

## Tropical Algebra

### A. Tropical Arithmetic

- **Tropical semiring**  $(\mathbb{R}, \oplus, \odot)$  is the set  $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  with

$$y \oplus x = \min\{x, y\} \quad \text{and} \quad x \odot y = x + y \quad x, y \in \bar{\mathbb{R}}$$

For example:

$$4 \oplus 5 = 4, \quad 4 \odot 5 = 9, \quad (-100) \oplus (-10) = -100, \quad x = 0 \odot x \quad \forall x \in \mathbb{R}$$

**Exercise:** What is the additive and multiplicative neutral element?

Find an  $x \in \mathbb{R} \cup \{\infty\}$  such that  $y = y \oplus x$  for all  $y \in \mathbb{R}$ .

Find an  $x \in \mathbb{R}$  such that  $y = y \odot x$  for all  $y \in \mathbb{R}$ .

**Exercise:** What is the division? How about subtraction?

Find an  $x \in \mathbb{R}$  such that  $10 = x \odot 3$ . Can we always do this?

Can we find  $x \in \mathbb{R}$  such that  $17 = 10 \oplus x$ ?

**Exercise:** Freshman's dream

If you are bored at home, try proving  $(x \oplus y)^{\odot 3} = x^{\odot 3} \oplus y^{\odot 3}$ .

- Vector operations:  $\oplus$  and  $\odot$  operate component-wise

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \min\{4, 2\} \\ \min\{2, 5\} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad 42 \odot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 45 \\ 44 \end{bmatrix}$$

- Matrix-vector operation:  $\begin{bmatrix} 2 & 3 \\ 5 & 11 \end{bmatrix} \odot \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

### B. Tropical Linear Algebra

- Tropical arithmetic occurs naturally in discrete math; for example

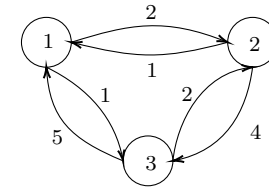
**Theorem: Shortest Path [2, Prop. 7.9]**

Let  $D_G$  be the adjacency matrix of a directed graph  $G$  which has only non-negative edges and contains no loops. Then, the length of the shortest path from node  $i$  to  $j$  is given by row  $i$  and column  $j$  of the matrix

$$D_G^{\odot(n-1)} = D_G \odot \dots \odot D_G.$$

*Proof.* See blackboard. □

**Example**



$$D_G = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 4 \\ 5 & 2 & 0 \end{bmatrix} \quad \text{and} \quad D_G^{\odot 2} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}.$$

Row 3 and column 1 of  $D_G^{\odot 2}$  yields that the shortest path from node 3 to 1 has length 3 (consider path  $3 \rightsquigarrow 2 \rightsquigarrow 1$  which is shorter than  $3 \rightsquigarrow 1$ ).

**Exercise:** Why does the Shortest Path Theorem fail for graphs with loops?

Note: the **Kleene plus**

$$D_G^+ = D_G \oplus D_G^{\odot 2} \oplus \dots \oplus D_G^{\odot n}.$$

generalizes the Shortest Path Theorem to graphs with *negative* edge lengths and *loops*; see [1, Exercise 1.9 (5)]

- **Tropical determinant** of an  $n \times n$ -matrix  $X = (x_{ij})$  is defined as

$$\text{tropdet}(X) = \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)},$$

where  $S_n = \{\pi : \pi \text{ is a permutation of } \{1, 2, \dots, n\}\}$

**Theorem: Assignment Problem [2, Prop. 7.10]**

The optimal cost of the assignment problem is given by the tropical determinant.

*Proof.* Follows directly from definition. □

**Example**

$$\text{tropdet} \begin{bmatrix} 1 & 5 & \mathbf{0} \\ \mathbf{2} & 7 & 4 \\ 0 & \mathbf{3} & 2 \end{bmatrix} = 0 \odot 2 \odot 3 = 5$$

- An **eigenvalue**  $\lambda \in \mathbb{R}$  of  $A \in \mathbb{R}^{n \times n}$  is a scalar that satisfies

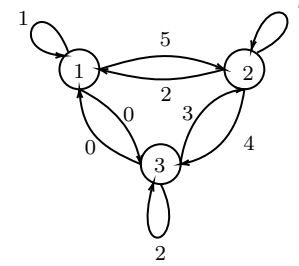
$$A \odot v = \lambda \odot v \quad \text{for some } v \in \mathbb{R}^n$$

**Theorem: Square matrices have exactly one eigenvalue [2, Thm. 7.11]**

Let  $A$  be an  $n \times n$ -matrix whose graph  $G(A)$  is strongly connected. Then,  $A$  has *precisely* one eigenvalue, denoted  $\lambda(A)$ . The eigenvalue  $\lambda(A)$  equals the minimum of normalized length over all directed cycles in  $G(A)$ .

*Proof.* See blackboard. We use the *Kleene plus*.

**Example**



Consider the 1-cycle  $2 \rightsquigarrow 2$ . It has normalized length  $\frac{7}{1} = 7$ .

Consider the 2-cycle  $1 \rightsquigarrow 2 \rightsquigarrow 1$ . It has normalized length  $\frac{5+2}{2} = 3.5$ .

Consider the 2-cycle  $1 \rightsquigarrow 3 \rightsquigarrow 1$ . It has normalized length  $\frac{0+0}{2} = 0$ . In fact,  $\lambda(A) = 0$ .

- The **eigenspace** is defined as  $\text{Eig}(A) = \{x \in \mathbb{R}^n : A \odot x = \lambda \odot x\}$ 
  - Define  $B = (-\lambda(A)) \odot A$
  - Simple to see:  $\text{Eig}(A) = \text{Eig}(B) = \{x : B \odot x = x\}$

**Theorem: The eigenspace of a tropical matrix [2, Thm. 7.13]**

Let  $B_0^+$  be the submatrix of  $B^+$  given by those columns whose diagonal entry  $B_{jj}^+$  is zero.

The (tropical) image of this matrix is equal to the eigenspace of  $A$

$$\text{Eig}(A) = \text{Eig}(B) = \text{Image}(B_0^+).$$

*Proof.* Omitted. □

□

## C. Literature

- [1] Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
- [2] M. Michałek and B. Sturmfels. *Invitation to Nonlinear Algebra*. Graduate Studies in Mathematics. American Mathematical Society, 2021.