Chipsplitting Games: A Combinatorial Approach to Classifying One-Dimensional Discrete Statistical Models with Rational Maximum Likelihood Estimator

by

Viet Duc Nguyen

A thesis submitted in partial satisfaction of the

requirements for the degree of

Master of Science

in

Mathematics

in the

Mathematics Department

of

Technische Universität Berlin

Supervisor and First Reviewer:
Prof. Dr. Carlos Enrique Améndola Cerón

Second Reviewer:
Prof. Dr. Christian Haase

Winter 2024

Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit eigenständig ohne Hilfe Dritter und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe. Alle Stellen die den benutzten Quellen und Hilfsmitteln unverändert oder sinngemäß entnommen sind, habe ich als solche kenntlich gemacht.

Sofern generative KI-Tools verwendet wurden, habe ich Produktnamen, Hersteller, die jeweils verwendete Softwareversion und die jeweiligen Einsatzzwecke (z.B. sprachliche Überprüfung und Verbesserung der Texte, systematische Recherche) benannt. Ich verantworte die Auswahl, die Übernahme und sämtliche Ergebnisse des von mir verwendeten KI-generierten Outputs vollumfänglich selbst.

Die Satzung zur Sicherung guter wissenschaftlicher Praxis an der TU Berlin vom 8. März 2017 habe ich zur Kenntnis genommen.

Ich erkläre weiterhin, dass ich die Arbeit in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegt habe.

Berlin, den

Zusammenfassung in deutscher Sprache

Diese Arbeit setzt die Arbeit von Bik und Marigliano zur Klassifizierung ein-dimensionaler diskreter statistischer Modelle mit rationalen Maximum Likelihood Schätzern mittels fundamentaler Modelle fort. Wir berechnen die Anzahl der fundamentalen Modelle im Simplex Δ_6 mit einem maximalen Grad elf. Darüber hinaus reduzieren wir die Anzahl der zu berücksichtigenden Fälle für den Beweis der endlichen Anzahl fundamentaler Modelle im Simplex Δ_5 mit einem maximalen Grad elf von 300.000 Fälle auf 12.000 Fälle; der zugrunde liegende Algorithmus ist in die Theorie der nicht-trivialen linearen Gleichungssysteme über Hyperkörper eingebettet, die wir speziell für diese Arbeit entwickeln. Der Code ist auf GitHub verfügbar.

Abstract

Chipsplitting Games: A Combinatorial Approach to Classifying One-Dimensional Discrete Statistical Models with Rational Maximum Likelihood Estimator

by

Viet Duc Nguyen

Technische Universität Berlin

This thesis continues the work of Bik and Marigliano on the classification of one-dimensional discrete statistical models with rational maximum likelihood estimators using fundamental models. We compute the number of fundamental models in the simplex Δ_6 with a maximum degree of eleven, a result that was previously unknown. Moreover, we reduce the number of cases to consider for proving the finite number of fundamental models with a maximum degree of eleven in the simplex Δ_5 from 300,000 to 12,000, enabling a feasible proof in the future. We embed the algorithm underpinning these key results in the framework of solving non-trivial hyperfield linear systems, which we develop specifically for this thesis. The code is publicly available on GitHub.

Contents

Chapter 1

Introduction

In statistics, we work with collections of probability distributions, such as the normal, Poisson, and binomial distributions, to model random variables in various applications. These collections are known as *statistical models*. Formally, a statistical model is defined as a set of probability distributions. In this thesis, we consider *discrete statistical models* with *finitely many state spaces* $n+1$, which we view as subsets of the probability simplex $\Delta_n \coloneqq \{p \in \mathbb{R}^{n+1} \mid \sum_{k=0}^{n} p_k = 1\}$; a distribution $p \in \Delta_n$ is a point in the probability simplex that assigns probabilities to the states $0, \ldots, n$.

Example 1.1. Say we have a binomial random variable X with $n+1$ states, then $p = (p_k)_{k=0}^n = (\binom{n}{k}\theta^k(1-\theta)^{n-k})_{k=0}^n$ computes the probability of observing k successes in n trials with success probability $\theta \in [0,1]$. The set $\mathcal{M} = \left\{(\binom{n}{k}\theta^k(1-\theta)^{n-k})_{k=0}^n \mid \theta \in [0,1]\right\}$ is our first example of a discrete statistical model, and it is known as the *binomial model*.

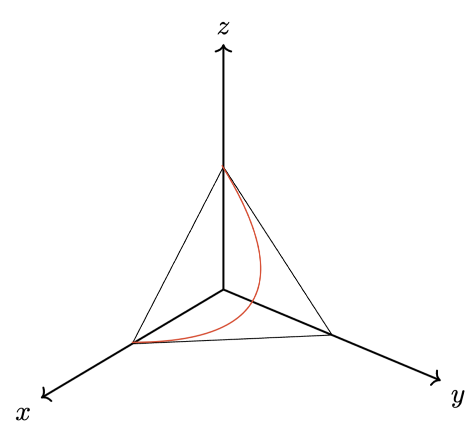

Figure 1.1: This figure shows the probability simplex Δ_2 with the binomial model (red curve). Every point on the curve is a binomial distribution.

Given a statistical model $\mathcal{M} \subset \Delta_n$ and data $u \in \mathbb{N}^{n+1}$, a common problem in statistics is determining the distribution within a statistical model that best fits the data. "Best" can mean a lot of things; in *maximum likelihood estimation*, it refers to finding the distribution that maximizes the probability of observing the data. The mapping

$$\Phi : \Delta_n \to \mathcal{M}, \quad u \mapsto \hat{p},$$

that assigns data u to a distribution $\hat{p} \in \mathcal{M}$ in the best possible way, in the sense of maximum likelihood estimation, is called the *maximum likelihood estimator (MLE)*. It is characterized by the property that $\hat{p}$ maximizes for all $p \in \mathcal{M}$ the log-likelihood function $\ell(p) = \sum_{k=0}^{n} u_k \log p_k$.

We focus on **one-dimensional discrete statistical models with rational MLE**. These are models $\mathcal{M}$ satisfying

- $\mathcal{M} = \text{image}(p)$ for some rational map $p = (p_0, \ldots, p_n) : I \to \Delta_n$ where p_k is a rational map and $I \subset \mathbb{R}$ is a union of closed intervals such that $p(\partial I) \subset \partial \Delta_n$, and
- all the $n + 1$ coordinates of the MLE Φ are rational functions in the data u.

We ask two intriguing questions about these statistical models: (1) which *form* do they take, and (2) can we *classify* them, i.e. can we divide them into easier to understand models? June Huh gave an answer to the first question; he showed that if Φ is rational, then each coordinate can be expressed as an alternating product of linear forms, where the numerator and denominator have the same degree [8, 7, 5]. For the second question, Arthur Bik and Orlando Marigliano provided a framework to classify discrete statistical models with rational MLE by introducing the concept of *fundamental models* and *chipsplitting games* [3].

This thesis builds upon the work of Bik and Marigliano. We present their findings on how fundamental models act as the building blocks of statistical models and investigate whether infinitely many fundamental models exist. It turns out that only finitely many fundamental models exist in Δ_n for $n \leq 4$; a result that we will prove using the techniques by Bik and Marigliano. For $n \geq 5$, the problem remains open due to the complexity of the problem. This thesis advances the understanding of $n = 5$ by reducing the number of cases to check from 300,000 to 12,000. Furthermore, we present new findings on the number of fundamental models in Δ_6 with a maximum degree of eleven. Moreover, we describe the algorithm for solving non-trivial hyperfield linear systems that underpins all the computational work discussed in this thesis.

The chapters are organized as follows:

- Chapter 2 provides tools to classify statistical models using fundamental models.
- Chapter 3 defines chipsplitting games and establishes the connection between fundamental statistical models and fundamental chipsplitting outcomes.
- Chapter 4 proves the finiteness of fundamental chipsplitting outcomes with positive support size $n \leq 3$ using the Invertibility Criterion.

- Chapter 5 proves the case $n = 4$ using the *Hyperfield Criterion* and the Invertibility Criterion.
- Chapter 6 introduces the final tool, the *Hexagon Criterion*, to show the case $n = 5$.
- Chapter 7 presents novel techniques to reduce the number of cases that need to be analyzed for $n = 6$.
- Chapter 8 computes the number of fundamental models.
- Chapter 9 concludes with a discussion on future research directions.

We could have opted to collect all the tools first and then apply them collectively to $n \leq 5$; however, we believe the current structure is more pedagogical as it emphasizes the challenges specific to each case.

The source code for the computations discussed in this thesis is available at [10].

Chapter 2

Classification with Fundamental Models

In this chapter, we introduce *reduced* and *fundamental models* to classify discrete statistical models with rational MLE; this classification is due to Bik and Marigliano [3].

2.1 Parametrization

First, we establish a parametrization for discrete statistical models with rational MLE, which is crucial for the classification of statistical models.

Proposition 2.1. *Let $\mathcal{M}$ be a one-dimensional discrete statistical models with rational MLE. Then, there exists a map of the form $p : [0,1] \to \Delta_n, \theta \mapsto (w_k\theta^{i_k}(1-\theta)^{j_k})_{k=0}^n$ with $i_k, j_k \in \mathbb{Z}_{\geq 0}$, $w_k \in \mathbb{R}_{>0}$ such that $\mathcal{M} = \operatorname{image}(p)$.*

We introduce some notation to simplify the proof of Proposition 2.1. Let $\mathcal{M} \subset \Delta_n$ be a one-dimensional discrete statistical model parametrized by rational functions $p_0 = \frac{g_0}{h_0}, \ldots, p_n = \frac{g_n}{h_n}$. Define $b \coloneqq \operatorname{lcm}(h_0, \ldots, h_n)$ and $a_i \coloneqq bp_i$. Then, we multiply $\sum_{k=0}^n p_k = 1$ by b to obtain $\sum_{k=0}^n a_k = b$. We use these polynomials $a_0, \ldots, a_n$ and b to determine the statistical model $\mathcal{M}$. The log-likelihood function then reads

$$\ell(p) = \sum_{k=0}^n u_k \log p_k = \sum_{k=0}^n u_k \log \frac{a_k}{b} = \sum_{k=0}^n u_k \log a_k - \sum_{k=0}^n u_k \log b.$$

To find the maximum likelihood estimator, we find all critical points of the log-likelihood function, which is equivalent to finding the critical points of

$$\ell(p(\theta))' = \sum_{k=0}^n u_k \frac{a_k'}{a_k} - \sum_{k=0}^n u_k \frac{b'}{b} = 0. \tag{2.1}$$

These equations are called the *score equations* in algebraic statistics.

Definition 2.2. The number of complex solutions to the score equations for general data $u \in \mathbb{C}^{n+1}$ is called the *maximum likelihood degree* (ML degree) of the statistical model.

This ML degree has an important meaning in algebraic statistics, as it is an algebraic measure of the complexity of the maximum likelihood estimation of the model [1, 4, 11]. We have the following relationship between MLE and ML degree.

Proposition 2.3. *A statistical model has rational maximum likelihood estimator if and only if the maximum likelihood degree of the model is one.*

Proof. Refer to [5] for a proof. $\square$

Next, we need the following lemma to prove Proposition 2.1,

Lemma 2.4. *Let $\mathcal{M}$ be a one-dimensional discrete statistical models with rational MLE. Then, there exist exactly two distinct complex linear factors in $a_0, \ldots, a_n$, and b.*

Proof. We prove the lemma in three steps:

- Let f be the product of all distinct complex linear factors in $a_0, \ldots, a_n$, and b. First, we multiply the score equations (2.1) by f to get $f\ell(p(\theta))' = \sum_{k=0}^{n} u_k f \frac{a_k'}{a_k} - \sum_{k=0}^{n} u_k f \frac{b'}{b} = 0$. Note that every linear factor of a_k with multiplicity m occurs in a_k' with multiplicity $m-1$. Thus, $\frac{a_k'}{a_k} = \frac{\lambda}{(x-\xi)}$ holds, where $\lambda \in \mathbb{R}$ and $x - \xi$ is some linear factor of a_k. Hence, $f \cdot \frac{\lambda}{(x-\xi)}$ is of degree $\deg(f) - 1$. Therefore, $f\ell(p(\theta))'$ is of degree $\deg(f) - 1$.

- We claim that the roots of $\ell(p(\theta))'$ are the same as the roots of $f\ell(p(\theta))'$. Assume that we have established this claim. By Proposition 2.3, the ML degree is one. So, $\ell(p(\theta))'$ has one root. Thus, $f\ell(p(\theta))'$ has one root. Therefore, $f\ell(p(\theta))'$ is of degree one. This implies that $\deg(f) = 2$. Thus, there are exactly two distinct complex linear factors in $a_0, \ldots, a_n$, and b.

- It remains to show that the roots of $\ell(p(\theta))'$ are the same as the roots of $f\ell(p(\theta))'$. Clearly, every root of $\ell(p(\theta))'$ is a root of $f\ell(p(\theta))'$. Conversely, we want to show that no new roots are introduced when multiplying by f, i.e. roots of f are not roots of $f \cdot \ell(p(\theta))'$. Let us rewrite $f\ell(p(\theta))' = \sum_{k=0}^{n} u_k f \frac{a_k'}{a_k} - \sum_{k=0}^{n} u_k f \frac{b'}{b} = \sum_{k=0}^{n+1} v_k f \frac{c_k'}{c_k}$ with $v_k := u_k, c_k := a_k$ for $k = 0, \ldots, n$, and $v_{n+1} := -\sum_{k=0}^{n} u_k$, $c_{n+1} := b$.

  Let q be a complex linear factor of f. We define polynomials $r_0, \ldots, r_{n+1}$ and r such that $c_k = q^{l_k} r_k$, $f = qr$ hold, and $r_0, \ldots, r_{n+1}, r$ do not have q as a factor. Then, we have

  $$f\frac{c_k'}{c_k} = qr \cdot \frac{l_k q^{l_k-1} q' r_k + q^{l_k} r_k'}{q^{l_k} r_k} = qr\frac{l_k q'}{q} + qr\frac{r_k'}{r_k} \equiv r l_k q' \pmod{q}$$

  for $k = 0, \ldots, n+1$. Thus, we obtain $f \cdot \ell(p(\theta))' \equiv rq' \sum_{k=0}^{n+1} v_k l_k \equiv rq' \sum_{k=0}^{n} v_k (l_k - l_{n+1})$ (mod q). Note that by definition of l_k, a value of $l_k = 0$ means that q is not a factor of c_k. By definition of f, at least one l_k is strictly positive. On the other hand, not all l_k can be strictly positive since $a_0, \ldots, a_n$, and b share no common factors. Hence, not all $l_k - l_{n+1}$ vanish. Hence, for generic data u we assume $\sum_{k=0}^{n} v_k(l_k - l_{n+1}) \neq 0$. This with $q'r \not\equiv 0 \pmod{q}$ implies that q is not a complex linear factor of $f\ell(p(\theta))'$. We showed that the roots of f are not roots of $f \cdot \ell(p(\theta))'$.

$\square$

We now prove Proposition 2.1.

Proof. First, we show that I is a single closed real interval and not a union of closed intervals. For the sake of contradiction, assume that $I = \bigcup_k I_k$ is a union of closed disjoint intervals. By definition of $\mathcal{M}$, we know that $p(\partial I) \subset \partial \Delta_n$. Thus, there exist $\theta_1, \theta_2 \in \partial I_0$ and $\theta_3, \theta_4 \in \partial I_1$ with $p_i(\theta_1) = p_i(\theta_2) = 0$ and $p_j(\theta_3) = p_j(\theta_4) = 0$ for some $i, j = 0, \ldots, n$. Note that θ_1 and θ_2 are roots of $\frac{a_i}{b}$; similarly, θ_3 and θ_4 are roots of $\frac{a_j}{b}$. By Lemma 2.4, exactly two distinct complex linear factors occur in $a_0, \ldots, a_n$, and b. Hence, $\theta_3 = \theta_1$ or $\theta_3 = \theta_2$ holds. We found a contradiction for I_0 and I_1 are disjoint.

We established that $I = [\alpha, \beta]$ is a real single closed interval. Thus, the roots of $a_0, \ldots, a_n$, and b are real and take values in $\partial I = \{\alpha, \beta\}$. By a suitable parametrization, we assume $I = [0, 1]$. Next, we write the polynomials $a_0, \ldots, a_n$, and b as $a_k(\theta) = w_k \theta^{i_k}(1-\theta)^{j_k}$ and $b(\theta) = w\theta^i(1-\theta)^j$ with $w_k, w \in \mathbb{R}_{>0}$, and $i_k, j_k, i, j \in \mathbb{Z}_{\geq 0}$. Since $a_0, \ldots, a_n$, and b share no common factors, there exists some $i_k = 0$ if $i > 0$; however, this contradicts $0 < w_k \leq a_0(0) + \cdots + a_n(0) = b(0) = 0$. So, $i = 0$ holds. Similarly, we have $j = 0$. Finally, we divide p by w to obtain $b \equiv 1$. $\square$

Corollary 2.5. *Any one-dimensional discrete statistical models with rational MLE can be represented by $(w_k, i_k, j_k)_{k=0}^n$ for $w_k \in \mathbb{R}_{>0}$ and $i_k, j_k \in \mathbb{Z}_{\geq 0}$.*

Definition 2.6. The *degree* $\deg(\mathcal{M})$ of a one-dimensional discrete statistical models with rational MLE $\mathcal{M}$ represented by $(w_k, i_k, j_k)_{k=0}^n$ is defined as $\max\{i_k + j_k : k = 0, \ldots, n\}$.

Remark 2.7. We view two models $(w_k, i_k, j_k)_{k=0}^n$ and $(w'_k, i'_k, j'_k)_{k=0}^n$ as the same model if they are equal up to a permutation of the coordinates.

Example 2.8. The sequence $((1,0,2),(2,1,1),(1,2,0))$ represents the binomial model with two trials. It has degree two. Its parametrization is given by $\theta \mapsto ((1-\theta)^2, 2\theta(1-\theta), \theta^2)$. See Figure 1.1 for a visualization of the binomial model within the probability simplex Δ_2. Note that we treat $((1,0,2),(2,1,1),(1,2,0))$, $((2,1,1),(1,0,2),(1,2,0))$, and $((2,1,1),(1,2,0),(1,0,2))$ as the same model, as coordinate order does not matter.

Definition 2.9. Let $\mathcal{M}$ be a model represented by $(w_k, i_k, j_k)_{k=0}^n$. The set of exponent pairs $(i_k, j_k)_{k=0}^n$ is called the support of $\mathcal{M}$, denoted by $\operatorname{supp}(\mathcal{M})$.

Throughout the remainder of this thesis, we adopt the convention that models refer to one-dimensional discrete statistical models with a rational MLE.

2.2 Reduced Models

The first building block for the classification of models are *reduced models*.

Definition 2.10. We call a model represented by $(w_k, i_k, j_k)_{k=0}^n$ *reduced* if $(i_k, j_k) \neq \mathbf{0}$ for all $k = 0, \ldots, n$, and $(i_k, j_k) \neq (i_l, j_l)$ for all $k \neq l$.

Next, we use functions to represent reduced models due to $(i_k, j_k) \neq (i_l, j_l)$.

Remark 2.11. A reduced model $\mathcal{M}$ represented by $(w_k, i_k, j_k)_{k=0}^n$ is identified by a function $f : \mathbb{Z}^2 \to \mathbb{R}_{\geq 0}, (i, j) \mapsto w$, where $w = w_k$ if $(i_k, j_k) = (i, j)$ and $w = 0$ otherwise. The support of f is the set of all pairs (i, j) with $f(i, j) > 0$. It coincides with the support of $\mathcal{M}$.

Reduced models are our first building blocks for the classification of statistical models because the following two propositions show that non-reduced models are transformed into a reduced model by a sequence of linear embeddings.

Proposition 2.12. *Let $n \in \mathbb{N}_{>0}$ and $\mathcal{M}$ be a statistical model represented by $(w_k, i_k, j_k)_{k=0}^n$. If $(i_l, j_l) = \mathbf{0}$ for some index $l \in \mathbb{N}$, then there exist $\mathcal{M}'$, $\lambda \in [0, 1]$ and $k \in \{0, \ldots, n\}$ such that $\mathcal{M} = \Psi_{\lambda,k}(\mathcal{M}')$, where $\Psi_{\lambda,k} : \Delta_{n-1} \to \Delta_n$ is defined as $p_i \mapsto \begin{cases} \lambda p_i & \text{if } k \neq i, \\ 1 - \lambda & \text{otherwise} \end{cases}$.*

Proof. Let $(i_l, j_l) = (0, 0)$ for some index $l \in \mathbb{N}$. If $w_l = 1$, then $w_m = 0$ for all $m \neq l$; this contradicts $w_m > 0$ by Proposition 2.1. Set $\lambda := 1 - w_l > 0$ and $k := l$. Define the statistical model $\mathcal{M}'$ by $\left(\frac{w_h}{1-w_l}, i_h, j_h\right)_{h=0, h\neq l}^n$. Then, $\mathcal{M} = \Psi_{\lambda,k}(\mathcal{M}')$ holds. □

Proposition 2.13. *Let $n \in \mathbb{N}_{>0}$. Let $\mathcal{M}$ be model represented by $(w_k, i_k, j_k)_{k=0}^n$. If $(i_m, j_m) = (i_l, j_l)$ for $m \neq l$, then there exist $\mathcal{M}'$, $\lambda \in [0, 1]$ and $k, h = 0, \ldots, n$ such that $\mathcal{M} = \Psi_{\lambda,k,h}(\mathcal{M}')$, where we define $\Psi_{\lambda,k,h} : \Delta_{n-1} \to \Delta_n, p_i \mapsto \begin{cases} p_i & \text{if } i \notin \{k, h\}, \\ \lambda p_k & \text{if } k = i, \\ (1-\lambda) p_k & \text{if } h = i. \end{cases}$*

Proof. Set $\lambda := \frac{w_m}{w_m + w_l}$, $k := m$, and $h := l$. Define $\mathcal{M}'$ by $(w_g + \delta_{gm} w_l, i_g, j_g)_{g=0, g\neq l}^n$. Then, $\mathcal{M} = \Psi_{\lambda,k}(\mathcal{M}')$ holds. □

Repeated application of these propositions transforms any model into a reduced model.

Corollary 2.14. *Let $n \in \mathbb{N}_{>0}$. If Δ_n contains a model of degree d, then there exists $m \leq n$ such that Δ_m contains a reduced model of degree d.*

2.3 Fundamental Models

We present *fundamental models*, which are building blocks for reduced models.

Definition 2.15. We call a model $\mathcal{M}$ represented by $(w_k, i_k, j_k)_{k=0}^n$ *fundamental* if it is reduced and the equation $\sum_{k=0}^n p_k = \sum_{k=0}^n x_k \theta^{i_k}(1-\theta)^{j_k} = 1, \forall \theta \in [0,1]$ has a unique solution $(x_k)_{k=0}^n$ with $x_k = w_k$.

Example 2.16. Consider the binomial model with two trials. Its exponent pairs read $(i_0, j_0) = (0,2)$, $(i_1, j_1) = (1,1)$, and $(i_2, j_2) = (2,0)$. Then, the equation is $p_0 + p_1 + p_2 = x_0\theta^2 + x_1\theta(1-\theta) + x_2(1-\theta)^2 \equiv 1$; it has a unique solution $w_0 = 1, w_1 = 2, w_2 = 1$. To see this, observe that the equation is equivalent to $x_0\theta^2 + x_1\theta - x_1\theta^2 + x_2 - 2x_2\theta + x_2\theta^2 = 1$, which in turn is equivalent to solving $x_2 - 1 + \theta(x_1 - 2x_2) + \theta^2(x_0 - x_1 + x_2) = 0$ for all $\theta \in [0,1]$. Thus, the binomial model is fundamental.

Example 2.17. The probability simplex Δ_0 only contains the model 1, which is fundamental.

Example 2.18. Consider Δ_1. It only contains the models $\theta \mapsto (\theta, 1-\theta)$ and $\theta \mapsto (1-\theta, \theta)$. Both models are equivalent and fundamental.

We use *composition* to construct reduced models from fundamental models.

Definition 2.19. Let $\mu \in (0,1)$. Let $\mathcal{M}$ and $\mathcal{M}'$ be reduced models represented by $f, g : \mathbb{Z}^2 \to \mathbb{R}_{\geq 0}$. The *composite* $\mathcal{M} *_\mu \mathcal{M}'$ is defined as the model represented by $(i,j) \mapsto \mu f(i,j) + (1-\mu) g(i,j)$. It is easy to see that $\mathcal{M} *_\mu \mathcal{M}'$ is a reduced model.

Proposition 2.20. *Every reduced model is the finite composite of fundamental models.*

Proof. The simplices Δ_0 and Δ_1 contain only fundamental models by Example 2.17 and 2.18.

Assume $n \geq 2$. Let $\mathcal{M} \in \Delta_n$ be a model that is not fundamental. We aim to show that $\mathcal{M}$ can be expressed as a composite of two models, $\mathcal{M}'$ and $\mathcal{M}''$, whose supports are proper subsets of $\operatorname{supp}(\mathcal{M})$. Assume this is indeed the case. Then, by applying the same argument to $\mathcal{M}'$ and $\mathcal{M}''$, we can recursively decompose each non-fundamental model into models with smaller supports. Since $\operatorname{supp}(\mathcal{M})$ is finite, this recursive decomposition eventually terminates, yielding a decomposition of $\mathcal{M}$ into fundamental models. Thus, we have shown that any reduced model is the composite of a finite number of fundamental models.

Let us prove that $\mathcal{M}$ is the composite of two models whose supports are proper subsets of $\operatorname{supp}(\mathcal{M})$. Since $\mathcal{M}$ is not fundamental, the equation $p_0 + \cdots + p_n = 1$ has distinct solutions $\mathbf{w}, \mathbf{w}' \in \mathbb{R}_{>0}^{n+1}$. Define $\mathbf{v} \coloneqq \mathbf{w} - \mathbf{w}' \neq \mathbf{0}$. Then, for all $\theta \in (0,1)$ we have $\sum_{k=0}^n v_k \theta^{i_k}(1-\theta)^{j_k} = 0$. Observe that there are strictly positive and negative coefficients v_k.

We define $\lambda \coloneqq \min\left\{\frac{w_k}{|v_k|} : k = 0, \ldots, n,\ v_k < 0\right\}$, $u_k \coloneqq w_k + \lambda v_k$ for $k = 0, \ldots, n$, and $S_1 \coloneqq \{(i_k, j_k) : k = 0, \ldots, n,\ u_k \neq 0\}$. Note that $\lambda > 0$ since all the coefficients w_k are strictly positive by definition. Also observe that $u_k \geq 0$ if $v_k \geq 0$. Moreover, we have $\frac{w_k}{|v_k|} \geq \lambda$ for all $k \geq 0$. Hence, if $v_k < 0$ holds, we have $\frac{u_k}{v_k} = \frac{w_k}{v_k} + \lambda \leq 0$. Multiplying by $v_k < 0$, we obtain $u_k \geq 0$. All in all, we have $u_k \geq 0$ for all $k = 0, \ldots, n$. Moreover, $u_k = 0$ holds if and only if $v_k < 0$ and $\lambda = \frac{w_k}{|v_k|}$ holds. This shows that $S_1 \subsetneq \operatorname{supp}(\mathcal{M})$. Since we have $u_0 + \ldots u_n = 1$, we found a reduced model $\mathcal{M}'$ represented by $(u_k, i_k, j_k)_{(i_k,j_k)\in S_1}$.

For the second model $\mathcal{M}''$, we define $\mu \coloneqq \min\left\{\frac{w_k}{u_k} : k = 0, \ldots, n,\ u_k \neq 0\right\}$, $t_k \coloneqq \frac{w_k - \mu u_k}{1-\mu}$ for $k = 0, \ldots, n$, and $S_2 \coloneqq \{(i_k, j_k) : k = 0, \ldots, n,\ t_k \neq 0\}$. As before, we have $\mu > 0$. Additionally, $\mu < 1$ holds because we have $v_k > 0$ implying $u_k > w_k$. By definition, we have $t_k \geq 0$ and $t_k = 0$ if and only if $u_k \neq 0$ and $\mu = \frac{w_k}{u_k}$. This shows that $S_2 \subsetneq \operatorname{supp}(\mathcal{M})$ and $S_1 \cup S_2 = \operatorname{supp}(\mathcal{M})$. Since $t_0 + \cdots + t_n = 1$, we found a reduced model $\mathcal{M}''$ represented by $(t_k, i_k, j_k)_{(i_k,j_k)\in S_2}$.

Finally, we see that $w_k = \mu u_k + (1-\mu)t_k$. This shows that $\mathcal{M} = \mathcal{M}' *_\mu \mathcal{M}''$. □

Applying the previous proposition with Corollary 2.14 yields the following corollary.

Corollary 2.21. *If Δ_n contains a non-fundamental model of degree d, then there exists a fundamental model of degree d in Δ_m for some $m < n$.*

Example 2.22. We classify all models for Δ_2. Note that the model $\mathcal{M}$ parametrized by $\theta \mapsto (\theta, 1-\theta)$ satisfies $\mathcal{M} *_\mu \mathcal{M} = \mathcal{M}$ for all $\mu \in (0,1)$. Since Δ_1 only contains this model, we conclude that Δ_2 only contains fundamental or non-reduced models.

To find all fundamental models in Δ_2, we check for all sets $S = \{(i_k, j_k)\}_{k=0}^2 \subset \mathbb{Z}^2_{>0}$ of size three if the equation $p_0 + p_1 + p_2 = 1$ has a unique solution. A priori, infinitely many sets S must be checked; we will see in the next section that it suffices to check only those sets S with degree $\deg(S) \coloneqq \max\{i + j : (i,j) \in S\}$ at most three. An exhaustive search yields the following supports for fundamental models: $\{(0,3),(1,1),(3,0)\}$, $\{(0,2),(1,1),(2,0)\}$, $\{(0,1),(1,1),(2,0)\}$, and $\{(0,2),(1,0),(1,1)\}$; they correspond to fundamental models $((1-\theta)^3, 3\theta(1-\theta), \theta^3)$, $((1-\theta)^2, 2\theta(1-\theta), \theta^2)$, $(1-\theta, \theta(1-\theta), \theta^2)$, and $((1-\theta)^2, \theta, \theta(1-\theta))$. Note that the last model is viewed as equivalent to the third model because we can use the parametrization $\theta \mapsto 1-\theta$ and a suitable permutation of the coordinates to obtain the third model.

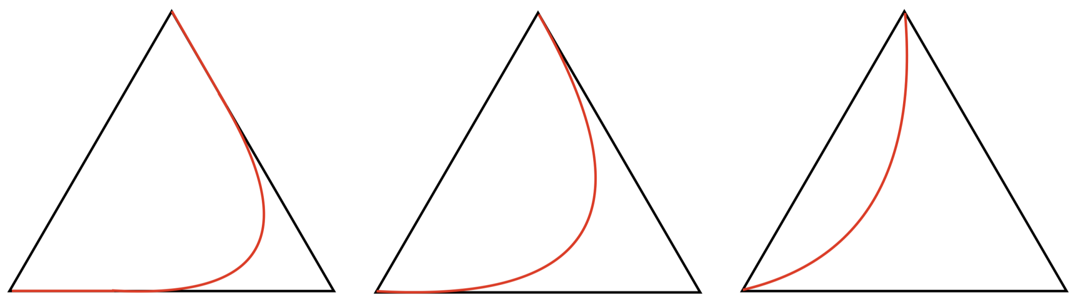

Figure 2.1: From left to right, the illustration depicts the models parametrized $((1-\theta)^3, 3\theta(1-\theta), \theta^3)$, $((1-\theta)^2, 2\theta(1-\theta), \theta^2)$, $(1-\theta, \theta(1-\theta), \theta^2)$, and $((1-\theta)^2, \theta, \theta(1-\theta))$.

We have computed all fundamental models of degree at most three in Δ_2. Of course, there exist non-reduced models, too. They come from linear embeddings $\Psi_{\lambda,k}$ and $\Psi_{\lambda,k,h}$, and for $\lambda = \frac{1}{3}$ we obtain the models $\theta \mapsto (\frac{2}{3}\theta, \frac{1}{3}, \frac{2}{3}(1-\theta))$ and $\theta \mapsto (1-\theta, \frac{1}{3}\theta, \frac{2}{3}\theta)$.

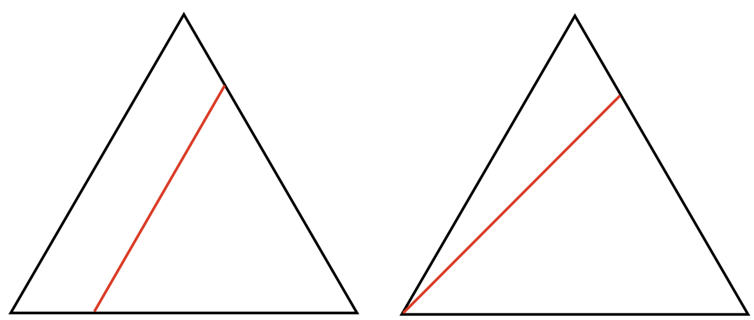

Figure 2.2: This illustration depicts two non-reduced models in Δ_2 parametrized by $\theta \mapsto (\frac{2}{3}\theta, \frac{1}{3}, \frac{2}{3}(1-\theta))$ and $\theta \mapsto (1-\theta, \frac{1}{3}\theta, \frac{2}{3}\theta)$.

Theorem 2.23. *Every model in Δ_n is the image of a reduced model in Δ_m under a linear embedding $\Delta_m \to \Delta_n$ for some $m \leq n$. Moreover, every reduced model $\mathcal{M} \subset \Delta$ can be written as a composite of finitely many fundamental models $\mathcal{M} = \mathcal{M}_1 *_{\mu_1} (\cdots *_{\mu_{m-2}} (\mathcal{M}_{m-1} *_{\mu_{m-1}} \mathcal{M}_m))$ for some $m < n$ and $\mu_1, \ldots, \mu_m \in (0,1)$.*

Proof. See Proposition 2.20, Proposition 2.12, and Proposition 2.13. $\square$

2.4 On the Finiteness of Fundamental Models

After establishing that fundamental models are building blocks for all models, we dedicate ourselves to the proof that only finitely many fundamental models in Δ_n for $n \leq 4$ exist – a result that was first established by Bik and Marigliano [3].

Theorem 2.24. *Let $\mathcal{M} \in \Delta_n$ be a model. For $n \leq 4$, we have $\deg(\mathcal{M}) \leq 2n - 1$.*

Given this theorem, it is easy to show the finiteness of fundamental models.

Theorem 2.25. *There are only finitely many fundamental models in Δ_n for all $n \leq 4$.*

Proof. Let $n \leq 4$. By Theorem 2.24, we know that the degree of a fundamental model is at most $2n-1$. Since the number of supports of a fundamental model of degree $2n-1$ is finite, there are only finitely many fundamental models in Δ_n. □

It turns out that proving Theorem 2.24 only for fundamental models is sufficient.

Theorem 2.26. *Let $N \in \mathbb{N}$. If the upper bound $\deg(\mathcal{M}) \leq 2n - 1$ holds for all $n \leq N$ and for all fundamental models $\mathcal{M} \in \Delta_n$, then this upper bound also holds for all statistical models, including non-fundamental ones, in Δ_n for all $n \leq N$.*

Proof. Let $N \in \mathbb{N}$ and $n \leq N$. Assume that there is a non-fundamental model $\mathcal{M}'$ in Δ_n of degree greater than $2n - 1$. By Corollary 2.21, there exists a fundamental model $\mathcal{M}$ in Δ_m of degree greater than $2m - 1$ for some $m < n$. This contradicts the assumption that the degree of fundamental models is at most $2k - 1$ for all $k \leq N$. □

This justifies that our north star is to prove the following theorem.

Theorem 2.27. *Let $\mathcal{M}$ be a fundamental model in Δ_n. For $n \leq 4$, we have $\deg(\mathcal{M}) \leq 2n-1$.*

The first step towards our north star is introducing a combinatorial puzzle to count fundamental models using the defining sequence $(w_k, i_k, j_k)_{k=0}^n$.

Chapter 3

Chipsplitting Games

The notion of a chipsplitting game was introduced in [3] as a combinatorial approach to count fundamental models. It was inspired by *chip-firing games* [9].

3.1 Basic Definitions

Definition 3.1. Let (V, E) be a directed graph without loops.

(1) A *chip configuration* is a vector $\mathbf{w} = (w_v)_{v \in V} \in \mathbb{Z}^V$ such that there are only finitely many nonzero components w_k.

(2) The *initial configuration* is the chip configuration $\mathbf{0} \in \mathbb{Z}^V$.

(3) A *splitting move* at $u \in V$, denoted by split_u, maps chip configurations $\mathbf{w}$ to chip configurations $\mathbf{w}'$ defined by $w'_v := \begin{cases} w_v - 1 & \text{if } v = u, \\ w_v + 1 & \text{if } (u, v) \in E, \\ w_v & \text{otherwise.} \end{cases}$

(4) An *unsplitting move* at $u \in V$, denoted by $\mathrm{unsplit}_u$, maps $\mathbf{w}'$ back to $\mathbf{w}$.

(5) A *chipsplitting game* is a finite sequence of splitting and unsplitting moves.

(6) An *outcome of a chipsplitting game* is the chip configuration obtained from applying the splitting and unsplitting moves defined by the game at the initial configuration.

Proposition 3.2. *The order of moves in a chipsplitting game does not affect the outcome.*

Proof. This follows from commutativity of addition. $\square$

Since all moves are reversible, we obtain the following corollary with Proposition 3.2.

Corollary 3.3. *Let* $\mathbf{w} \in \mathbb{Z}^V$ *be an outcome. Then, there exists a chipsplitting game such that its outcome equals* $\mathbf{w}$ *and where at no point both a splitting and an unsplitting move are applied at the same vertex in the game.*

Games that satisfy the condition in the corollary are called *reduced*. The map

$$\{\text{reduced games on } (V, E)\} / \sim \quad \to \quad \{g : V' \to \mathbb{Z} : \#\{p \in V' : g(p) \neq 0\} < \infty\} \tag{3.1}$$

$$f \mapsto (p \mapsto \text{number of moves at } p \text{ in game } f) \tag{3.2}$$

is a bijection, where $V' \subset V$ is the subset of vertices with at least one outgoing edge, and the map ($p \mapsto$ number of moves at p in game f) counts unsplitting moves negatively. The equivalence relation $\sim$ is defined by $f \sim g$ if and only if f and g are the same up to reordering. We identify a chipsplitting game with its corresponding function $V' \to \mathbb{Z}$. For every outcome $\mathbf{w} = (w_v)_{v \in V}$ we have $w_v = -f(v) + \sum_{u \in V', (u,v) \in E} f(u)$, where we set $f(t) = 0$ for $t \notin V$.

Now, we define the directed graphs we consider in this thesis.

Definition 3.4. For $d \in \mathbb{N} \cup \{\infty\}$, we define the vertices $V_d := \left\{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid i + j \leq d\right\}$ and the edges $E_d := \{(v, v + e) \mid v \in V_{d-1}, e \in \{(1,0), (0,1)\}\}$.

Definition 3.5. The degree $\deg(\mathbf{v})$ of a vertex $\mathbf{v} = (i,j)$ is defined as $i + j$.

Example 3.6. A chip configuration $\mathbf{w} = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ can be illustrated as a triangle of numbers where $w_{i,j}$ is placed at the position (i,j). For example, $w_{2,4} = 4$ means that the value 4 is placed in the third column and fifth row of the triangle (note that we start counting from index 0). The following is an example of a sequence of chip configurations for $d = 3$:

```
.           .           .           1           1           1
. .         . .         1 .         . 1         . 1         . .
. . .       1 . .       . 2 .       . 2 .       . 2 1       . 3 .
0 . . .    -1 1 . .    -1 . 1 .    -1 . 1 .    -1 . . 1    -1 . . 1
```

When we have $w_{i,j} = 0$, we omit the value and write `.` instead. The sequence above starts with the initial configuration and then applies splitting moves at the vertices $(0,0), (1,0), (0,1), (0,2)$ and $(2,0)$. Finally, we apply an unsplitting move at vertex $(1,1)$.

Definition 3.7. Let $\mathbf{w} = (w_{i,j})_{(i,j) \in V_d}$ be a chip configuration.

(1) The *positive support* of $\mathbf{w}$ is defined as $\text{supp}^+(\mathbf{w}) := \{(i,j) \in V_d \mid w_{i,j} > 0\}$.

(2) The *negative support* of $\mathbf{w}$ is defined as $\text{supp}^-(\mathbf{w}) := \{(i,j) \in V_d \mid w_{i,j} < 0\}$.

(3) The *support* of $\mathbf{w}$ is defined as the union of the positive and negative support.

(4) The *degree* of $\mathbf{w}$ is defined as $\deg(\mathbf{w}) := \max\{i + j \mid (i,j) \in \text{supp}(\mathbf{w})\}$.

(5) We say $\mathbf{w}$ is *valid* if its negative support is empty or only contains $(0,0)$.

We are interested in chip configurations that are *outcomes* and *valid* since they correspond to reduced models as we will see later. For that reason, it would be convenient to have a criterion for when a chip configuration is an outcome. The next section will provide an algebraic criterion using *Pascal equations*.

Example 3.8. Consider the following chip configuration:

```
 .
 ·   1
 1   ·   5
 ·   5   ·   2
 1   .   .   5   .
 ·   .   8   .   .   2
-2   ·   .   .   .   2   .
```

We clearly see that this configuration is valid, but is it also an outcome of a chipsplitting game? Currently, the only way to answer this question is to apply all possible sequences of splitting and unsplitting moves at the initial configuration, and check if the outcome equals the configuration above.

3.2 Pascal Forms

In this section, we establish that outcomes are roots of Pascal forms. To begin, we first define *linear forms*, as Pascal forms are a specific type of linear form.

Definition 3.9. A *linear form* on $\mathbb{Z}^{V_d}$ is a map $\mathbb{Z}^{V_d} \to \mathbb{Z}, \mathbf{w} \mapsto \sum_{(i,j)\in V_d} c_{i,j}w_{i,j}$.

Definition 3.10. A *Pascal form* on $\mathbb{Z}^{V_d}$ is a linear form $\sum_{(i,j)\in V_d} c_{i,j}x_{i,j}$ on $\mathbb{Z}^{V_d}$ satisfying $c_{i,j} = c_{i+1,j} + c_{i,j+1}$ for all $(i,j) \in V_{d-1}$.

Example 3.11. We can visualize a Pascal form as a triangle where $c_{i,j}$ is placed at the position (i,j). Here are examples of Pascal forms for $d = 2$:

```
0              1              0              0
1  1           1  0           0  0           1  1
2  1  0        1  0  0        1  1  1        0 -1 -2
```

Evaluating Pascal equations is invariant under splitting and unsplitting moves.

Proposition 3.12. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chip configuration. Let $p = \sum c_{i,j}x_{i,j}$ be a Pascal equation on $\mathbb{Z}^{V_d}$. Then, we have $p(\mathbf{w}) = p(\mathrm{split}_u(\mathbf{w})) = p(\mathrm{unsplit}_v(\mathbf{w}))$ for all $u, v \in V_{d-1}$.*

Proof. Let $u := (i', j') \in V_{d-1}$. By the Pascal property, we have $c_{i'+1,j'} + c_{i',j'+1} - c_{i',j'} = 0$. Thus, we have

$$p(\mathrm{split}_u(\mathbf{w})) = \sum_{(i,j)\in V_d} c_{i,j}(\mathrm{split}_u(\mathbf{w}))_{i,j} = \sum_{(i,j)\in V_d} c_{i,j} \begin{cases} w_{i,j} - 1 & \text{if } (i,j) = u, \\ w_{i,j} + 1 & \text{if } (i,j) \in \{(i'+1,j'),(i',j'+1)\} \\ w_{i,j} & \text{otherwise} \end{cases}$$
$$= \sum_{(i,j)\in V_d} c_{i,j} w_{i,j} = p(\mathbf{w}).$$

Similarly, we can show that $p(\mathrm{unsplit}_v(\mathbf{w})) = p(\mathbf{w})$ for all $v \in V_{d-1}$. □

Corollary 3.13. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be an outcome. Let $p = \sum c_{i,j} x_{i,j}$ be a Pascal equation on $\mathbb{Z}^{V_d}$. Then, $p(\mathbf{w}) = 0$.*

Proof. Clearly, we have $p(\mathbf{0}) = 0$. Then, we use Proposition 3.12 and the fact that $\mathbf{w}$ is obtained from the initial configuration $\mathbf{0}$ by a sequence of splitting and unsplitting moves. □

This proves that outcomes are roots of Pascal equations. The converse is also true.

Theorem 3.14. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chip configuration. Then, $\mathbf{w}$ is an outcome if and only if $\mathbf{w}$ is a root of all Pascal equations on $\mathbb{Z}^{V_d}$.*

First, let us introduce a basis from which we generate all Pascal forms through linear combinations. This helps us to prove the direction from right to left in Theorem 3.14.

Example 3.15. Let $d = 2$. We claim that the following set of Pascal forms is a basis:

```
0              0              1
0  0           1  1           0 -1
1  1  1,       0 -1 -2,       0  0  1.
```

Note that the zeroth column of each Pascal form is a unit vector in $\mathbb{R}^3$. Let us fix the zeroth row of each Pascal form to be a unit vector in $\mathbb{R}^3$:

```
1             -2              1
1  0          -1  1           0 -1
1  0  0,       0  1  0,       0  0  1.
```

We denote the first set by $\{\mathrm{col}(0), \mathrm{col}(1), \mathrm{col}(2)\}$ and the second one by $\{\mathrm{row}(0), \mathrm{row}(1), \mathrm{row}(2)\}$.

We generalize this example.

Proposition 3.16. *Let $\mathbf{a} = (a_0, \ldots, a_d)$ be any vector with integer entries. Then, the following two statements hold:*

(1) There exists a unique Pascal form $\sum c_{i,j}x_{i,j}$ *such that* $c_{0,\cdot} = \mathbf{a}$*.*

(2) There exists a unique Pascal form $\sum c_{i,j}x_{i,j}$ *such that* $c_{\cdot,0} = \mathbf{a}$*.*

Proof. Set $c_{0,\cdot} := \mathbf{a}$. Define $c_{i+1,j} := c_{i,j} - c_{i,j+1}$ for all $(i,j) \in V_d$ with $i = 0$. Then, we use the same formula to define $c_{i+1,j}$ for all $(i,j) \in V_d$ with $i = 1$. We repeat this process until we have defined all $c_{i,j}$ for $(i,j) \in V_d$.

For the second statement, we set $c_{\cdot,0} := \mathbf{a}$. Define $c_{i,j+1} := c_{i,j} - c_{i+1,j}$ for all $(i,j) \in V_d$ with $j = 0$. Then, we use the same formula to define $c_{i,j+1}$ for all $(i,j) \in V_d$ with $j = 1$. We repeat this process until we have defined all $c_{i,j}$ for $(i,j) \in V_d$. $\square$

Definition 3.17. Let $k = 0, \dots, d$ and $\mathbf{e}_k \in \mathbb{R}^{d+1}$ be the k-th unit vector. We define $\mathrm{col}(k)$ to be the unique Pascal form $\sum c_{i,j}x_{i,j}$ such that $c_{0,\cdot} = \mathbf{e}_k$. We define $\mathrm{row}(k)$ to be the unique Pascal form $\sum c_{i,j}x_{i,j}$ such that $c_{\cdot,0} = \mathbf{e}_k$.

Example 3.18. Consider the Pascal forms col(3) and row(3) for $d = 7$. We visualize it as follows:

·							
·	·						
·	·	·					
·	·	·	·				
1	1	1	1	1			
·	-1	-2	-3	-4	-5		
·	·	1	3	6	10	15	
·	·	·	-1	-4	-10	-20	-35.

-35							
-20	15						
-10	10	-4					
-4	6	-4	1				
-1	3	-3	1	.			
·	1	-2	1	.	.		
·	·	-1	1	.	.	.	
·	·	·	1	.	.	.	.

We set $\binom{a}{b} = 0$ for $b < 0$ or $b > a$.

Proposition 3.19. *For all integers* $k = 0, \dots, d$*, the following formulas hold:*

$$\mathrm{col}(k) = (-1)^k \sum_{(i,j)\in V_d} (-1)^j \binom{i}{k-j} x_{i,j} \quad \textit{and} \quad \mathrm{row}(k) = (-1)^k \sum_{(i,j)\in V_d} (-1)^i \binom{j}{k-i} x_{i,j}.$$

Proof. We claim that $(-1)^k \sum_{(i,j)\in V_d} (-1)^j \binom{i}{k-j} x_{i,j}$ is a Pascal equation. For all $(i,j) \in V_d$ we observe $(-1)^j\binom{i+1}{k-j} + (-1)^{j+1}\binom{i}{k-j-1} = (-1)^j\binom{i}{k-j}$ since we use $\binom{a}{b+1} + \binom{a}{b} = \binom{a+1}{b+1}$ with $a := i$ and $b := k-j-1$. Next, we see that $(-1)^{k+j}\binom{0}{k-j} = \delta_{jk}$. Thus, $(-1)^k \sum_{(i,j)\in V_d} (-1)^j \binom{i}{k-j} x_{i,j} = \mathrm{col}(k)$ holds.

By symmetry, we can use the same argument to show the second formula. $\square$

Next, we show that $\{\mathrm{col}(k)\}_{k=0}^d$ is a basis for Pascal forms on $\mathbb{Z}^{V_d}$.

Proposition 3.20. *Let* $d \in \mathbb{N}$ *and* p *be a Pascal form on* $\mathbb{Z}^{V_d}$*. The following statements hold:*

(1) *There exist unique coefficients* $\mu_0, \dots, \mu_d \in \mathbb{Z}$ *such that* $p = \mu_0 \mathrm{col}(0) + \cdots + \mu_d \mathrm{col}(d)$.

(2) *There exist unique coefficients* $\lambda_0, \dots, \lambda_d \in \mathbb{Z}$ *such that* $p = \mu_0 \mathrm{row}(0) + \cdots + \mu_d \mathrm{row}(d)$.

Proof. Let $p = \sum c_{i,j} x_{i,j}$ be a Pascal form on $\mathbb{Z}^{V_d}$. If we solve the equation

$$\sum_{(i,j) \in V_d} c_{i,j} x_{i,j} = \lambda_0 \mathrm{col}(0) + \cdots + \lambda_d \mathrm{col}(d) \tag{3.3}$$

for $\lambda_0, \dots, \lambda_d$, then due to Proposition 3.19 we have for all $(i,j) \in V_d$ that

$$\begin{aligned} c_{i,j} &= \lambda_0 (-1)^{0+j} \binom{i}{0-j} + \lambda_1 (-1)^{1+j} \binom{i}{1-j} + \cdots + \lambda_d (-1)^{d+j} \binom{i}{d-j} \\ &= \lambda_j (-1)^{2j} \binom{i}{0} + \lambda_{j+1} (-1)^{2j+1} \binom{i}{1} + \cdots + \lambda_{i+j} (-1)^{2j+i} \binom{i}{i}. \end{aligned}$$

We see $c_{0,\cdot} = (\lambda_0, \cdots, \lambda_d)$. Thus, we set $\boldsymbol{\mu} := c_{0,\cdot}$. We see that $\sum_{(i,j) \in V_d} c_{i,j} x_{i,j} = \mu_0 \mathrm{col}(0) + \cdots + \mu_d \mathrm{col}(d)$ by Proposition 3.16. Moreover, the same proposition shows that the coefficients $\lambda_0, \dots, \lambda_d$ in Equation 3.3 are uniquely determined.

For the second statement we use the same argument. $\square$

Corollary 3.21. *The sets* $\{\mathrm{col}(k)\}_{k=0}^{d}$ *and* $\{\mathrm{row}(k)\}_{k=0}^{d}$ *are a basis for all Pascal forms on* $\mathbb{Z}^{V_d}$.

Proof. This follows from the previous proposition. $\square$

Let us come back to Theorem 3.14. We now prove that roots of all Pascal equations on $\mathbb{Z}^{V_d}$ are outcomes.

Proposition 3.22. *Let* $d \in \mathbb{N}$ *and* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a chip configuration. If for all Pascal equations* p *on* $\mathbb{Z}^{V_d}$, *we have* $p(\mathbf{w}) = 0$, *then* $\mathbf{w}$ *is an outcome.*

Proof. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a chip configuration. By assumption, we have

$$\mathrm{col}(\deg(\mathbf{w}))(\mathbf{w}) = 0. \tag{3.4}$$

Note that by Proposition 3.19 for $\mathrm{col}(\deg(\mathbf{w})) = \sum c_{i,j} x_{i,j}$, we have $c_{i,\deg(\mathbf{w})-i} = (-1)^i$ for all $i = 0, \dots, \deg(\mathbf{w})$. Moreover, by Proposition 3.19 we have

$$c_{i,j} = 0 \quad \text{for all } i + j < \deg(\mathbf{w}). \tag{3.5}$$

Together with Equation 3.4 and 3.5, we obtain

$$\sum_{i=0}^{\deg(\mathbf{w})} (-1)^i w_{i,\deg(\mathbf{w})-i} = 0. \tag{3.6}$$

Furthermore, we know that there exists a unique minimal set of splitting or unsplitting moves at vertices (i,j) of degree $\deg(\mathbf{w}) - 1$ such that, when applied to $\mathbf{w}$, we obtain a chip configuration $\mathbf{w}'$ with $w'_{i,j} = 0$ for all $i = 0, ..., \deg(\mathbf{w})$. We call applying these set of moves to $\mathbf{w}$ *retraction*.

```
.                                         .
.   .                                     .   .
*   .   .                                 .   .   .
*   *   .   .          ------------->     *   .   .   .
*   *   *   .   .      retraction of w    *   *   .   .   .
*   *   *   *   .   .                     *   *   *   .   .   .
*   *   *   *   *   .   .                 *   *   *   *   .   .   .
*   *   *   *   *   *   .   .             *   *   *   *   *   .   .   .
```

Thus, $\deg(\mathbf{w}') < \deg(\mathbf{w})$. By Proposition 3.12, $\mathbf{w}'$ is also a root of all Pascal equations. We retract $\deg(\mathbf{w})$-many times until we obtain a chip configuration of degree zero, which is the initial configuration due to Equation 3.6. Thus, $\mathbf{w}$ is an outcome. □

We have successfully proved Theorem 3.14. Characterizing outcomes as roots of Pascal equations is a powerful tool to determine if a chip configuration is an outcome.

Algorithm 3.1 Validating outcomes

Require: chipsplitting configuration $\mathbf{w} \in \mathbb{Z}^{V_d}$
Ensure: `True` if $\mathbf{w}$ is an outcome, `False` otherwise

function IS_OUTCOME(A, n)
  for p of $\{\mathrm{col}(0), \ldots, \mathrm{col}(\deg(\mathbf{w}))\}$ **do**
    if $p(\mathbf{w}) \neq 0$ **then**
      return `False`
    end if
  end for
  return `True`
end function

Proof of correctness of Algorithm 3.1. This follows from Theorem 3.14. □

Example 3.23. Returning to Example 3.8, we see that the chip configuration is a root of all Pascal equations $\mathrm{col}(0), \ldots, \mathrm{col}(6)$ using Algorithm 3.1. Thus, the chip configuration is an outcome.

3.3 Valid Outcomes and Reduced Statistical Models

In the previous sections, we established that outcomes are roots of Pascal forms. Now, we will demonstrate that a subset of *valid outcomes* are in one-to-one correspondence with reduced statistical models. Thus, we can characterize reduced models algebraically through Pascal equations.

The function $\mathbf{w}(\cdot)$ maps reduced models $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^n$ to chip configurations $\mathbf{w}(\mathcal{M}) = (w_{i,j})_{(i,j)\in V_\infty}$ by

$$w_{i,j} \coloneqq \begin{cases} -1 & \text{if } (i,j) = (0,0), \\ w_k & \text{if } (i,j) = (i_k, j_k) \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathbf{w}(\mathcal{M})$ is a *real* chipsplitting game; the rules are exactly the same as for integer chipsplitting games.

Example 3.24. The binomial model $((1,3,0),(3,2,1),(3,1,2),(1,0,3))$ is mapped to

```
 1
 ·   3
 .   ·   3
-1   .   .   1.
```

Example 3.25. Does the following valid outcome from Example 3.8 induce a reduced statistical model via $\mathbf{w}^{-1}$?

```
  .
  ·    0.5
 0.5    ·    2.5
  ·    2.5    ·     1
 0.5    .     .    2.5    .
  ·     .     4     .     .     1
 -1     ·     .     .     .     1     .
```

The outcome would correspond to a reduced model in Δ_9 represented by

$$\mathcal{M} = ((0.5,2,0),(0.5,4,0),(2.5,1,3),(0.5,1,5),(4,2,1),(2.5,2,4), \\ (2.5,3,2),(1,3,3),(1,5,0),(1,5,1))$$

As it turns out, $\mathcal{M}$ is indeed a reduced model by the next theorem.

Theorem 3.26. *The map $\mathcal{M} \mapsto w(\mathcal{M})$ is a bijection between reduced statistical models and valid real outcomes $\mathbf{w} \in \mathbb{R}^{V_\infty}$ with $w_{0,0} = -1$.*

We first make some preparations. Let $\mathbb{R}[\theta]_{\le d}$ denote the vector space of polynomials in the variable θ of degree at most d with real coefficients. Similarly, we define $\mathbb{Z}[\theta]_{\le d}$ and

$\mathbb{Q}[\theta]_{\leq d}$. Next, we introduce the linear map $\alpha_d^{\mathbb{R}}$ that maps real chip configurations to real polynomials:

$$\alpha_d^{\mathbb{R}} : \mathbb{R}^{V_d} \to \mathbb{R}[\theta]_{\leq d}, \mathbf{w} \mapsto \sum_{(i,j)\in V_d} w_{i,j}\theta^i(1-\theta)^j.$$

We define the map $\alpha_d^{\mathbb{Z}}$ and $\alpha_d^{\mathbb{Q}}$ for integer and rational chip configurations analogously.

Lemma 3.27. *The following statements hold true for all $d \in \mathbb{N} \cup \{\infty\}$:*

(1) $\left\{\mathbf{w} \in \mathbb{R}^{V_d} \mid \mathbf{w} \text{ is an outcome}\right\} = \operatorname{kernel}(\alpha_d^{\mathbb{R}})$;

(2) $\left\{\mathbf{w} \in \mathbb{Z}^{V_d} \mid \mathbf{w} \text{ is an outcome}\right\} = \operatorname{kernel}(\alpha_d^{\mathbb{Z}})$;

(3) $\left\{\mathbf{w} \in \mathbb{Q}^{V_d} \mid \mathbf{w} \text{ is an outcome}\right\} = \operatorname{kernel}(\alpha_d^{\mathbb{Q}})$.

Proof. We prove the first statement; the other two are proven analogously. Note that it suffices to show the statement for $d < \infty$ since $\alpha_\infty^{\mathbb{R}}$ is the direct limit of $\alpha_0^{\mathbb{R}}, \alpha_1^{\mathbb{R}}, \alpha_2^{\mathbb{R}}, \ldots$.

Let $d < \infty$. By Corollary 3.21, the codimension of the outcome space is $d+1$, as it is defined by the roots of the Pascal forms $\operatorname{col}(0), \ldots, \operatorname{col}(d)$.

Let $f(\theta) = \lambda_0 + \lambda_1\theta + \cdots + \lambda_d\theta^d$ be a real polynomial of degree $\leq d$. Define a chipsplitting configuration $\mathbf{w}$ by $w_{i,j} \coloneqq \begin{cases} \lambda_i & \text{if } j = 0, \\ 0 & \text{otherwise} \end{cases}$. Then, $\alpha_d^{\mathbb{R}}(\mathbf{w}) = f$ holds, which shows that the map $\alpha_d^{\mathbb{R}}$ is surjective. Hence, the kernel of $\alpha_d^{\mathbb{R}}$ has codimension $d+1$; it has equal codimension as the space of outcomes.

Finally, we show that the space of outcomes is contained in $\operatorname{kernel}(\alpha_d^{\mathbb{R}})$. Since their codimensions coincide, the two spaces coincide. Let $\mathbf{w} \in \mathbb{R}^{V_d}$ be an outcome. The value of $\alpha_d^{\mathbb{R}}(\mathbf{w})$ remains the same if we apply splitting or unsplitting moves at arbitrary vertices $(i,j) \in V_{d-1}$ because we have $-\theta^i(1-\theta)^j + \theta^{i+1}(1-\theta)^j + \theta^i(1-\theta)^{j+1} = \theta^i(1-\theta)^j(-1+\theta+(1-\theta)) = 0$. The remaining claim follows from $\alpha_d^{\mathbb{R}}(\mathbf{0}) = 0$. □

We are ready to show Theorem 3.26.

Proof of Theorem 3.26. Let $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^n$ be a reduced model.

- First, we prove that $\mathbf{w} \coloneqq w(\mathcal{M})$ is an outcome; a-priori, we only know that it is a chip configuration. By definition of $w(\mathcal{M})$, we have $w_{0,0} = -1$. Since $\mathcal{M}$ is a model, we know that $\sum_{k=0}^n w_k\theta^{i_k}(1-\theta)^{j_k} = 1$. Thus, $\alpha_d^{\mathbb{R}}(\mathbf{w}) = \sum_{k=0}^n w_k\theta^{i_k}(1-\theta)^{j_k} - 1 = 0$. Thus, $\mathbf{w} \in \operatorname{kernel}(\alpha_d^{\mathbb{R}})$ holds. By Lemma 3.27, the chip configuration $\mathbf{w}$ is an outcome.
- **Injectivity:** Let $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^n$ and $\mathcal{M}' = (w'_k, i'_k, j'_k)_{k=0}^n$ be two distinct models. Then, we have $w(\mathcal{M}) \neq w(\mathcal{M}')$.
- **Surjectivity:** Let $\mathbf{w} \in \mathbb{R}^{V_\infty}$ be a valid real outcome with $w_{0,0} = -1$. We define $w_k \coloneqq w_{i_k,j_k}$ for all $k = 0, \ldots, n$. Then, $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^n$ is a reduced model by Lemma 3.27. We see that $w(\mathcal{M}) = \mathbf{w}$. Hence, $\mathcal{M} \mapsto w(\mathcal{M})$ is surjective.

□

Proposition 3.28. *The following statements hold for all reduced models $\mathcal{M}$:*

(1) $\operatorname{supp}^+(w(\mathcal{M})) = \operatorname{supp}^+(\mathcal{M})$.

(2) The map $\mathcal{M} \mapsto w(\mathcal{M})$ is degree-preserving.

(3) The outcome $w(\mathcal{M})$ is rational if and only if all the coefficients of $\mathcal{M}$ are rational.

Proof. All three statements follow directly from the definitions. □

Proposition 3.29. *Let $\mathbf{w} \in \mathbb{Q}^{V_\infty}$ be a valid rational outcome. Then, there exist positive $\lambda \in \mathbb{Q}$ and integral valid outcome $\mathbf{z} \in \mathbb{Z}^{V_\infty}$ such that $\mathbf{w} = \lambda\mathbf{z}$.*

Proof. Let $\mathbf{w}$ be a valid rational outcome; its support is finite. Thus, there exists $\mu \in \mathbb{N}$ such that $\mu\mathbf{w} \in \mathbb{Z}^{V_\infty}$. Define $\lambda := \frac{1}{\mu}$ and $\mathbf{v} := \mu\mathbf{w}$. Clearly, $\alpha^{\mathbb{Z}}_{\deg(\mathbf{v})}(\mathbf{v}) = \mu\alpha^{\mathbb{Q}}_{\deg(\mathbf{w})}(\mathbf{w}) \equiv 0$. By Lemma 3.27, $\mathbf{v}$ is an outcome. It is valid because $\mu\mathbf{w}$ is valid. □

Proposition 3.30. *Let $\mathbf{w} \in \mathbb{R}^{V_\infty}$ be a valid real outcome. If $w_{0,0} = 0$, then $\mathbf{w} = \mathbf{0}$.*

Proof. By Lemma 3.27, we have $\sum w_{i,j}\theta^i(1-\theta)^j \equiv 0$. By assumption, the negative support is empty. Hence, all the $w_{i,j}$ are non-negative. We evaluate at $\theta = \frac{1}{2}$ to conclude that the positive support of $\mathbf{w}$ is empty. Hence, $\mathbf{w} = 0$. □

Example 3.31. Let us reconsider Example 3.24. We have seen that the valid outcome on the left-hand side induces a reduced model by Theorem 3.26. It has degree three and positive support size four. Can we find another outcome with the same degree but smaller positive support size? Indeed, we unsplit at vertex $(1, 1)$ to get the outcome on the right-hand side.

```
1                          1
·   3                      ·   ·
.   ·   3                  ·   3   ·
-1   .   .   1            -1   ·   ·   1
```

Can we reduce the positive support size further? The answer is no, as we will find out.

Example 3.32. Let us fix the positive support size to be three. What is the largest degree of a valid outcome with positive support size three? We have already found

```
1
.   .
·   3   .
-1   ·   .   1.
```

However is there an even larger one? The answer is no.

Theorem 3.33. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome with* $|\mathrm{supp}^+(\mathbf{w})| \leq 5$*. Then, we have* $\deg(\mathbf{w}) \leq 2 \cdot |\mathrm{supp}^+(\mathbf{w})| - 3$.

Bik and Orlando established this theorem in [3]. Whether the upper bound extends to cases with larger positive support sizes, $|\mathrm{supp}^+(\mathbf{w})| > 5$ remains open. This theorem is of particular interest because it is equivalent to Theorem 2.24.

$$\text{Degree of outcomes } \mathbf{w} \longleftrightarrow \text{Degree of statistical models } \mathcal{M}$$
$$\mathrm{supp}^+(\mathbf{w}) \longleftrightarrow \text{Dimension of } \mathcal{M}$$

Proposition 3.34. *Theorem 2.24 and Theorem 3.33 are equivalent. In other words, proving that the degree of outcomes is bounded by their positive support size is equivalent to establishing that the degree of statistical models is bounded by their dimension.*

Proof. Assume Theorem 2.24 holds. We want to show Theorem 3.33. Let $\mathbf{w}$ be a valid integral outcome of positive support size $n \leq 5$. Normalize $\mathbf{w}$ such that $w_{0,0} = -1$. The degree and positive support size do not change. By Theorem 3.26, the outcome $\mathbf{w}$ induces a reduced statistical model $\mathcal{M}$ in Δ_{n-1}. Then, $\deg(\mathcal{M}) \leq 2(n-1) - 1 = 2n - 3$. Thus, applying Theorem 3.26 let us go back to the outcome $\mathbf{w}$, and Proposition 3.28 establishes $\deg(\mathbf{w}) \leq 2n - 3$.

For the converse direction, assume Theorem 3.33 holds. Let $n \leq 5$. We want to show Theorem 2.24. Let $\mathcal{M} = (w_k, i_k, j_k)_{k=0}^{n-1} \subset \Delta_{n-1}$ be a model with rational MLE. By Theorem 2.26, we assume that $\mathcal{M}$ is fundamental. We use Theorem 3.26 to map $\mathcal{M}$ to a valid real outcome $\mathbf{w} = (w_{i,j})$ with $w_{0,0} = -1$. Note that $\mathbf{w}$ is *rational* because by Definition 2.15 the weights $(w_k)_{k=0}^{n-1}$ of the fundamental model $\mathcal{M}$ are uniquely determined by the equation $p_0(\theta) + p_1(\theta) + \cdots + p_{n-1}(\theta) - 1 \equiv 0$ and for some $\theta \in [0, 1]$ this equation becomes rational. Next, we use Proposition 3.29 to find some integral valid outcome $\mathbf{z}$ such that $\mathbf{w} = \mu\mathbf{z}$ for some $\mu \in \mathbb{Q}_{>0}$. Again, scaling does not affect the degree or positive support size. By Proposition 3.28, the integral valid outcome $\mathbf{z}$ has positive support size n. By Theorem 3.33, we have $\deg(\mathbf{z}) \leq 2n - 3$. Thus, $\deg(\mathbf{w}) = \deg(\mu\mathbf{z}) = \deg(\mathbf{z}) \leq 2n - 3$ holds. Hence, the degree of $\mathcal{M}$ is smaller or equal to $2(n-1) - 1$ by Proposition 3.28. □

Recall that our north star is to prove that only finitely many fundamental models exist in Δ_n for $n \leq 4$. To achieve this, we originally aimed to prove Theorem 2.25 from Chapter 2. However, as established in Chapter 3, we may prove Theorem 3.33 instead. We focus on Theorem 3.33 because it allows us to address this problem combinatorically.

Chapter 4

Valid Outcomes of Positive Support Size One, Two, and Three

4.1 Invertibility Criterion

In this section, we study the supports of valid outcomes; knowing that some supports cannot be the supports of valid outcomes will help us to prove Theorem 3.33. For instance, configurations that have support in the entries below marked with an $*$ cannot be valid outcomes as we will see.

$$
\begin{array}{ccccccc}
\cdot & & & & & & \\
\cdot & \cdot & & & & & \\
* & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & * & \cdot & \\
* & \cdot & * & \cdot & \cdot & * & *
\end{array}
$$

The most important tool in the study of supports of valid outcomes is the *Invertibility Criterion*, which we will present in this chapter. It was first investigated in [3].

First, let us introduce a new basis for the space of Pascal forms.

Definition 4.1. Let $k = 0, \ldots, d$ and $\mathbf{e}_k \in \mathbb{R}^{d+1}$ be the k-th unit vector. We define $\operatorname{diag}(k)$ to be the unique Pascal form $\sum c_{i,j} x_{i,j}$ such that $c_{k,d-k} = \mathbf{e}_k$.

Proposition 4.2. *For all integers $k = 0, \ldots, d$ we have* $\operatorname{diag}(k) = \sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} x_{i,j}$.

Proof. Note that for all $(i,j) \in V_d$ with $i + j = d$ we have $\binom{d-i-j}{k-i} = 1$ if and only if $k = i$, and in all other cases $k \neq i$ the binomial coefficient is zero. Thus, it remains to show that $\sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} x_{i,j}$ is a Pascal form. We have $\binom{d-i-j}{k-i} = \binom{d-i-1-j}{k-i-1} + \binom{d-i-j-1}{k-i}$ for all $(i,j) \in V_{d-1}$ because $\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}$. $\square$

Example 4.3. Fix the degree $d = 7$. We visualize diag(3) by

```
  .
  .    .
  .    .    .
  1    1    1    1
  4    3    2    1    .
 10    6    3    1    .    .
 20   10    4    1    .    .    .
 35   15    5    1    .    .    .    .
```

Proposition 4.4. *Let p be a Pascal form on $\mathbb{Z}^{V_d}$. There exist unique coefficients $\mu_0, \dots, \mu_d \in \mathbb{Z}$ such that $p = \mu_0 \mathrm{diag}(0) + \cdots + \mu_d \mathrm{diag}(d)$.*

Proof. Let $p = \sum c_{i,j} x_{i,j}$. Choose $\mu_k = c_{k,d-k}$ for $k = 0, \dots, d$. Since p is a Pascal form, the coefficients $c_{i,j}$ satisfy the Pascal recurrence relation. Thus, the coefficients μ_k are uniquely determined. $\square$

Given $S \subset V_d$, the Invertibility Criterion uses the diagonal basis $(\mathrm{diag}(0), \dots, \mathrm{diag}(d))$ to determine whether a nonzero outcome with support in S exists.

Definition 4.5. Let $d \in \mathbb{N}$, $E \subset \{0, \dots, d\}$, and $S \subset V_d$ be non-empty sets such that $|E| = |S|$ holds. The *pairing matrix* of (E, S) is defined as $A^{(d)}_{E,S} := \left[\binom{d-i-j}{k-i}\right]_{k \in E, (i,j) \in S}$.

Example 4.6. Let $d = 2$, $S = \{(1, 1), (0, 0)\}$, and $E = \{0, 1\}$. The pairing matrix reads $A^{(d)}_{E,S} = \begin{bmatrix} \binom{2-1-1}{0-1} & \binom{2-0-0}{0-2} \\ \binom{2-1-1}{1-1} & \binom{2-0-0}{1-2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Now, assume that $\mathbf{w}$ is an outcome with support in S. Since it is an outcome, we have $\mathrm{diag}(k)(\mathbf{w}) = 0$ for all $k = 0, 1, 2, 3$. Thus, we have $A^{(d)}_{E,S}\mathbf{w} = \mathbf{0}$.

We make the following observation: if the pairing matrix $A^{(d)}_{E,S}$ is invertible (it is not for the example above), then $A^{(d)}_{E,S}\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{w} = \mathbf{0}$; so an invertible pairing matrix means that the initial configuration $\mathbf{0}$ is the *only* outcome with support in S. This is the Invertibility Criterion. For the example above, this criterion is inconclusive since the pairing matrix is not invertible.

Proposition 4.7 (Invertibility Criterion)**.** *Let $\mathbf{w}$ be an outcome with* $\mathrm{supp}(\mathbf{w}) \subset S$. *If $A^{(d)}_{E,S}$ is invertible, then $\mathbf{w} = \mathbf{0}$.*

Proof by Contraposition. Let $\mathbf{w} \neq \mathbf{0}$. Its support is non-empty. Then, $\mathbf{w}' := (w_{i,j})_{(i,j) \in S} \neq \mathbf{0}$. So, $A^{(d)}_{E,S} \cdot \mathbf{w}' = \mathbf{0}$. The kernel of the pairing matrix is non-trivial. Hence, the pairing matrix $A^{(d)}_{E,S}$ is not invertible. $\square$

Given a non-zero configuration $\mathbf{w}$, we try to construct sets $S \supset \operatorname{supp}(\mathbf{w})$ and E such that the pairing matrix $A^{(d)}_{E,S}$ is invertible. If we succeed, then $\mathbf{w}$ is *not* an outcome since the initial configuration is the only valid outcome with support in S.

The Invertibility Criterion inspires the following algorithm for determining if a set of indices $S \subset V_d$ can actually be the support of some nonzero outcome in $\mathbb{Z}^{V_d}$. It is later used in the proof of Theorem 5.49 and Proposition 6.3.

Algorithm 4.1 Only Zero Outcome

Require: Support set $S \subset V_d$, set $E \subset \{0, \dots, d\}$ with size $|S|$

Ensure: `True` only if $\mathbf{0}$ is the only outcome $\mathbf{w}$ with $\operatorname{supp}(\mathbf{w}) \subset S$, `False` if inconclusive

```
1: P ← build_pairing_matrix(d, E, S)
2: return rank(P) = |S|
```

The function `build_pairing_matrix` constructs the pairing matrix $A^{(d)}_{E,S}$ as defined in Definition 4.5.

4.2 Divide and Conquer

The Invertibility Criterion is a powerful tool to determine whether a configuration is an outcome. Unfortunately, it is not always easy to find suitable sets S and E. We will now introduce a method to construct such sets.

Divide

Instead of finding one set S with $\operatorname{supp}(\mathbf{w}) \subset S$, we divide S into smaller sets $S_1, \dots, S_l$. These smaller sets $S_1, \dots, S_k$ are implicitly defined by integers $\lambda_1, \dots, \lambda_l \in \mathbb{N}$. We choose $l \in \mathbb{N}$ and integers $\lambda_1, \dots, \lambda_l \in \mathbb{N}$ such that for all $i = 1, \dots, d$ we have

- $|S_i| \in \{0, \lambda_i\}$,
- $S_i := \{(i,j) \in \operatorname{supp}(\mathbf{w}) : i = c_{k-1}, \dots, c_k - 1\}$,
- $c_i := \lambda_1 + \dots + \lambda_i$, and
- $\lambda_1 + \dots + \lambda_l = d + 1$.

Moreover, for all $i = 1, \dots, l$ we define the sets $E_i := \begin{cases} \{c_{i-1}, \dots, c_i - 1\} & \text{if } |S_i| = \lambda_i, \\ \emptyset & \text{if } |S_i| = 0 \end{cases}.$

Remark 4.8. This decomposition works if $|\{(i,j) \in \operatorname{supp}(\mathbf{w}) : i \geq d - k\}| \leq k + 1$ for all $k = 0, \dots, d$. This is because we can always choose λ_1 minimal such that $|S_1| \in \{0, \lambda_1\}$. We repeat this process until $c_l = d + 1$.

Remark 4.9. We see that $|E_i| = |S_i|$ for all $i = 1, \ldots, l$.

Example 4.10. Fix the degree $d = 6$. Assume we have some configuration $\mathbf{w} \in \mathbb{Z}^{V_6}$ with support in the positions marked with an $*$ below. The first column contains two non-zero entries. So, we set $\lambda_1 = 2$. Then, we conclude that $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 1$.

```
.
. .
* . .
. . . .
. . . . .
. . . . * .
* . * . . * *
```

We present the following algorithm that implements this division rule.

Algorithm 4.2 Divide

Require: $S \subset V_d$
Ensure: $L \in \mathbb{Z}^k$ where $L_i = |S_i|$ if a division $(\lambda, (E_i)_{i=1}^k, (S_i)_{i=1}^k)$ is found; `None` otherwise
1: $R \leftarrow [0, \ldots, d]$, $L \leftarrow$ `list()`, and `col_start` $\leftarrow 0$
2: **for** `col_end` $\in \{0, \ldots, |R| - 1\}$ **do**
3:     `num_cols` $\leftarrow$ `col_end` $-$ `col_start` $+ 1$
4:     `points_in_col` $\leftarrow \{(x, y) \in S \mid x \in \{R[\texttt{col\_start}], \ldots, R[\texttt{col\_end}]\}\}$
5:     `num_points` $\leftarrow$ |`points_in_col`|
6:     **if** (`num_points` $= 0$) $\vee$ (`num_cols` $=$ `num_points`) **then**
7:         L.append(`num_points`)
8:         `col_start` $\leftarrow$ `col_end` $+ 1$
9:     **end if**
10: **end for**
11: **if** $\sum_{i=1}^k L_i \neq |S|$ **then**
12:     **return** `None`
13: **end if**
14: **return** L

Proof of Correctness. To construct λ from S, we set $\lambda_i = L_i$ if $L_i > 0$, otherwise $\lambda_i = 1$. Line four guarantees that $S_i := \{(i, j) \in S : i = c_{k-1}, \ldots, c_k - 1\}$. Line six states that $|S_i|$ is appended to L if and only if either $S_i = \emptyset$ or $|S_i| = \lambda_i$. Line eleven ensures that $\lambda_1 + \cdots + \lambda_l = d + 1$. □

Conquer

It remains to show how to conquer all the sets $S_1, \ldots, S_l$ and $E_1, \ldots, E_l$.

Proposition 4.11. *We have* $A^{(d)}_{E,S} = \begin{bmatrix} A^{(d)}_{E_1,S_1} & 0 & \dots & 0 \\ A^{(d)}_{E_2,S_1} & A^{(d)}_{E_2,S_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{(d)}_{E_l,S_1} & A^{(d)}_{E_l,S_2} & \dots & A^{(d)}_{E_l,S_l} \end{bmatrix}$.

Proof. First, note that

$$A^{(d)}_{E,S} = \begin{bmatrix} A^{(d)}_{E_1,S_1} & A^{(d)}_{E_1,S_2} & \dots & A^{(d)}_{E_1,S_l} \\ A^{(d)}_{E_2,S_1} & A^{(d)}_{E_2,S_2} & \dots & A^{(d)}_{E_2,S_l} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(d)}_{E_l,S_1} & A^{(d)}_{E_l,S_2} & \dots & A^{(d)}_{E_l,S_l} \end{bmatrix}.$$

Let $x, y = 1, \dots, l$ such that $x < y$. Let $k \in E_x$ and $(i, j) \in S_y$. Then, $k \le c_x - 1 < c_x \le c_{y-1} \le i$; so $k - i < 0$. Thus, $\binom{d-i-j}{k-i} = 0$ holds. This implies that the upper off-diagonal blocks are zero. $\square$

Corollary 4.12. *The matrix* $A^{(d)}_{E,S}$ *is invertible if and only if* $A^{(d)}_{E_1,S_1}, \dots, A^{(d)}_{E_l,S_l}$ *are invertible.*

Corollary 4.13. *Let* $\mathbf{w}$ *be an outcome with* $\operatorname{supp}(\mathbf{w}) \subset S$*. If the pairing matrices* $A^{(d)}_{E_1,S_1}, \dots, A^{(d)}_{E_l,S_l}$ *are invertible, then* $\mathbf{w} = \mathbf{0}$*.*

Example 4.14. We continue Example 4.10. Choose $\lambda = (2, 1, 1, 1, 1, 1)$. Then, we obtain

$$S_1 = \{(0,0), (0,4)\}, S_2 = \{(2,0)\}, S_3 = \emptyset, S_4 = \{(4,1)\}, S_5 = \{(5,0)\}, S_6 = \{(6,0)\},$$
$$E_1 = \{0,1\}, E_2 = \{2\}, E_3 = \emptyset, E_4 = \{4\}, E_5 = \{5\}, E_6 = \{6\}$$
$$A^{(d)}_{E,S} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 \end{bmatrix}.$$

The symbol $*$ stands for arbitrary entries. The pairing matrix is invertible, so no nonzero outcome with support in $S = \{(0,0), (0,4), (2,0), (4,1), (5,0), (6,0)\}$ exists.

4.3 Symmetry

With the Invertibility Criterion we exclude certain supports from being supports of valid outcomes. We now show that supports of valid outcomes are invariant under certain symmetries.

Proposition 4.15. *Let* $\mathbf{w} = (w_{i,j})_{(i,j)\in V_d}$ *be a configuration in* $\mathbb{Z}^{V_d}$*. Then* $\mathbf{w}$ *is an outcome if and only if the configuration* $\tilde{\mathbf{w}} \coloneqq (w_{j,i})_{(i,j)\in V_d}$ *is an outcome.*

Proof. Let $k = 0, \ldots, d$ and $\mathbf{w}$ be a valid outcome. Observe that

$$\operatorname{diag}(k) = \sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} x_{i,j} = \sum_{(i,j)\in V_d} \binom{d-i-j}{d-i-j-(k-i)} x_{i,j} = \sum_{(i,j)\in V_d} \binom{d-i-j}{d-k-j} x_{i,j}.$$

Then, $\operatorname{diag}(k)(\mathbf{w}) = 0$ holds. Thus, we have

$$\begin{aligned} \operatorname{diag}(k)(\tilde{\mathbf{w}}) = \sum_{(i,j)\in V_d} \binom{d-i-j}{k-j} w_{j,i} = \sum_{(i,j)\in V_d} \binom{d-i-j}{d-k-i} w_{j,i} &= \sum_{(i,j)\in V_d} \binom{d-i-j}{d-k-j} w_{i,j} \\ &= \sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} w_{i,j} \\ &= \operatorname{diag}(k)(\mathbf{w}) = 0. \end{aligned}$$

$\square$

Example 4.16. We have established that the support on the left-hand side cannot correspond to a valid outcome. By symmetry, the support on the right-hand side also does not correspond to a valid outcome.

```
.                    *
. .                  * .
* . .                . * .
. . . .              . . . .
. . . . .            * . . . .
. . . . * .          . . . . . .
* . * . . * *        * . . . * . .
```

Next, we introduce another kind of symmetry. Let $\mathbf{w} = (w_{i,j})$ be a configuration in $\mathbb{Z}^{V_d}$. We define $\mathbf{w} \mapsto \hat{\mathbf{w}} \coloneqq \big((-1)^{d-j} w_{j,d-i-j}\big)_{(i,j)\in V_d}$.

Proposition 4.17. *Let* $\mathbf{w} = (w_{i,j})_{(i,j)\in V_d}$ *be a configuration in* $\mathbb{Z}^{V_d}$*. Then* $\mathbf{w}$ *is an outcome if and only if* $\hat{\mathbf{w}} \coloneqq \big((-1)^{d-j} w_{j,d-i-j}\big)_{(i,j)\in V_d}$ *is an outcome.*

Proof. Let $k = 0, \ldots, d$. Then, we have $\operatorname{col}(k)(\hat{\mathbf{w}}) = (-1)^k \sum_{(i,j)\in V_d} (-1)^j \binom{i}{k-j} (-1)^{d-j} w_{j,d-i-j} = (-1)^{d-k} \sum_{(i,j)\in V_d} \binom{d-i-j}{k-i} w_{i,j} = (-1)^{d-k} \operatorname{diag}(k)(\mathbf{w}) = 0.$ $\square$

The symmetry just introduced is interpreted in the following way: we define a group action of the symmetry group S_3 on $\mathbb{Z}^{V_d}$ by $(12)\cdot\mathbf{w} = \tilde{\mathbf{w}}$ and $(123)\cdot\mathbf{w} = \hat{\mathbf{w}}$. The group actions (12), (13) and (23) can then be depicted in Figure 4.1.

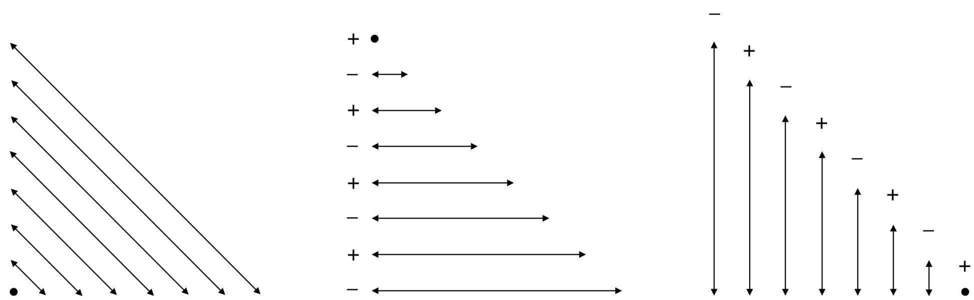

Figure 4.1: The left most illustration shows mirroring the configuration with respect to the main diagonal. The middle illustration shows switching the order on the same row while also alternating the signs of the row. The right most illustration shows switching the order on the same column while also alternating the signs of the column.

4.4 Impossible Supports

We show that certain supports cannot be the supports of valid outcomes.

Proposition 4.18. *Let $d \in \mathbb{N}$, and $i = 0, \dots, d$. If $S = \{(0, i)\}$ and $E = \{0\}$, then $A^{(d)}_{E,S}$ is invertible.*

Proof. We have $A^{(d)}_{E,S} = \begin{bmatrix} 1 \end{bmatrix}$, which is invertible. ☐

Proposition 4.19. *Let $d \in \mathbb{N}$. Assume $i, j = 0, \dots, d$ with $i < j$. If $S = \{(0, i), (0, j)\}$ and $E = \{0, 1\}$, then $A^{(d)}_{E,S}$ is invertible.*

Proof. We have $A^{(d)}_{E,S} = \begin{bmatrix} 1 & 1 \\ d-i & d-j \end{bmatrix}$, which is invertible. ☐

Proposition 4.20. *Let $d \in \mathbb{N}$. Assume $i, j, k = 0, \dots, d$ with $i < j < k$. If $S = \{(0, i), (0, j), (0, k)\}$ and $E = \{0, 1, 2\}$, then $A^{(d)}_{E,S}$ is invertible.*

Proof. We have
$$A^{(d)}_{E,S} = \begin{bmatrix} \binom{d-i}{0} & \binom{d-j}{0} & \binom{d-k}{0} \\ \binom{d-i}{1} & \binom{d-j}{1} & \binom{d-k}{1} \\ \binom{d-i}{2} & \binom{d-j}{2} & \binom{d-k}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ d-i & d-j & d-k \\ \frac{(d-i)(d-i-1)}{2} & \frac{(d-j)(d-j-1)}{2} & \frac{(d-k)(d-k-1)}{2} \end{bmatrix}.$$
We set $x = d - i, y = d - j$, and $z = d - k$. Then, we see $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} A^{(d)}_{E,S} = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}$. The matrix on the right-hand side is invertible because it is a Vandermonde matrix. Thus, $A^{(d)}_{E,S}$ is invertible. ☐

Proposition 4.21. *Let $d \in \mathbb{N}$, $i, j = 0, \dots, d$ with $i < j$, and $k = 0, \dots, d-1$. If we have $S = \{(0, i), (0, j), (1, k)\}$, $E = \{0, 1, 2\}$, and $i + j \neq 2k + 1$, then $A^{(d)}_{E,S}$ is invertible.*

Proof. We have

$$A_{E,S}^{(d)} = \begin{bmatrix} \binom{d-i}{0} & \binom{d-j}{0} & \binom{d-k-1}{-1} \\ \binom{d-i}{1} & \binom{d-j}{1} & \binom{d-k-1}{0} \\ \binom{d-i}{2} & \binom{d-j}{2} & \binom{d-k-1}{1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ d-i & d-j & d-k-1 \\ \frac{(d-i)(d-i-1)}{2} & \frac{(d-j)(d-j-1)}{2} & \frac{(d-k-1)(d-k-2)}{2} \end{bmatrix}.$$

We substitute $x = d-i, y = d-j$, and $z = d-k-1$. Then, we see

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} A_{E,S}^{(d)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{x-y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & x+y & 2z+1 \end{bmatrix}.$$

We see that the determinant is nonzero because $x + y \neq 2z + 1$ by $i + j \neq 2k + 1$. $\square$

Remark 4.22. We may assume $S \subset \{(i,j) \in V_d \mid i < |S|\}$ and $E = \{0, 1, \ldots, |S| - 1\}$ because $A_{E,S}^{(d)} = A_{E-\rho,S-\rho}^{(d-\rho)}$ holds for $\rho := \min\{E \cup \{i \mid (i,j) \in S\}\}$ and $E-\rho := \{(i-\rho, j) \mid (i,j) \in E\}$. This allows us to apply the previous propositions to more general S and E.

Example 4.23. Assume we have a configuration with support $S = \{(0,i), (0,j), (0,k)\}$ for $0 \leq i < j < k \leq d$. By Proposition 4.20, we know that no valid outcome has this support. Now, let us consider a configuration $\mathbf{w}$ with support $\operatorname{supp}(\mathbf{w}) \subset S$ such that S can be decomposed into $S_1, \ldots, S_l$ as described before. Let $\ell = 1, \ldots, l$. If $S_\ell = \{(x,i), (x,j), (x,k)\}$ for $0 \leq i < j < k \leq d$ and $x \in \mathbb{N}$, then $\mathbf{w} = \mathbf{0}$ by Proposition 4.20 and Remark 4.22. Hence, the configuration below is not an outcome because for $\lambda = (3,3,1,1)$ we have $S_2 = \{(3,0), (3,1), (3,3)\}$.

```
*
. .
. . *
. . . .
. . . * .
. . . . . .
. . . * . . .
* . . * . . . .
```

4.5 Proof

We have all the tools to prove our north star (Theorem 3.33) for the case of positive support size three or less.

Theorem 4.24. *No valid outcomes of positive support size one exists.*

Proof of Theorem 4.24. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome. Since it is valid, we either have an empty negative support or a negative support that only contains $(0,0)$. If the negative support is empty, then $\mathbf{w} = \mathbf{0}$ by Proposition 3.30. Hence, we assume $w_{0,0} < 0$.

Now, consider the Pascal form $\mathrm{diag}(0) = \sum c_{i,j}x_{i,j}$. We have $c_{0,0} = c_{0,1} = \cdots = c_{0,d} = 1$ and $c_{i,j} = 0$ for everything else. Similarly, we have for the Pascal form $\mathrm{diag}(d) = \sum c'_{i,j}x_{i,j}$ that $c'_{\cdot,0} = \mathbf{1}$ and $c'_{i,j} = 0$ for everything else. Since outcomes are roots of Pascal forms, we have $\mathrm{diag}(0)(\mathbf{w}) = \mathrm{diag}(d)(\mathbf{w}) = 0$. Since $w_{0,0} < 0$ we must have $w_{0,j} > 0$ and $w_{i,0} > 0$ for some $i, j > 0$. Hence, $\mathbf{w}$ has positive support size at least two. $\square$

Theorem 4.25. *For valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $|\mathrm{supp}^+(\mathbf{w})| = 2$ we have $\deg(\mathbf{w}) = 1$.*

Proof of Theorem 4.25. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be an outcome with positive support size two and degree d. By the previous proof, we see that $\mathrm{supp}^+(\mathbf{w}) = \{(0,j),(i,0)\}$. Without loss of generality, we assume $i = d$. We want to show that $j = d$.

Consider the Pascal form $\mathrm{row}(d) = \sum c_{i,j}x_{i,j}$; it satisfies $c_{i,j} \neq 0$ if and only if $i + j = d$. If d is odd, we have $c_{d,0} = 1$ and $c_{0,d} = -1$. Since $\mathrm{row}(d)(\mathbf{w}) = 0$, we have $j = d$. If d is even, we have $c_{d,0} = c_{0,d} = 1$. Thus, $\mathrm{row}(d)(\mathbf{w}) \neq 0$ for all $j = 0, \ldots, d$. Hence, valid outcomes with positive support size two do not exist for even degrees.

From now on, we assume $\mathrm{supp}^+(\mathbf{w}) = \{(0,d),(d,0)\}$. For sake of contradiction, let $d \geq 2$. Then, we divide $\mathrm{supp}(\mathbf{w}) = \{(0,0),(0,d),(d,0)\}$ via $\lambda = (2,1,\ldots,1)$ to obtain $S_1 = \{(0,0),(0,d)\}$, $S_k = \emptyset$, and $S_l = \{(d,0)\}$ for some $l \in \mathbb{N}$ and all $k \neq l, k \neq 1$. By Proposition 4.19, the pairing matrix induced by S_1 and $E_1 = \{0,1\}$ is invertible. For S_l we apply Proposition 4.18 to get that the induced pairing matrix is invertible. Hence, the outcome $\mathbf{w}$ is zero, which has an empty positive support. This is a contradiction to the assumption that the positive support size is two. Hence, the degree d equals one. $\square$

Example 4.26. The previous theorem shows that the only valid outcomes with positive support size two are multiples of

```
 1
-1  1
```

We now turn our attention to valid outcomes with positive support size three.

Theorem 4.27. *For valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $|\mathrm{supp}^+(\mathbf{w})| = 3$ we have $\deg(\mathbf{w}) \leq 3$.*

The following proposition gives us the possible supports of valid outcomes with positive support size three.

Proposition 4.28. *Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid integral outcome of degree $d \in \mathbb{N}$. If the positive support size of $\mathbf{w}$ is three, then one of the following holds:*

(1) We have $\mathrm{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,j)\}$ for some $i, j > 0$ with $i + j < d$.

(2) *We have* $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,d-i)\}$ *for some* $i = 1,\dots,d-1$.

(3) *We have* $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,0)\}$ *for some* $i = 1,\dots,d-1$.

(4) *We have* $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(0,i)\}$ *for some* $i = 1,\dots,d-1$.

(5) *We have* $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,e),(d-f,f)\}$ *for some* $e,f = 1,\dots,d-1$.

(6) *We have* $\operatorname{supp}(\mathbf{w}) = \{(0,0),(0,d),(e,0),(d-f,f)\}$ *for some* $e,f = 1,\dots,d-1$.

Proof. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid integral outcome of degree d. Assume $\{(0,0),(d,0),(0,d)\} \subset \operatorname{supp}(\mathbf{w})$. Clearly, statement 1, 2, 3, or 4 must hold.

So assume $(0,d) \notin \operatorname{supp}(\mathbf{w})$ and $(d,0) \notin \operatorname{supp}(\mathbf{w})$. As in the proof of Theorem 4.24, consider the Pascal form $\operatorname{diag}(0) = \sum c_{i,j}x_{i,j}$. We have $c_{0,\cdot} = \mathbf{1}$ and $c_{i,j} = 0$ for everything else. Similarly, we have for the Pascal form $\operatorname{diag}(d) = \sum c'_{i,j}x_{i,j}$ that $c'_{\cdot,0} = \mathbf{1}$ and $c'_{i,j} = 0$ for everything else. Since outcomes are roots of Pascal forms, we have $\operatorname{diag}(0)(\mathbf{w}) = \operatorname{diag}(d)(\mathbf{w}) = 0$. Due to $w_{0,0} < 0$, we conclude $w_{0,j} > 0$ and $w_{i,0} > 0$ for some $i,j > 0$. Thus, we have $\{(i,0),(0,j)\} \subset \operatorname{supp}(\mathbf{w})$ for some $i,j = 1,\dots,d-1$ using the assumption $(0,d) \notin \operatorname{supp}(\mathbf{w})$ and $(d,0) \notin \operatorname{supp}(\mathbf{w})$. Since $\mathbf{w}$ is of degree d, there exists $w_{k,d-k} > 0$ for some $k = 1,\dots,d-1$. However, $\operatorname{row}(d)(\mathbf{w}) = 0$ implies that there must be some $w_{h,d-h} > 0$ for some $h \neq k$; this h cannot equal 0 or d. Thus, the positive support size of $\mathbf{w}$ is at least four, which is a contradiction. Hence, we must have $(d,0) \in \operatorname{supp}(\mathbf{w})$ or $(0,d) \in \operatorname{supp}(\mathbf{w})$.

Let $(d,0) \in \operatorname{supp}(\mathbf{w})$ and $(e,0) \in \operatorname{supp}(\mathbf{w})$ for some $e = 1,\dots,d-1$. Now using the same argument as before, there must exist some $w_{f,d-f} > 0$ for some $f = 1,\dots,d-1$; otherwise $\operatorname{row}(d)(\mathbf{w}) > 0$ which is a contradiction since $\mathbf{w}$ is a root of all Pascal forms. This proves statement 5.

The proof for statement 6 is analogous. □

We apply the Invertibility Criterion to each possible support.

Proposition 4.29. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome. If* $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,j)\}$ *for some* $i,j > 0$ *with* $i+j < d$*, then* $d = 3$ *and* $(i,j) = (1,1)$.

Proof. Let $i > 1$. Set $\lambda = (2,1,\dots,1)$. We get $E_1 = \{0,1\}, S_1 = \{(0,0),(0,d)\}$, $E_{i-1} = \{i\}, S_{i-1} = \{(i,j)\}$, $E_{d-1} = \{d\}, S_{d-1} = \{(d,0)\}$, and $E_k = S_k = \emptyset$ for all $k \in \{1,\dots,d-1\} \setminus \{1,i-1,d-1\}$. The pairing matrices $A^{(d)}_{E_n,S_n}$ are all invertible for $n = 1,\dots,d-1$. Hence, the pairing matrix $A^{(d)}_{\{0,1,i,d\},\operatorname{supp}(\mathbf{w})}$ is also invertible. By the Invertibility Criterion, $\mathbf{w}$ is the zero configuration, which is a contradiction. Thus, we have $i = 1$. Hence, by symmetry we also have $j = 1$.

Finally, we show that $d = 3$. For the sake of contradiction, assume $d > 3$. Then, we choose $\lambda = (3,1,\dots,1)$. We obtain $E_1 = \{0,1,2\}$ and $S_1 = \{(0,0),(0,d),(1,1)\}$. By Proposition 4.21 this pairing matrix $A^{(d)}_{E_1,S_1}$ is invertible. The other relevant pairing matrix $A^{(d)}_{\{d\},\{(d,0)\}}$ is also invertible. Thus, the pairing matrix $A^{(d)}_{\{0,1,2,d\},\operatorname{supp}(\mathbf{w})}$ is invertible. Hence, the configuration $\mathbf{w}$ is the zero configuration, which is a contradiction. □

Proposition 4.30. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome. Assume the outcome* $\mathbf{w}$ *satisfies one of the following conditions:*

(1) $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,d-i)\}$ *for some* $i = 1,\dots,d-1$*,*

(2) $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(i,0)\}$ *for some* $i = 1,\dots,d-1$*,*

(3) $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,d),(0,i)\}$ *for some* $i = 1,\dots,d-1$*.*

Then, $d = 2$ *and* $i = 1$ *hold.*

Proof. Assume $d > 2$. Let $\mathbf{w}$ satisfy the third condition. Choose $\lambda = (3,1,\dots,1)$. Then, apply Proposition 4.20. So, the pairing matrix is invertible. Thus, $\mathbf{w} = 0$ holds, which is a contradiction. Thus, $d = 2$.

By symmetry, we have the same result for the second condition.

We want to show $d = 2$ for all outcomes $\mathbf{w}$ satisfying the first condition. Let $\mathbf{w}'$ satisfy the second condition. Then $\mathbf{w} = (123)\mathbf{w}'$ holds. Assume $d > 2$. By Proposition 4.17, we found an outcome $\mathbf{w}'$ of degree at least three. This contradicts Proposition 4.30 (2). Thus, $d = 2$ holds.

Finally, we have $i = 1$ because $i = 1,\dots,d-1$ and $d = 2$. □

Proposition 4.31. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome. If there exist* $e, f \in \{1,\dots,d-1\}$ *such that* $\operatorname{supp}(\mathbf{w}) = \{(0,0),(d,0),(0,e),(d-f,f)\}$ *or* $\operatorname{supp}(\mathbf{w}) = \{(0,0),(0,d),(e,0),(d-f,f)\}$*, then* $d = 2$ *and* $e = f = 1$ *holds.*

Proof. By symmetry, it suffices to show the statement for outcomes $\mathbf{w}$ satisfying the first condition. Let $d > 2$. If $f = d-1$, then choose $\lambda = (3,1,\dots,1)$. We apply Proposition 4.21 because $0 + e \neq 2d-1$ for $d > 1$. So, $\mathbf{w} = \mathbf{0}$ holds, which is a contradiction. Thus, we have $f < d-1$. Then, we choose $\lambda = (2,1,\dots,1)$. Use Proposition 4.19 to get a contradiction. Hence, we have $d = 2$.

Let $d = 2$. Then, we have $e = f = 1$ by definition of e and f. □

Finally, we can prove Theorem 4.27.

Proof of Theorem 4.27. Use Proposition 4.28. For each case, either apply Proposition 4.29, Proposition 4.30, or Proposition 4.31. □

Chapter 5

Valid Outcomes of Positive Support Size Four

5.1 Hyperfield Criterion

We introduce the *Hyperfield Criterion*. This criterion can be interpreted as constraints on the support of valid outcomes.

Definition 5.1. Let $H \coloneqq \{-1, 0, 1\}$. We define the addition $+ : H \times H \to 2^H \setminus \{\emptyset\}$ on H as follows: $0 + x = \{x\}, 1 + 1 = \{1\}, 1 + (-1) = H$, and $(-1) + (-1) = \{-1\}$ for all $x \in H$. Multiplication $\times : H \times H \to H$ is defined as usual. We call H the *sign hyperfield*.

For singleton sets $\{x\}$, we often write x instead of $\{x\}$; thus, $1+1 = 1$ and $(-1)+0 = -1$.

Remark 5.2. The tuple $(H, +, \cdot, 0, 1)$ is called a *hyperfield*. For more details, see [2]. In summary, a hyperfield satisfies the following properties:

(1) The maps $+$ and $\cdot$ are symmetric;

(2) $(H \setminus \{0\}, \cdot, 1)$ is a group;

(3) $0 \cdot x = 0$ and $0 + x = x$ hold for all $x \in H$;

(4) $\bigcup_{q \in x+y}(q + z) = \bigcup_{q \in x+y}(x + q)$ hold for all $x, y, z \in H$;

(5) $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$ hold for all $a, x, y \in H$.

(6) An inverse element $y \in H$ exists for every $x \in H$ such that the set $x + y$ contains 0. This inverse element y is unique for every x and is denoted by $-x$.

Definition 5.3. A polynomial in n variables $x_1, \ldots, x_n$ over H is a formal sum $f = \sum_{\mathbf{k} \in \mathbb{Z}^n_{\geq 0}} \lambda_\mathbf{k} \mathbf{x}^\mathbf{k}$ with $\lambda_\mathbf{k} \in H$, where only a finite number of coefficients $\lambda_\mathbf{k}$ are non-zero, and $\mathbf{x}^\mathbf{k} \coloneqq x_1^{k_1} \cdots x_n^{k_n}$. The set of all polynomials in n variables over H is denoted by $H[x_1, \ldots, x_n]$.

Definition 5.4. Let $\mathbf{x} \in H^n$. We define $f(\mathbf{x}) \coloneqq \sum_{\mathbf{k}\in\mathbb{Z}^n_{\geq 0}} \lambda_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} \subset H$.

Definition 5.5. We say that f *vanishes* at $\mathbf{x} \in H^n$ if $0 \in f(\mathbf{x})$, and call $\mathbf{x}$ a *hyperfield root* of f.

Any *real* polynomial can be turned into a polynomial over the sign hyperfield by replacing the coefficients with their sign.

Definition 5.6. Let $f = \sum \lambda_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} \in \mathbb{R}[\mathbf{x}]$. We call $\operatorname{sign}(f) \coloneqq \sum_{\mathbf{k}\in\mathbb{Z}^n_{\geq 0}} \operatorname{sign}(\lambda_{\mathbf{k}})\mathbf{x}^{\mathbf{k}} \in H[\mathbf{x}]$ the polynomial over H induced by f.

For sake of simplicity, we write $\operatorname{sign}(\mathbf{w}) \coloneqq (\operatorname{sign}(w_1), \ldots, \operatorname{sign}(w_n))$ for any $\mathbf{w} \in \mathbb{R}^n$.

Definition 5.7. A hyperfield Pascal form is a polynomial over H induced by a Pascal form.

Example 5.8. We illustrate the hyperfield versions of $\operatorname{diag}(i)$, $\operatorname{col}(i)$, and $\operatorname{row}(i)$ on V_5 for $i = 0, \ldots, 5$. A dot represents zero, $+$ represents plus one, and $-$ represents minus one.

$$
\begin{array}{cccccc}
+ &&&&& \\
+ & \cdot &&&& \\
+ & \cdot & \cdot &&& \\
+ & \cdot & \cdot & \cdot && \\
+ & \cdot & \cdot & \cdot & \cdot & \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
+ & + &&&& \\
+ & + & \cdot &&& \\
+ & + & \cdot & \cdot && \\
+ & + & \cdot & \cdot & \cdot & \\
+ & + & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
\cdot & \cdot &&&& \\
+ & + & + &&& \\
+ & + & + & \cdot && \\
+ & + & + & \cdot & \cdot & \\
+ & + & + & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
\cdot & \cdot &&&& \\
\cdot & \cdot & \cdot &&& \\
+ & + & + & + && \\
+ & + & + & + & \cdot & \\
+ & + & + & + & \cdot & \cdot
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
\cdot & \cdot &&&& \\
\cdot & \cdot & \cdot &&& \\
\cdot & \cdot & \cdot & \cdot && \\
+ & + & + & + & + & \\
+ & + & + & + & + & \cdot
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
\cdot & \cdot &&&& \\
\cdot & \cdot & \cdot &&& \\
\cdot & \cdot & \cdot & \cdot && \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
+ & + & + & + & + & +
\end{array}
$$

$$
\begin{array}{cccccc}
\cdot &&&&& \\
\cdot & \cdot &&&& \\
\cdot & \cdot & \cdot &&& \\
\cdot & \cdot & \cdot & \cdot && \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
+ & + & + & + & + & +
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
\cdot & \cdot &&&& \\
\cdot & \cdot & \cdot &&& \\
\cdot & \cdot & \cdot & \cdot && \\
+ & + & + & + & + & \\
\cdot & - & - & - & - & -
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
\cdot & \cdot &&&& \\
\cdot & \cdot & \cdot &&& \\
+ & + & + & + && \\
\cdot & - & - & - & - & \\
\cdot & \cdot & + & + & + & +
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
\cdot & \cdot &&&& \\
+ & + & + &&& \\
\cdot & - & - & - && \\
\cdot & \cdot & + & + & + & \\
\cdot & \cdot & \cdot & - & - & -
\end{array}
\quad
\begin{array}{cccccc}
\cdot &&&&& \\
+ & + &&&& \\
\cdot & - & - &&& \\
\cdot & \cdot & + & + && \\
\cdot & \cdot & \cdot & - & - & \\
\cdot & \cdot & \cdot & \cdot & + & +
\end{array}
\quad
\begin{array}{cccccc}
+ &&&&& \\
\cdot & - &&&& \\
\cdot & \cdot & + &&& \\
\cdot & \cdot & \cdot & - && \\
\cdot & \cdot & \cdot & \cdot & + & \\
\cdot & \cdot & \cdot & \cdot & \cdot & -
\end{array}
$$

$$
\begin{array}{cccccc}
+ &&&&& \\
+ & \cdot &&&& \\
+ & \cdot & \cdot &&& \\
+ & \cdot & \cdot & \cdot && \\
+ & \cdot & \cdot & \cdot & \cdot & \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{cccccc}
- &&&&& \\
- & + &&&& \\
- & + & \cdot &&& \\
- & + & \cdot & \cdot && \\
- & + & \cdot & \cdot & \cdot & \\
\cdot & + & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{cccccc}
+ &&&&& \\
+ & - &&&& \\
+ & - & + &&& \\
+ & - & + & \cdot && \\
\cdot & - & + & \cdot & \cdot & \\
\cdot & \cdot & + & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{cccccc}
- &&&&& \\
- & + &&&& \\
- & + & - &&& \\
\cdot & + & - & + && \\
\cdot & \cdot & - & + & \cdot & \\
\cdot & \cdot & \cdot & + & \cdot & \cdot
\end{array}
\quad
\begin{array}{cccccc}
+ &&&&& \\
+ & - &&&& \\
\cdot & - & + &&& \\
\cdot & \cdot & + & - && \\
\cdot & \cdot & \cdot & - & + & \\
\cdot & \cdot & \cdot & \cdot & + & \cdot
\end{array}
\quad
\begin{array}{cccccc}
- &&&&& \\
\cdot & + &&&& \\
\cdot & \cdot & - &&& \\
\cdot & \cdot & \cdot & + && \\
\cdot & \cdot & \cdot & \cdot & - & \\
\cdot & \cdot & \cdot & \cdot & \cdot & +
\end{array}
$$

The sign hyperfield was introduced to disregard the specific numerical values of a Pascal form's coefficients and concentrate solely on their signs.

Proposition 5.9. *Let $f \in \mathbb{R}[\mathbf{x}]$ be a real polynomial and $\mathbf{w} \in \mathbb{R}^n$ be a root of f. Then,* $\operatorname{sign}(\mathbf{w})$ *is a root of* $\operatorname{sign}(f)$.

Proof. Define $\mathbf{s} \coloneqq \operatorname{sign}(\mathbf{w})$. Write $f = \sum \lambda_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ with real coefficients $\lambda_{\mathbf{k}}$. If $\lambda_{\mathbf{k}} \mathbf{w}^{\mathbf{k}} = 0$ for all $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$, then clearly the sign of $\lambda_{\mathbf{k}} \mathbf{w}^{\mathbf{k}}$ is zero; hence the sign of f is the singleton set $\{0\}$ when evaluated at $\mathbf{s}$. So, $\mathbf{s}$ is a root of $\operatorname{sign}(f)$.

Now, suppose that there exists some $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ such that $\lambda_{\mathbf{k}} \mathbf{w}^{\mathbf{k}} \neq 0$. Assume $\lambda_{\mathbf{k}} \mathbf{w}^{\mathbf{k}} > 0$. Then, there also exists some $\mathbf{j} \in \mathbb{Z}^n_{\geq 0}$ such that we have $\lambda_{\mathbf{j}} \mathbf{w}^{\mathbf{j}} < 0$; otherwise $f(\mathbf{w}) > 0$ which is a contradiction to $\mathbf{w}$ being a root of f. Thus, $\operatorname{sign}(f)(\mathbf{s})$ includes summands with both positive and negative signs, and hence $\operatorname{sign}(f)(\mathbf{s}) = H$. So $0 \in \operatorname{sign}(f)(\mathbf{s})$ holds. Therefore, $\mathbf{s}$ is a root of $\operatorname{sign}(f)$. □

Taking the contrapositive of the above proposition, we get the *Hyperfield Criterion*.

Proposition 5.10 (Hyperfield Criterion)**.** *Let* $\mathbf{s} = (s_{i,j})_{(i,j) \in V_d} \in H^{V_d}$*. Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a configuration with* $\operatorname{sign}(\mathbf{w}) = \mathbf{s}$*. If* $\mathbf{s}$ *is not a root of a hyperfield Pascal form, then* $\mathbf{w}$ *is not an outcome.*

Proof. Follows from Proposition 5.9 and Theorem 3.14. □

We call a vector $\mathbf{s} \in H^{V_d}$ a *sign configuration* or *hyperfield configuration*.

Definition 5.11. Let $\mathbf{s} \in H^{V_d}$ be a sign configuration. We define:

(1) The positive support is defined as $\operatorname{supp}^+(\mathbf{s}) \coloneqq \{(i,j) \in V_d \mid s_{i,j} = 1\}$.

(2) The negative support is defined as $\operatorname{supp}^-(\mathbf{s}) \coloneqq \{(i,j) \in V_d \mid s_{i,j} = -1\}$.

(3) The support is defined as $\operatorname{supp}(\mathbf{s}) \coloneqq \operatorname{supp}^+(\mathbf{s}) \cup \operatorname{supp}^-(\mathbf{s})$.

(4) The degree of $\mathbf{s}$ is defined as $\deg(\mathbf{s}) \coloneqq \max \{i + j \mid (i,j) \in \operatorname{supp}(\mathbf{s})\}$.

(5) We call $\mathbf{s}$ *valid* if its support is empty or $\operatorname{supp}^-(\mathbf{s}) = \{(0,0)\}$.

Lemma 5.12. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a chipsplitting configuration. Then, we have* $\operatorname{supp}^+(\operatorname{sign}(\mathbf{w})) = \operatorname{sign}^+(\mathbf{w})$, $\operatorname{supp}^-(\operatorname{sign}(\mathbf{w})) = \operatorname{sign}^-(\mathbf{w})$*, and* $\deg(\operatorname{sign}(\mathbf{w})) = \deg(\mathbf{w})$*.*

Proof. Follows from the definitions. □

We investigate hyperfield forms induced by Pascal forms $\operatorname{col}(k), \operatorname{row}(k)$, and $\operatorname{diag}(k)$.

Proposition 5.13. *Let* $k = 0, \ldots, d$*. Define*

$$\begin{aligned} A_k^+ &\coloneqq \{(i,j) \in V_d \mid j = 0, \ldots, k \textit{ and } i = k - j, \ldots, d - j \textit{ with } j \equiv k \pmod 2\}, \\ A_k^- &\coloneqq \{(i,j) \in V_d \mid j = 0, \ldots, k \textit{ and } i = k - j, \ldots, d - j \textit{ with } j \not\equiv k \pmod 2\}. \end{aligned}$$

Then, the following statements hold:

(1) We have $\operatorname{sign}(\operatorname{diag}(k)) = \sum_{i=0}^{k} \sum_{j=0}^{d-k} x_{i,j}$*.*

(2) We have $\operatorname{sign}(\operatorname{col}(k)) = \sum_{(i,j) \in A_k^+} x_{i,j} - \sum_{(i,j) \in A_k^-} x_{i,j}$*.*

(3) *We have* $\mathrm{sign}(\mathrm{row}(k)) = \sum_{(i,j)\in A_k^+} x_{j,i} - \sum_{(i,j)\in A_k^-} x_{j,i}$.

Proof. The first statement follows directly from Proposition 4.2 since $i \leq k$ and $d-i-j \geq k-i$ must hold for the binomial coefficient to be non-zero. The second and third statement follow similarly from Proposition 3.19. □

Proposition 5.14. *Let* $\mathbf{s} \in H^{V_d}$ *be a valid nonzero sign configuration. The following statements hold:*

(1) *Let* $k = 0, \ldots, d$. *If* $0 \in \mathrm{sign}(\mathrm{diag}(k))(\mathbf{s})$, *then* $\mathrm{sign}(\mathrm{diag}(k))(\mathbf{s}) = H$.

(2) *If* $0 \in \mathrm{sign}(\mathrm{col}(k))(\mathbf{s})$ *for all* $k = 0, \ldots, d$, *then* $\mathrm{sign}(\mathrm{col}(k))(\mathbf{s}) = H$.

(3) *If* $0 \in \mathrm{sign}(\mathrm{row}(k))(\mathbf{s})$ *for all* $k = 0, \ldots, d$, *then* $\mathrm{sign}(\mathrm{row}(k))(\mathbf{s}) = H$.

Proof. We see that $\mathbf{s}$ has at least degree $d \geq 1$ since it is nonzero and valid. All $s_{i,j}$ equal one for $i + j > 0$, and there exists $s_{k,d-k} = 1$ for some $k = 0, \ldots, d$.

(1) Let $0 \in \mathrm{sign}(\mathrm{diag}(k))(\mathbf{s})$. By Proposition 5.13, $0 \in \mathrm{sign}(\mathrm{diag}(k))(\mathbf{s}) = \sum_{i=0}^{k} \sum_{j=0}^{d-k} s_{i,j}$ holds. We know that $s_{0,0} = -1$. So, we have $s_{i,j} = 1$ for some i, j with $i + j > 0$. Thus, we have $\mathrm{sign}(\mathrm{diag}(k))(\mathbf{s}) = H$.

(2) First note that $\mathrm{col}(0) = \mathrm{diag}(d)$. So, the case $k = 0$ is proven. Let $k > 0$. We start with $k = d$. Then, the union of A_d^+ and A_d^- consists exactly of vertices of degree d. Since $\mathrm{sign}(\mathrm{col}(d))(\mathbf{s}) = \sum_{(i,j)\in A_d^+} s_{i,j} - \sum_{(i,j)\in A_d^-} s_{i,j}$ contains zero, we have $s_{i,j} = 1$ for some $(i, j) \in A_d^+$, and $s_{i',j'} = -1$ for some $(i', j') \in A_d^-$. Hence, $\mathrm{sign}(\mathrm{col}(d))(\mathbf{s}) = H$.

   Let $k = d - 1$. Then, $s_{i,j} = 1$ for some $(i, j) \in A_{k+1}^+$, and $s_{i',j'} = -1$ for some $(i', j') \in A_{k+1}^-$. Note that $A_{k+1}^- \subset A_k^+$ by definition. Since $\mathrm{sign}(\mathrm{col}(k))(\mathbf{s}) = \sum_{(i,j)\in A_k^+} s_{i,j} - \sum_{(i,j)\in A_k^-} s_{i,j}$ contains zero, we have $s_{i'',j''} = -1$ for some $(i'', j'') \in A_k^-$. Hence, $\mathrm{sign}(\mathrm{col}(k))(\mathbf{s}) = H$.

   Repeat this argument for $k = d - 2, \ldots, 1$ to show that $\mathrm{sign}(\mathrm{col}(k))(\mathbf{s}) = H$.

(3) The proof is analogous to the previous case.

□

Corollary 5.15. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome. Then, we have* $\mathrm{sign}(p)(\mathrm{sign}(\mathbf{w})) = H$ *for all* $p \in \{\mathrm{diag}(k), \mathrm{col}(k), \mathrm{row}(k) \mid k = 0, \ldots, d\}$.

Proof. This follows from Theorem 3.14, Proposition 5.9, and Proposition 5.14. □

Example 5.16. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome of degree $d = 5$. By the previous corollary and Example 5.8, we know that the outcome $\mathbf{w}$ has at least one positive entry $w_{i,j}$ in each of the following marked areas + because $w_{0,0} < 0$:

+	·	·	·	·	·
+ ·	+ +	· ·	· ·	· ·	· ·
+ · ·	+ + ·	+ + +	· · ·	· · ·	· · ·
+ · · ·	+ + · ·	+ + + ·	+ + + +	· · · ·	· · · ·
+ · · · ·	+ + · · ·	+ + + · ·	+ + + + ·	+ + + + +	· · · · ·
+ · · · · ·	+ + · · · ·	+ + + · · ·	+ + + + · ·	+ + + + + ·	+ + + + + +

Moreover, for each triangle below the outcome $\mathbf{w}$ has some $w_{i,j} > 0$ for (i, j) in the plus area and $w_{i,j} > 0$ for (i', j') in the minus area because $\mathrm{sign}(\mathrm{col})(\mathrm{sign}(\mathbf{w})) = H$.

·	·	·	·	+
· ·	· ·	· ·	+ +	· −
· · ·	· · ·	+ + +	· − −	· · +
· · · ·	+ + + +	· − − −	· · + +	· · · −
+ + + + +	· − − − −	· · + + +	· · · − −	· · · · +
· − − − − −	· · + + + +	· · · − − −	· · · · + +	· · · · · −

An analogous statement holds for sign(row):

−	+	−	+	−
− +	+ −	− +	+ −	· +
− + ·	+ − +	− + −	· − +	· · −
− + · ·	+ − + ·	· + − +	· · + −	· · · +
− + · · ·	· − + · ·	· · − + ·	· · · − +	· · · · −
· + · · · ·	· · + · · ·	· · · + · ·	· · · · + ·	· · · · · +

The examples above demonstrate that we can view Corollary 5.15 as constraints on the support of an outcome $\mathbf{w}$. Configurations violating these constraints are not outcomes.

Corollary 5.17. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome of degree* $d \geq 1$*. Then, all the following constraints hold:*

(1) For all $k = 0, \ldots, d$*, the positive support of* $\mathbf{w}$ *contains the vertex* (i, j) *for at least one* $i = 0, \ldots, k$ *and* $j = 0, \ldots, d - k$*.*

(2) For all $k = 1, \ldots, d$*, the positive support of* $\mathbf{w}$ *contains at least one* $(i, j) \in A_k^+$ *and* $(i', j') \in A_k^-$*.*

(3) For all $k = 1, \ldots, d$*, the positive support of* $\mathbf{w}$ *contains at least one* (i, j) *and* (i', j') *from* $(j, i) \in A_k^+$ *and* $(j', i') \in A_k^-$*.*

We use these constraints to compute the supports of all outcomes of degree d.

5.2 Solving Hyperfield Linear Systems

We diverge from the approach of Bik and Marigliano by dedicating attention to solving hyperfield linear systems. Specifically, we are interested in valid configurations $\mathbf{w} \in \mathbb{Z}^{V_d}$ satisfying $0 \in \operatorname{sign}(p)(\operatorname{sign}(\mathbf{w}))$ for all $p \in \{\operatorname{diag}(k), \operatorname{row}(k), \operatorname{col}(k)\}_{k=0}^{d}$. These configurations are potential valid outcomes due to the Hyperfield Criterion.

Problem: Given a set of linear forms $A = \{p_1, \ldots, p_k\}$, compute the solution set $S_n(A) \coloneqq V(A) \cap \{\mathbf{x} \in H^{V_d} : \operatorname{supp}^-(\mathbf{x}) = \{(0,0)\}, |\operatorname{supp}^+(\mathbf{x})| = n\}$, where $V(A) \coloneqq \{\mathbf{x} \in H^{V_d} : 0 \in \operatorname{sign}(p_i)(\mathbf{x}) \quad \forall i = 1, \ldots, k\}$.

Note that $S_n(A)$ is a superset of supports of valid outcomes with positive support size n.

A Naive Approach

To compute $S_n(A)$, we use a brute force algorithm: just iterate over all supports of positive support size n, and check if they are hyperfield roots of some hyperfield Pascal basis. It has exponential time complexity as it checks $\binom{(d+1)(d+2)/2}{n}$ supports.

Algorithm 5.1 Brute Force Algorithm

Require: Positive support size n, a set of linear forms $A = \{p_1, \ldots, p_k\}$
Ensure: $S_n(A)$

function SOLVE(A, n)
    initialize empty list `solutions`
    for n-combination $S = \{(c_i, r_i) : i = 1, \ldots, n\}$ of V_d **do**
        initialize $\mathbf{x} \in H^{V_d}$ with positive support S and $x_{0,0} = -1$
        if $\mathbf{x}$ is a hyperfield root of every $p \in A$ **then**
            add S to `solutions`
        end if
    end for
    return `solutions`
end function

Efficient Algorithm

For *non-trivial* systems A, we greatly speed up the computation of $S_n(A)$.

Definition 5.18. Let p be a linear form in H^{V_d}, and $\mathbf{x} \in H^{V_d}$ be a hyperfield root of p. If $\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(p) = \emptyset$, then the root $\mathbf{x}$ is called a *trivial root* of p. Otherwise, the root $\mathbf{x}$ is called a *non-trivial root* of p.

Definition 5.19. Let A be a system of linear forms in H^{V_d}. We say $S_n(A)$ is *non-trivial* if $S_n(A) \neq \emptyset$ and every $\mathbf{x} \in S_n(A)$ is a non-trivial root for all $p \in A$. We say A is *non-trivial* if $S_n(A)$ is non-trivial.

Proposition 5.20. *Let A be a system of linear forms in H^{V_d}, $p \in A$ and $\mathbf{x} \in S_n(A)$. Then, the following statements hold:*

(1) If $(0,0) \in \operatorname{supp}^+(p)$, then $x_{i,j} = 1$ for some $(i,j) \in \operatorname{supp}^+(p)$.

(2) If $(0,0) \in \operatorname{supp}^-(p)$, then $x_{i,j} = 1$ for some $(i,j) \in \operatorname{supp}^-(p)$.

Proof. Assume $(0,0) \in \operatorname{supp}^+(p)$. Since $x_{0,0} = -1$, we have $-1 \in \operatorname{sign}(p)(x)$. By assumption, $\mathbf{x}$ is a hyperfield root of p, so $0 \in \operatorname{sign}(p)(\mathbf{x})$. This can only happen if $x_{i,j} = 1$ for some $(i,j) \in \operatorname{supp}^+(p)$. The case $(0,0) \in \operatorname{supp}^-(p)$ is similar. $\square$

The next proposition assumes that A is non-trivial.

Proposition 5.21. *Let A be a non-trivial system of linear forms, $p \in A$ and $\mathbf{x} \in S_n(A)$. If $(0,0) \notin \operatorname{supp}(p)$, then we have $x_{i,j} = x_{i',j'} = 1$ for some $(i,j) \in \operatorname{supp}^+(p)$ and $(i',j') \in \operatorname{supp}^-(p)$.*

Proof. Assume $(0,0) \notin \operatorname{supp}(p)$. First, $\operatorname{supp}(p) \neq \emptyset$ because $S_n(A)$ is non-empty and consists only of non-trivial roots. If $\operatorname{supp}^+(p) = \emptyset$, then $\operatorname{supp}^+(x) \subset \operatorname{supp}^-(p) = \operatorname{supp}(p) \neq \emptyset$. Hence, $\operatorname{sign}(p)(x) = \{-1\}$, which contradicts $\mathbf{x}$ being a root. Thus, $\operatorname{supp}^+(p)$ is non-empty. Similarly, $\operatorname{supp}^-(p)$ is non-empty.

By non-triviality, $x_{i,j} = 1$ for some $(i,j) \in \operatorname{supp}(p)$. Assume $(i,j) \in \operatorname{supp}^+(p)$. Hence, $1 \in \operatorname{sign}(p)(\mathbf{x})$. Since $\mathbf{x}$ is a root, we also have $0 \in \operatorname{sign}(p)(\mathbf{x})$. This can only occur if $x_{i',j'} = 1$ for some $(i',j') \in \operatorname{supp}^-(p)$. The case $(i,j) \in \operatorname{supp}^-(p)$ is similar. $\square$

Both propositions allow us to interpret hyperfield linear forms in a non-trivial system A as constraints on the positive supports of roots in $S_n(A)$.

Example 5.22. Let $d = 3$. Assume $p = \operatorname{diag}(0) \in A$. The support of p is visualized below on the left-hand side. Since $x_{0,0}$ is negative, we see that any nonzero hyperfield root $\mathbf{x}$ of A satisfies $x_{i,j} = 1$ for some $(i,j) \in \{(0,0),(0,1),(0,2),(0,3)\}$. Now, assume A is non-trivial. Consider $q = \operatorname{row}(3) \in A$. Its support is depicted on the right-hand side.

$$
\begin{array}{cccc}
+ & & & \\
+ & . & & \\
+ & . & . & \\
+ & . & . & .
\end{array}
\qquad\qquad
\begin{array}{cccc}
- & & & \\
. & + & & \\
. & . & - & \\
. & . & . & +
\end{array}
$$

For $\mathbf{x}$ with $\operatorname{supp}^-(\mathbf{x}) = \{(0,0)\}$ to be a hyperfield root of q, we must have either $x_{i,j} = x_{i',j'} = 1$ for some $(i,j) \in \{(0,3),(2,1)\}$ and $(i',j') \in \{(3,0),(1,2)\}$, or $\operatorname{supp}(\mathbf{x}) \subset V_3 \setminus \operatorname{supp}(q)$. By considering only non-trivial roots $\mathbf{x}$, we exclude the latter case. Thus, for a non-trivial system A with $p, q \in A$, computing $S_n(A)$ requires checking only those hyperfield roots whose support intersects non-trivially with each of the following three regions:

```
+          +          .
+ .        . .        . +
+ . .      . . +      . . .
+ . . .    . . . .    . . . +
```

Here are examples of supports satisfying the constraints above:

```
+          .
. +        . .
. . .      + . +
. . . .    . . . +
```

Definition 5.23. To each linear form p in H^{V_d}, we associate a finite set of supports, denoted by $\operatorname{constraints}(p) \coloneqq \{\operatorname{supp}^+(p) \setminus \{(0,0)\}, \operatorname{supp}^-(p) \setminus \{(0,0)\}\} \subset 2^{V_d}$.

Proposition 5.24. *Let p be a linear form in $\mathbb{H}^{V_d}$, and $\mathbf{x} \in H^{V_d}$ with $\operatorname{supp}^-(\mathbf{x}) = \{(0,0)\}$. Then, $\mathbf{x}$ is a non-trivial hyperfield root of p if and only if $\operatorname{supp}^+(\mathbf{x}) \cap S \neq \emptyset$ for all $S \in \operatorname{constraints}(p)$.*

Proof. Let $S \in \operatorname{constraints}(p)$. Since $\mathbf{x}$ is non-trivial, we have $\operatorname{supp}^+(\mathbf{x}) \cap S \neq \emptyset$. The converse direction is also clear since $p(\mathbf{x}) = 1 - 1 = H$. □

We present an algorithm for computing $S_n(A)$ of non-trivial systems A.

Algorithm 5.2 Algorithm for Non-Trivial Systems

Require: Positive support size n, non-trivial system A
Ensure: $S_n(A)$

function SOLVE(A, n)
  C $\leftarrow \bigcup_{p \in A} \operatorname{constraints}(p)$
  `solutions` $\leftarrow \{\mathbf{x} \in H^{V_d} \mid \forall S \in C : \operatorname{supp}^+(\mathbf{x}) \cap S \neq \emptyset, |\operatorname{supp}^+(\mathbf{x})| = n, \operatorname{supp}^-(\mathbf{x}) = \{(0,0)\}\}$
  return `solutions`
end function

Proof of correctness. The correctness follows from Proposition 5.24. □

Remark 5.25. The algorithm can also be used for linear forms on H^{Ξ} with arbitrary index set $\Xi \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and solutions $\mathbf{x} \in H^{\Xi}$. The proof is analogous.

5.3 Implementation of the Hyperfield Criterion

The Hyperfield Criterion asserts that only the common hyperfield roots of all Pascal forms can serve as supports for valid outcomes. Initially, the system of all Pascal forms is an *infinite* and *non-trivial* system. However, we have identified several bases of Pascal forms, including the row, column, and diagonal Pascal bases. These bases enable us to restrict our consideration to *finite* systems. Specifically, we define the finite system $A := \{\mathrm{diag}(i)\}_{i=0}^{d} \cup \{\mathrm{row}(i)\}_{i=0}^{d} \cup \{\mathrm{col}(i)\}_{i=0}^{d}$.

Proposition 5.26. *The system A is non-trivial.*

Proof. First, $S_n(A)$ is non-empty because $\mathbf{x} = (x_i)_{i=0}^{n}$ defined as $x_{i,d-i} = \binom{n}{i}$ is a solution of A. Let $\mathbf{x} \in S_n(A)$ and $i = 0, \ldots, n$. Consider the following cases.

- Assume, $\mathbf{x} \notin \mathrm{supp}(\mathrm{diag}(i))$; then $\mathrm{diag}(i)(\mathbf{x}) < 0$ but $\mathbf{x}$ is a root of $\mathrm{diag}(i)$.

- Assume, $\mathbf{x} \notin \mathrm{supp}(\mathrm{row}(i))$. If $i = d$, then $\mathbf{x}$ is not of degree n. Therefore, we assume $i < d$. Then, either $\mathbf{x}$ is a trivial root for $\mathrm{row}(i+1)$ or we have $\mathrm{row}(i+1)(\mathbf{x}) \neq 0$. In the latter case, we found a contradiction to $\mathbf{x}$ being a root. For the former case that $\mathbf{x}$ is a trivial root, we conclude that there exists nonzero $x_{u,d-u}$ for some $u = i+2, \ldots, d$ since $\mathbf{x}$ is of degree n; now we just repeat the argument for $\mathrm{row}(i+1)$. More precisely, if $\mathbf{x}$ is again a trivial root for $\mathrm{row}(i+2)$, we repeat the argument for $\mathrm{row}(i+2)$ until we will end up with a contradiction $\mathrm{row}(u)(\mathbf{x}) \neq 0$.

- For the case $\mathrm{col}(i)$, we can argue by symmetry.

□

Corollary 5.27. *Configurations $\mathbf{x} \in \mathbb{Z}^{V_d}$ of the form $\mathrm{supp}(\mathbf{x}) \subset \{(i,j) \in \mathbb{Z}^{V_d} : i + j \leq k \text{ or } i > k+1\}$ are not valid outcomes for any $k = 0, \ldots, d-1$. Neither are configurations $\mathbf{x} \in \mathbb{Z}^{V_d}$ of the form $\mathrm{supp}(\mathbf{x}) \subset \{(i,j) \in \mathbb{Z}^{V_d} : i + j \leq k \text{ or } j > k+1\}$ for $k = 0, \ldots, d-1$ due to symmetry.*

Proof. Let $k = 0, \ldots, d-1$. Since the system A is non-trivial, we have that supports of valid outcomes intersect the support of $\mathrm{row}(k+1)$ non-trivially. □

Example 5.28. This is not a valid outcome by the previous corollary.

```
.
.  .
.  .  .
.  .  .  *
*  .  .  *  *
*  *  .  *  *  *
```

Now that we have shown A to be a trivial system, we have an efficient implementation of Algorithm 5.2. Specifically, the computation of $\{\mathbf{x} \in H^{V_d} \mid \forall S \in C : \operatorname{supp}^+(\mathbf{x}) \cap S \neq \emptyset, |\operatorname{supp}^+(\mathbf{x})| = n, \operatorname{supp}^-(\mathbf{x}) = \{(0,0)\}\}$ on line three can be efficiently implemented with a breadth-first search algorithm as shown below.

Algorithm 5.3 Efficient Implementation of Algorithm 5.2 for non-trivial A

Require: positive support size n, non-trivial system A
Ensure: $S_n(A)$

```
 1: constraints ← build constraints(A)
 2: queue ← deque()
 3: queue.append(empty tuple())
 4: for each constr ∈ constraints do
 5:     for _ in range(|queue|) do
 6:         conf ← queue.popleft()
 7:         if conf intersects constr then
 8:             queue.append(conf)
 9:         else if |conf| < n then
10:             for each j ∈ constr do
11:                 queue.append(conf ∪ {j})
12:             end for
13:         end if
14:     end for
15: end for
16: return queue
```

Proof of correctness. After each iteration of line four, `queue` contains only configurations that satisfy `constr` due to line seven and line eleven. Thus, the algorithm returns all configurations that satisfy all constraints in `constraints`. □

Proposition 5.29. *Let* $A = \{\operatorname{diag}(i)\}_{i=0}^{d} \cup \{\operatorname{row}(i)\}_{i=0}^{d} \cup \{\operatorname{col}(i)\}_{i=0}^{d}$ *for some degree* $d \in \mathbb{N}$. *Let* $\mathbf{x} \in H^{V_d}$ *be nonzero with* $\operatorname{supp}^-(\mathbf{x}) = \{(0,0)\}$. *The following statements hold:*

(1) If $d = 6$, *then we have* $\mathbf{x} \in S_4(A)$ *if and only if* $\operatorname{supp}^+(\mathbf{x})$ *is one of the following sets:*

$$\{(0,3),(1,5),(4,1),(6,0)\}, \{(0,5),(1,1),(3,3),(6,0)\}, \{(0,6),(1,1),(3,3),(5,0)\},$$
$$\{(0,6),(1,1),(3,3),(6,0)\}, \{(0,6),(1,4),(3,0),(5,1)\}.$$

(2) If $d = 7$, *then we have* $\mathbf{x} \in S_4(A)$ *if and only if* $\operatorname{supp}^+(\mathbf{x})$ *is one of the following sets:*

$$\{(0,7),(1,1),(3,3),(7,0)\}, \{(0,7),(1,3),(5,1),(7,0)\}, \{(0,7),(1,5),(3,1),(7,0)\}.$$

(3) If $d = 8, 9, 10, 11$, *then the solution set* $S_4(A)$ *is empty.*

Proof. We compute the set of $S_4(A)$ for $d = 6, 7, 8, 9, 10$, and 11 using Algorithm 5.3. The code is found in [10] under `chapter05.ipynb`. □

Corollary 5.30. *There are no valid outcomes with positive support size four for* $d = 8, 9, 10, 11$.

The case $\deg(\mathbf{w}) > 11$ is still open. We will address this next.

5.4 Contractions

We aim to show that for every valid integral outcome $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 4$, its degree is at most five. The challenge arises from the need to check an infinite number of possible supports. To overcome this, we introduce *contractions* and *fixed-contractable forms*. The term *contractions* was introduced in [3], while *fixed-contractable forms* is a term specifically developed for this thesis. The concept of contraction involves *contracting* or *consolidating* vertices in V_d by merging rows or columns into a single vertex. This is achieved by introducing new formal variables $b_i, c_i, d_i, e_i, y_{i,j}$, and $z_{i,j}$, referred to as *contraction variables*.

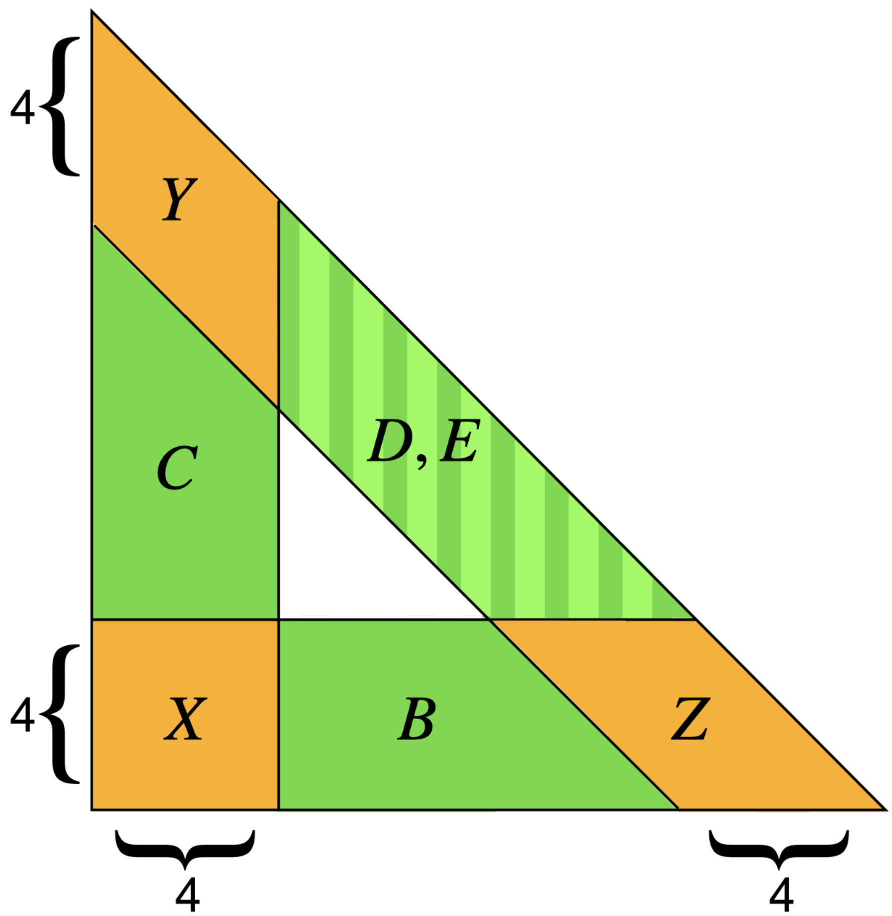

Figure 5.1: This figure illustrates the contraction variables. The yellow areas X, Y, Z represent formal variables $x_{i,j}$ that remain unaffected by the contraction. Each green area B, C, D, E represents rows or columns of vertices that are merged into a single vertex.

Definition 5.31. Let $x_{i,j}$ be formal variables indexed by V_d. We merge a subset of rows and columns into a single vertex by defining *contraction variables*:

$$
\begin{aligned}
y_{i,j} &:= x_{i,d-3-i+j} \quad \text{for } i,j = 0,\dots,3,\\
z_{i,j} &:= x_{d-3-j+i,j} \quad \text{for } i,j = 0,\dots,3,\\
b_j &:= x_{4,j} + \cdots + x_{d-4-j,j} \quad \text{for } j = 0,\dots,3,\\
c_i &:= x_{i,4} + \cdots + x_{i,d-4-i} \quad \text{for } i = 0,\dots,3,\\
d_k &:= \begin{cases} x_{4,d-4-k} + x_{6,d-6-k} + \cdots + x_{d-4-k,4} & \text{if } d+k \text{ is even}\\ x_{4,d-4-k} + x_{6,d-6-k} + \cdots + x_{d-5-k,5} & \text{if } d+k \text{ is odd}\end{cases} \qquad \text{for } k = 0,\dots,3,\\
e_k &:= \begin{cases} x_{5,d-5-k} + x_{7,d-7-k} + \cdots + x_{d-5-k,5} & \text{if } d+k \text{ is even}\\ x_{5,d-5-k} + x_{7,d-7-k} + \cdots + x_{d-4-k,4} & \text{if } d+k \text{ is odd}\end{cases} \qquad \text{for } k = 0,\dots,3.
\end{aligned}
$$

Let us visualize the contraction variables for $d = 16$ in the following figure.

$$
\begin{array}{ccccccccccccccccc}
y_{0,3} \\
y_{0,2} & y_{1,3} \\
y_{0,1} & y_{1,2} & y_{2,3} \\
y_{0,0} & y_{1,1} & y_{2,2} & y_{3,3} \\
c_0 & y_{1,0} & y_{2,1} & y_{3,2} & d_0 \\
c_0 & c_1 & y_{2,0} & y_{3,1} & d_1 & e_0 \\
c_0 & c_1 & c_2 & y_{3,0} & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & * & e_3 & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & e_3 & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & * & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & * & * & e_3 & d_2 & e_1 & d_0 \\
x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} & b_3 & b_3 & b_3 & b_3 & b_3 & b_3 & z_{0,3} & z_{1,3} & z_{2,3} & z_{3,3} \\
x_{0,2} & x_{1,2} & x_{2,2} & x_{3,2} & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & z_{0,2} & z_{1,2} & z_{2,2} & z_{3,2} \\
x_{0,1} & x_{1,1} & x_{2,1} & x_{3,1} & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & z_{0,1} & z_{1,1} & z_{2,1} & z_{3,1} \\
x_{0,0} & x_{1,0} & x_{2,0} & x_{3,0} & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & z_{0,0} & z_{1,0} & z_{2,0} & z_{3,0}
\end{array}
$$

As we can see, the vertices $x_{0,4}, \dots, x_{0,d-4}$ are merged into the contraction variable c_0 by summing them up. The key insight is that some hyperfield Pascal forms $\sum_{(i,j)\in V_d} \lambda_{i,j} x_{i,j}$ can be expressed in terms of contraction variables.

Example 5.32. Consider sign(diag(1)) in $H^{V_{16}}$. It is depicted in the following figure:

```
.
+ +
+ + .
+ + . .
+ + . . .
+ + . . . .
+ + . . . . .
+ + . . . . . .
+ + . . . . . . .
+ + . . . . . . . .
+ + . . . . . . . . .
+ + . . . . . . . . . .
+ + . . . . . . . . . . .
+ + . . . . . . . . . . . .
+ + . . . . . . . . . . . . .
+ + . . . . . . . . . . . . . .
+ + . . . . . . . . . . . . . . .
```

We see that $\mathrm{sign}(\mathrm{diag}(1)) = x_{0,0} + x_{0,1} + x_{0,2} + x_{0,3} + x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + y_{0,0} + y_{0,1} + y_{0,2} + y_{1,0} + y_{1,1} + y_{1,2} + y_{1,3} + c_0 + c_1$. Note that this expression is *independent* of the degree d,

Definition 5.33. Let $d \in \mathbb{N}_{\geq 11}$ and p be a hyperfield linear form on H^{V_d}. We say p is *contractable* for d if we can write $p = \hat{p}$ for some linear form $\hat{p} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$.

Definition 5.34. Let t be a formal variable. Define $I = \{0, 1, 2, 3, 4, t-4, t-3, t-2, t-1, t\}$. Let T be a formal linear combination of

$$\{\mathrm{diag}(k), \mathrm{row}(k), \mathrm{col}(k)\}_{k\in I} \text{ or } \{\mathrm{sign}(\mathrm{diag}(k)), \mathrm{sign}(\mathrm{row}(k)), \mathrm{sign}(\mathrm{col}(k))\}_{k\in I}\,.$$

Let $d \in \mathbb{N}_{\geq 11}$. We write p_d for the realization of T at $t = d$; the realization p_d is just a linear form on $\mathbb{Z}^{V_d}$ or H^{V_d}, respectively, where the formal variable t is replaced by the actual value d.

Example 5.35. Let $d = 15$. Consider the formal linear combination $T = \mathrm{row}(3) - \mathrm{row}(t-2)$. The realization p_d of T at $t = d$ is the linear form $p = \mathrm{row}(3) - \mathrm{row}(13)$. It is depicted in the figure below. Note that the realization p_d is contractable for all $d \in \mathbb{N}$ with $d \geq 15$. So, we ask whether the linear form $\hat{p}_d \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ change with d?

```
-350
-350   .
-285  65  65
-220  65   . -65
-165  55 -10 -10  55
-120  45 -10   .  10 -45
 -84  36  -9   1   1  -9  36
 -56  28  -8   1   .  -1   8 -28
 -35  21  -7   1   .   .   1  -7  21
 -20  15  -6   1   .   .   .  -1   6 -15
 -10  10  -5   1   .   .   .   .   1  -5  10
  -4   6  -4   1   .   .   .   .   .  -1   4  -6
  -1   3  -3   1   .   .   .   .   .   .   1  -3   3
   .   1  -2   1   .   .   .   .   .   .   .  -1   2  -1
   .   .  -1   1   .   .   .   .   .   .   .   .   1  -1   .
   .   .   .   1   .   .   .   .   .   .   .   .   .  -1   .   .
```

Definition 5.36. Let $D \in \mathbb{N}$ and T be a formal linear combination of

$$\{\operatorname{sign}(\operatorname{diag}(k)), \operatorname{sign}(\operatorname{row}(k)), \operatorname{sign}(\operatorname{col}(k))\}_{k\in\{0,1,2,3,4,t-4,t-3,t-2,t-1,t\}}.$$

We say T is *fixed-contractable* starting from degree D if all the following statements hold:

(1) The realization p_d is contractable for all $d \geq D$;

(2) There exists a linear form $\hat{p}^{\text{even}} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ such that $p_d = \hat{p}^{\text{even}}$ for all even degrees $d \geq D$;

(3) There exists a linear form $\hat{p}^{\text{odd}} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ such that $p_d = \hat{p}^{\text{odd}}$ for all odd degrees $d \geq D$.

Proposition 5.37. *Let* $T \in \{\operatorname{sign}(\operatorname{diag}(k)), \operatorname{sign}(\operatorname{row}(k)), \operatorname{sign}(\operatorname{col}(k))\}_{k\in\{0,1,2,3,4,t-4,t-3,t-2,t-1,t\}}$ *be a formal form. Then, T is fixed-contractable starting from degree $D = 11$.*

Proof. Let $d \geq 11$. We have the following cases:

(1) Let $T \in \{\operatorname{sign}(\operatorname{row}(k)), \operatorname{sign}(\operatorname{diag}(k)) \mid k = 0, 1, 2, 3\}$. We observe that the hyperfield form p_d has support contained in the areas X, C, and Y from Figure 5.1. This follows directly from Proposition 5.13. We also see that p_d depends only on the column sums on C.

(2) Let $T \in \{\operatorname{sign}(\operatorname{col}(k)), \operatorname{sign}(\operatorname{diag}(t-k)) \mid k = 0, 1, 2, 3\}$. We see that p_d has support contained in the areas X, B, and Z from Figure 5.1 by Proposition 5.13. We conclude that p_d only depends on the row sums on B.

(3) Let $T \in \{\mathrm{sign}(\mathrm{col}(t-k)), \mathrm{sign}(\mathrm{row}(t-k)) \mid k = 0,1,2,3\}$. All the hyperfield forms p_d depend on entries in the area Y, D, and Z from Figure 5.1 by Proposition 5.13. We see that p_d depends in area D on alternating diagonal sums. This shows that p_d is a sum of the contraction variables $y_{i,j}$, $z_{i,j}$, and $d_k - e_k$.

□

Example 5.38. The support of $\mathrm{sign}(\mathrm{col}(d-3))$ is depicted for $d = 12$ and $d = 13$:

$$\begin{array}{ccccccccccccc}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
+ & + & + & + \\
\cdot & - & - & - & - \\
\cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & -
\end{array}
\qquad
\begin{array}{cccccccccccccc}
\cdot \\
\cdot & \cdot \\
\cdot & \cdot & \cdot \\
+ & + & + & + \\
\cdot & - & - & - & - \\
\cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & - & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & + & +
\end{array}$$

For $d = 12$, we write $\mathrm{sign}(\mathrm{col}(d-3)) = \sum_{i=0}^{3} \sum_{j=0}^{i} (-1)^{i+j} y_{i,j} - \sum_{k=0}^{3} (-1)^k d_k + \sum_{k=0}^{3} (-1)^k e_k - \sum_{i,j=0}^{3} (-1)^j z_{i,j}$. For $d = 13$ we write $\mathrm{sign}(\mathrm{col}(d-3)) = \sum_{i=0}^{3} \sum_{j=0}^{i} (-1)^{i+j} y_{i,j} - \sum_{k=0}^{3} (-1)^k d_k + \sum_{k=0}^{3} (-1)^k e_k + \sum_{i,j=0}^{3} (-1)^j z_{i,j}$.

We have merged formal variables $x_{i,j}$ indexed by vertices (i,j) in the areas B, C, D into contraction variables. We also see that our row, column, and diagonal Pascal forms can be expressed by contraction variables; even more, this contraction form is independent of the degree d. Let us apply these contractions to concrete elements $\mathbf{s} \in H^{V_d}$.

Definition 5.39. We define the index set $\Xi := \{0,1,2,3\}^2 \sqcup \{0,1,2,3\}^2 \sqcup \{0,1,2,3\}^2 \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\}$. Let $\mathbf{s} \in H^{\Xi}$. We call $\mathbf{s}$ a *contracted hyperfield configuration*, and write $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = (x_{i,j}, y_{i,j}, z_{i,j}, b_j, c_i, d_k, e_k)$.

Definition 5.40. Let $\mathbf{s} \in H^{\Xi}$ be a contracted hyperfield configuration. We say $\mathbf{s}$ is *valid* if one of the following holds:

(1) $\mathbf{s} = \mathbf{0}$ or

(2) $s_{0,0} = -1$, $x_{i,j} \geq 0$ for all $i + j > 0$, and $y_{i,j}, z_{i,j}, b_j, c_i, d_k, e_k \geq 0$ for all $i, j, k = 0, 1, 2, 3$.

Going from the world of hyperfield configurations to the world of *contracted* hyperfield configurations is achieved via the following map.

Definition 5.41. Let $d \geq 11$. Let $\mathbf{s} \in H^{V_d}$ be a hyperfield configuration. We define

$$\mathrm{contr}_d(\mathbf{s}) : \mathbf{s} \mapsto (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = (x_{i,j}, y_{i,j}, z_{i,j}, b_j, c_i, d_k, e_k)$$

where we set

$$\begin{aligned}
x_{i,j} &\coloneqq s_{i,j} \quad \text{for } i, j = 0, \ldots, 3, \\
y_{i,j} &\coloneqq s_{i,d-3-i+j} \quad \text{for } i, j = 0, \ldots, 3, \\
z_{i,j} &\coloneqq s_{d-3-j+i,j} \quad \text{for } i, j = 0, \ldots, 3, \\
b_j &\coloneqq s_{4,j} + \cdots + s_{d-4-j,j} \quad \text{for } j = 0, \ldots, 3, \\
c_i &\coloneqq s_{i,4} + \cdots + s_{i,d-4-i} \quad \text{for } i = 0, \ldots, 3, \\
d_k &\coloneqq \begin{cases} s_{4,d-4-k} + s_{6,d-6-k} + \cdots + s_{d-4-k,4} & \text{if } d+k \text{ is even} \\ s_{4,d-4-k} + s_{6,d-6-k} + \cdots + s_{d-5-k,5} & \text{if } d+k \text{ is odd} \end{cases} \quad \text{for } k = 0, \ldots, 3, \\
e_k &\coloneqq \begin{cases} s_{5,d-5-k} + s_{7,d-7-k} + \cdots + s_{d-5-k,5} & \text{if } d+k \text{ is even} \\ s_{5,d-5-k} + s_{7,d-7-k} + \cdots + s_{d-4-k,4} & \text{if } d+k \text{ is odd} \end{cases} \quad \text{for } k = 0, \ldots, 3.
\end{aligned}$$

The contraction map contr_d maps hyperfield configurations $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) \in H^{V_d}$ to elements in H^{Ξ} if $\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \geq 0$. If one of the entries is negative, the map may output to some element $(2^H)^{\Xi}$. To make life easier, we only consider *weakly valid* configuration; these are configurations whose negative support is only contained in the yellow area below.

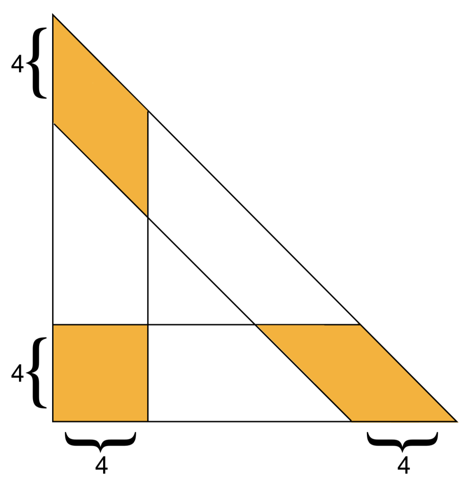

Figure 5.2: A hyperfield configuration is weakly valid if its negative support is contained in the yellow area.

Definition 5.42. Let $\mathbf{s} \in H^{V_d}$ be a hyperfield configuration. We say $\mathbf{s}$ is *weakly valid* if for all $(i, j) \in \mathrm{supp}^-(\mathbf{s})$ one of the following holds:

(1) $i, j = 0, \ldots, 3$, or

(2) $i = 0, \ldots, 3$ and $i + j \geq d - 3$, or

(3) $j = 0, \ldots, 3$ and $i + j \geq d - 3$.

Definition 5.43. Let $\mathbf{s} \in H^{\Xi}$ be a contracted hyperfield configuration. The *positive support* of $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$ is defined as the set of all symbols $x_{i,j}, y_{i,j}, z_{i,j}, b_j, c_i, d_k, e_k$ such that the corresponding coefficients of $\mathbf{s}$ equal to one.

Example 5.44. Let $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) \in H^{\Xi}$ be a contracted hyperfield configuration defined by $x_{0,0} = -1, x_{0,3} = 1, x_{1,1} = 1, x_{3,0} = 1, d_0 = 1, e_0 = 1$, and all other entries are zero. Then, the positive support of $\mathbf{s}$ is given by $\mathrm{supp}^+(\mathbf{s}) = \{x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}$.

5.5 Proof

We now prove that for every valid outcome $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 4$, we have $\deg(\mathbf{w}) \leq 5$. Define two systems of Pascal forms that valid outcomes must satisfy:

(1) $\Phi_1 := \{\mathrm{col}(i), \mathrm{row}(i), \mathrm{diag}(i), \mathrm{diag}(d - i)\}_{i=1}^{3}$,

(2) $\Phi_2 := \{\mathrm{row}(d - i), \mathrm{col}(d - i)\}_{i=0}^{3}$, and

(3) $\Phi := \Phi_1 \cup \Phi_2$.

By Proposition 5.37, we write all forms p in Φ_1 as $\mathrm{sign}(p) = \hat{p}$ for some linear form $\hat{p} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ if $d \geq 11$. This linear form is independent of the degree d. To make notations consistent later, we set $\hat{p}^{\mathrm{even}} := \hat{p}^{\mathrm{odd}} := \hat{p}$. Similarly, by Proposition 5.37, we write all forms p in Φ_2 as $\mathrm{sign}(p) = \begin{cases} \hat{p}^{\mathrm{even}} & \text{if } d \text{ is even} \\ \hat{p}^{\mathrm{odd}} & \text{if } d \text{ is odd} \end{cases}$, where $\hat{p}^{\mathrm{even}}, \hat{p}^{\mathrm{odd}} \in H[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}]$ if $d \geq 12$. Again, these linear forms $\hat{p}^{\mathrm{even}}, \hat{p}^{\mathrm{odd}}$ are independent of the degree d.

Definition 5.45. We define the following three solution sets:

(1) Define Γ_d to be the set of all valid hyperfield configurations $\mathbf{s} \in H^{V_d}$ of degree d such that $\mathrm{sign}(p)(\mathbf{s}) = H$ for all $p \in \Phi$.

(2) Define Γ^{even} to be the set of all valid contracted hyperfield configurations $\mathbf{s} \in H^{\Xi}$ such that $\hat{p}^{\mathrm{even}}(\mathbf{s}) = H$ for all $p \in \Phi$.

(3) Define Γ^{odd} to be the set of all valid contracted hyperfield configurations $\mathbf{s} \in H^{\Xi}$ such that $\hat{p}^{\mathrm{odd}}(\mathbf{s}) = H$ for all $p \in \Phi$.

By Proposition 5.10, valid chipsplitting outcomes of degree d have supports in Γ_d.

Proposition 5.46. *Let $d \geq 12$. If d is even, then $\Gamma_d = \mathrm{contr}_d^{-1}(\Gamma^{\mathrm{even}})$ holds. If d is odd, then $\Gamma_d = \mathrm{contr}_d^{-1}(\Gamma^{\mathrm{odd}})$ holds.*

Proof. Let $d \geq 12$ be even. Let $\mathbf{s} \in H^{V_d}$ be a hyperfield configuration and $p \in \Phi$. Then, we have $\mathrm{sign}(p)(\mathbf{s}) = \hat{p}^{\mathrm{even}}(\mathrm{contr}_d(\mathbf{s}))$ by definition of $\hat{p}^{\mathrm{even}}$. If $\mathbf{s} \in \Gamma_d$, then $H = \mathrm{sign}(p)(\mathbf{s}) = \hat{p}^{\mathrm{even}}(\mathrm{contr}_d(\mathbf{s}))$. Hence, $\mathrm{contr}_d(\mathbf{s})$ is contained in Γ^{even}. If $\mathrm{contr}_d(\mathbf{s}) \in \Gamma^{\mathrm{even}}$ holds, using the equation above we also see that $\mathbf{s} \in \Gamma_d$. This shows that $\Gamma_d = \mathrm{contr}_d^{-1}(\Gamma^{\mathrm{even}})$.

The second statement for odd degrees d follows analogously. □

Corollary 5.47. *Let $d \geq 12$ and $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome. Then, we have $\mathrm{contr}_d(\mathrm{sign}(\mathbf{w})) \in \Gamma^{\mathrm{even}} \cup \Gamma^{\mathrm{odd}}$.*

Proof. Define $\mathbf{s} := \mathrm{sign}(\mathbf{w})$. By Proposition 5.14 we have $\mathbf{s} \in \Gamma_d$. If d is even, then $\mathrm{contr}_d(\mathbf{s}) \in \Gamma^{\mathrm{even}}$ by the previous proposition. If d is odd, then $\mathrm{contr}_d(\mathbf{s}) \in \Gamma^{\mathrm{odd}}$ by the previous proposition. □

This corollary allows us to exclude certain supports as supports of valid outcomes. Assume we have a contracted hyperfield configuration $\xi \in H^{\Xi}$ that is not a root of some $\hat{p}$ for $p \in \Phi$. Then, any chipsplitting configuration $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $\mathrm{contr}_d(\mathrm{sign}(\mathbf{w})) = \xi$ is not a valid outcome.

Proposition 5.48. *Let $\mathbf{s} \in H^{V_d}$ be a valid hyperfield configuration of degree d with positive support size four or less. If $d \geq 12$, then $\mathbf{s} \notin \Gamma_d$.*

Proof. Let $d \geq 12$. For computing Γ_d we could use Algorithm 5.2 for all $d = 12, 13, 14, \ldots$ and so on, which is not feasible since we would compute solutions sets for many infinitely many degrees d. Instead, we show that $\Gamma^{\mathrm{even}} \cup \Gamma^{\mathrm{odd}}$ is empty. By Proposition 5.46, Γ_d is empty as well for all $d \geq 12$.

To show that Γ^{even} is empty, we use Algorithm 5.2 and Remark 5.25 with $A := \{\hat{p}^{\mathrm{even}} \mid p \in \Phi\}$. Similarly, to compute that Γ^{odd} is empty, we define $A := \left\{\hat{p}^{\mathrm{odd}} \mid p \in \Phi\right\}$ and use Algorithm 5.2. The implementation details are available on GitHub in the file `chapter05.ipynb` [10]. □

Theorem 5.49. *For valid integral outcomes $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 4$ we have $\deg(\mathbf{w}) \leq 5$.*

Proof. Let $d \geq 6$. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome with $|\mathrm{supp}^+(\mathbf{w})| = 4$ and degree d. We have $\mathrm{sign}(\mathbf{w}) \in \Gamma_d$. By the previous proposition, there is no such $\mathrm{sign}(\mathbf{w})$ for $d \geq 12$. By Proposition 5.29, the degree of $\mathrm{sign}(\mathbf{w}) = d$ is six or seven. So, we check eight cases. Of these eight cases, we exclude all of them using the Invertibility Criterion, see the file `chapter05.ipynb` in [10]. □

Chapter 6

Valid Outcomes of Positive Support Size Five

6.1 Hexagon Criterion

We introduce the *Hexagon Criterion*, first presented in [3], to determine whether subconfigurations of a chipsplitting outcome qualify as outcomes. This criterion applies to configurations supported within the yellow area below; we say the support is not contained inside the "hexagon".

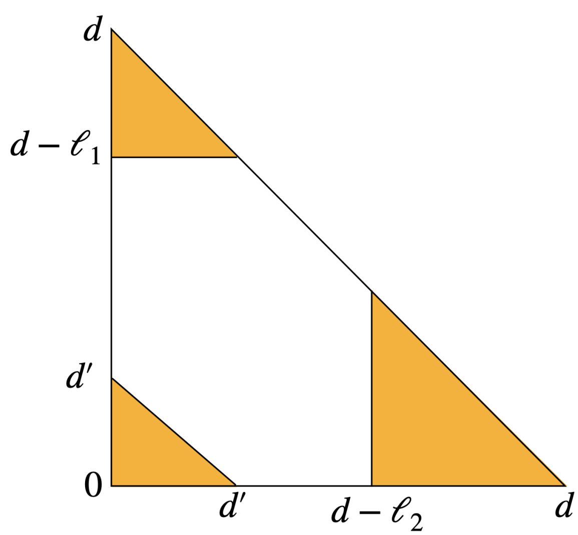

Figure 6.1: A configuration's support lies outside the hexagon if it is contained in the yellow area spanned by parameters ℓ_1, ℓ_2, and d'.

First, we need the following lemma to compute the determinant of matrix that we will encounter in the proof of the Hexagon Criterion.

Lemma 6.1. *Let $a, b, c \in \mathbb{Z}_{\geq 0}$. Define the map $H : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}, n \mapsto 0!1! \cdots (n-1)!$; note that $H(0) = 1$. Then, the following holds:*

$$\det\left(\left[\binom{a+b}{b-i+j}\right]_{i,j=1}^{c}\right) = \frac{H(a)H(b)H(c)H(a+b+c)}{H(b+c)H(c+a)H(a+b)}.$$

Proof. See Theorem 8 of [6]. □

We state the Hexagon Criterion.

Proposition 6.2. *Let $d, d', \ell_1, \ell_2 \in \mathbb{N}$ with $d' \geq 1$, $\ell_1, \ell_2 \geq d'$, and $d' + \ell_1 + \ell_2 \leq d$. Let $\mathbf{w} = (w_{i,j})_{(i,j)\in V_d} \in \mathbb{Z}^{V_d}$ be a chip configuration. Define the subconfiguration $\mathbf{w}' := (w_{i,j})_{(i,j)\in V_{d'}} \in \mathbb{Z}^{V_{d'}}$. Assume the support of $\mathbf{w}$ is not contained inside the "hexagon" (see Figure 6.1), i.e.* $\operatorname{supp}(\mathbf{w}) \subset V_{d'} \cup \{(i,j) \in V_d \mid j > d - \ell_1\} \cup \{(i,j) \in V_d \mid i > d - \ell_2\}$. *Then, the following holds:*

(1) If $\mathbf{w}$ is an outcome, then also its subconfiguration $\mathbf{w}'$ is an outcome.

(2) If $\mathbf{w}$ is a valid outcome, then $\deg(\mathbf{w}) \leq d'$.

Proof. We prove the first statement. Assume $\mathbf{w}$ is an outcome. Let $k = 0, \ldots, d'$ and $\operatorname{diag}(\ell_1 + k) = \sum_{(i,j)\in V_d} \mu_{i,j} x_{i,j}$. Consider the restricted linear form $l_k = \sum_{(i,j)\in V_{d'}} \lambda_{i,j} x_{i,j} : \mathbb{Z}^{V_{d'}} \to \mathbb{Z}$ with $\lambda_{i,j} := \mu_{i,j}$ for $(i,j) \in V_{d'}$. Then, $l_0, \ldots, l_{d'}$ are Pascal forms on $\mathbb{Z}^{V_{d'}}$ with $l_k(\mathbf{w}') = \operatorname{diag}(\ell_1 + k)(\mathbf{w}) = 0$. By Proposition 3.14 it suffices to show that $l_0, \ldots, l_{d'}$ are linearly independent to show that $\mathbf{w}'$ is an outcome.

Let $a = 0, \ldots, d'$. Define

$$e^{(a)}_{i,j} := \begin{cases} 1 & \text{if } i = a \text{ and } j = d' - a, \\ 0 & \text{otherwise.} \end{cases}$$

$$A := \left[l_k(\mathbf{e}^{(a)})\right]_{k,a=0}^{d'} = \left[\binom{d-d'}{\ell_1+k-a}\right]_{k,a=0}^{d'} = \left[\binom{(d-d'-\ell_1)+\ell_1}{\ell_1+k-a}\right]_{k,a=0}^{d'}.$$

We want to show that the matrix A is invertible because this implies that the linear forms $l_0, \ldots, l_{d'}$ are linearly independent. We observe that

(1) all entries of A are nonzero because $0 \leq \ell_1 + k - a \leq d - d'$, and

(2) $d - d' - \ell_1 \geq \ell_2 \geq 0$.

This allows us to use Lemma 6.1 with $a := d - d' - \ell_1, b := \ell_1$, and $c := d' + 1$. We obtain a nonzero determinant $\det(A) = \frac{H(d-d'-\ell_1)H(\ell_1)H(d'+1)H(d+1)}{H(\ell_1+d'+1)H(1+d-\ell_1)H(d-d')} \neq 0$. Hence, $l_0, \ldots, l_{d'}$ are linearly independent.

For the second statement, let $\mathbf{w}$ be a valid outcome. By the previous statement, the subconfiguration $\mathbf{w}'$ is an outcome, as well. We extend $\mathbf{w}'$ to some configuration $\mathbf{v} \in \mathbb{Z}^{V_d}$ by $v_{i,j} := \begin{cases} w'_{i,j} & \text{if } (i,j) \in V_{d'}, \\ 0 & \text{otherwise} \end{cases}$. Clearly, $\mathbf{v}$ is a *valid* outcome of degree at most d'. Then, $\mathbf{v} - \mathbf{w}$ is an outcome with empty negative support. By Proposition 3.30, $\mathbf{v} - \mathbf{w}$ is zero. Hence, $\deg(\mathbf{w}) \leq d'$. $\square$

Next, we prove that for all valid integral outcomes $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 5$ we have $\deg(\mathbf{w}) \leq 7$. We use the Invertibility Criterion, Hyperfield Criterion, and the Hexagon Criterion.

6.2 Proof $d = 8, \ldots, 41$

First, we show that no outcome of degree $d = 8, \ldots, 41$ exists with $|\mathrm{supp}^+(\mathbf{w})| = 5$; this is similar to the proof of Proposition 5.29.

Proposition 6.3. *No outcome of degree* $d = 8, \ldots, 41$ *exists with* $|\mathrm{supp}^+(\mathbf{w})| = 5$.

Proof. Let $d = 8, \ldots, 41$. We define the system $A = \{\mathrm{diag}(i)\}_{i=0}^{d} \cup \{\mathrm{row}(i)\}_{i=0}^{d} \cup \{\mathrm{col}(i)\}_{i=0}^{d}$. Then, we compute the set $S_5(A)$ using Algorithm 5.3. Next, we apply Algorithm 4.1 to each $\mathbf{s} \in S_5(A)$. The implementation is found in [10] under `chapter06.ipynb`. $\square$

6.3 Proof $d \geq 42$

We show that no valid outcome of degree $d \geq 42$ exists using contractions. As in Proposition 5.48, we compute the sets $\Gamma^{\text{even}} \cap \big\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| \leq 5\big\}$ and $\Gamma^{\text{odd}} \cap \big\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| \leq 5\big\}$ using Algorithm 5.2. By Proposition 5.48, we just need to check the case $|\mathrm{supp}^+(\mathbf{s})| = 5$.

Definition 6.4. We define the sets $\Gamma_5^{\text{even}} := \Gamma^{\text{even}} \cap \big\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| = 5\big\}$ and $\Gamma_5^{\text{odd}} := \Gamma^{\text{odd}} \cap \big\{\mathbf{s} \in H^{\Xi} : |\mathrm{supp}^+(\mathbf{s})| = 5\big\}$.

Proposition 6.5. *We have* $|\Gamma_5^{\text{even}}| = 1283$ *and* $|\Gamma_5^{\text{odd}}| = 1265$.

Proof. This is verified by computer; see [10], specifically the file `chapter06.ipynb`. $\square$

Corollary 6.6. *Let* $d \geq 42$. *Let* $\mathbf{w} \in \mathbb{Z}^{V_d}$ *be a valid outcome of degree* d *with* $|\mathrm{supp}^+(\mathbf{w})| = 5$. *Then,* $\mathrm{contr}_d(\mathrm{sign}(\mathbf{w})) \in \Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}$.

Proof. This follows from Corollary 5.47. $\square$

For each $\mathbf{s} \in \Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}$, we want to show that any valid outcome $\mathbf{w} \in \mathbb{Z}^{V_d}$ that maps to $\mathbf{s}$ under $\mathrm{contr}_d \circ \mathrm{sign}$ is the initial configuration. Note that $|\Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}| = 2318$; so we have to check 2318 cases. To make life easier when checking these cases, we simplify the index set Ξ.

Definition 6.7. Define the index set

$$\Xi' := \{0,1,2,3\}^2 \sqcup \{0,1,2,3\}^2 \sqcup \{0,1,2,3\}^2 \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\} \sqcup \{0,1,2,3\}$$

with the map $\chi : H^{\Xi} \to H^{\Xi'}$ defined by $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) \mapsto (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \underbrace{\mathbf{d}+\mathbf{e}}_{:=\mathbf{f}})$. The new coordinate system $H^{\Xi'}$ is depicted below.

$$\begin{array}{ccccccccccccccccc}
y_{0,3} \\
y_{0,2} & y_{1,3} \\
y_{0,1} & y_{1,2} & y_{2,3} \\
y_{0,0} & y_{1,1} & y_{2,2} & y_{3,3} \\
c_0 & y_{1,0} & y_{2,1} & y_{3,2} & f_0 \\
c_0 & c_1 & y_{2,0} & y_{3,1} & f_1 & f_0 \\
c_0 & c_1 & c_2 & y_{3,0} & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & * & f_3 & f_2 & f_1 & f_0 \\
c_0 & c_1 & c_2 & c_3 & * & * & * & * & * & f_3 & f_2 & f_1 & f_0 \\
x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} & b_3 & b_3 & b_3 & b_3 & b_3 & b_3 & z_{0,3} & z_{1,3} & z_{2,3} & z_{3,3} \\
x_{0,2} & x_{1,2} & x_{2,2} & x_{3,2} & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & z_{0,2} & z_{1,2} & z_{2,2} & z_{3,2} \\
x_{0,1} & x_{1,1} & x_{2,1} & x_{3,1} & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & z_{0,1} & z_{1,1} & z_{2,1} & z_{3,1} \\
x_{0,0} & x_{1,0} & x_{2,0} & x_{3,0} & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & z_{0,0} & z_{1,0} & z_{2,0} & z_{3,0}
\end{array}$$

Figure 6.2: This figure shows the simplified coordinate system $H^{\Xi'}$.

Definition 6.8. We define $\mathrm{contr}'_d := \chi \circ \mathrm{contr}_d$.

Definition 6.9. We define $\Lambda := \chi(\Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}) \subset H^{\Xi'}$.

Proposition 6.10. *Let $\mathbf{s}' \in \Lambda$. Then, $\mathbf{s}'$ has positive support of size five or four.*

Proof. Let $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) \in \Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}$. Since $\mathbf{d}, \mathbf{e}$ are non-negative, we see from the definition of χ that $\mathbf{f} \in \{0, 1\}$ holds. Hence, $\mathbf{s}'$ has positive support of size five or four. $\square$

Proposition 6.11. *Λ contains 2290 elements.*

Proof. This is verified by computer; see [10] file `chapter06.ipynb`. $\square$

6.4 Proof $d \geq 42$ Continued: $\mathbf{s}' \in \Lambda$ with Positive Support Size Four

Corollary 6.12. *From Proposition 6.10, we see that $\mathbf{s}' \in \Lambda$ has positive support size four if and only if there exist $i, j \in \{0,1,2,3\}$ such that $\mathbf{s} = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$ satisfies $d_i = 1$ and $e_j = 1$.*

Corollary 6.13. *Let $\mathbf{s} \in \Gamma_5^{\text{even}} \cup \Gamma_5^{\text{odd}}$. The element $\mathbf{s}$ maps to some $\mathbf{s}' \in \Lambda$ with positive support size four under χ if and only if $\operatorname{supp}(\mathbf{s}) = \{x_{0,0}, x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}$.*

Proof. This is easily verified by computer. □

We exclude this one case from the 2290 cases in Λ with the following proposition.

Proposition 6.14. *Let $d \geq 42$ and $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a weakly valid outcome. Then, we have*

$$\operatorname{supp}^+(\operatorname{contr}_d(\operatorname{sign}(\mathbf{w}))) \neq \{x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}.$$

Proof. Assume that $\operatorname{supp}^+(\operatorname{contr}_d(\operatorname{sign}(\mathbf{w}))) = \{x_{0,3}, x_{1,1}, x_{3,0}, d_0, e_0\}$. Then, $\mathbf{w}$ is an outcome with support $\operatorname{supp}(\mathbf{w}) = \{(0,0), (0,3), (1,1), (3,0), (i, d-i), (j, d-j)\}$ for some even $i \in \{4, \dots, d-4\}$ and odd $j \in \{5, \dots, d-4\}$. Let $\mathbf{u} \in \mathbb{Z}^{V_3}$ be an outcome with $u_{0,0} = -1$, $u_{1,1} = 3$, $u_{0,3} = u_{3,0} = 1$, and $u_{i,j} = 0$ for everything else. It has support in $\operatorname{supp}(\mathbf{u}) = \{(0,0), (0,3), (1,1), (3,0)\} \subset \operatorname{supp}(\mathbf{w})$. Define the outcome $\mathbf{v} := \mathbf{w} + w_{0,0}\mathbf{u}$. Then, $v_{0,0} = 0$ and $\mathbf{v} \neq \mathbf{0}$. However, if we apply the Invertibility Criterion with $\lambda = \mathbf{1}$ on $\mathbf{v}$, we see that $\mathbf{v}$ is zero. This is a contradiction. □

6.5 Proof $d \geq 42$ Continued: $\mathbf{s}' \in \Lambda$ with Positive Support Size Five

It remains to show the other 2289 cases of $\mathbf{s}' \in \Lambda$ using *relative coordinates*.

Relative Coordinates and the Invertibility Criterion

Definition 6.15. Let $d \geq 42$. Let M be a sentinel value with no further significance other than to encode integers from $4, \dots, d-7$. Define the maps

$$\begin{aligned}
\operatorname{relcoord} : \{0, \dots, d\} &\to \{0,1,2,3,d-6,d-5,d-4,d-3,d-2,d-1,d,M\}, \\
x &\mapsto \begin{cases} x & \text{if } x \in \{0,1,2,3,d-6,d-5,d-4,d-3,d-2,d-1,d\}, \\ M & \text{if } x \in \{4, \dots, d-7\}, \end{cases} \\
\operatorname{relset} : \mathbb{Z}^{V_d} &\to 2^{\{0,\dots,3,M,d-6,\dots,d\} \times \{0,\dots,3,M,d-6,\dots,d\}}, \\
\mathbf{w} &\mapsto \{(\operatorname{relcoord}(i), \operatorname{relcoord}(j)) \mid (i,j) \in \operatorname{supp}(\mathbf{w})\}.
\end{aligned}$$

Proposition 6.16. *Let $d \geq 42$, $i, j \in \{0, 1, 2, 3\}$, and $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid outcome with positive support size five and degree d. Write $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f})$. Then, all the following conditions hold:*

(1) We have $(i, j) \in \mathrm{relset}(\mathbf{w})$ if and only if $x_{i,j} \neq 0$;

(2) We have $(i, d-3+j-i) \in \mathrm{relset}(\mathbf{w})$ if and only if $y_{i,j} \neq 0$;

(3) We have $(d-3+i-j, j) \in \mathrm{relset}(\mathbf{w})$ if and only if $z_{i,j} \neq 0$;

(4) We have $\mathrm{relset}(\mathbf{w}) \cap \{(M, i), (d-6, i), \ldots, (d-4-i, i)\} \neq \emptyset$ if and only if $b_i \neq 0$;

(5) We have $\mathrm{relset}(\mathbf{w}) \cap \{(i, M), (i, d-6), \ldots, (i, d-4-i)\} \neq \emptyset$ if and only if $c_i \neq 0$;

(6) We have $\mathrm{relset}(\mathbf{w}) \cap \{(M, d-4-i), \ldots, (M, d-6), (M, M), (d-6, M), \ldots, (d-4-i, M)\} \neq \emptyset$ if and only if $f_i \neq 0$.

Proof. Let $\mathbf{w}$ be some valid outcome with $x_{i,j} \neq 0$. Then, $w_{i,j} \neq 0$ with $i, j = 0, 1, 2, 3$. By definition of relcoord, $i \mapsto i$ and $j \mapsto j$. So $(i, j) \in \mathrm{relset}(\mathbf{w})$ since $(i, j) \in \mathrm{supp}(\mathbf{w})$.

Assume $y_{i,j} \neq 0$. Then, $w_{i,d-3+j-i} \neq 0$. By definition of relcoord, $i \mapsto i$ and $d-3+j-i \mapsto d-3+j-i$. So $(i, d-3+j-i) \in \mathrm{relset}(\mathbf{w})$ since $(i, d-3+j-i) \in \mathrm{supp}(\mathbf{w})$. The case $z_{i,j} \neq 0$ is similar.

Assume $b_i \neq 0$. Then, there must exist some nonzero $w_{k,i}$ for some $k = 4, \ldots, d-4-i$. Clearly, k maps to some element in $\{M, d-6, \ldots, d-4-i\}$. This shows the claim. The case for c_i is similar.

Assume $f_i \neq 0$. Then, there must exist some nonzero $w_{k,d-i-k}$ for some $k = 4, \ldots, d-4-i$. Clearly, k and $d-i-k$ map to some element in $\{M, d-6, \ldots, d-4-i\}$. ☐

Remark 6.17. Given $\mathbf{s}' \in H^{\Xi'}$, we compute all $\mathrm{relset}(\mathbf{w})$ for $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = \mathbf{s}'$ using Proposition 6.16; we view all the six conditions (1), (2), (3), (4), (5) and (6) in Proposition 6.16 as constraints on the relative support of $\mathbf{w}$.

For instance, if $\mathbf{s}' = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f})$ satisfies $b_0 \neq 0$, then we know that corresponding outcomes $\mathbf{w}$ satisfy $w_{i,j} \neq 0$ for some $(i, j) \in \{(M, 0), (d-6, 0), \ldots, (d-4, 0)\}$.

Relative coordinates help us to apply the Invertibility Criterion.

Example 6.18. Let $d \geq 42$. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be some valid configuration with support size six and $\mathrm{relset}(\mathbf{w}) = \{(0, 0), (0, d), (1, 3), (M, 2), (M, d-6), (d-5, M)\}$. Can such a configuration exist?

We see $\mathrm{supp}(\mathbf{w}) = \{(0, 0), (0, d), (1, 3)\} \cup \{(i, 2), (j, d-6)\} \cup \{(d-5, k)\}$ for $i, j, k = 4, \ldots, d-7$. When $i = j$, we apply the Invertibility Criterion with $\lambda = (3, 1, \ldots, 1, 2, 1, \ldots, 1)$. So, we assume $i \neq j$. Then, we use $\lambda = (3, 1, \ldots, 1)$. Hence, $\mathbf{w}$ cannot be an outcome.

```
d    X
d-1  .  .
d-2  .  .  .
d-3  .  .  .  .
d-4  .  .  .  .  .
d-5  .  .  .  .  .  .
d-6  .  .  .  .  .  X  X
M    .  .  .  .  .  .  .  .
M    ............................
M    ...............................
M    ..................................
M    .  .  .  .  .  .  .  .  .  .  .  .  .
M    .  .  .  .  .  .  .  .  .  .  .  .  .  X
M    .  .  .  .  .  .  .  .  .  .  .  .  .  X  .
3    .  X  .  .  .  .  .  .  .  .  .  .  .  .  .  .
2    .  .  .  .  X  X  X  X  X  X  X  X  .  .  .  .  .
1    .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
0    X  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .

     0  1  2  3  M  M  M  M  M  M  M  M d-6 d-5 d-4 d-3 d-2 d-1 d
```

Let us generalize this example to elements $\mathbf{s}' \in \Lambda$.

Proposition 6.19. *Let $\mathbf{s}' \in \Lambda$. All the following conditions hold: $|\mathrm{supp}^+(\mathbf{s}') \cap \{b_0, b_1, b_2, b_3\}| \leq 1$, $|\mathrm{supp}^+(\mathbf{s}') \cap \{c_0, c_1, c_2, c_3\}| \leq 1$, and $|\mathrm{supp}^+(\mathbf{s}') \cap \{f_0, f_1, f_2, f_3\}| \leq 1$. Moreover, this immediately implies $|\{(x, y) \in \mathrm{relset}(\mathbf{s}') : x = M\}| \leq 2$ and $|\{(x, y) \in \mathrm{relset}(\mathbf{s}') : y = M\}| \leq 2$.*

Proof. This is verified by computer, see [10]. □

Proposition 6.20. *Let $d \geq 42$. Let $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a valid configuration with positive support size five and $|\{(x, y) \in \mathrm{relset}(\mathbf{w}) : x = M\}| = 2$. Denote these elements by (M, x) and (M, y) for $x \neq y$. Write $(i, x), (i', y) \in \mathrm{supp}(\mathbf{w})$ with $i, i' = 4, \ldots, d-7$ for the elements that map to (M, x) or (M, y) under* relcoord*, respectively. If we successfully apply the Invertibility Criterion for the case $i = i'$ to show the contradiction $\mathbf{w} = \mathbf{0}$ with $\lambda = (\mathbf{a}, 1, \ldots, 1, 2, 1, \ldots, 1, \mathbf{b})$ for some $\mathbf{a} \in \mathbb{Z}^k_{\geq 1}, \mathbf{b} \in \mathbb{Z}^h_{\geq 1}$, and $1 \leq k, h \leq 4$, then we can also apply the Invertibility Criterion for the case $i \neq i'$ with $\lambda' = (\mathbf{a}, 1, \ldots, 1, \mathbf{b})$ to show the same contradiction.*

Proof. Assume $i \neq i'$. Then, we just apply Proposition 4.18 as long as $S'_{l'} \in \{0, \lambda'_l\}$ is satisfied for all l'. These conditions are satisfied because the sets S_l induced by λ satisfy $S_l \in \{0, \lambda_l\}$ by assumption. □

Corollary 6.21. *In the divide step of the Invertibility Criterion, we may specialize to the case $i = i'$ when $(i, x), (i', y) \in \mathrm{supp}(\mathbf{w})$ for $i, i' = 4, \ldots, d-7$ and $x \neq y$ occurs*

Similar statements hold for the case $|\{(x, y) \in \mathrm{relset}(\mathbf{w}) : y = M\}| = 2$.

Example 6.22. Returning to Example 6.18, we see that it suffices to consider the case $i = j$. The case $i \neq j$ then follows from the Corollary 6.21.

Corollary 6.23. *Let $d \geq 42$. Let $S = \{(i, x), (i, y), (i + 1, z)\} \subset V_d$. If* $\mathrm{relcoord}(x) \in \{0, 1, 2, 3\}$, $\mathrm{relcoord}(y) \in \{d - 6, \ldots, d\}$ *and* $\mathrm{relcoord}(z) \neq M$*, then $A^{(d)}_{E,S}$ is invertible.*

Proof. Use Proposition 4.21; we only need to make sure that $x + y \neq 2z + 1$. Assume $x + y = 2z + 1$. Then, $z = \frac{x+y-1}{2}$. The smallest value for z is $\frac{d-6-1}{2} = 17.5$, and the largest value is $\frac{d+3-1}{2} = 22$ for $d = 42$. Hence, z does map to M under relative coordinates for all $d = 42$ and for $d > 42$ we well. By assumption, $\mathrm{relcoord}(z) \neq M$. This is a contradiction. □

We have the following rules for computing the midpoint in relative coordinates.

Proposition 6.24. *Let $d \geq 42$. Let $a, b = 0, \ldots, d$ with $b \geq a$ Then, the following hold:*

(1) If $b = 0, 1, 2, 3$ or $a = d - 6, \ldots, d$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \neq M$.

(2) Let $b = 4, \ldots, d - 7$.

   a) If $a = 0, 1$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \in \{2, 3, M\}$.

   b) If $a = 2, 3$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \in \{3, M\}$.

(3) Let $a = 4, \ldots, d - 7$.

   a) If $b = d - 6, d - 5$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) = M$.

   b) If $b = d - 4, d - 3$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \in \{M, d - 6\}$.

   c) If $b = d - 2, d - 1$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \in \{M, d - 6, d - 5\}$.

   d) If $b = d$, then $\mathrm{relcoord}(\frac{a+b-1}{2}) \in \{M, d - 6, d - 5, d - 4\}$.

Proof. Let $d \geq 42$.

(1) Let $a = b = 3$. Then $\frac{6-1}{2} = 2.5$. Thus, for all $a \leq b \leq 3$ we have $\frac{a+b-1}{2} \leq 2.5$. Therefore, $\mathrm{relcoord}(\frac{a+b-1}{2}) \neq M$. Let $a = b = d - 6$. Then $\frac{a+b-1}{2} > d - 7$. Thus, for all $d - 6 \leq a \leq b \leq d$ we have $\mathrm{relcoord}(\frac{a+b-1}{2}) \neq M$.

(2) Let $b = 4, \ldots, d - 7$.

   a) For $a = 0$ and $b = 4$ we obtain $\frac{a+b-1}{2} > 1.5$. So, $\mathrm{relcoord}(\frac{a+b-1}{2}) \notin \{0, 1\}$. For $a = 1$ and $b = d - 7$, we see that $\frac{a+b-1}{2} = \frac{d-7}{2}$, which maps to M under relative coordinates for $d \geq 42$. So the midpoint maps to values between two and M under relcoord, i.e. $\{2, 3, M\}$. This shows the claim.

   b) The proof of case $a = 2, 3$ is similar to the previous proof.

(3) Let $a = 4, \ldots, d - 7$.

a) Let $b = d-6, d-5$. Then, $\frac{a+b-1}{2} \geq \frac{d-3}{2} \geq 19.5$ for all $d \geq 42$. This maps to M under relative coordinates. Moreover, $\frac{a+b-1}{2} \leq \frac{d-7+d-5-1}{2} = \frac{2d-13}{2} = d-6.5$. Thus, $\mathrm{relcoord}(\frac{a+b-1}{2}) = M$.

b) The proof of cases $b = d-4, d-3, \ldots, d$ are similar to the previous proof.

$\square$

Let us apply the Invertibility Criterion to all the 2289 cases. First, we compute the set $R(\mathbf{s}') := \{\mathrm{relset}(\mathbf{w}) \mid \mathbf{w} \in \mathbb{Z}^{V_d}$ such that $\mathrm{contr}'(\mathrm{sign}(\mathbf{w})) = \mathbf{s}'\}$ for each $\mathbf{s}' \in \Lambda$. Here is a simple depth first search algorithm to compute $R(\mathbf{s}')$.

Algorithm 6.1 Compute $R(\mathbf{s}')$

Require: $\mathbf{s}' \in H^{\Xi}$
Ensure: $R(\mathbf{s}')$

1: **function** COMPUTE_RELSETS($\mathbf{s}'$)
2:     `constraints` $\leftarrow$ build constraints as in Proposition 6.16
3:     `accu` $\leftarrow$ `list()`, `res` $\leftarrow$ `list()`
4:     **function** DFS(`i`)
5:         **if** `i` $\geq$ |`constraints`| **then**
6:             `res.append(accu.copy())`
7:             **return**
8:         **end if**
9:         **for** $x \in$ `constraints[i]` **do**
10:             `accu.append(`x`)`
11:             `dfs(i + 1)`
12:             `accu.pop()`
13:         **end for**
14:     **end function**
15:     `dfs(0)`
16:     **return** `res`
17: **end function**

Proof of correctness. Let $\mathbf{s}' = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f})$. By Proposition 6.16, $\mathbf{m} \in R(\mathbf{s}')$ if and only if $\mathbf{m} \cap C \neq \emptyset$ for all $C \in \{$`make_rel_constraints(t)` $\mid t \in \{x_{i,j}, y_{i,j}, z_{i,j}, b_i, c_i, f_i\}, t \neq 0\}$. It just remains to show that $\mathbf{m} \in$ `res` if and only if $\mathbf{m}$ satisfies all of these constraints. First note that $x \in \mathbf{m}$ if and only if $x \in$ constraints[i] for some i by line eleven. Since $\mathbf{m} \in$ `res` if and only if $\mathbf{m}$ satisfies all constraints by line six, the algorithm is correct. $\square$

For all $\mathbf{m} \in R(\mathbf{s}')$ we apply the Invertibility Criterion as in Example 6.18. If each application of the Invertibility Criterion to $\mathbf{m} \in R(\mathbf{s}')$ is successful, we exclude $\mathbf{s}'$. After this procedure, we are left with 1107 cases of $\mathbf{s}' \in \Lambda$, where the Invertibility Criterion was inconclusive. The implementation details are public in [10] under the file name `chapter06.ipynb`.

Symmetry

We use symmetry to reduce the number of cases further. Let $\mathbf{s}' \in \Lambda$ be one of the 1107 cases. Apply the following symmetries to $\mathbf{s}'$ similar to Section 4.3:

$$(12) \cdot (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f}) := ((x_{j,i})_{i,j=0}^{3}, (y_{j,i})_{i,j=0}^{3}, (z_{j,i})_{i,j=0}^{3}, \mathbf{c}, \mathbf{b}, \mathbf{f}),$$
$$(13) \cdot (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{f}) := ((z_{3-i,j})_{i,j=0}^{3}, (y_{3-j,3-i})_{i,j=0}^{3}, (x_{3-i,j})_{i,j=0}^{3}, \mathbf{f}, \mathbf{c}, \mathbf{b}).$$

Define the group S_3 generated by these two symmetries. For all weakly valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$, the actions $\sigma \in S_3$ satisfy $\sigma \cdot \mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = \mathrm{contr}'_d(\mathrm{sign}(\sigma \cdot \mathbf{w}))$.

Proposition 6.25. *Let $d \in \mathbb{N}$ and $\mathbf{s}' \in H^{\Xi'}$. If there are no weakly valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$ of degree d with $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = \sigma\mathbf{s}'$ for some $\sigma \in S_3 \setminus \{(1)\}$, then there are no weakly valid outcomes $\mathbf{v} \in \mathbb{Z}^{V_d}$ of degree d with $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{v})) = \mathbf{s}'$.*

Proof by Contraposition. Let $\sigma \in S_3 \setminus \{(1)\}$. Assume there is some valid outcome $\mathbf{v} \in \mathbb{Z}^{V_d}$ of degree d such that $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{v})) = \mathbf{s}'$. Define $\mathbf{w} := \sigma\mathbf{v}$. Then, we have $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = \sigma \cdot \mathrm{contr}'_d(\mathrm{sign}(\mathbf{v})) = \sigma\mathbf{s}'$. Note that $\mathbf{w}$ is weakly valid since the symmetry group S_3 preserves the weak validity of outcomes. $\square$

We take the 1107 cases of $\mathbf{s}' \in \Lambda$, apply the symmetry (12) to each case, and use the Invertibility Criterion. Then, the inconclusive cases are checked with the symmetry (13). We are left with 349 cases. Next, we consider the equivalence relation $\sim$ defined by $w \sim v \iff w = (12)v$ or $w = (13)v$. We compute the equivalence classes, and see there are 348 cases left.

Hexagon Criterion

Let $d \geq 42$. We want to apply the Hexagon Criterion (Proposition 6.2) with $\ell_1 = \ell_2 = 7, d' = 6$ to the remaining 348 cases. First, we check that the requirements of the Hexagon Criterion are met: $d' + \ell_1 + \ell_2 = 20 \leq 42 = d$. Next, we find all contracted configurations $\mathbf{s}' \in \Lambda$ from the 348 cases whose support lies outside the hexagon spanned by d', ℓ_1 and ℓ_2.

Proposition 6.26. *Let $d \geq 42$, $\mathbf{s}' \in H^{\Xi'}$, and $\mathbf{w} \in \mathbb{Z}^{V_d}$ be a configuration with $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = \mathbf{s}'$. If $\mathrm{supp}^+(\mathbf{s}') \cap \{\mathbf{b}, \mathbf{c}, \mathbf{f}\} = \emptyset$, then $\mathrm{supp}(\mathbf{w})$ lies outside the hexagon spanned by d', ℓ_1 and ℓ_2.*

Proof. This follows immediately from the definition of the hexagon spanned by d', ℓ_1 and ℓ_2, and the definition of the map contr'_d. $\square$

We compute that 325 cases satisfy the proposition above, so we apply the Hexagon Criterion. Let $\mathbf{s}' \in \Lambda$ be one of these cases. By the Hexagon Criterion, valid outcomes $\mathbf{w}$ with $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = \mathbf{s}'$ have degree at most $d' = 20$; however, by assumption $\deg(\mathbf{w}) = d \geq 42$.

It remains to check 23 cases; we apply the Hexagon Criterion one by one.

Example 6.27. The support $(y_{0,3}, z_{2,0}, z_{2,2}, z_{3,1}, c_1)$ belongs to one of the 23 cases.

```
y03 z20 z22 z31 c1

*
y  y
y  y  y
y  y  y  y
c  y  y  y  d
c  *  y  y  d  d
c  *  c  y  d  d  d
c  *  c  c  d  d  d  d
c  *  c  c     d  d  d  d
c  *  c  c        d  d  d  d
c  *  c  c           d  d  d  d
x  x  x  x  b  b  b  b  z  z  z  z
x  x  x  x  b  b  b  b  b  z  z  *  z
x  x  x  x  b  b  b  b  b  b  z  z  z  *
*  x  x  x  b  b  b  b  b  b  b  z  z  *  z
```

To apply the Hexagon Criterion to configurations $\mathbf{w}$ with support as above, we need to know where the nonzero entry of $\mathbf{w}$ in the c_1-column is roughly located; here the c_1-column denotes the entries $w_{1,k}$ with $k = 4, \dots, d-1-4$. There exists only one nonzero entry in the c_1-column since $\mathbf{w}$ has positive support size five. Let us denote the nonzero entry in the c_1-column by $(a, b) \in V_d$ where $a = 1$. There are two cases:

(1) Let $a + b \leq \text{floor}(\frac{d}{3})$. Set $\ell_1 = \ell_2 = d' = \text{floor}(\frac{d}{3}) \geq \frac{42}{3} = 14$. We see that the other non-entries $y_{0,3}, z_{2.0}, z_{2,2}, z_{3,1}$ lie outside the hexagon.

(2) Let $b \geq \text{floor}(\frac{d}{3})$. Set $d' = 6$, $\ell_2 = 7$ and $\ell_1 = d + 1 - \text{floor}(\frac{d}{3}) \geq 43 - 14 = 29$. We see that $d' + \ell_1 + \ell_2 \leq d$ since $d \geq 42$. We easily see that the non-entries $y_{0,3}, z_{2.0}, z_{2,2}, z_{3,1}$ lie outside the hexagon. It remains to check that (a, b) also lies outside the hexagon. We have $d - \ell_1 = d - (d+1) + \text{floor}(\frac{d}{3}) = \text{floor}(\frac{d}{3}) - 1 < \text{floor}(\frac{d}{3}) \leq b$. Thus, (a, b) lies outside the hexagon.

Hence, we apply the Hexagon Criterion showing that the degree of $\mathbf{w}$ is at most $d' \leq \text{floor}(\frac{d}{3})$. However, we assume that $\mathbf{w}$ is of degree d.

We generalize Example 6.27 to apply the Hexagon Criterion to the remaining 23 cases. However, one case, $\{y_{0,3}, z_{3,0}, b_1, c_1, d_1\}$, cannot be ruled out by the Hexagon Criterion. We will handle this case separately and focus on the remaining 22 cases for now.

Note that all 22 cases satisfy $|\text{supp}^+(\mathbf{s}') \cap \{b_0, b_1, c_0, c_1, f_0, f_1\}| = 1$ as well as $|\text{supp}^+(\mathbf{s}') \cap \{b_2, c_2, f_2, b_3, c_3, f_3\}| = 0$. Therefore, for all valid outcomes $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $\text{contr}'_d(\text{sign}(\mathbf{w})) =$

$\mathbf{s}'$ we have $\mathrm{supp}(\mathbf{w}) \setminus \{(a,b)\} \subset V_6 \cup \{(i,j) \in V_d \mid j > d-7\} \cup \{(i,j) \in V_d \mid i > d-7\}$ for some vertex $(a,b) \in V_d$ with $a = 0, 1$ or $b = 0, 1$ or $a + b \geq d - 1$.

Proposition 6.28. *We either have $a + b \leq \mathrm{floor}(\frac{d}{3})$, or $a \geq \mathrm{floor}(\frac{d}{3})$, or $b \geq \mathrm{floor}(\frac{d}{3})$.*

Proof. If $a + b \leq \mathrm{floor}(\frac{d}{3})$, then the claim holds. If $a = 0$ with $a + b > \mathrm{floor}(\frac{d}{3})$, then $b > \mathrm{floor}(\frac{d}{3})$; let $a = 1$ with $a + b > \mathrm{floor}(\frac{d}{3})$; then, $b \geq \mathrm{floor}(\frac{d}{3})$ holds. If $b = 0$ with $a + b > \mathrm{floor}(\frac{d}{3})$, then $a > \mathrm{floor}(\frac{d}{3})$; let $b = 1$ with $a + b > \mathrm{floor}(\frac{d}{3})$; then, $a \geq \mathrm{floor}(\frac{d}{3})$ holds. If $a + b \geq d - 1$, then we have $a > \mathrm{floor}(\frac{d}{3})$ or $b > \mathrm{floor}(\frac{d}{3})$ since $d - 1 > 2\mathrm{floor}(\frac{d}{3})$. ☐

Proposition 6.29. *All the 22 cases are impossible.*

Proof. We see that all the 22 cases satisfy Proposition 6.28. When the first condition holds, we let $d' = \ell_1 = \ell_2 = \mathrm{floor}(\frac{d}{3}) \geq 14 > 6$. When the second condition holds, we choose $d' = 6, \ell_1 = 7$ and $\ell_2 = d + 1 - \mathrm{floor}(\frac{d}{3})$. Note that $d' + \ell_1 + \ell_2 \leq 42$ since $d \geq 42$. When the third condition holds, we choose $d' = 6, \ell_2 = 7$ and $\ell_1 = d + 1 - \mathrm{floor}(\frac{d}{3})$. Then, we apply the Hexagon Criterion with these choices of d', ℓ_1 and ℓ_2 to each of the 22 cases. ☐

The Final Case

We are left with one case $\{y_{0,3}, z_{3,0}, b_1, c_1, d_1\}$.

Proposition 6.30. *Let $d \geq 42$. There is no weakly valid outcome $\mathbf{w} \in \mathbb{Z}^{V_d}$ with*

$$\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = \{x_{0,0}, y_{0,3}, z_{3,0}, b_1, c_1, d_1\}.$$

Proof by Contradiction. We assume that $d \geq 42$. Assume that a weakly valid outcome $\mathbf{w} \in \mathbb{Z}^{V_d}$ with $\mathrm{contr}'_d(\mathrm{sign}(\mathbf{w})) = \{x_{0,0}, y_{0,3}, z_{3,0}, b_1, c_1, d_1\}$ exists. Then, the support of $\mathbf{w}$ reads $S = \{(0,0), (d,0), (0,d), (i,1), (1,j), (k, d-1-k)\}$ for some integers i, j, k.

Let us define $e \in \mathbb{Z}$ such that we can write $d = 2e + 1$. If $j \neq e$, we use Proposition 4.21 to show that such a $\mathbf{w}$ cannot exist. So, we assume that $j = e$. By symmetry $(12) \in S_3$ and $(13) \in S_3$, we conclude that $i = e$ and $k = e$. This shows that the support of $\mathbf{w}$ reads $S = \{(0,0), (d,0), (0,d), (e,1), (1,e), (e, d-1-e)\} = \{(0,0), (d,0), (0,d), (e,1), (1,e), (e,e)\}$. We apply the Invertibility Criterion with $E \coloneqq \{0, 1, 3, e, d-1, d\}$ which leads to

$$A^{(d)}_{E,S} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 1 & 0 \\ \binom{d}{3} & 0 & 0 & 0 & \binom{e}{2} & 0 \\ \binom{d}{e} & 0 & 0 & 1 & e & 1 \\ d & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This pairing matrix has determinant $\frac{(2e+1)(e+1)e}{6} \neq 0$. ☐

We proved that for all valid outcomes $\mathbf{w}$ with $|\mathrm{supp}^+(\mathbf{w})| = 5$ we have $\deg(\mathbf{w}) \leq 7$.

Chapter 7

Valid Outcomes of Positive Support Size Six

We want to prove that all valid outcomes $\mathbf{w}$ with $|\text{supp}^+(\mathbf{w})| = 6$ have $\deg(\mathbf{w}) \leq 9$. This thesis makes a contribution towards a proof by reducing the number of cases to check. In theory, we could use the same approach as in the previous chapters; specifically, we begin by computing $\Gamma^{\text{even}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\text{supp}^+(\mathbf{s})| = 6\right\}$ and $\Gamma^{\text{odd}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\text{supp}^+(\mathbf{s})| = 6\right\}$, albeit with a slight modification to these sets. While we previously considered contractions of size four, we now consider contractions of size five, i.e. we work with contraction variables as depicted below.

$$
\begin{array}{cccccccccccccccccc}
y_{0,4} \\
y_{0,3} & y_{1,4} \\
y_{0,2} & y_{1,3} & y_{2,4} \\
y_{0,1} & y_{1,2} & y_{2,3} & y_{3,4} \\
y_{0,0} & y_{1,1} & y_{2,2} & y_{3,3} & y_{4,4} \\
c_0 & y_{1,0} & y_{2,1} & y_{3,2} & y_{4,3} & d_0 \\
c_0 & c_1 & y_{2,0} & y_{3,1} & y_{4,2} & d_1 & e_0 \\
c_0 & c_1 & c_2 & y_{3,0} & y_{4,1} & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & y_{4,0} & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & d_4 & e_3 & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & * & e_4 & d_3 & e_2 & d_1 & e_0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & * & * & d_4 & e_3 & d_2 & e_1 & d_0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & * & * & * & e_4 & d_3 & e_2 & d_1 & e_0 \\
x_{0,4} & x_{1,4} & x_{2,4} & x_{3,4} & x_{4,4} & b_4 & b_4 & b_4 & b_4 & z_{0,4} & z_{1,4} & z_{2,4} & z_{3,4} & z_{4,4} \\
x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} & x_{4,3} & b_3 & b_3 & b_3 & b_3 & b_3 & z_{0,3} & z_{1,3} & z_{2,3} & z_{3,3} & z_{4,3} \\
x_{0,2} & x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & z_{0,2} & z_{1,2} & z_{2,2} & z_{3,2} & z_{4,2} \\
x_{0,1} & x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & z_{0,1} & z_{1,1} & z_{2,1} & z_{3,1} & z_{4,1} \\
x_{0,0} & x_{1,0} & x_{2,0} & x_{3,0} & x_{4,0} & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & b_0 & z_{0,0} & z_{1,0} & z_{2,0} & z_{3,0} & z_{4,0}
\end{array}
$$

From this point forward, when we refer to $\mathbf{s} \in H^{\Xi}$, we will mean contractions of size five. We increase the contraction size to gain more information about potential supports, which we hope will prove useful when handling cases manually.

Definition 7.1. Define Γ_6^{even} and Γ_6^{odd} analogously to Definition 5.45 with

$$\Phi_1 = \{\text{col}(i), \text{row}(i), \text{diag}(i), \text{diag}(d-i)\}_{i=0}^{4} \text{ and } \Phi_2 = \{\text{col}(d-i), \text{row}(d-i)\}_{i=0}^{4}.$$

Proposition 7.2. *We have* $|\Gamma_6^{\text{even}}| = 150032$ *and* $|\Gamma_6^{\text{odd}}| = 154177$.

Proof. This is verified by a computer program, which is available on GitHub [10] under tile file `chapter07_intro.ipynb`. □

The number of cases to check has increased by a factor of hundred compared to the previous chapters. If we were able to reduce the cases, we could apply the same techniques as in the previous chapters. We will spend the remaining chapter to reduce the number of cases to check to around 12,000 cases in the hope to make the proof computationally feasible. Due to time constraints of this thesis, we were not able to attempt a complete proof.

7.1 Finding Fixed-Contractable Pascal Forms

To reduce the number of cases, we generate *fixed-contractable* linear combinations of hyperfield Pascal forms.

Definition 7.3. Let $d \in \mathbb{N}_{\geq 15}$, $k = 0, \ldots, 4$, and p be a hyperfield linear form on H^{V_d}. We say p is *contractable* for d on b_k if we can write

$$p = \sum_{(i,j) \in V_d \setminus \{(5,k), \ldots, (d-k-5,k)\}} \lambda_{i,j} x_{i,j} + \lambda b_k \quad \text{for} \quad b_k \coloneqq x_{5,k} + \cdots + x_{d-5-k,k}$$

for $\lambda_{i,j}, \lambda \in H$. In other words, we can write $p = \hat{p}$ for some linear form $\hat{p} \in H[\mathbf{x}, b_k]$. Similarly, we define p is *contractable* on c_k, d_k, and e_k if we can write $p = \hat{p}$ for some linear form $\hat{p} \in H[\mathbf{x}, c_k]$, $\hat{p} \in H[\mathbf{x}, d_k]$, and $\hat{p} \in H[\mathbf{x}, e_k]$, respectively.

Remark 7.4. Clearly, p is contractable for d if and only if it is contractable for d on b_k, c_k, d_k, and e_k for all $k = 0, \ldots, 4$.

Definition 7.5. Let $d \in \mathbb{N}_{\geq 15}, i \in \{0, \ldots, 4, d-4, \ldots, d\}$, and $p = \sum \lambda_{i,j} x_{i,j}$ be a hyperfield linear form on H^{V_d}. We define the i-th b-row of p as $p_{b_i} \coloneqq \begin{bmatrix} \lambda_{5,i} & \ldots & \lambda_{d-i-5,i} \end{bmatrix} \in H^{d-i-9}$. Similarly, we define p_{c_i}, p_{d_i} and p_{e_i} to denote the i-th b-column, d-diagonal and e-diagonal of p, respectively.

Proposition 7.6. *Let $d \in \mathbb{N}_{\geq 15}, i \in \{0, \ldots, 4\}$, and T be a formal linear combination of*

$$\{\operatorname{sign}(\operatorname{row}(j)), \operatorname{sign}(\operatorname{col}(j)), \operatorname{sign}(\operatorname{diag}(j)) \mid j \in \{0, \ldots, 4\} \cup \{t-4, \ldots, t\}\}.$$

Then, the following statements hold for all realizations p_d of T:

(1) The c-column of $(p_d)_{c_i}$ only depends on $\{\operatorname{sign}(\operatorname{row}(k)), \operatorname{sign}(\operatorname{diag}(k))\}_{k=0}^4$.

(2) The b-row of $(p_d)_{b_i}$ only depends on $\{\operatorname{sign}(\operatorname{col}(k)), \operatorname{sign}(\operatorname{diag}(d-k))\}_{k=0}^4$.

(3) The d-diagonal of $(p_d)_{d_i}$ only depends on $\{\operatorname{sign}(\operatorname{row}(d-k)), \operatorname{sign}(\operatorname{col}(d-k))\}_{k=0}^4$. A similar statement holds for e-diagonals.

Proof. This follows immediately from the definition of row, col, and diag. □

Proposition 7.7. *Let $d \in \mathbb{N}_{\geq 15}$ and $i = 0, \ldots 4$. The following statements hold:*

(1) Let $T \in \{\operatorname{sign}(\operatorname{row}(k)), \operatorname{sign}(\operatorname{diag}(k))\}_{k=0,\ldots,4}$. The c_i-column of p_d is a constant vector.

(2) Let $T \in \{\operatorname{sign}(\operatorname{col}(k)), \operatorname{sign}(\operatorname{diag}(d-k))\}_{k=0,\ldots,4}$. The b_i-row of p_d is a constant vector.

(3) Let $T \in \{\operatorname{sign}(\operatorname{row}(d-k)), \operatorname{sign}(\operatorname{col}(d-k))\}_{k=0,\ldots,4}$. The d_i-diagonal of p_d is a constant vector; similarly for the e_i-diagonal.

Proof. This also follows easily from the definition of row, col, and diag. □

Let us investigate how realizations of formal form change when increasing the degree. We fix the following notations:

- Let $T = \sum_{i=0}^4 \lambda_i \operatorname{row}(i)$ or $T = \sum_{i=0}^4 \lambda_{t-i} \operatorname{row}(t-i)$;

- Write the realizations of T as $p \coloneqq \sum p_{i,j}x_{i,j} \coloneqq p_d$ and $q \coloneqq \sum q_{i,j}x_{i,j} \coloneqq p_{d+1}$ for some degree $t = d$;
- Define $r \coloneqq \max\{i \mid \lambda_i \neq 0\}$ if $\max\{i \mid \lambda_i \neq 0\} \leq 4$, otherwise $r \coloneqq \min\{i \mid \lambda_i \neq 0\}$;
- Write $\mathrm{row}(r) = \sum r_{i,j}x_{i,j}$.

Lemma 7.8. *Let $d \in \mathbb{N}_{\geq 9}$. Then, $q_{i,j} = p_{i,j}$ holds for all $(i,j) \in V_d$.*

Proof. Let $d \in \mathbb{N}$, $\ell = 0,\dots,4,t-4,\dots,t$ and $T_\ell = \mathrm{row}(\ell)$. We denote its realizations in d and $d+1$ by $p_d^{(\ell)} = \sum p_{i,j}^{(\ell)}x_{i,j}$ and $p_{d+1}^{(\ell)} = \sum q_{i,j}^{(\ell)}x_{i,j}$, respectively. By Proposition 3.19, we see that $p_{i,j}^{(\ell)} = q_{i,j}^{(\ell)}$ for all $(i,j) \in V_d$. Next, assume $T = \sum_{i=0}^{4} \lambda_i \mathrm{row}(i) + \sum_{i=0}^{4} \lambda_{t-i}\mathrm{row}(t-i)$. Then, $p_{i,j} = \sum_\ell \lambda_\ell p_{i,j}^{(\ell)} = \sum_\ell \lambda_\ell q_{i,j}^{(\ell)} = q_{i,j}$ for all $(i,j) \in V_d$. $\square$

Example 7.9. Let $T = \text{row}(3) + \text{row}(2)$. We visualize p_8 and p_9:

```
                           -48
-28                        -28 20
-14 14                     -14 14 -6
 -5  9 -5                   -5  9 -5  1
  .  5 -4  1                 .  5 -4  1  .
  2  2 -3  1  .              2  2 -3  1  .  .
  2  . -2  1  .  .           2  . -2  1  .  .  .
  1 -1 -1  1  .  .  .        1 -1 -1  1  .  .  .  .
  . -1  .  1  .  .  .  .     . -1  .  1  .  .  .  .  .
  .  .  1  1  .  .  .  .  .  .  .  1  1  .  .  .  .  .  .
```

Lemma 7.10. *Let $d \in \mathbb{N}_{\geq 9}$. If there exists $k \in \{0, \dots, r\}$ such that $\text{sign}(r_{i,d-i}) = \text{sign}(p_{i,d-i})$ for all $i = k, \dots, r$, then $\text{sign}(q_{i,d+1-i}) = \text{sign}(p_{i,d-i})$ for all $i = k, \dots, r$.*

Proof. Without loss of generality, we assume that $\lambda_r > 0$. First, we see that $q_{r,\cdot} := (q_{r,j})_{j=0}^{d-r} = \lambda_r \cdot \mathbf{1}$ and $q_{i,\cdot} = \mathbf{0}$ for all $i > r$. By the Pascal property, we have $q_{r-1,d+1-(r-1)} = q_{r-1,d-(r-1)} - q_{r,d+1-r} = q_{r-1,d-(r-1)} - \lambda_r$. This shows $q_{r-1,d+1-(r-1)} < q_{r-1,d-(r-1)} = p_{r-1,d-(r-1)} < 0$, where the last inequality follows from the assumption $\text{sign}(r_{i,d-i}) = \text{sign}(p_{i,d-i})$. Thus, we have $\text{sign}(q_{r-1,d+1-(r-1)}) = \text{sign}(q_{r-1,d-(r-1)}) = -1$.

Next, use the Pascal property $q_{r-2,d+1-(r-2)} = q_{r-2,d-(r-2)} - q_{r-1,d+1-(r-1)}$. Note $q_{r-2,d+1-(r-2)} > 0$ as $q_{r-2,d-(r-2)} > 0$ and $q_{r-1,d+1-(r-1)} < 0$. Hence, we have $\text{sign}(q_{r-2,d+1-(r-2)}) = \text{sign}(q_{r-2,d-(r-2)}) = 1$. We continue this argument for $r-3, r-4, \dots, k$. This shows that $\text{sign}(q_{i,d+1-i}) = \text{sign}(q_{i,d-i}) = \text{sign}(p_{i,d-i})$ for all $i = k, \dots, r$. □

Lemma 7.11. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \leq 4$. If there exists $k \in \{0, \dots, r\}$ such that for all $i = k, \dots, r$ we have $\text{sign}(\text{row}(r))_{c_i} = \text{sign}(p)_{c_i}$, then for all $i = k, \dots, r$ we either have $p_{c_i} > 0, q_{c_i} > 0$ or $p_{c_i} < 0, q_{c_i} < 0$.*

Proof. Let $i = k, \dots, r$. If we show $\text{sign}(p_{i,d-i}) = \text{sign}(p_{i,d-i-5})$, then $\text{sign}(r_{i,d-i}) = \text{sign}(p_{i,d-i})$ holds by assumption $\text{sign}(\text{row}(r))_{c_i} = \text{sign}(p)_{c_i}$. So, we can use Lemma 7.10 to prove the statement.

We see that $\text{sign}(p_{r,d-r}) = \text{sign}(p_{r,d-r-5})$ since $\text{sign}(\text{row}(r))_{c_r} = \text{sign}(p)_{c_r}$ and $\text{sign}(r_{r,d-r}) = \text{sign}(r_{r,d-r-5})$. For $r-1$, we then use the Pascal property. We repeat this argument for $r-2, r-3, \dots, k$. This shows that $\text{sign}(p_{i,d-i}) = \text{sign}(p_{i,d-i-5})$ for all $i = k, \dots, r$. □

Proposition 7.12. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \leq 4$. If there exists $k \in \{0, \dots, r\}$ such that for all $i = k, \dots, r$ we have $\text{sign}(\text{row}(r))_{c_i} = \text{sign}(p)_{c_i}$, then $\text{sign}(T)$ is fixed-contractable on c_i for all $i = k, \dots, r$.*

Proof. Let $i = k, \dots, r$ and $d \in \mathbb{N}_{\geq 15}$ First, it is easy to see that p is contractable on c_i because $\text{row}(r)$ is contractable and $\text{sign}(\text{row}(r))_{c_i} = \text{sign}(p)_{c_i}$. By Lemma 7.11 the sign does not change when increasing the degree $d \rightsquigarrow d+1$. Hence, the contractability of p on c_i is preserved for all degrees greater or equal to d. Therefore, there exists one $\hat{p} \in H[\mathbf{x}, c_i]$ for all $d' \geq d$ such that $\hat{p} = p_{d'}$. □

There exist similar propositions for contractability on d_i and e_i.

Proposition 7.13. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \geq d-4$. If there exists $k \in \{r, \dots, d\}$ such that for all $i = r, \dots, k$ we have $\operatorname{sign}(\operatorname{row}(r))_{d_i} = \operatorname{sign}(p)_{d_i}$, then $\operatorname{sign}(T)$ is fixed-contractable on d_i for all $i = r, \dots, k$. The analogous statement holds for e_i.*

Proof. We use the same proof as before, but now the sign of the entire diagonal d_i changes whenever we increase the dimension by one. Fortunately, the contractability on d_i is not affected by this. □

We state analogous propositions for $\operatorname{col}(\cdot)$ of Proposition 7.12 and 7.13 but skip the proofs since they are similar. Let $T = \sum_{i=0}^{4} \lambda_i \operatorname{col}(i) + \sum_{i=0}^{4} \lambda_{t-i} \operatorname{col}(t-i)$.

Proposition 7.14. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \leq 4$. If there exists $k \in \{0, \dots, r\}$ such that for all $i = k, \dots, r$ we have $\operatorname{sign}(\operatorname{col}(r))_{b_i} = \operatorname{sign}(p)_{b_i}$, then $\operatorname{sign}(T)$ is fixed-contractable on b_i for all $i = k, \dots, r$.*

Proposition 7.15. *Let $d \in \mathbb{N}_{\geq 15}$ and $r \geq d-4$. If there exists $k \in \{r, \dots, d\}$ such that for all $i = r, \dots, k$ we have $\operatorname{sign}(\operatorname{col}(r))_{d_i} = \operatorname{sign}(p)_{d_i}$, then $\operatorname{sign}(T)$ is fixed-contractable on d_i for all $i = r, \dots, k$. The analogous statement holds for e_i.*

Here is an analogous version of Lemma 7.8 but for $\operatorname{diag}(\cdot)$.

Lemma 7.16. *Let $d \in \mathbb{N}_{\geq 9}$. Then, $q_{i,j+1} = p_{i,j}$ holds for all $(i,j) \in V_d$.*

Proof. Just use Lemma 7.8 and symmetries $\sigma \in S_3$. □

Example 7.17. Consider $T = \operatorname{diag}(3) + \operatorname{diag}(2)$. Then, p_8 is represented by the triangle on the left and p_9 is represented by the triangle on the right.

```
  .                                  .
  .   .                              .   .
  1   1   1                          1   1   1
  4   3   2   1                      4   3   2   1   .
 10   6   3   1   .                 10   6   3   1   .   .
 20  10   4   1   .   .             20  10   4   1   .   .   .
 35  15   5   1   .   .   .         35  15   5   1   .   .   .   .
 56  21   6   1   .   .   .   .     56  21   6   1   .   .   .   .   .
 84  28   7   1   .   .   .   .   . 84  28   7   1   .   .   .   .   .   .
                                   120  36   8   1   .   .   .   .   .   .   .
```

Not surprisingly, we have analogous propositions for $\operatorname{diag}(\cdot)$ of Proposition 7.12. Consider the formal linear combination $T = \sum_{i=0}^{4} \lambda_i \operatorname{diag}(i) + \sum_{i=0}^{4} \lambda_{t-i} \operatorname{diag}(t-i)$.

Proposition 7.18. *Let $d \in \mathbb{N}_{\geq 15}$. The following statements hold:*

(1) Assume $r \leq 4$. If there is $k \in \{0, \dots, r\}$ such that for all $i = k, \dots, r$ we have $\operatorname{sign}(\operatorname{diag}(r))_{c_i} = \operatorname{sign}(p)_{c_i}$, then $\operatorname{sign}(T)$ is fixed-contractable on c_i for all $i = k, \dots, r$.

(2) Assume $r \geq d - 4$. If there exists $k \in \{r, \dots, d\}$ such that for all $i = r, \dots, d$ we have $\operatorname{sign}(\operatorname{diag}(r))_{b_i} = \operatorname{sign}(p)_{b_i}$, then $\operatorname{sign}(T)$ is fixed-contractable on b_i for all $i = r, \dots, d$.

Proof. The proofs are analogous to the proof of Proposition 7.12. □

We now provide more Propositions that will later help us to automatically prove that formal forms are fixed-contractable.

Proposition 7.19. *Let $d \in \mathbb{N}_{\geq 15}$, $T = \sum_{i=0}^{4} \lambda_i \operatorname{row}(i)$ with realization p_d, and $T' = T - \operatorname{diag}(0)$ with realization p'_d. If $(p_d)_{c_0} \geq \mathbf{2}$, then we have $(p'_{d'})_{c_0} \geq \mathbf{1}$ for all $d' \geq d$.*

Proof. We see that $(\operatorname{diag}(0))_{c_0} = \mathbf{1}$ is a constant vector for all degrees. Note that $(p_{d'})_{c_0} \geq \mathbf{2}$ for all $d' \geq d$ by Lemma 7.8. So, we have $(p_{d'} - \operatorname{diag}(0))_{c_0} \geq \mathbf{1}$ for all $d' \geq d$. □

Example 7.20. Let $T' = \operatorname{row}(1) + \operatorname{row}(2) - \operatorname{diag}(0)$. Then, $(p'_d)_{c_0} \geq \mathbf{1}$ for all dimensions $d \geq 15$. Let us visualize $\operatorname{row}(1) + \operatorname{row}(2)$ for $d = 18$:

```
135
119  -16
104  -15   1
 90  -14   1   .
 77  -13   1   .   .
 65  -12   1   .   .   .
 54  -11   1   .   .   .   .
 44  -10   1   .   .   .   .   .
 35   -9   1   .   .   .   .   .   .
 27   -8   1   .   .   .   .   .   .   .
 20   -7   1   .   .   .   .   .   .   .   .
 14   -6   1   .   .   .   .   .   .   .   .   .
  9   -5   1   .   .   .   .   .   .   .   .   .   .
  5   -4   1   .   .   .   .   .   .   .   .   .   .   .
  2   -3   1   .   .   .   .   .   .   .   .   .   .   .   .
  .   -2   1   .   .   .   .   .   .   .   .   .   .   .   .   .
 -1   -1   1   .   .   .   .   .   .   .   .   .   .   .   .   .   .
 -1    .   1   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .
  .    1   1   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .
```

As we can see, it is c_0-contractable since its c_0-column is positive. Subtracting $\operatorname{diag}(0)$ from $\operatorname{row}(1) + \operatorname{row}(2)$ will not change the sign of the c_0-column.

Proposition 7.21. *Let $d \in \mathbb{N}_{\geq 15}$, $T = \sum_{i=0}^{4} \lambda_i \operatorname{row}(i)$ with realization p_d, and $T' = T + \operatorname{diag}(0)$ with realization p'_d. If $(p_d)_{c_0} \geq \mathbf{0}$, then we have $(p'_{d'})_{c_0} \geq \mathbf{1}$ for all $d' \geq d$.*

Proof. We see that $(\text{diag}(0))_{c_0} = \mathbf{1}$ is a constant vector for all degrees. Note that $(p_{d'})_{c_0} \geq \mathbf{0}$ for all $d' \geq d$ by Lemma 7.8. So, we have $(p_{d'} + \text{diag}(0))_{c_0} \geq \mathbf{1}$ for all $d' \geq d$. □

Example 7.22. Let $T' = \text{row}(2) + \text{row}(3) - \text{diag}(0)$. Then, $(p'_d)_{c_0} < \mathbf{0}$ for all $d \geq 15$. Moreover, we can use Proposition 7.12 to show $(p'_d)_{c_1} > \mathbf{0}$ for all $d \geq 15$.

Proposition 7.23. *Let $T = \sum_{i=d-4}^{d} \lambda_i \text{col}(i)$. Assume that $(p_d)_{d_0} \geq \mathbf{0}$ for some degree $d \in \mathbb{N}_{\geq 15}$. Then $(p_{d'} - \text{col}(d'))_{d_0} \geq \mathbf{1}$ for all degrees $d' \geq d$*

Proof. We see that $(\text{col}(d))_{d_0} = -\mathbf{1}$ is a constant vector for all degrees. Note that $(p_{d'})_{d_0} \geq \mathbf{0}$ for all $d' \geq d$ by Lemma 7.16. So, we have $(p_{d'} - \text{col}(d'))_{d_0} \geq \mathbf{1}$ for all dimensions $d' \geq d$. □

7.2 An Extended Trivial System

To compute Γ_6^{even}, we defined the system $\Phi = \Phi_1 \cup \Phi_2$. We proved that this system is non-trivial. If we can find a system Ψ that is a superset of Φ and is also non-trivial, then we can reduce the number of cases to check.

Proposition 7.24. *Define $\Psi = \Phi \cup \{\text{diag}(i) - \text{diag}(j) \mid (i,j) \in Z\}$, where*

$$\begin{gathered} Z \coloneqq \{(0,1),(0,2),(0,3),(0,4),(0,d-1),(0,d-2),(0,d-3),(0,d-4), \\ (1,2),(1,3),(1,d),(1,d-4),(1,d-2),(1,d-3),(2,d),(2,d-1), \\ (2,d-3),(2,d-4),(3,d),(3,d-1),(3,d-2),(3,d-4)(d-4,d),(d-3,d), \\ (d-2,d),(d-2,d-1),(d-1,d-2),(d-1,d-3),(d-1,d),(1,d-1)\}. \end{gathered}$$

The system Ψ is non-trivial.

Proof. Let $T \in \Psi$. The case $T \in \Phi$ has already been covered in previous chapters.

- Let $T = \text{diag}(1) - \text{diag}(d-1)$ be a formal linear combination. Let $d \in \mathbb{N}_{\geq 15}$ be odd and $\mathbf{w}$ be a root of Ψ. Then, the realization $p_d = \sum \lambda_{i,j} x_{i,j}$ of T for $t = d$ satisfies $\lambda_{0,0} = 0$ and $\lambda_{0,k} < 0$ for all $k = 1, \ldots, d$.

  If $\mathbf{w}$ is a trivial root of p_d, then it satisfies $w_{0,k} = 0$ for all $k = 1, \ldots, d$. Then, $\text{diag}(0)(\mathbf{w}) < 0$; this is a contradiction because $\mathbf{w}$ is supposed to be a root of $\text{diag}(0)$.

  Let $d \in \mathbb{N}_{\geq 15}$ be even and $\mathbf{w}$ be a root of Ψ. The realization $p_d = \sum \lambda_{i,j} x_{i,j}$ satisfies $\lambda_{i,d-i} \neq 0$ if and only if $i \in \{0, d\}$. If $\mathbf{w}$ is a trivial solution of p_d, then it satisfies $w_{d,0} > 0$ since it is a root of $\text{diag}(d)$. However, $\text{col}(d-1)(\mathbf{w}) < 0$, which is a contradiction since $\mathbf{w}$ is a root of $\text{col}(d-1)$.

- In every other case, it is easy to see that the realization $p = \sum \lambda_{i,j} x_{i,j} \coloneqq p_d$ of T satisfies $\lambda_{0,0} \neq 0$, $\text{supp}^+(p) \neq \emptyset$, and $\text{supp}^-(p) \neq \emptyset$ for any degree $d \in \mathbb{N}_{\geq 15}$. Thus, any root $\mathbf{w}$ of p satisfies $\text{supp}^+(p) \cap \text{supp}^+(w) \neq \emptyset$ or $\text{supp}^-(p) \cap \text{supp}^+(w) \neq \emptyset$.

□

Proposition 7.25. *Let $T \in \Psi$. Then, T is fixed-contractable.*

Proof. Let $T \in \Phi$. This case has already been covered in previous chapters. Let $T \notin \Phi$, i.e. $T = \operatorname{diag}(i) - \operatorname{diag}(j)$. Then, use Proposition 7.18 if $i, j \leq 4$ or $i, j \geq d-4$. Otherwise, the claim follows immediately from Proposition 7.6. □

Definition 7.26. Define Γ^{even} to be the set of all valid contracted hyperfield configurations $\mathbf{s} \in H^{\Xi}$ such that $\hat{p}^{\text{even}}(\mathbf{s}) = H$ for all $p \in \Psi$, and Γ^{odd} to be the set of all valid contracted hyperfield configurations $\mathbf{s} \in H^{\Xi}$ such that $\hat{p}^{\text{odd}}(\mathbf{s}) = H$ for all $p \in \Psi$.

Definition 7.27. We define $\Gamma_6^{\text{even}} := \Gamma^{\text{even}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\operatorname{supp}^+(\mathbf{s})| = 6\right\}$. Additionally, let us define $\Gamma_6^{\text{odd}} := \Gamma^{\text{odd}} \cap \left\{\mathbf{s} \in H^{\Xi} : |\operatorname{supp}^+(\mathbf{s})| = 6\right\}$.

Proposition 7.28. *We have $|\Gamma_6^{\text{even}}| = 106806$ and $|\Gamma_6^{\text{odd}}| = 110272$.*

Proof. This is verified by computer. □

We excluded around 100,000 cases; there are still around 217,000 cases left to check.

7.3 Reducing Cases with Fixed-Contractable Forms

The final reduction step relies on using fixed-contractable forms as a filter. Let G be a set of formal linear combinations that contains the expressions

$$\begin{gathered}
\mathrm{col}(i_1)+\mathrm{row}(i_2), \mathrm{col}(i_1)-\mathrm{col}(i_2), \mathrm{col}(i_1)-\mathrm{diag}(i_2), \mathrm{row}(i_1)+\mathrm{row}(i_2), \mathrm{row}(i_1)-\mathrm{row}(i_2),\\
\mathrm{row}(i_1)-\mathrm{col}(i_2), \mathrm{row}(i_1)-\mathrm{diag}(i_2), \mathrm{diag}(i_1)-\mathrm{diag}(i_2), \mathrm{diag}(i_1)+\mathrm{row}(i_2)+\mathrm{col}(i_3),\\
\mathrm{row}(i_1)-\mathrm{diag}(i_2)+\mathrm{col}(i_3), \mathrm{col}(i_1)+\mathrm{row}(i_2)+\mathrm{col}(i_3), \mathrm{col}(i_1)+\mathrm{row}(i_2)-\mathrm{col}(i_3),\\
\mathrm{col}(i_1)-\mathrm{row}(i_2)-\mathrm{col}(i_3), \mathrm{col}(i_1)+\mathrm{col}(i_2)-\mathrm{col}(i_3), \mathrm{diag}(i_1)-\mathrm{diag}(i_2)+\mathrm{col}(i_3)+\mathrm{row}(i_4),\\
\mathrm{row}(i_1)+\mathrm{row}(i_2)+\mathrm{col}(i_3)+\mathrm{diag}(i_4), \mathrm{row}(i_1)+\mathrm{row}(i_2)+\mathrm{col}(i_3)-\mathrm{diag}(i_4),\\
\mathrm{row}(i_1)+\mathrm{row}(i_2)-\mathrm{col}(i_3)-\mathrm{diag}(i_4), \mathrm{diag}(i_1)-\mathrm{diag}(i_2)+\mathrm{col}(i_3)+\mathrm{row}(i_4)+\mathrm{col}(i_5),\\
\mathrm{diag}(i_1)-\mathrm{diag}(i_2)+\mathrm{col}(i_3)+\mathrm{row}(i_4)-\mathrm{col}(i_5), \mathrm{diag}(i_1)-\mathrm{diag}(i_2)+\mathrm{col}(i_3)-\mathrm{row}(i_4)-\mathrm{col}(i_5),\\
\mathrm{diag}(i_1)-\mathrm{diag}(i_2)+\mathrm{diag}(i_3)+\mathrm{row}(i_4)+\mathrm{col}(i_5), \mathrm{diag}(i_1)-\mathrm{diag}(i_2)+\mathrm{diag}(i_3)+\mathrm{row}(i_4)-\mathrm{col}(i_5),\\
\mathrm{diag}(i_1)-\mathrm{diag}(i_2)+\mathrm{col}(i_3)+\mathrm{col}(i_4)-\mathrm{col}(i_5), \mathrm{row}(i_1)+\mathrm{row}(i_2)+\mathrm{row}(i_3)+\mathrm{row}(i_4),\\
\mathrm{row}(i_1)+\mathrm{row}(i_2)+\mathrm{row}(i_3)-\mathrm{row}(i_4), \mathrm{row}(i_1)+\mathrm{row}(i_2)-\mathrm{row}(i_3)-\mathrm{row}(i_4),\\
\mathrm{row}(i_1)-\mathrm{row}(i_2)-\mathrm{row}(i_3)-\mathrm{row}(i_4), \mathrm{col}(i_1)+\mathrm{col}(i_2)+\mathrm{col}(i_3)+\mathrm{col}(i_4),\\
\mathrm{col}(i_1)+\mathrm{col}(i_2)+\mathrm{col}(i_3)-\mathrm{col}(i_4), \mathrm{col}(i_1)+\mathrm{col}(i_2)-\mathrm{col}(i_3)-\mathrm{col}(i_4),\\
\mathrm{col}(i_1)-\mathrm{col}(i_2)-\mathrm{col}(i_3)-\mathrm{col}(i_4), \mathrm{diag}(i_1)+\mathrm{diag}(i_2)+\mathrm{diag}(i_3)+\mathrm{diag}(i_4),\\
\mathrm{diag}(i_1)+\mathrm{diag}(i_2)+\mathrm{diag}(i_3)-\mathrm{diag}(i_4), \mathrm{diag}(i_1)+\mathrm{diag}(i_2)-\mathrm{diag}(i_3)-\mathrm{diag}(i_4),\\
\mathrm{diag}(i_1)-\mathrm{diag}(i_2)-\mathrm{diag}(i_3)-\mathrm{diag}(i_4), \mathrm{col}(i_1)+\mathrm{col}(i_2)+\mathrm{col}(i_3)+\mathrm{col}(i_4)-\mathrm{col}(i_5),\\
\mathrm{col}(i_1)+\mathrm{col}(i_2)+\mathrm{col}(i_3)-\mathrm{col}(i_4)-\mathrm{col}(i_5), \mathrm{col}(i_1)+\mathrm{col}(i_2)-\mathrm{col}(i_3)-\mathrm{col}(i_4)-\mathrm{col}(i_5),\\
\mathrm{row}(i_1)+\mathrm{row}(i_2)+\mathrm{row}(i_3)+\mathrm{row}(i_4)-\mathrm{row}(i_5), \mathrm{row}(i_1)+\mathrm{row}(i_2)+\mathrm{row}(i_3)-\mathrm{row}(i_4)-\mathrm{row}(i_5),\\
\mathrm{row}(i_1)+\mathrm{row}(i_2)-\mathrm{row}(i_3)-\mathrm{row}(i_4)-\mathrm{row}(i_5)
\end{gathered}$$

for all $i_1, \ldots, i_5 \in \{0, 1, 2, 3, 4, t-4, t-3, t-2, t-1, t\}$.

The choice of linear combinations in G is arbitrary; the more forms we include, the more cases we can exclude later. We develop an algorithm to computationally prove that some forms in G are fixed-contractable.

Step 1: Realization

The first step of the algorithm is to realize all the forms in G at a fixed degree D and check if it is contractable at degree D. We choose $D = 40$; the higher the degree, the more likely it is that a form is fixed-contractable from that degree onwards.

Algorithm 7.1 Realize

Require: a set of formal forms G and degree D
Ensure: $G' \subset G$ is a set containing all forms whose realization at degree D is contractable.

function REALIZE
  $G' \leftarrow \emptyset$
  for $T \in G$ **do**
    $p \leftarrow$ realization of T at degree D
    if `is_contractable`(p) **then**
      $G' \leftarrow G' \cup \{T\}$
    end if
  end for
  return G'
end function

Computing G' took around two hours on a MacBook Air with an M1 chip. The details can be found in the source code [10] under the file `chapter07_step1_realize.ipynb`.

Step 2: Automatic Proof of Fixed-Contractability

The set G' contains formal linear combinations of Pascal forms whose realizations at degree D are contractable. However, this does not imply that they are fixed-contractable. We need to prove that they are fixed-contractable. For that purpose, we develop an algorithm that automatically proves fixed-contractability. Its pseudocode is given below.

Algorithm 7.2 Automatic Proof

Require: G'
Ensure: $G'' \subset G'$ is a set containing fixed-contractable forms

function PROVE
  $G'' \leftarrow \emptyset$
  for $T \in G'$ **do**
    if `prove_fixed_contractable`(T) **then**
      $G'' \leftarrow G'' \cup \{T\}$
    end if
  end for
  return G''
end function

The function `prove_fixed_contractable`(T) just checks if Proposition 7.12, 7.13, 7.14, 7.15, 7.18, 7.19, 7.21, or Proposition 7.23 can be applied to T; if they can, we proved that T is fixed-contractable. For details, we refer to the implementation found in [10] under the file `chapter07_step2_prove.ipynb`.

Step 3: Filtering Invalid Cases

Once the filter set G'' has been constructed, we proceed as follows: For each $\mathbf{s} \in \Gamma_6^{\text{even}}$, we check whether for all $T \in G''$ its realization $p \coloneqq p_D$ with $D = 40$ satisfies $0 \in \hat{p}^{\text{even}}(\mathbf{s})$. If there exists any T for which this condition fails, $\mathbf{s}$ is excluded. By applying this method, we were able to reduce the number of cases from 106,806 to just 6,700.

Algorithm 7.3 Apply Filter (even)

```
function FILTER_EVEN
    for s ∈ Γ_6^even do
        for T ∈ G'' do
            p ← realization of T at degree D = 40
            if 0 ∉ p̂^even(s) then
                exclude s from further consideration
            end if
        end for
    end for
end function
```

The process is also repeated for $\mathbf{s} \in \Gamma_6^{\text{odd}}$ with $D = 41$, resulting in 8737 cases. In total, this amounts to 15,437 cases. The computations required three hours on a MacBook Pro equipped with an M3 chip. Further details can be found in the source code [10] under the file `chapter07_step3_apply_filter.ipynb`.

7.4 Carrying Out the Proof

To finalize the proof, it is necessary to check the $15,437$ cases. One approach is to reduce the number of cases further by considering a larger set G, which requires more computational power. Once the number of cases is sufficiently small, we can apply the techniques developed in previous chapters, such as the Invertibility Criterion, the Hyperfield Criterion, and the Hexagon Criterion. Then, we deal with each case individually, i.e. applying the Invertibility Criterion with hand-picked λ. Due to time constraints, we were unable to complete the proof.

Chapter 8

Computation of Fundamental Models

In the final chapter, we compute the number of fundamental models. The implementation details are publicly available in the repository [10]. The results of these computations are summarized in the following table.

$n \setminus d$	1	2	3	4	5	6	7	8	9	10	11
2	1										
3		3	1								
4			12	4	2						
5				82	38	10	4				
6					602	254	88	24	2		
7						6710	2421	643	198	32	4

Figure 8.1: The number of fundamental outcomes for each positive support size n and degree d. For $n = 2, 3, 4, 5$ there are no additional columns beyond those shown as we have proven that the degree is bounded.

To construct this table, we employed the following methodology. First, we calculated the set of all supports of valid outcomes for a fixed degree $d \in \mathbb{N}$ using Algorithm 5.2. For each chipsplitting support generated, we mapped it back to a statistical model and computed the rank of the corresponding linear system to determine whether it yields a unique solution. If the system is of full rank, the associated statistical model is fundamental, as defined in Definition 2.15. By using this approach, we computed all valid outcomes for positive support sizes $n = 1, \ldots, 7$ and degree $d \leq 11$. This extends the results of Bik and Marigliano [3] by one additional support size, made possible by a more efficient implementation.

Chapter 9

Discussion

This thesis establishes a connection between the classification of discrete statistical models (Theorem 2.24) and a combinatorial puzzle related to chipsplitting games (Theorem 3.33). Specifically, the puzzle investigates whether the degree of a valid chipsplitting outcome can grow indefinitely while its support size remains fixed. For outcomes with positive support sizes up to five, we prove that the degree cannot grow indefinitely, providing a definitive negative answer.

For outcomes with a positive support size of six, progress was made toward a similar conclusion. By employing systematic reductions, the number of cases requiring analysis was reduced from approximately 300,000 to 12,000, indicating that a negative answer may hold for this case as well. With additional computational resources, one can reduce the number of cases even further, potentially leading to a number of cases that can be analyzed using the techniques described in this thesis. With even greater computational power, one could potentially extend the results to support size seven.

We would like to conclude this thesis by discussing some possible directions for future research. First, it would be interesting to investigate a better criterion for determining fixed-contractables forms. This could potentially lead to a reduction in the number of cases that need to be analyzed for positive support size six. Second, investigating a larger contraction size than five could provide further insights into the degree of valid outcomes of positive support size six. Finally, finding a larger non-trivial system would allow us to exclude even more cases.

Bibliography

[1] Carlos Améndola et al. “The maximum likelihood degree of toric varieties”. In: *Journal of Symbolic Computation* 92 (2019), pp. 222–242.

[2] Matt Baker and Nathan Bowler. “Matroids over hyperfields”. In: *AIP Conference Proceedings*. Vol. 1978. 1. AIP Publishing. 2018.

[3] Arthur Bik and Orlando Marigliano. “Classifying one-dimensional discrete models with maximum likelihood degree one”. In: *arXiv preprint arXiv:2205.09547* (2022).

[4] Fabrizio Catanese et al. “The maximum likelihood degree”. In: *American Journal of Mathematics* 128.3 (2006), pp. 671–697.

[5] Eliana Duarte, Orlando Marigliano, and Bernd Sturmfels. “Discrete statistical models with rational maximum likelihood estimator”. In: (2021).

[6] Darij Grinberg. *A hyperfactorial divisibility, version of 27 July 2015.*

[7] June Huh. “The maximum likelihood degree of a very affine variety”. In: *Compositio Mathematica* 149.8 (2013), pp. 1245–1266.

[8] June Huh. “Varieties with maximum likelihood degree one”. In: *arXiv preprint arXiv:1301.2732* (2013).

[9] Caroline J Klivans. *The mathematics of chip-firing*. Chapman and Hall/CRC, 2018.

[10] Viet Duc Nguyen. *chipsplitting-game*. `https://github.com/geniegeist/chipsplitting-game/tree/master-thesis`. 2024.

[11] Seth Sullivant. *Algebraic statistics*. Vol. 194. American Mathematical Society, 2023.