2

LINEAR DYNAMIC SYSTEMS

> All the effects of nature are only mathematical results of a small number of immutable laws.
>
> —Pierre-Simon, marquis de Laplace (1749–1827)

2.1 CHAPTER FOCUS

This chapter is about the dynamic models used in Kalman filtering, and especially those represented by systems of linear differential equations.

The objective will be to demonstrate, using specific examples, how one goes about building such models and how one can go from a model using differential equations to one suitable for Kalman filtering.

Where one gets these differential equations depends on the application. For modeling electromechanical systems, these differential equations generally come from the laws of physics.

For example, the differential equations for modeling many mechanical systems come from Newton's laws of mechanics, such as $F = ma$, its rotational companion $T = M\dot{\omega}$, or Newton's laws of gravitation (used in Example 2.1).

2.1.1 The Bigger Picture

How the dynamic models of this chapter fit into the overall estimation problem is illustrated in the simplified schematic of Figure 2.1, with variables as defined in

Kalman Filtering: Theory and Practice Using MATLAB®, Fourth Edition.
Mohinder S. Grewal and Angus P. Andrews.

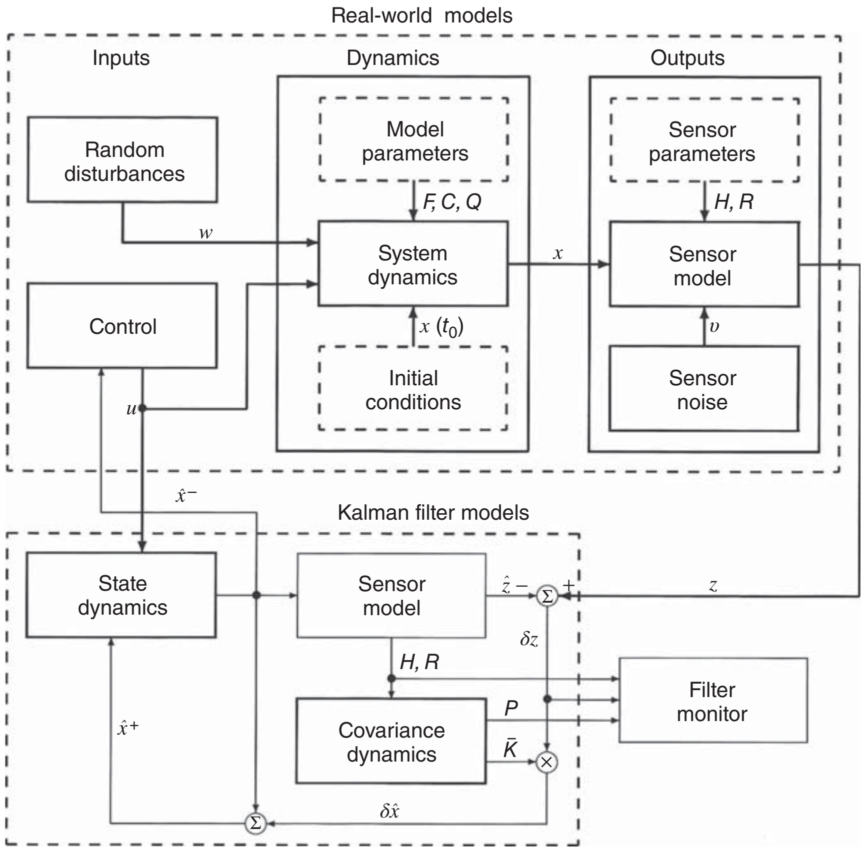

Figure 2.1 Schematic of Kalman filter implementation.

Table 2.1. The schematic has been simplified to represent either continuous-time or discrete-time applications, the relationships between which are covered in this chapter.

The schematic shows the following three dynamic models.

- The one labeled "System Dynamics" models the real-world dynamics in the intended application of the Kalman filter. This model may contain state variables for representing time-correlated random processes such as
  - time-correlated random dynamic disturbances, such as winds and (for ships) currents;
  - time-correlated sensor noise, such as the effects of ambient temperature;
  - in some applications, the model may also contain as additional state variables the slowly varying "parameters" of its own dynamic model.

  Ideally, this model would be based on the laws of physics and the results of calibrating (with a Kalman filter) the real-world system dynamic parameters.

TABLE 2.1 Mathematical Models for Dynamic Systems

Model Type	Continuous-Time Model	Discrete-Time Model
Time invariant		
Linear	$\dot{x}(t) = Fx(t) + Cu(t) + w(t)$	$x_k = \Phi x_{k-1} + \Gamma u_{k-1} + w_{k-1}$
	$z(t) = Hx(t) + Du(t) + v(t)$	$z_k = Hx_k + Du_k + v_k$
General	$\dot{x}(t) = f(x(t), u(t)) + w(t)$	$x_k = f(x_{k-1}, u_{k-1}) + w_{k-1}$
	$z(t) = h(x(t), u(t)) + v(t)$	$z_k = h(x_k, u_k) + v_k$
Time varying		
Linear	$\dot{x}(t) = F(t)x(t) + C(t)u(t) + w(t)$	$x_k = \Phi_{k-1}x_{k-1} + \Gamma_{k-1}u_{k-1} + w_{k-1}$
	$z(t) = H(t)x(t) + D(t)u(t) + v(t)$	$z_k = H_k x_k + D_k u_k + v_k$
General	$\dot{x}(t) = f(t, x(t), u(t)) + w(t)$	$x_k = f_k(x_{k-1}, u_{k-1}) + w_{k-1}$
	$z(t) = h(t, x(t), u(t)) + v(t)$	$z_k = h_k(x_k, u_k) + v_k$

System inputs:
u represents known control inputs.
w represents random dynamic disturbances.
v represents random sensor noise.
System outputs:
z represents sensor outputs.

- The one labeled "State Dynamics" in the Kalman filter should contain a fairly faithful replication of the true system dynamics, except that it does not know the values of the random inputs w in the real-world model. The Kalman filter model may also include other variables to be estimated in addition to the true system state vector x. These additional variables might include estimates of parameters shown in the dashed boxes in Figure 2.1:
  - The initial conditions of the true system state vector $x(t_0)$. (The estimator used for this application is called a *fixed-point smoother,* which is described in Chapter 6.)
  - Matrix parameters F (or its discrete-time equivalent Φ) and Q (and possibly C) of the true system dynamic model, used as shown in Table 2.1. The estimator in this case becomes nonlinear, as discussed in Chapter 8.
  - Matrix parameters H and R of the sensor model, in which case the filter also becomes nonlinear. These sensor parameters may also include sensor output biases, for example.
  - The filter state variables may also include the means (biases) of the random disturbances w and/or sensor noise v. Additionally, the Kalman filter state variables may, in some applications, include the variances of the "random disturbances" (Q) and/or "sensor noise" (R) models, in which case the filter becomes nonlinear.
- The one labeled "covariance dynamics," is a dynamic model for the second moment (covariance matrix P) of estimation errors. This is a *matrix Riccati equation* when the other models are linear or "quasilinear" (i.e., close enough to being linear that the errors due to nonlinearity do not matter). Otherwise, the

propagation over time of the estimated mean and covariance of the state variable distribution may be implemented by using structured sampling methods (e.g., the unscented transform). Either way, the covariance update model includes the state dynamics model, and the implementation generates the Kalman gain matrix $\overline{K}$ as a partial result.

The system input u (from "control") is covered in this chapter, although no distinction is made between u and random dynamic disturbance w until Chapter 4. The even-bigger picture (outside the scope of this book) includes the implementation inside the box labeled control, which uses the estimated state $\hat{x}$ as input.

The other parts of this higher level schematic are covered in the following chapters.

- Random process models for the boxes labeled random disturbances and sensor noise are addressed in Chapter 4.
- Sensor models are addressed in Chapters 4 and 8 (for nonlinear sensors).
- The Riccati equation is addressed in Chapters 5 and 9, and the equivalent methods for nonlinear applications are covered in Chapter 8.
- Methods for Kalman filter health monitoring (in the box labeled filter monitor) are discussed in Chapter 9.

2.1.2 Models for Dynamic Systems

2.1.2.1 Differential Equations and State Variables Since their introduction by Isaac Newton and Gottfried Leibniz in the seventeenth century, differential equations have provided concise and faithful mathematical models for many dynamic systems of importance to humans. By this device, Newton was able to model the motions of the planets in our solar system with a small number of variables and parameters. Given a finite number of initial conditions (accurate masses and initial positions and velocities of the sun and planets will do) and these equations, one can determine the positions and velocities of the planets relatively accurately for many years. The finite-dimensional representation of a problem (in this example, the problem of predicting the future course of the planets) is the basis for the so-called state-space approach to the representation of differential equations and their solutions, which is the focus of this chapter. The dependent variables of the differential equations become *state variables* of the dynamic system. They explicitly represent all the important characteristics of the dynamic system at any time.

2.1.2.2 Other Approaches The whole of dynamic system theory is a subject of considerably more scope than one needs for the present undertaking (Kalman filtering). This chapter will stick to just those concepts that are essential for that purpose, which is the development of the state-space representation for dynamic systems described by the systems of linear differential equations. These are given a somewhat heuristic treatment, without the mathematical rigor often accorded the subject, but including development and use of the transform methods of functional analysis for solving differential equations in the derivation of the Kalman filter. The interested reader will find a more formal and thorough presentation in most upper-level and graduate-level textbooks on ordinary differential equations. The objective of the more engineering-oriented treatments of dynamic systems is usually to solve the *controls problem*, which is the problem of defining the *inputs* (i.e., control settings) that will bring the state of the dynamic system to a desirable condition. That is not the objective here, however.

2.1.3 Main Points to Be Covered

The objective in this chapter is to characterize the measurable *outputs*[1] of dynamic systems as functions of the internal *states* and *inputs* of the system. The treatment here is deterministic, in order to define functional relationships between inputs and outputs.

In Chapter 4, the inputs are allowed to be nondeterministic (i.e., random), and the focus of the Chapter 5 is on how to estimate the state variables of a dynamic system in this context.

Dynamic Systems and Differential Equations. In the context of Kalman filtering, a *dynamic system* has come to be synonymous with a system of ordinary differential equations describing the evolution over time of the state of a physical system. This mathematical model is used to derive its solution, which specifies the functional dependence of the state variables on their initial values and the system inputs. This solution defines the functional dependence of the measurable outputs on the inputs and the coefficients of the model.

Mathematical Models for Continuous and Discrete Time. The principal dynamic system models of interest in this book are summarized in Table 2.1. These include linear models (the focus of this chapter) and nonlinear models (the focus in Chapter 8). They also include models in continuous time, as defined by differential equations, and discrete time, which is more suitable for computer implementation.

Modeling in Continuous Time. How one goes about building linear differential equation models for real-world dynamic systems is the focus of Section 2.2. This is how we have learned to apply the laws of physics to represent dynamic systems.

Transforming to Discrete Time. How the resulting linear differential equation models can be transformed into equivalent models in discrete time is the focus of Sections 2.3 and 2.4. Having derived trustworthy models in continuous time, this is how it gets reformulated for Kalman filtering.

Observability from Outputs. Whether or not the state of a dynamic system model can be determined from its outputs is the focus of Section 2.5.

[1]The italicized terms in this sentence will be defined more precisely further along.

2.2 DETERMINISTIC DYNAMIC SYSTEM MODELS

2.2.1 Dynamic Systems Modeled by Differential Equations

A *system* is an assemblage of interrelated entities that can be considered as a whole. If the attributes of interest of a system are changing with time, then it is called a *dynamic system*. A *process* is the evolution over time of a dynamic system.

Our *solar system*, consisting of the sun and its planets, is a physical example of a dynamic system. The motions of these bodies are governed by laws of motion that depend only upon their current relative positions and velocities. Sir Isaac Newton (1642–1727) discovered these laws and expressed them as a system of differential equations—another of his discoveries. From the time of Newton, engineers and scientists have learned to define dynamic systems in terms of the differential equations that govern their behavior. They have also learned how to solve many of these differential equations to obtain formulas for predicting the future behavior of dynamic systems.

2.2.2 Newtonian Models

The most fundamental state-space models for vehicles or all sorts come from Newtonian mechanics of "rigid" (i.e., infinitely stiff) bodies characterized by their mass and rotational moments of inertia,[2] with any applied forces supplying both accelerations and torques. The distributions of applied forces can usually be decoupled into translational forces and rotational torques, which can be treated independently. Each results in a system of first-order linear differential equations.

2.2.2.1 Rigid-Body Translational Mechanics If x is a position vector for the center of mass of an object, then a state-space model for the dynamics of that point mass has the general form

$$\dot{x} = v \text{ (velocity)} \tag{2.1}$$

$$\dot{v} = a \text{ (acceleration)} \tag{2.2}$$

$$\dot{a} = j_e \text{ (jerk)} \tag{2.3}$$

$$\dot{j}_e = j_o \text{ (jounce)} \tag{2.4}$$

$$\vdots$$

where the series of derivatives can continue as long as the application requires, and the right-hand sides can also represent input accelerations, jerk, and so on. This is where Newton's Laws come in, with $F = ma$, and so on.

[2]Interestingly enough, the Kalman filter also characterizes its error dynamics in terms of their means and moments about their means.

2.2.2.2 Rigid-Body Rotational Mechanics Newtonian rotational mechanics are a bit more complicated because attitude is three dimensional, but its domain folds back on itself, in effect. For example, if you rotate yourself about the vertical axis by 360°, you end up where you started without changing the direction of rotation. Orientation has three independent degrees of freedom (represented by, e.g., roll, pitch, and yaw angles), but its topology is the same as the surface of the unit sphere in four dimensions. It has been quite successfully modeled as such by using four-dimensional quaternians of unit length. Attitude can also be represented by other mathematical objects, including 3×3 orthogonal matrices A, where $A = I$ would represent the reference attitude or origin. The linear dynamic model with A as the orientation variable would then be

$$\dot{A} = A\omega\otimes \text{(angular velocity)} \tag{2.5}$$

$$\omega\otimes \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2.6}$$

$$\dot{\omega} = \alpha \text{ (angular acceleration)} \tag{2.7}$$

$$\dot{\alpha} = \text{? (angular jerk)} \tag{2.8}$$

$$\vdots$$

where, as in the translational case, the series of derivatives can continue as long as the application requires. In this case, the rotational analog of $F = ma$ is $\tau = M\alpha$, where τ is the applied torque and M (the analog of m in translational mechanics) is the positive-definite matrix of body moments of inertia about its center of gravity.

2.2.2.3 Nonrigid Body Dynamic Models Because real-world bodies are not infinitely rigid, they generally have vibration modes with some amount of damping. These vibration modes have their own state-space models, which can be either translational or rotational and generally including coupled vibration modes. These modes can be important dynamic subsystems for some space applications (e.g., space-based telescopes) and for applications with acceleration or rotation sensors attached to the host vehicle frame.[3] There can be similar vibrational issues with fuel sloshing in rockets and aircraft as the tanks empty. In these cases, it is usually necessary to model the dynamics in order to control it.

Example 2.1 (Newton's Model for a Dynamic System of *n* Massive Bodies) For a planetary system with *n bodies,* as illustrated in Figure 2.2(a) for $n = 4$, the acceleration of the ith body in any *inertial* (i.e., nonrotating and nonaccelerating) Cartesian

[3]For example, early Atlas missile testing showed that the bending modes of the monocoque frame gave the closed-loop missile attitude control system a pole in the right-half plane.

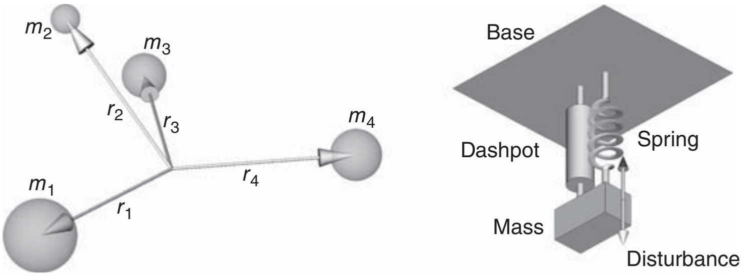

Figure 2.2 Dynamical system examples. (a) Gravitational dynamics and (b) mechanical oscillator.

coordinate system is given by Newton's third law as the second-order differential equation

$$\frac{d^2 r_i}{dt^2} = C_g \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j[r_j - r_i]}{|r_j - r_i|^3}, 1 \leq i \leq n,$$

where r_j is the position coordinate vector of the jth body, m_j is the mass of the jth body, and C_g is the gravitational constant. This set of n differential equations plus the associated initial conditions of the bodies (i.e., their initial positions and velocities) theoretically determines the future history of the planetary system.

This differential equation is *homogeneous* in the sense that it contains no terms that do not depend on the dependent variables r_j.

Example 2.2 (The Harmonic Resonator with Linear Damping) Consider idealized apparatus in Figure 2.2(b) with a mass m attached through a spring to an immovable base and its frictional contact to its support base represented by a dashpot.[4] Let δ be the displacement of the mass from its position at rest, $d\delta/dt$ be the velocity of the mass, and $a(t) = d^2\delta/dt^2$ its acceleration. The force F acting on the mass can be represented by Newton's second law as

$$\begin{aligned} F(t) &= ma(t) \\ &= m\left[\frac{d^2\delta}{dt^2}(t)\right] \\ &= -k_s\delta(t) - k_d\frac{d\delta}{dt}(t), \end{aligned}$$

where k_s is the spring constant and k_d is the drag coefficient of the dashpot. This relationship can be written as a differential equation

$$m\frac{d^2\delta}{dt^2} = -k_s\delta - k_d\frac{d\delta}{dt}$$

[4]Dashpots operate in the regime of "low Reynolds number" hydrodynamics, where damping is linear.

in which time (t) is the differential variable and displacement (δ) is the dependent variable. This equation constrains the dynamical behavior of the damped harmonic resonator. The *order* of a differential equation is the order of the highest derivative, which is *2* in this example. This one is called a *linear* differential equation, because both sides of the equation are linear combinations of δ and its derivatives. (That of Example 2.1 is a *nonlinear* differential equation.)

Not all dynamic systems can be modeled by differential equations There are other types of dynamic system models, such as Petri nets, inference nets or tables of experimental data. However, the only types of dynamic systems considered in this book will be modeled by differential equations or discrete-time linear state dynamic equations derived from linear differential or difference equations.

2.2.3 State Variables and State Equations for Deterministic Systems

The second-order differential equation of the previous example can be transformed to a system of two first-order differential equations in the two dependent variables $x_1 = \delta$ and $x_2 = d\delta/dt$. In this way, one can reduce the form of any system of higher order differential equations to an equivalent system of first-order differential equations. These systems are generally classified into the types shown in Table 2.1, with the most general type being a *time-varying* differential equation for representing a dynamic system with time-varying dynamic characteristics. This is represented in vector form as

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{2.9}$$

where Newton's "dot" notation is used as a shorthand for the derivative with respect to time and a vector-valued function f is used to represent a system of n equations

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, x_2, x_3, \dots, x_n, u_1, u_2, u_3, \dots, u_r), \\
\dot{x}_2 &= f_2(t, x_1, x_2, x_3, \dots, x_n, u_1, u_2, u_3, \dots, u_r), \\
\dot{x}_3 &= f_3(t, x_1, x_2, x_3, \dots, x_n, u_1, u_2, u_3, \dots, u_r), \\
&\vdots \\
\dot{x}_n &= f_n(t, x_1, x_2, x_3, \dots, x_n, u_1, u_2, u_3, \dots, u_r)
\end{aligned} \tag{2.10}$$

in the independent variable t (time), n dependent variables $\{x_i | 1 \le i \le n\}$, and r known inputs $\{u_i | 1 \le i \le r\}$. These are called the *state equations* of the dynamic system.

2.2.3.1 Homogeneous and Nonhomogeneous Differential Equations The variables u_i in Equations 2.10 can be independent of the state variables x_i, in which case the system of equations is considered *nonhomogeneous.* This distinction can become a bit muddled in the context of control theory, however, where the u_i may become control inputs that are functions of the x_i (or of the dynamic system outputs, which are functions of the x_i).

In general, differential equations containing terms that do not depend on the dependent variables and their derivatives are called *nonhomogeneous,* and those without such "nonhomogeneous terms" (i.e., the u_i) are called *homogeneous.*

2.2.3.2 State Variables Represent the Degrees of Freedom of Dynamic Systems The variables $x_1, \ldots, x_n$ are called the *state variables* of the dynamic system defined by Equation 2.10. They are collected into a single n-vector

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & \cdots & x_n(t) \end{bmatrix}^{\mathrm{T}} \tag{2.11}$$

called the *state vector* of the dynamic system. The n-dimensional domain of the state vector is called the *state space* of the dynamic system. Subject to certain continuity conditions on the functions f_i and u_i, the values $x_i(t_0)$ at some initial time t_0 will uniquely determine the values of the solutions $x_i(t)$ on some closed time interval $t \in [t_0, t_f]$ with initial time t_0 and final time t_f [1]. In that sense, the initial value of each state variable represents an independent degree of freedom of the dynamic system. The n values $x_1(t_0), x_2(t_0), x_3(t_0), \ldots, x_n(t_0)$ can be varied independently, and they uniquely determine the state of the dynamic system over the time interval $t_0 \le t \le t_f$.

Example 2.3 (State-Space Model of the Harmonic Resonator) For the second-order differential equation introduced in Example 2.2, let the state variables $x_1 = \delta$ and $x_2 = \dot{\delta}$. The first state variable represents the displacement of the mass from static equilibrium, and the second state variable represents the instantaneous velocity of the mass. The system of first-order differential equations for this dynamic system can be expressed in matrix form as

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = F_c \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

$$F_c = \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{k_d}{m} \end{bmatrix},$$

where F_c is called the *coefficient matrix* of the system of first-order linear differential equations. This is an example of what is called the *companion form* for higher order linear differential equations expressed as a system of first-order differential equations.

2.2.4 Continuous Time and Discrete Time

The dynamic system defined by Equation 2.10 is an example of a *continuous* system, so called because it is defined with respect to an independent variable t that varies continuously over some real interval $t \in [t_0, t_f]$. For many practical problems, however, one is only interested in knowing the state of a system at a discrete set of times $t \in \{t_1, t_2, t_3, \ldots\}$. These discrete times may, for example, correspond to the times at which the outputs of a system are sampled (such as the times at which Piazzi recorded the direction to Ceres). For problems of this type, it is convenient to order the times t_k according to their integer subscripts:

$$t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < t_{k+1} < \cdots .$$

That is, the time sequence is ordered according to the subscripts and the subscripts take on all successive values in some range of integers. For problems of this type, it suffices to define the state of the dynamic system as a recursive relation,

$$x(t_{k+1}) = f(x(t_k), t_k, t_{k+1}), \tag{2.12}$$

by means of which the state is represented as a function of its previous state. This is a definition of a *discrete dynamic system*. For systems with *uniform time intervals* Δt,

$$t_k = t[sub\ 0] + k\Delta t.$$

2.2.4.1 Shorthand Notation for Discrete-Time Systems It uses up a lot of ink if one writes $x(t_k)$ when all one cares about is the sequence of values of the state variable x. It is more efficient to shorten this to x_k, as long as it is understood that it stands for $x(t_k)$ and not the kth component of x. If one must talk about a particular component at a particular time, one can always resort to writing $x_i(t_k)$ to remove any ambiguity. Otherwise, let us drop t as a symbol whenever it is clear from the context that we are talking about discrete-time systems.

2.2.5 Time-Varying Systems and Time-Invariant Systems

The term *physical plant* or *plant* is sometimes used in place of "dynamic system," especially for applications in manufacturing. In many such applications, the dynamic system under consideration is literally a physical plant—a fixed facility used in the manufacture of materials. Although the input $u(t)$ may be a function of time, the functional dependence of the state dynamics on u and x does not depend upon time. Such systems are called *time invariant* or *autonomous*. Their solutions are generally easier to obtain than those of time-varying systems.

2.3 CONTINUOUS LINEAR SYSTEMS AND THEIR SOLUTIONS

2.3.1 Input–Output Models of Linear Dynamic Systems

The blocks labeled "State dynamics" and "Sensor model" in Figure 2.1 model a dynamic system with the equation models listed in Table 2.1. These model represent dynamic systems using three types of variables:

- *Control inputs* u_i, which can be under our control, and therefore known to us.

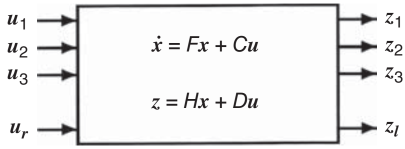

Figure 2.3 Block diagram of a linear dynamic system.

- Internal *state variables* x_i, which were described in the previous section. In most applications, these are "hidden variables," in the sense that they cannot generally be measured directly but must be somehow inferred from what can be measured.
- *Outputs* z_i, also called *observations* or *measurements,* which are those things that can be known from sensors used for measuring some of the internal state variables x_i.

These concepts are discussed in greater detail in the following subsections.

The common model for a generic linear dynamic systems with inputs and outputs is shown schematically in Figure 2.3.

2.3.2 Dynamic Coefficient Matrices and Input Coupling Matrices

The dynamics of linear systems are represented by a set of n first-order linear differential equations expressible in vector form as

$$\begin{aligned}\dot{x}(t) &= \frac{d}{dt}x(t)\\ &= F(t)x(t) + C(t)u(t),\end{aligned} \tag{2.13}$$

where the elements and components of the matrices and vectors can be functions of time:

$$F(t) = \begin{bmatrix} f_{11}(t) & f_{12}(t) & f_{13}(t) & \cdots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & f_{23}(t) & \cdots & f_{2n}(t) \\ f_{31}(t) & f_{32}(t) & f_{33}(t) & \cdots & f_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1}(t) & f_{n2}(t) & f_{n3}(t) & \cdots & f_{nn}(t) \end{bmatrix},$$

$$C(t) = \begin{bmatrix} c_{11}(t) & c_{12}(t) & c_{13}(t) & \cdots & c_{1r}(t) \\ c_{21}(t) & c_{22}(t) & c_{23}(t) & \cdots & c_{2r}(t) \\ c_{31}(t) & c_{32}(t) & c_{33}(t) & \cdots & c_{3r}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1}(t) & c_{n2}(t) & c_{n3}(t) & \cdots & c_{nr}(t) \end{bmatrix},$$

$$u(t) = \begin{bmatrix} u_1(t) & u_2(t) & u_3(t) & \cdots & u_r(t) \end{bmatrix}^{\mathrm{T}}.$$

The matrix $F(t)$ is called the *dynamic coefficient matrix*, or simply the *dynamic matrix*. Its elements are called the *dynamic coefficients*. The matrix $C(t)$ is called the *input coupling matrix*, and its elements are called *input coupling coefficients*. The r-vector u is called the *input vector*.

Nonhomogeneous part The term $C(t)u(t)$ in Equation 2.13 is called the *nonhomogeneous part* of the differential equation, because it does not depend upon the dependent variable x.

Example 2.4 (Dynamic Equation for a Heating/Cooling System) Consider the temperature T in a heated enclosed room or building as the state variable of a dynamic system. A simplified plant model for this dynamic system is the linear equation

$$\dot{T}(t) = -k_c[T(t) - T_o(t)] + k_h u(t),$$

where the constant "cooling coefficient" k_c depends on the quality of thermal insulation from the outside, T_o is the temperature outside, k_h is the heating/cooling rate coefficient of the heater or cooler, and u is an input function that is either $u = 0$ (off) or $u = 1$ (on) and can be defined as a function of any measurable quantities. The outside temperature T_o, on the other hand, is an example of an input function which may be directly measurable at any time but is not predictable in the future. It is effectively a random process.

2.3.3 Companion Form for Higher Order Derivatives

In general, the nth-order linear differential equation

$$\frac{d^n y(t)}{dt^n} + f_1(t)\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + f_{n-1}(t)\frac{dy(t)}{dt} + f_n(t)y(t) = u(t) \tag{2.14}$$

can be rewritten as a system of n first-order differential equations. Although the state variable representation as a first-order system is not unique [2], there is a unique way of representing it called the *companion form*.

2.3.3.1 Companion Form of the State Vector For the nth-order linear dynamic system shown above, the companion form of the state vector is

$$x(t) = \left[y(t), \quad \frac{d}{dt}y(t), \quad \frac{d^2}{dt^2}y(t), \quad \ldots, \quad \frac{d^{n-1}}{dt^{n-1}}y(t)\right]^{\mathrm{T}}. \tag{2.15}$$

2.3.3.2 Companion Form of the Differential Equation The nth-order linear differential equation can be rewritten in terms of the above state vector $x(t)$ as the vector differential equation

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -f_n(t) & -f_{n-1}(t) & -f_{n-2}(t) & \cdots & -f_1(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t). \tag{2.16}$$

When Equation 2.16 is compared with Equation 2.13, the matrices $F(t)$ and $C(t)$ are easily identified.

Although it simplifies the relationship between higher order linear differential equations and first-order systems of differential equations, the companion matrix is not recommended for implementation. Studies by Kenney and Liepnik [3] have shown that it is poorly conditioned for solving differential equations.

2.3.4 Outputs and Measurement Sensitivity Matrices

2.3.4.1 Measurable Outputs and Measurement Sensitivities Only the inputs and outputs of the system can be measured, and it is usual practice to consider the variables z_i as the measured values. For linear problems, they are related to the state variables and the inputs by a system of linear equations that can be represented in vector form as

$$z(t) = H(t)x(t) + D(t)u(t), \tag{2.17}$$

where

$$z(t) = \begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & \cdots & z_\ell(t) \end{bmatrix}^{\mathrm{T}},$$

$$H(t) = \begin{bmatrix} h_{11}(t) & h_{12}(t) & h_{13}(t) & \cdots & h_{1n}(t) \\ h_{21}(t) & h_{22}(t) & h_{23}(t) & \cdots & h_{2n}(t) \\ h_{31}(t) & h_{32}(t) & h_{33}(t) & \cdots & h_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\ell 1}(t) & h_{\ell 2}(t) & h_{\ell 3}(t) & \cdots & h_{\ell n}(t) \end{bmatrix},$$

$$D(t) = \begin{bmatrix} d_{11}(t) & d_{12}(t) & d_{13}(t) & \cdots & d_{1r}(t) \\ d_{21}(t) & d_{22}(t) & d_{23}(t) & \cdots & d_{2r}(t) \\ d_{31}(t) & d_{32}(t) & d_{33}(t) & \cdots & d_{3r}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{\ell 1}(t) & d_{\ell 2}(t) & d_{\ell 3}(t) & \cdots & d_{\ell r}(t) \end{bmatrix}.$$

The ℓ-vector $z(t)$ is called the *measurement vector* or the *output vector* of the system. The coefficient $h_{ij}(t)$ represents the *sensitivity* (measurement sensor scale factor) of the ith measured output to the jth internal state. The matrix $H(t)$ of these values is called the *measurement sensitivity matrix*, and $D(t)$ is called the *input–output coupling matrix*. The *measurement sensitivities* $h_{ij}(t)$ and input/output coupling coefficients $d_{ij}(t), 1 \le i \le \ell, 1 \le j \le r$, are known functions of time. The state equation 2.13 and the output equation 2.17 together form the dynamic equations of the system shown in Table 2.1.

2.3.5 Difference Equations and State-Transition Matrices (STMs)

Difference equations are the discrete-time versions of differential equations. They are usually written in terms of *forward differences* $x(t_{k+1}) - x(t_k)$ of the state variable (the dependent variable), expressed as a function ψ of all independent variables or of the forward value $x(t_{k+1})$ as a function ϕ of all independent variables (including the previous value as an independent variable):

$$x(t_{k+1}) - x(t_k) = \psi(t_k, x(t_k), u(t_k)),$$

or

$$\begin{aligned} x(t_{k+1}) &= \phi(t_k, x(t_k), u(t_k)), \\ \phi(t_k, x(t_k), u(t_k)) &= x(t_k) + \psi(t_k, x(t_k), u(t_k)). \end{aligned} \tag{2.18}$$

The second of these (Equation 2.18) has the same general form of the recursive relation shown in Equation 2.12, which is the one that is usually implemented for discrete-time systems.

For linear dynamic systems, the functional dependence of $x(t_{k+1})$ on $x(t_k)$ and $u(t_k)$ can be represented by matrices:

$$\begin{aligned} x(t_{k+1}) - x(t_k) &= \Psi(t_k)x(t_k) + C(t_k)u(t_k), \\ x_{k+1} &= \Phi_k x_k + C_k u_k, \\ \Phi_k &= I + \Psi(t_k), \end{aligned} \tag{2.19}$$

where the matrices Ψ and Φ replace the functions ψ and ϕ, respectively. The matrix Φ is called the *state-transition matrix* (*STM*). The matrix C is called the *discrete-time input coupling matrix* or simply the *input coupling matrix*—if the discrete-time context is already established.

2.3.6 Solving Differential Equations for STMs

An STM is a solution of what is called the *homogeneous*[5] matrix equation associated with a given linear dynamic system. Let us first define what homogeneous equations are and then show how their solutions are related to the solutions of a given linear dynamic system.

2.3.6.1 Homogeneous Systems The equation $\dot{x}(t) = F(t)x(t)$ is called the *homogeneous part* of the linear differential equation $\dot{x}(t) = F(t)x(t) + C(t)u(t)$. The solution of the homogeneous part can be obtained more easily than that of the full equation, and its solution is used to define the solution to the general (nonhomogeneous) linear equation.

[5]This terminology comes from the notion that every term in the expression so labeled contains the dependent variable. That is, the expression is *homogeneous* with respect to the dependent variable.

2.3.6.2 Fundamental Solutions of Homogeneous Equations An $n \times n$ matrix-valued function $\Phi(t)$ is called a *fundamental solution* of the homogeneous equation $\dot{x}(t) = F(t)x(t)$ on the interval $t \in [0, T]$ if $\dot{\Phi}(t) = F(t)\Phi(t)$ and $\Phi(0) = I_n$, the $n \times n$ identity matrix. Note that for any possible initial vector $x(0)$, the vector $x(t) = \Phi(t)x(0)$ satisfies the equation

$$\dot{x}(t) = \frac{d}{dt}[\Phi(t)x(0)] \tag{2.20}$$

$$= \left[\frac{d}{dt}\Phi(t)\right] x(0) \tag{2.21}$$

$$= [F(t)\Phi(t)]x(0) \tag{2.22}$$

$$= F(t)[\Phi(t)x(0)] \tag{2.23}$$

$$= F(t)x(t). \tag{2.24}$$

That is, $x(t) = \Phi(t)x(0)$ is the solution of the homogeneous equation $\dot{x} = Fx$ with initial value $x(0)$.

Example 2.5 (Toeplitz matrices) The unit upper triangular Toeplitz[6] matrix

$$\Phi(t) = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{1\cdot 2\cdot 3}t^3 & \cdots & \frac{1}{(n-1)!}t^{n-1} \\ 0 & 1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{(n-2)!}t^{n-2} \\ 0 & 0 & 1 & t & \cdots & \frac{1}{(n-3)!}t^{n-3} \\ 0 & 0 & 0 & 1 & \cdots & \frac{1}{(n-4)!}t^{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is the fundamental solution for the nth-order derivative, $\left(\frac{df}{dt}\right)^n$, the state-space form of which is $\dot{x} = Fx$ with the strictly upper triangular Toeplitz dynamic coefficient matrix

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

which can be verified by showing that $\Phi(0) = I$ and $\dot{\Phi} = F\Phi$. This dynamic coefficient matrix, in turn, is the companion matrix for the nth-order linear homogeneous differential equation $(d/dt)^n y(t) = 0$.

[6]Named after Otto Toeplitz (1881–1940).

2.3.6.3 Existence and Nonsingularity of Fundamental Solutions If the elements of the matrix $F(t)$ are continuous functions on some interval $0 \leq t \leq T$, then the fundamental solution matrix $\Phi(t)$ is guaranteed to exist and to be nonsingular on an interval $0 \leq t \leq \tau$ for some $\tau > 0$. These conditions also guarantee that $\Phi(t)$ will be nonsingular on some interval of nonzero length, as a consequence of the continuous dependence of the solution $\Phi(t)$ of the matrix equation on its (nonsingular) initial conditions ($\Phi(0) = I$) [1].

2.3.6.4 State-Transition Matrices Note that the fundamental solution matrix $\Phi(t)$ transforms any initial state $x(0)$ of the dynamic system to the corresponding state $x(t)$ at time t. If $\Phi(t)$ is nonsingular, then the products $\Phi^{-1}(t)x(t) = x(0)$ and $\Phi(\tau)\Phi^{-1}(t)x(t) = x(\tau)$. That is, the matrix product

$$\Phi(\tau, t) = \Phi(\tau)\Phi^{-1}(t) \tag{2.25}$$

transforms a solution from time t to the corresponding solution at time τ, as diagrammed in Figure 2.4. Such a matrix is called the *state-transition matrix*[7] for the associated linear homogeneous differential equation. The STM $\Phi(\tau, t)$ represents the transition to the state at time τ from the state at time t.

2.3.6.5 Properties of STMs and Fundamental Solution Matrices The same symbol (Φ) has been used for *fundamental solution matrices* and STMs, the distinction being made by the number of arguments. By convention, then,

$$\Phi(\tau, 0) = \Phi(\tau).$$

Other useful properties of Φ include the following:

1. $\Phi(\tau, \tau) = \Phi(0) = I$,
2. $\Phi^{-1}(\tau, t) = \Phi(t, \tau)$,
3. $\Phi(\tau, \sigma)\Phi(\sigma, t) = \Phi(\tau, t)$,
4. $(\partial/\partial\tau)\Phi(\tau, t) = F(\tau)\Phi(\tau, t)$,
5. $(\partial/\partial t)\Phi(\tau, t) = -\Phi(\tau, t)F(t)$.

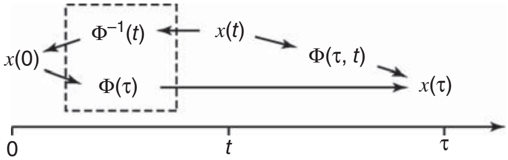

Figure 2.4 The STM as a composition of fundamental solution matrices.

[7]Formally, an operator $\Phi(t, t_0, x(t_0))$ such that $x(t) = \Phi(t, t_0, x(t_0))$ is called an *evolution operator* for a dynamic system with state x. An STM is a linear evolution operator.

Example 2.6 (State-Transition Matrix for the Underdamped Harmonic Resonator Model) In the general solution of the differential equation in Examples 2.2 and 2.3, the displacement δ of the damped harmonic resonator was modeled by the state equation

$$x = \begin{bmatrix} \delta \\ \dot{\delta} \end{bmatrix},$$

$$\dot{x} = Fx,$$

$$F = \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{k_d}{m} \end{bmatrix}.$$

The characteristic values of the dynamic coefficient matrix F are the roots of its characteristic polynomial

$$\det(\lambda I - F) = \lambda^2 + \frac{k_d}{m}\lambda + \frac{k_s}{m},$$

which is a quadratic polynomial with roots

$$\lambda_1 = \frac{1}{2}\left(-\frac{k_d}{m} + \sqrt{\frac{k_d^2}{m^2} - \frac{4k_s}{m}}\right),$$

$$\lambda_2 = \frac{1}{2}\left(-\frac{k_d}{m} - \sqrt{\frac{k_d^2}{m^2} - \frac{4k_s}{m}}\right).$$

The general solution for the displacement δ can then be written in the form

$$\delta(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t},$$

where α and β are (possibly complex) free variables.

The Underdamped Solution The resonator is considered *underdamped* if the discriminant

$$\frac{k_d^2}{m^2} - \frac{4k_s}{m} < 0,$$

in which case the roots are a conjugate pair of nonreal complex numbers and the general solution can be rewritten in "real form" as

$$\delta(t) = ae^{-t/\tau}\cos(\omega t) + be^{-t/\tau}\sin(\omega t),$$

$$\tau = \frac{2m}{k_d},$$

$$\omega = \sqrt{\frac{k_s}{m} - \frac{k_d^2}{4m^2}},$$

where a and b are now real variables, τ is the decay time constant, and ω is the resonator resonant frequency. This solution can be expressed in state-space form in terms of the real variables a and b as

$$\begin{bmatrix} \delta(t) \\ \dot{\delta}(t) \end{bmatrix} = e^{-t/\tau} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\dfrac{\cos(\omega t)}{\tau} - \omega \sin(\omega t) & \omega \cos(\omega t) - \dfrac{\sin(\omega t)}{\tau} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

The initial values

$$\delta(0) = a, \quad \dot{\delta}(0) = -\frac{a}{\tau} + \omega b$$

can be solved for a and b as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{\omega\tau} & \dfrac{1}{\omega} \end{bmatrix} \begin{bmatrix} \delta(0) \\ \dot{\delta}(0) \end{bmatrix}.$$

This can then be combined with the solution for $x(t)$ in terms of a and b to yield the fundamental solution

$$\begin{aligned} x(t) &= \Phi(t)x(0), \\ \Phi(t) &= \frac{e^{-t/\tau}}{\omega\tau^2} \begin{bmatrix} \tau[\omega\tau\cos(\omega t) + \sin(\omega t)] & \tau^2 \sin(\omega t) \\ -(1+\omega^2\tau^2)\sin(\omega\tau) & [\omega\tau^2\cos(\omega t) - \tau\sin(\omega t)] \end{bmatrix} \end{aligned}$$

in terms of the damping time constant and the resonant frequency.

2.3.7 Solution of Nonhomogeneous Equations

The solution of the nonhomogeneous state equation 2.13 is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)C(\tau)u(\tau)\, d\tau \tag{2.26}$$

$$= \Phi(t)\Phi^{-1}(t_0)x(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(\tau)C(\tau)u(\tau)\, d\tau, \tag{2.27}$$

where $x(t_0)$ is the initial value and $\Phi(t, t_0)$ is the STM of the dynamic system defined by $F(t)$. (This can be verified by taking derivatives and using the properties of STMs given above.)

2.3.8 Closed-Form Solutions of Time-Invariant Systems

2.3.8.1 Using Matrix Exponentials In this case, the coefficient matrix F is a constant function of time. The solution will still be a function of time, but the associated STMs $\Phi(t, \tau)$ will only depend on the differences $t - \tau$. In fact, one can show that

$$\Phi(t, \tau) = e^{F(t-\tau)} \tag{2.28}$$

$$= \sum_{i=0}^{\infty} \frac{(t-\tau)^i}{i!} F^i, \tag{2.29}$$

where $F^0 = I$, by definition. Equation 2.28 uses the matrix exponential function, defined by Equation 2.29.

The solution of the nonhomogeneous equation in this case will be

$$x(t) = e^{F(t-\tau)} x(\tau) + \int_{\tau}^{t} e^{F(t-\sigma)} Cu(\sigma)\, d\sigma \tag{2.30}$$

$$= e^{F(t-\tau)} x(\tau) + e^{Ft} \int_{\tau}^{t} e^{-F\sigma} Cu(\sigma)\, d\sigma. \tag{2.31}$$

2.3.8.2 Using Laplace Transforms Equation 2.29 can be used to derive the STM $\Phi(t)$ using inverse Laplace transforms. The Laplace transform of $\Phi(t) = \exp\ (Ft)$ will be

$$\begin{aligned}
\mathcal{L}\Phi(t) &= \mathcal{L}[\exp\ (Ft)] \\
&= \mathcal{L}\left[\sum_{k=0}^{\infty} \frac{1}{k!} F^k t^k\right] \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} F^k\ [\mathcal{L}t^k] \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} F^k\ \left[\frac{k!}{s^{k+1}}\right] \\
&= \frac{1}{s} \sum_{k=0}^{\infty} (s^{-1}F)^k \\
&= \frac{1}{s} [I - s^{-1}F]^{-1} \\
&= [sI - F]^{-1},
\end{aligned} \tag{2.32}$$

and consequently $\Phi(t)$ can be derived analytically in terms of the inverse Laplace transform $\mathcal{L}^{-1}$ as

$$\Phi(t) = \mathcal{L}^{-1}\{[sI - F]^{-1}\}. \tag{2.33}$$

Equations 2.29 (matrix exponential) and 2.33 (inverse Laplace transform) can be used to derive the STM in "closed form" (i.e., as a formula) for dynamic systems modeled by linear time-invariant differential equations.

2.3.8.3 Computing Matrix Exponentials The following methods have been used for computing matrix exponentials numerically:

1. A "scaling and squaring" method combined with a Padé approximation is the recommended general-purpose method. This method is discussed in greater detail in this section. The MATLAB® function `expm` uses this method to compute the matrix exponential numerically.
2. Numerical integration of the homogeneous part of the differential equation,

$$\frac{d}{dt}\Phi(t) = F\Phi(t), \tag{2.34}$$

   with initial value $\Phi(0) = I$. This method also works for time-varying systems.
3. The approximation of e^{Ft} by a truncated power series expansion is *not* a recommended general-purpose method. Its convergence is poor unless the characteristic values of Ft are well inside the unit circle in the complex plane.

There are many other methods for computing matrix exponentials,[8] but these are the most important.

Example 2.7 (Laplace Transform Method) We will now demonstrate a Laplace transform method for deriving the STM for nonhomogeneous differential equations and the total solution.

The differential equation model is given as follows for a nonhomogeneous equation:

$$\dot{x}_1(t) = -2x_2(t)$$
$$\dot{x}_2(t) = x_1(t) - 3\,x_2(t) + u(t).$$

The output equation is given as follows:

$$z(t) = x_1(t).$$

These equations can be rewritten in state variable form, as in Equations 2.13 and 2.17.

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

[8]See, for example, Brockett [2], DeRusso et al. [4], Timothy and Bona [5], Evangelisti [6], or Kreindler and Sarachik [7].

$$\dot{x}(t) = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + [0]u(t),$$

where

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0 \end{cases}$$

is a unit step function at $t = 0$, and the state-vector model coefficient matrices are

$$F = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = [0].$$

State-Transition Matrix As an alternative to using the matrix exponential, one can obtain the STM by applying the inverse Laplace transform $\mathcal{L}^{-1}$ to the Laplace transformed matrix $sI - F$:

$$(sI - F) = \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}$$

$$(sI - F)^{-1} = \frac{1}{(s^2 + 3s + 2)} \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}[sI - F]^{-1}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{-2}{(s+1)(s+2)} \\ \dfrac{2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & 2(e^{-2t} - e^{-t}) \\ e^{-t} - e^{-2t} & 2e^{-2t} - e^{-t} \end{bmatrix}.$$

General Solution Equation 2.30 with $\tau = 0$ becomes

$$x(t) = \Phi(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \Phi(t - \tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} 2(e^{-2(t-\tau)}) - e^{-(t-\tau)} \\ 2e^{-2(t-\tau)} - e^{-(t-\tau)} \end{bmatrix} d\tau$$

$$
\begin{aligned}
z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} 2(e^{-2(t-\tau)} - e^{-(t-\tau)}) \\ 2e^{-2(t-\tau)} - e^{-(t-\tau)} \end{bmatrix} d\tau \right\} \\
&= (2e^{-t} - e^{-2t}) + \int_0^t 2[e^{-2(t-\tau)} - e^{-(t-\tau)}] d\tau \\
&= 2e^{-t} - e^{-2t} + 2e^{-t} - e^{-2t} - 1,\ t \geq 0 \\
&= 4e^{-t} - 2e^{-2t} - 1,\ t \geq 0.
\end{aligned}
$$

2.3.9 Time-Varying Systems

If $F(t)$ is not constant, the dynamic system is called *time varying*. If $F(t)$ is a piecewise smooth function of t, the $n \times n$ homogeneous matrix differential equation 2.34 can be solved numerically by the fourth-order Runge–Kutta method.[9] The ordinary differential equation solvers in MATLAB make use of Runge–Kutta integration methods.

2.4 DISCRETE LINEAR SYSTEMS AND THEIR SOLUTIONS

2.4.1 Discretized Linear Systems

If one is only interested in the system state at discrete times, then one can use the formula

$$x(t_k) = \Phi(t_k, t_{k-1})x(t_{k-1}) + \int_{t_{k-1}}^{t_k} \Phi(t_k, \sigma)C(\sigma)u(\sigma)\, d\sigma \tag{2.35}$$

to propagate the state vector between the times of interest.

2.4.1.1 Simplification for Constant u If u is constant over the interval $[t_{k-1}, t_k]$, then the above integral can be simplified to the form

$$x(t_k) = \Phi(t_k, t_{k-1})x(t_{k-1}) + \Gamma(t_{k-1})u(t_{k-1}) \tag{2.36}$$

$$\Gamma(t_{k-1}) = \int_{t_{k-1}}^{t_k} \Phi(t_k, \sigma)C(\sigma)\, d\sigma. \tag{2.37}$$

2.4.1.2 Shorthand Discrete-time Notation For discrete-time systems, the indices k in the time sequence $\{t_k\}$ characterize the times of interest. One can save some ink by using the shorthand notation

$$
\begin{aligned}
&x_k \stackrel{\text{def}}{=} x(t_k), \quad z_k \stackrel{\text{def}}{=} z(t_k), \quad u_k \stackrel{\text{def}}{=} u(t_k), \quad H_k \stackrel{\text{def}}{=} H(t_k), \\
&D_k \stackrel{\text{def}}{=} D(t_k), \quad \Phi_{k-1} \stackrel{\text{def}}{=} \Phi(t_k, t_{k-1}), \quad G_k \stackrel{\text{def}}{=} G(t_k)
\end{aligned}
$$

[9]Named after the German mathematicians Karl David Tolme Runge (1856–1927) and Wilhelm Martin Kutta (1867–1944).

for discrete-time systems, eliminating t entirely. Using this notation, one can represent the discrete-time state equations in the more compact form

$$x_k = \Phi_{k-1} x_{k-1} + \Gamma_{k-1} u_{k-1}, \tag{2.38}$$

$$z_k = H_k x_k + D_k u_k. \tag{2.39}$$

2.4.2 Discrete-Time Solution for Time-Invariant Systems

For continuous-time-invariant systems that have been discretized using fixed time intervals, the matrices $\Phi,\ \Gamma,\ H,\ u$, and D in Equations 2.38 and 2.39 are independent of the discrete-time index k as well. In that case, the solution can be written in closed form as

$$x_k = \Phi^k x_0 + \sum_{i=0}^{k-1} \Phi^{k-i-1} \Gamma u_i, \tag{2.40}$$

where Φ^k is the kth power of Φ.

Inverse z-transform solution The matrix powers Φ^k in Equation 2.40 can be computed using z-transforms as

$$\Phi^k = \mathcal{Z}^{-1}[(zI - \Phi)^{-1} z^k], \tag{2.41}$$

where z is the z-transform variable and $\mathcal{Z}^{-1}$ is the inverse z-transform.

This is, in some ways, analogous to the inverse Laplace transform method used in Section 2.3.8.2 and Example 2.7 to compute the STM ($\Phi(t)$) for linear time-invariant systems in continuous time. The analogous formulas for the inverse transforms are

$$\begin{aligned}
\Phi(t) &= \mathcal{L}^{-}1(sI - F)^{-1} \text{ (inverse Laplace transform)} \\
\Phi^k &= \mathcal{Z}^{-1}(zI - F)^{-1} z^k \text{ (inverse z-transform)} \\
&= \sum \text{residuals of } [(zI - F)^{-1} z^k] \text{ (Cauchy's residue theorem)}
\end{aligned}$$

Example 2.8 (Solution using z-transforms) Consider Equation 2.38 with

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Phi = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \Gamma u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The matrices

$$[zI - \Phi]^{-1} = \begin{bmatrix} z & -1 \\ 6 & z+5 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{z+5}{(z+2)(z+3)} & \dfrac{1}{(z+2)(z+3)} \\ \dfrac{-6}{(z+2)(z+3)} & \dfrac{z}{(z+2)(z+3)} \end{bmatrix}$$

$$[zI - \Phi]^{-1} z^k = \begin{bmatrix} \dfrac{z^{k+1} + 5z^k}{(z+2)(z+3)} & \dfrac{z^k}{(z+2)(z+3)} \\ \dfrac{-6z^k}{(z+2)(z+3)} & \dfrac{z^{k+1}}{(z+2)(z+3)} \end{bmatrix}$$

$$\mathcal{Z}^{-1}\{[zI - \Phi]^{-1} z^k\} = \begin{bmatrix} 3\,(-2)^k - 2\,(-3)^k & (-2)^k - (-3)^k \\ -6\,[(-2)^k - (-3)^k] & -2\,(-2)^k + 3\,(-3)^k \end{bmatrix}$$
$$= \Phi^k,$$

so that the general solution of Equation 2.38 with the given values for x_0, Φ, and Gu becomes

$$x_k = \begin{bmatrix} 3\,(-2)^k - 2\,(-3)^k \\ -6\,[(-2)^k - (-3)^k] \end{bmatrix} + \sum_{i=0}^{k-1} \begin{bmatrix} (-2)^{k-i-1} - (-3)^{k-i-1} \\ -2\,(-2)^{k-i-1} + 3\,(-3)^{k-i-1} \end{bmatrix}.$$

2.5 OBSERVABILITY OF LINEAR DYNAMIC SYSTEM MODELS

Observability is the issue of whether the state of a dynamic system *with a known model* is uniquely determinable from its inputs and outputs. It is essentially a property of the given *system model*. A given linear dynamic system model with a given linear input/output model is considered *observable* if and only if its state is *uniquely* determinable from the model definition, its inputs, and its outputs. If the system state is *not* uniquely determinable from the system inputs and outputs, then the system model is considered *unobservable*.

2.5.1 How to Determine Whether a Given Dynamic System Model Is Observable

If the measurement sensitivity matrix is invertible at any (continuous or discrete) time, then the system state can be uniquely determined (by inverting it) as $x = H^{-1}z$. In this case, the system model is considered to be *completely observable* at that time. However, the system can still be *observable over a time interval* even if H is not invertible at *any* time. In the latter case, the unique solution for the system state can be defined by using the least-squares methods of Chapter 1, including those of Sections 1.3.2.3 and 1.3.2.4 (pages 12 and 13). These use the so-called *Gramian matrix* to characterize whether or not a vector variable is determinable from a given linear model. When applied to the problem of the determinacy of the state of a linear dynamic system, the Gramian matrix is called the *observability matrix* of the given system model.

The observability matrix for dynamic system models in continuous time has the form

$$M(H, F, t_0, t_f) = \int_{t_0}^{t_f} \Phi^{\mathrm{T}}(t) H^{\mathrm{T}}(t) H(t) \Phi(t)\, dt \tag{2.42}$$

for a linear dynamic system with fundamental solution matrix $\Phi(t)$ and measurement sensitivity matrix $H(t)$, defined over the continuous-time interval $t_0 \le t \le t_f$. Note that this depends on the interval over which the inputs and outputs are observed but not on the inputs and outputs per se. In fact, the observability matrix of a dynamic system model *does not* depend on the inputs u, the input coupling matrix C, or the input–output coupling matrix D—even though the outputs and the state vector depend on them. Because the fundamental solution matrix Φ depends only on the dynamic coefficient matrix F, the observability matrix depends *only* on H and F.

The observability matrix of a linear dynamic system model over a discrete-time interval $t_0 \le t \le t_{k_f}$ has the general form

$$M(H_k, \Phi_k, 1 \le k \le k_f) = \left\{ \sum_{k=1}^{k_f} \left[\prod_{i=0}^{k-1} \Phi_{k-i} \right]^{\mathrm{T}} H_k^{\mathrm{T}} H_k \left[\prod_{i=0}^{k-1} \Phi_{k-i} \right] \right\}, \tag{2.43}$$

where H_k is the observability matrix at time t_k and Φ_k is the STM from time t_k to time t_{k+1} for $0 \le k \le k_f$. Therefore, the observability of discrete-time system models depends only on the values of H_k and Φ_k over this interval. As in the continuous-time case, observability does not depend on the system inputs.

The derivations of these formulas are left as exercises for the reader.

2.5.2 Observability of Time-Invariant Systems

The formulas defining observability are simpler when the dynamic coefficient matrices or STMs of the dynamic system model are time invariant. In that case, observability can be characterized by the rank of the matrices

$$M = \begin{bmatrix} H^{\mathrm{T}} & \Phi^{\mathrm{T}} H^{\mathrm{T}} & (\Phi^{\mathrm{T}})^2 H^{\mathrm{T}} & \ldots & (\Phi^{\mathrm{T}})^{n-1} H^{\mathrm{T}} \end{bmatrix} \tag{2.44}$$

for discrete-time systems and

$$M = \begin{bmatrix} H^{\mathrm{T}} & F^{\mathrm{T}} H^{\mathrm{T}} & (F^{\mathrm{T}})^2 H^{\mathrm{T}} & \ldots & (F^{\mathrm{T}})^{n-1} H^{\mathrm{T}} \end{bmatrix} \tag{2.45}$$

for continuous-time systems. The systems are observable if these have rank n, the dimension of the system state vector. The first of these matrices can be obtained by representing the *initial state* of the linear dynamic system as a function of the system inputs and outputs. The initial state can then be shown to be uniquely determinable if and only if the rank condition is met. The derivation of the latter matrix is not as straightforward. Ogata [8] presents a derivation obtained by using properties of the characteristic polynomial of F.

2.5.2.1 Practicality of the Formal Definition of Observability Singularity of the observability matrix is a concise mathematical characterization of observability. This can be too fine a distinction for practical application—especially in finite-precision arithmetic—because arbitrarily small changes in the elements of a singular matrix can render it nonsingular. The following practical considerations should be kept in mind when applying the formal definition of observability:

- It is important to remember that the model is only an approximation to a real system, and we are primarily interested in the properties of the real system, not the model. Differences between the real system and the model are called *model truncation errors*. The art of system modeling depends on knowing where to truncate, but there will almost surely be some truncation error in any model.
- Computation of the observability matrix is subject to model truncation errors and *roundoff errors*, which could make the difference between singularity and nonsingularity of the result. Even if the computed observability matrix is *close* to being singular, it is cause for concern. One should consider a system as *poorly observable* if its observability matrix is close to being singular. For that purpose, one can use the *singular-value decomposition* or the *condition number* of the observability matrix to define a more *quantitative* measure of unobservability. The reciprocal of its condition number measures how close the system is to being unobservable.
- Real systems tend to have some amount of unpredictability in their behavior, due to unknown or neglected exogenous inputs. Although such effects cannot be modeled deterministically, they are not always negligible. Furthermore, the process of measuring the outputs with physical sensors introduces some amount of sensor noise, which will cause errors in the estimated state. It would be better to have a quantitative characterization of observability that takes these types of uncertainties into account. An approach to these issues (pursued in Chapter 5) uses a *statistical* characterization of observability, based on a statistical model of the uncertainties in the measured system outputs and the system dynamics. The degree of uncertainty in the estimated values of the system states can be characterized by an *information matrix*, which is a statistical generalization of the observability matrix.

Example 2.9 Consider the following continuous system:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).\end{aligned}$$

The observability matrix, using Equation 2.45, is

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ rank of } M = 2.$$

Here, M has rank equal to the dimension of $x(t)$. Therefore, the system is observable.

Example 2.10 Consider the following continuous system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t).$$

The observability matrix, using Equation 2.45, is

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{rank of } M = 1.$$

Here, M has rank less than the dimension of $x(t)$. Therefore, the system is not observable.

Example 2.11 Consider the following discrete system:

$$x_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} x_{k-1} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u_{k-1},$$
$$z_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x_k.$$

The observability matrix, using Equation 2.44, is

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{rank of } M = 2.$$

The rank is less than the dimension of x_k. Therefore, the system is not observable.

Example 2.12 Consider the following discrete system:

$$x_k = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u_{k-1},$$
$$z_k = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} x_k.$$

The observability matrix, using Equation 2.44, is

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \text{rank of} M = 2$$

The system is observable.

2.5.3 Controllability of Time-Invariant Linear Systems

2.5.3.1 Controllability in Continuous Time The concept of observability in estimation theory has algebraic relationships to the concept of *controllability* in control theory. These concepts and their relationships were discovered by Kalman as what he called the *duality* and *separability* of the estimation and control problems for linear dynamic systems. Kalman's[10] dual concepts are presented here and in the next subsection, although they are not issues for the estimation problem.

A dynamic system defined on the finite interval $t_0 \leq t \leq t_f$ by the linear model

$$\dot{x}(t) = Fx(t) + Cu(t), \quad z(t) = Hx(t) + Du(t) \tag{2.46}$$

and with initial state vector $x(t_0)$ is said to be *controllable* at time $t = t_0$ if, for any desired final state $x(t_f)$, there exists a piecewise continuous input function $u(t)$ that drives to state $x(t_f)$. If every initial state of the system is controllable in some finite time interval, then the *system* is said to be controllable.

The system given in Equation 2.46 is controllable if and only if matrix S has n linearly independent columns,

$$S = \begin{bmatrix} C & FC & F^2C & \cdots & F^{n-1}C \end{bmatrix}. \tag{2.47}$$

2.5.3.2 Controllability in Discrete Time Consider the time-invariant system model given by the equations

$$x_k = \Phi x_{k-1} + \Gamma u_{k-1}, \tag{2.48}$$

$$z_k = Hx_k + Du_k. \tag{2.49}$$

This system model is considered controllable[11] if there exists a set of control signals u_k defined over the discrete interval $0 \leq k \leq N$ that bring the system from an initial state x_0 to a given final state x_N in N sampling instants, where N is a finite positive integer. This condition can be shown to be equivalent to the matrix

$$S = \begin{bmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{N-1}\Gamma \end{bmatrix} \tag{2.50}$$

having rank n.

Example 2.13 Determine the controllability of Example 2.9. The controllability matrix, using Equation 2.47, is

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{rank of} \quad S = 2.$$

[10]The dual relationships between estimation and control given here are those originally defined by Kalman. These concepts have been refined and extended by later investigators to include concepts of *reachability* and *reconstructibility* as well. The interested reader is referred to the more recent textbooks on "modern" control theory for further exposition of these other "-ilities."

[11]This condition is also called *reachability*, with controllability restricted to $x_N = 0$.

Here, S has rank equal to the dimension of $x(t)$. Therefore, the system is controllable.

Example 2.14 Determine the controllability of Example 2.11. The controllability matrix, using Equation 2.50, is

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad \text{rank of} \quad S = 2.$$

The system is not controllable.

2.6 SUMMARY

1. A *system* is a collection of interrelated objects treated as a whole for the purpose of modeling its behavior. It is called *dynamic* if attributes of interest are changing with time. A *process* is the evolution over time of a system.
2. Although it is sometimes convenient to model time as a continuum, it is often more practical to consider it as taking on discrete values. (Most clocks, e.g., advance in discrete-time steps.)
3. The *state* of a dynamic system at a given instant of time is characterized by the instantaneous values of its attributes of interest. For the problems of interest in this book, the attributes of interest can be characterized by real numbers, such as the electric potentials, temperatures, or positions of its component parts—in appropriate units. A *state variable* of a system is the associated real number. The *state vector* of a system has state variables as its component elements. The system is considered *closed* if the future state of the system for all time is uniquely determined by its current state. For example, neglecting the gravity fields from other massive bodies in the universe, the solar system could be considered as a closed system. If a dynamic system is not closed, then the exogenous causes are called *inputs* to the system. This state vector of a system must be *complete* in the sense that the future state of the system is uniquely determined by its current state *and its future inputs*. In order to obtain a complete state vector for a system, one can extend the state variable components to include derivatives of other state variables. This allows one to use velocity (the derivative of position) or acceleration (the derivative of velocity) as state variables, for example.
4. In order that the future state of a system may be determinable from its current state and future inputs, the dynamical behavior of each state variable of the system must be a known function of the instantaneous values of other state variables and the system inputs. In the canonical example of our solar system, for instance, the acceleration of each body is a known function of the relative positions of the other bodies. The *state-space model* for a dynamic system represents these functional dependencies in terms of first-order *differential equations* (in continuous time) or *difference equations* (in discrete time).

The differential or difference equations representing the behavior of a dynamic system are called its *state equations*. If these can be represented by *linear* functions, then it is called a *linear dynamic system*.

5. The model for a linear dynamic system in continuous time can be expressed in general form as a first-order vector differential equation

$$\frac{d}{dt}x(t) = F(t)x(t) + C(t)u(t),$$

where $x(t)$ is the n-dimensional *system state vector* at time t, $F(t)$ is its $n \times n$ *dynamic coefficient matrix,* $u(t)$ is the r-dimensional *system input vector*, and $C(t)$ is the $n \times r$ *input coupling matrix*. The corresponding model for a linear dynamic system in discrete time can be expressed in the general form

$$x_k = \Phi_{k-1}x_{k-1} + \Gamma_{k-1}u_{k-1},$$

where x_{k-1} is the n-dimensional system state vector at time t_{k-1}, x_k is its value a time $t_k > t_{k-1}, \Phi_{k-1}$ is the $n \times n$ *STM* for the system at time t_k, u_k is the input vector to the system a time t_k, and Γ_k is the corresponding input coupling matrix.

6. If F and C (or Φ and C) do not depend upon t (or k), then the continuous (or discrete) model is called *time invariant*. Otherwise, the model is *time varying*.

7. The equation

$$\frac{d}{dt}x(t) = F(t)x(t)$$

is called the *homogeneous part* of the model equation

$$\frac{d}{dt}x(t) = F(t)x(t) + C(t)u(t).$$

A solution $\Phi(t)$ to the corresponding $n \times n$ *matrix equation*

$$\frac{d}{dt}\Phi(t) = F(t)\Phi(t)$$

on an interval starting at time $t = t_0$ and with initial condition

$$\Phi(t_0) = I \quad \text{(the identity matrix)}$$

is called a *fundamental solution matrix* to the homogeneous equation on that interval. It has the property that if the elements of $F(t)$ are bounded, then $\Phi(t)$ cannot become singular on a finite interval. Furthermore, for any initial value $x(t_0)$,

$$x(t) = \Phi(t)x(t_0)$$

is the solution to the corresponding homogeneous equation.

8. For a *homogeneous* system, the STM Φ_{k-1} from time t_{k-1} to time t_k can be expressed in terms of the fundamental solution $\Phi(t)$ as

$$\Phi_{k-1} = \Phi(t_k)\Phi^{-1}(t_{k-1})$$

for times $t_k > t_{k-1} > t_0$.

9. The model for a dynamic system in continuous time can be transformed into a model in discrete time using the above formula for the STM and the following formula for the equivalent discrete-time inputs:

$$u_{k-1} = \Phi(t_k) \int_{t_{k-1}}^{t_k} \Phi^{-1}(\tau)C(\tau)u(\tau)\, d\tau.$$

10. An *output* of a dynamic system is something we can measure directly, such as directions of the lines of sight to the planets (viewing conditions permitting) or the temperature at a thermocouple. A dynamic system model is said to be *observable* from a given set of outputs if it is feasible to determine the state of the system from those outputs. If the dependence of an output z on the system state x is linear, it can be expressed in the form

$$z = Hx,$$

where H is called the *measurement sensitivity matrix*. It can be a function of continuous time ($H(t)$) or discrete time (H_k). Observability can be characterized by the rank of an *observability matrix* associated with a given system model. The observability matrix is defined as

$$M = \begin{cases} \int_{t_0}^{t} \Phi^{\mathrm{T}}(\tau)H^{\mathrm{T}}(\tau)H(\tau)\Phi(\tau)\, d\tau & \text{for continuous-time models,} \\ \sum_{i=0}^{m} \left[\left(\prod_{k=0}^{i-1} \Phi_k^{\mathrm{T}} \right) H_i^{\mathrm{T}} H_i \left(\prod_{k=0}^{i-1} \Phi_k^{\mathrm{T}} \right)^{\mathrm{T}} \right] & \text{for discrete-time models.} \end{cases}$$

The system is observable if and only if its observability matrix has full rank (n) for some integer $m \geq 0$ or time $t > t_0$. (The test for observability can be simplified for time-invariant systems.) Note that the determination of observability depends on the (continuous or discrete) interval over which the observability matrix is determined.

11. The closed-form solution of a system of first-order differential equations with constant coefficients can be expressed symbolically in terms of the exponential function of a matrix, but the problem of numerical approximation of the exponential function of a matrix is notoriously ill-conditioned.

PROBLEMS

2.1 What is a state vector model for the linear dynamic system $\frac{dy(t)}{dt} = u(t)$, expressed in terms of y? (Assume the companion form of the dynamic coefficient matrix.)

2.2 What is the companion matrix for the nth-order differential equation $(d/dt)^n y(t) = 0$? What are its dimensions?

2.3 What is the companion matrix of the above problem when $n = 1$? When $n = 2$?

2.4 What is the fundamental solution matrix of Exercise 2.2 when $n = 1$? When $n = 2$?

2.5 What is the STM of the above problem when $n = 1$? When $n = 2$?

2.6 Find the fundamental solution matrix $\Phi(t)$ for the system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and also the solution $x(t)$ for the initial conditions

$$x_1(0) = 1 \quad \text{and} \quad x_2(0) = 2.$$

2.7 Find the total solution and STM for the system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 2$.

2.8 *The reverse problem: from a discrete-time model to a continuous-time model.* For the discrete-time dynamic system model

$$x_k = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

find the STM for continuous time and the solution for the continuous-time system with initial conditions

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

2.9 Use the same general approach as in Example 2.7 (Laplace transform method) for deriving the state-transition matrix for the models used in Examples 2.2, 2.3, and 2.6. (damped harmonic resonator), starting with a linear differential equation model in the form

$$\ddot{\delta}(t) + 2\zeta w_n \dot{\delta}(t) + w_n^2 \delta(t) = u(t).$$

Assume that $u(t) = 1$, a constant disturbing force, and

$$\dot{\delta}(t) = \frac{d\delta}{dt}, \qquad \ddot{\delta}(t) = \frac{d^2\delta}{dt^2}, \qquad \zeta = \frac{k_d}{2\sqrt{mk_s}} = \frac{1}{2}, \qquad \omega_n = \sqrt{\frac{k_s}{m}} = 1.$$

The transformed parameter ζ is now a unitless damping coefficient and ω_n the "natural" (i.e., undamped) frequency of the resonator.
Transform this model to state-space form in continuous time and then use the Laplace transform to obtain the general solution in terms of a state-transition matrix.

2.10 Find conditions on c_1, c_2, h_1, and h_2 such that the following system is completely observable and controllable:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} u(t),$$

$$z(t) = \begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

2.11 Determine the controllability and observability of the dynamic system model given below:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

2.12 Derive the STM of the time-varying system

$$\dot{x}(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} x(t).$$

2.13 Find the STM for

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

2.14 For the system of three first-order differential equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = 0,$$

(a) What is the companion matrix F?

(b) What is the fundamental solution matrix $\Phi(t)$ such that $(d/dt)\Phi(t) = F\Phi(t)$ and $\Phi(0) = I$?

2.15 Show that the matrix exponential of an antisymmetric matrix is an orthogonal matrix.

2.16 Derive the formula of Equation 2.42 for the observability matrix of a linear dynamic system model in continuous time. (*Hint*: Use the approach of Example 1.3 for estimating the initial state of a system and Equation 2.27 for the state of a system as a linear function of its initial state and its inputs.)

2.17 Derive the formula of Equation 2.43 for the observability matrix of a dynamic system in discrete time. (*Hint*: Use the method of least squares of Example 1.1 for estimating the initial state of a system, and compare the resulting Gramian matrix to the observability matrix of Equation 2.43.)

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