, Arthur Bik and Orlando Marigliano classified all one-dimensional discrete statistical models with rational MLE using *fundamental models* [2].

This thesis continues the work of Bik and Marigliano. In the first half, we present their classification results on how fundamental models serve as the building blocks of one-dimensional discrete models with rational MLE. In the second half, we establish and extend their finding that there are only finitely many fundamental models within the probability simplices Δ_n for $n \leq 4$. Due to the complexity of the problem, the cases $n \geq 5$ were left open. We make progress for $n = 5$ by reducing the number of cases to check from 300,000 to 12,000. Additionally, this thesis introduces new results on the number of fundamental models in Δ_6, with a maximum degree of eleven, and provides an algorithm for solving non-trivial hyperfield linear systems, which is essential to all the computational work presented.

The outline of this thesis is as follows:

- Chapter 2 provides a classification of statistical models using fundamental models.
- Chapter 3 introduces chipsplitting games and establishes the connection to fundamental models via chipsplitting outcomes.
- Chapter 4 develops tools for analyzing valid chipsplitting outcomes, and Chapter 5 applies these tools to prove that the degree of valid outcomes with positive support size up to three is bounded.

- Chapter 6 extends the tools from Chapter 4 to prove the boundedness of degree for valid outcomes with positive support size four, and Chapter 7 uses these tools to establish a bound on the degree for positive support size four outcomes.
- Chapter 8 introduces the final tool to tackle outcomes with positive support size five, and Chapter 9 completes the proof for positive support size five outcomes.
- Chapter 10 presents new techniques to reduce the number of cases that need to be analyzed for proving that the degree of valid outcomes with support size six is bounded.
- Chapter 11 computes the number of fundamental outcomes.
- Chapter 12 concludes with a discussion on future research directions and the implications of the findings.

The source code for the computations discussed in this thesis is available at [7].

Chapter 2

Classification with Fundamental Models

In this chapter we present the classification of one-dimensional discrete statistical models with rational maximum likelihood estimator (MLE) using fundamental models. The classification is due to Arthur Bik and Orlando Marigliano [2].

> **Problem statement:** Can we find a class of easy to understand models that serve as building blocks for all one-dimensional discrete statistical models with rational MLE?

The answer to this question are *reduced* and *fundamental models*.

2.1 Parametrization

It turns out that one-dimensional discrete statistical models with rational MLE admit the following parametrization.

Proposition 2.1. *Let $\mathcal{M}$ be a one-dimensional discrete statistical models with rational maximum likelihood estimator. Then, there exists a map of the form*

$$p : [0,1] \to \Delta_n, \quad \theta \mapsto (w_k \theta^{i_k}(1-\theta)^{j_k})_{k=0}^n$$
$$i_k, j_k \in \mathbb{Z}_{\geq 0},\ w_k \in \mathbb{R}_{>0} \quad \forall k = 0, \dots, n$$

such that $\mathcal{M} = \operatorname{image}(p)$.

We introduce some notation to simplify the proof of Proposition 2.1. Let $\mathcal{M} \subset \Delta_n$ be a one-dimensional discrete statistical model parametrized by rational functions $p_0 = \frac{g_0}{h_0}, \dots, p_n = \frac{g_n}{h_n}$. Define b to be the least common multiple of $h_0, \dots, h_n$ and $a_i := bp_i$. Since $\sum p_k = 1$, we can multiply by b to obtain $\sum a_k = b$. We see that the polynomials $a_0, \dots, a_n, b$ determine the statistical model $\mathcal{M}$, and have no common factors. The log-likelihood function is then given by $\ell(p) = \sum u_i \log p_i = \sum u_i \log \frac{a_i}{b} = \sum u_i \log a_i - \sum u_i \log b$.

To find the maximum likelihood estimator, we need find all critical points of the log-likelihood function. This is equivalent to finding the roots of the gradient of the log-likelihood function

$$\ell(p(\theta))' = \sum u_k \frac{a_k'}{a_k} - \sum u_k \frac{b'}{b} = 0. \tag{2.1}$$

These equations are called the *score equations* in algebraic statistics, and the number of complex solutions to these equations for general data $u \in \mathbb{C}^{n+1}$ is called the *maximum likelihood degree* of the statistical model. This ML degree has an important meaning in algebraic statistics, as it determines the complexity of the model. We have the following relationship between the ML estimator and the ML degree.

Proposition 2.2. *Having rational maximum likelihood estimator can be expressed equivalently by saying that the maximum likelihood degree of the statistical model is one.*

Proof. Refer to [3] for a proof. □

To prove Proposition 2.1, we need the following lemma.

Lemma 2.3. *If $\mathcal{M}$ has rational MLE, then there are exactly two distinct complex linear factors in $a_0, \dots, a_n$, and b.*

Proof. We prove the lemma in three steps:

- Let f be the product of all distinct complex linear factors in $a_0, \dots, a_n, b$. If we multiply the score equations (2.1) by f, we get $f \cdot \ell(p(\theta))' = \sum u_k f \frac{a_k'}{a_k} - \sum u_k f \frac{b'}{b} = 0$. Note that every linear factor of a_k with multiplicity m occurs in a_k' with multiplicity $m-1$; thus every summand of $\frac{a_k'}{a_k}$ is of the form $\frac{\lambda}{(x-\xi)}$, where $\lambda \in \mathbb{R}$ and $x - \xi$ is some linear factor of a_k; hence $f \cdot \frac{\lambda}{(x-\xi)}$ is of degree $\deg(f) - 1$, and therefore $f \cdot \ell(p(\theta))'$ is of degree $\deg(f) - 1$.

- We claim that the roots of $\ell(p(\theta))'$ are the same as the roots of $f \cdot \ell(p(\theta))'$. Assume we have shown this claim. By Proposition 2.2 the ML degree is one. So, $\ell(p(\theta))'$ has one root. Thus, $f \cdot \ell(p(\theta))'$ has one root, and therefore $f \cdot \ell(p(\theta))'$ is of degree one. This implies that $\deg(f) = 2$ with the previous step. Thus, there are exactly two distinct complex linear factors in $a_0, \dots, a_n$, and b.

- It remains to show that the roots stay the same. Clearly, every root of $\ell(p(\theta))'$ is a root of $f \cdot \ell(p(\theta))'$. Conversely, we want to show that no new roots are introduced when multiplying by f, i.e. roots of f are not roots of $f \cdot \ell(p(\theta))'$. To do so, we rewrite $f \cdot \ell(p(\theta))' = \sum_{k=0}^{n} u_k f \frac{a_k'}{a_k} - \sum_{k=0}^{n} u_k f \frac{b'}{b} = \sum_{k=0}^{n+1} v_k f \frac{c_k'}{c_k}$ with $v_k := u_k, c_k := a_k$ for $k = 0, \dots, n$, and $v_{n+1} := -\sum_{k=0}^{n} u_k$, $c_{n+1} := b$.

  Let q be a complex linear factor of f. We define polynomials $r_0, \dots, r_{n+1}$ and r such that $c_k = q^{l_k} r_k$, $f = qr$, and $r_0, \dots, r_{n+1}, r$ do not have q as a factor. Then, for

$k = 0, \ldots, n+1$ we have $f\frac{c_k'}{c_k} = qr \cdot \frac{l_k q^{l_k - 1} q' r_k + q^{l_k} r_k'}{q^{l_k} r_k} = qr\frac{l_k q'}{q} + qr\frac{r_k'}{r_k} \equiv rl_k q' \pmod q$. Thus, we obtain $f \cdot \ell(p(\theta))' \equiv rq' \sum_{k=0}^{n+1} v_k l_k \equiv rq' \sum_{k=0}^{n} v_k(l_k - l_{n+1}) \pmod q$. Note that by definition of l_k, a value of $l_k = 0$ means that q is not a factor of c_k. By definition of f, at least one $l_k > 0$. On the other hand, not all l_k can be positive since $a_0, \ldots, a_n, b$ share no common factors. Hence, not all $l_k - l_{n+1} = 0$ vanish. Hence, for generic data u we assume $\sum_{k=0}^{n} v_k(l_k - l_{n+1}) \neq 0$. This with $q'r \not\equiv 0 \pmod q$ implies that q is not a complex linear factor of $f \cdot \ell(p(\theta))'$. We showed that the roots of f are not roots of $f \cdot \ell(p(\theta))'$.

□

Equipped with the lemma, we can now prove Proposition 2.1.

Proof. First, we show that I is a single closed real interval and not a union of closed intervals. For the sake of contradiction assume that $I = \bigcup_k I_k$ is a union of closed disjoint intervals. By definition of $\mathcal{M}$ we know that $p(\partial I) \subset \partial \Delta_n$. Thus, there exist $\theta_1, \theta_2 \in \partial I_0$ and $\theta_3, \theta_4 \in \partial I_1$ with $p_i(\theta_1) = p_i(\theta_2) = 0$ and $p_j(\theta_3) = p_j(\theta_4) = 0$ for some $i, j = 0, \ldots, n$. Note that θ_1, θ_2 are roots of $\frac{a_i}{b}$ and θ_3, θ_4 are roots of $\frac{a_j}{b}$. By Lemma 2.3 exactly two distinct complex linear factors occur in $a_0, \ldots, a_n, b$. Hence, $\theta_3 = \theta_1$ or $\theta_3 = \theta_2$. Contradiction for I_0 and I_1 are disjoint.

The previous argument shows that $I = [\alpha, \beta]$ is a real single closed interval. Thus, the roots of $a_0, \ldots, a_n, b$ are real and take values in $\partial I = \{\alpha, \beta\}$. By a suitable parametrization, we can assume without loss of generality that $I = [0, 1]$. We can now write the polynomials $a_0, \ldots, a_n, b$ as $a_k(\theta) = w_k \theta^{i_k}(1-\theta)^{j_k}$, $b(\theta) = w\theta^i(1-\theta)^j$ with $w_k, w \in \mathbb{R}_{>0}$, and $i_k, j_k, i, j \in \mathbb{Z}_{\geq 0}$ for all $k = 0, \ldots, n$. Since $a_0, \ldots, a_n, b$ share no common factors, there exists some $i_k = 0$ if $i > 0$; however this would contradict $0 < w_k \leq a_0(0) + \cdots + a_n(0) = b(0) = 0$. So $i = 0$. Similarly, $j = 0$. Finally, we divide p by w to obtain $b \equiv 1$. □

Corollary 2.4. *Any one-dimensional discrete statistical models with rational MLE can be represented by $(w_k, i_k, j_k)_{k=0}^n$ for $w_k \in \mathbb{R}_{>0}$ and $i_k, j_k \in \mathbb{Z}_{\geq 0}$.*

From now on, we only consider one-dimensional discrete statistical models with rational MLE; we call them *models* for short.

Definition 2.5. The degree $\deg(\mathcal{M})$ of a model $\mathcal{M}$ represented by $(w_k, i_k, j_k)_{k=0}^n$ is defined as $\max\{i_k + j_k : k = 0, \ldots, n\}$.

Remark 2.6. We view two models $(w_k, i_k, j_k)_{k=0}^n$ and $(w_k', i_k', j_k')_{k=0}^n$ as the same model if they are equal up to a permutation of the coordinates.

Example 2.7. The sequence $((1,0,2),(2,1,1),(1,2,0))$ represents the binomial model with two trials. It has degree two. Its parametrization is given by $\theta \mapsto ((1-\theta)^2, 2\theta(1-\theta), \theta^2)$. See Figure 1.1 for a visualization of the binomial model within the probability simplex Δ_2. Note that we treat $((1,0,2),(2,1,1),(1,2,0))$, $((2,1,1),(1,0,2),(1,2,0))$, and $((2,1,1),(1,2,0),(1,0,2))$ as the same model, as coordinate order does not matter.

Definition 2.8. Let $\mathcal{M}$ be a model represented by $(w_k, i_k, j_k)_{k=0}^n$. The set of exponent pairs $(i_k, j_k)_{k=0}^n$ is called the support of $\mathcal{M}$, denoted by $\operatorname{supp}(\mathcal{M})$.

This was our first step towards understanding the structure of models. The next step is to introduce the concept of reduced models.

2.2 Reduced Models

Models in this section refer to one-dimensional discrete statistical models with rational MLE.

Definition 2.9. We call a model represented by $(w_k, i_k, j_k)_{k=0}^n$ *reduced* if $(i_k, j_k) \neq \mathbf{0}$ for all $k = 0, \dots n$, and $(i_k, j_k) \neq (i_l, j_l)$ for all $k \neq l$.

Due to $(i_k, j_k) \neq (i_l, j_l)$, we can use functions to represent reduced models.

Remark 2.10. A reduced model $\mathcal{M}$ represented by $(w_k, i_k, j_k)_{k=0}^n$ can also be identified by a function $f : \mathbb{Z}^2 \to \mathbb{R}_{\geq 0}, (i, j) \mapsto w$, where $w = w_k$ if $(i_k, j_k) = (i, j)$ and $w = 0$ otherwise. The support of f is the set of all pairs (i, j) with $f(i, j) > 0$. It coincides with the support of $\mathcal{M}$.

Reduced models are our first building blocks for the classification of models. This statement is justified by the following two propositions. They show that every non-reduced model can be transformed into a reduced model by a sequence of linear embeddings.

Proposition 2.11. *Let $n \in \mathbb{N}_{>0}$. Let $\mathcal{M}$ be a model represented by $(w_k, i_k, j_k)_{k=0}^n$. If $(i_l, j_l) = \mathbf{0}$ for some index l, then there exist a model $\mathcal{M}'$, $\lambda \in [0, 1]$ and $k = 0, \dots, n$ such that $\mathcal{M} = \Psi_{\lambda,k}(\mathcal{M}')$, where $\Psi_{\lambda,k} : \Delta_{n-1} \to \Delta_n$ is defined as* $p_i \mapsto \begin{cases} \lambda p_i & \text{if } k \neq i, \\ 1 - \lambda & \text{otherwise} \end{cases}$

Proof. Let $(i_l, j_l) = \mathbf{0}$ for some index l. If $w_l = 1$, then $w_m = 0$ for all $m \neq l$; this contradicts $w_m > 0$ by Proposition 2.1. Set $\lambda = 1 - w_l > 0$ and $k = l$. Define the model $\mathcal{M}'$ represented by $\left(\frac{w_h}{1-w_l}, i_h, j_h\right)_{h=0, h\neq l}^n$. Then, $\mathcal{M} = \Psi_{\lambda,k}(\mathcal{M}')$. □

Proposition 2.12. *Let $n \in \mathbb{N}_{>0}$. Let $\mathcal{M}$ be model represented by $(w_k, i_k, j_k)_{k=0}^n$. If $(i_m, j_m) = (i_l, j_l)$ for $m \neq l$, then there exist a model $\mathcal{M}'$, $\lambda \in [0, 1]$ and $k, h = 0, \dots, n$ such that $\mathcal{M} = \Psi_{\lambda,k,h}(\mathcal{M}')$, where $\Psi_{\lambda,k,h} : \Delta_{n-1} \to \Delta_n$ is defined as* $p_i \mapsto \begin{cases} p_i & \text{if } i \notin \{k, h\}, \\ \lambda p_k & \text{if } k = i, \\ (1 - \lambda) p_k & \text{if } h = i. \end{cases}$

Proof. Define $\lambda = \frac{w_m}{w_m + w_l}$, $k = m$, and $h = l$. Define the model $\mathcal{M}'$ represented by $(w_g + \delta_{gm} w_l, i_g, j_g)_{g=0, g\neq l}^n$. Then, $\mathcal{M} = \Psi_{\lambda,k}(\mathcal{M}')$. □

Repeated application of the two propositions transforms any model into a reduced model.

Corollary 2.13. *If Δ_n contains a model of degree d, then there also exists a reduced model of degree d in Δ_m for some $m \leq n$.*

2.3 Fundamental Models

As before, models refer to one-dimensional discrete statistical models with rational MLE. The main building blocks for the classification of models are *fundamental models*; we will see that reduced models come from fundamental models.

Definition 2.14. We call a model represented by $(w_k, i_k, j_k)_{k=0}^n$ *fundamental* if it is reduced and the equation $p_0 + \ldots p_n \equiv 1$ for given $(i_k, j_k)_{k=0}^n$ uniquely determines the weights $(w_k)_{k=0}^n$.

Example 2.15. The binomial model with two trials is fundamental. Given $(i_0, j_0) = (0, 2)$, $(i_1, j_1) = (1, 1)$, and $(i_2, j_2) = (2, 0)$, the equation $p_0+p_1+p_2 = w_0\theta^2+w_1\theta(1-\theta)+w_2(1-\theta)^2 \equiv 1$ uniquely determines the weights $w_0 = 1, w_1 = 2, w_2 = 1$. To see this observe that this equation is equivalent to $w_0\theta^2 + w_1\theta - w_1\theta^2 + w_2 - w_2 2\theta + w_2\theta^2 = 1$ which is equivalent to solving $w_2 - 1 + \theta(w_1 - 2w_2) + \theta^2(w_0 - w_1 + w_2) = 0$ for all $\theta \in \mathbb{R}$.

Example 2.16. Consider the probability simplex Δ_0. It only contains the model 1 which is fundamental.

Example 2.17. Now, consider the probability simplex Δ_1. It only contains the models $\theta \mapsto (\theta, 1-\theta)$ and $\theta \mapsto (1-\theta, \theta)$ which are equivalent. They are fundamental.

We will see that fundamental models like the ones above are building blocks for all reduced models by *composition*.

Definition 2.18. Let $\mathcal{M}$ and $\mathcal{M}'$ be reduced models which are represented by functions $f, g : \mathbb{Z}^2 \to \mathbb{R}_{\geq 0}$, see Remark 2.10. Let $\mu \in (0, 1)$. The *composite* $\mathcal{M} *_\mu \mathcal{M}'$ of $\mathcal{M}$ and $\mathcal{M}'$ is the reduced model represented by the function $(i, j) \mapsto \mu f(i, j) + (1-\mu)g(i, j)$.

We are about to show that every reduced model is the composite of finitely many fundamental models.

Proposition 2.19. *Let $\mathcal{M}$ be a reduced model. Then $\mathcal{M}$ is the composite of finitely many fundamental models.*

Proof. For Δ_0 and Δ_1 we know that they only contain fundamental models, see Examples 2.16 and 2.17.

Assume we are given Δ_n with $n \geq 2$, and let $\mathcal{M}$ be a model that is not fundamental. We aim to show that $\mathcal{M}$ can be expressed as a composite of two models, $\mathcal{M}'$ and $\mathcal{M}''$, whose supports are proper subsets of $\text{supp}(\mathcal{M})$. Assume this is indeed the case. Then, by applying the same argument to $\mathcal{M}'$ and $\mathcal{M}''$, we can recursively decompose each non-fundamental model into models with smaller supports. Since $\text{supp}(\mathcal{M})$ is finite, this recursive decomposition must eventually terminate, yielding a decomposition of $\mathcal{M}$ into fundamental models. Thus, we have shown that any reduced model is the composite of a finite number of fundamental models.

Let us prove that $\mathcal{M}$ is the composite of two models whose supports are proper subsets of $\operatorname{supp}(\mathcal{M})$. Since $\mathcal{M}$ is not fundamental, the equation $p_0 + \cdots + p_n = 1$ has distinct solutions $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^{n+1}_{>0}$. Define $\mathbf{v} := \mathbf{w} - \mathbf{w}' \neq \mathbf{0}$. Then, for all $\theta \in (0,1)$ we have $\sum_{k=0}^{n} v_k \theta^{i_k}(1-\theta)^{j_k} = 0$. Observe that there are strictly positive and negative coefficients v_k.

Define $\lambda := \min\left\{\frac{w_k}{|v_k|} : k = 0,\dots,n,\ v_k < 0\right\}$, $u_k := w_k + \lambda v_k$ for $k = 0,\dots,n$, and $S_1 := \{(i_k, j_k) : k = 0,\dots,n,\ u_k \neq 0\}$. Note that $\lambda > 0$ since all the coefficients w_k are strictly positive by definition. Also observe that $u_k \geq 0$ if $v_k \geq 0$. Moreover, by definition $\frac{w_k}{|v_k|} \geq \lambda$ for all $k \geq 0$. Hence, if $v_k < 0$, we also have $\frac{u_k}{v_k} = \frac{w_k}{v_k} + \lambda \leq 0$. Multiplying by $v_k < 0$ we obtain $u_k \geq 0$. All in all, we have $u_k \geq 0$ for all $k = 0,\dots,n$. Moreover, $u_k = 0$ if and only if $v_k < 0$ and $\lambda = \frac{w_k}{|v_k|}$. This shows that $S_1 \subsetneq \operatorname{supp}(\mathcal{M})$. Since $u_0 + \dots u_n = 1$, we have found a reduced model $\mathcal{M}'$ represented by $(u_k, i_k, j_k)_{(i_k,j_k)\in S_1}$.

For the second model, we define

$$\begin{aligned} \mu &:= \min\left\{\frac{w_k}{u_k} : k = 0,\dots,n,\ u_k \neq 0\right\}, \\ t_k &:= \frac{w_k - \mu u_k}{1-\mu} \quad \text{for } k = 0,\dots,n, \\ S_2 &:= \{(i_k, j_k) : k = 0,\dots,n,\ t_k \neq 0\}. \end{aligned}$$

Similarly, $\mu > 0$. We have $\mu < 1$ because some v_k is positive implying $u_k > w_k$. By definition, we have $t_k \geq 0$, and $t_k = 0$ if and only if $u_k \neq 0$ and $\mu = \frac{w_k}{u_k}$. This shows that $S_2 \subsetneq \operatorname{supp}(\mathcal{M})$ and $S_1 \cup S_2 = \operatorname{supp}(\mathcal{M})$. Since $t_0 + \cdots + t_n = 1$, we have found a reduced model $\mathcal{M}''$ represented by $(t_k, i_k, j_k)_{(i_k,j_k)\in S_2}$.

Finally, we see that $w_k = \mu u_k + (1-\mu)t_k$. This shows that $\mathcal{M} = \mathcal{M}' *_\mu \mathcal{M}''$. □

Applying the previous proposition with Corollary 2.13 yields the following corollary.

Corollary 2.20. *If Δ_n contains a non-fundamental model of degree d, then there exists a fundamental model of degree d in Δ_m for some $m < n$.*

Example 2.21. For the two-dimensional probability simplex Δ_2, we can classify all models. Again, models refer to one-dimensional discrete statistical models with rational MLE. Note that the model $\mathcal{M}$ parametrized by $\theta \mapsto (\theta, 1-\theta)$ satisfies $\mathcal{M} *_\mu \mathcal{M} = \mathcal{M}$ for all μ. Since Δ_1 only contains the model $\theta \mapsto (\theta, 1-\theta)$, we can conclude that Δ_2 only contains fundamental models or models that are not reduced.

To find all the fundamental models in Δ_2, we need to check for all sets $S = \{(i_k, j_k)\}_{k=0}^{2} \subset \mathbb{Z}^2_{>0}$ of size three if the equation $p_0 + p_1 + p_2 = \sum_{k=0}^{2} w_k \theta^{i_k}(1-\theta)^{j_k} = 1$ has a unique solution (w_0, w_1, w_2). As we can see, a priori infinitely many sets S need to be checked. However, as we will see in the next section, only those sets S with $\max\{i + j : (i,j) \in S\} \leq 2n - 1 = 3$

need to be considered. Clearly, this reduces the number of sets S to be checked to a finite number.

We compute that only the following supports uniquely determine the weights (w_0, w_1, w_2):

$$\{(0,3),(1,1),(3,0)\}, \{(0,2),(1,1),(2,0)\}, \{(0,1),(1,1),(2,0)\}, \{(0,2),(1,0),(1,1)\}.$$

They correspond to the fundamental models $((1-\theta)^3, 3\theta(1-\theta), \theta^3)$, $((1-\theta)^2, 2\theta(1-\theta), \theta^2)$, $(1-\theta, \theta(1-\theta), \theta^2)$, and $((1-\theta)^2, \theta, \theta(1-\theta))$. The fourth model is equivalent to the third model by a parametrization $\theta \mapsto 1-\theta$ and permutation of the coordinates.

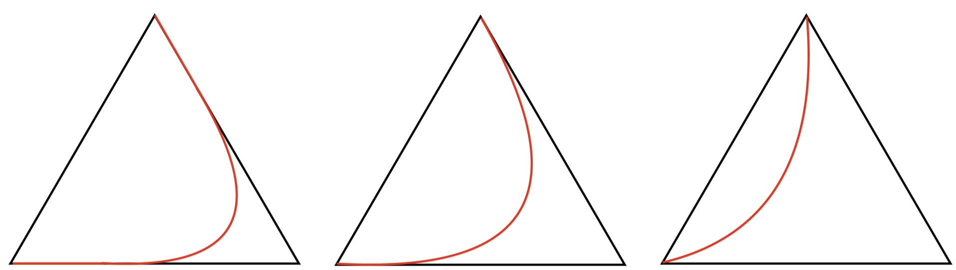

Figure 2.1: From left to right, the illustration depicts the models parametrized $((1-\theta)^3, 3\theta(1-\theta), \theta^3)$, $((1-\theta)^2, 2\theta(1-\theta), \theta^2)$, $(1-\theta, \theta(1-\theta), \theta^2)$, and $((1-\theta)^2, \theta, \theta(1-\theta))$.

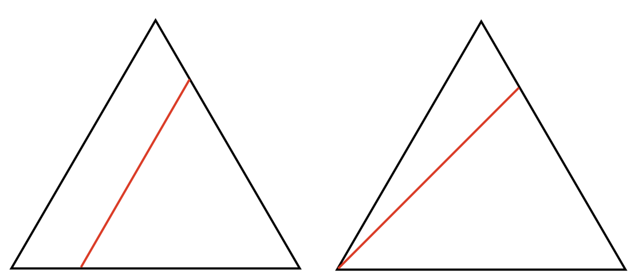

Figure 2.2: This illustration depicts two non-reduced models in Δ_2 for $\lambda = \frac{1}{3}$. They are parametrized by $\theta \mapsto (\frac{2}{3}\theta, \frac{1}{3}, \frac{2}{3}(1-\theta))$ and $\theta \mapsto (1-\theta, \frac{1}{3}\theta, \frac{2}{3}\theta)$. All other non-reduced models can be obtained by varying λ.

We just computed all fundamental models of degree three or less in Δ_2. We will see shortly that these are all models in the probability simplex Δ_2. Of course, Δ_2 contains non-reduced models, too. These are models that come from linear embeddings $\Psi_{\lambda,k}$ and $\Psi_{\lambda,k,h}$, see Proposition 2.11 and Proposition 2.12. There are infinitely many of them, and for $\lambda = \frac{1}{3}$ we obtain the models $\theta \mapsto (\frac{2}{3}\theta, \frac{1}{3}, \frac{2}{3}(1-\theta))$ and $\theta \mapsto (1-\theta, \frac{1}{3}\theta, \frac{2}{3}\theta)$.

Let us summarize the results of this section. It is the first part of our classification theorem.

Theorem 2.22. *Every one-dimensional discrete statistical model with rational MLE in* Δ_n *is the image of a reduced model in* Δ_m *under a linear embedding* $\Delta_m \to \Delta_n$ *for some* $m \leq n$.

Moreover, every reduced model $\mathcal{M} \subset \Delta$ *can be written as a composite of finitely many fundamental models* $\mathcal{M} = \mathcal{M}_1 *_{\mu_1} (\cdots *_{\mu_{m-2}} (\mathcal{M}_{m-1} *_{\mu_{m-1}} \mathcal{M}_m))$ *for some* $m < n$ *and* $\mu_1, \dots, \mu_m \in (0,1)$.

Proof. See Proposition 2.19, Proposition 2.11, and Proposition 2.12. □

2.4 On the Finiteness of Fundamental Models

We have established the first part of our classification theorem, namely that fundamental models are building blocks for all models. The second part is showing that there are only finitely many fundamental models in Δ_n given $n \in \mathbb{N}$. Artuhr Bik and Orlando Marigliano proved that there are only finitely many fundamental models in Δ_n for $n \leq 4$ [2]. We will later make significant progress towards proving the case $n = 5$. For $n \geq 6$ no attempt has been made yet to the best of our knowledge.

Arthur Bik and Orlando Marigliano first proved the following proposition.

Theorem 2.23. *Let* $\mathcal{M}$ *be a one dimensional discrete statistical model with rational MLE in* Δ_n. *For* $n \leq 4$ *we have* $\deg(\mathcal{M}) \leq 2n - 1$.

Given Theorem 2.23 it is easy to show the second part of our classification.

Theorem 2.24. *There are only finitely many fundamental models in* Δ_n *for all* $n \leq 4$.

Proof. By Theorem 2.23 we know that the degree of a fundamental model is at most $2n-1$. Since the number of supports of a fundamental model of degree $2n-1$ is finite, there are only finitely many fundamental models in Δ_n for all $n \leq 4$. □

We will now spend the rest of this thesis on proving Theorem 2.23. The idea is to use the building blocks of fundamental models that we have established so far. Namely, it suffices to show the theorem for fundamental models.

Theorem 2.25. *Let* $N \in \mathbb{N}$. *If for all* $n \leq N$ *and for all fundamental models* $\mathcal{M} \in \Delta_n$ *the upper bound* $\deg(\mathcal{M}) \leq 2n-1$ *holds, then the upper bound also holds for all statistical models in* $\Delta_{n'}$ *for all* $n' \leq N$.

Proof. Let $N \in \mathbb{N}$ and $n \leq N$. Assume there is some non-fundamental model $\mathcal{M}'$ in Δ_n of degree greater than $2n-1$. By Corollary 2.20 there exists a fundamental model $\mathcal{M}$ in Δ_m for some $m < n$ of degree greater than $2m-1$. This contradicts the assumption that the degree of fundamental models is at most $2n'-1$ for all $n' \leq N$. □

$$\begin{tikzcd}
. \arrow[r] & . \arrow[r] & . \arrow[r] & . \\
1 \arrow[r] \arrow[u] & . \arrow[r] \arrow[u] & . \arrow[r] \arrow[u] & . \arrow[u] \\
. \arrow[r] \arrow[u] & 2 \arrow[r] \arrow[u] & . \arrow[r] \arrow[u] & . \arrow[u] \\
. \arrow[r] \arrow[u] & . \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & . \arrow[u]
\end{tikzcd}$$

Figure 2.3: The binomial model with two trials visualized in a directed graph with vertices in $\{0, 1, 2, 3\}^2$.

Counting all *fundamental* models in Δ_n for $n \leq 4$ is our guiding objective. As a first step, we introduce a combinatorial game that aids in counting fundamental models. We know that every reduced model can be represented by the sequence of triples $(w_k, i_k, j_k)_{k=0}^n$, where $w_k \in \mathbb{R}_{>0}$ and $i_k, j_k \in \mathbb{Z}_{\geq 0}$. The model can be visualized in a directed graph with vertices in $\mathbb{Z}^2$, where we can place values w_k on vertices (i_k, j_k). Each vertex (i, j) is connected by directed edges to $(i + 1, j)$ and $(i, j + 1)$.

Surprisingly, we can derive a combinatorial game from this graph by defining a specific set of rules. This game, called the *chipsplitting game*, will be rigorously introduced in the next chapter. After that, we will explore the game's properties and show how it can be used to count fundamental models in Δ_n for $n \leq 4$.

Chapter 3

Chipsplitting Games

The notion of a chipsplitting game was introduced by [2] as a combinatorial approach to classifying one-dimensional discrete statistical models with rational maximum likelihood estimator. It was inspired by *chipfiring games* and for a subset of chipfiring games, the chipsplitting game is equivalent to the chipfiring game. We refer to [6] for a comprehensive introduction to chipfiring games.

3.1 Basic Definitions

Let us define the notion of a chipsplitting game.

Definition 3.1. Let (V, E) be a directed graph without loops.

(1) A *chip configuration* is a vector $\mathbf{w} = (w_v)_{v \in V} \in \mathbb{Z}^V$ such that there are only finitely many nonzero components w_k.

(2) The *initial configuration* is the chip configuration $\mathbf{0} \in \mathbb{Z}^V$.

(3) A *splitting move* at $u \in V$ maps a chip configuration $\mathbf{w}$ to some chip configuration $\mathbf{w}'$ defined by

$$w'_v := \begin{cases} w_v - 1 & \text{if } v = u, \\ w_v + 1 & \text{if } (u, v) \in E \\ w_v & \text{otherwise.} \end{cases}$$

This map is denoted by split_u.

(4) An *unsplitting move* at $u \in V$ maps $\mathbf{w}'$ back to $\mathbf{w}$. This map is denoted by $\mathrm{unsplit}_u$.

(5) A *chipsplitting game* is a finite sequence of splitting and unsplitting moves.

(6) An *outcome of a chipsplitting game* is the chip configuration obtained from applying the sequence of splitting and unsplitting moves defined by the game at the initial configuration.

(7) Any outcome of a chipsplitting game is called an *outcome*.

Proposition 3.2. *The order of the moves in a chipsplitting game does not affect the outcome.*

Proof. This follows from commutativity of addition. □

Note that all moves are reversible. Thus, we obtain the following corollary with Proposition 3.2.

Corollary 3.3. *Let* $\mathbf{w}$ *be an outcome. Then, there exists a chipsplitting game whose outcome is* $\mathbf{w}$ *and where at no point both a splitting and an unsplitting move are applied at the same vertex.*

Games that satisfy the condition in the corollary are called *reduced*. We will only consider reduced games in this thesis for simplicity. The map

$$\begin{aligned} \{\text{reduced games on } (V,E)\} / \sim \quad &\to \quad \{g : V' \to \mathbb{Z} : \#\{p \in V' : g(p) \neq 0\} < \infty\} \\ f &\mapsto (p \mapsto \text{number of moves at } p \text{ in game } f) \end{aligned}$$

is a bijection, where $V' \subset V$ is the subset of vertices with at least one outgoing edge. The equivalence relation $\sim$ is defined by $f \sim g$ if f and g are the same up to reordering. Unsplitting moves are counted negatively by $p \mapsto$ number of moves at p in game f. Using the map above we identify a chipsplitting game with its corresponding function $V' \to \mathbb{Z}$. For every outcome $\mathbf{w} = (w_v)_{v \in V}$ we have $w_v = -f(v) + \sum_{u \in V', (u,v) \in E} f(u)$, where we set $f(t) = 0$ for $t \notin V$.

Now, we define the directed graphs that we will consider in this thesis. For $d \in \mathbb{N} \cup \{\infty\}$ we write

$$\begin{aligned} V_d &\coloneqq \left\{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid i + j \leq d\right\}, \\ E_d &\coloneqq \{(v, v+e) \mid v \in V_{d-1}, e \in \{(1,0),(0,1)\}\}. \end{aligned}$$

Definition 3.4. The degree $\deg(\mathbf{v})$ of a vertex $\mathbf{v} = (i,j)$ is defined as $i + j$.

Example 3.5. A chip configuration $\mathbf{w} = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ can be illustrated as a triangle of numbers where $w_{i,j}$ is placed at the position (i,j) in the triangle. For example, $w_{2,4} = 4$ means that the value 4 is placed in the second column and fourth row of the triangle. The following is an example of a sequence of chip configurations for $d = 3$:

```
.            .            .            1            1            1
. .          . .          1 .          . 1          . 1          . .
. . .        1 . .        . 2 .        . 2 .        . 2 1        . 3 .
0 . . .     -1 1 . .     -1 . 1 .     -1 . 1 .     -1 . . 1     -1 . . 1
```

When $w_{i,j} = 0$, we omit the value in the triangle and write a dot instead. The sequence above starts with the initial configuration and then applies a splitting move at the vertex $(0,0), (1,0), (0,1), (0,2)$ and $(2,0)$. Finally, we apply an unsplitting move at the vertex $(1,1)$ to obtain the final configuration. Coming back to figure 2.4, we see that it is represented as the third configuration of the triangle above.

Let us define some more terminology.

Definition 3.6. Let $\mathbf{w} = (w_{i,j})_{(i,j)\in V_d}$ be a chip configuration.

(1) The *positive support* of $\mathbf{w}$ is defined as $\operatorname{supp}^+(\mathbf{w}) \coloneqq \{(i,j) \in V_d \mid w_{i,j} > 0\}$.

(2) The *negative support* of $\mathbf{w}$ is defined as $\operatorname{supp}^-(\mathbf{w}) \coloneqq \{(i,j) \in V_d \mid w_{i,j} < 0\}$.

(3) The *support* of $\mathbf{w}$ is defined as the union of the positive and negative support.

(4) The *degree* of $\mathbf{w}$ is defined as $\deg(\mathbf{w}) \coloneqq \max\{i + j \mid (i,j) \in \operatorname{supp}(\mathbf{w})\}$.

(5) We say $\mathbf{w}$ is *valid* if its negative support is empty or only contains $(0,0)$.

We are interested in *outcomes* that are *valid* since they will correspond to reduced models as we will see later. For that reason, it would be convenient to have a criterion for when a chip configuration is an outcome. The next section will provide such a criterion with the help of *Pascal equations*.

Example 3.7. Consider the following chip configuration:

```
 ·
 ·   1
 1   ·   5
 ·   5   ·   2
 1   .   .   5   .
 ·   .   8   .   .   2
-2   ·   .   .   .   2   .
```

We clearly see that this configuration is valid, but is it also an outcome of a chipsplitting game? Currently, the only way to answer this question is to apply all possible sequences of splitting and unsplitting moves to the initial configuration and check if the outcome is the given configuration. In the next section, we present an easily computable characterization to answer this question.

3.2 Pascal Equations

In this chapter we will establish that outcomes are roots of Pascal equations. So let us first define Pascal equations which are special cases of *linear forms*.

Definition 3.8. A *linear form* on $\mathbb{Z}^{V_d}$ is a map of the form $\mathbb{Z}^{V_d} \to \mathbb{Z}, \quad \mathbf{w} \mapsto \sum_{(i,j)\in V_d} c_{i,j} w_{i,j}$, denoted by $\sum_{(i,j)\in V_d} c_{i,j} x_{i,j}$.

Definition 3.9. A *Pascal form* on $\mathbb{Z}^{V_d}$ is a linear form $\sum_{(i,j)\in V_d} c_{i,j} x_{i,j}$ on $\mathbb{Z}^{V_d}$ satisfying $c_{i,j} = c_{i+1,j} + c_{i,j+1}$ for all $(i,j) \in V_{d-1}$.

Example 3.10. We can visualize a Pascal form as a triangle of numbers where $c_{i,j}$ is placed at the position (i,j) in the triangle. Here are examples of Pascal forms for $d = 2$:

```
0            1            0            0
1  1         1  0         0  0         1  1
2  1  0      1  0  0      1  1  1      0 -1 -2
```

Evaluating Pascal equations is invariant under splitting and unsplitting moves.

Proposition 3.11. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chip configuration. Let $p = \sum c_{i,j} x_{i,j}$ be a Pascal equation on $\mathbb{Z}^{V_d}$. Then, we have $p(\mathbf{w}) = p(\mathrm{split}_u(\mathbf{w})) = p(\mathrm{unsplit}_v(\mathbf{w}))$ for all $u, v \in V_{d-1}$.*

Proof. Let $u := (i', j') \in V_{d-1}$. By the Pascal property, we have $c_{i'+1,j'} + c_{i',j'+1} - c_{i',j'} = 0$. Thus, we have

$$\begin{aligned}
p(\mathrm{split}_u(\mathbf{w})) &= \sum_{(i,j)\in V_d} c_{i,j} (\mathrm{split}_u(\mathbf{w}))_{i,j} \\
&= \sum_{(i,j)\in V_d} c_{i,j} \begin{cases} w_{i,j} - 1 & \text{if } (i,j) = u, \\ w_{i,j} + 1 & \text{if } (i,j) \in \{(i'+1, j'), (i', j'+1)\} \\ w_{i,j} & \text{otherwise} \end{cases} \\
&= \sum_{(i,j)\in V_d} c_{i,j} w_{i,j} = p(\mathbf{w}).
\end{aligned}$$

Similarly, we can show that $p(\mathrm{unsplit}_v(\mathbf{w})) = p(\mathbf{w})$ for all $v \in V_{d-1}$. □

Corollary 3.12. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be an outcome. Let $p = \sum c_{i,j} x_{i,j}$ be a Pascal equation on $\mathbb{Z}^{V_d}$. Then, $p(\mathbf{w}) = 0$.*

Proof. Clearly, we have $p(\mathbf{0}) = 0$. Then, we use Proposition 3.11 and the fact that $\mathbf{w}$ is obtained from the initial configuration $\mathbf{0}$ by a sequence of splitting and unsplitting moves. □

This demonstrates that outcomes are roots of Pascal equations. The converse is also true as we will see now. This is one of the most important results; so let us state it now.

Theorem 3.13. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chip configuration. Then, $\mathbf{w}$ is an outcome if and only if $\mathbf{w}$ is a root of all Pascal equations on $\mathbb{Z}^{V_d}$.*

The direction left to right is the content of the previous corollary. For the other direction life would be easier if we had not to deal with infinitely many Pascal equations. So let us fix this first by introducing a basis from which we can generate all Pascal equations through linear combinations.

Example 3.14. Fix the degree $d = 2$. We later claim that the following set of Pascal forms is a basis:

```
0           0            1
0  0        1  1         0 -1
1  1  1,    0 -1 -2,     0  0  1.
```

Note that the first column of each Pascal form is a unit vector in $\mathbb{R}^3$. We can also fix the first row of each Pascal form to be a unit vector in $\mathbb{R}^3$:

```
1          -2            1
1  0       -1  1         0 -1
1  0  0,    0  1  0,     0  0  1.
```

We will denote the first set of Pascal forms by $\{\mathrm{col}(0), \mathrm{col}(1), \mathrm{col}(2)\}$ and the second set by $\{\mathrm{row}(0), \mathrm{row}(1), \mathrm{row}(2)\}$.

To generalize the example above to an arbitrary degree $d \in \mathbb{N}$ and to vectors beyond unit vectors, we assert that there exists a unique Pascal form whose first column is any chosen vector.

Proposition 3.15. *Let $\mathbf{a} = (a_0, \ldots, a_d)$ be any vector with integer entries. Then, the following two statements hold:*

(1) There exists a unique Pascal form $\sum c_{i,j} x_{i,j}$ such that $c_{0,\cdot} = \mathbf{a}$.

(2) There exists a unique Pascal form $\sum c_{i,j} x_{i,j}$ such that $c_{\cdot,0} = \mathbf{a}$.

Proof. Set $c_{0,\cdot} := \mathbf{a}$. Define $c_{i+1,j} := c_{i,j} - c_{i,j+1}$ for all $(i,j) \in V_d$ with $i = 0$. Then, we use the same formula to define $c_{i+1,j}$ for all $(i,j) \in V_d$ with $i = 1$. We repeat this process until we have defined all $c_{i,j}$ for $(i,j) \in V_d$.

For the second statement, we set $c_{\cdot,0} := \mathbf{a}$. Define $c_{i,j+1} := c_{i,j} - c_{i+1,j}$ for all $(i,j) \in V_d$ with $j = 0$. Then, we use the same formula to define $c_{i,j+1}$ for all $(i,j) \in V_d$ with $j = 1$. We repeat this process until we have defined all $c_{i,j}$ for $(i,j) \in V_d$. $\square$

Let us define our first two Pascal form bases.

Definition 3.16. Let $k = 0, \dots, d$ and $\mathbf{e}_k \in \mathbb{R}^{d+1}$ be the k-th unit vector.

- We define $\mathrm{col}(k)$ to be the unique Pascal form $\sum c_{i,j} x_{i,j}$ such that $c_{0,\cdot} = \mathbf{e}_k$.
- We define $\mathrm{row}(k)$ to be the unique Pascal form $\sum c_{i,j} x_{i,j}$ such that $c_{\cdot,0} = \mathbf{e}_k$.

For examples of the Pascal forms $\mathrm{col}(k)$ and $\mathrm{row}(k)$ for $d = 2$ see Example 3.14. We provide another example for $d = 7$.

Example 3.17. Let us consider the Pascal form col(3) for $d = 7$. We visualize this Pascal form as follows:

```
·
·   .
·   ·   .
·   ·   ·   .
1   1   1   1   1
·  -1  -2  -3  -4  -5
·   ·   1   3   6  10  15
·   ·   ·  -1  -4 -10 -20 -35.
```

The Pascal form row(3) is visualized as follows:

```
-35
-20  15
-10  10  -4
-4    6  -4   1
-1    3  -3   1   .
 ·    1  -2   1   .   .
 ·    ·  -1   1   .   .   .
 ·    ·   ·   1   .   .   .   .
```

Proposition 3.18. *For all integers $k = 0, \dots, d$ the following formulas hold:*

$$\mathrm{col}(k) = (-1)^k \sum_{(i,j) \in V_d} (-1)^j \binom{i}{k-j} x_{i,j} \text{ and } \mathrm{row}(k) = (-1)^k \sum_{(i,j) \in V_d} (-1)^i \binom{j}{k-i} x_{i,j}.$$

Note that $\binom{a}{b} = 0$ for $b < 0$ or $b > a$.

Proof. We claim that $(-1)^k \sum_{(i,j) \in V_d} (-1)^j \binom{i}{k-j} x_{i,j}$ is a Pascal equation. To see that observe

$$(-1)^j \binom{i+1}{k-j} + (-1)^{j+1} \binom{i}{k-j-1} = (-1)^j \binom{i}{k-j}$$

for all $(i,j) \in V_d$ due to $\binom{a}{b+1} + \binom{a}{b} = \binom{a+1}{b+1}$ where we set $a = i$ and $b = k - j - 1$. Next, we see that $(-1)^{k+j}\binom{0}{k-j} = \delta_{jk}$. Thus, $(-1)^k \sum_{(i,j)\in V_d} (-1)^j \binom{i}{k-j} x_{i,j}$ is indeed $\mathrm{col}(k)$.

By symmetry of the binomial coefficients, we can use the same argument to show the second formula. □

We now show that $\{\mathrm{col}(k)\}_{k=0}^{d}$ is indeed a basis for all Pascal forms on $\mathbb{Z}^{V_d}$.

Proposition 3.19. *Let p be a Pascal form on $\mathbb{Z}^{V_d}$. The following statements hold:*

(1) There exist unique coefficients $\mu_0, \ldots, \mu_d \in \mathbb{Z}$ such that $p = \mu_0 \mathrm{col}(0) + \cdots + \mu_d \mathrm{col}(d)$.

(2) There exist unique coefficients $\lambda_0, \ldots, \lambda_d \in \mathbb{Z}$ such that $p = \mu_0 \mathrm{row}(0) + \cdots + \mu_d \mathrm{row}(d)$.

Proof. Let $p = \sum c_{i,j} x_{i,j}$ be a Pascal form on $\mathbb{Z}^{V_d}$. If we try to solve the equation

$$\sum_{(i,j)\in V_d} c_{i,j} x_{i,j} = \lambda_0 \mathrm{col}(0) + \cdots + \lambda_d \mathrm{col}(d) \tag{3.1}$$

for $\lambda_0, \ldots, \lambda_d$, then due to Proposition 3.18 we have for all $(i,j) \in V_d$ that

$$\begin{aligned} c_{i,j} &= \lambda_0 (-1)^{0+j} \binom{i}{0-j} + \lambda_1 (-1)^{1+j} \binom{i}{1-j} + \cdots + \lambda_d (-1)^{d+j} \binom{i}{d-j} \\ &= \lambda_j (-1)^{2j} \binom{i}{0} + \lambda_{j+1} (-1)^{2j+1} \binom{i}{1} + \cdots + \lambda_{i+j} (-1)^{2j+i} \binom{i}{i}. \end{aligned}$$

We see $c_{0,\cdot} = (\lambda_0, \cdots, \lambda_d)$. Thus we set the coefficients $\boldsymbol{\mu} := c_{0,\cdot}$ and by Proposition 3.15 we see that $\sum_{(i,j)\in V_d} c_{i,j} x_{i,j} = \mu_0 \mathrm{col}(0) + \cdots + \mu_d \mathrm{col}(d)$. Moreover, the same proposition shows that the coefficients $\lambda_0, \ldots, \lambda_d$ in Equation 3.1 are uniquely determined.

For the second statement we use the same argument. □

Corollary 3.20. *The set $\{\mathrm{col}(k)\}_{k=0}^{d}$ is a basis for all Pascal forms on $\mathbb{Z}^{V_d}$. The same holds for $\{\mathrm{row}(k)\}_{k=0}^{d}$.*

Proof. This follows from the previous proposition. □

Let us come back to Theorem 3.13. We can now prove the other direction; namely that roots of all Pascal equations on $\mathbb{Z}^{V_d}$ are outcomes.

Proposition 3.21. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chip configuration. If for all Pascal equations p on $\mathbb{Z}^{V_d}$ we have $p(\mathbf{w}) = 0$, then $\mathbf{w}$ is an outcome.*

Proof. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chip configuration. By assumption, we have

$$\mathrm{col}(\deg(\mathbf{w}))(\mathbf{w}) = 0. \tag{3.2}$$

Note that by Proposition 3.18 for $\text{col}(\deg(\mathbf{w})) = \sum c_{i,j} x_{i,j}$ we have $c_{i,\deg(\mathbf{w})-i} = (-1)^i$ for all $i = 0, \ldots, \deg(\mathbf{w})$. Moreover, we have

$$c_{i,j} = 0 \quad \text{for all } i + j < \deg(\mathbf{w}) \tag{3.3}$$

by Proposition 3.18. Together with Equation 3.2 and 3.3 we obtain

$$\sum_{i=0}^{\deg(\mathbf{w})} (-1)^i w_{i,\deg(\mathbf{w})-i} = 0. \tag{3.4}$$

Furthermore, we know that there exists a unique minimal set of splitting or unsplitting moves at vertices (i, j) of degree $\deg(\mathbf{w}) - 1$ such that when applied to $\mathbf{w}$ we obtain a chip configuration $\mathbf{w}'$ with $w'_{i,j} = 0$ for all $i = 0, ..., \deg(\mathbf{w})$. We call applying these set of moves to $\mathbf{w}$ *retraction.*

```
.
.    .
.    .    .
*    .    .    .
*    *    .    .    .
*    *    *    .    .    .
*    *    *    *    .    .    .
*    *    *    *    *    .    .    .
*    *    *    *    *    *    .    .    .
     |
     |
     | retraction of w of degree five
     |
     |
     V
.
.    .
.    .    .
.    .    .    .
*    .    .    .    .
*    *    .    .    .    .
*    *    *    .    .    .    .
*    *    *    *    .    .    .    .
*    *    *    *    *    .    .    .    .
```

Thus, $\mathbf{w}'$ has degree less than $\deg(\mathbf{w})$. By Proposition 3.11 $\mathbf{w}'$ is also a root of all Pascal equations. We repeat the retraction process $\deg(\mathbf{w})$ many times until we obtain some chip configuration of degree 0. This chip configuration is the initial configuration due to Equation 3.4. Thus, $\mathbf{w}$ is an outcome. □

We have shown Theorem 3.13. Characterizing outcomes as roots of Pascal equations is a powerful tool to determine if a chip configuration is an outcome.

Algorithm 1 Validating outcomes

```
Require: chipsplitting configuration w ∈ Z^{V_d}
Ensure: True if w is an outcome, False otherwise
function isOutcome(A, n)
    initialize set S = {col(0), ..., col(deg(w))}
    for p of S do
        if p(w) ≠ 0 then
            return False
        end if
    end for
    return True
end function
```

Proof of correctness of Algorithm 1. This follows from Theorem 3.13. □

Example 3.22. Returning to Example 3.7, we see that the chip configuration is a root of all Pascal equations $\mathrm{col}(0), \dots, \mathrm{col}(6)$ using Algorithm 1. Thus, the chip configuration is an outcome.

3.3 Valid Outcomes and Reduced Statistical Models

In the previous sections, we have established that outcomes are roots of Pascal forms. Now, we will demonstrate that a subset of *valid outcomes* are in one-to-one correspondence with reduced statistical models. Thus, we obtain not only a combinatorial characterization of reduced statistical models through chip-splitting games but also an algebraic characterization through Pascal equations. As before, statistical models mean one-dimensional discrete statistical models with rational maximum likelihood estimator.

We remind that valid chipsplitting configurations are those where the negative support is empty or only contains the vertex $(0,0)$. Hence, valid outcomes are roots of Pascal equations whose negative supports are empty or only contain the vertex $(0,0)$.

The function $\mathbf{w}(\mathcal{M})$ maps reduced models $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^n$ to chip configurations $\mathbf{w}(\mathcal{M}) = (w_{i,j})_{(i,j)\in V_\infty}$ by

$$w_{i,j} \coloneqq \begin{cases} -1 & \text{if } (i,j) = (0,0), \\ w_k & \text{if } (i,j) = (i_k, j_k) \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the map $\mathbf{w}(\mathcal{M})$ defines a *real* chipsplitting games; the rules of the game are the same as for integer chipsplitting games.

Example 3.23. The binomial model $((1,3,0),(3,2,1),(3,1,2),(1,0,3))$ with three trials is mapped to the chip configuration below:

```
  1
  ·   3
  .   ·   3
 -1   .   .   1.
```

Example 3.24. Does the following valid real outcome from Example 3.7 induce a reduced statistical model through the inverse map $\mathbf{w}^{-1}$?

```
  .
  ·    0.5
 0.5    ·    2.5
  ·    2.5    ·    1
 0.5    .     .   2.5    .
  ·     .     4    .     .    1
 -1     ·     .    .     .    1    .
```

The outcome would correspond to the reduced model

$$\mathcal{M} = ((0.5,2,0),(0.5,4,0),(2.5,1,3),(0.5,1,5),(4,2,1),(2.5,2,4),\\ (2.5,3,2),(1,3,3),(1,5,0),(1,5,1))$$

in the probability simplex Δ_9. As it turns out $\mathcal{M}$ is indeed a reduced statistical model by the next theorem.

Theorem 3.25. *The map $\mathcal{M} \mapsto w(\mathcal{M})$ is a bijection between reduced statistical models and valid real outcomes $\mathbf{w} \in \mathbb{R}^{V_\infty}$ with $w_{0,0} = -1$.*

$$\text{reduced models} \twoheadleftarrow\!\!\!\longrightarrow \text{valid outcomes}$$

Figure 3.1: Bijection between reduced models and valid real outcomes $\mathbf{w}$ with $w_{0,0} = -1$

To show this theorem, we need to do some preparations. Let $\mathbb{R}[\theta]_{\leq d}$ denote the vector space of polynomials in the variable θ of degree at most d with real coefficients. Similarly, we define $\mathbb{Z}[\theta]_{\leq d}$ and $\mathbb{Q}[\theta]_{\leq d}$. Next, we introduce the linear map $\alpha_d^{\mathbb{R}}$ that maps real chip configurations to real polynomials:

$$\alpha_d^{\mathbb{R}} : \mathbb{R}^{V_d} \to \mathbb{R}[\theta]_{\leq d}, \mathbf{w} \mapsto \sum_{(i,j)\in V_d} w_{i,j}\theta^i(1-\theta)^j.$$

We define the map $\alpha_d^{\mathbb{Z}}$ and $\alpha_d^{\mathbb{Q}}$ for integer and rational chip configurations analogously.

Lemma 3.26. *The following statements hold true for all $d \in \mathbb{N} \cup \{\infty\}$:*

(1) $\{\mathbf{w} \in \mathbb{R}^{V_d} \mid \mathbf{w} \textit{ is an outcome}\} = \operatorname{kernel}(\alpha_d^{\mathbb{R}})$*;*

(2) $\{\mathbf{w} \in \mathbb{Z}^{V_d} \mid \mathbf{w} \textit{ is an outcome}\} = \operatorname{kernel}(\alpha_d^{\mathbb{Z}})$*;*

(3) $\{\mathbf{w} \in \mathbb{Q}^{V_d} \mid \mathbf{w} \textit{ is an outcome}\} = \operatorname{kernel}(\alpha_d^{\mathbb{Q}})$*.*

Proof. We only prove the first statement. The other two statements are proven analogously. Note that it suffices to show the statement for $d < \infty$ since $\alpha_\infty^{\mathbb{R}}$ is the direct limit of $\alpha_0^{\mathbb{R}}, \alpha_1^{\mathbb{R}}, \alpha_2^{\mathbb{R}}, \ldots$, and so on.

Let $d < \infty$. By Corollary 3.20, the codimension of the outcome space is $d+1$, as it is defined by the roots of the Pascal forms $\operatorname{col}(0), \ldots, \operatorname{col}(d)$.

Let $f(\theta) = \lambda_0 + \lambda_1\theta + \cdots + \lambda_d\theta^d$ be a polynomial in $\mathbb{R}$ of degree at most d. Define a chipsplitting configuration $\mathbf{w}$ by

$$w_{i,j} := \begin{cases} \lambda_i & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\alpha_d^{\mathbb{R}}(\mathbf{w}) = f$, which shows that the map $\alpha_d^{\mathbb{R}}$ is surjective. Hence, the kernel of $\alpha_d^{\mathbb{R}}$ has codimension $d+1$; it has equal codimension as the space of outcomes.

Finally, we just need to show that the space of outcomes is contained in the kernel of $\alpha_d^{\mathbb{R}}$. Since their codimensions are equal, the two spaces must be equal. Let $\mathbf{w} \in \mathbb{R}^{V_d}$ be an outcome. The value of $\alpha_d^{\mathbb{R}}(\mathbf{w})$ remains the same if apply splitting or unsplitting moves at arbitrary vertices $(i,j) \in V_{d-1}$ because we have

$$-\theta^i(1-\theta)^j + \theta^{i+1}(1-\theta)^j + \theta^i(1-\theta)^{j+1} = \theta^i(1-\theta)^j(-1+\theta+(1-\theta)) = 0.$$

The remaining claim follows from $\alpha_d^{\mathbb{R}}(\mathbf{0}) = 0$. □

We now show Theorem 3.25.

Proof of Theorem 3.25. Let $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^n$ be a reduced model.

- First, we need to show that $\mathbf{w} := w(\mathcal{M})$ is an outcome; a-priori we only know that it is some chip configuration. By definition of $w(\mathcal{M})$, we have that $w_{0,0} = -1$. Since $\mathcal{M}$ is a statistical model, we know that $\sum_{k=0}^n w_k\theta^{i_k}(1-\theta)^{j_k} \equiv 1$. Thus, $\alpha_d^{\mathbb{R}}(\mathbf{w}) = \sum_{k=0}^n w_k\theta^{i_k}(1-\theta)^{j_k} - 1 \equiv 0$. Thus, $\mathbf{w} \in \operatorname{kernel}(\alpha_d^{\mathbb{R}})$. By Lemma 3.26, the chip configuration $\mathbf{w}$ is an outcome.

- **Injectivity:** Let $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^n$ and $\mathcal{M}' = (w'_k, i'_k, j'_k)_{k=0}^n$ be two distinct models. Then, $w(\mathcal{M}) \neq w(\mathcal{M}')$ (see Remark 2.6).
- **Surjectivity:** Let $\mathbf{w} \in \mathbb{R}^{V_\infty}$ be a valid real outcome with $w_{0,0} = -1$. We define $w_k := w_{i_k,j_k}$ for all $k = 0, \dots, n$. Then, $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^n$ is a reduced model by Lemma 3.26. We see that $w(\mathcal{M}) = \mathbf{w}$. Hence, $\mathcal{M} \mapsto w(\mathcal{M})$ is surjective.

□

Proposition 3.27. *The following statements hold for all reduced models $\mathcal{M}$:*

(1) $\mathrm{supp}^+(w(\mathcal{M})) = \mathrm{supp}^+(\mathcal{M})$*.*

(2) The map $\mathcal{M} \mapsto w(\mathcal{M})$ is degree-preserving.

(3) The outcome $w(\mathcal{M})$ is a rational outcome if and only if all the coefficients of $\mathcal{M}$ are rational.

Proof. All three statements follow directly from definitions. □

Proposition 3.28. *Let $\mathbf{w} \in \mathbb{Q}^{V_\infty}$ be a valid rational outcome. Then, there exist positive $\lambda \in \mathbb{Q}$ and integral valid outcome $\mathbf{z} \in \mathbb{Z}^{V_\infty}$ such that $\mathbf{w} = \lambda \mathbf{z}$.*

Proof. Let $\mathbf{w}$ be a valid rational outcome. Its support is finite. Thus, there exist $\mu \in \mathbb{N}$ such that $\mu\mathbf{w} \in \mathbb{Z}^{V_\infty}$. Define $\lambda := \frac{1}{\mu}$ and $\mathbf{v} := \mu\mathbf{w}$. Clearly, $\alpha^{\mathbb{Z}}_{\deg(\mathbf{v})}(\mathbf{v}) = \mu\alpha^{\mathbb{Q}}_{\deg(\mathbf{w})}(\mathbf{w}) \equiv 0$. By Lemma 3.26, $\mathbf{v}$ is an outcome. It is valid because $\mu\mathbf{w}$ is valid. □

Proposition 3.29. *Let $\mathbf{w} \in \mathbb{R}^{V_\infty}$ be a valid real outcome. If $w_{0,0} = 0$, then $\mathbf{w} = \mathbf{0}$.*

Proof. By Lemma 3.26, we have $\sum w_{i,j}\theta^i(1-\theta)^j \equiv 0$. By assumption, the negative support is empty. Hence, all the $w_{i,j}$ are non-negative. We evaluate at $\theta = \frac{1}{2}$ to conclude that the positive support of $\mathbf{w}$ is empty. Hence, $\mathbf{w} = 0$. □

Let us go back to Example 3.23.

Example 3.30. We have seen that the valid real outcome below induces a reduced statistical model by Theorem 3.25.

```
1
·    3
.    ·    3
-1   .    .    1.
```

The outcome has degree three and positive support size four. Can we find another outcome with the same degree but smaller positive support size? Indeed we can unsplitt at vertex $(1, 1)$ to get

```
 1
 .   .
 ·   3   .
-1   ·   .   1.
```

Can we reduce the positive support size further? As it will turn out, we cannot. The positive support size is minimal for a degree three outcome.

Example 3.31. Let us fix the positive support size to be three. What is the largest degree of a valid real outcome with positive support size three? We have already found

```
 1
 .   .
 ·   3   .
-1   ·   .   1.
```

However is there an even larger one? Once again, the answer is no: by the following theorem, the degree of a valid real outcome with positive support of size three is at most three.

Theorem 3.32. *For valid integral outcomes* $\mathbf{w}$ *with* $|\mathrm{supp}^+(\mathbf{w})| \leq 5$ *we have* $\deg(\mathbf{w}) \leq 2 \cdot |\mathrm{supp}^+(\mathbf{w})| - 3$.

Arthur Bik and Marigliano Orlando established this theorem in [2], and we will provide a proof in the following sections. The question of whether the upper bound extends to cases with larger positive support sizes, $|\mathrm{supp}^+(\mathbf{w})| > 5$, remains open. A primary contribution of this thesis is the significant advancement in proving this bound for the case $|\mathrm{supp}^+(\mathbf{w})| = 6$.

Pos. Support Size	**Max Degree**
2	1
3	3
4	5
5	7

Table 3.1: The following table shows the largest degree of a valid real outcome with positive support size $2, 3, 4$ and 5.

Theorem 3.32 is of particular interest because it is equivalent to Theorem 2.23, i.e. proving that the degree of outcomes is bounded by their positive support size is equivalent to establishing that the degree of statistical models is bounded by their dimension.

$$\text{degree of outcomes } \mathbf{w} \longleftrightarrow \text{degree of statistical models } \mathcal{M}$$
$$\mathrm{supp}^+(\mathbf{w}) \longleftrightarrow \text{dimension of } \mathcal{M}$$

Proposition 3.33. *Theorem 2.23 and Theorem 3.32 are equivalent.*

Proof. Assume Theorem 2.23 holds. We want to show Theorem 3.32. Let $\mathbf{w}$ be a valid integral outcome of positive support size $n \leq 5$. Normalize $\mathbf{w}$ such that $w_{0,0} = -1$. The degree and positive support size do not change. By Theorem 3.25, the outcome $\mathbf{w}$ induces a reduced statistical model $\mathcal{M}$ in Δ_{n-1}. Then, $\deg(\mathcal{M}) \leq 2(n-1) - 1 = 2n - 3$. Thus, applying Theorem 3.25 let us go back to the outcome $\mathbf{w}$, and Proposition 3.27 establishes $\deg(\mathbf{w}) \leq 2n - 3$.

For the converse direction, assume Theorem 3.32 holds. Let $n \leq 5$. We want to show Theorem 2.23. Let $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^{n-1} \subset \Delta_{n-1}$ be a one-dimensional discrete statistical model with rational MLE. By Theorem 2.25 we may assume that $\mathcal{M}$ is fundamental. We use Theorem 3.25 to map $\mathcal{M}$ to some valid real outcome $\mathbf{w} = (w_{i,j})$ with $w_{0,0} = -1$. Note that $\mathbf{w}$ is even a *rational* outcome because by Definition 2.14 the weights $(w_k)_{k=0}^{n-1}$ of the fundamental model $\mathcal{M}$ are uniquely determined by the equation $p_0(\theta) + p_1(\theta) + \cdots + p_{n-1}(\theta) - 1 \equiv 0$, and for some $\theta \in [0, 1]$ this equation becomes rational. Next, we use Proposition 3.28 to find some integral valid outcome $\mathbf{z}$ such that $\mathbf{w} = \mu\mathbf{z}$ for some positive scalar $\mu \in \mathbb{Q}_{>0}$. Again, scaling does not affect the degree or size of the positive support. By Proposition the integral valid outcome 3.27 $\mathbf{z}$ has positive support size n. By Theorem 3.32 we have $\deg(\mathbf{z}) \leq 2n-3$. Thus, $\deg(\mathbf{w}) = \deg(\mu\mathbf{z}) = \deg(\mathbf{z}) \leq 2n - 3$. Hence, the degree of $\mathcal{M}$ is smaller or equal to $2(n-1) - 1$ by Proposition 3.27. We proved Theorem 2.23. $\square$

Recall that our goal is to demonstrate that only finitely many fundamental statistical models exist in Δ_n for $n \leq 4$. To achieve this, we originally aimed to prove Theorem 2.24 from Chapter 2. However, as established in Chapter 3, we can alternatively prove Theorem 3.32, which serves as an equivalent approach to resolving the problem. We have chosen to focus on Theorem 3.32 because it allows us to address the problem from a combinatorial perspective.

Chapter 4

Supports of Valid Outcomes

Let us devote the remaining chapters to the study of Theorem 3.32, which for the sake of convenience we restate below.

Theorem. *The upper bound* $\deg(\mathbf{w}) \leq 2 \cdot |\mathrm{supp}^{+}(\mathbf{w})| - 3$ *holds for valid integral outcomes* $\mathbf{w}$ *with* $|\mathrm{supp}^{+}(\mathbf{w})| \leq 5$.

From now on, valid outcomes $\mathbf{w}$ refer to *integral* valid outcomes in $\mathbb{Z}^{V_d}$ for *finite* $d \in \mathbb{N}$. To show the theorem, we will first study the supports of valid outcomes; knowing that some kinds of supports cannot be the supports of valid outcomes will help us to prove the theorem. For instance, integer configurations that have support in the entries below marked with an $*$ cannot be valid outcomes:

$$\begin{array}{ccccccc}
\cdot & & & & & & \\
\cdot & \cdot & & & & & \\
* & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & * & \cdot & \\
* & \cdot & * & \cdot & \cdot & * & *
\end{array}$$

4.1 Invertibility Criterion

Let $d \in \mathbb{N}$. One of the most important tools in the study of outcomes is the *Invertibility Criterion* first introduced in [2]. By Theorem 3.13 we can characterize outcomes as the roots of all Pascal forms on $\mathbb{Z}^{V_d}$. In the previous chapter we have already found two bases for the space of Pascal forms, namely $(\mathrm{row}(0), \ldots, \mathrm{row}(d))$ and $(\mathrm{col}(0), \ldots, \mathrm{col}(d))$ (see Definition 3.16). Let us introduce a *new* basis for the space of Pascal forms.

Definition 4.1. Let $k = 0, \ldots, d$ and $\mathbf{e}_k \in \mathbb{R}^{d+1}$ be the k-th unit vector. We define $\mathrm{diag}(k)$ to be the unique Pascal form $\sum c_{i,j} x_{i,j}$ such that $c_{k,d-k} = \mathbf{e}_k$.

Example 4.2. Fix the degree $d = 7$. We visualize diag(3) by

```
 .
 .   .
 .   .   .
 1   1   1   1
 4   3   2   1   .
10   6   3   1   .   .
20  10   4   1   .   .   .
35  15   5   1   .   .   .   .
```

Proposition 4.3. *For all integers $k = 0, \ldots, d$ we have* $\mathrm{diag}(k) = \sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} x_{i,j}$. *Note that $\binom{a}{b} = 0$ for $b < 0$ or $b > a$.*

Proof. Note that for all $(i, j) \in V_d$ with $i + j = d$ we have $\binom{d-i-j}{k-i} = 1$ if and only if $k = i$, and in all other cases $k \neq i$ the binomial coefficient is zero. Thus, it remains to show that $\sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} x_{i,j}$ is a Pascal form. We have $\binom{d-i-j}{k-i} = \binom{d-i-1-j}{k-i-1} + \binom{d-i-j-1}{k-i}$ for all $(i, j) \in V_{d-1}$ because $\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}$. □

Proposition 4.4. *Let p be a Pascal form on $\mathbb{Z}^{V_d}$. There exist unique coefficients $\mu_0, \ldots, \mu_d \in \mathbb{Z}$ such that $p = \mu_0 \mathrm{diag}(0) + \cdots + \mu_d \mathrm{diag}(d)$.*

Proof. Let $p = \sum c_{i,j} x_{i,j}$. Choose $\mu_k = c_{k,d-k}$ for $k = 0, \ldots, d$. Since p is a Pascal form, the coefficients $c_{i,j}$ satisfy the Pascal recurrence relation. Thus, the coefficients μ_k are uniquely determined. □

Given some set of vertices $S \subset V_d$ the invertibility criterion uses the diagonal basis $(\mathrm{diag}(0), \ldots, \mathrm{diag}(d))$ to determine whether a nonzero outcome with support in S exists.

Definition 4.5. Let $E \subset \{0, \ldots, d\}$ and $S \subset V_d$ with $|E| = |S| \neq 0$. The *pairing matrix* of (E, S) is definded as $A^{(d)}_{E,S} := \left[\binom{d-i-j}{k-i}\right]_{k\in E,(i,j)\in S}$.

Example 4.6. Let $d = 2$, $S = \{(1, 1), (0, 0)\}$ and $E = \{0, 1\}$. Then the pairing matrix is

$$A^{(d)}_{E,S} = \begin{bmatrix} \binom{2-1-1}{0-1} & \binom{2-0-0}{0-2} \\ \binom{2-1-1}{1-1} & \binom{2-0-0}{1-2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now, assume $\mathbf{w}$ is an outcome with support in S. Since it is an outcome, we have $\mathrm{diag}(k)(\mathbf{w}) = 0$ for all $k = 0, 1, 2, 3$. Thus, we have $A^{(d)}_{E,S}\mathbf{w} = \mathbf{0}$.

We make the following observation: if the matrix $A^{(d)}_{E,S}$ were invertible (it is not for the given example), then we would have $\mathbf{w} = \mathbf{0}$; so in this case the initial configuration $\mathbf{0}$ is the *only* outcome with support in S. This is the invertibility criterion.

Proposition 4.7 (Invertibility Criterion)**.** *Let* $\mathbf{w}$ *be an outcome with* $\operatorname{supp}(\mathbf{w}) \subset S$*. If* $A^{(d)}_{E,S}$ *is invertible, then* $\mathbf{w} = \mathbf{0}$*.*

Proof by Contraposition. Let $\mathbf{w} \neq \mathbf{0}$. Its support is non-empty. Then, $\mathbf{w}' \coloneqq (w_{i,j})_{(i,j)\in S} \neq \mathbf{0}$. So, $A^{(d)}_{E,S} \cdot \mathbf{w}' = \mathbf{0}$. The kernel of the pairing matrix is non-trivial. Hence, the pairing matrix $A^{(d)}_{E,S}$ is not invertible. □

Given a non-zero configuration $\mathbf{w}$ we try to construct sets $S \supset \operatorname{supp}(\mathbf{w})$ and E such that the pairing matrix $A^{(d)}_{E,S}$ is invertible. If we succeed, then $\mathbf{w}$ is *not* an outcome since the initial configuration is the only valid outcome with support in S.

4.2 Divide and Conquer

The invertibility criterion is a powerful tool to determine whether a given configuration is an outcome. However, it is not always easy to find suitable sets S and E such that the pairing matrix is invertible. We will now introduce a method to construct such sets.

Divide

Instead of finding one large set S with $\operatorname{supp}(\mathbf{w}) \subset S$, we divide S into smaller sets $S_1, \dots, S_l$. These smaller sets $S_1, \dots, S_k$ will be implicitly defined by integers $\lambda_1, \dots, \lambda_l \in \mathbb{N}$ as we will shortly see. We choose $l \in \mathbb{N}$ and integers $\lambda_1, \dots, \lambda_l \in \mathbb{N}$ such that for all $i = 1, \dots, d$ we have

$$\begin{aligned} |S_i| \in \{0, \lambda_i\}\,,\ S_i &\coloneqq \{(i,j) \in \operatorname{supp}(\mathbf{w}) : i = c_{k-1}, \dots, c_k - 1\} \\ c_i &\coloneqq \lambda_1 + \dots + \lambda_i,\ \lambda_1 + \dots + \lambda_l = d + 1. \end{aligned}$$

Remark 4.8. The above decomposition works when $|\{(i,j) \in \operatorname{supp}(\mathbf{w}) : i \geq d - k\}| \leq k+1$ for all $k = 0, \dots, d$. This is because we can always choose λ_1 minimal such that $|S_1| \in \{0, \lambda_1\}$. We repeat this process until $c_l = d + 1$. This decomposition is illustrated in the following example.

Example 4.9. Fix the degree $d = 6$. Assume we have some configuration $\mathbf{w} \in \mathbb{Z}^{V_6}$ with support in the positions marked with an $*$ below.

```
·
· ·
* · ·
· · · ·
· · · · ·
· · · · * ·
* · * · · * *
```

The first column contains two non-zero entries. So we see $\lambda_1 = 2$. Then, we conclude that $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 1$; otherwise $|S_i| \notin \{0, \lambda_i\}$ for $i > 1$.

Next with S_i defined, we define for all $i = 1, \dots, l$ the sets

$$E_i \coloneqq \begin{cases} \{c_{i-1}, \dots, c_i - 1\} & \text{if } |S_i| = \lambda_i, \\ \emptyset & \text{if } |S_i| = 0. \end{cases}$$

We see that $|E_i| = |S_i|$ for all $i = 1, \dots, l$.

Conquer

Given some support S, we divide it into smaller sets $S_1, \dots, S_l$ as described above. We also define sets $E_1, \dots, E_l$. Write $E \coloneqq E_1 \cup \dots \cup E_l$.

Proposition 4.10. *We have* $A^{(d)}_{E,S} = \begin{bmatrix} A^{(d)}_{E_1,S_1} & 0 & \dots & 0 \\ A^{(d)}_{E_2,S_1} & A^{(d)}_{E_2,S_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{(d)}_{E_l,S_1} & A^{(d)}_{E_l,S_2} & \dots & A^{(d)}_{E_l,S_l} \end{bmatrix}$.

Proof. First, note that

$$A^{(d)}_{E,S} = \begin{bmatrix} A^{(d)}_{E_1,S_1} & A^{(d)}_{E_1,S_2} & \dots & A^{(d)}_{E_1,S_l} \\ A^{(d)}_{E_2,S_1} & A^{(d)}_{E_2,S_2} & \dots & A^{(d)}_{E_2,S_l} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(d)}_{E_l,S_1} & A^{(d)}_{E_l,S_2} & \dots & A^{(d)}_{E_l,S_l} \end{bmatrix}.$$

Let $x, y = 1, \dots, l$ such that $x < y$. Let $k \in E_x$ and $(i, j) \in S_y$. Then, $k \le c_x - 1 < c_x \le c_{y-1} \le i$; so $k - i < 0$. Thus, $\binom{d-i-j}{k-i} = 0$. This implies that the upper off-diagonal blocks are zero. $\square$

Corollary 4.11. *The pairing matrix* $A^{(d)}_{E,S}$ *is invertible if and only if* $A^{(d)}_{E_1,S_1}, \dots, A^{(d)}_{E_l,S_l}$ *are invertible.*

Corollary 4.12 (Invertibility Criterion, Divide and Conquer)**.** *Let* $\mathbf{w}$ *be an outcome with* $\mathrm{supp}(\mathbf{w}) \subset S$. *If* $A^{(d)}_{E_1,S_1}, \dots, A^{(d)}_{E_l,S_l}$ *are invertible, then* $\mathbf{w} = \mathbf{0}$.

Example 4.13. We continue Example 4.9. With $\lambda = (2,1,1,1,1,1)$ we obtain the following decomposition and pairing matrix

$$S_1 = \{(0,0),(0,4)\}, S_2 = \{(2,0)\}, S_3 = \emptyset, S_4 = \{(4,1)\}, S_5 = \{(5,0)\}, S_6 = \{(6,0)\},$$
$$E_1 = \{0,1\}, E_2 = \{2\}, E_3 = \emptyset, E_4 = \{4\}, E_5 = \{5\}, E_6 = \{6\}$$
$$A_{E,S}^{(d)} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 \end{bmatrix}.$$

The symbol $*$ stands for arbitrary entries. The pairing matrix is invertible, so no nonzero outcome with support in $S = \{(0,0),(0,4),(2,0),(4,1),(5,0),(6,0)\}$ exists.

4.3 Implementation of the Hyperfield Criterion

The Invertibility Criterion inspires the following algorithm for determining if a set of indices $S \subset V_d$ can actually be the support of some nonzero outcome in $\mathbb{Z}^{V_d}$. It is later used in the proof of Theorem 7.5 and Proposition 9.2.

Algorithm 2 Only Zero Outcome

Require: Positive support set $S \subset V_d$
Ensure: `True` only if $\mathbf{0}$ is the only outcome $\mathbf{w}$ with $\mathrm{supp}^+(\mathbf{w}) \subset S$, `False` if unconclusive

1: `support` $\leftarrow \{(0,0)\} \cup$ `S`
2: `E` $\leftarrow \{0, \ldots, |$`support`$| - 1\}$
3: `P` $\leftarrow$ `build_pairing_matrix`$(d,$ `E`, `support`$)$
4: **return** `rank`(`P`) $\neq |$`support`$|$

The function `build_pairing_matrix` constructs the pairing matrix $A_{E,S}^{(d)}$ as defined in Definition 4.5. If this Algorithm is inconclusive, we can try to use another set $E \subset \{0, \ldots, d\}$. This leads to the following generalization:

Algorithm 3 Only Zero Outcome (Generalized)

Require: Positive support set $S \subset V_d$, set $E \subset \{0, \ldots, d\}$ with size $|S| + 1$
Ensure: `True` only if $\mathbf{0}$ is the only outcome $\mathbf{w}$ with $\mathrm{supp}^+(\mathbf{w}) \subset S$, `False` if unconclusive

1: `support` $\leftarrow \{(0,0)\} \cup$ `S`
2: `P` $\leftarrow$ `build_pairing_matrix`$(d,$ `E`, `support`$)$
3: **return** `rank`(`P`) $\neq |$`support`$|$

Algorithm `Only Zero Outcome` is clearly a special case of Algorithm `Only Zero Outcome (Generalized)` with $E = \{0, \ldots, |S|\}$.

4.4 Symmetry

With the Invertibility Criterion we can exclude certain supports from being the supports of valid outcomes. We will now show that the supports of valid outcomes are symmetric with respect to the main diagonal.

Proposition 4.14. *Let* $\mathbf{w} = (w_{i,j})_{(i,j)\in V_d}$ *be a configuration in* $\mathbb{Z}^{V_d}$*. Then* $\mathbf{w}$ *is an outcome if and only if* $\tilde{\mathbf{w}} \coloneqq (w_{j,i})_{(i,j)\in V_d}$ *is an outcome.*

Proof. Observe that

$$\operatorname{diag}(k) = \sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} x_{i,j} = \sum_{(i,j)\in V_d} \binom{d-i-j}{d-i-j-(k-i)} x_{i,j} = \sum_{(i,j)\in V_d} \binom{d-i-j}{d-k-j} x_{i,j}.$$

Let $\mathbf{w}$ be a valid outcome. Then, $\operatorname{diag}(k)(\mathbf{w}) = 0$ for all $k = 0, \dots, d$. Thus, for all $k = 0, \dots, d$ we have

$$\begin{aligned}\operatorname{diag}(k)(\tilde{\mathbf{w}}) = \sum_{(i,j)\in V_d} \binom{d-i-j}{k-j} w_{j,i} = \sum_{(i,j)\in V_d} \binom{d-i-j}{d-k-i} w_{j,i} &= \sum_{(i,j)\in V_d} \binom{d-i-j}{d-k-j} w_{i,j} \\ &= \sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} w_{i,j} \\ &= \operatorname{diag}(k)(\mathbf{w}) = 0.\end{aligned}$$

∎

Example 4.15. We have established that the support on the left-hand side cannot correspond to a valid outcome. By symmetry, the support on the right-hand side also cannot correspond to a valid outcome.

$$\begin{array}{lllllll} \cdot \\ \cdot & \cdot \\ * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & * & \cdot \\ * & \cdot & * & \cdot & \cdot & * & * \end{array} \qquad\qquad \begin{array}{lllllll} * \\ * & \cdot \\ \cdot & * & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot & * & \cdot & \cdot \end{array}$$

Next, we introduce another kind of symmetry. Let $\mathbf{w} = (w_{i,j})$ be a configuration in $\mathbb{Z}^{V_d}$. We define

$$\mathbf{w} \mapsto \hat{\mathbf{w}} \coloneqq \left((-1)^{d-j} w_{j,d-i-j}\right)_{(i,j)\in V_d}.$$

Proposition 4.16. *Let* $\mathbf{w} = (w_{i,j})_{(i,j)\in V_d}$ *be a configuration in* $\mathbb{Z}^{V_d}$. *Then* $\mathbf{w}$ *is an outcome if and only if* $\hat{\mathbf{w}} := \big((-1)^{d-j} w_{j,d-i-j}\big)_{(i,j)\in V_d}$ *is an outcome.*

Proof. Let $k = 0, \dots, d$. We have

$$\begin{aligned}
\operatorname{col}(k)(\hat{\mathbf{w}}) &= (-1)^k \sum_{(i,j)\in V_d} (-1)^j \binom{i}{k-j} (-1)^{d-j} w_{j,d-i-j} \\
&= (-1)^{d-k} \sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} w_{i,j} \\
&= (-1)^{d-k} \operatorname{diag}(k)(\mathbf{w}) = 0.
\end{aligned}$$

□

The symmetry just introduced can be interpreted in the following way: we define a group action of the symmetry group S_3 on $\mathbb{Z}^{V_d}$ by

$$(12)\cdot\mathbf{w} = \tilde{\mathbf{w}} \quad \text{and} \quad (123)\cdot\mathbf{w} = \hat{\mathbf{w}}.$$

Then, the group actions (12), (13) and (23) can be depicted in Figure 4.1.

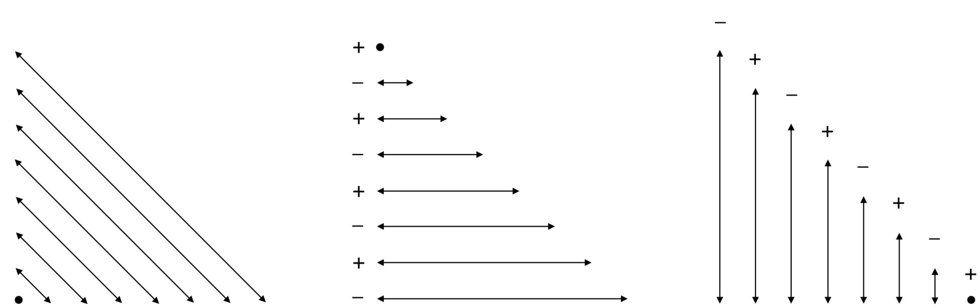

Figure 4.1: The left most illustration shows mirroring the configuration with respect to the main diagonal. The middle illustration shows switching the order on the same row while also alternating the signs of the row. The right most illustration shows switching the order on the same column while also alternating the signs of the column.

4.5 Impossible Supports

Now, we show that specific supports cannot be the supports of valid integral outcomes. Hence, the title of this section *Impossible Supports*. For instance, can we have an outcome whose support is only contained in $S = \{(0,0),(0,i)\}$ for some $i \in \mathbb{N}$? We will show that this is not possible.

Proposition 4.17. *Let $d \in \mathbb{N}$, and $i = 0, \dots, d$. If $S = \{(0,i)\}$ and $E = \{0\}$, then $A^{(d)}_{E,S}$ is invertible.*

Proof. We have $A^{(d)}_{E,S} = \begin{bmatrix} 1 \end{bmatrix}$, which is invertible. □

Proposition 4.18. *Let $d \in \mathbb{N}$. Assume $i, j = 0, \dots, d$ with $i < j$. If $S = \{(0,i),(0,j)\}$ and $E = \{0,1\}$, then $A^{(d)}_{E,S}$ is invertible.*

Proof. We have $A^{(d)}_{E,S} = \begin{bmatrix} 1 & 1 \\ d-i & d-j \end{bmatrix}$, which is invertible. □

Proposition 4.19. *Let $d \in \mathbb{N}$. Assume $i, j, k = 0, \dots, d$ with $i < j < k$. If $S = \{(0,i),(0,j),(0,k)\}$ and $E = \{0,1,2\}$, then $A^{(d)}_{E,S}$ is invertible.*

Proof. We have

$$A^{(d)}_{E,S} = \begin{bmatrix} \binom{d-i}{0} & \binom{d-j}{0} & \binom{d-k}{0} \\ \binom{d-i}{1} & \binom{d-j}{1} & \binom{d-k}{1} \\ \binom{d-i}{2} & \binom{d-j}{2} & \binom{d-k}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ d-i & d-j & d-k \\ \frac{(d-i)(d-i-1)}{2} & \frac{(d-j)(d-j-1)}{2} & \frac{(d-k)(d-k-1)}{2} \end{bmatrix}.$$

We substitute $x = d-i, y = d-j$, and $z = d-k$. Then, we see

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} A^{(d)}_{E,S} = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}.$$

The matrix on the right-hand side is invertible because it is a Vandermonde matrix. Thus, the pairing matrix $A^{(d)}_{E,S}$ is invertible. □

Proposition 4.20. *Let $d \in \mathbb{N}$. Assume $i, j = 0, \dots, d$ with $i < j$. Moreover, let $k = 0, \dots, d-1$. If $S = \{(0,i),(0,j),(1,k)\}$, $E = \{0,1,2\}$, and $i + j \neq 2k + 1$, then $A^{(d)}_{E,S}$ is invertible.*

Proof. We have

$$A^{(d)}_{E,S} = \begin{bmatrix} \binom{d-i}{0} & \binom{d-j}{0} & \binom{d-k-1}{-1} \\ \binom{d-i}{1} & \binom{d-j}{1} & \binom{d-k-1}{0} \\ \binom{d-i}{2} & \binom{d-j}{2} & \binom{d-k-1}{1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ d-i & d-j & d-k-1 \\ \frac{(d-i)(d-i-1)}{2} & \frac{(d-j)(d-j-1)}{2} & \frac{(d-k-1)(d-k-2)}{2} \end{bmatrix}.$$

We substitute $x = d-i, y = d-j$, and $z = d-k-1$. Then, we see

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} A^{(d)}_{E,S} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{x-y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & x+y & 2z+1 \end{bmatrix}.$$

We see that the determinant is nonzero because $x + y \neq 2z + 1$ by $i + j \neq 2k + 1$. □

Remark 4.21. Without loss of generality, we may assume

$$S \subset \{(i,j) \in V_d \mid i < |S|\} \quad \text{and} \quad E = \{0,1,\ldots,|S|-1\}$$

because the matrices $A^{(d)}_{E,S} = A^{(d-\rho)}_{E-\rho,S-\rho}$ are equal, where $\rho \coloneqq \min\{E \cup \{i \mid (i,j) \in S\}\}$ and $E-\rho \coloneqq \{(i-\rho,j) \mid (i,j) \in E\}$. This assumption allows us to apply the previous propostions to more general S and E.

Example 4.22. Assume we have a configuration with support $S = \{(0,i),(0,j),(0,k)\}$ for $0 \le i < j < k \le d$. By Proposition 4.19, we know that no valid integral nonzero outcome can have this support.

Now, let us consider a configuration $\mathbf{w}$ with support $\operatorname{supp}(\mathbf{w}) \subset S$ such that S can be decomposed into $S_1,\ldots,S_l$ as described before. Let $\ell = 1,\ldots,l$. If $S_\ell = \{(x,i),(x,j),(x,k)\}$ for $0 \le i < j < k \le d$ and $x \in \mathbb{N}$, then $\mathbf{w} = \mathbf{0}$ by Proposition 4.19 and the previous comment on the generality of S and E.

For instance, this configuration is not an outcome

```
*
.  .
.  .  *
.  .  .  .
.  .  .  *  .
.  .  .  .  .  .
.  .  .  *  .  .  .
*  .  .  *  .  .  .  .
```

where $*$ denotes a non-zero entry. This is because for $\lambda = (3,3,1,1)$ we have $S_2 = \{(3,0),(3,1),(3,3)\}$. Similarly, these configurations are not outcomes

```
*                          *
.  .                       .  .
.  .  *                    .  .  *
.  .  .  .                 .  .  .  .
.  .  .  .  .              .  .  .  .  .
.  .  .  *  .  .           .  .  .  .  .  *
.  .  .  *  .  .  .        .  .  .  .  .  *  .
*  .  .  *  .  .  .  .     *  .  .  .  .  *  .  .
```

Chapter 5

Valid Outcomes of Positive Support Size One, Two, and Three

All the tools are ready to show the following three theorems, which were first proved in [2].

Theorem 5.1. *No valid integral outcomes of positive support size one exists.*

Theorem 5.2. *For valid integral outcomes* $\mathbf{w}$ *with* $|\mathrm{supp}^+(\mathbf{w})| = 2$ *we have* $\deg(\mathbf{w}) = 1$.

Theorem 5.3. *For valid integral outcomes* $\mathbf{w}$ *with* $|\mathrm{supp}^+(\mathbf{w})| = 3$ *we have* $\deg(\mathbf{w}) \leq 3$.

This proves our Main Theorem 3.32 for the case of positive support size three or less, i.e.

$$\deg(\mathbf{w}) \leq 2 \cdot |\mathrm{supp}^+(\mathbf{w})| - 3$$

for all valid integral outcomes $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| \leq 3$.

We start with the proof of the first theorem.

Proof of Theorem 5.1. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid integral outcome. Since it is valid, we either have an empty negative support or a negative support that only contains $(0,0)$. If the negative support is empty, then $\mathbf{w} = \mathbf{0}$ by Proposition 3.29. Hence, we assume $w_{0,0} < 0$.

Now, consider the Pascal form $\mathrm{diag}(0) = \sum c_{i,j} x_{i,j}$. We have $c_{0,0} = c_{0,1} = \cdots = c_{0,d} = 1$ and $c_{i,j} = 0$ for everything else. Similarly, we have for the Pascal form $\mathrm{diag}(d) = \sum c'_{i,j} x_{i,j}$ that $c'_{\cdot,0} = \mathbf{1}$ and $c'_{i,j} = 0$ for everything else. Since outcomes are roots of Pascal forms, we have $\mathrm{diag}(0)(\mathbf{w}) = \mathrm{diag}(d)(\mathbf{w}) = 0$. Since $w_{0,0} < 0$ we must have $w_{0,j} > 0$ and $w_{i,0} > 0$ for some $i, j > 0$. Hence, $\mathbf{w}$ has positive support size at least two. $\square$

Next, we prove the second theorem.

Proof of Theorem 5.2. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be an integral outcome with positive support size two and degree d. By the previous proof, we see that $\mathrm{supp}^+(\mathbf{w}) = \{(0,j),(i,0)\}$.

Without loss of generality, we assume $i = d$. We want to show that $j = d$. Consider the Pascal form $\mathrm{row}(d) = \sum c_{i,j} x_{i,j}$, which has only nonzero coefficients $c_{i,j}$ for $i + j = d$.

- If d is odd, we have $c_{d,0} = 1$ and $c_{0,d} = -1$. Since $\operatorname{row}(d)(\mathbf{w}) = 0$, we have $j = d$.
- If d is even, we have $c_{d,0} = c_{0,d} = 1$. Thus, $\operatorname{row}(d)(\mathbf{w}) \neq 0$ for all $j = 0, \dots, d$. Hence, valid outcomes with positive support size two do not exist for even degrees.

From now on, we assume $\operatorname{supp}^+(\mathbf{w}) = \{(0,d),(d,0)\}$. For sake of contradiction, let $d \geq 2$. Then, we can divide the support $\operatorname{supp}(\mathbf{w}) = \{(0,0),(0,d),(d,0)\}$ via $\lambda = (2,1,\dots,1)$ as in Chapter 4.2 to obtain $S_1 = \{(0,0),(0,d)\}$, $S_k = \emptyset$, and $S_l = \{(d,0)\}$. By Proposition 4.18, the pairing matrix induced by S_1 and $E_1 = \{0,1\}$ is invertible. For S_l we apply Proposition 4.17 and Remark 4.21 to get that the induced pairing matrix is invertible. By Corollary 4.12, the outcome $\mathbf{w}$ is zero, which has an empty positive support. This is a contradiction to the assumption that the positive support size is two. Hence, the degree d equals one. $\square$

Example 5.4. The previous theorem shows that the only valid integral outcomes with positive support size two are multiples of

```
 1
-1  1.
```

It remains to prove Theorem 5.3. For that consider the following lemma which characterizes the possible supports of valid integral outcomes with positive support size three.

Proposition 5.5. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid integral outcome of degree d. If the positive support size of $\mathbf{w}$ is three, then one of the following holds:*

(1) We have $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,j)\}$ *for some* $i,j > 0$ *with* $i+j < d$.

(2) We have $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,d-i)\}$ *for some* $i = 1,\dots,d-1$.

(3) We have $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,0)\}$ *for some* $i = 1,\dots,d-1$.

(4) We have $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(0,i)\}$ *for some* $i = 1,\dots,d-1$.

(5) We have $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,e),(d-f,f)\}$ *for some* $e,f = 1,\dots,d-1$.

(6) We have $\operatorname{supp}(\mathbf{w}) = \{(0,0),(0,d),(e,0),(d-f,f)\}$ *for some* $e,f = 1,\dots,d-1$.

Proof. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid integral outcome of degree d. Assume $\{(0,0),(d,0),(0,d)\} \subset \operatorname{supp}(\mathbf{w})$. Clearly, statement 1, 2, 3, or 4 must hold.

So assume $(0,d) \notin \operatorname{supp}(\mathbf{w})$ and $(d,0) \notin \operatorname{supp}(\mathbf{w})$. As in the proof of Theorem 5.1, consider the Pascal form $\operatorname{diag}(0) = \sum c_{i,j}x_{i,j}$. We have $c_{0,\cdot} = \mathbf{1}$ and $c_{i,j} = 0$ for everything else. Similarly, we have for the Pascal form $\operatorname{diag}(d) = \sum c'_{i,j}x_{i,j}$ that $c'_{\cdot,0} = \mathbf{1}$ and $c'_{i,j} = 0$ for everything else. Since outcomes are roots of Pascal forms, we have $\operatorname{diag}(0)(\mathbf{w}) = \operatorname{diag}(d)(\mathbf{w}) = 0$. Due to $w_{0,0} < 0$, we conclude $w_{0,j} > 0$ and $w_{i,0} > 0$ for some $i,j > 0$. Thus, we have $\{(i,0),(0,j)\} \subset \operatorname{supp}(\mathbf{w})$ for some $i,j = 1,\dots,d-1$ using the assumption

$(0,d) \notin \operatorname{supp}(\mathbf{w})$ and $(d,0) \notin \operatorname{supp}(\mathbf{w})$. Since $\mathbf{w}$ is of degree d, there exists $w_{k,d-k} > 0$ for some $k = 1, \dots, d-1$. However, $\operatorname{row}(d)(\mathbf{w}) = 0$ implies that there must be some $w_{h,d-h} > 0$ for some $h \neq k$; this h cannot equal 0 or d. Thus, the positive support size of $\mathbf{w}$ is at least four, which is a contradiction. Hence, we must have $(d,0) \in \operatorname{supp}(\mathbf{w})$ or $(0,d) \in \operatorname{supp}(\mathbf{w})$.

Let $(d,0) \in \operatorname{supp}(\mathbf{w})$ and $(e,0) \in \operatorname{supp}(\mathbf{w})$ for some $e = 1, \dots, d-1$. Now using the same argument as before, there must exist some $w_{f,d-f} > 0$ for some $f = 1, \dots, d-1$; otherwise $\operatorname{row}(d)(\mathbf{w}) > 0$ which is a contradiction since $\mathbf{w}$ is a root of all Pascal forms. This proves statement 5.

The proof for statement 6 is analogous. □

Knowing the possible supports of valid integral outcomes with positive support size three, we apply the Invertibility Criterion 4.12 to each possible support to prove Theorem 5.3.

Proposition 5.6. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid integral outcome of degree d. If $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,j)\}$ for some $i,j > 0$ with $i+j < d$, then $d = 3$ and $(i,j) = (1,1)$.*

Proof. Let $i > 1$. Choose $\lambda = (2,1,\dots,1)$. Then, $E_1 = \{0,1\}, S_1 = \{(0,0),(0,d)\}$, $E_{i-1} = \{i\}, S_{i-1} = \{(i,j)\}$, $E_{d-1} = \{d\}, S_{d-1} = \{(d,0)\}$ and, $E_k = S_k = \emptyset$ for all $k \in \{1,\dots,d-1\} \setminus \{1, i-1, d-1\}$. The pairing matrices $A^{(d)}_{E_n,S_n}$ are all invertible for $n = 1,\dots,d-1$. Hence, the pairing matrix $A^{(d)}_{\{0,1,i,d\},\operatorname{supp}(\mathbf{w})}$ is also invertible. By the Invertibility Criterion, $\mathbf{w}$ is the zero configuration, which is a contradiction. Thus, we have $i = 1$.

Now, we assume $j > 1$. The configuration $\tilde{\mathbf{w}} = (w_{ji})_{(i,j)\in V_d}$ is an outcome by Proposition 4.14 because $\mathbf{w}$ is an outcome. Then $\tilde{\mathbf{w}}$ has support $\{(0,0),(d,0),(0,d),(1,\cdot)\}$ by the previous argument. However, then we have $j = 1$ which is a contradiction. So, we have $j = 1$.

Finally, we need to show that the degree d equals three. For the sake of contradiction, assume $d > 3$. Then, we can choose $\lambda = (3,1,\dots,1)$. We obtain $E_1 = \{0,1,2\}$ and $S_1 = \{(0,0),(0,d),(1,1)\}$. By Proposition 4.20 this pairing matrix $A^{(d)}_{E_1,S_1}$ is invertible. The other relevant pairing matrix $A^{(d)}_{\{d\},\{(d,0)\}}$ is also invertible. Thus, the pairing matrix $A^{(d)}_{\{0,1,2,d\},\operatorname{supp}(\mathbf{w})}$ is invertible. By the Invertibility Criterion, the configuration $\mathbf{w}$ is the zero configuration, which is a contradiction. Hence, we have $d = 3$. □

Proposition 5.7. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid integral outcome of degree d. Assume the outcome $\mathbf{w}$ satisfies one of the following conditions:*

(1) $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,d-i)\}$ *for some $i = 1,\dots,d-1$,*

(2) $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,0)\}$ *for some $i = 1,\dots,d-1$,*

(3) $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(0,i)\}$ *for some $i = 1,\dots,d-1$.*

Then, $d = 2$ and $i = 1$ hold.

Proof. Assume $d > 2$. Let $\mathbf{w}$ satisfy the third condition. Choose $\lambda = (3, 1, \dots, 1)$. Then, apply Proposition 4.19. So, we have that the pairing matrix is invertible. So, $\mathbf{w} = 0$ which is a contradiction. Thus, $d = 2$. By symmetry we have the same result for the second condition.

We want to show $d = 2$ for all outcomes $\mathbf{w}$ satisfying the first condition. Let $\mathbf{w}'$ satisfy the second condition. Then $\mathbf{w} = (123)\mathbf{w}'$ holds. Assume $d > 2$. By Proposition 4.16, we have found an outcome $\mathbf{w}'$ of degree at least three. This contradicts Proposition 5.7 that we have just shown for the second condition. Thus, $d = 2$ holds.

Finally, we have $i = 1$ because $i = 1, \dots, d - 1$ and $d = 2$. □

Proposition 5.8. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid integral outcome of degree* d. *If there exist* $e, f \in \{1, \dots, d-1\}$ *such that* $\operatorname{supp}(\mathbf{w}) = \{(0,0), (d,0), (0,e), (d-f, f)\}$ *or* $\operatorname{supp}(\mathbf{w}) = \{(0,0), (0,d), (e,0), (d-f,f)\}$, *then* $d = 2$ *and* $e = f = 1$ *holds.*

Proof. By Proposition 4.14, it suffices to show the statement for outcomes $\mathbf{w}$ satisfying the first condition. Let $d > 2$. If $f = d - 1$, then choose $\lambda = (3, 1, \dots, 1)$. This allows us to apply Proposition 4.20 because $0 + e \neq 2d - 1$ for $d > 1$. So, $\mathbf{w} = \mathbf{0}$ which is a contradiction. Thus, we have $f < d - 1$. Then, we can choose $\lambda = (2, 1, \dots, 1)$. Use Proposition 4.18 to get that $\mathbf{w} = \mathbf{0}$. This is a contradiction. Hence, we have $d = 2$.

Let $d = 2$. Then, we have $e = f = 1$ by definition of e and f. □

Finally, we can prove Theorem 5.3.

Proof of Theorem 5.3. Use Proposition 5.5. For each case, either apply Proposition 5.6, Proposition 5.7, or Proposition 5.8 to show that the degree d equals two or three. □

Chapter 6

Constraints on Supports of Valid Outcomes

We introduce the *Hyperfield Criterion* and *Contractions*, along with the *Invertibility Criterion*, to prove the degree bound of valid outcomes. These criteria can be interpreted as constraints on the support of valid outcomes.

6.1 Hyperfield Criterion

Let us define the sign hyperfield. For some set A, the set 2^A denotes the power set of A.

Definition 6.1. Let $H \coloneqq \{-1, 0, 1\}$. We define the addition $+ : H \times H \to 2^H \setminus \{\emptyset\}$ on H as follows: $0 + x = \{x\}, 1 + 1 = \{1\}, 1 + (-1) = H$, and $(-1) + (-1) = \{-1\}$ for all $x \in H$. Multiplication $\times : H \times H \to H$ is defined as usual. We call H the *sign hyperfield*.

For singleton sets $\{x\}$, we often write x instead of $\{x\}$; thus, $1+1 = 1$ and $(-1)+0 = -1$.

Remark 6.2. The tuple $(H, +, \cdot, 0, 1)$ is called a *hyperfield*. For more details, see Section 6.1 of [2] or [1]. In summary, a hyperfield satisfies the following properties:

(1) The maps $+$ and $\cdot$ are symmetric;

(2) $(H \setminus \{0\}, \cdot, 1)$ is a group;

(3) $0 \cdot x = 0$ and $0 + x = x$ hold for all $x \in H$;

(4) $\bigcup_{q \in x+y}(q + z) = \bigcup_{q \in x+y}(x + q)$ hold for all $x, y, z \in H$;

(5) $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$ hold for all $a, x, y \in H$.

(6) An inverse element $y \in H$ exists for every $x \in H$ such that the set $x + y$ contains 0. This inverse element y is unique for every x and is denoted by $-x$.

Definition 6.3. A polynomial in n variables $x_1, \dots, x_n$ over H is a formal sum $f = \sum_{\mathbf{k} \in \mathbb{Z}^n_{\geq 0}} \lambda_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ with $\lambda_{\mathbf{k}} \in H$, where only a finite number of coefficients $\lambda_{\mathbf{k}}$ are non-zero, and $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_n^{k_n}$. The set of all polynomials in n variables over H is denoted by $H[x_1, \dots, x_n]$.

Definition 6.4. Let $\mathbf{x} \in H$. We define $f(\mathbf{x}) \coloneqq \sum_{\mathbf{k} \in \mathbb{Z}^n_{\geq 0}} \lambda_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \subset H$.

Definition 6.5. We say that f *vanishes* at $\mathbf{x} \in H$ if $0 \in f(\mathbf{x})$. We call $\mathbf{x}$ a *hyperfield root* of f in this case

Any *real* polynomial can be turned into a polynomial over the sign hyperfield by replacing the coefficients with elements of H. We can then evaluate the polynomial at any point in H.

Definition 6.6. Let $f = \sum \lambda_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \in \mathbb{R}[\mathbf{x}]$ be a polynomial over $\mathbb{R}$. We call $\operatorname{sign}(f) \coloneqq \sum_{\mathbf{k} \in \mathbb{Z}^n_{\geq 0}} \operatorname{sign}(\lambda_{\mathbf{k}}) \mathbf{x}^{\mathbf{k}} \in H[\mathbf{x}]$ the polynomial over H induced by f.

For sake of simplicity, we also write $\operatorname{sign}(\mathbf{w}) \coloneqq (\operatorname{sign}(w_1), \dots, \operatorname{sign}(w_n))$ for any $\mathbf{w} \in \mathbb{R}^n$.

Definition 6.7. A hyperfield Pascal form is a polynomial over H induced by a Pascal form.

Example 6.8. We illustrate the hyperfield versions of $\operatorname{diag}(i)$, $\operatorname{col}(i)$, and $\operatorname{row}(i)$ on V_5 for $i = 0, \dots, 5$. A dot represents zero, $+$ represents plus one, and $-$ represents minus one.

$$
\begin{array}{llllll} + \\ + & \cdot \\ + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}
\quad
\begin{array}{llllll} \cdot \\ + & + \\ + & + & \cdot \\ + & + & \cdot & \cdot \\ + & + & \cdot & \cdot & \cdot \\ + & + & \cdot & \cdot & \cdot & \cdot \end{array}
\quad
\begin{array}{llllll} \cdot \\ \cdot & \cdot \\ + & + & + \\ + & + & + & \cdot \\ + & + & + & \cdot & \cdot \\ + & + & + & \cdot & \cdot & \cdot \end{array}
\quad
\begin{array}{llllll} \cdot \\ \cdot & \cdot \\ \cdot & \cdot & \cdot \\ + & + & + & + \\ + & + & + & + & \cdot \\ + & + & + & + & \cdot & \cdot \end{array}
\quad
\begin{array}{llllll} \cdot \\ \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & + & + & + & + \\ + & + & + & + & + & \cdot \end{array}
\quad
\begin{array}{llllll} \cdot \\ \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & + & + & + & + & + \end{array}
$$

$$
\begin{array}{llllll} \cdot \\ \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & + & + & + & + & + \end{array}
\quad
\begin{array}{llllll} \cdot \\ \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & + & + & + & + \\ \cdot & - & - & - & - & - \end{array}
\quad
\begin{array}{llllll} \cdot \\ \cdot & \cdot \\ \cdot & \cdot & \cdot \\ + & + & + & + \\ \cdot & - & - & - & - \\ \cdot & \cdot & + & + & + & + \end{array}
\quad
\begin{array}{llllll} \cdot \\ \cdot & \cdot \\ + & + & + \\ \cdot & - & - & - \\ \cdot & \cdot & + & + & + \\ \cdot & \cdot & \cdot & - & - & - \end{array}
\quad
\begin{array}{llllll} \cdot \\ + & + \\ \cdot & - & - \\ \cdot & \cdot & + & + \\ \cdot & \cdot & \cdot & - & - \\ \cdot & \cdot & \cdot & \cdot & + & + \end{array}
\quad
\begin{array}{llllll} + \\ \cdot & - \\ \cdot & \cdot & + \\ \cdot & \cdot & \cdot & - \\ \cdot & \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot & \cdot & - \end{array}
$$

$$
\begin{array}{llllll} + \\ + & \cdot \\ + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}
\quad
\begin{array}{llllll} - \\ - & + \\ - & + & \cdot \\ - & + & \cdot & \cdot \\ - & + & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot & \cdot \end{array}
\quad
\begin{array}{llllll} + \\ + & - \\ + & - & + \\ + & - & + & \cdot \\ \cdot & - & + & \cdot & \cdot \\ \cdot & \cdot & + & \cdot & \cdot & \cdot \end{array}
\quad
\begin{array}{llllll} - \\ - & + \\ - & + & - \\ \cdot & + & - & + \\ \cdot & \cdot & - & + & \cdot \\ \cdot & \cdot & \cdot & + & \cdot & \cdot \end{array}
\quad
\begin{array}{llllll} + \\ + & - \\ \cdot & - & + \\ \cdot & \cdot & + & - \\ \cdot & \cdot & \cdot & - & + \\ \cdot & \cdot & \cdot & \cdot & + & \cdot \end{array}
\quad
\begin{array}{llllll} - \\ \cdot & + \\ \cdot & \cdot & - \\ \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot & - \\ \cdot & \cdot & \cdot & \cdot & \cdot & + \end{array}
$$

The reason we introduced the sign hyperfield is that it allows us to neglect the concrete values of the coefficients of a polynomial and focus on their signs. This makes reasoning about roots easier, which is helpful since we saw in earlier chapters that chipsplitting outcomes are roots of Pascal forms.

Proposition 6.9. *Let $f \in \mathbb{R}[\mathbf{x}]$ be a real polynomial. Let $\mathbf{w} \in \mathbb{R}^n$ be a root of f. Then,* $\operatorname{sign}(\mathbf{w})$ *is a root of* $\operatorname{sign}(f)$*.*

Proof. Define $\mathbf{s} := \operatorname{sign}(\mathbf{w})$. Write $f = \sum \lambda_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ with real coefficients $\lambda_{\mathbf{k}}$. If $\lambda_{\mathbf{k}} \mathbf{w}^{\mathbf{k}} = 0$ for all $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$, then clearly the sign of $\lambda_{\mathbf{k}} \mathbf{w}^{\mathbf{k}}$ is zero; hence the sign of f is the singleton set $\{0\}$ when evaluated at $\mathbf{s}$. So, $\mathbf{s}$ is a root of $\operatorname{sign}(f)$.

Now, suppose that there exists some $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ such that $\lambda_{\mathbf{k}} \mathbf{w}^{\mathbf{k}} \neq 0$. Assume $\lambda_{\mathbf{k}} \mathbf{w}^{\mathbf{k}} > 0$. Then, there also exists some $\mathbf{j} \in \mathbb{Z}^n_{\geq 0}$ such that we have $\lambda_{\mathbf{j}} \mathbf{w}^{\mathbf{j}} < 0$; otherwise $f(\mathbf{w}) > 0$ which is a contradiction to $\mathbf{w}$ being a root of f. Thus, $\operatorname{sign}(f)(\mathbf{s})$ has summands of both signs, and hence $\operatorname{sign}(f)(\mathbf{s}) = H$. So $0 \in \operatorname{sign}(f)(\mathbf{s})$ holds. Therefore, $\mathbf{s}$ is a root of $\operatorname{sign}(f)$. □

Taking the contrapositive of the above proposition, we get the *Hyperfield Criterion* which was first presented by Bik and Marigliano in [2].

Proposition 6.10 (Hyperfield Criterion)**.** *Let $\mathbf{s} = (s_{i,j})_{(i,j)\in V_d} \in H^{V_d}$. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chipsplitting configuration with* $\operatorname{sign}(\mathbf{w}) = \mathbf{s}$*. If $\mathbf{s}$ is not a root of a hyperfield Pascal form, then $\mathbf{w}$ is not a chipsplitting outcome.*

Proof. Follows from Proposition 6.9 and Theorem 3.13. □

We call a vector $\mathbf{s} \in H^{V_d}$ a *sign configuration* or *hyperfield configuration.* We state definitions for sign configurations $\mathbf{s} \in H^{V_d}$ similar to those in Definition 3.6.

Definition 6.11. Let $\mathbf{s} \in H^{V_d}$ be a sign configuration. We define:

(1) The positive support is defined as $\operatorname{supp}^+(\mathbf{s}) := \{(i,j) \in V_d \mid s_{i,j} = 1\}$.

(2) The negative support is defined as $\operatorname{supp}^-(\mathbf{s}) := \{(i,j) \in V_d \mid s_{i,j} = -1\}$.

(3) The support is defined as $\operatorname{supp}(\mathbf{s}) := \operatorname{supp}^+(\mathbf{s}) \cup \operatorname{supp}^-(\mathbf{s})$.

(4) The degree of $\mathbf{s}$ is defined as $\deg(\mathbf{s}) := \max\{i+j \mid (i,j) \in \operatorname{supp}(\mathbf{s})\}$.

(5) We call $\mathbf{s}$ *valid* if its support is empty or $\operatorname{supp}^-(\mathbf{s}) = \{(0,0)\}$.

Lemma 6.12. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chipsplitting configuration. Then, we have:*

(1) $\operatorname{supp}^+(\operatorname{sign}(\mathbf{w})) = \operatorname{sign}^+(\mathbf{w})$,

(2) $\operatorname{supp}^-(\operatorname{sign}(\mathbf{w})) = \operatorname{sign}^-(\mathbf{w})$,

(3) $\deg(\operatorname{sign}(\mathbf{w})) = \deg(\mathbf{w})$.

Proof. Follows from the definitions. □

To make use of the Hyperfield Criterion, we investigate hyperfield forms induced by the Pascal forms $\mathrm{col}(k), \mathrm{row}(k)$, and $\mathrm{diag}(k)$.

Proposition 6.13. *Let $k = 0, \dots, d$. Define*

$$A_k^+ := \{(i,j) \in V_d \mid j = 0, \dots, k \text{ and } i = k-j, \dots, d-j \text{ with } j \equiv k \pmod 2\},$$
$$A_k^- := \{(i,j) \in V_d \mid j = 0, \dots, k \text{ and } i = k-j, \dots, d-j \text{ with } j \not\equiv k \pmod 2\}.$$

Then, the following statements hold:

(1) We have $\mathrm{sign}(\mathrm{diag}(k)) = \sum_{i=0}^{k} \sum_{j=0}^{d-k} x_{i,j}$.

(2) We have $\mathrm{sign}(\mathrm{col}(k)) = \sum_{(i,j)\in A_k^+} x_{i,j} - \sum_{(i,j)\in A_k^-} x_{i,j}$.

(3) We have $\mathrm{sign}(\mathrm{row}(k)) = \sum_{(i,j)\in A_k^+} x_{j,i} - \sum_{(i,j)\in A_k^-} x_{j,i}$.

Proof. The first statement follows directly from Proposition 4.3 since $i \leq k$ and $d-i-j \geq k-i$ must hold for the binomial coefficient to be non-zero. The second and third statement follow similarly from Proposition 3.18. □

Proposition 6.14. *Let $\mathbf{s} \in H^{V_d}$ be a valid nonzero sign configuration. The following statements hold:*

(1) Let $k = 0, \dots, d$. If $0 \in \mathrm{sign}(\mathrm{diag}(k))(\mathbf{s})$, then $\mathrm{sign}(\mathrm{diag}(k))(\mathbf{s}) = H$.

(2) If $0 \in \mathrm{sign}(\mathrm{col}(k))(\mathbf{s})$ for all $k = 0, \dots, d$, then $\mathrm{sign}(\mathrm{col}(k))(\mathbf{s}) = H$.

(3) If $0 \in \mathrm{sign}(\mathrm{row}(k))(\mathbf{s})$ for all $k = 0, \dots, d$, then $\mathrm{sign}(\mathrm{row}(k))(\mathbf{s}) = H$.

Proof. We see that $\mathbf{s}$ has at least degree $d \geq 1$ since it is nonzero and valid. All $s_{i,j}$ equal one for $i + j > 0$, and there exists $s_{k,d-k} = 1$ for some $k = 0, \dots, d$.

(1) Assume $0 \in \mathrm{sign}(\mathrm{diag}(k))(\mathbf{s})$. By Proposition 6.13, we have $0 \in \mathrm{sign}(\mathrm{diag}(k))(\mathbf{s}) = \sum_{i=0}^{k} \sum_{j=0}^{d-k} s_{i,j}$. We know that $s_{0,0} = -1$. So, we have $s_{i,j} = 1$ for some i, j with $i + j > 0$. Thus, $\mathrm{sign}(\mathrm{diag}(k))(\mathbf{s}) = H$.

(2) First note that $\mathrm{col}(0) = \mathrm{diag}(d)$. So, the case $k = 0$ is proven. Let $k > 0$. We start with $k = d$. Then, the union of A_d^+ and A_d^- consists exactly of vertices of degree d. Since $\mathrm{sign}(\mathrm{col}(d))(\mathbf{s}) = \sum_{(i,j)\in A_d^+} s_{i,j} - \sum_{(i,j)\in A_d^-} s_{i,j}$ contains zero, we have $s_{i,j} = 1$ for some $(i,j) \in A_d^+$, and $s_{i',j'} = -1$ for some $(i',j') \in A_d^-$. Hence, $\mathrm{sign}(\mathrm{col}(d))(\mathbf{s}) = H$.

Let $k = d-1$. Then, $s_{i,j} = 1$ for some $(i,j) \in A_{k+1}^+$, and $s_{i',j'} = -1$ for some $(i',j') \in A_{k+1}^-$. Note that $A_{k+1}^- \subset A_k^+$ by definition. Since $\mathrm{sign}(\mathrm{col}(k))(\mathbf{s}) = \sum_{(i,j)\in A_k^+} s_{i,j} - \sum_{(i,j)\in A_k^-} s_{i,j}$ contains zero, we have $s_{i'',j''} = -1$ for some $(i'',j'') \in A_k^-$. Hence, $\mathrm{sign}(\mathrm{col}(k))(\mathbf{s}) = H$.

Repeat this argument for $k = d-2, \dots, 1$ to show that $\mathrm{sign}(\mathrm{col}(k))(\mathbf{s}) = H$.

(3) The proof is analogous to the previous case.

□

Corollary 6.15. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome. Then, we have* $\operatorname{sign}(p)(\operatorname{sign}(\mathbf{w})) = H$ *for all* $p \in \{\operatorname{diag}(k), \operatorname{col}(k), \operatorname{row}(k) \mid k = 0, \ldots, d\}$.

Proof. This follows from Theorem 3.13, Proposition 6.9, and Proposition 6.14. □

Example 6.16. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome of degree $d = 5$. By the previous corollary and Example 6.8, we know that the outcome $\mathbf{w}$ has at least one positive entry $w_{i,j}$ in each of the following marked areas $+$ because $w_{0,0} < 0$:

$$
\begin{array}{llllll}
+ \\
+ & \cdot \\
+ & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{llllll}
\cdot \\
+ & + \\
+ & + & \cdot \\
+ & + & \cdot & \cdot \\
+ & + & \cdot & \cdot & \cdot \\
+ & + & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{llllll}
\cdot \\
\cdot & \cdot \\
+ & + & + \\
+ & + & + & \cdot \\
+ & + & + & \cdot & \cdot \\
+ & + & + & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{llllll}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
+ & + & + & + \\
+ & + & + & + & \cdot \\
+ & + & + & + & \cdot & \cdot
\end{array}
\quad
\begin{array}{llllll}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
+ & + & + & + & + \\
+ & + & + & + & + & \cdot
\end{array}
\quad
\begin{array}{llllll}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
+ & + & + & + & + & +
\end{array}
$$

Moreover, for each triangle below the outcome $\mathbf{w}$ must have some $w_{i,j} > 0$ for one vertex (i, j) in the plus area $+$ and $w_{i,j} > 0$ for another vertex (i', j') in the minus area $-$ because $\operatorname{sign}(\operatorname{col})(\operatorname{sign}(\mathbf{w})) = H$.

$$
\begin{array}{llllll}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
+ & + & + & + & + \\
\cdot & - & - & - & - & -
\end{array}
\quad
\begin{array}{llllll}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
+ & + & + & + \\
\cdot & - & - & - & - \\
\cdot & \cdot & + & + & + & +
\end{array}
\quad
\begin{array}{llllll}
\cdot \\
\cdot & \cdot \\
+ & + & + \\
\cdot & - & - & - \\
\cdot & \cdot & + & + & + \\
\cdot & \cdot & \cdot & - & - & -
\end{array}
\quad
\begin{array}{llllll}
\cdot \\
+ & + \\
\cdot & - & - \\
\cdot & \cdot & + & + \\
\cdot & \cdot & \cdot & - & - \\
\cdot & \cdot & \cdot & \cdot & + & +
\end{array}
\quad
\begin{array}{llllll}
+ \\
\cdot & - \\
\cdot & \cdot & + \\
\cdot & \cdot & \cdot & - \\
\cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & -
\end{array}
$$

Similarly, the statement holds for sign(row):

$$
\begin{array}{llllll}
- \\
- & + \\
- & + & \cdot \\
- & + & \cdot & \cdot \\
- & + & \cdot & \cdot & \cdot \\
\cdot & + & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{llllll}
+ \\
+ & - \\
+ & - & + \\
+ & - & + & \cdot \\
\cdot & - & + & \cdot & \cdot \\
\cdot & \cdot & + & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{llllll}
- \\
- & + \\
- & + & - \\
\cdot & + & - & + \\
\cdot & \cdot & - & + & \cdot \\
\cdot & \cdot & \cdot & + & \cdot & \cdot
\end{array}
\quad
\begin{array}{llllll}
+ \\
+ & - \\
\cdot & - & + \\
\cdot & \cdot & + & - \\
\cdot & \cdot & \cdot & - & + \\
\cdot & \cdot & \cdot & \cdot & + & \cdot
\end{array}
\quad
\begin{array}{llllll}
- \\
\cdot & + \\
\cdot & \cdot & - \\
\cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & +
\end{array}
$$

The above example demonstrates that we can view Corollary 6.15 as constraints on the support of a valid outcome $\mathbf{w}$. Configurations that do not satisfy these constraints are not valid outcomes.

Corollary 6.17. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome of degree $d \geq 1$. Then, all of the following constraints hold:*

(1) For all $k = 0, \ldots, d$, the positive support of $\mathbf{w}$ contains the vertex (i, j) for at least one $i = 0, \ldots, k$ and $j = 0, \ldots, d - k$.

(2) For all $k = 1, \ldots, d$, the positive support of $\mathbf{w}$ contains at least one $(i, j) \in A_k^+$ and $(i', j') \in A_k^-$.

(3) For all $k = 1, \ldots, d$, the positive support of $\mathbf{w}$ contains at least one (i, j) and (i', j') from $(j, i) \in A_k^+$ and $(j', i') \in A_k^-$.

We will use these constraints to efficiently compute all outcomes of degree some d.

6.2 Solving Hyperfield Linear Systems

We deviate from the approach by Bik and Marigliano in [2] by spending some considerate time in solving hyperfield linear systems. It will help us path the way for some breakthrough for the efficient computation of valid outcomes.

Fix some degree d. Due to the Hyperfield Criterion, we are interested in valid configurations $\mathbf{w} \in \mathbb{Z}^{V_d}$ satisfying $0 \in \operatorname{sign}(p)(\operatorname{sign}(\mathbf{w}))$ for all $p \in \{\operatorname{diag}(k), \operatorname{row}(k), \operatorname{col}(k)\}_{k=0}^{d}$. These configurations are potential valid outcomes. Let us consider how we can efficiently compute such valid configurations.

Problem: Given a set of linear forms $A = \{p_1, \ldots, p_k\}$, compute the solution set $V(A) := \{\mathbf{x} \in H^{V_d} : 0 \in \operatorname{sign}(p_i)(\mathbf{x}) \quad \forall i = 1, \ldots, k\}$.

We further simplify the problem by only considering configurations $\mathbf{x}$ with $\operatorname{supp}^-(\mathbf{x}) = \{(0, 0)\}$ and $|\operatorname{supp}^+(\mathbf{x})| = n$ for some fixed $n \in \mathbb{N}$.

Problem: Given a set of linear forms $A = \{p_1, \ldots, p_k\}$, compute the solution set $S_n(A) := V(A) \cap \{\mathbf{x} \in H^{V_d} : \operatorname{supp}^-(\mathbf{x}) = \{(0, 0)\}, |\operatorname{supp}^+(\mathbf{x})| = n\}$.

Note that $S_n(A)$ is a superset of valid outcomes of positive support size n, which will be useful in finding all valid outcomes.

A Naive Approach

To compute $S_n(A)$ a simple brute force algorithm can be used; just iterate over all positive support size n supports and check if they are hyperfield roots of some hyperfield Pascal basis.

The naive approach has exponential time complexity as it checks $\binom{(d+1)(d+2)/2}{n}$ supports.

Algorithm 4 Brute Force Algorithm

Require: Positive support size n, a set of linear forms $A = \{p_1, \ldots, p_k\}$
Ensure: $S_n(A)$

function SOLVE(A, n)
    initialize empty list `solutions`
    for n-combination $S = \{(c_i, r_i) : i = 1, \ldots, n\}$ of V_d **do**
        initialize $\mathbf{x} \in H^{V_d}$ with positive support S and $x_{0,0} = -1$
        if $\mathbf{x}$ is a hyperfield root of every $p \in A$ **then**
            add S to `solutions`
        end if
    end for
    return `solutions`
end function

Efficient Algorithm

For some systems A, we can greatly speed up the computation of $S_n(A)$.

Definition 6.18. Let p be a linear form in H^{V_d}, and let $\mathbf{x} \in H^{V_d}$ be some hyperfield root of p. If $\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(p) = \emptyset$, then the root $\mathbf{x}$ is called a *trivial root* of p. Otherwise, the root $\mathbf{x}$ is called a *non-trivial root* of p.

Definition 6.19. Let A be a system of linear forms in H^{V_d}. We say $S_n(A)$ is *non-trivial* if $S_n(A) \neq \emptyset$ and every $\mathbf{x} \in S_n(A)$ is a non-trivial root for every form $p \in A$. We say A is *non-trivial* if $S_n(A)$ is non-trivial.

Proposition 6.20. *Let A be a system of linear forms in H^{V_d}, $p \in A$ and $\mathbf{x} \in S_n(A)$. Then, the following statements hold:*

(1) If $(0,0) \in \operatorname{supp}^+(p)$, then $x_{i,j} = 1$ for some $(i,j) \in \operatorname{supp}^+(p)$.

(2) If $(0,0) \in \operatorname{supp}^-(p)$, then $x_{i,j} = 1$ for some $(i,j) \in \operatorname{supp}^-(p)$.

Proof. Assume $(0,0) \in \operatorname{supp}^+(p)$. Since $x_{0,0} = -1$, we have $-1 \in \operatorname{sign}(p)(x)$. By assumption, $\mathbf{x}$ is a hyperfield root of p, so $0 \in \operatorname{sign}(p)(\mathbf{x})$. This can only happen if $x_{i,j} = 1$ for some $(i,j) \in \operatorname{supp}^+(p)$. The case $(0,0) \in \operatorname{supp}^-(p)$ is similar. □

The next proposition assumes that A is non-trivial.

Proposition 6.21. *Let A be a non-trivial system of linear forms, $p \in A$ and $\mathbf{x} \in S_n(A)$. If $(0,0) \notin \operatorname{supp}(p)$, then $\operatorname{supp}^+(p) \neq \emptyset$, $\operatorname{supp}^-(p) \neq \emptyset$ as well as $x_{i,j} = x_{i',j'} = 1$ for some $(i,j) \in \operatorname{supp}^+(p)$ and $(i',j') \in \operatorname{supp}^-(p)$.*

Proof. Assume $(0,0) \notin \operatorname{supp}(p)$. First, $\operatorname{supp}(p) \neq \emptyset$ because $S_n(A)$ is non-empty and consists only of non-trivial roots. If $\operatorname{supp}^+(p) = \emptyset$, then $\operatorname{supp}^+(x) \subset \operatorname{supp}^-(p) = \operatorname{supp}(p) \neq \emptyset$. Hence, $\operatorname{sign}(p)(x) = \{-1\}$, which contradicts $\mathbf{x}$ being a root. Thus, $\operatorname{supp}^+(p)$ is non-empty. Similarly, $\operatorname{supp}^-(p)$ is non-empty.

By non-triviality, $x_{i,j} = 1$ for some $(i,j) \in \operatorname{supp}(p)$. Assume $(i,j) \in \operatorname{supp}^+(p)$. Hence, $1 \in \operatorname{sign}(p)(\mathbf{x})$. Since $\mathbf{x}$ is a root, we also have $0 \in \operatorname{sign}(p)(\mathbf{x})$. This can only occur if $x_{i',j'} = 1$ for some $(i',j') \in \operatorname{supp}^-(p)$. The case $(i,j) \in \operatorname{supp}^-(p)$ is similar. $\square$

Both propositions allow us to interpret hyperfield linear forms in a non-trivial system A as constraints on the positive supports of roots in $S_n(A)$.

Example 6.22. Fix the degree $d = 3$. Assume a system A and some linear form $p \in A$. Further, assume $p = \operatorname{diag}(0)$. The support of p is represented by the following diagram:

```
+
+ .
+ . .
+ . . .
```

Since $x_{0,0}$ is negative, we see that any nonzero hyperfield root $\mathbf{x}$ of A satisfies $x_{i,j} = 1$ for some $(i,j) \in \{(0,0),(0,1),(0,2),(0,3)\}$

Now, assume A is non-trivial. Consider a row pascal equation $q = \operatorname{row}(3)$ which is contained in A. Its support is depicted by the following diagram:

```
-
. +
. . -
. . . +
```

For some $\mathbf{x}$ with $\operatorname{supp}^-(\mathbf{x}) = \{(0,0)\}$ to be a hyperfield root of q, we must have either

(1) $x_{i,j} = x_{i',j'} = 1$ for some $(i,j) \in \{(0,3),(2,1)\}$ and $(i',j') \in \{(3,0),(1,2)\}$, or

(2) $\operatorname{supp}(\mathbf{x}) \subset V_3 \setminus \operatorname{supp}(q)$.

Considering only non-trivial roots $\mathbf{x}$ lets us exclude the latter case. Thus, if we have a non-trivial system A with $p, q \in A$, to compute $S_n(A)$, it suffices to check only those hyperfield roots whose support intersected with each of the three following regions is non-empty:

```
+            +            .
+ .          . .          . +
+ . .        . . +        . . .
+ . . .      . . . .      . . . +
```

Here are examples of such hyperfield roots:

```
+                .
.  +             .  .
.  .  .          +  .  +
.  .  .  .       .  .  .  +
```

Definition 6.23. To each linear form p in H^{V_d} we can associate a finite set of supports, which we call $\text{constraints}(p) := \{\text{supp}^+(p) \setminus \{(0,0)\}, \text{supp}^-(p) \setminus \{(0,0)\}\} \subset 2^{V_d}$.

The name is justified by the following proposition.

Proposition 6.24. *Let p be a linear form in $\mathbb{H}^{V_d}$, and $\mathbf{x} \in H^{V_d}$ with $\text{supp}^-(\mathbf{x}) = \{(0,0)\}$. Then, $\mathbf{x}$ is a non-trivial hyperfield root of p if and only if $\text{supp}^+(\mathbf{x}) \cap S \neq \emptyset$ for all $S \in \text{constraints}(p)$.*

Proof. Since $\mathbf{x}$ is non-trivial, we clearly have non-empty intersection of $\text{supp}^+(\mathbf{x})$ and $S \in \text{constraints}(p)$. The converse direction is also clear since $p(\mathbf{x}) = 1 - 1 = H$. □

We present an algorithm for computing $S_n(A)$ of non-trivial systems A.

Algorithm 5 Algorithm for Non-Trivial Systems

Require: Positive support size n, non-trivial system A

Ensure: $S_n(A)$

function SOLVE(A, n)
  C $\leftarrow \bigcup_{p \in A} \text{constraints}(p)$
  `solutions` $\leftarrow \{\mathbf{x} \in H^{V_d} \mid \forall S \in C : \text{supp}^+(\mathbf{x}) \cap S \neq \emptyset, |\text{supp}^+(\mathbf{x})| = n, \text{supp}^-(\mathbf{x}) = \{(0,0)\}\}$
  return `solutions`
end function

Proof of correctness. The correctness of `solutions` $= S_n(A)$ follows from Proposition 6.24 and the assumption that A is non-trivial. □

Remark 6.25. The algorithm can also be used for linear forms on H^{Ξ} with general index set $\Xi \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and solutions $\mathbf{x} \in H^{\Xi}$. The proof is analogous.

6.3 Implementation of the Hyperfield Criterion

The Hyperfield Criterion states that only the common hyperfield roots of all Pascal forms can be supports of valid outcomes. The system of all Pascal forms is a-priori an *infinite* and *non-trivial* system. However, we found out that several bases of Pascal forms exist such as the row, col and diag Pascal basis. This allows us to consider *finite* systems. Define the finite system $A := \{\text{diag}(i)\}_{i=0}^d \cup \{\text{row}(i)\}_{i=0}^d \cup \{\text{col}(i)\}_{i=0}^d$.

Proposition 6.26. *The system A is non-trivial.*

Proof. First, $S_n(A)$ is non-empty because $\mathbf{x} = (x_i)_{i=0}^n$ defined as $x_{i,d-i} = \binom{n}{i}$ is a solution of the system A.

Let $\mathbf{x} \in S_n(A)$ and $i = 0, \ldots, n$. Consider the following cases.

- Assume, $\mathbf{x} \notin \operatorname{supp}(\operatorname{diag}(i))$; then $\operatorname{diag}(i)(\mathbf{x}) < 0$ but $\mathbf{x}$ is a root of $\operatorname{diag}(i)$.
- Assume, $\mathbf{x} \notin \operatorname{supp}(\operatorname{row}(i))$. If $i = d$, then $\mathbf{x}$ is not of degree n. Therefore, we assume $i < d$. Then, either $\mathbf{x}$ is a trivial root for $\operatorname{row}(i+1)$ or we have $\operatorname{row}(i+1)(\mathbf{x}) \neq 0$. In the latter case, we found a contradiction to $\mathbf{x}$ being a root. For the former case that $\mathbf{x}$ is a trivial root, we conclude that there exists nonzero $x_{u,d-u}$ for some $u = i+2, \ldots, d$ since $\mathbf{x}$ is of degree n; now we just repeat the argument for $\operatorname{row}(i+1)$. More precisely, if $\mathbf{x}$ is again a trivial root for $\operatorname{row}(i+2)$, we repeat the argument for $\operatorname{row}(i+2)$ until we will end up with a contradiction $\operatorname{row}(u)(\mathbf{x}) \neq 0$.
- For the case $\operatorname{col}(i)$, we can argue by symmetry.

$\square$

Corollary 6.27. *Configurations $\mathbf{x} \in \mathbb{Z}^{V_d}$ of the form $\operatorname{supp}(\mathbf{x}) \subset \{(i,j) \in \mathbb{Z}^{V_d} : i+j \leq k \text{ or } i > k+1\}$ are not valid outcomes for any $k = 0, \ldots, d-1$. Neither are configurations $\mathbf{x} \in \mathbb{Z}^{V_d}$ of the form $\operatorname{supp}(\mathbf{x}) \subset \{(i,j) \in \mathbb{Z}^{V_d} : i+j \leq k \text{ or } j > k+1\}$ for $k = 0, \ldots, d-1$ due to symmetry.*

Proof. Since the previously defined system A is non-trivial, we must have that supports of valid outcomes intersect the support of $\operatorname{row}(k+1)$ non-trivially. $\square$

Example 6.28. This is not a valid outcome:

```
.
.  .
.  .  .
.  .  .  *
*  .  .  *  *
*  *  .  *  *  *
```

Now that we have shown that A is a trivial system, we have found an efficient way to apply the Hyperfield Criterion. Here is a detailed breakdown of an implementation of the algorithm.

We implement the line of code $C = \bigcup_{p \in A} \operatorname{constraints}(p)$ of Algorithm 5 using the procedure described in Algorithm 6. We can optimize the constraints by removing redundant constraints that are contained in each other, see Algorithm 7. To compute the solutions

$$\texttt{solutions} = \{\mathbf{x} \in H^{V_d} \mid \forall S \in C : \operatorname{supp}^+(\mathbf{x}) \cap S \neq \emptyset, |\operatorname{supp}^+(\mathbf{x})| = n, \operatorname{supp}^-(\mathbf{x}) = \{(0,0)\}\}$$

from Algorithm 5, we use the implementation detailed in Algorithm 8. To further enhance efficiency, we employ a heuristic that sorts the constraints by size and processes smaller constraints first (see line two of Algorithm 8). The complete implementation is written in Python 3 and included in the appendix TODO. Using this implementation, we can efficiently compute $S_4(A)$.

Algorithm 6 Make Constraints

Require: some hyperfield linear form p

Ensure: constraints(p)

```
constraints ← list()
for each (pos, neg) in support(p) do
    if 0 ∈ pos then
        new_constr ← {i | i ∈ pos ∧ i > 0}
        if new_constr ∉ constraints then
            constraints.append(new_constr)
        end if
    else if 0 ∈ neg then
        new_constr ← {i | i ∈ neg ∧ i > 0}
        if new_constr ∉ constraints then
            constraints.append(new_constr)
        end if
    else if len(pos) > 0 ∧ len(neg) > 0 then
        if pos ∉ constraints then
            constraints.append(pos)
        end if
        if neg ∉ constraints then
            constraints.append(neg)
        end if
    end if
end for
return constraints
```

Proposition 6.29. *Let* $A = \{\mathrm{diag}(i)\}_{i=0}^{d} \cup \{\mathrm{row}(i)\}_{i=0}^{d} \cup \{\mathrm{col}(i)\}_{i=0}^{d}$ *for some degree* $d \in \mathbb{N}$. *Let* $\mathbf{x} \in H^{V_d}$ *be nonzero with* $\mathrm{supp}^-(\mathbf{x}) = \{(0,0)\}$. *The following statements hold:*

(1) If $d = 6$, *then we have* $\mathbf{x} \in S_4(A)$ *if and only if* $\mathrm{supp}^+(\mathbf{x})$ *is one of the following sets:*

$$\{(0,3),(1,5),(4,1),(6,0)\}, \{(0,5),(1,1),(3,3),(6,0)\}, \{(0,6),(1,1),(3,3),(5,0)\},$$
$$\{(0,6),(1,1),(3,3),(6,0)\}, \{(0,6),(1,4),(3,0),(5,1)\}.$$

(2) If $d = 7$, *then we have* $\mathbf{x} \in S_4(A)$ *if and only if* $\mathrm{supp}^+(\mathbf{x})$ *is one of the following sets:*

$$\{(0,7),(1,1),(3,3),(7,0)\}, \{(0,7),(1,3),(5,1),(7,0)\}, \{(0,7),(1,5),(3,1),(7,0)\}.$$

Algorithm 7 Remove Redundant Constraints

Require: constraints
Ensure: nonredundant constraints

```
to_remove ← list()
for each c ∈ constraints do
    for each d ∈ constraints do
        if c ⊃ d then
            to_remove.append(c)
        end if
        if d ⊃ c then
            to_remove.append(d)
        end if
    end for
end for
return {x | x ∈ constraints ∧ x ∉ to_remove}
```

Algorithm 8 Solve

Require: positive support size n, non-trivial system A
Ensure: $S_n(A)$

```
constraints ← remove_redundant_constraints(make_constraints(A))
constraints.sort by length
queue ← deque([()])
for each constr ∈ constraints do
    for _ in range(|queue|) do
        conf ← queue.popleft()
        if conf intersects constr then
            queue.append(conf)
        else if |conf| < support_size then
            for each j ∈ constr do
                queue.append(conf ∪ {j})
            end for
        end if
    end for
end for
return queue
```

(3) If $d = 8, 9, 10, 11$, then the solution set $S_4(A)$ is empty.

Proof. We compute the set of $S_4(A)$ for $d = 6, 7, 8, 9, 10$, and 11 using the implementation of Algorithm 8 which is included in the appendix TODO. This gives us the results stated in the proposition. □

We obtain the following corollary.

Corollary 6.30. *For all valid integral outcomes $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 4$ and $\deg(\mathbf{w}) \leq 11$ we have $\deg(\mathbf{w}) \leq 5$.*

The case $\deg(\mathbf{w}) > 11$ is still open. We will address this in the next section.

6.4 Contractions

We aim to show that for every valid integral outcome $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 4$, its degree is at most five. This has been proven for outcomes of degree at most eleven. The challenge lies in checking the infinite set of valid outcomes with degree at least twelve. To address this, we introduce the technique of *contractions*, which reduces the problem to finitely many cases. *Contractions* were first presented in [2].

The concept of contraction involves *contracting* or *consolidating* vertices in V_d by merging rows or columns into a single vertex. This is achieved by introducing new formal variables $b_i, c_i, d_i, e_i, y_{i,j}$, and $z_{i,j}$, referred to as *contraction variables*.

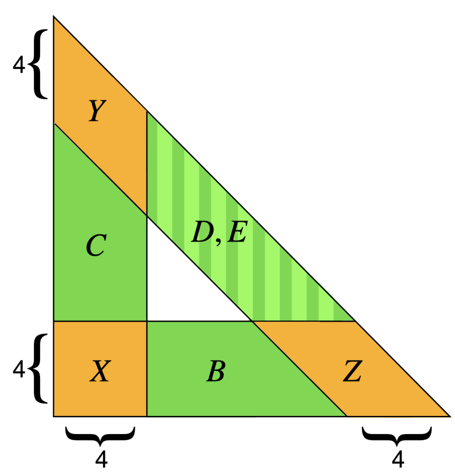

Figure 6.1: This figure illustrates the contraction variables. The yellow areas X, Y, Z represent formal variables $x_{i,j}$ that remain unaffected by the contraction. Each green area B, C, D, E represents rows or columns of vertices that are merged into a single vertex.

Definition 6.31. Let $x_{i,j}$ be formal variables indexed by V_d. We merge a subset of rows and columns of these formal variables in V_d into a single vertex by defining the following *contraction variables*:

$$
\begin{aligned}
y_{i,j} &:= x_{i,d-3-i+j} \quad \text{for } i,j = 0,\dots,3,\\
z_{i,j} &:= x_{d-3-j+i,j} \quad \text{for } i,j = 0,\dots,3,\\
b_j &:= x_{4,j} + \cdots + x_{d-4-j,j} \quad \text{for } j = 0,\dots,3,\\
c_i &:= x_{i,4} + \cdots + x_{i,d-4-i} \quad \text{for } i = 0,\dots,3,\\
d_k &:= \begin{cases} x_{4,d-4-k} + x_{6,d-6-k} + \cdots + x_{d-4-k,4} & \text{if } d+k \text{ is even} \\ x_{4,d-4-k} + x_{6,d-6-k} + \cdots + x_{d-5-k,5} & \text{if } d+k \text{ is odd} \end{cases} \quad \text{for } k = 0,\dots,3,\\
e_k &:= \begin{cases} x_{5,d-5-k} + x_{7,d-7-k} + \cdots + x_{d-5-k,5} & \text{if } d+k \text{ is even} \\ x_{5,d-5-k} + x_{7,d-7-k} + \cdots + x_{d-4-k,4} & \text{if } d+k \text{ is odd} \end{cases} \quad \text{for } k = 0,\dots,3.
\end{aligned}
$$

Let us visualize the contraction variables for $d = 16$ in the following figure.

$$
\begin{array}{ccccccccccccccccc}
y_{0,3} \\
y_{0,2} & y_{1,3} \\
y_{0,1} & y_{1,2} & y_{2,3} \\
y_{0,0} & y_{1,1} & y_{2,2} & y_{3,3} \\
c_0 & y_{1,0} & y_{2,1} & y_{3,2} & d_0 \\
c_0 & c_1 & y_{2,0} & y_{3,1} & d_1 & e_0 \\
c_0 & c_1 & c_2 & y_{3,0} & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & * & e_3 & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & e_3 & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & * & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & * & * & e_3 & d_2 & e_1 & d_0 \\
x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} & b_3 & b_3 & b_3 & b_3 & b_3 & b_3 & z_{0,3} & z_{1,3} & z_{2,3} & z_{3,3} \\
x_{0,2} & x_{1,2} & x_{2,2} & x_{3,2} & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & z_{0,2} & z_{1,2} & z_{2,2} & z_{3,2} \\
x_{0,1} & x_{1,1} & x_{2,1} & x_{3,1} & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & z_{0,1} & z_{1,1} & z_{2,1} & z_{3,1} \\
x_{0,0} & x_{1,0} & x_{2,0} & x_{3,0} & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & z_{0,0} & z_{1,0} & z_{2,0} & z_{3,0}
\end{array}
$$

As we can see, the vertices $x_{0,4},\dots,x_{0,d-4}$ are merged into the contraction variable c_0 by summing them up.

Recall that hyperfield Pascal forms are expressed as sums $\sum_{(i,j)\in V_d} \lambda_{i,j} x_{i,j}$ with $\lambda_{i,j} \in H$. The key insight is that some Pascal forms can be rewritten in terms of the contraction variables c_0, c_1, c_2, c_3 instead of the original variables $x_{i,j}$.

Example 6.32. Consider the Pascal form diag(1) in $\mathbb{Z}^{V_{16}}$. Its support is depicted in the following figure:

```
.
+ +
+ + .
+ + . .
+ + . . .
+ + . . . .
+ + . . . . .
+ + . . . . . .
+ + . . . . . . .
+ + . . . . . . . .
+ + . . . . . . . . .
+ + . . . . . . . . . .
+ + . . . . . . . . . . .
+ + . . . . . . . . . . . .
+ + . . . . . . . . . . . . .
+ + . . . . . . . . . . . . . .
+ + . . . . . . . . . . . . . . .
```

We see that $\mathrm{diag}(1) = x_{0,0} + x_{0,1} + x_{0,2} + x_{0,3} + x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + y_{0,0} + y_{0,1} + y_{0,2} + y_{1,0} + y_{1,1} + y_{1,2} + y_{1,3} + c_0 + c_1$.

The previous example also demonstrates that the expression $\mathrm{diag}(1) = x_{0,0} + x_{0,1} + x_{0,2} + x_{0,3} + x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + y_{0,0} + y_{0,1} + y_{0,2} + y_{1,0} + y_{1,1} + y_{1,2} + y_{1,3} + c_0 + c_1$ is *independent* of the degree d, i.e. if we were to consider the Pascal form diag(1) in $\mathbb{Z}^{V_d}$ for some arbitrary d, the expression would still hold. This is great news since it allows us to express Pascal forms in terms of contraction variables for all degrees d at once.

So far, we have only considered the contraction variables c_0, c_1, c_2, and c_3. As we might expect, we can also express some Pascal forms in terms of the contraction variables b_0, b_1, b_2, b_3, d_0, d_1, d_2, d_3, e_0, e_1, e_2, e_3, $y_{i,j}$, and $z_{i,j}$. We will now find these kinds of Pascal forms that can be represented in terms of the contraction variables independent of the degree d. A good set of Pascal forms to consider are the Pascal forms $\mathrm{diag}(k), \mathrm{col}(k)$ and $\mathrm{row}(k)$ for $k = 0, 1, 2, 3, d-3, d-2, d-1, d$.

Proposition 6.33. *Let $d \geq 11$. Let p be a hyperfield form induced by one of the following Pascal forms on $\mathbb{Z}^{V_d}$:*

(1) $\mathrm{col}(1), \mathrm{col}(2), \mathrm{col}(3)$, *or*

(2) $\mathrm{row}(1), \mathrm{row}(2), \mathrm{row}(3)$, *or*

(3) $\mathrm{diag}(1), \mathrm{diag}(2), \mathrm{diag}(3)$*, or*

(4) $\mathrm{diag}(d-1), \mathrm{diag}(d-2), \mathrm{diag}(d-3)$*.*

Then, we have

$$p = \sum_{i,j=0}^{3} \lambda_{i,j}^{(x)} x_{i,j} + \sum_{i,j=0}^{3} \lambda_{i,j}^{(y)} y_{i,j} + \sum_{i,j=0}^{3} \lambda_{i,j}^{(z)} z_{i,j} + \sum_{j=0}^{3} \lambda_{j}^{(b)} b_j + \sum_{i=0}^{3} \lambda_{i}^{(c)} c_i + \sum_{k=0}^{3} \lambda_{k}^{(d)} d_k + \sum_{k=0}^{3} \lambda_{k}^{(e)} e_k$$

with coefficients $\lambda_{i,j}^{(x)}, \lambda_{i,j}^{(y)}, \lambda_{i,j}^{(z)}, \lambda_{j}^{(b)}, \lambda_{i}^{(c)}, \lambda_{k}^{(d)}, \lambda_{k}^{(e)} \in H$*.*

Define $\lambda(p,d) := \left(\lambda_{i,j}^{(x)}, \lambda_{i,j}^{(y)}, \lambda_{i,j}^{(z)}, \lambda_{j}^{(b)}, \lambda_{i}^{(c)}, \lambda_{k}^{(d)}, \lambda_{k}^{(e)}\right)$ *for the coefficients of* p *on* H^{V_d}*. Then, we have*

$$\lambda(p,d) = \lambda(p,d+1) = \lambda(p,d+2) = \lambda(p,d+3) = \dots$$

In other words, all the coefficients $\lambda_{i,j}^{(x)}, \lambda_{i,j}^{(y)}, \lambda_{i,j}^{(z)}, \lambda_{j}^{(b)}, \lambda_{i}^{(c)}, \lambda_{k}^{(d)}, \lambda_{k}^{(e)}$ *are independent of the degree* d*.*

Proof. For case two and three we observe that the hyperfield form p has support contained in the areas X, C, and Y from Figure 6.1. This follows directly from Proposition 6.13. We also see that p depends only on the column sums on C.

For case one and four we see that p has support contained in the areas X, B, and Z from Figure 6.1 by Proposition 6.13. We conclude that p depends only on the row sums on B. □

Here is another example.

Example 6.34. Consider diag(3) and $d = 11$. Write $\mathrm{sign}(\mathrm{diag}(3)) = \sum_{i,j=0}^{3} x_{i,j} + \sum_{i=0}^{3} c_i + \sum_{i=0}^{3} \sum_{j=0}^{i} y_{i,j}$, and we see that this linear form is independent of the degree d.

Proposition 6.35. *Let* $d \geq 12$*. Let* p *be a hyperfield form induced by one of the following Pascal forms on* $\mathbb{Z}^{V_d}$*:*

(1) $\mathrm{col}(d), \mathrm{col}(d-1), \mathrm{col}(d-2), \mathrm{col}(d-3)$*, or*

(2) $\mathrm{row}(d), \mathrm{row}(d-1), \mathrm{row}(d-2), \mathrm{row}(d-3)$*.*

Then, we have

$$p = \sum_{i,j=0}^{3} \lambda_{i,j}^{(x)} x_{i,j} + \sum_{i,j=0}^{3} \lambda_{i,j}^{(y)} y_{i,j} + \sum_{i,j=0}^{3} \lambda_{i,j}^{(z)} z_{i,j} + \sum_{j=0}^{3} \lambda_{j}^{(b)} b_j + \sum_{i=0}^{3} \lambda_{i}^{(c)} c_i + \sum_{k=0}^{3} \lambda_{k}^{(d)} d_k + \sum_{k=0}^{3} \lambda_{k}^{(e)} e_k$$

with coefficients $\lambda_{i,j}^{(x)}, \lambda_{i,j}^{(y)}, \lambda_{i,j}^{(z)}, \lambda_{j}^{(b)}, \lambda_{i}^{(c)}, \lambda_{k}^{(d)}, \lambda_{k}^{(e)} \in H$*.*

Define $\lambda(p,d) := \left(\lambda_{i,j}^{(x)}, \lambda_{i,j}^{(y)}, \lambda_{i,j}^{(z)}, \lambda_{j}^{(b)}, \lambda_{i}^{(c)}, \lambda_{k}^{(d)}, \lambda_{k}^{(e)}\right)$ *for the coefficients of* p *on* H^{V_d}. *Then, we have*

$$\lambda(p,d) = \lambda(p,d+2) = \lambda(p,d+4) = \lambda(p,d+6) = \dots$$
$$\lambda(p,d+1) = \lambda(p,d+3) = \lambda(p,d+5) = \lambda(p,d+7) = \dots$$

In other words, all the coefficients of forms on even degrees are independent, and all the coefficients of forms on odd degrees are independent of the degree d.

Proof. All the hyperfield forms p depend on entries in the area Y, D, and Z from Figure 6.1 by Proposition 6.13. We see that p depends only on the alternating diagonal sums on D for $x_{i,j}$ in the area D. This shows that p is a sum of the contraction variables $y_{i,j}$, $z_{i,j}$, and $d_k - e_k$. $\square$

Example 6.36. Consider $\mathrm{col}(d-3)$. Let $d = 12$. The support of $\mathrm{sign}(\mathrm{col}(d-3))$ is depicted in the following figure:

$$\begin{array}{ccccccccccccc}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
+ & + & + & + \\
\cdot & - & - & - & - \\
\cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & -
\end{array}$$

We write

$$\mathrm{sign}(\mathrm{col}(d-3)) = \sum_{i=0}^{3}\sum_{j=0}^{i}(-1)^{i+j}y_{i,j} - \sum_{k=0}^{3}(-1)^k d_k + \sum_{k=0}^{3}(-1)^k e_k - \sum_{i,j=0}^{3}(-1)^j z_{i,j}.$$

For $d = 13$, the support of $\mathrm{sign}(\mathrm{col}(d-3))$ is depicted in the following figure:

$$\begin{array}{cccc}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
+ & + & + & +
\end{array}$$

$$
\begin{array}{cccccccccccccc}
\cdot & - & - & - & - & & & & & & & & & \\
\cdot & \cdot & + & + & + & + & & & & & & & & \\
\cdot & \cdot & \cdot & - & - & - & - & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & + & + & + & + & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & + & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & + & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & +
\end{array}
$$

We write

$$\operatorname{sign}(\operatorname{col}(d-3)) = \sum_{i=0}^{3}\sum_{j=0}^{i}(-1)^{i+j}y_{i,j} - \sum_{k=0}^{3}(-1)^k d_k + \sum_{k=0}^{3}(-1)^k e_k + \sum_{i,j=0}^{3}(-1)^j z_{i,j}.$$

We have merged formal variables $x_{i,j}$ indexed by vertices (i,j) in the areas B, C, D into contraction variables. Now, we apply these contractions to concrete elements $\mathbf{s} \in H^{V_d}$.

Definition 6.37. Define the index set

$$\Xi \coloneqq \{0,1,2,3\}^2 \sqcup \{0,1,2,3\}^2 \sqcup \{0,1,2,3\}^2 \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\}.$$

Let $\mathbf{s} \in H^{\Xi}$. We call $\mathbf{s}$ a *contracted hyperfield configuration*, and write $\mathbf{s}$ as

$$\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = (x_{i,j}, y_{i,j}, z_{i,j}, b_j, c_i, d_k, e_k).$$

Definition 6.38. Let $\mathbf{s} \in H^{\Xi}$ be a contracted hyperfield configuration. We say $\mathbf{s}$ is *valid* if one of the following holds:

(1) $\mathbf{s} = \mathbf{0}$ or

(2) $s_{0,0} = -1$, $x_{i,j} \geq 0$ for all $i+j > 0$, and $y_{i,j}, z_{i,j}, b_j, c_i, d_k, e_k \geq 0$ for all $i,j,k = 0,1,2,3$.

Going from the world of hyperfield configurations to the world of *contracted* hyperfield configurations is done via the following map.

Definition 6.39. Let $d \geq 11$. Let $\mathbf{s} \in H^{V_d}$ be a hyperfield configuration. We define

$$\operatorname{contr}_d(\mathbf{s}) : \mathbf{s} \mapsto (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = (x_{i,j}, y_{i,j}, z_{i,j}, b_j, c_i, d_k, e_k)$$

where we set

$$
\begin{aligned}
x_{i,j} &\coloneqq s_{i,j} \quad \text{for } i,j = 0,\dots,3,\\
y_{i,j} &\coloneqq s_{i,d-3-i+j} \quad \text{for } i,j = 0,\dots,3,\\
z_{i,j} &\coloneqq s_{d-3-j+i,j} \quad \text{for } i,j = 0,\dots,3,\\
b_j &\coloneqq s_{4,j} + \cdots + s_{d-4-j,j} \quad \text{for } j = 0,\dots,3,\\
c_i &\coloneqq s_{i,4} + \cdots + s_{i,d-4-i} \quad \text{for } i = 0,\dots,3,\\
d_k &\coloneqq \begin{cases} s_{4,d-4-k} + s_{6,d-6-k} + \cdots + s_{d-4-k,4} & \text{if } d+k \text{ is even} \\ s_{4,d-4-k} + s_{6,d-6-k} + \cdots + s_{d-5-k,5} & \text{if } d+k \text{ is odd} \end{cases} \quad \text{for } k = 0,\dots,3,\\
e_k &\coloneqq \begin{cases} s_{5,d-5-k} + s_{7,d-7-k} + \cdots + s_{d-5-k,5} & \text{if } d+k \text{ is even} \\ s_{5,d-5-k} + s_{7,d-7-k} + \cdots + s_{d-4-k,4} & \text{if } d+k \text{ is odd} \end{cases} \quad \text{for } k = 0,\dots,3.
\end{aligned}
$$

exactly as in Definition 6.31.

The contraction map contr_d maps hyperfield configurations $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) \in H^{V_d}$ to elements in H^{Ξ} if $\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \geq 0$. If one of the entries is negative, the map may output to some element $(2^H)^{\Xi}$. To make life easier, we only consider weakly valid configuration; that are configurations whose negative support may only be contained in the yellow area below.

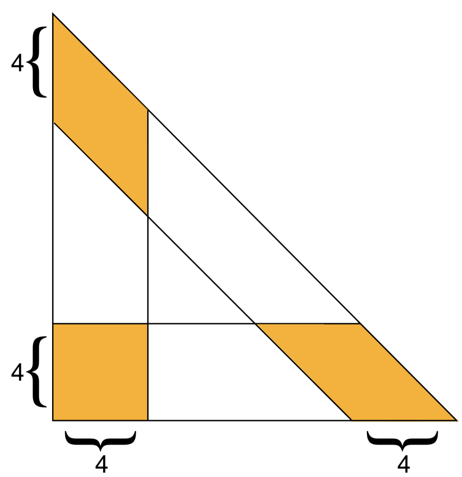

Figure 6.2: A hyperfield configuration is weakly valid if its negative support is contained in the yellow area.

Definition 6.40. Let $\mathbf{s} \in H^{V_d}$ be a hyperfield configuration. We say $\mathbf{s}$ is *weakly valid* if for all $(i,j) \in \mathrm{supp}^-(\mathbf{s})$ one of the following holds:

(1) $i, j = 0, \dots, 3$, or

(2) $i = 0, \dots, 3$ and $i + j \geq d - 3$, or

(3) $j = 0, \dots, 3$ and $i + j \geq d - 3$.

From now on, we only consider *weakly valid* hyperfield configurations because in this case the contraction map contr_d always outputs elements in H^{Ξ}.

Definition 6.41. Let $\mathbf{s} \in H^{\Xi}$ be a contracted hyperfield configuration. The *positive support* of $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$ is defined as the set of all symbols $x_{i,j}, y_{i,j}, z_{i,j}, b_j, c_i, d_k, e_k$ such that the corresponding coefficients of $\mathbf{s}$ equal to one.

Example 6.42. Let $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) \in H^{\Xi}$ be a contracted hyperfield configuration defined by

$$x_{0,0} = -1, \quad x_{0,3} = 1, \quad x_{1,1} = 1, \quad , x_{3,0} = 1, \quad d_0 = 1, \quad e_0 = 1$$

where all other entries are zero. Then, the positive support of $\mathbf{s}$ is given by

$$\mathrm{supp}^{+}(\mathbf{s}) = \{x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}.$$

Chapter 7

Valid Outcomes of Positive Support Size Four

We are now ready to prove that for every valid integral outcome $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 4$, we have $\deg(\mathbf{w}) \leq 5$. As in previous chapters, outcomes are characterized as roots of Pascal forms. We define the two systems of Pascal forms that valid outcomes must satisfy: $\Phi_1 := \{\mathrm{col}(i), \mathrm{row}(i), \mathrm{diag}(i), \mathrm{diag}(d-i)\}_{i=1}^{3}$, $\Phi_2 := \{\mathrm{row}(d-i), \mathrm{col}(d-i)\}_{i=0}^{3}$, and $\Phi := \Phi_1 \cup \Phi_2$.

By Proposition 6.33, we can write all hyperfield forms induced by Pascal forms p in Φ_1 as $\mathrm{sign}(p) = \hat{p}$ for some linear form $\hat{p} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ if $d \geq 11$. This linear form is independent of the degree d. To make notations consistent later, we set $\hat{p}^{\mathrm{even}} := \hat{p}^{\mathrm{odd}} := \hat{p}$.

Similarly, by Proposition 6.35, we can write all hyperfield forms induced by Pascal forms p in Φ_2 as $\mathrm{sign}(p) = \begin{cases} \hat{p}^{\mathrm{even}} & \text{if } d \text{ is even} \\ \hat{p}^{\mathrm{odd}} & \text{if } d \text{ is odd} \end{cases}$, where $\hat{p}^{\mathrm{even}}, \hat{p}^{\mathrm{odd}} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ if $d \geq 12$. These linear forms $\hat{p}^{\mathrm{even}}, \hat{p}^{\mathrm{odd}}$ are independent of the degree d.

Definition 7.1. We define the following three solution sets:

(1) Define Γ_d to be the set of all valid hyperfield configurations $\mathbf{s} \in H^{V_d}$ of degree d such that $\mathrm{sign}(p)(\mathbf{s}) = H$ for all $p \in \Phi$.

(2) Define Γ^{even} to be the set of all valid contracted hyperfield configurations $\mathbf{s} \in H^{\Xi}$ such that $\hat{p}^{\mathrm{even}}(\mathbf{s}) = H$ for all $p \in \Phi$.

(3) Define Γ^{odd} to be the set of all valid contracted hyperfield configurations $\mathbf{s} \in H^{\Xi}$ such that $\hat{p}^{\mathrm{odd}}(\mathbf{s}) = H$ for all $p \in \Phi$.

By Proposition 6.10, valid chipsplitting outcomes of degree d have supports in Γ_d. This is the reason why we have defined Γ_d in the first place.

Proposition 7.2. *Let $d \geq 12$. If d is even, then $\Gamma_d = \mathrm{contr}_d^{-1}(\Gamma^{\mathrm{even}})$ holds. If d is odd, then $\Gamma_d = \mathrm{contr}_d^{-1}(\Gamma^{\mathrm{odd}})$ holds.*

Proof. Let $d \geq 12$ be even. Let $\mathbf{s} \in H^{V_d}$ be a hyperfield configuration and $p \in \Phi$. Then, we have $\operatorname{sign}(p)(\mathbf{s}) = \hat{p}^{\text{even}}(\operatorname{contr}_d(\mathbf{s}))$ by definition of $\hat{p}^{\text{even}}$. If $\mathbf{s} \in \Gamma_d$, then $H = \operatorname{sign}(p)(\mathbf{s}) = \hat{p}^{\text{even}}(\operatorname{contr}_d(\mathbf{s}))$. Hence, $\operatorname{contr}_d(\mathbf{s})$ is contained in Γ^{even}. If $\operatorname{contr}_d(\mathbf{s}) \in \Gamma^{\text{even}}$ holds, using the equation above we also see that $\mathbf{s} \in \Gamma_d$. This shows that $\Gamma_d = \operatorname{contr}_d^{-1}(\Gamma^{\text{even}})$.

The second statement for odd degrees d follows analogously. □

Corollary 7.3. *Let $d \geq 12$ and $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome. Then,* $\operatorname{contr}_d(\operatorname{sign}(\mathbf{w})) \in \Gamma^{\text{even}} \cup \Gamma^{\text{odd}}$ *holds.*

Proof. Define $\mathbf{s} \coloneqq \operatorname{sign}(\mathbf{w})$. By Proposition 6.14 we have $\mathbf{s} \in \Gamma_d$. If d is even, then $\operatorname{contr}_d(\mathbf{s}) \in \Gamma^{\text{even}}$ by the previous proposition. If d is odd, then $\operatorname{contr}_d(\mathbf{s}) \in \Gamma^{\text{odd}}$ by the previous proposition. This shows the claim. □

This corollary allows us to exclude certain outcomes as valid outcomes. Assume we have some contracted hyperfield configuration $\xi \in H^{\Xi}$ that is not a root of some of the linear forms $\hat{p}^{\text{even}}, \hat{p}^{\text{odd}}$ for $p \in \Phi$. Then, any chipsplitting configuration $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $\operatorname{contr}_d(\operatorname{sign}(\mathbf{w})) = \xi$ is not a valid outcome.

Proposition 7.4. *Let $\mathbf{s} \in H^{V_d}$ be a valid hyperfield configuration of degree d with positive support size four or less. If $d \geq 12$, then $\mathbf{s} \notin \Gamma_d$.*

Proof. Let $d \geq 12$. For computing Γ_d we could use Algorithm 5 for all $d = 12, 13, 14, \ldots$ and so on, which is not feasible since we would compute solutions sets for many infinitely many degrees d. Instead, we show that $\Gamma^{\text{even}} \cup \Gamma^{\text{odd}}$ is empty. By Proposition 7.2, Γ_d is empty as well for all $d \geq 12$.

To show that Γ^{even} is empty, we can just use Algorithm 5 and Remark 6.25 with $A \coloneqq \{\hat{p}^{\text{even}} \mid p \in \Phi\}$. Similarly, we compute $\Gamma^{\text{odd}} = \emptyset$ with $A \coloneqq \left\{\hat{p}^{\text{odd}} \mid p \in \Phi\right\}$ and Algorithm 5. The results are in Appendix TODO. This shows the claim. TODO: show that A is non-trivial. □

Theorem 7.5. *For valid integral outcomes $\mathbf{w}$ with $|\operatorname{supp}^+(\mathbf{w})| = 4$ we have $\deg(\mathbf{w}) \leq 5$.*

Proof. Let $d \geq 6$. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome with $|\operatorname{supp}^+(\mathbf{w})| = 4$ and degree d. We have $\operatorname{sign}(\mathbf{w}) \in \Gamma_d$. By the previous proposition, there is no such $\operatorname{sign}(\mathbf{w})$ for $d \geq 12$. By Proposition 6.29, the degree of $\operatorname{sign}(\mathbf{w}) = d$ is six or seven. So, we just need to check eight cases. Of these eight cases, we can exclude all of them by applying Algorithm 2. The result of this algorithm is that only the zero outcome is possible for all these cases. This shows that the degree of $\mathbf{w}$ is at most five. □

Chapter 8

Hexagon Criterion

We introduce the *Hexagon Criterion*, first presented in [2], to determine whether subconfigurations of a chipsplitting outcome qualify as outcomes. This criterion applies to configurations supported within the yellow area below, we say its support is not contained inside the "hexagon".

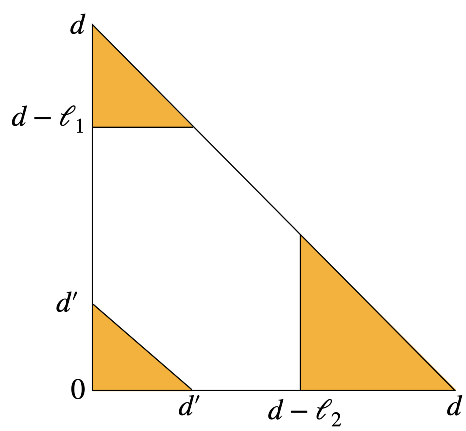

Figure 8.1: A configuration's support lies outside the hexagon if it is contained in the yellow area spanned by parameters ℓ_1, ℓ_2, and d'.

First, we need the following lemma to compute the determinant of matrix that we will encounter in the proof of the Hexagon Criterion.

Lemma 8.1. *Let $a, b, c \in \mathbb{Z}_{\geq 0}$. Define the map $H : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}, n \mapsto 0!1! \cdots (n-1)!$; note that $H(0) = 1$. Then, the following holds:*

$$\det \left[\binom{a+b}{b-i+j} \right]_{i,j=1}^{c} = \frac{H(a)H(b)H(c)H(a+b+c)}{H(b+c)H(c+a)H(a+b)}$$

Proof. See Theorem 8 of [4]. $\square$

Proposition 8.2 (Hexagon Criterion). *Let $d, d', \ell_1, \ell_2 \in \mathbb{N}$ with $d' \geq 1$, $\ell_1, \ell_2 \geq d'$, and $d' + \ell_1 + \ell_2 \leq d$. Let $\mathbf{w} = (w_{i,j})_{(i,j)\in V_d} \in \mathbb{Z}^{V_d}$ be a chip configuration. Define the subconfiguration $\mathbf{w}' := (w_{i,j})_{(i,j)\in V_{d'}} \in \mathbb{Z}^{V_{d'}}$. Assume the support of $\mathbf{w}$ is not contained inside the "hexagon" (see Figure 8.1), or equivalently, assume that*

$$\operatorname{supp}(\mathbf{w}) \subset V_{d'} \cup \{(i,j) \in V_d \mid j > d - \ell_1\} \cup \{(i,j) \in V_d \mid i > d - \ell_2\}.$$

Then, the following holds:

(1) If $\mathbf{w}$ is an outcome, then also its subconfiguration $\mathbf{w}'$ is an outcome.

(2) If $\mathbf{w}$ is a valid outcome, then $\deg(\mathbf{w}) \leq d'$.

Proof. We prove the first statement. Assume $\mathbf{w}$ is an outcome. Let $k = 0, \dots, d'$ and $\operatorname{diag}(\ell_1 + k) = \sum_{(i,j)\in V_d} \mu_{i,j} x_{i,j}$. Consider the restricted linear form $l_k = \sum_{(i,j)\in V_{d'}} \lambda_{i,j} x_{i,j} : \mathbb{Z}^{V_{d'}} \to \mathbb{Z}$ with $\lambda_{i,j} := \mu_{i,j}$ for $(i,j) \in V_{d'}$. Then, $l_0, \dots, l_{d'}$ are Pascal forms on $\mathbb{Z}^{V_{d'}}$ with $l_k(\mathbf{w}') = \operatorname{diag}(\ell_1 + k)(\mathbf{w}) = 0$. By Proposition 3.13 it suffices to show that $l_0, \dots, l_{d'}$ are linearly independent to show that $\mathbf{w}'$ is an outcome.

Let $a = 0, \dots, d'$. Define

$$e_{i,j}^{(a)} := \begin{cases} 1 & \text{if } i = a \text{ and } j = d' - a, \\ 0 & \text{otherwise.} \end{cases}$$

$$A := \left[l_k(\mathbf{e}^{(a)})\right]_{k,a=0}^{d'} = \left[\binom{d-d'}{\ell_1+k-a}\right]_{k,a=0}^{d'} = \left[\binom{(d-d'-\ell_1)+\ell_1}{\ell_1+k-a}\right]_{k,a=0}^{d'}.$$

We want to show that the matrix A is invertible because this implies that the linear forms $l_0, \dots, l_{d'}$ are linearly independent. We observe that

(1) all entries of A are nonzero because $0 \leq \ell_1 + k - a \leq d - d'$, and

(2) $d - d' - \ell_1 \geq \ell_2 \geq 0$.

This allows us to use Lemma 8.1 with $a := d - d' - \ell_1, b := \ell_1$, and $c := d' + 1$. We obtain a nonzero determinant $\det(A) = \frac{H(d-d'-\ell_1)H(\ell_1)H(d'+1)H(d+1)}{H(\ell_1+d'+1)H(1+d-\ell_1)H(d-d')} \neq 0$. Hence, $l_0, \dots, l_{d'}$ are linearly independent.

For the second statement, let $\mathbf{w}$ be a valid outcome. By the previous statement, the subconfiguration $\mathbf{w}'$ is an outcome, as well. We extend $\mathbf{w}'$ to some configuration $\mathbf{v} \in \mathbb{Z}^{V_d}$ by $v_{i,j} := \begin{cases} w'_{i,j} & \text{if } (i,j) \in V_{d'}, \\ 0 & \text{otherwise} \end{cases}$. Clearly, $\mathbf{v}$ is a *valid* outcome of degree at most d'. Then, $\mathbf{v} - \mathbf{w}$ is an outcome with empty negative support. By Proposition 3.29, $\mathbf{v} - \mathbf{w}$ is zero. Hence, $\mathbf{w}$ has degree at most d'. $\square$

Chapter 9

Valid Outcomes of Positive Support Size Five

In this chapter, we prove that for all valid integral outcomes $\mathbf{w}$ with $|\text{supp}^+(\mathbf{w})| = 5$ we have

$$\deg(\mathbf{w}) \leq 7.$$

Tools like the Invertibility Criterion, Hyperfield Criterion, and the Hexagon Criterion will be used to prove this result. We follow the same proof as Bik and Marigliano in [2].

9.1 Case $d = 8, \ldots, 41$

First, we show similar to Proposition 6.29 that no outcome of degree $d = 8, \ldots, 41$ exists with $|\text{supp}^+(\mathbf{w})| = 5$.

Proposition 9.1. *Let* $A = \{\text{diag}(i)\}_{i=0}^{d} \cup \{\text{row}(i)\}_{i=0}^{d} \cup \{\text{col}(i)\}_{i=0}^{d}$ *for some degree* $d \in \mathbb{N}$. *Let* $\mathbf{x} \in H^{V_d}$ *be nonzero with* $\text{supp}^-(\mathbf{x}) = \{(0,0)\}$. *Then, the number of solutions* $|S_5(A)|$ *for* $d = 8, \ldots, 41$ *is depicted in Table* **??**.

Proof. We compute the set of $S_5(A)$ for $d = 8, \ldots, 40$, and 41 using the implementation of Algorithm 8 which is included in the appendix TODO. □

Proposition 9.2. *No outcome of degree* $d = 8, \ldots, 41$ *exists with* $|\text{supp}^+(\mathbf{w})| = 5$.

Proof. We use Algorithm 2. The result of this algorithm is that only the zero outcome is possible for all these cases except for the following eight:

(1) $\{(4,1), (5,0), (1,6), (0,8), (3,5)\}$,

(2) $\{(3,0), (0,4), (5,1), (1,7), (4,4)\}$,

(3) $\{(3,0), (0,5), (5,1), (1,7), (4,4)\}$,

(4) $\{(1,1),(3,0),(3,4),(0,8),(5,3)\}$,

(5) $\{(3,0),(0,6),(5,1),(1,7),(4,4)\}$,

(6) $\{(0,3),(3,0),(5,1),(1,7),(4,4)\}$,

(7) $\{(1,1),(4,0),(3,4),(0,8),(5,3)\}$,

(8) $\{(3,1),(5,0),(1,6),(0,8),(3,5)\}$.

Valid configurations that have one of these eight sets as positive support are of degree eight or less.

Finally, we apply Algorithm 3 to these eight cases with $E := \{3,4,5,6,7,8\}$. The result is that only the zero outcome is possible. This shows the claim. □

9.2 Case $d \geq 42$

We have proven that no valid outcome of degree $d = 8, \ldots, 41$ exists with $|\mathrm{supp}^+(\mathbf{w})| = 5$. Next, we show that no valid outcome of degree $d \geq 42$ exists using contractions.

As in Proposition 7.4, we compute the sets

$$\Gamma^{\mathrm{even}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| \leq 5\right\} \text{ and } \Gamma^{\mathrm{odd}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| \leq 5\right\}$$

using Algorithm 5. By Proposition 7.4, we just need to check the case $|\mathrm{supp}^+(\mathbf{s})| = 5$.

Definition 9.3. We define $\Gamma_5^{\mathrm{even}} := \Gamma^{\mathrm{even}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| = 5\right\}$ and $\Gamma_5^{\mathrm{odd}} := \Gamma^{\mathrm{odd}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| = 5\right\}$.

Proposition 9.4. *We have* $|\Gamma_5^{\mathrm{even}}| = 1283$ *and* $|\Gamma_5^{\mathrm{odd}}| = 1265$.

Proof. This is verified by computer, see [7]. □

Corollary 9.5. *Let* $d \geq 42$. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome of degree* d *with* $|\mathrm{supp}^+(\mathbf{w})| = 5$. *Then,* $\mathrm{contr}_d(\mathrm{sign}(\mathbf{w})) \in \Gamma_5^{\mathrm{even}} \cup \Gamma_5^{\mathrm{odd}}$.

Proof. This follows from Corollary 7.3. □

Let us pick some $\mathbf{s} \in \Gamma_5^{\mathrm{even}} \cup \Gamma_5^{\mathrm{odd}}$. We want to show that any configuration $\mathbf{w} \in \mathbb{Z}^{V_d}$ that maps to $\mathbf{s}$ under $\mathrm{contr}_d \circ \mathrm{sign}$ must be a zero outcome which is a contradiction. So, we need to check $1283 + 1265 = 2548$ cases. Some of these cases occur multiple times; when we exclude these, we have 2318 cases to check.

Remark 9.6. Comparing this proof with the proof of Theorem 7.5, you will notice that the number of cases in $\Gamma^{\mathrm{even}} \cup \Gamma^{\mathrm{odd}}$ to check has increased from zero to 2318.

We make the following simplification to the index set Ξ.

Definition 9.7. Define the index set

$$\Xi' \coloneqq \{0,1,2,3\}^2 \sqcup \{0,1,2,3\}^2 \sqcup \{0,1,2,3\}^2 \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\}$$

with the map $\chi : H^{\Xi} \to H^{\Xi'}, \mathbf{s} = (\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}) \mapsto (\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{b},\mathbf{c},\underbrace{\mathbf{d}+\mathbf{e}}_{\coloneqq\mathbf{f}})$. The new coordinate system $H^{\Xi'}$ is visualized below.

$$\begin{array}{ccccccccccccccccc}
y_{0,3} \\
y_{0,2} & y_{1,3} \\
y_{0,1} & y_{1,2} & y_{2,3} \\
y_{0,0} & y_{1,1} & y_{2,2} & y_{3,3} \\
c_0 & y_{1,0} & y_{2,1} & y_{3,2} & f_0 \\
c_0 & c_1 & y_{2,0} & y_{3,1} & f_1 & f_0 \\
c_0 & c_1 & c_2 & y_{3,0} & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & * & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & * & * & f_3 & f_2 & f_1 & f_0 \\
x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} & b_3 & b_3 & b_3 & b_3 & b_3 & b_3 & z_{0,3} & z_{1,3} & z_{2,3} & z_{3,3} \\
x_{0,2} & x_{1,2} & x_{2,2} & x_{3,2} & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & z_{0,2} & z_{1,2} & z_{2,2} & z_{3,2} \\
x_{0,1} & x_{1,1} & x_{2,1} & x_{3,1} & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & z_{0,1} & z_{1,1} & z_{2,1} & z_{3,1} \\
x_{0,0} & x_{1,0} & x_{2,0} & x_{3,0} & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & z_{0,0} & z_{1,0} & z_{2,0} & z_{3,0}
\end{array}$$

Definition 9.8. We define $\mathrm{contr}'_d \coloneqq \chi \circ \mathrm{contr}_d$.

Definition 9.9. We define $\Lambda \coloneqq \chi(\Gamma_5^{\mathrm{even}} \cup \Gamma_5^{\mathrm{odd}}) \subset H^{\Xi'}$.

Proposition 9.10. *Let $\mathbf{s}' \in \Lambda$. Then, $\mathbf{s}'$ has positive support of size five or four.*

Proof. Let $\mathbf{s} = (\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}) \in \Gamma_5^{\mathrm{even}} \cup \Gamma_5^{\mathrm{odd}}$. Since $\mathbf{d},\mathbf{e}$ are non-negative, we see from the definition of χ that $\mathbf{f} \in \{0,1\}$ holds. Hence, $\mathbf{s}'$ has positive support of size five or four. □

Proposition 9.11. *Λ contains 2290 elements.*

Proof. This is verified by computer, see [7]. □

9.3 Case $d \geq 42$ Continued: $\mathbf{s}' \in \Lambda$ with Positive Support Size Four

Corollary 9.12. *From Proposition 9.10, we see that* $\mathbf{s}' \in \Lambda$ *has positive support size four if and only if* $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$ *satisfies* $d_i = 1$ *and* $e_j = 1$ *for some* $i, j = 0, 1, 2, 3$.

Corollary 9.13. *Let* $\mathbf{s} \in \Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}$*. The element* $\mathbf{s}$ *maps to some* $\mathbf{s}' \in \Lambda$ *with positive support size four under* χ *if and only if* $\operatorname{supp}(\mathbf{s}) = \{x_{0,0}, x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}$.

Proof. This is easily verified by computer using Algorithm 9. TODO TAG JAIKAJ. ☐

Algorithm 9 Check Configurations for Positive Support

Ensure: a subset $X \subset \Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}$ consisting of configurations that map to $\mathbf{s}' \in \Lambda$ with positive support size four under χ.

$X \leftarrow$ `list()`
$D \leftarrow \{d_0, d_1, d_2, d_3\}$
$E \leftarrow \{e_0, e_1, e_2, e_3\}$
for $\mathbf{s} \in \Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}$ **do**
  if $\operatorname{supp}(\mathbf{s}) \cap D \neq \emptyset$ **and** $\operatorname{supp}(\mathbf{s}) \cap E \neq \emptyset$ **then**
    `X.append(s)`
  end if
end for
return X

We exclude this one case from the 2290 cases in Λ with the following proposition.

Proposition 9.14. *Let* $d \geq 42$*. Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a weakly valid outcome. Then, we have* $\operatorname{supp}^+(\operatorname{contr}_d(\operatorname{sign}(\mathbf{w}))) \neq \{x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}$.

Proof. Assume that $\operatorname{supp}^+(\operatorname{contr}_d(\operatorname{sign}(\mathbf{w}))) = \{x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}$. Then, $\mathbf{w}$ is an outcome with support $\operatorname{supp}(\mathbf{w}) = \{(0,0), (0,3), (1,1), (3,0), (i, d-i), (j, d-j)\}$ for some even $i = 4, 6, 8, \ldots, d-4$ and odd $j = 5, 7, 9, \ldots, d-4$.

Let $\mathbf{u} \in \mathbb{Z}^{V_3}$ be the following outcome

```
 1
 .  .
 .  3  .
-1  .  .  1
```

It has support in $\operatorname{supp}(\mathbf{u}) = \{(0,0), (0,3), (1,1), (3,0)\} \subset \operatorname{supp}(\mathbf{w})$. Define the outcome $\mathbf{v} := \mathbf{w} + w_{0,0}\mathbf{u}$. Then, $v_{0,0} = 0$ and $\mathbf{v} \neq \mathbf{0}$. However, if we apply the Invertibility Criterion with $\lambda = \mathbf{1}$ on $\mathbf{v}$, we see that $\mathbf{v}$ is zero. This is a contradiction. Hence, $\operatorname{supp}^+(\operatorname{contr}_d(\operatorname{sign}(\mathbf{w}))) \neq \{x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}$. ☐

9.4 Case $d \geq 42$ Continued: $\mathbf{s}' \in \Lambda$ with Positive Support Size Five

It remains to show the other 2289 cases of $\mathbf{s}' \in \Lambda$ with positive support size five. From now on, assume that $\mathbf{s}' \in \Lambda$ has positive support size five. We will show that all these cases are invalid. For that we introduce *relative coordinates* defined below to make use of the Invertibility Criterion.

Relative Coordinates and the Invertibility Criterion

Definition 9.15. Let $d \geq 42$. Let M be a sentinel value with no further significance. We use it to encode integers from $4, \ldots, d-7$. Define the map

$$\begin{aligned}
\mathrm{relcoord} : \{0, \ldots, d\} &\to \{0, 1, 2, 3, d-6, d-5, d-4, d-3, d-2, d-1, d, M\}\,, \\
x &\mapsto \begin{cases} x & \text{if } x \in \{0, 1, 2, 3, d-6, d-5, d-4, d-3, d-2, d-1, d\}\,, \\ M & \text{if } x \in \{4, \ldots, d-7\}\,. \end{cases}
\end{aligned}$$

Define the map *relative set* as follows:

$$\begin{aligned}
\mathrm{relset} : \mathbb{Z}^{V_d} &\to 2^{\{0,\ldots,3,M,d-6,\ldots,d\} \times \{0,\ldots,3,M,d-6,\ldots,d\}}, \\
\mathbf{w} &\mapsto \{(\mathrm{relcoord}(i), \mathrm{relcoord}(j)) \mid (i,j) \in \mathrm{supp}(\mathbf{w})\}\,.
\end{aligned}$$

Proposition 9.16. *Let $d \geq 42$. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome with positive support size five and degree d. Write $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f})$. Let $i, j = 0, 1, 2, 3$. Then, all of the following conditions hold:*

(1) $(i,j) \in \mathrm{relset}(\mathbf{w})$ *if and only if* $x_{i,j} \neq 0$*;*

(2) $(i, d-3+j-i) \in \mathrm{relset}(\mathbf{w})$ *if and only if* $y_{i,j} \neq 0$*;*

(3) $(d-3+i-j, j) \in \mathrm{relset}(\mathbf{w})$ *if and only if* $z_{i,j} \neq 0$*;*

(4) $\mathrm{relset}(\mathbf{w}) \cap \{(M,i), (d-6,i), \ldots, (d-4-i,i)\} \neq \emptyset$ *if and only if* $b_i \neq 0$*;*

(5) $\mathrm{relset}(\mathbf{w}) \cap \{(i,M), (i,d-6), \ldots, (i,d-4-i)\} \neq \emptyset$ *if and only if* $c_i \neq 0$*;*

(6) $\mathrm{relset}(\mathbf{w}) \cap \{(M,d-4-i), \ldots, (M,d-6), (M,M), (d-6,M), \ldots, (d-4-i,M)\} \neq \emptyset$ *if and only if* $f_i \neq 0$.

Proof. Let $\mathbf{w}$ be some valid outcome with $x_{i,j} \neq 0$. Then, $w_{i,j} \neq 0$ with $i, j = 0, 1, 2, 3$. By definition of relcoord, $i \mapsto i$ and $j \mapsto j$. So $(i,j) \in \mathrm{relset}(\mathbf{w})$ since $(i,j) \in \mathrm{supp}(\mathbf{w})$.

Assume $y_{i,j} \neq 0$. Then, $w_{i,d-3+j-i} \neq 0$. By definition of relcoord, $i \mapsto i$ and $d-3+j-i \mapsto d-3+j-i$. So $(i, d-3+j-i) \in \mathrm{relset}(\mathbf{w})$ since $(i, d-3+j-i) \in \mathrm{supp}(\mathbf{w})$. The case $z_{i,j} \neq 0$ is similar.

Assume $b_i \neq 0$. Then, there must exist some nonzero $w_{k,i}$ for some $k = 4, \ldots, d-4-i$. Clearly, k maps to some element in $\{M, d-6, \ldots, d-4-i\}$. This shows the claim. The case for c_i is similar.

Assume $f_i \neq 0$. Then, there must exist some nonzero $w_{k,d-i-k}$ for some $k = 4, \ldots, d-4-i$. Clearly, k and $d-i-k$ map to some element in $\{M, d-6, \ldots, d-4-i\}$. This shows the claim. □

Remark 9.17. Given some $\mathbf{s}' \in H^{\Xi'}$ we can compute all the relative supports $\operatorname{relset}(\mathbf{w})$ for all $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $\operatorname{contr}'_d(\operatorname{sign}(\mathbf{w})) = \mathbf{s}'$. Proposition 9.16 gives us an easy way to compute this set of relative supports by interpreting all the six conditions 1., 2., 3., 4., 5., and 6. in Proposition 9.16 as constraints on the relative support of $\mathbf{w}$.

Algorithm 10 Make relative constraints

Require: some nonzero coordinate $t \in \{x_{i,j}, y_{i,j}, z_{i,j}, b_i, c_i, f_i\}$ of $\mathbf{s}' = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f}) \in H^{\Xi'}$
Ensure: set of constraints on the relative support of $\mathbf{w}$ that correspond to the nonzero coordinate t as in Proposition 9.16.

function MAKE_REL_CONSTRAINTS(t)
  if $t = x_{i,j}$ **then**
    return $[(i,j)]$
  end if
  if $t = y_{i,j}$ **then**
    return $[(i, d-3+j-i)]$
  end if
  if $t = z_{i,j}$ **then**
    return $[(d-3+i-j, j)]$
  end if
  if $t = b_i$ **then**
    return $\{(M,i), (d-6,i), \ldots, (d-4-i,i)\}$
  end if
  if $t = c_i$ **then**
    return $\{(i,M), (i,d-6), \ldots, (i,d-4-i)\}$
  end if
  if $t = f_i$ **then**
    return $\{(M,d-4-i), \ldots, (M,d-6), (M,M), (d-6,M), \ldots, (d-4-i,M)\}$
  end if
end function

Relative coordinates help us to apply the Invertibility Criterion.

Example 9.18. Let $d \geq 42$. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be some valid configuration with support size six. Now, assume that $\mathbf{w}$ is $\operatorname{relset}(\mathbf{w}) = \{(0,0), (0,d), (1,3), (M,2), (M,d-6), (d-5,M)\}$. Can such a configuration exist?

```
d       X
d-1     .   .
d-2     .   .   .
d-3     .   .   .   .
d-4     .   .   .   .   .
d-5     .   .   .   .   .   .
d-6     .   .   .   .   .   X   X
M       .   .   .   .   .   .   .   .
M       ........................................
M       ..........................................
M       ............................................
M       .   .   .   .   .   .   .   .   .   .   .   .
M       .   .   .   .   .   .   .   .   .   .   .   .   .   X
M       .   .   .   .   .   .   .   .   .   .   .   .   .   X   .
3       .   X   .   .   .   .   .   .   .   .   .   .   .   .   .   .
2       .   .   .   .   X   X   X   X   X   X   X   X   .   .   .   .   .
1       .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .
0       X   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .

        0   1   2   3   M   M   M   M   M   M   M   M  d-6 d-5 d-4 d-3 d-2 d-1  d
```

We see that $\mathrm{supp}(\mathbf{w}) = \{(0,0),(0,d),(1,3)\} \cup \{(i,2),(j,d-6)\} \cup \{(d-5,k)\}$ for $i,j,k = 4,\dots,d-7$. When $i = j$, we can apply the Invertibility Criterion (Proposition 4.20 and Proposition 4.18) with $\lambda = (3,1,\dots,1,2,1,\dots,1)$ to see that $\mathbf{w} = \mathbf{0}$, which is a contradiction. So assume $i \neq j$. Then, we use $\lambda = (3,1,\dots,1)$ to see that $\mathbf{w} = \mathbf{0}$, which is a contradiction. Hence, $\mathbf{w}$ cannot be an outcome.

Let us generalize this example to elements $\mathbf{s}' \in \Lambda$.

Proposition 9.19. *Let $\mathbf{s}' \in \Lambda$ with positive support size five. Then, we have $|\mathrm{supp}^+(\mathbf{s}') \cap \{b_0,b_1,b_2,b_3\}| \leq 1$, $|\mathrm{supp}^+(\mathbf{s}') \cap \{c_0,c_1,c_2,c_3\}| \leq 1$, and $|\mathrm{supp}^+(\mathbf{s}') \cap \{f_0,f_1,f_2,f_3\}| \leq 1$.*

Proof. This is verified by computer, see [7]. □

Corollary 9.20. *Let $\mathbf{s}' \in \Lambda$. If $|\mathrm{supp}^+(\mathbf{s}')| = 5$, then $|\{(x,y) \in \mathrm{relset}(\mathbf{s}') : x = M\}| \leq 2$ and $|\{(x,y) \in \mathrm{relset}(\mathbf{s}') : y = M\}| \leq 2$ hold.*

Proof. Follows immediately from the previous proposition. □

Proposition 9.21. *Let $d \geq 42$. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid configuration with positive support size five and $|\{(x,y) \in \mathrm{relset}(\mathbf{w}) : x = M\}| = 2$. Denote these elements by (M,x) and (M,y) for $x \neq y$. Write $(i,x),(i',y) \in \mathrm{supp}(\mathbf{w})$ with $i,i' = 4,\dots,d-7$ for the elements that map to (M,x) or (M,y) under* relcoord*, respectively. If we successfully apply the Invertibility Criterion for the case $i = i'$ to show the contradiction $\mathbf{w} = \mathbf{0}$ with $\lambda = (\mathbf{a},1,\dots,1,2,1,\dots,1,\mathbf{b})$*

for some $\mathbf{a} \in \mathbb{Z}^k_{\geq 1}, \mathbf{b} \in \mathbb{Z}^h_{\geq 1}$*, and* $1 \leq k, h \leq 4$*, then we can also apply the Invertibility Criterion for the case* $i \neq i'$ *with* $\lambda' = (\mathbf{a}, 1, \ldots, 1, \mathbf{b})$ *to show the same contradiction.*

Proof. Assume $i \neq i'$. Then, we can just apply Proposition 4.17 as long as $S'_{l'} \in \{0, \lambda'_l\}$ is satisfied for all l', see Section 4.2 on the divide and conquer approach of the Invertibility Criterion. Theses conditions are satisfied because the sets S_l induced by λ satisfy $S_l \in \{0, \lambda_l\}$ by assumption. □

Corollary 9.22. *The previous propostion allows us to specialize to the case* $i = i'$ *when* $(i, x), (i', y) \in \operatorname{supp}(\mathbf{w})$ *for* $i, i' = 4, \ldots, d - 7$ *and* $x \neq y$ *occurs. A similar statement holds for the case* $|\{(x, y) \in \operatorname{relset}(\mathbf{w}) : y = M\}| = 2$*.*

Example 9.23. Returning to Example 9.18, we see that it suffices to consider the case $i = j$. The case $i \neq j$ then follows from the previous corollary.

Corollary 9.24. *Let* $d \geq 42$*. Let* $S = \{(i, x), (i, y), (i + 1, z)\} \subset V_d$*. If* $\operatorname{relcoord}(x) \in \{0, 1, 2, 3\}$*,* $\operatorname{relcoord}(y) \in \{d - 6, \ldots, d\}$ *and* $\operatorname{relcoord}(z) \neq M$*, then* $A^{(d)}_{E,S}$ *is invertible.*

Proof. Use Proposition 4.20. We just need to make sure that $x + y \neq 2z + 1$. Assume $x + y = 2z + 1$. Then, $z = \frac{x+y-1}{2}$. The smallest value for z is $\frac{d-6-1}{2} = 17.5$, and the largest value is $\frac{d+3-1}{2} = 22$ for $d = 42$. Hence, z does map to M under relative coordinates for all $d = 42$ and for $d > 42$ we well. By assumption, $\operatorname{relcoord}(z) \neq M$. This is a contradiction, so $x + y \neq 2z + 1$ for all $d \geq 42$. □

The corollary above can be extended to several other cases which allows us to implement an algorithm to exclude certain cases of $\mathbf{s}' \in \Lambda$. First, we need an algorithm for the division step.

Algorithm 11 Divide

Require: `relset`: Support of a configuration in relative coordinates
Ensure: $L \in \mathbb{Z}^k$ where $L_i = |S_i|$ if a division $(\lambda, (E_i)_{i=1}^k, (S_i)_{i=1}^k)$ is found; `None` otherwise

$M \leftarrow 500$, $d \leftarrow 1000$
$R \leftarrow [0, 1, 2, 3, M, d-6, d-5, d-4, d-3, d-2, d-1, d]$
$L \leftarrow$ `list()`, `col_start` $\leftarrow 0$
for `col_end` $\in \{0, \dots, |R|-1\}$ **do**
    `num_cols` $\leftarrow$ `col_end` $-$ `col_start` $+ 1$
    `points_in_col` $\leftarrow \{p = (x, y) \in$ `relset` $\mid x \in [R[$`col_start`$], R[$`col_end`$]]\}$
    `num_points` $\leftarrow |$`points_in_col`$|$
    if $($`num_points` $= 0) \vee ($`num_cols` $=$ `num_points`$) \vee (R[$`col_end`$] = M)$ **then**
        L.append(`num_points`)
        `col_start` $\leftarrow$ `col_end` $+ 1$
    end if
end for
if `sum`$(L) \neq 6$ **then**
    return `None`
end if
return L

This division algorithm just implements the division rule to choose λ_i minimal such that $|S_i| \in \{0, \lambda_i\}$ as explained in Remark 4.8.

Example 9.25. Here are some examples of the algorithm in action.

(1) $L = [3, 0, 2, 0, 0, 0, 0, 0, 0, 1]$ for support $[(0, 0), (0, 3), (1, d-1), (M, 1), (M, M), (d, 0)]$.

(2) $L = [3, 0, 2, 0, 0, 0, 0, 0, 0, 1]$ for support $[(0, 0), (0, 3), (1, d-1), (M, 1), (M, d-6), (d, 0)]$.

(3) $L = [3, 0, 0, 0, 2, 0, 0, 0, 1]$ for support $[(0, 0), (0, 3), (1, d-1), (d, 0), (d-5, 1), (d-5, M)]$.

Proposition 9.26. *Let $d \geq 42$. Let $a, b = 0, \dots, d$ with $b \geq a$ Then, the following hold:*

(1) If $b = 0, 1, 2, 3$ or $a = d-6, \dots, d$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \neq M$.

(2) Let $b = 4, \dots, d-7$.

a) If $a = 0, 1$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \in \{2, 3, M\}$.

b) If $a = 2, 3$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \in \{3, M\}$.

(3) Let $a = 4, \dots, d-7$.

a) If $b = d-6, d-5$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) = M$.

b) If $b = d-4, d-3$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \in \{M, d-6\}$.

c) *If* $b = d-2, d-1$, *then* $\text{relcoord}(\frac{a+b-1}{2}) \in \{M, d-6, d-5\}$.

d) *If* $b = d$, *then* $\text{relcoord}(\frac{a+b-1}{2}) \in \{M, d-6, d-5, d-4\}$.

Proof. Let $d \geq 42$.

(1) Let $a = b = 3$. Then $\frac{6-1}{2} = 2.5$. Thus, for all $a \leq b \leq 3$ we have $\frac{a+b-1}{2} \leq 2.5$, so $\text{relcoord}(\frac{a+b-1}{2}) \neq M$. Let $a = b = d-6$. Then $\frac{a+b-1}{2} > d-7$. Thus, for all $d-6 \leq a \leq b \leq d$ we have $\text{relcoord}(\frac{a+b-1}{2}) \neq M$.

(2) Let $b = 4, \dots, d-7$.

a) For $a = 0$ and $b = 4$ we obtain $\frac{a+b-1}{2} > 1.5$. So, $\text{relcoord}(\frac{a+b-1}{2}) \notin \{0, 1\}$. For $a = 1$ and $b = d-7$, we see that $\frac{a+b-1}{2} = \frac{d-7}{2}$, which maps to M under relative coordinates for $d \geq 42$. So the midpoint maps to values between two and M under relcoord, i.e. $\{2, 3, M\}$. This shows the claim.

b) The proof of case $a = 2, 3$ is similar to the previous proof.

(3) Let $a = 4, \dots, d-7$.

a) Let $b = d-6, d-5$. Then, $\frac{a+b-1}{2} \geq \frac{d-3}{2} \geq 19.5$ for all $d \geq 42$. This maps to M under relative coordinates. Moreover, $\frac{a+b-1}{2} \leq \frac{d-7+d-5-1}{2} = \frac{2d-13}{2} = d - 6.5$. Thus, $\text{relcoord}(\frac{a+b-1}{2}) = M$.

b) The proof of cases $b = d-4, d-3, \dots, d$ are similar to the previous proof.

$\square$

This proposition gives Algorithm 12 to compute the midpoint of two points in relative coordinates. Next, we define the order in relative coordinates:

$$0 < 1 < 2 < 3 < M < d-6 < d-5 < d-4 < d-3 < d-2 < d-1 < d.$$

The `succ` function is defined as the successor in this order, see Algorithm 13. We can now implement the conquer step of the divide and conquer approach of the Invertibility Criterion, see Algorithm 14. This leads us to an implementation of the Invertibility Criterion for relative coordinates, see Algorithm 15.

Let us apply the Invertibility Criterion for relative coordinates to all of the 2289 cases of $\mathbf{s}' \in \Lambda$ with a positive support size of five. We want to compute the set

$$R(\mathbf{s}') := \{\text{relset}(\mathbf{w}) \mid \mathbf{w} \in \mathbb{Z}^{V_d} \text{ such that } \text{contr}'(\text{sign}(\mathbf{w})) = \mathbf{s}'\}.$$

Here is a simple depth first search algorithm to compute $R(\mathbf{s}')$ using Proposition 9.16, see Algorithm 17. Now, we can apply the Invertibility Criterion for relative coordinates to all of $\mathbf{m} \in R(\mathbf{s}')$. If all of these applications to $\mathbf{m} \in R(\mathbf{s}')$ are successful, then $\mathbf{s}'$ is impossible. Otherwise, we cannot exclude $\mathbf{s}'$ as a possible outcome.

Algorithm 12 Compute midpoint $\frac{a+b-1}{2}$

Require: Constants: `SENTINEL_M` $\leftarrow 500$, `SENTINEL_D` $\leftarrow 1000$

```
function MIDPOINT(a, b)
    low, high ← min(a, b), max(a, b)
    if low > SENTINEL_M or high < SENTINEL_M then
        return {(low+high−1)/2}
    else if high = SENTINEL_M then
        if low ∈ {0, 1} then
            return {2, 3, SENTINEL_M}
        else if low ∈ {2, 3} then
            return {3, SENTINEL_M}
        end if
    else if low = SENTINEL_M then
        if high ∈ {SENTINEL_D − 6, SENTINEL_D − 5} then
            return {SENTINEL_M}
        else if high ∈ {SENTINEL_D − 4, SENTINEL_D − 3} then
            return {SENTINEL_M, SENTINEL_D − 6}
        else if high ∈ {SENTINEL_D − 2, SENTINEL_D − 1} then
            return {SENTINEL_M, SENTINEL_D − 6, SENTINEL_D − 5}
        else if high = SENTINEL_D then
            return {SENTINEL_M, SENTINEL_D − 6, SENTINEL_D − 5, SENTINEL_D − 4}
        end if
    end if
    return {SENTINEL_M}
end function
```

Algorithm 13 Successor

Require: Constants: `SENTINEL_M` $\leftarrow 500$, `SENTINEL_D` $\leftarrow 1000$

```
function SUCC(x)
    if x = 3 then
        return SENTINEL_M
    else if x = SENTINEL_M then
        return SENTINEL_D − 6
    else
        return x + 1
    end if
end function
```

Algorithm 14 Conquer Algorithm

Require: support `relset` in relative coordinates

Ensure: `True` only if the Invertibility Criterion is successful; `False` if inconclusive

```
function CONQUER(relset)
    relset ← relset.sort_by_column()
    length ← |relset|
    if length ≤ 2 then
        return True                                         ▷ Proposition 4.17 and 4.18
    else if length = 3 then
        x, y, z ← relset[0], relset[1], relset[2]
        same_column ← (col(x) = col(y) = col(z))
        if same_column then                                 ▷ Proposition 4.19
            return True
        else if col(x) = col(y) and succ(col(x)) = col(z) then
            if row(z) ∉ midpoint(row(x), row(y)) then       ▷ Proposition 4.20
                return True
            end if
        end if
        return False
    else
        return False
    end if
end function
```

Algorithm 15 Invertibility Criterion for relative coordinates

Require: support `relset` in relative coordinates

Ensure: `True` only if the Invertibility Criterion is successful; `False` if inconclusive

```
function INVERTIBILITY_CRITERION(relset)
    L ← divide(relset)
    return all_is_true([conquer(s) for s in make_subrelset(relset, L)])
end function
```

Algorithm 16 Make subrelset

```
function MAKE_SUBRELSET(relset, L)
    i ← 0
    for n ∈ {l ∈ L | l > 0} do
        yield relset[i : i + n]
        i ← i + n
    end for
end function
```

Of the 2289 cases of $\mathbf{s}' \in \Lambda$ with positive support size five, we rule out 1182 cases by applying the Invertibility Criterion for relative coordinates using Algorithm 18. The remaining 1107 cases are inconclusive. The details of the implementation can be found in the Appendix TODO.

Algorithm 17 Find Relative Coordinates

Require: $\mathbf{s}' \in H^{\Xi}$
Ensure: $R(\mathbf{s}')$

```
1: function RELSETS(s')
2:     constraints ← [make_rel_constraints(t) for t in s' if t != 0]
3:     accu ← list()
4:     res ← list()
5:     function DFS(i)
6:         if i ≥ |constraints| then
7:             res.append(accu.copy())
8:             return
9:         end if
10:        for x ∈ constraints[i] do
11:            accu.append(x)
12:            dfs(i + 1)
13:            accu.pop()
14:        end for
15:    end function
16:    dfs(0)
17:    return res
18: end function
```

Proof of correctness. Let $\mathbf{s}' = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f})$. By Proposition 9.16, $\mathbf{m} \in R(\mathbf{s}')$ if and only if $\mathbf{m} \cap C \neq \emptyset$ for all $C \in \{\texttt{make\_rel\_constraints(t)} \mid t \in \{x_{i,j}, y_{i,j}, z_{i,j}, b_i, c_i, f_i\}, t \neq 0\}$.

It just remains to show that $\mathbf{m} \in \texttt{res}$ if and only if $\mathbf{m}$ satisfies all of these constraints. First note that $x \in \mathbf{m}$ if and only if $x \in$ constraints[i] for some i by line eleven. Since $\mathbf{m} \in \texttt{res}$ if and only if $\mathbf{m}$ satisfies all constraints by line six, the algorithm is correct. $\square$

Symmetry

It remains to consider 1107 cases thanks to the Invertibility Criterion for relative coordinates. We can use symmetry to reduce the number of cases further. Let $\mathbf{s}' \in \Lambda$ be one of the 1107 cases. We can apply the following symmetries to $\mathbf{s}'$ similar to the symmetries in Section 4.4:

$$(12) \cdot (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f}) \coloneqq ((x_{j,i})_{i,j=0}^{3}, (y_{j,i})_{i,j=0}^{3}, (z_{j,i})_{i,j=0}^{3}, \mathbf{c}, \mathbf{b}, \mathbf{f}),$$
$$(13) \cdot (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f}) \coloneqq ((z_{3-i,j})_{i,j=0}^{3}, (y_{3-j,3-i})_{i,j=0}^{3}, (x_{3-i,j})_{i,j=0}^{3}, \mathbf{f}, \mathbf{c}, \mathbf{b}).$$

Algorithm 18 Check Configurations Against Invertibility Criterion

Require: Λ

Ensure: `left_to_check`: supports not satisfying the Invertibility Criterion

```
left_to_check ← list()
for each s′ ∈ Λ do
    res ← [invertibility_criterion(R) for R ∈ relsets(s′)]
    if ¬all(res) then
        left_to_check.append(s′)
    end if
end for
return left_to_check
```

Define the group S_3 generated by these two symmetries. For all weakly valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$, the actions $\sigma \in S_3$ satisfy $\sigma \cdot \operatorname{contr}'_d(\operatorname{sign}(\mathbf{w})) = \operatorname{contr}'_d(\operatorname{sign}(\sigma \cdot \mathbf{w}))$. We can use this symmetry to reduce the number of cases to consider.

Proposition 9.27. *Let $d \in \mathbb{N}$ and $\mathbf{s}' \in H^{\Xi'}$. If there are no weakly valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$ of degree d with $\operatorname{contr}'_d(\operatorname{sign}(\mathbf{w})) = \sigma\mathbf{s}'$ for some $\sigma \in S_3 \setminus \{(1)\}$, then there are no weakly valid outcomes $\mathbf{v} \in \mathbb{Z}^{V_d}$ of degree d with $\operatorname{contr}'_d(\operatorname{sign}(\mathbf{v})) = \mathbf{s}'$.*

Proof by Contraposition. Let $\sigma \in S_3 \setminus \{(1)\}$. Assume there is some valid outcome $\mathbf{v} \in \mathbb{Z}^{V_d}$ of degree d such that $\operatorname{contr}'_d(\operatorname{sign}(\mathbf{v})) = \mathbf{s}'$. Define $\mathbf{w} := \sigma\mathbf{v}$. Then, $\operatorname{contr}'_d(\operatorname{sign}(\mathbf{w})) = \sigma \cdot \operatorname{contr}'_d(\operatorname{sign}(\mathbf{v})) = \sigma\mathbf{s}'$. Note that $\mathbf{w}$ is weakly valid since the symmetry group S_3 preserves the weak validity of outcomes. This shows the claim. $\square$

We take the 1107 cases of $\mathbf{s}' \in \Lambda$, apply the symmetry (12) to each case and check if the Invertibility Criterion for relative coordinates is satisfied. The inconclusive cases are then checked with the symmetry (13). We find that 758 cases are impossible and 349 cases are inconclusive. A pseudo code implementation of this symmetry reduction is given in Algorithm 19. More details can be found in Appendix TODO.

Of the 349 configurations, we consider the equivalence relation $\sim$ defined by $w \sim v \iff w = (12)v$ or $w = (13)v$. We compute the equivalence classes of $\sim$, and see there are 179 equivalence classes. By Proposition 4.14 amd 4.16, it suffices to consider one contracted configuration $\mathbf{s}'$ from each of the 179 equivalence classes.

Hexagon Criterion

We have made significant progress in reducing the number of cases to consider. We are left with 179 cases of $\mathbf{s}' \in \Lambda$ out of 2290 initial cases. It is now time to use our third and last tool from our toolbox, the Hexagon Criterion.

Let $d \geq 42$. We want to apply the Hexagon Criterion (Proposition 8.2) with $\ell_1 = \ell_2 = 7, d' = 6$ to a subset of the 179 cases. First, we check that the requirements of the Hexagon

Algorithm 19 Further Reduciton by Symmetry

Require: 1107 inconclusive cases $\mathbf{s}' \in \Lambda$

Ensure: `left_to_check`: supports not satisfying the Invertibility Criterion

```
tmp ← list(), left_to_check ← list()
for each s' ∈ {(12)t' : t' ∈ Λ} do
    res ← [invertibility_criterion(R) for R ∈ relsets(s')]
    if ¬all(res) then
        tmp.append((21)s')
    end if
end for
for each s' ∈ {(13)t' : t' ∈ tmp} do
    res ← [invertibility_criterion(R) for R ∈ relsets(s')]
    if ¬all(res) then
        left_to_check.append((31)s')
    end if
end for
return left_to_check
```

Criterion are met: $d' + \ell_1 + \ell_2 = 20 \leq 42 = d$. Next, we need to find all contracted configurations from the 179 cases of $\mathbf{s}' \in \Lambda$ whose support lies outside the hexagon spanned by d', ℓ_1 and ℓ_2.

Proposition 9.28. *Let $d \geq 42$. Let $\mathbf{s}' \in H^{\Xi'}$ be some contracted configuration and $\mathbf{w} \in \mathbb{Z}^{V_d}$ be some configuration with $\operatorname{contr}'_d(\operatorname{sign}(\mathbf{w})) = \mathbf{s}'$. If $\operatorname{supp}^+(\mathbf{s}') \cap \{\mathbf{b}, \mathbf{c}, \mathbf{f}\} = \emptyset$, then $\operatorname{supp}(\mathbf{w})$ lies outside the hexagon spanned by d', ℓ_1 and ℓ_2.*

Proof. This follows immediately from the definition of the hexagon spanned by d', ℓ_1 and ℓ_2, and the definition of the map contr'_d. $\square$

A simple computation shows that 166 cases out of the 179 cases satisfy the Proposition above (for details see Appendix TODO). By the Hexagon Criterion, valid outcomes $\mathbf{w}$ with $\operatorname{contr}'_d(\operatorname{sign}(\mathbf{w})) = \mathbf{s}'$ where $\mathbf{s}'$ is one of these 166 cases have degree at most $d' = 20$. However, we already showed that there are no valid outcomes of degree ≤ 41; we found a contradiction. Thus, it remains to check the thirteen cases where the Hexagon Criterion could not be applied.

Here are the thirteen cases of $\mathbf{s}' \in \Lambda$ that remain:

(1) $\{y_{0,3}, z_{2,0}, z_{2,2}, z_{3,1}, c_1\}$

(2) $\{y_{0,3}, y_{1,2}, y_{2,1}, z_{3,0}, b_1\}$

(3) $\{y_{0,3}, y_{1,1}, y_{1,3}, z_{2,0}, b_1\}$

(4) $\{y_{0,3}, y_{1,3}, y_{2,2}, z_{2,0}, b_1\}$

(5) $\{x_{0,1}, x_{2,1}, y_{1,3}, z_{3,0}, d_1\}$

(6) $\{y_{0,3}, z_{1,0}, z_{2,2}, z_{3,1}, c_1\}$

(7) $\{y_{0,2}, y_{1,1}, y_{1,3}, z_{3,0}, b_1\}$

(8) $\{y_{0,3}, z_{2,2}, z_{3,0}, z_{3,1}, c_1\}$

(9) $\{y_{0,3}, z_{1,1}, z_{3,0}, z_{3,1}, c_1\}$

(10) $\{x_{0,3}, x_{1,1}, x_{3,0}, z_{3,3}, d_0\}$

(11) $\{x_{0,2}, x_{2,1}, y_{1,3}, z_{3,0}, d_1\}$

(12) $\{y_{0,3}, z_{3,0}, b_1, c_1, d_1\}$

(13) $\{x_{1,2}, x_{2,1}, y_{0,3}, z_{3,0}, d_1\}$

Let $d \geq 42$. We can go through these cases one by one and apply the Hexagon Criterion.

Example 9.29. Consider the first case.

```
y03 z20 z22 z31 c1

*
y  y
y  y  y
y  y  y  y
c  y  y  y  d
c  *  y  y  d  d
c  *  c  y  d  d  d
c  *  c  c  d  d  d  d
c  *  c  c     d  d  d  d
c  *  c  c        d  d  d  d
c  *  c  c           d  d  d  d
x  x  x  x  b  b  b  b  z  z  z  z
x  x  x  x  b  b  b  b  b  z  z  *  z
x  x  x  x  b  b  b  b  b  b  z  z  z  *
*  x  x  x  b  b  b  b  b  b  b  z  z  *  z
```

To apply the Hexagon Criterion to some configuration $\mathbf{w}$ with the support indicated as above, we need to know where the nonzero entry of $\mathbf{w}$ in the c_1-column is roughly located; here the c_1-column denotes the entries $w_{1,k}$ with $k = 4, \dots, d-1-4$. There can only exist one nonzero entry in the c_1-column since $\mathbf{w}$ has positive support size five. Let us denote the nonzero entry in the c_1-column by $(a, b) \in V_d$ where $a = 1$. We differentiate between three cases:

(1) Let $a+b \leq \text{floor}(\frac{d}{3})$. Set $\ell_1 = \ell_2 = d' = \text{floor}(\frac{d}{3}) \geq \frac{42}{3} = 14$. We easily see that the other non-entries $y_{0,3}, z_{2.0}, z_{2,2}, z_{3,1}$ lie outside the hexagon.

(2) Let $b \geq \text{floor}(\frac{d}{3})$. Set $d' = 6$, $\ell_2 = 7$ and $\ell_1 = d+1-\text{floor}(\frac{d}{3}) \geq 43-14 = 29$. We see that $d' + \ell_1 + \ell_2 \leq d$ since $d \geq 42$. We easily see that the non-entries $y_{0,3}, z_{2.0}, z_{2,2}, z_{3,1}$ lie outside the hexagon. It remains to check that (a,b) also lies outside the hexagon. We have $d - \ell_1 = d-(d+1)+\text{floor}(\frac{d}{3}) = \text{floor}(\frac{d}{3}) - 1 < \text{floor}(\frac{d}{3}) \leq b$. Thus, (a,b) lies outside the hexagon.

Both these cases show that we can apply the Hexagon Criterion; the criterion shows that the degree of $\mathbf{w}$ is at most $d' \leq \text{floor}(\frac{d}{3})$. However, we assume that $\mathbf{w}$ is of degree d, which is a contradiction since $d' < d$. We rule out $\{y_{0,3}, z_{2,0}, z_{2,2}, z_{3,1}, c_1\}$.

We generalize Example 9.29 to apply the Hexagon Criterion to the remaining twelve cases. Unfortunately, there is one case $\{y_{0,3}, z_{3,0}, b_1, c_1, d_1\}$ which cannot be ruled out by the Hexagon Criterion. We need to consider this case separately; so let us ignore this case for now and consider the remaining eleven cases.

First, we observe that all the eleven cases satisfy $|\text{supp}^+(\mathbf{s}') \cap \{b_0, b_1, c_0, c_1, f_0, f_1\}| = 1$ and $|\text{supp}^+(\mathbf{s}') \cap \{b_2, c_2, f_2, b_3, c_3, f_3\}| = 0$. Hence, for all valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $\text{contr}'_d(\text{sign}(\mathbf{w})) = \mathbf{s}'$ there exists some vertex $(a,b) \in V_d$ with $a = 0,1$ or $b = 0,1$ or $a+b \geq d-1$ such that

$$\text{supp}(\mathbf{w}) \setminus \{(a,b)\} \subset V_6 \cup \{(i,j) \in V_d \mid j > d-7\} \cup \{(i,j) \in V_d \mid i > d-7\}\,.$$

Proposition 9.30. *One of the following statements holds:*

(1) $a+b \leq \text{floor}(\frac{d}{3})$,

(2) $a \geq \text{floor}(\frac{d}{3})$, *or*

(3) $b \geq \text{floor}(\frac{d}{3})$.

Proof. We make a case distinction:

- Let $a+b \leq \text{floor}(\frac{d}{3})$. Our claim holds.
- Let $a = 0$ with $a+b > \text{floor}(\frac{d}{3})$. Then, $b > \text{floor}(\frac{d}{3})$. Let $a = 1$ with $a+b > \text{floor}(\frac{d}{3})$. Then, $b \geq \text{floor}(\frac{d}{3})$.
- Let $b = 0$ with $a+b > \text{floor}(\frac{d}{3})$. Then, $a > \text{floor}(\frac{d}{3})$. Let $b = 1$ with $a+b > \text{floor}(\frac{d}{3})$. Then, $a \geq \text{floor}(\frac{d}{3})$.
- Let $a+b \geq d-1$. Since $d-1 > 2\text{floor}(\frac{d}{3})$, we have $a > \text{floor}(\frac{d}{3})$ or $b > \text{floor}(\frac{d}{3})$.

This proves the proposition. □

Proposition 9.31. *All the eleven cases are impossible.*

Proof. It is easy to see that all of the eleven cases either satisfy 1., 2., or 3. of Proposition 9.30. When 1. holds, we let $d' = \ell_1 = \ell_2 = \text{floor}(\frac{d}{3}) \geq 14 > 6$. When 2. holds, we choose $d' = 6, \ell_1 = 7$ and $\ell_2 = d + 1 - \text{floor}(\frac{d}{3})$. Note that $d' + \ell_1 + \ell_2 \leq 42$ since $d \geq 42$. When 3. holds, we choose $d' = 6, \ell_2 = 7$ and $\ell_1 = d + 1 - \text{floor}(\frac{d}{3})$.

We apply the Hexagon Criterion with these choices of d', ℓ_1 and ℓ_2 to each of the eleven cases. We find that the degree of the valid outcomes $\mathbf{w}$ is at most d'. However, we assumed that the degree of the valid outcomes is d, which is a contradiction. This shows that all of the eleven cases are impossible. □

The Final Case

We are left with one case $\{y_{0,3}, z_{3,0}, b_1, c_1, d_1\}$.

Proposition 9.32. *There is no weakly valid outcome* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *such that* $\text{contr}'_d(\text{sign}(\mathbf{w})) = \{x_{0,0}, y_{0,3}, z_{3,0}, b_1, c_1, d_1\}$.

Proof by Contradiction. We assume that $d \geq 42$. Assume that a weakly valid outcome $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $\text{contr}'_d(\text{sign}(\mathbf{w})) = \{x_{0,0}, y_{0,3}, z_{3,0}, b_1, c_1, d_1\}$ exists. Then, the support of $\mathbf{w}$ reads

$$S = \{(0,0), (d,0), (0,d), (i,1), (1,j), (k, d-1-k)\}$$

for some integers i, j, k. Let us define $e \in \mathbb{Z}$ such that we can write $d = 2e + 1$. If $j \neq e$, we use Proposition 4.20 to show that such a $\mathbf{w}$ cannot exist. So, we assume that $j = e$. By symmetry $(12) \in S_3$ and $(13) \in S_3$, we conclude that $i = e$ and $k = e$. This shows that the support of $\mathbf{w}$ reads

$$\begin{aligned} S &= \{(0,0), (d,0), (0,d), (e,1), (1,e), (e, d-1-e)\} \\ &= \{(0,0), (d,0), (0,d), (e,1), (1,e), (e,e)\}. \end{aligned}$$

We apply the Invertibility Criterion with $E \coloneqq \{0, 1, 3, e, d-1, d\}$ which leads to the following pairing matrix

$$A^{(d)}_{E,S} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 1 & 0 \\ \binom{d}{3} & 0 & 0 & 0 & \binom{e}{2} & 0 \\ \binom{d}{e} & 0 & 0 & 1 & e & 1 \\ d & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This pairing matrix has determinant $\frac{(2e+1)(e+1)e}{6} \neq 0$. Thus, S cannot be the support of a valid nonzero outcome. □

With the last case excluded, we proved that for all valid integral outcomes $\mathbf{w}$ with $|\text{supp}^+(\mathbf{w})| = 5$ we have $\deg(\mathbf{w}) \leq 7$.

Chapter 10

Valid Outcomes of Positive Support Size Six

We want to prove that for all valid integral outcomes $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 6$ we have $\deg(\mathbf{w}) \leq 9$. The original paper by Bik and Marigliano [2] does not provide a proof of this result.

This thesis makes a contribution towards proving it by reducing the number of cases to check, thereby making the problem computationally feasible. In principle, we could use the same approach as in the previous chapters to find all valid outcomes with positive support size six. Specifically, we begin by computing $\Gamma^{\text{even}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| = 6\right\}$ and $\Gamma^{\text{odd}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| = 6\right\}$, albeit with a slight modification to these sets. While we previously considered contractions of size four, here we consider contractions of size five. In other words, we now work with contraction variables as depicted in Figure 10.1. From this point forward, when we refer to $\mathbf{s} \in H^{\Xi}$, we will mean contractions of size five.

Definition 10.1. We define

$$\Gamma_6^{\text{even}} := \Gamma^{\text{even}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| = 6\right\} \text{ and } \Gamma_6^{\text{odd}} := \Gamma^{\text{odd}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| = 6\right\},$$

where Γ^{even} and Γ^{odd} are defined analogously to Definition 7.1, but now they contain hyperfield configurations of contraction size five for

$$\Phi_1 := \{\mathrm{col}(1), \mathrm{col}(2), \mathrm{col}(3), \mathrm{col}(3), \mathrm{row}(1), \mathrm{row}(2), \mathrm{row}(3), \mathrm{row}(4),$$
$$\mathrm{diag}(1), \mathrm{diag}(2), \mathrm{diag}(3), \mathrm{diag}(4), \mathrm{diag}(d-1), \mathrm{diag}(d-2), \mathrm{diag}(d-3), \mathrm{diag}(d-4)\},$$

and

$$\Phi_2 := \{\mathrm{col}(d), \mathrm{col}(d-1), \mathrm{col}(d-2), \mathrm{col}(d-3), \mathrm{col}(d-4),$$
$$\mathrm{row}(d), \mathrm{row}(d-1), \mathrm{row}(d-2), \mathrm{row}(d-3), \mathrm{row}(d-4)\}.$$

We increased the contraction size to gain more information about potential supports, which we hope will prove useful when handling cases manually.

$$
\begin{array}{cccccccccccccccccc}
y_{0,4} \\
y_{0,3} & y_{1,4} \\
y_{0,2} & y_{1,3} & y_{2,4} \\
y_{0,1} & y_{1,2} & y_{2,3} & y_{3,4} \\
y_{0,0} & y_{1,1} & y_{2,2} & y_{3,3} & y_{4,4} \\
c_0 & y_{1,0} & y_{2,1} & y_{3,2} & y_{4,3} & d_0 \\
c_0 & c_1 & y_{2,0} & y_{3,1} & y_{4,2} & d_1 & e_0 \\
c_0 & c_1 & c_2 & y_{3,0} & y_{4,1} & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & y_{4,0} & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & d_4 & e_3 & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & * & e_4 & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & * & * & d_4 & e_3 & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & * & * & * & e_4 & d_3 & e_2 & d_1 & e_0 \\
x_{0,4} & x_{1,4} & x_{2,4} & x_{3,4} & x_{4,4} & b_4 & b_4 & b_4 & b_4 & z_{0,4} & z_{1,4} & z_{2,4} & z_{3,4} & z_{4,4} \\
x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} & x_{4,3} & b_3 & b_3 & b_3 & b_3 & b_3 & z_{0,3} & z_{1,3} & z_{2,3} & z_{3,3} & z_{4,3} \\
x_{0,2} & x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & z_{0,2} & z_{1,2} & z_{2,2} & z_{3,2} & z_{4,2} \\
x_{0,1} & x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & z_{0,1} & z_{1,1} & z_{2,1} & z_{3,1} & z_{4,1} \\
x_{0,0} & x_{1,0} & x_{2,0} & x_{3,0} & x_{4,0} & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & z_{0,0} & z_{1,0} & z_{2,0} & z_{3,0} & z_{4,0}
\end{array}
$$

Figure 10.1: This figure shows contraction variables for $d = 16$ of contraction size five.

Proposition 10.2. *We have* $|\Gamma_6^{\text{even}}| = 150032$ *and* $|\Gamma_6^{\text{odd}}| = 154177$.

Proof. This is verified by a computer program, which is available on GitHub [7]. □

The number of cases to check has increased by a factor of hundred compared to the previous chapters. If we would be able to reduce the cases immensely, we could in theory apply the same techniques as in the previous chapters to prove $\deg(\mathbf{w}) \leq 9$. We will spend the remaining part of this chapter to find a way to reduce the number of cases to check to around 10,000 cases; this is a reduction by a factor of 30. We hope that this reduction is sufficient to make the problem computationally feasible for future proofs. Due to time constraints of this thesis, we will not be able to prove $\deg(\mathbf{w}) \leq 9$ for all valid outcomes of positive support size six.

10.1 Fixed-Contractable Pascal Forms

To reduce the number of cases to check, we want to generate a set of hyperfield Pascal forms that are *fixed-contractable*.

Definition 10.3. Let $d \in \mathbb{N}_{\geq 15}$ and p be a hyperfield linear form on H^{V_d}. We say p is *contractable* for d if we can write $p = \hat{p}$ for some linear form $\hat{p} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$.

Definition 10.4. Let $d \in \mathbb{N}_{\geq 15}$, $k = 0, \ldots, 4$, and p be a hyperfield linear form on H^{V_d}. We say p is *contractable* for d on b_k if we can write

$$p = \sum_{(i,j) \in V_d \setminus \{(5,k), \ldots, (d-k-5,k)\}} \lambda_{i,j} x_{i,j} + \lambda b_k \quad \text{for} \quad b_k \coloneqq x_{5,k} + \cdots + x_{d-5-k,k}$$

for $\lambda_{i,j}, \lambda \in H$. In other words, we can write $p = \hat{p}$ for some linear form $\hat{p} \in H[\mathbf{x}, b_k]$. Similarly, we define p is *contractable* on c_k, d_k, and e_k if we can write $p = \hat{p}$ for some linear form $\hat{p} \in H[\mathbf{x}, c_k]$, $\hat{p} \in H[\mathbf{x}, d_k]$, and $\hat{p} \in H[\mathbf{x}, e_k]$, respectively.

Remark 10.5. Clearly, p is contractable for d if and only if it is contractable for d on b_k, c_k, d_k, and e_k for all $k = 0, \ldots, 4$.

We are interested in linear combinations of hyperfield Pascal forms p that are contractable such that $\hat{p}$ is independent of the degree d. We call such linear combinations *fixed-contractable* because $\hat{p}$ is *fixed* over all degrees d. To formalize this, we introduce the following definition.

Definition 10.6. Let t be a formal variable and T be a formal linear combination of

$$\{\operatorname{sign}(\operatorname{diag}(k)), \operatorname{sign}(\operatorname{row}(k)), \operatorname{sign}(\operatorname{col}(k))\}_{k \in \{0,1,2,3,4,t-4,t-3,t-2,t-1,t\}}.$$

Let $d \in \mathbb{N}_{\geq 15}$. We write p_d for the realization of T for $t = d$; the realization p_d is just a linear form on H^{V_d} where we replace the formal variable t by an actual value d.

Example 10.7. Consider the formal linear combination $T = \text{row}(3) - row(t-2)$. Let $d = 15$. The realization p_d of T for $t = d$ is the linear form depicted in the following figure

-350															
-350	.														
-285	65	65													
-220	65	.	-65												
-165	55	-10	-10	55											
-120	45	-10	.	10	-45										
-84	36	-9	1	1	-9	36									
-56	28	-8	1	.	-1	8	-28								
-35	21	-7	1	.	.	1	-7	21							
-20	15	-6	1	.	.	.	-1	6	-15						
-10	10	-5	1	.	.	.	.	1	-5	10					
-4	6	-4	1	.	.	.	.	.	-1	4	-6				
-1	3	-3	1	.	.	.	.	.	.	1	-3	3			
.	1	-2	1	.	.	.	.	.	.	.	-1	2	-1		
.	.	-1	1	.	.	.	.	.	.	.	.	1	-1	.	
.	.	.	1	.	.	.	.	.	.	.	.	.	-1	.	.

The realization p_d is contractable for all $d \in \mathbb{N}$ with $d \geq 15$, but does the linear form $\hat{p}_d \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ change with d? We will show that it does not change.

Definition 10.8. Let T be a formal linear combination of

$$\{\text{sign}(\text{diag}(k)), \text{sign}(\text{row}(k)), \text{sign}(\text{col}(k))\}_{k \in \{0,1,2,3,4,t-4,t-3,t-2,t-1,t\}}.$$

We say T is *fixed-contractable* if all of the following statements hold:

(1) the realization p_d is contractable for all $d \in \mathbb{N}$ with $d \in \mathbb{N}_{\geq 15}$;

(2) there exists a linear form $\hat{p}^{\text{even}} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ such that $p_d = \hat{p}^{\text{even}}$ for all even degrees $d \in \mathbb{N}_{\geq 15}$;

(3) there exists a linear form $\hat{p}^{\text{odd}} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ such that $p_d = \hat{p}^{\text{odd}}$ for all odd degrees $d \in \mathbb{N}_{\geq 15}$.

Similarly, we define p is *fixed-contractable* on b_i if $\hat{p}^{\text{even}} \in H[\mathbf{x}, b_i]$ such that $p_d = \hat{p}^{\text{even}}$ for all even degrees $d \in \mathbb{N}_{\geq 15}$ and $\hat{p}^{\text{odd}} \in H[\mathbf{x}, b_i]$ such that $p_d = \hat{p}^{\text{odd}}$ for all odd degrees $d \in \mathbb{N}_{\geq 15}$.

The definition of *fixed-contractable* on c_i, d_i, and e_i is analogous.

Example 10.9. By Proposition 6.33 and 6.35, the following formal linear combinations are fixed-contractable: $\text{col}(1), \text{col}(2), \text{col}(3)$, $\text{row}(1), \text{row}(2), \text{row}(3)$, $\text{diag}(1), \text{diag}(2), \text{diag}(3)$,

$\mathrm{diag}(d-1), \mathrm{diag}(d-2), \mathrm{diag}(d-3)$, $\mathrm{col}(d), \mathrm{col}(d-1), \mathrm{col}(d-2), \mathrm{col}(d-3)$, $\mathrm{row}(d), \mathrm{row}(d-1), \mathrm{row}(d-2), \mathrm{row}(d-3)$.

We are looking for more fixed-contractable formal linear forms.

Definition 10.10. Let $d \in \mathbb{N}_{\geq 15}, i \in \{0, \ldots, 4, d-4, \ldots, d\}$, and $p = \sum \lambda_{i,j} x_{i,j}$ be a hyperfield linear form on H^{V_d}. We define the i-th b-row of p as $p_{b_i} := \begin{bmatrix} \lambda_{5,i} & \ldots & \lambda_{d-i-5,i} \end{bmatrix} \in H^{d-i-9}$.

Similarly, we define p_{c_i}, p_{d_i} and p_{e_i} to denote the i-th b-column, d-diagonal and e-diagonal of p, respectively.

Proposition 10.11. *Let $d \in \mathbb{N}_{\geq 15}, i = 0, \ldots, 4$, and T be a formal linear combination of $\{\mathrm{sign}(\mathrm{row}(j)), \mathrm{sign}(\mathrm{col}(j)), \mathrm{sign}(\mathrm{diag}(j)) \mid j \in \{0, \ldots, 4\} \cup \{t-4, \ldots, t\}\}$. Then, the following statements hold for all realizations p_d of T:*

(1) The c-column of $(p_d)_{c_i}$ only depends on $\{\mathrm{sign}(\mathrm{row}(k)), \mathrm{sign}(\mathrm{diag}(k))\}_{k=0}^{4}$.

(2) The b-row of $(p_d)_{b_i}$ only depends on $\{\mathrm{sign}(\mathrm{col}(k)), \mathrm{sign}(\mathrm{diag}(d-k))\}_{k=0}^{4}$.

(3) The d-diagonal of $(p_d)_{d_i}$ only depends on $\{\mathrm{sign}(\mathrm{row}(d-k)), \mathrm{sign}(\mathrm{col}(d-k))\}_{k=0}^{4}$. A similar statement holds for e-diagonals.

Proof. This follows immediately from the definition of row, col, and diag. □

Proposition 10.12. *Let $d \in \mathbb{N}_{\geq 15}$ and $i = 0, \ldots 4$. The following statements hold:*

(1) Let $T \in \{\mathrm{sign}(\mathrm{row}(k)), \mathrm{sign}(\mathrm{diag}(k))\}_{k=0,\ldots,4}$. The c_i-column of p_d is a constant vector.

(2) Let $T \in \{\mathrm{sign}(\mathrm{col}(k)), \mathrm{sign}(\mathrm{diag}(d-k))\}_{k=0,\ldots,4}$. The b_i-row of p_d is a constant vector.

(3) Let $T \in \{\mathrm{sign}(\mathrm{row}(d-k)), \mathrm{sign}(\mathrm{col}(d-k))\}_{k=0,\ldots,4}$. The d_i-diagonal of p_d is a constant vector; similarly for the e_i-diagonal.

Proof. This also follows easily from the definition of row, col, and diag. □

Let us investigate how realizations of certain formal linear combinations change when we increase the degree d.

Example 10.13. Consider $T = \mathrm{row}(3) + \mathrm{row}(2)$. Then, p_8 is represented by the triangle on the left and p_9 is represented by the triangle on the right.

-28								
-14	14							
-5	9	-5						
.	5	-4	1					
2	2	-3	1	.				
2	.	-2	1	.	.			
1	-1	-1	1	.	.	.		
.	-1	.	1	.	.	.	.	
.	.	1	1	.	.	.	.	.

-48									
-28	20								
-14	14	-6							
-5	9	-5	1						
.	5	-4	1	.					
2	2	-3	1	.	.				
2	.	-2	1	.	.	.			
1	-1	-1	1	.	.	.	.		
.	-1	.	1	.	.	.	.	.	
.	.	1	1	.	.	.	.	.	.

For the following propositions and lemmas, we fix the following notations:

- Let T be a formal linear combination of $\{\mathrm{row}(i)\}_{i=0}^4$ or $\{\mathrm{row}(i)\}_{i=t-4}^t$, i.e. $T = \sum_{i=0}^4 \lambda_i \mathrm{row}(i)$ or $T = \sum_{i=0}^4 \lambda_{t-i} \mathrm{row}(t-i)$;
- Write the realizations of T as $p \coloneqq \sum p_{i,j} x_{i,j} \coloneqq p_d$ and $q \coloneqq \sum q_{i,j} x_{i,j} \coloneqq p_{d+1}$ for some degree $t = d$;
- Define $r \coloneqq \max\{i \mid \lambda_i \neq 0\}$ if $\max\{i \mid \lambda_i \neq 0\} \leq 4$, otherwise $r \coloneqq \min\{i \mid \lambda_i \neq 0\}$;
- Write $\mathrm{row}(r) = \sum r_{i,j} x_{i,j}$.

Lemma 10.14. *Let $d \in \mathbb{N}_{\geq 9}$. Then, $q_{i,j} = p_{i,j}$ holds for all $(i,j) \in V_d$.*

Proof. Let $d \in \mathbb{N}$, $\ell = 0, \dots, 4, t-4, \dots, t$ and $T_\ell = \mathrm{row}(\ell)$. We denote its realizations in d and $d+1$ by $p_d^{(\ell)} = \sum p_{i,j}^{(\ell)} x_{i,j}$ and $p_{d+1}^{(\ell)} = \sum q_{i,j}^{(\ell)} x_{i,j}$, respectively. By Proposition 3.18, we see that $p_{i,j}^{(\ell)} = q_{i,j}^{(\ell)}$ for all $(i,j) \in V_d$.

Next, assume $T = \sum_{i=0}^4 \lambda_i \mathrm{row}(i) + \sum_{i=0}^4 \lambda_{t-i} \mathrm{row}(t-i)$. Then, $p_{i,j} = \sum_\ell \lambda_\ell p_{i,j}^{(\ell)} = \sum_\ell \lambda_\ell q_{i,j}^{(\ell)} = q_{i,j}$ for all $(i,j) \in V_d$. $\square$

Lemma 10.15. *Let $d \in \mathbb{N}_{\geq 9}$. If there exists $k \in \{0, \dots, r\}$ such that $\mathrm{sign}(r_{i,d-i}) = \mathrm{sign}(p_{i,d-i})$ for all $i = k, \dots, r$, then $\mathrm{sign}(q_{i,d+1-i}) = \mathrm{sign}(p_{i,d-i})$ for all $i = k, \dots, r$.*

Proof. Without loss of generality, we assume that $\lambda_r > 0$. First, we see that $q_{r,\cdot} \coloneqq (q_{r,j})_{j=0}^{d-r} = \lambda_r \cdot \mathbf{1}$ and $q_{i,\cdot} = \mathbf{0}$ for all $i > r$. By the Pascal property, we have $q_{r-1,d+1-(r-1)} = q_{r-1,d-(r-1)} - q_{r,d+1-r} = q_{r-1,d-(r-1)} - \lambda_r$. This shows $q_{r-1,d+1-(r-1)} < q_{r-1,d-(r-1)} = p_{r-1,d-(r-1)} < 0$, where the last inequality follows from the assumption $\mathrm{sign}(r_{i,d-i}) = \mathrm{sign}(p_{i,d-i})$. Thus, we have $\mathrm{sign}(q_{r-1,d+1-(r-1)}) = \mathrm{sign}(q_{r-1,d-(r-1)}) = -1$.

Next, we again use the Pascal property $q_{r-2,d+1-(r-2)} = q_{r-2,d-(r-2)} - q_{r-1,d+1-(r-1)}$. We see that $q_{r-2,d+1-(r-2)} > 0$ because $q_{r-2,d-(r-2)} > 0$ and $q_{r-1,d+1-(r-1)} < 0$. Thus, we have

$\mathrm{sign}(q_{r-2,d+1-(r-2)}) = \mathrm{sign}(q_{r-2,d-(r-2)}) = 1$. We can continue this argument for $r-3, r-4, \ldots, k$. This shows that $\mathrm{sign}(q_{i,d+1-i}) = \mathrm{sign}(q_{i,d-i}) = \mathrm{sign}(p_{i,d-i})$ for all $i = k, \ldots, r$. □

Lemma 10.16. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \leq 4$. If there exists $k \in \{0, \ldots, r\}$ such that for all $i = k, \ldots, r$ we have $\mathrm{sign}(\mathrm{row}(r))_{c_i} = \mathrm{sign}(p)_{c_i}$, then for all $i = k, \ldots, r$ we either have $p_{c_i} > 0, q_{c_i} > 0$ or $p_{c_i} < 0, q_{c_i} < 0$.*

Proof. Let $i = k, \ldots, r$. If we show $\mathrm{sign}(p_{i,d-i}) = \mathrm{sign}(p_{i,d-i-5})$, then we have $\mathrm{sign}(r_{i,d-i}) = \mathrm{sign}(p_{i,d-i})$ by assumption $\mathrm{sign}(\mathrm{row}(r))_{c_i} = \mathrm{sign}(p)_{c_i}$. So, we can use Lemma 10.15 to prove the statement.

It is easy to see that $\mathrm{sign}(p_{r,d-r}) = \mathrm{sign}(p_{r,d-r-5})$ since $\mathrm{sign}(\mathrm{row}(r))_{c_r} = \mathrm{sign}(p)_{c_r}$ and $\mathrm{sign}(r_{r,d-r}) = \mathrm{sign}(r_{r,d-r-5})$. For $r-1$, we then use the Pascal property. We repeat this argument for $r-2, r-3, \ldots, k$. This shows that $\mathrm{sign}(p_{i,d-i}) = \mathrm{sign}(p_{i,d-i-5})$ for all $i = k, \ldots, r$. □

Proposition 10.17. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \leq 4$. If there exists $k \in \{0, \ldots, r\}$ such that for all $i = k, \ldots, r$ we have $\mathrm{sign}(\mathrm{row}(r))_{c_i} = \mathrm{sign}(p)_{c_i}$, then $\mathrm{sign}(T)$ is fixed-contractable on c_i for all $i = k, \ldots, r$.*

Proof. Let $i = k, \ldots, r$ and $d \in \mathbb{N}_{\geq 15}$ First, it is easy to see that p is contractable on c_i because $\mathrm{row}(r)$ is contractable and $\mathrm{sign}(\mathrm{row}(r))_{c_i} = \mathrm{sign}(p)_{c_i}$. By Lemma 10.16 the sign does not change when increasing the degree $d \rightsquigarrow d+1$. Hence, the contractability of p on c_i is preserved for all degrees greater or equal to d. Therefore, there exists one $\hat{p} \in H[\mathbf{x}, c_i]$ for all $d' \geq d$ such that $\hat{p} = p_{d'}$. □

There exist similar propositions for contractability on d_i and e_i.

Proposition 10.18. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \geq d-4$. If there exists $k \in \{r, \ldots, d\}$ such that for all $i = r, \ldots, k$ we have $\mathrm{sign}(\mathrm{row}(r))_{d_i} = \mathrm{sign}(p)_{d_i}$, then $\mathrm{sign}(T)$ is fixed-contractable on d_i for all $i = r, \ldots, k$. The analogous statement holds for e_i.*

Proof. We can use the same proof as before, but now the sign of the entire diagonal d_i changes whenever we increase the dimension by one. Fortunately, the contractability on d_i is not affected by this. □

We state analogous propositions for $\mathrm{col}(\cdot)$ of Proposition 10.17 and 10.18 but skip the proofs since they are similar. Let $T = \sum_{i=0}^{4} \lambda_i \mathrm{col}(i) + \sum_{i=0}^{4} \lambda_{t-i} \mathrm{col}(t-i)$.

Proposition 10.19. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \leq 4$. If there exists $k \in \{0, \ldots, r\}$ such that for all $i = k, \ldots, r$ we have $\mathrm{sign}(\mathrm{col}(r))_{b_i} = \mathrm{sign}(p)_{b_i}$, then $\mathrm{sign}(T)$ is fixed-contractable on b_i for all $i = k, \ldots, r$.*

Proposition 10.20. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \geq d-4$. If there exists $k \in \{r, \ldots, d\}$ such that for all $i = r, \ldots, k$ we have $\mathrm{sign}(\mathrm{col}(r))_{d_i} = \mathrm{sign}(p)_{d_i}$, then $\mathrm{sign}(T)$ is fixed-contractable on d_i for all $i = r, \ldots, k$. The analogous statement holds for e_i.*

Here is an analogous version of Lemma 10.14 but for diag(·).

Lemma 10.21. *Let $d \in \mathbb{N}_{\geq 9}$. Then, $q_{i,j+1} = p_{i,j}$ holds for all $(i, j) \in V_d$.*

Proof. Just use Lemma 10.14 and symmetries $\sigma \in S_3$. □

Example 10.22. Consider $T = \mathrm{diag}(3) + \mathrm{diag}(2)$. Then, p_8 is represented by the triangle on the left and p_9 is represented by the triangle on the right.

```
  .
  .   .
  1   1   1
  4   3   2   1
 10   6   3   1   .
 20  10   4   1   .   .
 35  15   5   1   .   .   .
 56  21   6   1   .   .   .   .
 84  28   7   1   .   .   .   .   .
```

```
  .
  .   .
  1   1   1
  4   3   2   1   .
 10   6   3   1   .   .
 20  10   4   1   .   .   .
 35  15   5   1   .   .   .   .
 56  21   6   1   .   .   .   .   .
 84  28   7   1   .   .   .   .   .   .
120  36   8   1   .   .   .   .   .   .   .
```

Not surprisingly, we have analogous propositions for diag(·) of Proposition 10.17. Consider the formal linear combination $T = \sum_{i=0}^{4} \lambda_i \mathrm{diag}(i) + \sum_{i=0}^{4} \lambda_{t-i} \mathrm{diag}(t-i)$.

Proposition 10.23. *Let $d \in \mathbb{N}_{\geq 15}$. The following statements hold:*

(1) Assume $r \leq 4$. If there exists $k \in \{0, \ldots, r\}$ such that for all $i = k, \ldots, r$ we have $\mathrm{sign}(\mathrm{diag}(r))_{c_i} = \mathrm{sign}(p)_{c_i}$, then $\mathrm{sign}(T)$ is fixed-contractable on c_i for all $i = k, \ldots, r$.

(2) Assume $r \geq d-4$. If there exists $k \in \{r, \ldots, d\}$ such that for all $i = r, \ldots, d$ we have $\mathrm{sign}(\mathrm{diag}(r))_{b_i} = \mathrm{sign}(p)_{b_i}$, then $\mathrm{sign}(T)$ is fixed-contractable on b_i for all $i = r, \ldots, d$.

Proof. The proofs are analogous to the proof of Proposition 10.17. □

10.2 An Extended Trivial System

To compute Γ_6^{even}, we defined the system $\Phi = \Phi_1 \cup \Phi_2$. We proved that this system is nontrivial. If we can find a system Ψ that is a superset of Φ and is also nont-rivial, then we can reduce the number of cases to check.

Proposition 10.24. *Define $\Psi' = \Phi \cup \{\mathrm{diag}(i) - \mathrm{diag}(j) \mid (i, j) \in Z\}$, where*

$$\begin{gathered} Z \coloneqq \{(0,1), (0,2), (0,3), (0,4), (0,d-1), (0,d-2), (0,d-3), (0,d-4), \\ (1,2), (1,3), (1,d), (1,d-4), (1,d-2), (1,d-3), \\ (2,d), (2,d-1), (2,d-3), (2,d-4), \\ (3,d), (3,d-1), (3,d-2), (3,d-4) \\ (d-4,d), (d-3,d), (d-2,d), (d-2,d-1), (d-1,d-2), (d-1,d-3), (d-1,d)\}. \end{gathered}$$

The system Ψ' is non-trivial.

Proof. It is easy to see that every $p = \sum \lambda_{i,j} x_{i,j} = \operatorname{diag}(q) - \operatorname{diag}(r)$ for $(q, r) \in Z$ satisfies $\lambda_{0,0} \neq 0$, $\operatorname{supp}^+(p) \neq \emptyset$, and $\operatorname{supp}^-(p) \neq \emptyset$. Thus, any root $\mathbf{w}$ of p satisfies $\operatorname{supp}^+(p) \cap \operatorname{supp}^+(w) \neq \emptyset$ or $\operatorname{supp}^-(p) \cap \operatorname{supp}^+(w) \neq \emptyset$. □

Proposition 10.25. *Let $T \in \{\operatorname{diag}(i) - \operatorname{diag}(j) \mid (i, j) \in Z\}$. Then, T is fixed-contractable.*

Proof. Let $(i, j) \in Z$. Use Proposition 10.23 if $i, j \leq 4$ or $i, j \geq d - 4$. Otherwise, the claim follows immediately from Proposition 10.11. □

Proposition 10.26. *Define $\Psi \coloneqq \Psi \cup \{\operatorname{diag}(1) - \operatorname{diag}(d - 1)\}$. The system Ψ is non-trivial.*

Proof. Let $T = \operatorname{diag}(1) - \operatorname{diag}(d - 1)$ be a formal linear combination. Let $d \in \mathbb{N}_{\geq 15}$ be odd and $\mathbf{w}$ be a root of Ψ. Then, the realization $p_d = \sum \lambda_{i,j} x_{i,j}$ of T for $t = d$ satisfies $\lambda_{0,0} = 0$ and $\lambda_{0,k} < 0$ for all $k = 1, \ldots, d$.

If $\mathbf{w}$ is a trivial root of p_d, then it satisfies $w_{0,k} = 0$ for all $k = 1, \ldots, d$. Then, $\operatorname{diag}(0)(\mathbf{w}) < 0$; this is a contradiction because $\mathbf{w}$ is supposed to be a root of $\operatorname{diag}(0)$.

Let $d \in \mathbb{N}_{\geq 15}$ be even and $\mathbf{w}$ be a root of Ψ. The realization $p_d = \sum \lambda_{i,j} x_{i,j}$ satisfies $\lambda_{i,d-i} \neq 0$ if and only if $i \in \{0, d\}$. If $\mathbf{w}$ is a trivial solution of p_d, the it satisfies $w_{d,0} > 0$ since it is a root of $\operatorname{diag}(d)$. However, $\operatorname{col}(d - 1)(\mathbf{w}) < 0$, which is a contradiction since $\mathbf{w}$ is a root of $\operatorname{col}(d - 1)$. □

Proposition 10.27. *Let $T = \operatorname{diag}(1) - \operatorname{diag}(d - 1)$. Then, T is fixed-contractable.*

Proof. Use Proposition 10.11. □

We redefine the following sets.

Definition 10.28. We define the following three solution sets:

(1) Define Γ^{even} to be the set of all valid contracted hyperfield configurations $\mathbf{s} \in H^{\Xi}$ such that $\hat{p}^{\text{even}}(\mathbf{s}) = H$ for all $p \in \Psi$.

(2) Define Γ^{odd} to be the set of all valid contracted hyperfield configurations $\mathbf{s} \in H^{\Xi}$ such that $\hat{p}^{\text{odd}}(\mathbf{s}) = H$ for all $p \in \Psi$.

Definition 10.29. We define

$$\Gamma_6^{\text{even}} \coloneqq \Gamma^{\text{even}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\operatorname{supp}^+(\mathbf{s})| = 6\right\} \text{ and } \Gamma_6^{\text{odd}} \coloneqq \Gamma^{\text{odd}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\operatorname{supp}^+(\mathbf{s})| = 6\right\}.$$

Proposition 10.30. *All valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$ with positive support size six satisfy $\operatorname{contr}_d(\operatorname{sign}(\mathbf{w})) \in \Gamma_6^{\text{even}} \cup \Gamma_6^{\text{odd}}$.*

Proof. This follows easily since outcomes are characterized as roots of Pascal forms. □

Proposition 10.31. *We have $|\Gamma_6^{\text{even}}| = 106806$ and $|\Gamma_6^{\text{odd}}| = 110272$.*

Proof. This is verified by computer. □

We excluded around 100,000 cases; there are still around 217,000 cases left to check.

10.3 Reducing Cases with Fixed-Contractable Pascal Forms

The final reduction step relies on using fixed-contractable forms as a filter. We define the following set.

Proposition 10.32. *s*

Starting with a set of formal linear combinations of row, col, and diag, our goal is to extract a subset that consists only of fixed-contractable forms. To achieve this, we fix a degree $D \in \mathbb{N}$, realize each form at degree D, and algorithmically verify that it is fixed-contractable. Propositions 10.17 and 10.18 provide examples of such algorithmic proofs. The corresponding source code is available on [7]. Using this method, we computed a total of 18,273 fixed-contractable forms.

Once a set G' containing only fixed-contractable forms is obtained, we proceed as follows: For each $\mathbf{s} \in \Gamma_6^{\text{even}}$, we check whether $0 \in \text{contr}(p_D)(\mathbf{s})$ for all $T \in G'$. If there exists any realization p_D of T such that $\mathbf{s}$ is not a hyperfield root, then $\mathbf{s}$ can be excluded. This exclusion is valid because T is a linear combination of Pascal forms, and all valid outcomes on $\mathbb{Z}^{V_d}$ must be roots of every Pascal form.

By applying this method, we were able to reduce the number of cases from 106,806 to just 6,700. TODO COMPUTE.

Chapter 11

Computation of Fundamental Models

In the final chapter, we compute the number of fundamental models. The implementation details are publicly available in the repository [7]. The results of these computations are summarized in the following table.

$n \setminus d$	1	2	3	4	5	6	7	8	9	10	11
2	1										
3		3	1								
4			12	4	2						
5				82	38	10	4				
6					602	254	88	24	2		
7						6710	2421	643	198	32	4

Figure 11.1: The number of fundamental outcomes for each positive support size n and degree d. For $n = 2, 3, 4, 5$ there are no additional columns beyond those shown as we have proven that the degree is bounded.

To construct this table, we employed the following methodology. First, we calculated the set of all supports of valid outcomes for a fixed degree $d \in \mathbb{N}$ using Algorithm 5. For each chipsplitting support generated, we mapped it back to a statistical model and computed the rank of the corresponding linear system to determine whether it yields a unique solution. If the system is of full rank, the associated statistical model is fundamental, as defined in Definition 2.14. By using this approach, we computed all valid outcomes for positive support sizes $n = 1, \ldots, 7$ and degree $d \leq 11$. This extends the results of Bik and Marigliano [2] by one additional support size, made possible by a more efficient implementation.

Chapter 12

Discussion

This thesis establishes a connection between the classification of discrete statistical models (Theorem 2.23) and a combinatorial puzzle related to chipsplitting games (Theorem 3.32). Specifically, the puzzle investigates whether the degree of a valid chipsplitting outcome can grow indefinitely while its support size remains fixed. For outcomes with positive support sizes up to five, we prove that the degree cannot grow indefinitely, providing a definitive negative answer.

For outcomes with a positive support size of six, significant progress was made toward a similar conclusion. By employing systematic reductions, the number of cases requiring analysis was reduced from approximately 300,000 to 12,000, indicating that a negative answer may hold for this case as well. With additional computational resources, one can reduce the number of cases even further, potentially leading to a number of cases that can be analyzed using the techniques described in this thesis. With even greater computational power, one could potentially extend the results to support size seven.

We would like to conclude this thesis by discussing some possible directions for future research. First, it would be interesting to investigate a better criterion for determining fixed-contractables forms. This could potentially lead to a reduction in the number of cases that need to be analyzed for positive support size six. Second, investigating a larger contraction size than four or five could provide further insights into the degree of valid outcomes of positive support size six. Finally, finding a larger non-trivial system would allow us to exclude even more cases.

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