# Tropical Algebra

## I. Tropical Arithmetic

- ***Tropical semiring*** $(\bar{\mathbb{R}}, \oplus, \odot)$ is the set $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with

$$y \oplus x = \min\{x, y\} \quad \text{and} \quad x \odot y = x + y \qquad x, y \in \bar{\mathbb{R}}$$

For example:

$$4 \oplus 5 = 4, \quad 4 \odot 5 = 9, \quad (-100) \oplus (-10) = -100, \quad x = 0 \odot x \quad \forall x \in \mathbb{R}$$

**Exercise: Warmup**

Calculate (i) $10 \oplus -5$, (ii) $8 \odot -4$, (iii) $10 \oplus \infty$, and (iv) $4^{\odot 5}$.

**Exercise: What is the additive and multiplicative neutral element?**

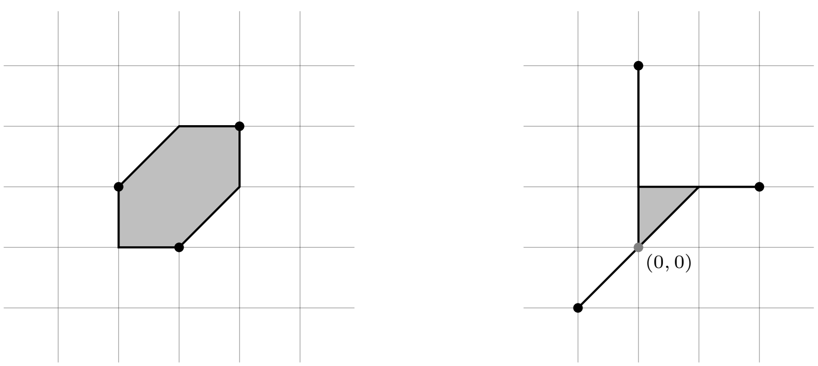
Find an $x \in \mathbb{R} \cup \{\infty\}$ such that $y = y \oplus x$ for all $y \in \mathbb{R}$.
Find an $x \in \mathbb{R}$ such that $y = y \odot x$ for all $y \in \mathbb{R}$.

**Exercise: What is the division? How about subtraction?**

Find an $x \in \mathbb{R}$ such that $10 = x \odot 3$. Can we always do this?
Can we find $x \in \mathbb{R}$ such that $17 = 10 \oplus x$?

- Matrix-Vector operations: $\oplus$ and $\odot$ operate in $\bar{\mathbb{R}}^{n \times n}$ as we would expect it

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \min\{4,2\} \\ \min\{2,5\} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad 42 \odot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 45 \\ 44 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 5 & 11 \end{bmatrix} \odot \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$




(a) The tropical image of a $3 \times 3$-matrix in $\mathbb{R}^3/\mathbf{1}\mathbb{R}$

(b) A tropical triangle in $\mathbb{R}^3/\mathbf{1}\mathbb{R}$

## II. Tropical Linear Algebra

### A. Tropical Matrices and Shortest Path

- Tropical arithmetic occurs naturally in discrete math; for example

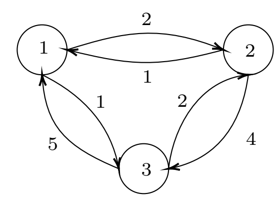
**Theorem: Shortest Path [2, Prop. 7.9]**

Let $D_G$ be the adjacency matrix of a directed graph $G$ which has only non-negative edges and contains no loops. Then, the length of the shortest path from node $i$ to $j$ is given by row $i$ and column $j$ of the matrix

$$D_G^{\odot(n-1)} = D_G \odot \cdots \odot D_G.$$

*Proof.* See blackboard. □

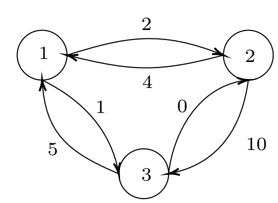
**Example**




$$D_G = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 4 \\ 5 & 2 & 0 \end{bmatrix} \quad \text{and} \quad D_G^{\odot 2} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}.$$

Row 3 and column 1 of $D_G^{\odot 2}$ yields that the shortest path from node 3 to 1 has length 3 (consider path $3 \rightsquigarrow 2 \rightsquigarrow 1$ which is shorter that $3 \rightsquigarrow 1$).

**Exercise: Find the length of each shortest path.**



- Theorem fails for graphs with loops; however, the ***Kleene plus*** $D_G^+ = D_G \oplus D_G^{\odot 2} \oplus \ldots \oplus D_G^{\odot n}$ generalizes the Shortest Path Theorem to graphs with *negative* edge lengths and *loops* [1, Exercise 1.9 (5)]

## B. Determinant and the Assignment Problem

- The ***tropical determinant*** of an $n \times n$-matrix $X = (x_{ij})$ is defined as

$$\operatorname{tropdet}(X) = \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)},$$

where $S_n = \{\pi : \pi \text{ is a permutation of } \{1, 2, \ldots, n\}\}$

Theorem: Assignment Problem [2, Prop. 7.10]

The optimal cost of the assignment problem is given by the tropical determinant.

*Proof.* Follows directly from definition. □

Example

$$\operatorname{tropdet}\begin{bmatrix} 1 & 5 & \mathbf{0} \\ \mathbf{2} & 7 & 4 \\ 0 & \mathbf{3} & 2 \end{bmatrix} = 0 \odot 2 \odot 3 = 5$$

Exercise: Calculate the tropical determinant.

$$\operatorname{tropdet}\begin{bmatrix} 0 & 4 & 1 \\ 1 & 8 & 6 \\ 1 & 2 & 0 \end{bmatrix}$$

## C. Eigenvalues and Cycle Lengths

- An ***eigenvalue*** $\lambda \in \mathbb{R}$ of $A \in \bar{\mathbb{R}}^{n\times n}$ is a scalar that satisfies

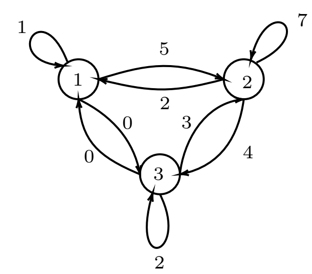
$$A \odot v = \lambda \odot v \quad \text{for some } v \in \mathbb{R}^n$$

Theorem: Square matrices have exactly one eigenvalue [2, Thm. 7.11]

Let $A$ be an $n \times n$-matrix whose graph $G(A)$ is strongly connected. Then, $A$ has *precisely* one eigenvalue, denoted $\lambda(A)$. The eigenvalue $\lambda(A)$ equals the minimum of normalized length over all directed cycles in $G(A)$.

*Proof.* Use the *Kleene plus*. □

Example




Consider the 1-cycle $2 \rightsquigarrow 2$. It has normalized length $\frac{7}{1} = 7$.
Consider the 2-cycle $1 \rightsquigarrow 2 \rightsquigarrow 1$. It has normalized length $\frac{5+2}{2} = 3.5$.
Consider the 2-cycle $1 \rightsquigarrow 3 \rightsquigarrow 1$. It has normalized length $\frac{0+0}{2} = 0$. In fact, $\lambda(A) = 0$.

Exercise: Calculate the eigenvalue

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \end{bmatrix}, \; C = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 4 & 3 \end{bmatrix}$$

# C. Literature

[1] Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

[2] M. Michałek and B. Sturmfels. *Invitation to Nonlinear Algebra*. Graduate Studies in Mathematics. American Mathematical Society, 2021.