

Tropical Algebra

A. Tropical Arithmetic

- **Tropical semiring** $(\bar{\mathbb{R}}, \oplus, \odot)$ is the set $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with

$$y \oplus x = \min\{x, y\} \quad \text{and} \quad x \odot y = x + y \quad x, y \in \bar{\mathbb{R}}$$

For example:

$$4 \oplus 5 = 4, \quad 4 \odot 5 = 9, \quad (-100) \oplus (-10) = -100, \quad x = 0 \odot x \quad \forall x \in \bar{\mathbb{R}}$$

Exercise: What is the additive and multiplicative neutral element?

Find an $x \in \mathbb{R} \cup \{\infty\}$ such that $y = y \oplus x$ for all $y \in \bar{\mathbb{R}}$.

Find an $x \in \mathbb{R}$ such that $y = y \odot x$ for all $y \in \bar{\mathbb{R}}$.

Exercise: What is the division? How about subtraction?

Find an $x \in \bar{\mathbb{R}}$ such that $10 = x \odot 3$. Can we always do this?

Can we find $x \in \bar{\mathbb{R}}$ such that $17 = 10 \oplus x$?

Exercise: Freshman's dream

If you are bored at home, try proving $(x \oplus y)^{\odot 3} = x^{\odot 3} \oplus y^{\odot 3}$.

- Vector operations: \oplus and \odot operate component-wise

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \min\{4, 2\} \\ \min\{2, 5\} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad 42 \odot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 45 \\ 44 \end{bmatrix}$$

- Matrix-vector operation: $\begin{bmatrix} 2 & 3 \\ 5 & 11 \end{bmatrix} \odot \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

B. Tropical Linear Algebra

- Tropical arithmetic occurs naturally in discrete math; for example

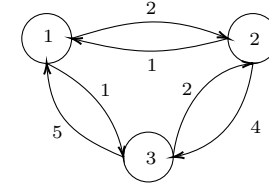
Theorem: Shortest Path [2, Prop. 7.9]

Let D_G be the adjacency matrix of a directed graph G which has only non-negative edges and contains no loops. Then, the length of the shortest path from node i to j is given by row i and column j of the matrix

$$D_G^{\odot(n-1)} = D_G \odot \cdots \odot D_G.$$

Proof. See blackboard. □

Example



$$D_G = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 4 \\ 5 & 2 & 0 \end{bmatrix} \quad \text{and} \quad D_G^{\odot 2} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}.$$

Row 3 and column 1 of $D_G^{\odot 2}$ yields that the shortest path from node 3 to 1 has length 3 (consider path $3 \rightsquigarrow 2 \rightsquigarrow 1$ which is shorter than $3 \rightsquigarrow 1$).

Exercise: Why does the Shortest Path Theorem fail for graphs with loops?

Note: the **Kleene plus**

$$D_G^+ = D_G \oplus D_G^{\odot 2} \oplus \cdots \oplus D_G^{\odot n}.$$

generalizes the Shortest Path Theorem to graphs with *negative* edge lengths and *loops*; see [1, Exercise 1.9 (5)]

- **Tropical determinant** of an $n \times n$ -matrix $X = (x_{ij})$ is defined as

$$\text{tropdet}(X) = \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)},$$

where $S_n = \{\pi : \pi \text{ is a permutation of } \{1, 2, \dots, n\}\}$

Theorem: Assignment Problem [2, Prop. 7.10]

The optimal cost of the assignment problem is given by the tropical determinant.

Proof. Follows directly from definition. □

Example

$$\text{tropdet} \begin{bmatrix} 1 & 5 & \mathbf{0} \\ \mathbf{2} & 7 & 4 \\ 0 & \mathbf{3} & 2 \end{bmatrix} = 0 \odot 2 \odot 3 = 5$$

- An **eigenvalue** $\lambda \in \mathbb{R}$ of $A \in \mathbb{R}^{n \times n}$ is a scalar that satisfies

$$A \odot v = \lambda \odot v \quad \text{for some } v \in \mathbb{R}^n$$

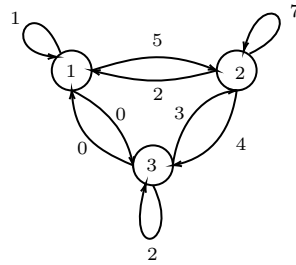
Theorem: Square matrices have exactly one eigenvalue [2, Thm. 7.11]

Let A be an $n \times n$ -matrix whose graph $G(A)$ is strongly connected. Then, A has *precisely* one eigenvalue, denoted $\lambda(A)$. The eigenvalue $\lambda(A)$ equals the minimum of normalized length over all directed cycles in $G(A)$.

Proof. See blackboard. We use the *Kleene plus*.

□

Example



Consider the 1-cycle $2 \rightsquigarrow 2$. It has normalized length $\frac{7}{1} = 7$.

Consider the 2-cycle $1 \rightsquigarrow 2 \rightsquigarrow 1$. It has normalized length $\frac{5+2}{2} = 3.5$.

Consider the 2-cycle $1 \rightsquigarrow 3 \rightsquigarrow 1$. It has normalized length $\frac{0+0}{2} = 0$. In fact, $\lambda(A) = 0$.

- The **eigenspace** is defined as $\text{Eig}(A) = \{x \in \mathbb{R}^n : A \odot x = \lambda \odot x\}$
 - Define $B = (-\lambda(A)) \odot A$
 - Simple to see: $\text{Eig}(A) = \text{Eig}(B) = \{x : B \odot x = x\}$

Theorem: The eigenspace of a tropical matrix [2, Thm. 7.13]

Let B_0^+ be the submatrix of B^+ given by those columns whose diagonal entry B_{jj}^+ is zero. The (tropical) image of this matrix is equal to the eigenspace of A

$$\text{Eig}(A) = \text{Eig}(B) = \text{Image}(B_0^+).$$

Proof. Omitted.

□

C. Literature

- [1] Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
- [2] M. Michałek and B. Sturmfels. *Invitation to Nonlinear Algebra*. Graduate Studies in Mathematics. American Mathematical Society, 2021.