Separable Differential Equations

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1 Introduction

Separable differential equations are a type of nonlinear first order differential equations. They are first order differential equations as they only involve first derivative terms and are nonlinear since they may be, say, a quadratic function of the first derivative. Let y be a function of x, (i.e. $y = y(x)$).

Definition: A separable differential equation is of the form

$$y'(x) = \frac{f(x)}{g(y(x))}.$$

2 Derivation

We now derive the method for solving separable differential equations. Suppose we have a separable differential equation $y'(x) = f(x)/g(y(x))$. We can rewrite this using Leibniz notation as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y(x))}.$$

We can now multiply both sides by $g(y(x))$ and integrate with respect to x to get

$$\int g(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int f(x)\,\mathrm{d}x.$$

We integrate this using a substitution: $u = y(x)$. This implies that $\mathrm{d}u = y'(x)\,\mathrm{d}x = \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x$. Plugging this in then gives

$$\int g(u)\,\mathrm{d}u = \int f(x)\,\mathrm{d}x.$$

The differential equation thus reduces to a problem of integration. Although this is the mathematically correct way of solving the differential equation, there is an easier method that gives the same result. Given a separable differential equation, multiply both sides by $g(y(x))\,\mathrm{d}x$ to get

$$g(y(x))\,\mathrm{d}y = f(x)\,\mathrm{d}x.$$

All that remains to be done is to integrate both sides,

$$\int g(y(x))\,\mathrm{d}y = \int f(x)\,\mathrm{d}y.$$

This is how we will compute the examples.

3 Examples

Example 1: We solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$$

using the method of separation of variables. In this example, $f(x) = x$ and $g(y(x)) = y$. We multiply both sides of the equation by $g(y(x))\,\mathrm{d}x = y\,\mathrm{d}x$ to get $y\,\mathrm{d}y = x\,\mathrm{d}x$. We now integrate both sides:

$$\int y\,\mathrm{d}y = \int x\,\mathrm{d}x \implies \frac{y^2}{2} + c_1 = \frac{x^2}{2} + c_2.$$

We multiply both sides by 2 to get $y^2 + 2c_1 = x^2 + 2c_2$. Note that $2c_2 - 2c_1$ is also a constant, call it k. This reduces the prior equation to $y^2 = x^2 + k$ which is the simplified result.

Example 2: We solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(x)}{\cos(y)}.$$

Multiply both sides by $\cos(y)\,\mathrm{d}x$ to get $\cos(y)\,\mathrm{d}y = \sin(x)\,\mathrm{d}x$. Integrate both sides to get

$$\int \cos(y)\,\mathrm{d}y = \int \sin(x)\,\mathrm{d}x \implies \sin(y) + c_1 = -\cos(x) + c_2.$$

We can write $k = c_2 - c_1$ to give the solution $\sin(y) = k - \cos(x)$.

Example 3: Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2.$$

One might wonder how this can be a separable differential equation since there appears to be no $f(x)$ and we are multiplying by $g(y)$ rather than dividing by it. This is in fact separable since we can let $f(x) = 1$ and $g(x) = 1/y^2$. We can verify that $f(x)/g(y) = 1/(1/y^2) = y^2$. We multiply both sides of the differential equation by $g(y)\,\mathrm{d}x = (1/y^2)\,\mathrm{d}x$ to get

$$\frac{1}{y^2}\,\mathrm{d}y = 1\,\mathrm{d}x$$

We can integrate both sides by using the power rule to get

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int 1\,\mathrm{d}x \implies -y^{-1} + c_1 = x + c_2.$$

We can let $k = c_2 - c_1$ to get $-y^{-1} = x + k$ as the solution.

Example 4: Let us compute one more example where we are given the initial conditions. The differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{4y^3}$$

and $y(0) = 100$. We multiply both sides by $4y^3\,\mathrm{d}x$ to get $4y^3\,\mathrm{d}y = 2x\,\mathrm{d}x$. Integrating both sides and applying the power rule gives

$$\int 4y^3\,\mathrm{d}y = \int 2x\,\mathrm{d}x \implies y^4 + c_1 = x^2 + c_2.$$

We can write $k = c_2 - c_1$ to get $y^4 = x^2 + k$. Plugging in our initial condition forces $(100)^4 = 0^2 \cdot k$ which implies that $k = \sqrt[4]{100} = \sqrt{10}$. Thus, the solution simplifies to $y^4 = x^2 + \sqrt{10}$.

Example 5: The most illuminating example is solving the differential equation from exponential growth:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky.$$

This differential equation requires the knowledge of the derivatives of e^x and $\log(x)$ which is solved in the handout on exponential and logarithmic growth.