# Tropical Algebra

## A.

- $S \subset \mathbb{R}^n$ is ***tropically convex*** if $x, y \in S$ and $a, b \in \mathbb{R}$ implies that $a \odot x \oplus b \odot y \in S$
  - in other words: *tropical convex combinations are just tropical linear combinations*
  - if $S$ is tropically convex, then $S + \mathbb{R}1 \subset S$; hence we identify $S$ with its image in $\mathbb{R}^n/\mathbb{R}1 \subset \mathbb{TP}^{n-1}$
  - Example: $V = \{(-2,1,1), (-2,-2,-2), (-1,-1,-2), (0,0,0)\}$, the image of $V$ in $\mathbb{TP}^{n-1}$ is $\{(-3,0,0), (0,0,0), (1,1,0)\}$

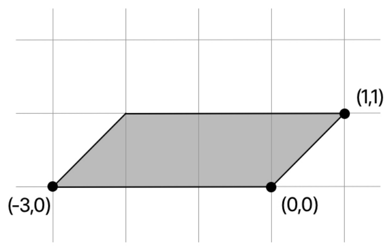



Figure 1: The tropical convex hull of $V$

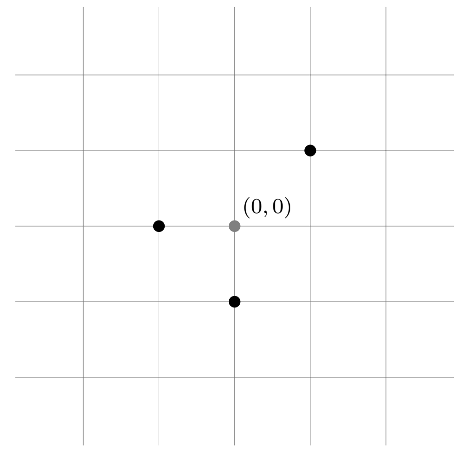
- ***Tropical polynomial function*** $p : \mathbb{R}^n \to \mathbb{R}$ is the minimum of a finite set of linear functions
- ***Hypersurface*** $V(p)$ is the set of all points where $p$ attains its minimum at least twice; equivalently $p$ is not linear at $w$

## B. Tropical Linear Algebra

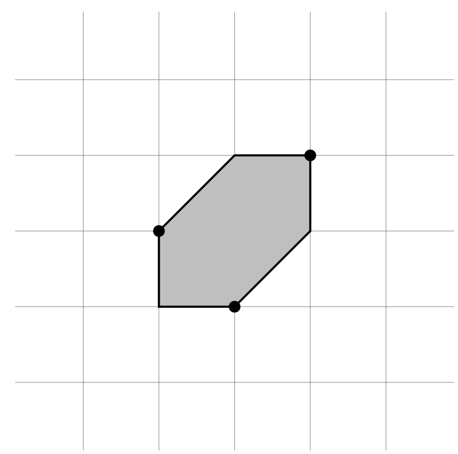
- We want to compute the eigenspace of $A = \begin{bmatrix} 3 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \end{bmatrix}$. Its eigenvalue $\lambda(A)$ is 3 because of the loop. Then, $B = B^+ = B_0^+ = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The eigenspace is the image of $B_0^+$ under tropical arithmetic. Multiplying with the unit vectors $e_1 = [0, \infty, \infty]^T$, $e_2 = [\infty, 0, \infty]^T$, $e_3 = [\infty, \infty, 0]^T$ from the right yields the vectors $[0,1,1]^T$, $[1,0,1]^T$, $[1,1,0]^T$, respectively. We normalize the vectors to draw them in the projective space $\mathbb{TP}^2$ (normalizing means to multiply each vector by a scalar such that the third coordinate vanishes). This gives us

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Here is the picture in $\mathbb{TP}^2$:




Next, we draw the tropical convex hull of these three points to obtain the image of $B_0^+$ in $\mathbb{TP}^2$. The eigenspace is a hexagon



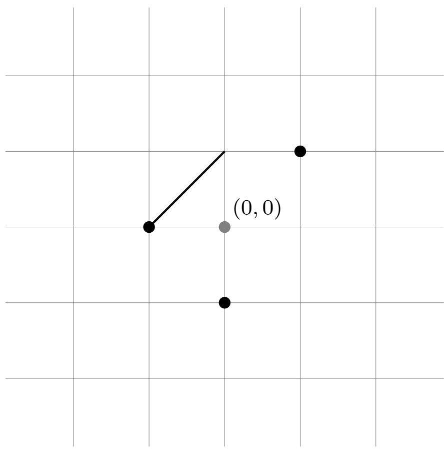
Here is a computation for the convex hull of $\{[-1,0,0]^T, [1,1,0]^T\}$. We want to compute $a \odot [-1,0,0]^T \oplus b \odot [1,1,0]^T = [\min(a-1, b+1), \min(a, b+1), \min(a,b)]^T$ for any real scalars $a$ and $b$. Consider the case that the convex hull yields the point

$$\begin{bmatrix} a-1 \\ a \\ b \end{bmatrix} \overset{\text{normalize}}{\rightsquigarrow} \begin{bmatrix} a-1-b \\ a-b \\ 0 \end{bmatrix}.$$

This case occurs if and only if

$$\begin{cases} a-1 & \leq b+1 \\ a & \leq b+1 \\ b & \leq a \end{cases} \iff \begin{cases} a-2 & \leq b \\ a-1 & \leq b \\ b & \leq a \end{cases}.$$

So, $b \in [a-1, a]$. Next, by assumption we have $x = a-1-b$ and $y = a-b$. Subtracting the first equation with the second gives $x - y = -1$. Hence, $x + 1 = y$. Since $b \in [a-1, a]$, we obtain $x \in [-1, 0]$. With this information, we draw a line segment $x + 1 = y$ for $x \in [-1, 0]$:




However, this does not finish the computation of the convex hull yet. We still need to consider the other cases. Some lead to contradictions, some lead to valid cases. If we compute everything correctly, we will obtain the following picture for the convex hull of $\{[-1,0,0]^T, [1,1,0]^T\}$:

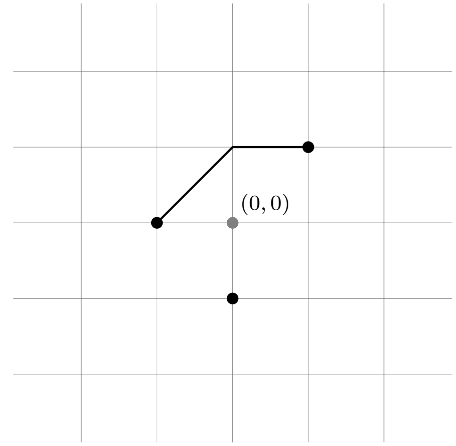



Figure 2: Convex hull of $\{[-1,0,0]^T, [1,1,0]^T\}$