# Tropical Algebra

## A. Tropical Arithmetic

- ***Tropical semiring*** $(\bar{\mathbb{R}}, \oplus, \odot)$ is the set $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with

$$y \oplus x = \min\{x, y\} \quad \text{and} \quad x \odot y = x + y \qquad x, y \in \bar{\mathbb{R}}$$

For example:

$$4 \oplus 5 = 4, \quad 4 \odot 5 = 9, \quad (-100) \oplus (-10) = -100, \quad x = 0 \odot x \quad \forall x \in \mathbb{R}$$

Exercise: What is the additive and multiplicative neutral element?

Find an $x \in \mathbb{R} \cup \{\infty\}$ such that $y = y \oplus x$ for all $y \in \mathbb{R}$.
Find an $x \in \mathbb{R}$ such that $y = y \odot x$ for all $y \in \mathbb{R}$.

Exercise: What is the division? How about subtraction?

Find an $x \in \mathbb{R}$ such that $10 = x \odot 3$. Can we always do this?
Can we find $x \in \mathbb{R}$ such that $17 = 10 \oplus x$?

Exercise: Freshman's dream

If you are bored at home, try proving $(x \oplus y)^{\odot 3} = x^{\odot 3} \oplus y^{\odot 3}$.

- Vector operations: $\oplus$ and $\odot$ operate component-wise

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \min\{4,2\} \\ \min\{2,5\} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad 42 \odot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 45 \\ 44 \end{bmatrix}$$

- Matrix-vector operation: $\begin{bmatrix} 2 & 3 \\ 5 & 11 \end{bmatrix} \odot \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

## B. Tropical Linear Algebra

- Tropical arithmetic occurs naturally in discrete math; for example

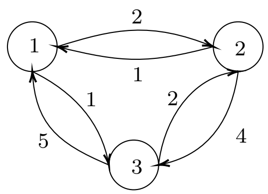
Theorem: Shortest Path [2, Prop. 7.9]

Let $D_G$ be the adjacency matrix of a directed graph $G$ which has only non-negative edges and contains no loops. Then, the length of the shortest path from node $i$ to $j$ is given by row $i$ and column $j$ of the matrix

$$D_G^{\odot(n-1)} = D_G \odot \cdots \odot D_G.$$

*Proof.* See blackboard. □

Example




$$D_G = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 4 \\ 5 & 2 & 0 \end{bmatrix} \quad \text{and} \quad D_G^{\odot 2} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}.$$

Row 3 and column 1 of $D_G^{\odot 2}$ yields that the shortest path from node 3 to 1 has length 3 (consider path $3 \rightsquigarrow 2 \rightsquigarrow 1$ which is shorter that $3 \rightsquigarrow 1$).

Exercise: Why does the Shortest Path Theorem fail for graphs with loops?

Note: the ***Kleene plus***

$$D_G^{+} = D_G \oplus D_G^{\odot 2} \oplus \ldots \oplus D_G^{\odot n}.$$

generalizes the Shortest Path Theorem to graphs with *negative* edge lengths and *loops*; see [1, Exercise 1.9 (5)]

- ***Tropical determinant*** of an $n \times n$-matrix $X = (x_{ij})$ is defined as

$$\text{tropdet}(X) = \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)},$$

where $S_n = \{\pi : \pi \text{ is a permutation of } \{1, 2, ..., n\}\}$

**Theorem: Assignment Problem [2, Prop. 7.10]**

The optimal cost of the assignment problem is given by the tropical determinant.

*Proof.* Follows directly from definition. □

**Example**

$$\text{tropdet} \begin{bmatrix} 1 & 5 & \mathbf{0} \\ \mathbf{2} & 7 & 4 \\ 0 & \mathbf{3} & 2 \end{bmatrix} = 0 \odot 2 \odot 3 = 5$$

- An ***eigenvalue*** $\lambda \in \mathbb{R}$ of $A \in \bar{\mathbb{R}}^{n \times n}$ is a scalar that satisfies

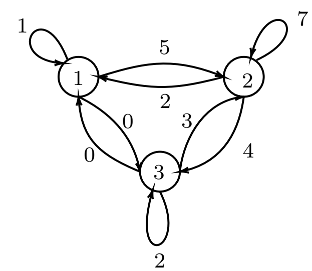
$$A \odot v = \lambda \odot v \quad \text{for some } v \in \mathbb{R}^n$$

**Theorem: Square matrices have exactly one eigenvalue [2, Thm. 7.11]**

Let $A$ be an $n \times n$-matrix whose graph $G(A)$ is strongly connected. Then, $A$ has *precisely* one eigenvalue, denoted $\lambda(A)$. The eigenvalue $\lambda(A)$ equals the minimum of normalized length over all directed cycles in $G(A)$.

*Proof.* See blackboard. We use the *Kleene plus*. □

**Example**




Consider the 1-cycle $2 \rightsquigarrow 2$. It has normalized length $\frac{7}{1} = 7$.
Consider the 2-cycle $1 \rightsquigarrow 2 \rightsquigarrow 1$. It has normalized length $\frac{5+2}{2} = 3.5$.
Consider the 2-cycle $1 \rightsquigarrow 3 \rightsquigarrow 1$. It has normalized length $\frac{0+0}{2} = 0$. In fact, $\lambda(A) = 0$.

- The ***eigenspace*** is defined as $\text{Eig}(A) = \{x \in \mathbb{R}^n : A \odot x = \lambda \odot x\}$
  - Define $B = (-\lambda(A)) \odot A$
  - Simple to see: $\text{Eig}(A) = \text{Eig}(B) = \{x : B \odot x = x\}$

**Theorem: The eigenspace of a tropical matrix [2, Thm. 7.13]**

Let $B_0^+$ be the submatrix of $B^+$ given by those columns whose diagonal entry $B_{jj}^+$ is zero. The (tropical) image of this matrix is equal to the eigenspace of $A$

$$\text{Eig}(A) = \text{Eig}(B) = \text{Image}(B_0^+).$$

*Proof.* Omitted. □

## C. Literature

[1] Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

[2] M. Michałek and B. Sturmfels. *Invitation to Nonlinear Algebra*. Graduate Studies in Mathematics. American Mathematical Society, 2021.