Exoplanets

Harsh Ambardekar

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Table of Contents

1 Introduction **2**

2 Exoplanet Detection **2**

2.1 Direct Imaging 2

2.2 Radial Velocity Method 2

2.3 Transit Method 3

3 Planetary Equilibrium Temperature **3**

4 Worked Problems **5**

5 Practice Problems **7**

6 Conclusion **7**

1 Introduction

Exoplanets are planets outside the Solar System; they are planets that orbit a star *other* than the Sun. Studying exoplanets is important because it can tell us more about our own Solar System. For the USAAAO, most exoplanet questions are math-based and there aren't very specific "topics" related to exoplanets. We'll thus mainly focus on solving problems in this handout to give you an idea of the types of questions that can be asked.

2 Exoplanet Detection

Before we get to the mathy stuff, let's talk about something conceptual: exoplanet detection. There are three main methods to detect exoplanets, and we'll explore each one.

2.1 Direct Imaging

We'll start with the most obvious method: **direct imaging**. As you can probably guess, detection by direct imaging means we discovered them by taking an actual picture with a telescope. This most often occurs when we are imaging a star and and notice an exoplanet in the image. Now, why don't we just detect all exoplanets using direct imaging? Well, exoplanets are *extremely* faint so it's hard to detect their light. In fact, only about 200 exoplanets (less than 1 percent of all discovered exoplanets) have been discovered using direct imaging.
Below you can see an example of a direct image.

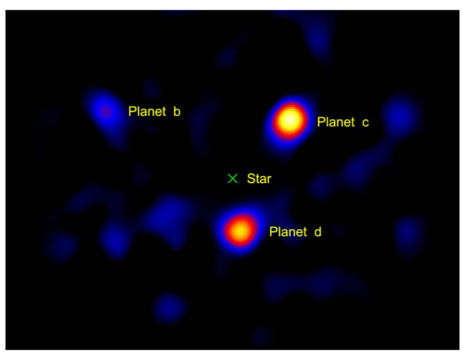

Figure 1: An example of a direct image (Source: Wikipedia)

2.2 Radial Velocity Method

Another method to detect exoplanets is called the **radial velocity** method. This method relies on the Doppler effect to detect exoplanets. More specifically, periodic variations in a star's spectrum can be used to create a graph of the star's radial velocity vs time. The radial velocity of the star changes over time because it is affected by the gravitational force from the exoplanet; we normally neglect the mass of the exoplanet in some calculations, but in reality, every exoplanet causes some sort of variation in its host star's radial velocity, though the amplitude of the variations might be very small and thus hard to detect. With the radial velocity graph, we can figure out the period of the exoplanet's orbit (see Figure 2). Then, we use this period to figure out many other quantities, such as the distance between the star and exoplanet, the velocity of the exoplanet, and the mass of the exoplanet. You can see examples of this in the Worked Problems section.

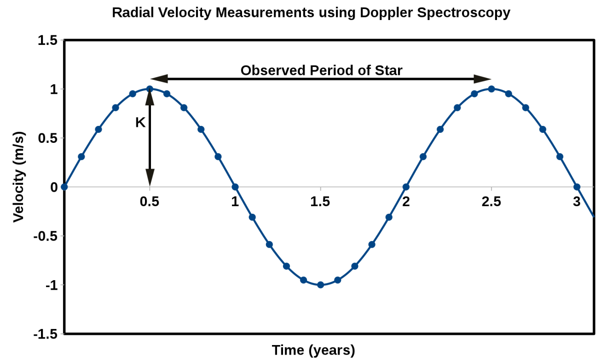

Figure 2: An example of a radial velocity curve (Source: Wikipedia)

2.3 Transit Method

A **transit** occurs when an exoplanet passes between us and its host star. This means the exoplanet will cover part of the star, reducing its apparent brightness. We can track a star's brightness using a **light curve**, which is a graph of the star's **apparent magnitude** vs time (look through the Stellar Flux Relationships handout if you are unfamiliar with magnitudes). This light curve will show a periodic "dip" whenever the exoplanet transits its host star. Based on the size of this dip, we can deduce the relative sizes of the host star and its exoplanet. The light curve can also be used to determine the orbital period of the exoplanet, since the dip will occur periodically. An example of such a curve is shown in Figure 3. Notice the dip when the planet is directly between us and the star.

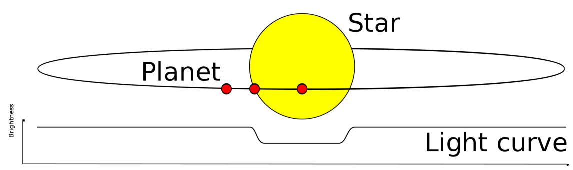

Figure 3: A star's light curve before and after the transit (Source: Wikipedia)

An important quantity to know for transits is the **transit depth**, which is defined as

$$\text{Transit depth} = \left(\frac{R_{\text{planet}}}{R_{\text{star}}}\right)^2$$

An example problem with transit depth can be found in the Worked Problems section.

3 Planetary Equilibrium Temperature

When a planet orbits around a star, it absorbs some radiation from the star and heats up. But at the same time, the planet is itself radiating energy. We say that the planet is in **thermal equilibrium** if at any given time, the amount of radiation absorbed by the planet is equal to the amount of radiation emitted by the planet. In other words, energy in = energy out. The planet's temperature in this equilibrium is called the **planetary equilibrium temperature**.

Let's derive the formula for the planetary equilibrium temperature. To start, we need to find a formula for the radiation incident on the planet. The star's luminosity can be written using the Stefan-Boltzmann law:

$$L = 4\pi R^2 \sigma T^4$$

If the planet orbits at a distance d from its host star, then the incident flux on the planet is

$$\frac{L}{4\pi d^2}$$

To understand this, you can think of a sphere propagating from the star. We're looking for power per unit area, so we have to divide the luminosity of the star by the surface area of the sphere ($4\pi d^2$). However, we have to account for the planet's **albedo** and spherical shape. The albedo is a quantity that ranges from 0 to 1 and represents the fraction of incident light *reflected* by a planet. If the albedo is 0, that means all incoming light is absorbed and nothing is reflected. Conversely, if the albedo is 1, that means all incoming light is reflected. Earth's albedo is about 0.3. Taking albedo and the spherical nature of the planet into account, we get that the incident flux is

$$\frac{L(1-a)}{16\pi d^2}$$

The extra factor of $\frac{1}{4}$ (we went from $\frac{1}{4}$ to $\frac{1}{16}$) comes from the fact that only a single hemisphere is lit at any moment in time (creates a factor of 1/2), and from integrating over angles of incident sunlight on the lit hemisphere (creating another factor of 1/2) (Wikipedia). The $(1-a)$ represents the fraction of incident light *absorbed* by the planet; remember that albedo is the fraction of light that is *reflected* but we're looking for the amount of radiation absorbed. The flux emitted by the planet is simply σT_{eq}^4, so setting these two equal yields

$$\frac{L(1-a)}{16\pi d^2} = \sigma T_{eq}^4$$

Finally, we can rearrange for T_{eq} to get

$$T_{eq} = (\frac{L(1-a)}{16\sigma\pi d^2})^{\frac{1}{4}}$$

This formula can also be written in terms of the star's radius and temperature since we know that $L = 4\pi R^2 \sigma T^4$. Substituting this in (try doing the algebra yourself), we get

$$T_{eq} = T_{star}\sqrt{\frac{R_{star}}{2d}}(1-a)^{\frac{1}{4}}$$

4 Worked Problems

Example 1: (2021 USAAAO First Round #5) HD 209458b has a radius of 1.35 Jupiter radii, while the radius of HD 209458 is 1.20 Solar radii. What is the transit depth of HD 209458b, in percent?

Solution: We simply use the formula for transit depth. If we want a percent, we multiply by 100. So converting the radii to meters, we get

$$100 * (\frac{1.35 * 7.1492 * 10^7}{1.20 * 6.96 * 10^8})^2 = \mathbf{1.3\%} \quad \square$$

Example 2: (2023 USAAAO First Round #13) The apparent magnitude of a star of radius $0.41R_\odot$ as observed from Earth appears to fluctuate by 0.037. That is, the difference between the maximum and minimum apparent magnitudes is 0.037. This fluctuation is caused by an exoplanet that orbits the star. Determine the radius of the exoplanet.

Solution: If you are unfamiliar with flux and magnitude, please go through the Stellar Flux Relationships handout first. Now, as we discussed in the Transit Method section, this fluctuation in the star's apparent magnitude occurs when the exoplanet transits the star. When the exoplanet transits (or "blocks" if that helps you visualize it better) the star, the star's apparent magnitude *increases*; remember that a higher apparent magnitude corresponds to a lower apparent brightness, and the star's apparent brightness will be lower when it is partially blocked by the exoplanet. Now, we can use the formula

$$m_2 - m_1 = -2.5\log(\frac{F_2}{F_1}),$$

where m_2 and m_1 represent the apparent magnitudes of the star at two different occasions (transit and non-transit), and F_2 and F_1 represent the flux from the star at these two occasions. If we let m_2 be the apparent magnitude of the star during the transit, we know that $m_2 > m_1$ so $m_2 - m_1 = 0.037$ (that was given in the problem). The flux received from the star depends on the star's luminosity, which itself depends on the star's surface area and temperature (by the Stefan-Boltzmann law). When the exoplanet transits the star, it covers a fraction of $(\frac{R_{\text{planet}}}{R_{\text{star}}})^2$. If we represent the flux from the star when it is *not* in transit as F, then the flux from the star *during* transit is

$$F(1 - (\frac{R_{\text{planet}}}{R_{\text{star}}})^2)$$

because the flux comes from the fraction of the star's surface that is *not* covered by the exoplanet. Substituting this into our formula above, we get that

$$0.037 = -2.5\log(\frac{F(1 - (\frac{R_{\text{planet}}}{R_{\text{star}}})^2)}{F})$$

Solving for R_{planet} yields $R_{\text{planet}} = 0.075R\odot \quad \square$

Example 3: (2023 USAAAO First Round #19) In 1995, researchers at the University of Geneva discovered an exoplanet in the main-sequence star 51 Pegasi. This was the first-ever discovery of an exoplanet orbiting a Sun-like star! When they observed the star, a periodic Doppler shifting of its stellar spectrum indicated that its radial velocity was varying sinusoidally; this wobbling could be explained if the star was being pulled in a circle by the gravity of an exoplanet. The radial velocity sinusoid of 51 Pegasi was measured to have a semi-amplitude of 56 m/s with a period of 4.2 days, and the mass of the star is known to be 1.1M⊙. Assuming that the researchers at Geneva viewed the planet's orbit edge-on and that the orbit was circular, what is the mass of the exoplanet in Jupiter masses?

Solution: Wow. That's a lot. Let's break it down. The first couple of sentences just tell us about the radial velocity method, which we already know about. It's the numbers that are important. The orbit being viewed "edge-on" essentially means we see the orbits as circles in a plane parallel to our line of sight. This is important because we can then use equations of circular motions; if the orbit was not edge-on (if the **inclination** was not 90 degrees), we would only be measuring a component of the star's velocity. But we don't need to worry about that. We also have the period of the radial velocity sinusoid, which we also know to be the period of the exoplanet's orbit. Whenever you have a period, you should immediately think of Kepler's Third Law. We also have the mass of the star. How do we put all this together? Well, remember that for binary systems,

$$\frac{M_s}{M_p} = \frac{a_p}{a_s} = \frac{v_p}{v_s},$$

where M represents the mass of the star/planet, a is the distance from the star or planet to the *center of mass*, and v is the velocity of the star or planet around the center of mass. We already have the mass of the star and the velocity of the star around the center of mass, so we just need the velocity of the planet around the center of mass to determine the planet's mass. We know that

$$\frac{T^2}{a_p^3} = \frac{4\pi^2}{GM_s}$$

and

$$v_p = \frac{2\pi a_p}{T} \rightarrow a_p = \frac{v_p T}{2\pi}$$

Substituting this expression for a_p into Kepler's 3rd law, we get

$$\frac{8\pi^3}{Tv_p^3} = \frac{4\pi^2}{GM_s}$$

Plugging everything in (using SI units) and solving for v_p, we get $v_p = 1.357 \cdot 10^5$ m/s. You may get a slightly different answer if you rounded differently. Finally, we can use

$$\frac{M_s}{M_p} = \frac{v_p}{v_s}$$

to solve for M_p and get that (again using SI units and then converting to Jupiter masses) $M_p = 0.47 M_J$ □

Example 4: (2022 USAAAO First Round #23) An exoplanet discovered by the radial velocity method is found to have an orbital period of 2.45 days around a Sun-like star. Assuming the planet has zero albedo (i.e., absorbs all incoming starlight) and perfectly transports heat across its surface, estimate the temperature at the photosphere of the planet.

Solution: This is a simple application of the formula we previously derived for planetary equilibrium temperature:

$$T_{eq} = (\frac{L(1-a)}{16\sigma\pi d^2})^{\frac{1}{4}}$$

We know the luminosity ("sun-like star" tells us that it's the same as the Sun's luminosity) and the albedo (0), so we just need to find the distance between the star and exoplanet. Luckily, we have the orbital period, so we can use Kepler's 3rd Law. Let's use the simple version of the law, for which we need T to be in years and a to be in AU (we know the mass of the star is one solar mass because it's sun-like). Thus, we get

$$T^2 = a^3 \rightarrow a = (\frac{2.45}{365})^{\frac{2}{3}}\text{AU} = 0.036\text{AU}$$

Now, plugging everything in using SI units, we get that $T =$ **1476 K** $\square$.

5 Practice Problems

I highly recommend you try all of the following problems by yourself. They are very good practice, and it is likely that similar problems will show up on future exams (there's already so many similar problems!).

Go to the past exams page and try the following problems:

- **2023 First Round:** Number 3 and Number 22
- **2022 First Round:** Number 7 and Number 8
- **2021 First Round:** Number 3 and Number 18
- **2020 First Round:** Numbers 2, 3, 18, and 19

6 Conclusion

Hopefully this handout helped you understand some of the concepts and math about exoplanets. As always, Google is your best friend if you want to learn more about these topics. If you couldn't already tell, exoplanet problems get really repetitive when you master the basics, so thoroughly understanding the Worked Problems and doing the Practice Problems will greatly help you solve future exoplanet-related problems on the USAAAO.