Hyperfield Criterion

Viet Duc Nguyen

October 24, 2024

1 A Necessary Condition

The Hyperfield Criterion is a necessary condition for a *valid* outcome.

Proposition 1 (Hyperfield Criterion). *Let w be a valid outcome. Then* $\operatorname{sign}(w)$ *is a hyperfield root of* $\operatorname{sign}(p)$ *for every pascal form p.*

With the Hyperfield Criterion, we can exclude many supports from consideration when searching for valid outcomes; more precisely, supports that are not hyperfield roots of some pascal form cannot be the support of a valid outcome. Hence, we just need to check the supports that are common roots of all hyperfield pascal forms. Since these hyperfield pascal forms are just linear forms, we are essentially solving a homogeneous linear system in a hyperfield.

2 Solving Homogeneous Hyperfield Linear Systems

Problem: Given a set of linear forms $A = \{p_1, \dots, p_k\}$, compute the solution set $V(A) \coloneqq \{x \in H^{V_d} : 0 \in \operatorname{sign}(p_i)(x) \quad \forall i = 1, \dots, k\}$.

We further simplify the problem by only considering solutions x with $\operatorname{supp}^-(x) = \{(0,0)\}$ and $|\operatorname{supp}^+(x)| = n$ for some fixed $n \in \mathbb{N}$.

Problem: Given a set of linear forms $A = \{p_1, \dots, p_k\}$, compute the solution set $S_n(A) \coloneqq V(A) \cap \{x \in H^{V_d} : \operatorname{supp}^-(x) = \{(0,0)\}, |\operatorname{supp}^+(x)| = n\}$.

Note that $S_n(A)$ is a superset of valid outcomes of positive support size n, which will be useful in finding all valid outcomes.

A Naive Approach

To compute $S_n(A)$ a simple brute force algorithm can be used; just iterate over all positive support size n supports and check if they are hyperfield roots of some pascal basis.

Algorithm 1 Brute Force Algorithm

Input: Positive support size n, a set of linear forms $A = \{p_1, \dots, p_k\}$
Output: $S_n(A)$

function SOLVE(A, n)
  initialize empty list `solutions`
  for n-combination $S = \{(c_i, r_i) : i = 1, \dots, n\}$ of V_d **do**
    initialize $x \in H^{V_d}$ with positive support S and $x_{0,0} = -1$
    if x is a hyperfield root of every $p \in A$ **then**
      add S to `solutions`
    end if
  end for
  return `solutions`
end function

The naive approach has an exponential time complexity since we need to check $\binom{(d+1)(d+2)/2}{n}$ many supports.

Efficient Algorithm

For a specific type of system of linear forms A, we can greatly speed up the computation of $S_n(A)$.

Definition 2 (Trivial Root)**.** *Let $p(x) = \sum_{i=1}^{n} \lambda_i x_i$ be a linear form, and let $x = (x_1, \dots, x_n)$ be some root of p, i.e. $p(x) = 0$. If $\operatorname{supp}(x) \cap \operatorname{supp}(p) = \emptyset$, then the root x is called a* ***trivial root*** *of p. Otherwise, the root x is called a* ***non-trivial root*** *of p.*

Let A be a system of linear forms. We say $S_n(A)$ is ***non-trivial*** *if $S_n(A) \neq \emptyset$ and every $x \in S_n(A)$ is a non-trivial root for every form $p \in A$. We say A is* ***non-trivial*** *if $S_n(A)$ is non-trivial.*

Proposition 3. *Let A be a system of linear forms, $p \in A$ and $x \in S_n(A)$. Then, the following statements hold:*

1. *If $(0,0) \in \operatorname{supp}^+(p)$, then $x_i = 1$ for some $i \in \operatorname{supp}^+(p)$.*
2. *If $(0,0) \in \operatorname{supp}^-(p)$, then $x_i = 1$ for some $i \in \operatorname{supp}^-(p)$.*

Proof. Assume $(0,0) \in \operatorname{supp}^+(p)$. Since $x_{0,0} = -1$, we have $-1 \in \operatorname{sign}(p)(x)$. By assumption, x is a hyperfield root of p, so $0 \in \operatorname{sign}(p)(x)$. This can only happen if $x_i = 1$ for some $i \in \operatorname{supp}^+(p)$. The case $(0,0) \in \operatorname{supp}^-(p)$ is similar. $\square$

The next proposition assumes that A is non-trivial.

Proposition 4. *Let A be a non-trivial system of linear forms, $p \in A$ and $x \in S_n(A)$. If $(0,0) \notin \operatorname{supp}(p)$, then $\operatorname{supp}^+(p) \neq \emptyset$, $\operatorname{supp}^-(p) \neq \emptyset$ as well as $x_i = x_j = 1$ for some $i \in \operatorname{supp}^+(p)$ and $j \in \operatorname{supp}^-(p)$.*

Proof. Assume $(0,0) \notin \operatorname{supp}(p)$. First, $\operatorname{supp}(p) \neq \emptyset$ because $S_n(A)$ is non-empty and consists only of non-trivial roots. If $\operatorname{supp}^+(p) = \emptyset$, then $\operatorname{supp}^+(x) \subset \operatorname{supp}^-(p) = \operatorname{supp}(p) \neq \emptyset$. Hence, $\operatorname{sign}(p)(x) = \{-1\}$, which contradicts x being a root. Thus, $\operatorname{supp}^+(p)$ is non-empty. Similarly, $\operatorname{supp}^-(p)$ is non-empty.

By non-triviality, $x_i = 1$ for some $i \in \operatorname{supp}(p)$. Assume $i \in \operatorname{supp}^+(p)$. Hence, $1 \in \operatorname{sign}(p)(x)$. Since x is a root, we also have $0 \in \operatorname{sign}(p)(x)$. This can only occur if $x_j = 1$ for some $j \in \operatorname{supp}^-(p)$. The case $i \in \operatorname{supp}^-(p)$ is similar. $\square$

Both propositions allow us to interpret linear forms in a non-trivial system A as constraints on the positive supports of roots in $S_n(A)$.

Example 5. *Fix the degree $d = 3$. Assume a system A and some linear form $p \in A$. Further assume p is a diagonal pascal equation of order 0. The support of p is represented by the following diagram:*

```
+
+ .
+ . .
+ . . .
```

We see that any hyperfield root $x \neq 0$ of A satisfies $x_i = 1$ for some $i \in \{(0,0),(0,1),(0,2),(0,3)\}$ because x_{00} is negative.

Now, assume A is non-trivial. Consider a row pascal equation $q \in A$ of order 3. Its support is depicted by the following diagram:

```
−
. +
. . −
. . . +
```

For some x with $\mathrm{supp}^-(x) = \{(0,0)\}$ *to be a hyperfield root of q, we must have either*

1. $x_{i,j} = x_{i',j'} = 1$ *for some* $(i,j) \in \{(0,3),(2,1)\}$ *and* $(i',j') \in \{(3,0),(1,2)\}$, *or*

2. $\mathrm{supp}(x) \subset V_3 \setminus \mathrm{supp}(q)$.

Considering only non-trivial roots x lets us exclude the latter case.

Thus, if we have a non-trivial system A with $p, q \in A$, to compute $S_n(A)$, it suffices to check only those hyperfield roots whose support intersected with each of the three following regions is non-empty:

```
+
+ .
+ . .
+ . . .
```

```
+
. .
. . +
. . . .
```

```
.
. +
. . .
. . . +
```

Here are examples of such roots:

```
+
. +
. . .
. . . .
```

```
.
. .
+ . +
. . . +
```

Definition 6 (Constraints of a linear form)**.** *To each linear form p we can associate a finite set of supports, which we call* $\text{constraints}(p) \subset 2^{V_d} := \{\text{supp}^+(p) \setminus \{0\}, \text{supp}^-(p)\{0\}\}$.

The name is justified by the following proposition.

Proposition 7. *Let p be a linear form in a hyperfield, and $x \in H^{V_d}$ with $\text{supp}^-(x) = \{(0,0)\}$. Then, x is a non-trivial hyperfield root of p if and only if $\text{supp}^+(x) \cap S \neq \emptyset$ for all $S \in \text{constraints}(p)$.*

Proof. Since x is non-trivial, we clearly have non-empty intersection of $\text{supp}^+(x)$ and $S \in \text{constraints}(p)$. The converse direction is also clear since $p(x) = 1 - 1 = H$ in a hyperfield. ☐

We present an algorithm for computing $S_n(A)$ of non-trivial systems A.

Algorithm 2 Algorithm for Non-Trivial Systems

Input: Positive support size n, non-trivial system A
Output: $S_n(A)$

function SOLVE(A, n)
    $\text{C} \leftarrow \bigcup_{p \in A} \text{constraints}(p)$
    `solutions` $\leftarrow \{x \in H^{V_d} \mid \forall S \in C : \text{supp}^+(x) \cap S \neq \emptyset, |\text{supp}^+(x)| = n, \text{supp}^-(x) = \{(0,0)\}\}$
    return `solutions`
end function

Proof of correctness. The correctness of `solutions` $= S_n(A)$ follows from Proposition 7 and the assumption that A is non-trivial. ☐

3 Implementing the Hyperfield Criterion

The Hyperfield Criterion states that only the common hyperfield roots of all pascal forms can be supports of valid outcomes. The system of all pascal forms is a-priori an infinite and non-trivial system. However, we found out that several bases of pascal forms exist such as the row, col and diag pascal basis, which let us consider finite systems. Define the finite system $A = \{\text{diag}(i)\}_{i=0}^{d} \cup \{\text{row}(i)\}_{i=0}^{d} \cup \{\text{col}(i)\}_{i=0}^{d}$.

Proposition 8. *The system A is non-trivial.*

Proof. First, $S_n(A)$ is non-empty because $x = (x_i)_{i=0}^n$ defined as $x_{i,d-i} = \binom{n}{i}$ is a solution of the system A.

Let $x \in S_n(A)$ and $i = 0, \ldots, n$. Consider the following cases.

- Assume, $x \notin \operatorname{supp}(\operatorname{diag}(i))$; then $\operatorname{diag}(i)(x) < 0$; we found a contradiction to x being a root.

- Assume, $x \notin \operatorname{supp}(\operatorname{row}(i))$. If $i = d$, then x is not of degree n. Therefore, we assume $i < d$. Then, either x is a trivial root for $\operatorname{row}(i+1)$ or we have $\operatorname{row}(i+1)(x) \neq 0$. In the latter case, we found a contradiction to x being a root. For the former case that x is a trivial root, we conclude that there exists nonzero $x_{u,d-u}$ for some $u = i+2, \ldots, d$ since x is of degree n; now we just repeat the argument for $\operatorname{row}(i+1)$. More precisely, if x is again a trivial root for $\operatorname{row}(i+2)$, we repeat the argument for $\operatorname{row}(i+2)$ until we will end up with a contradiction $\operatorname{row}(u)(x) \neq 0$.

- For the case col, we can argue by symmetry.

$\square$

Corollary 9. *Configurations $x \in \mathbb{Z}^{V_d}$ of the form* $\operatorname{supp}(x) \subset \{(i,j) \in \mathbb{Z}^{V_d} : i + j \leq k \text{ or } i > k+1\}$ *are not valid outcomes for any $k = 0, \ldots, d-1$. Neither are configurations $x \in \mathbb{Z}^{V_d}$ of the form* $\operatorname{supp}(x) \subset \{(i,j) \in \mathbb{Z}^{V_d} : i + j \leq k \text{ or } j > k+1\}$ *for $k = 0, \ldots, d-1$ due to symmetry.*

Proof. Since the previously defined system A is non-trivial, we must have that supports of valid outcomes intersect the support of $\operatorname{row}(k+1)$ non-trivially (i.e. the intersection is non-empty). $\square$

Example 10. *This is not a valid outcome:*

$$\begin{array}{cccccc}
. & & & & & \\
. & . & & & & \\
. & . & . & & & \\
. & . & . & * & & \\
* & . & . & * & * & \\
* & * & . & * & * & *
\end{array}$$

Now that we have shown that A is a trivial system, we have found an efficient way to apply the Hyperfield Criterion. Here is a detailed breakdown of an implementation of the algorithm.

To-Do...

4 Contractions

The Hyperfield Criterion is useful to narrow down the search space for valid outcomes $w \in \mathbb{Z}^{V_d}$ for *fixed* $d \in \mathbb{N}$. However, it is our goal to show that certain outcomes in $\mathbb{Z}^{V_d}$ cannot exist for all $d \geq d'$ for some fixed $d' \in \mathbb{N}$; it remains unclear how to apply the Hyperfield Criterion in this case, where infinitely many degrees need to be checked. To solve this problem, we introduce the notion of contractions.

Definition 11. *Let $k \in \mathbb{N}$ be a positive integer that we will call* contraction size. *Let $d \in \mathbb{N}$. Let $p : H^{V_d} \to 2^H, x \mapsto \sum \lambda_{ij} x_{ij}$ be a linear form in a hyperfield. We say that p is k-contractable in H^{V_d} if p can be expressed in the following way:*

$$p(x) = \sum_{\substack{(i,j)\in V_d\\ 0\leq i+j<k}} \lambda_{ij}x_{ij} + \sum_{\substack{(i,j)\in V_d\\ 0\leq i<k\\ i+j>d-k}} \lambda_{ij}x_{ij} + \sum_{\substack{(i,j)\in V_d\\ 0\leq j<k\\ i+j>d-k}} \lambda_{ij}x_{ij}$$
$$+ \sum_{i=0}^{k-1} \beta_i b_i + \sum_{i=0}^{k-1} \gamma_i c_i + \sum_{i=0}^{k-1} \delta_i d_i + \sum_{i=0}^{k-1} \epsilon_i e_i \quad \forall x \in H^{V_d},$$

where $\beta_i, \gamma_i, \delta_i, \epsilon_i \in H$ and

$$b_i \coloneqq \sum_{j=k}^{d-i-k} x_{ij}, \quad c_i \coloneqq \sum_{j=k}^{d-i-k} x_{ji},$$
$$d_i \coloneqq x_{k,d-k-i} + x_{k+2,d-(k+2)-i} + \cdots + x_{d-k-i,k},$$
$$e_i \coloneqq x_{k+1,d-(k+1)-i} + x_{k+3,d-(k+3)-i} + \cdots + x_{d-k-i,k}.$$

Definition 12. *We say that p is k-contractable if there exists $d' \in \mathbb{N}$ such that p is k-contractable in H^{V_d} for all $d \geq d'$.*

Definition 13. *Let $i = 0, \dots, k-1$. We say that p is k-contractable on b_i if $\lambda_{i,k} = \lambda_{i,k+1} = \cdots = \lambda_{i,d-i-k}$. TODO: add definition for c,d, e.*

To simplify notation, we introduce some more notations.

Definition 14. *Let k be some contraction size. Let $p(x) = \sum \lambda_{ij} x_{ij}$ be any hyperfield linear form. For $i = 0, \dots, k-1$ we write*

$$p_{c_i} \coloneqq \begin{bmatrix} \lambda_{i,k} & \dots & \lambda_{i,d-i-k} \end{bmatrix}.$$

We call this the i-th c-column of p.

Similarly, we define p_{b_i}, p_{d_i} and p_{e_i} to denote the i-th b-row, d-diagonal and e-diagonal of p, respectively.

Proposition 15. *Let p be a linear combination of $\{\mathrm{row}(i), \mathrm{col}(i), \mathrm{diag}(i) : i \in \{0, \ldots, k-1\} \cup \{d-k+1, \ldots, d\}\}$ in the hyperfield H^{V_d}. Then, the following statements hold:*

- *The c-columns of p only depend on $\{\mathrm{row}(i), \mathrm{diag}(i)\}_{i=0,\ldots,k-1}$.*
- *The b-rows of p only depend on $\{\mathrm{col}(i), \mathrm{diag}(d-i)\}_{i=0,\ldots,k-1}$.*
- *The d-diagonals and e-diagonals of p only depend on $\{\mathrm{row}(d-i), \mathrm{col}(d-i)\}_{i=0,\ldots,k-1}$.*

Proof. This follows easily from the definition of $\mathrm{row}, \mathrm{col}, \mathrm{diag}$. □

Proposition 16. *The following statements hold:*

- *Let $p \in \{\mathrm{row}(i), \mathrm{diag}(i)\}_{i=0,\ldots,k-1}$. For any $i = 0, \ldots, k-1$ the c_i-column of p is a constant vector, i.e. $p_{c_i} \in \{-1, 0, 1\}$.*
- *Let $p \in \{\mathrm{col}(i), \mathrm{diag}(d-i)\}_{i=0,\ldots,k-1}$. For any $i = 0, \ldots, k-1$ the b_i-row of p is a constant vector, i.e. $p_{b_i} \in \{-1, 0, 1\}$.*
- *Let $p \in \{\mathrm{row}(d-i), \mathrm{col}(d-i)\}_{i=0,\ldots,k-1}$. For any $i = 0, \ldots, k-1$ the d_i-diagonal of p is a constant vector, i.e. $p_{d_i} \in \{-1, 0, 1\}$; similarly for the e_i-diagonal.*

Proof. This also follows easily from the definition of $\mathrm{row}, \mathrm{col}, \mathrm{diag}$. □

To-do: DOnt need this lemma.

Lemma 17. *Fix some dimension $d \in \mathbb{N}$. Let $u, q, r \in \mathbb{N}$ with $k \leq q < r$. Define v to be the c_u-column of $\mathrm{row}(q)$ and w to be the c_u-column of $\mathrm{row}(r)$ in H^{V_d}. Then, there exists an index $N \in \mathbb{N}$ such that for all ngeqN, we have $|v_n| < |w_n|$. Moreover, this index N is independent of d.*

Proof. This follows from the definition of $\mathrm{row}(h)$:

$$\mathrm{row}(h)(x) = (-1)^h \sum_{(i,j) \in V_d} (-1)^i \binom{j}{h-i} x_{ij}.$$

By fixing i in the formula above, we fix the c_i-column of $\mathrm{row}(h)$; thus only j changes in $\binom{j}{h-i}$ when we iterate through the c_u-column of $\mathrm{row}(h)$. Thus if we compare the c_u-column of $\mathrm{row}(q)$ against $\mathrm{row}(r)$, we compare $\binom{j}{q-i}$ against $\binom{j}{r-i}$. We see that for almost all j the latter is larger than the former since $q - i < r - i$. □

Lemma 18. *Fix some dimension $d \in \mathbb{N}$. Let $u, q, r \in \mathbb{N}$ with $k \leq q < r$. Define v to be the b_u-row of $\operatorname{col}(q)$ and w to be the b_u-row of $\operatorname{col}(r)$ in H^{V_d}. Then, there exists an index $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|v_n| < |w_n|$. Moreover, this index N is independent of d.*

Proof. Use symmetry. $\square$

Proposition 19. *Let $p : \mathbb{Z}^{V_d} \to \mathbb{Z}, q : \mathbb{Z}^{V_{d+1}} \to \mathbb{Z}$ be Pascal forms that can be expressed as $p = q = \sum_{i=0}^{d} \lambda_i \operatorname{row}(i)$. Let $(p_{ij})_{(i,j)\in V_d}$ denote the coefficients of the linear form $p : x \mapsto \sum_{(i,j)\in V_d} p_{ij} x_{ij}$, and let $(q_{ij})_{(i,j)\in V_{d+1}}$ denote the coefficients of the linear form $q : x \mapsto \sum_{(i,j)\in V_{d+1}} q_{ij} x_{ij}$. Then, $q_{ij} = p_{ij}$ for all $(i,j) \in V_d$.*

In other words, if the visualization of the coefficients $(p_{ij})_{(i,j)\in V_d}$ of the linear form $p : x \mapsto \sum_{(i,j)\in V_d} p_{ij} x_{ij}$ on the grid V_d looks like this

$$
\begin{array}{cccccccc}
p_{0,d} & & & & & & & \\
\vdots & p_{1,d-1} & & & & & & \\
\vdots & \vdots & \ddots & & & & & \\
\vdots & \vdots & \vdots & \ddots & & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\
p_{0,0} & p_{1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & p_{d,0}
\end{array}
.
$$

Then, the visualization of the coefficients $(q_{ij})_{(i,j)\in V_d}$ of the linear form $q : x \mapsto \sum_{(i,j)\in V_d} q_{ij} x_{ij}$ on the grid V_{d+1} looks like this

$$
\begin{array}{ccccccccc}
q_{0,d+1} & & & & & & & & \\
p_{0,d} & q_{1,d} & & & & & & & \\
\vdots & p_{1,d-1} & q_{2,d-1} & & & & & & \\
\vdots & \vdots & \ddots & \ddots & & & & & \\
\vdots & \vdots & \vdots & \ddots & \ddots & & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \\
p_{0,0} & p_{1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & p_{d,0} & q_{d+1,0}
\end{array}
.
$$

Proof. First, the statement follows immediately for $p = q = \lambda \text{row}(i)$ for all $i = 0, \ldots, d$ from the definition of $\text{row}(i)$.

Now, assume $p = q = \sum_{i=0}^{d} \lambda_i \text{row}(i)$. Let $(p_{ij}^{(l)})_{(i,j)\in V_d}$ and $(q_{ij}^{(l)})_{(i,j)\in V_{d+1}}$ denote the coefficients of the linear forms $\text{row}(l)$. Since we know that $p_{ij}^{(l)} = q_{ij}^{(l)}$ for all $l = 0, \ldots, d$ and $(i, j) \in V_d$, we find that $q_{ij} = q_{ij}^{(l)} = \sum p_{ij}^{(l)} = p_{ij}$ for all $(i, j) \in V_d$. $\square$

Corollary 20. *The same statement holds for* $p = q = \sum_{i=0}^{d} \lambda_i \text{col}(i)$.

Proof. Use symmetry. $\square$

Example 21. *Consider* $p = q = \text{row}(3) + \text{row}(2)$ *in* $\mathbb{Z}^{V_8}$ *and* $\mathbb{Z}^{V_9}$*, respectively. Then,* p *is represented by*

-28								
-14	14							
-5	9	-5						
.	5	-4	1					
2	2	-3	1	.				
2	.	-2	1	.	.			
1	-1	-1	1	.	.	.		
.	-1	.	1	.	.	.	.	
.	.	1	1	.	.	.	.	.

and q *is represented by*

-48									
-28	20								
-14	14	-6							
-5	9	-5	1						
.	5	-4	1	.					
2	2	-3	1	.	.				
2	.	-2	1	.	.	.			
1	-1	-1	1	.	.	.	.		
.	-1	.	1	.	.	.	.	.	
.	.	1	1	.	.	.	.	.	.

Proposition 22. *Assume we have the assumptions as in Proposition 19. Let* $u = 0, \ldots, d$. *If* $\text{sign}(\text{row}(r))_{i,d-i} = \text{sign}(p)_{i,d-i}$ *for all* $i = u, \ldots, d$*, then* $\text{sign}(q_{i,d+1-i}) = \text{sign}(p_{i,d-i})$ *for all* $i = u, \ldots, d$.

Proof. Without loss of generality, we assume that $\lambda_r > 0$ (otherwise we multiply p by -1). First, we see that $q_{r,\cdot} = \lambda_r \cdot \mathbf{1}$ and $q_{i,\cdot} = \mathbf{0}$ for all $i > r$. By the Pascal property, we have $q_{r-1,d+1-(r-1)} = q_{r-1,d-(r-1)} - q_{r,d+1-r} = q_{r-1,d-(r-1)} - \lambda_r$. This shows $q_{r-1,d+1-(r-1)} < q_{r-1,d-(r-1)} = p_{r-1,d-(r-1)} < 0$, where the last inequality follows from assumption. Thus, we have $\operatorname{sign}(q_{r-1,d+1-(r-1)}) = \operatorname{sign}(q_{r-1,d-(r-1)}) = -1$. Next, we again use the Pascal property $q_{r-2,d+1-(r-2)} = q_{r-2,d-(r-2)} - q_{r-1,d+1-(r-1)}$. We see that $q_{r-2,d+1-(r-2)} > 0$ because $q_{r-2,d-(r-2)} > 0$ and $q_{r-1,d+1-(r-1)} < 0$. Thus, we have $\operatorname{sign}(q_{r-2,d+1-(r-2)}) = \operatorname{sign}(q_{r-2,d-(r-2)}) = 1$. We can continue this argument until we reach $q_{r-(r-u),d+1-(r-(r-u))} = q_{u,d+1-u}$. This shows that $\operatorname{sign}(q_{i,d+1-i}) = \operatorname{sign}(q_{i,d-i}) = \operatorname{sign}(p_{i,d-i})$ for all $i = u, \dots, d$. $\square$

Proposition 23. *Assume we have the assumptions as in Proposition 19. Let* $u = 0, \dots, d$. *If* $\operatorname{sign}(\operatorname{row}(r))_{c_i} = \operatorname{sign}(p)_{c_i}$ *for all* $i = u, \dots, k-1$, *then the* c_i*-column of* q *has the same sign as the* c_i*-column of* p *for all* $i = u, \dots, k-1$.

Proof. Let $i = u, \dots, k-1$. If we show $\operatorname{sign}(p_{i,d-i}) = \operatorname{sign}(p_{i,d-i-k})$, then $\operatorname{sign}(\operatorname{row}(r))_{i,d-i} = \operatorname{sign}(p)_{i,d-i}$, and we can use Proposition 22 to prove the statement.

We will now prove $p_{i,d-i} = p_{i,d-i-k}$ for all $i = u, \dots, d$. For that we consider the restriction of p on $\mathbb{Z}^{V_{d-k}}$, and call this restriction $\tilde{p}$. Note that $\tilde{p}_{ij} = p_{ij}$ for all $(i,j) \in V_{d-k}$. Then, we apply Proposition 22 on $\tilde{p}$ to show that $\operatorname{sign}(\tilde{p}_{i,d-i-k}) = \operatorname{sign}(p_{i,d-i-k}) = \operatorname{sign}(p_{i,d-i-k+1})$. We repeat this argument until we reach $\operatorname{sign}(p_{i,d-i}) = \operatorname{sign}(p_{i,d-i-k})$. $\square$

Proposition 24. *Let* k *be some contraction size,* $u = 0, \dots, k-1$, *and* $d' \in \mathbb{N}$. *Let* $p = \sum_{i=0}^{k-1} \gamma_i \operatorname{row}(i)$ *with* $\gamma_i \in H$ *be a hyperfield linear form and* $r := \max\{i : \gamma_i \neq 0\}$. *If* $\operatorname{sign}(\operatorname{row}(r))_{c_i} = \operatorname{sign}(p)_{c_i}$ *for all* $i = u, \dots, d'$, *then* p *is* k*-contractable on* c_i *in* H^{V_d} *for all* $d \geq d'$ *and all* $i = u, \dots, k-1$.

Proof. Let $i = u, \dots, k-1$. First, it is easy to see that p is k-contractable on c_i in $H^{V_{d'}}$ because $\operatorname{row}(r)$ is k-contractable and $\operatorname{sign}(\operatorname{row}(r))_{c_i} = \operatorname{sign}(p)_{c_i}$. By Proposition 23 the sign does not change when increasing the degree $d \rightsquigarrow d+1$. Hence, the contractability of p on c_i is preserved for all $d \geq d'$. $\square$

Proposition 25. *Let* k *be some contraction size,* $u = 0, \dots, k-1$, *and* $d' \in \mathbb{N}$. *Let* $p = \sum_{i=0}^{k-1} \gamma_i \operatorname{row}(i)$ *with* $\gamma_i \in H$ *be a hyperfield linear form and* $r := \max\{i : \gamma_i \neq 0\}$. *The following statements hold for all* $i = u, \dots, d'$:

- *If* $\operatorname{sign}(\operatorname{row}(r))_{d_i} = \operatorname{sign}(p)_{d_i}$, *then* p *is* k*-contractable on* d_i *in* H^{V_d} *for all* $d \geq d'$.

- *If* $\operatorname{sign}(\operatorname{row}(r))_{e_i} = \operatorname{sign}(p)_{e_i}$*, then* p *is* k*-contractable on* e_i *in* H^{V_d} *for all* $d \geq d'$*.*

Proof. We can use the same proof as before, but now the sign of the entire diagonal d_u changes whenever we increase the dimension by one. The contractability on d_u is not affected by this.

For e_u, we use the same argument. □

Corollary 26. *By symmetry, we have an analogous statement for* $p = \sum_{i=0}^{k-1} \gamma_i \operatorname{col}(i)$ *and the* d_i*-diagonals as well as the* e_i*-diagonals.*

Proposition 27. *Let* $p : \mathbb{Z}^{V_d} \to \mathbb{Z}, q : \mathbb{Z}^{V_{d+1}} \to \mathbb{Z}$ *be Pascal forms that can be expressed as* $p = q = \sum_{i=0}^{d} \lambda_i \operatorname{diag}(i)$*. Let* $(p_{ij})_{(i,j)\in V_d}$ *denote the coefficients of the linear form* $p : x \mapsto \sum_{(i,j)\in V_d} p_{ij}x_{ij}$*, and let* $(q_{ij})_{(i,j)\in V_{d+1}}$ *denote the coefficients of the linear form* $q : x \mapsto \sum_{(i,j)\in V_{d+1}} q_{ij}x_{ij}$*. Then,* $q_{i,j+1} = p_{i,j}$ *for all* $(i,j) \in V_d$*.*

In other words, if the visualization of the coefficients $(p_{ij})_{(i,j)\in V_d}$ *of the linear form* $p : x \mapsto \sum_{(i,j)\in V_d} p_{ij}x_{ij}$ *on the grid* V_d *looks like this*

$$\begin{array}{cccccccc}
p_{0,d} & & & & & & & \\
\vdots & p_{1,d-1} & & & & & & \\
\vdots & \vdots & \ddots & & & & & \\
\vdots & \vdots & \vdots & \ddots & & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\
p_{0,0} & p_{1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & p_{d,0}
\end{array}$$

Then, the visualization of the coefficients $(q_{ij})_{(i,j)\in V_d}$ *of the linear form* q *:*

$x \mapsto \sum_{(i,j)\in V_d} q_{ij} x_{ij}$ *on the grid* V_{d+1} *looks like this*

$$
\begin{array}{ccccccccc}
p_{0,d} & & & & & & & & \\
\vdots & p_{1,d-1} & & & & & & & \\
\vdots & \vdots & \ddots & & & & & & \\
\vdots & \vdots & \vdots & \ddots & & & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\
p_{0,0} & p_{1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & p_{d,0} & \\
q_{0,0} & q_{1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & q_{d,0} & q_{d+1,0}
\end{array}.
$$

Proof. As in Proposition 19, we first show it for $p = \lambda \mathrm{diag}(i)$ and then for the sum. $\square$

Example 28. *Consider* $p = q = \mathrm{diag}(3) + \mathrm{diag}(2)$ *in* $\mathbb{Z}^{V_8}$ *and* $\mathbb{Z}^{V_9}$, *respectively. Then, p is represented by*

```
 .
 .    .
 1    1    1
 4    3    2    1
10    6    3    1    .
20   10    4    1    .    .
35   15    5    1    .    .    .
56   21    6    1    .    .    .    .
84   28    7    1    .    .    .    .    .
```

and q is represented by

```
 .
 .    .
 1    1    1
 4    3    2    1
10    6    3    1    .
20   10    4    1    .    .
35   15    5    1    .    .    .
56   21    6    1    .    .    .    .
```

84	28	7	1	.	.	.	.	.	
120	36	8	1	.	.	.	.	.	.

Proposition 29 (Sufficient condition for FC on c of uni-diag Pascal forms)**.** *Let k be some contraction size, $u = 0, \dots, k-1$, and $d' \in \mathbb{N}$. Let $p = \sum_{i=0}^{k-1} \gamma_i \mathrm{diag}(i)$ with $\gamma_i \in H$ be a hyperfield linear form and $r := \max\{i : \gamma_i \neq 0\}$. If $\mathrm{sign}(\mathrm{diag}(r))_{c_i} = \mathrm{sign}(p)_{c_i}$ for all $i = u, \dots, d'$, then p is k-contractable on c_i in H^{V_d} for all $d \geq d'$ and all $i = u, \dots, k-1$.*

Proof. Using the assumptions it is easy to see that $\mathrm{sign}(p)_{c_i} = 1$ for all $i = u, \dots r$. Therefore, by extending p from degree d' to $d'+1$, we see that $\mathrm{sign}(q_{i,0}) = 1$ for all $i \leq r$. Hence, the contractability is preserved. $\square$

Proposition 30 (Sufficient condition for FC on b of uni-diag Pascal forms)**.** *The same statement holds not only for c-columns but also for b-rows.*

Proof. Use symmetry. $\square$

We want to find contractable Pascal forms whose contractions stay fixed for all dimensions.

Definition 31. *We say a hyperfield Pascal form p is fixed-contractable for contraction size k if there exists $n \in \mathbb{N}$ such that*

- *p is k-contractable in H^{V_d} for all $d \geq n$,*
- *$\mathrm{contr}_{d'}(p) = \mathrm{contr}_{d''}(p)$ for all even $d', d'' \geq d$, and*
- *$\mathrm{contr}_{d'}(p) = \mathrm{contr}_{d''}(p)$ for all odd $d', d'' \geq d$.*

Definition 32 (Fixed-contractable Pascal forms)**.** *Given a set of hyperfield Pascal forms, contraction size k and dimension d, define the set of fixed-contractables $\mathrm{FC}(B)$ to be*

$$\begin{aligned}\mathrm{FC}(B) := \{f \in \mathrm{span}_H(B) \mid & f \text{ is } k\text{-contractable for all } d' \geq d,\\ & \mathrm{contr}_{d'}(f) = \mathrm{contr}_{d''}(f) \text{ for all even } d', d'' \geq d\\ & \mathrm{contr}_{d'}(f) = \mathrm{contr}_{d''}(f) \text{ for all odd } d', d'' \geq d\}.\end{aligned}$$

Let $k = 5$ be our contraction size from now on. If not otherwise stated, we will consider the degree $d = 40$.

Proposition 33. *Let $p = \sum \lambda_i \mathrm{row}(i)$. Assume that $p_{c_0} \geq \mathbf{2}$ for some degree $d \in \mathbb{N}$. Then $(p - \mathrm{diag}(0))_{c_0} \geq \mathbf{1}$ for all degrees greater or equal to d.*

Proof. We see that $(\mathrm{diag}(0))_{c_0} = \mathbf{1}$ is a constant vector for all degrees. Note that $p_{c_0} \geq \mathbf{2}$ for all $d' \geq d$ by Proposition 19. So, we have $(p - \mathrm{diag}(0))_{c_0} \geq \mathbf{1}$ for all dimensions $d' \geq d$. $\square$

Example 34. *Let* $p = \mathrm{row}(1) + \mathrm{row}(2) - \mathrm{diag}(0)$. *Then,* $\mathrm{sign}(p_{c_0}) = \mathbf{1}$ *for all dimensions* $d \geq 40$ *by using Proposition 33 and* $d = 40$.

Let us visualize $\mathrm{row}(1) + \mathrm{row}(2)$ *for* $d = 18$*:*

```
135
119  -16
104  -15    1
 90  -14    1    .
 77  -13    1    .    .
 65  -12    1    .    .    .
 54  -11    1    .    .    .    .
 44  -10    1    .    .    .    .    .
 35   -9    1    .    .    .    .    .    .
 27   -8    1    .    .    .    .    .    .    .
 20   -7    1    .    .    .    .    .    .    .    .
 14   -6    1    .    .    .    .    .    .    .    .    .
  9   -5    1    .    .    .    .    .    .    .    .    .    .
  5   -4    1    .    .    .    .    .    .    .    .    .    .    .
  2   -3    1    .    .    .    .    .    .    .    .    .    .    .    .
  .   -2    1    .    .    .    .    .    .    .    .    .    .    .    .    .
 -1   -1    1    .    .    .    .    .    .    .    .    .    .    .    .    .    .
 -1    .    1    .    .    .    .    .    .    .    .    .    .    .    .    .    .    .
  .    1    1    .    .    .    .    .    .    .    .    .    .    .    .    .    .    .    .
```

As we can see, it is c_0*-contractable since its* c_0*-column is positive. Subtracting* $\mathrm{diag}(0)$ *from* $\mathrm{row}(1) + \mathrm{row}(2)$ *will not change the sign of the* c_0*-column. For comparison, here is the visualization of* $\mathrm{row}(1)+\mathrm{row}(2)-\mathrm{diag}(0)$ *for* $d = 18$*:*

```
134
118  -16
103  -15    1
 89  -14    1    .
 76  -13    1    .    .
 64  -12    1    .    .    .
 53  -11    1    .    .    .    .
 43  -10    1    .    .    .    .    .
 34   -9    1    .    .    .    .    .    .
 26   -8    1    .    .    .    .    .    .    .
 19   -7    1    .    .    .    .    .    .    .    .
 13   -6    1    .    .    .    .    .    .    .    .    .
  8   -5    1    .    .    .    .    .    .    .    .    .    .
  4   -4    1    .    .    .    .    .    .    .    .    .    .    .
  1   -3    1    .    .    .    .    .    .    .    .    .    .    .    .
 -1   -2    1    .    .    .    .    .    .    .    .    .    .    .    .    .
 -2   -1    1    .    .    .    .    .    .    .    .    .    .    .    .    .    .
 -2    .    1    .    .    .    .    .    .    .    .    .    .    .    .    .    .    .
```

```
-1   1   1   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .
```

Proposition 35. *Let* $p = \sum \lambda_i \mathrm{row}(i)$*. Assume that* $p_{c_0} \geq \mathbf{0}$ *for some degree* $d \in \mathbb{N}$*. Then* $(p + \mathrm{diag}(0))_{c_0} \geq \mathbf{1}$ *for all degrees greater or equal to* d*.*

Proof. We see that $(\mathrm{diag}(0))_{c_0} = \mathbf{1}$ is a constant vector for all degrees. Note that $p_{c_0} \geq \mathbf{0}$ for all $d' \geq d$ by Proposition 19. So, we have $(p+\mathrm{diag}(0))_{c_0} \geq \mathbf{1}$ for all dimensions $d' \geq d$. $\square$

Example 36. *Let* $p = \mathrm{row}(2) + \mathrm{row}(3) - \mathrm{diag}(0)$*. Then,* $\mathrm{sign}(p_{c_0}) = -\mathbf{1}$ *by using Proposition 35 and* $d = 40$*. Moreover, we can use Proposition 24 to show* $\mathrm{sign}(p_{c_1}) = \mathbf{1}$ *since for* $d = 40$ *we have that* p *is* k*-contractable.*

Proposition 37. *Let* $p = \mathrm{diag}(0) - \mathrm{diag}(1) + \mathrm{row}(1)$*. Then,* $\mathrm{sign}(p_{c_0}) = -\mathbf{1}$ *and* $\mathrm{sign}(p_{c_1}) = \mathbf{0}$ *for all degrees* $d \geq 2$*.*

Proof. By using the definition of row and diag it is easy to see that the c_1-column of $-\mathrm{diag}(1) + \mathrm{row}(1)$ vanishes and that the c_0-column is a constant vector of value $-d$ for all degrees d. Thus, adding diag(0) does not affect the sign of the c_0-column if $d \geq 2$. $\square$

We have similar statements for the d and e-diagonals.

Proposition 38. *Let* $p = \sum \lambda_i \mathrm{col}(i)$*. Assume that* $p_{d_0} \geq \mathbf{0}$ *for some degree* $d \in \mathbb{N}$*. Then* $(p - \mathrm{col}(d))_{d_0} \geq \mathbf{1}$ *for all degrees greater or equal to* d*.*

Proof. We see that $(\mathrm{col}(d))_{d_0} = -\mathbf{1}$ is a constant vector for all degrees. Note that $p_{d_0} \geq \mathbf{0}$ for all $d' \geq d$ by Proposition 27. So, we have $(p - \mathrm{col}(d))_{d_0} \geq \mathbf{1}$ for all dimensions $d' \geq d$. $\square$

Example 39. *Let* $p = \mathrm{col}(d-3) + \mathrm{col}(d-2) - \mathrm{col}(d)$*. Then,* $\mathrm{sign}(p_{d_0}) = \mathbf{1}$ *by using Proposition 38 for all degrees* $d \geq 18$*.*

Here is a visualization of $\mathrm{col}(d-3) + \mathrm{col}(d-2)$ *for better understanding:*

```
 .
 .    .
 1    1    1
 1    .   -1   -2
 .   -1   -1    .    2
 .    .    1    2    2    .
 .    .    .   -1   -3   -5   -5
 .    .    .    .    1    4    9   14
 .    .    .    .    .   -1   -5  -14  -28
 .    .    .    .    .    .    1    6   20   48
 .    .    .    .    .    .    .   -1   -7  -27  -75
 .    .    .    .    .    .    .    .    1    8   35  110
```

```
    .    .    .    .    .    .    .    .    .   -1   -9  -44 -154
    .    .    .    .    .    .    .    .    .    .    1   10   54  208
    .    .    .    .    .    .    .    .    .    .    .   -1  -11  -65 -273
    .    .    .    .    .    .    .    .    .    .    .    .    1   12   77  350
    .    .    .    .    .    .    .    .    .    .    .    .    .   -1  -13  -90 -440
```

As we can see its $d_{0,0}$ value is zero, and thus it is not contractable. Here is a visualization of $\mathrm{col}(d)$*:*

```
    1
    .   -1
    .    .    1
    .    .    .   -1
    .    .    .    .    1
    .    .    .    .    .   -1
    .    .    .    .    .    .    1
    .    .    .    .    .    .    .   -1
    .    .    .    .    .    .    .    .    1
    .    .    .    .    .    .    .    .    .   -1
    .    .    .    .    .    .    .    .    .    .    1
    .    .    .    .    .    .    .    .    .    .    .   -1
    .    .    .    .    .    .    .    .    .    .    .    .    1
    .    .    .    .    .    .    .    .    .    .    .    .    .   -1
    .    .    .    .    .    .    .    .    .    .    .    .    .    .    1
    .    .    .    .    .    .    .    .    .    .    .    .    .    .    .   -1
    .    .    .    .    .    .    .    .    .    .    .    .    .    .    .    .    1
```

By subtracting, we get a contractable form:

```
   -1
    .    1
    1    1    .
    1    .   -1   -1
    .   -1   -1    .    1
    .    .    1    2    2    1
    .    .    .   -1   -3   -5   -6
    .    .    .    .    1    4    9   15
    .    .    .    .    .   -1   -5  -14  -29
    .    .    .    .    .    .    1    6   20   49
    .    .    .    .    .    .    .   -1   -7  -27  -76
    .    .    .    .    .    .    .    .    1    8   35  111
    .    .    .    .    .    .    .    .    .   -1   -9  -44 -155
    .    .    .    .    .    .    .    .    .    .    1   10   54  209
    .    .    .    .    .    .    .    .    .    .    .   -1  -11  -65 -274
    .    .    .    .    .    .    .    .    .    .    .    .    1   12   77  351
    .    .    .    .    .    .    .    .    .    .    .    .    .   -1  -13  -90 -441
```

Proposition 40. *Let $i = 0, \ldots, k-1$. Assume $p = \sum \lambda_i \mathrm{row}(i)$ is fixed-contractable on c_i. Then, $p + \lambda \mathrm{diag}(j)$ is fixed-contractable on c_i for all $j \in \{0, \ldots, i-1\} \cup \{d-k+1, \ldots, d\}$ and $\lambda \in \mathbb{Z}$.*

Proof. It is easy to see that $\mathrm{diag}(j)_{c_i} = \mathbf{0}$. Hence, contractability on c_i is preserved. $\square$

Example 41. *Let $d = 40$. Consider $p = \mathrm{diag}(0) - \mathrm{diag}(d) + \mathrm{row}(3) + \mathrm{row}(4) - \mathrm{row}(1)$. We see that $\mathrm{row}(3) + \mathrm{row}(4) - \mathrm{row}(1)$ is fixed-contractable on c_1 by Proposition 24. By applying Proposition 40 we see that p is also c_1-fixed-contractable.*

Proposition 42. *Assume $p = \sum \lambda_i \mathrm{row}(i)$ is fixed-contractable on c_0 and c_1. If $\mathrm{sign}(p)_{c_0} = \mathbf{1}$ and $p_{c_1} < -\mathbf{1}$, then $p + \mathrm{diag}(1)$ is fixed-contractable on c_0 and c_1.*

Proof. Since $\mathrm{diag}(1)_{c_0} \geq \mathbf{1}$, we have $(p+\mathrm{diag}(1))_{c_0} > \mathbf{1}$ for all degrees. Since $\mathrm{diag}(1)_{c_1} = \mathbf{1}$, we have $(p + \mathrm{diag}(1))_{c_1} < \mathbf{0}$ for all degrees. $\square$

Example 43. *Let $d = 16$. Consider $p = \mathrm{diag}(1) + \mathrm{row}(1) + \mathrm{row}(2)$. We see that $\mathrm{row}(1) + \mathrm{row}(2)$ is fixed-contractable on c_0 and c_1 by Proposition 24. By applying Proposition 42 we see that p is also c_0 and c_1-fixed-contractable.*

Here is $\mathrm{row}(1) + \mathrm{row}(2)$*:*

```
104
 90  -14
 77  -13    1
 65  -12    1    .
 54  -11    1    .    .
 44  -10    1    .    .    .
 35   -9    1    .    .    .    .
 27   -8    1    .    .    .    .    .
 20   -7    1    .    .    .    .    .    .
 14   -6    1    .    .    .    .    .    .    .
  9   -5    1    .    .    .    .    .    .    .    .
  5   -4    1    .    .    .    .    .    .    .    .    .
  2   -3    1    .    .    .    .    .    .    .    .    .    .
  .   -2    1    .    .    .    .    .    .    .    .    .    .    .
 -1   -1    1    .    .    .    .    .    .    .    .    .    .    .    .
 -1    .    1    .    .    .    .    .    .    .    .    .    .    .    .    .
  .    1    1    .    .    .    .    .    .    .    .    .    .    .    .    .    .
```

Here is p:

```
104
 91  -13
 79  -12    1
 68  -11    1    .
 58  -10    1    .    .
 49   -9    1    .    .    .
 41   -8    1    .    .    .    .
 34   -7    1    .    .    .    .    .
```

```
28  -6   1   .   .   .   .   .   .
23  -5   1   .   .   .   .   .   .   .
19  -4   1   .   .   .   .   .   .   .   .
16  -3   1   .   .   .   .   .   .   .   .   .
14  -2   1   .   .   .   .   .   .   .   .   .   .
13  -1   1   .   .   .   .   .   .   .   .   .   .   .
13   .   1   .   .   .   .   .   .   .   .   .   .   .   .
14   1   1   .   .   .   .   .   .   .   .   .   .   .   .   .
16   2   1   .   .   .   .   .   .   .   .   .   .   .   .   .   .
```

Proposition 44. *Let $d = 40$, $k = 5$ and $D = \{0, 1, 2, 3, 4, 36, 37, 38, 39, 40\}$. Let p be a hyperfield Pascal form which satisfies one of the following conditions:*

1. $p = \mathrm{row}(i_1) + \mathrm{row}(i_2)$ *for* $i \in D^2$,

If p is k-contractable in H^{V_d}, then $p \in \mathrm{FC}(B)$.

Algorithm 3 Computing a subset of $\mathrm{CC}(B)$

Input: Set $B \subset \{\mathrm{col}(i), \mathrm{row}(i), \mathrm{diag}(i)\}$, contraction size k, dimension d

Output: set C that is a subset of $\mathrm{CC}(B)$

```
function GENERATE(B, k, d)
    initialize empty list C
    for p ∈ span_H(B) do
        if p is k-contractable in H^{V_d} then
            if proveCC(p) then
                add p to C
            end if
        end if
    end for
    return C
end function

function PROVECC(p)
    initialize empty list C
    for p ∈ span_H(B) do
        if p is k-contractable in H^{V_d} then
            if proveCC(p) then
                add p to C
            end if
        end if
    end for
    return C
end function
```