Kepler's Laws & Orbital Geometry

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1 Introduction

Kepler's laws make up the basis of astrophysics, and remain tremendously important for understanding all of celestial mechanics. They allow us to understand the shapes of orbits, predict hard-to-observe parameters like distance from easy-to-observe ones like period, and to set up key equations to understand the rate of movement!

2 Kepler's First Law

2.1 Law and Importance

Kepler's first law says that all planets orbit in ellipses with the sun at one focus of the ellipse, which can be generalized to the fact that all objects in closed orbits orbit their central bodies in ellipses with the central body at a focus of the ellipse. Open orbits are either hyperbolas or parabolas, and also have the central body at a focus.

Kepler's first law is important for understanding the orbital behavior of objects, as well as providing the basis for the patched conics technique, which involves combining the paths of conic orbits to create more advanced orbital maneuvers.

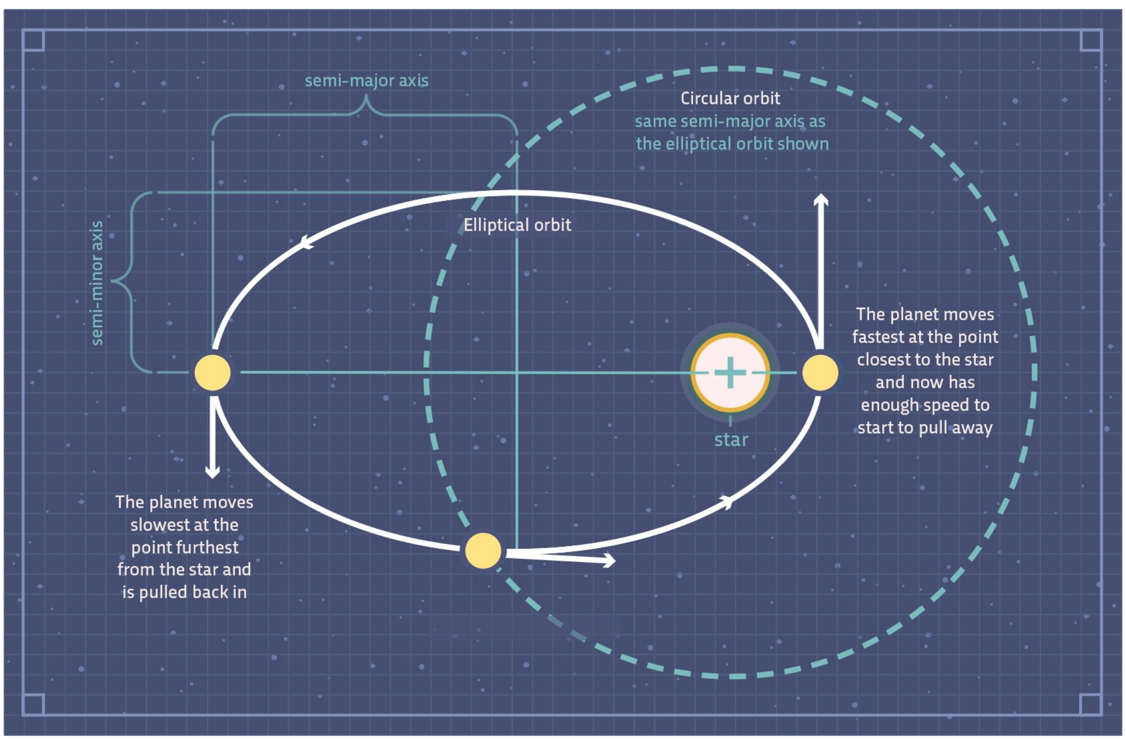


Figure 1: A diagram of the closed orbital behavior of a body (Source: BBC)

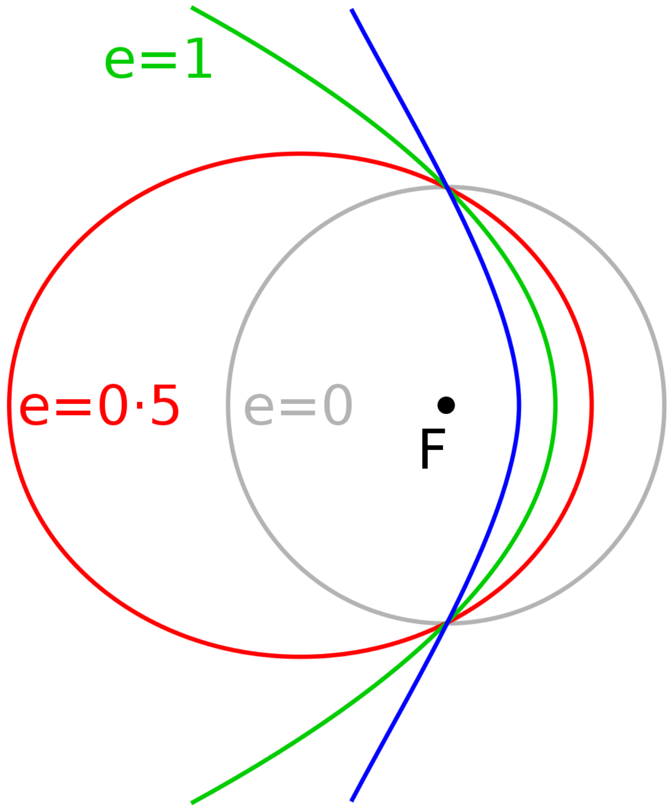


Figure 2: A diagram of the open orbital behavior of a body (Source: Wikipedia)

2.2 Derivation

2.2.1 Equation of an ellipse

Before we can prove Kepler's First Law, we first need to understand what an elliptical path looks like. For this, we will find the equation of an ellipse in polar coordinates. If you are not familiar with polar coordinates, all we are doing is defining points as (r,θ) instead of (x,y) where if we draw a radius from our point to the origin, r is the length of this radius and θ is the angle it makes with the positive x-axis.

For our ellipse, we can set one of our foci as the origin. In this example, if the major axis is horizontal, then I'm setting the right focus as the origin to make the following math easier. By definition, the distance from a point on the ellipse to this focus is r. Let's call the distance to the other focus r'. This means our point is at (r,θ) in polar coordinates and $(r\cos\theta, r\sin\theta)$ in rectangular coordinates. Since the distance between the two foci in an ellipse is $2c = 2ae$, we can use the Pythagorean theorem to find r':

$$r'^2 = r^2\sin^2\theta + (2ae + r\cos\theta)^2$$

By expanding and using the identity $\sin^2\theta + \cos^2\theta = 1$, it can be rewritten as

$$r'^2 = r^2 + 4ae(ae + r\cos\theta)$$

By definition, we know that for an ellipse $r + r' = 2a$, so we can substitute $r' = 2a - r$ to solve for r. After simplifying, we get:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

2.2.2 Using gravity

Warning: this section involves excessive amounts of calculus. Feel free to skip this part.

To find the path of an orbit around a central body, we begin by using a polar coordinate system with the Sun at the origin. The only force acting on our planet is gravity ($-\frac{GM}{r^2}$), directed towards the origin. The acceleration from this force will contribute to two components:

1. **Centripetal acceleration:** $-\omega^2 r$ (negative since it acts inwards) where ω is the angular velocity
2. The change in radius: $\frac{d^2r}{dt^2}$

Thus, we have the equation

$$-\frac{GM}{r^2} = \frac{d^2r}{dt^2} - \omega^2 r$$

However, ω also varies with r (due to conservation of angular momentum), so we must express it in terms of r by using the fact that angular momentum $L = \mu\omega r^2$ where μ is the reduced mass.

$$-\frac{GM}{r^2} = \frac{d^2r}{dt^2} - (\frac{L}{\mu r^2})^2 r = \frac{d^2r}{dt^2} - \frac{L^2}{\mu^2 r^3}$$

Here, we see a lot of r in the denominator, so we make the sub $r = \frac{1}{u}$. However, we need to be able to convert $\frac{d^2r}{dt^2}$ into a usable form by abusing $\omega = \frac{d\theta}{dt}$.

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{dt}(\frac{dt}{d\theta})(\omega) = \frac{1}{u^2}\frac{du}{d\theta}(\frac{Lu^2}{\mu}) = -\frac{L}{\mu}\frac{du}{d\theta}$$

$$\frac{-L}{\mu}\frac{d^2u}{d\theta^2} = \frac{d\frac{dr}{dt}}{d\theta} = \frac{d^2r}{(dt)(d\theta)}(\frac{d\theta}{dt})(\frac{1}{\omega}) = \frac{\mu}{Lu^2}\frac{d^2r}{dt^2}$$

$$\frac{d^2r}{dt^2} = -\frac{L^2u^2}{\mu^2}\frac{d^2u}{d\theta^2}$$

Therefore, our equation becomes

$$-GMu^2 = -\frac{L^2u^2}{\mu^2}\frac{d^2u}{d\theta^2} - \frac{L^2u^3}{\mu^2}$$

This leaves us with the simple differential equation:

$$\frac{GM\mu^2}{L^2} = \frac{d^2u}{d\theta^2} + u$$

$$u = \frac{1}{r} = \frac{GM\mu^2}{L^2} + A\cos\theta$$

This still looks ugly, but we can combine most of the constants using contrived substitutions:

$$r = \frac{(\frac{L}{\mu})^2}{GM(1 + e\cos\theta)} = \frac{a(1-e^2)}{1+e\cos\theta}$$

This is the polar equation for an ellipse! Therefore, we have proven Kepler's First Law.

2.2.3 Other types of Orbits

In an orbit, there are two types of energy we mainly have to worry about. Potential energy, which we say is negative because it keeps the object bound in the orbit, and kinetic energy, which we say is positive because it can help an object escape orbit. An orbit is bound (elliptical) if the energy overall is negative, which means that the gravitational potential energy is greater than the kinetic energy. A circular orbit is the very special case where gravitational potential energy is two times kinetic energy.

For a parabola, where the overall energy is 0 (meaning the gravitational potential perfectly balances the kinetic energy) and the orbital eccentricity is 1,

$$r = \frac{2p}{1 + \cos\theta}$$

For a hyperbola, which have positive orbital energies (meaning the gravitational potential is less than the kinetic energy) and eccentricities greater than 1,

$$r = \frac{a(e^2 - 1)}{1 + e\cos\theta}$$

3 Kepler's Second Law

3.1 Law and Importance

Kepler's second law says that planets sweep out equal areas in equal times. This is a restatement of the conservation of momentum, only one that Kepler came up with before the conservation of angular momentum!

The area of an entire ellipse is $ab\pi$, so with the period, the area that is swept out over a certain period of time can be easily calculated. This is useful for deriving various important orbital relations, especially when used together with the conservation of angular momentum.

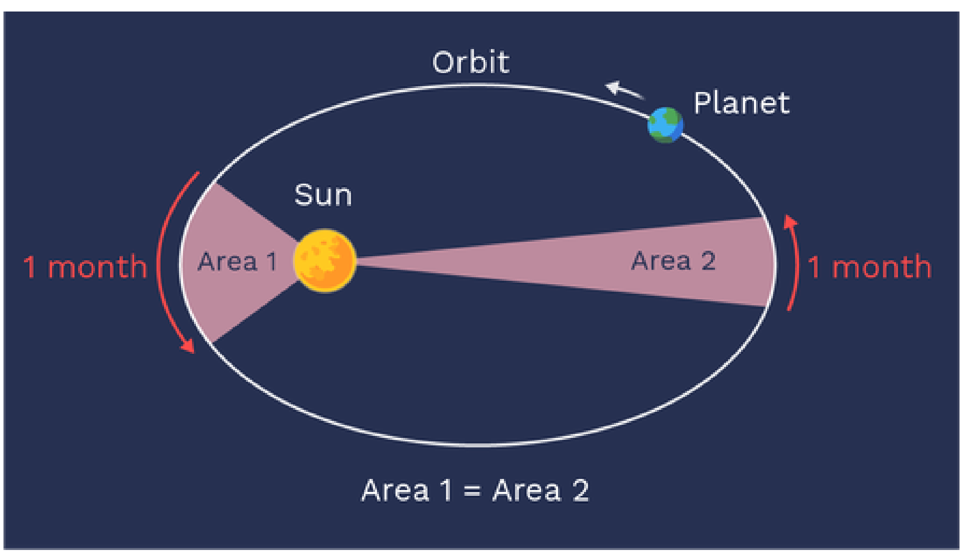
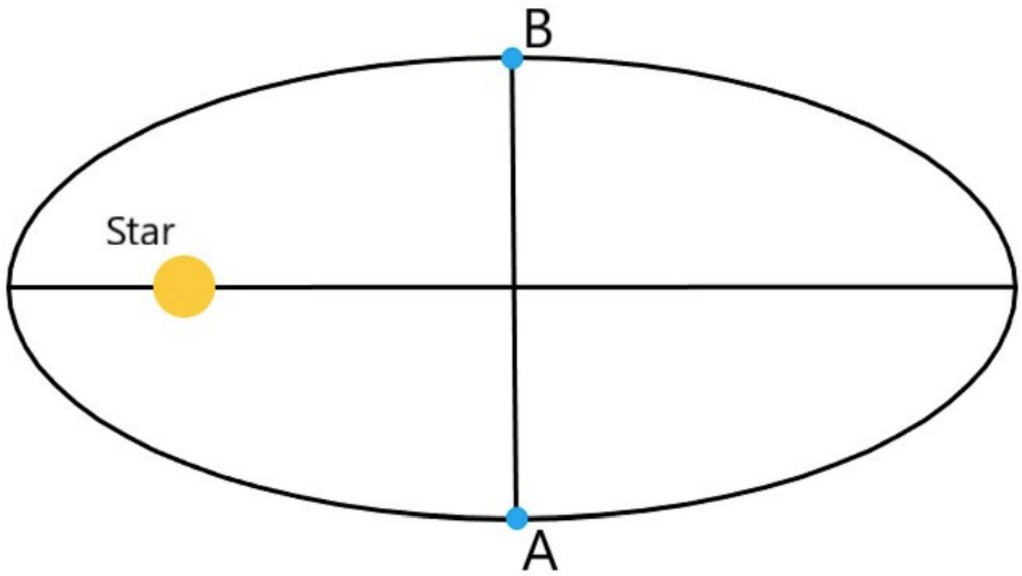


Figure 3: An illustration of Kepler's second law (Source: Labster Theory)

Example 2.1: (USAAAO Round One 2022)

Consider the following elliptical orbit of a comet around a star:



Which of the following expressions corresponds to the time that the comet takes to go from point A to point B as a function of the period of the comet (T) and the eccentricity of the orbit (e)?

Assume that the direction of the orbit is **counterclockwise**.

(a) $\frac{T}{2}$

(b) $\left(\frac{e}{\pi} + \frac{1}{2}\right) * T$

(c) $\left(\frac{1}{2} - \frac{e}{\pi}\right) * T$

(d) $(1 + e) * \frac{T}{2}$

(e) $\frac{T * e}{2}$

Solution: From Kepler's second law, equal areas are swept out in equal times, regardless of where in the orbit that area is, and using Δt as the amount of time we are looking for:

$$\frac{\Delta t}{T} = \frac{A_{movement}}{A_{total}}$$

using the total area formula and the area that is swept out as half the total area $\frac{\pi ab}{2}$ plus a triangular area $2 \times \frac{abe}{2}$ that is formed by the semiminor axis and the lines connecting the star to points A and B,

$$\frac{\Delta t}{T} = \frac{2 \times \frac{abe}{2} + \frac{\pi ab}{2}}{\pi ab}$$

$$\frac{\Delta t}{T} = \frac{abe + \frac{\pi ab}{2}}{\pi ab}$$

Dividing through,

$$\Delta t = (\frac{e}{\pi} + 0.5)T$$

which is **answer choice B**.

3.2 Derivation

Warning: This section uses significant amounts of calculus! If you want to attempt reading this section and you do not have a solid calculus background, please check out our Calculus Primer handouts!

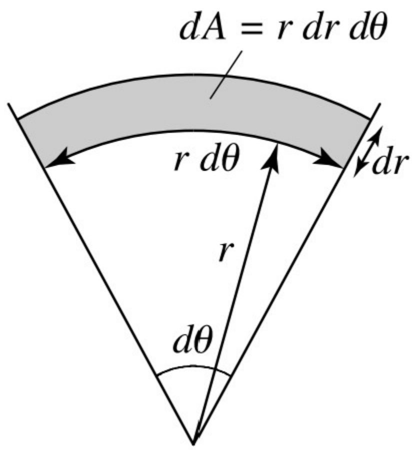


Figure 4: Diagram of a small angular area (Source: Caroll and Ostlie)

From Figure 4,

$$\mathrm{d}A = \mathrm{d}r(r\mathrm{d}\theta) = r\mathrm{d}r\mathrm{d}\theta$$

Integrating from the principal focus (the focus that the central body is at) to r,

$$\mathrm{d}A = \frac{1}{2}r^2\mathrm{d}\theta$$

Taking the time derivative

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

The orbital velocity can be expressed in terms of the radial and tangential velocities,

$$v = v_r + v_t = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\theta}{t}\hat{\theta}$$

Solving for v_t and using substituting in,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}rv_t$$

As v_t is the tangential velocity,

$$rv_t = |r \times v| = \frac{L}{\mu}$$

thus,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{L}{2\mu}$$

4 Kepler's Third Law

4.1 Law and Importance

Kepler's third law states that **ONLY in the solar system**

$$P^2 \propto a^3$$

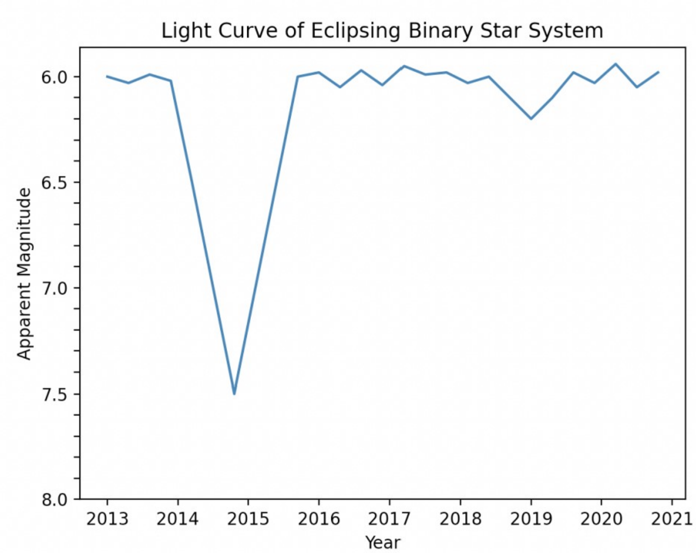
where P is the period in **years** and a is the semi-major axis in **AU**. This expression is a specific version of Kepler's third law, known as Newton's version of Kepler's third law, that only applies to our solar system, because the proportionality constant in the solar system evaluates to 1. This is not a coincidence; this reduction comes from the fundamental definition of our units. Newton's version of Kepler's third law is

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}$$

where all values are in SI units. A common way of reducing this is using the approximation that if M_1 is the mass of a star and M_2 is a planet, $M_1 \approx M_1 + M_2$. Kepler's third law is an incredibly important basis for many relations and formulas in orbital mechanics, and all of astronomy in general.

Example 3.1:(USAAAO Round One 2022)

An astronomer observes an eclipsing binary star system from Earth, and he plots the following light curve.



Suppose that both stars have circular orbits and the distance between the stars is 14.8 AU. What is the total mass of the binary star system in terms of solar masses?

(a) $2.3M_\odot$
(b) $5.7M_\odot$
(c) $6.8M_\odot$
(d) $23M_\odot$
(e) $46M_\odot$

Solution: From the graph, there is 4.2 years between the primary and secondary eclipse, which are when the stars eclipse each other (see the section on binaries in the Stellar Evolution handout for more info). The deeper eclipse is the primary; though this is irrelevant for the problem. Since both stars follow circular orbits, the orbital period is simply double this time difference or 8.4 years. Using Kepler's third law,

$$(8.4yrs)^2 = \frac{4\pi^2(14.8AU)^3}{G(M_1 + M_2)}$$

using the appropriate unit conversions to SI and solving for the mass, this results in 46 solar masses, or **answer choice E**.

> **Example 3.2:** (USAAAO Round One 2020) Planet Nine is a hypothetical planet in the outer Solar System, with a semimajor axis between 400 and 800 AU. Which of the following is a possible orbital period for Planet Nine?
> A. 71.1 years
> B. 600 years
> C. 1,500 years
> D. 15,000 years
> E. 360,000 years

Solution: Use Kepler's third law for the solar system, where

$$P^2 \propto a^3$$

and they are equal if P is in years and a is in AU. Plugging in for 400 AU and 800 AU, the only choice greater than the period for 400 AU and less than the period for 800 AU is 15,000 years, or **answer choice D**.

> **Example 2.3:** Does Kepler's third law in the solar system work for Earth?

Solution: **Yes**, as in the solar system,

$$P^2 \propto a^3$$

and they are equal if P is in years and a is in AU. Plugging in for one year and one AU, we see that it works and the proportionality constant must by 1!

4.2 Derivation

There are two derivations in this section: a simpler version that does not involve calculus and is independent of Kepler's second law, and another more rigorous one based on the second law.

4.2.1 Circular orbits

First, set centripetal force equal to the gravitational force:

$$m\frac{v^2}{r} = G\frac{mM}{r^2}$$

$$v^2 = \frac{GM}{r}$$
$$v = \sqrt{\frac{GM}{r}}$$

In physics, velocity × time = distance. In an orbit, the distance is the circumference ($2\pi r$) and the time is the period:

$$vP = 2r\pi$$
$$v = \sqrt{\frac{GM}{r}} = \frac{2r\pi}{P}$$

Squaring both sides and rearranging:

$$P^2 = \frac{4\pi^2 a^3}{GM}$$

which is the version of Kepler's third law used when the central body mass is much larger than the orbiting body mass. Note: we substituted a in place of r since the two are equivalent in a circle.

4.2.2 Generalized

For the second derivation, starting with the integrated version of Kepler's second law,

$$A = \frac{L}{2\mu}P$$

Rearranging and substituting $A = ab\pi$:

$$P^2 = (\frac{2\mu ab\pi}{L})^2 = \frac{\mu^2 4\pi^2}{L^2}a^2b^2$$

Then using Kepler's first law forms

$$r = \frac{L^2/\mu^2}{GM(1 + ecos\theta)}$$

$$r = \frac{a(1-e^2)}{1 + ecos\theta}$$

Setting them equal to each other and solving for L

$$L = \mu\sqrt{GMa(1-e^2)}$$

From the geometry of an ellipse

$$b^2 = a^2(1-e^2)$$

Plugging both of those into the equation derived from Kepler's second law

$$P^2 = \frac{4\pi^2}{GM}a^3 = \frac{4\pi^2}{G(m_1 + m_2)}a^3$$

which is the full, more accurate version of Kepler's third law.

5 Orbital Geometry

5.1 Ellipses

All closed orbits are ellipses. An ellipse is defined as the locus of points where the distances from two foci sum to a constant.

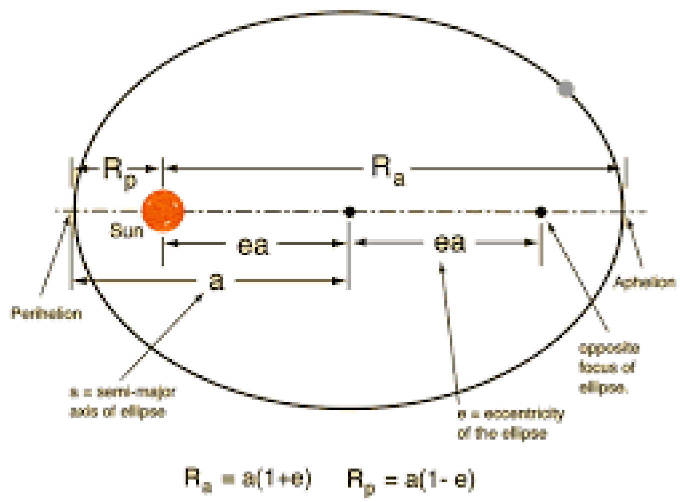


Figure 5: Ellipse diagram (Source: Hyperphysics)

From the diagram, we can define points F_1 and F_2 as the two foci, b as the semi-minor axis, and a as the semimajor axis. Alternatively, it can be defined as the points a given summed distance from a focus and a directrix, or a specific line. The eccentricity of an ellipse, a measure of how "circular" it is, ranges from 0 to 1 in an ellipse and can be defined as

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

or through

$$b^2 = a^2(1 - e^2)$$

which is often more useful. A circle has an eccentricity of 0. Most planets have extremely low eccentricities. Eccentricity can also be expressed in terms of the focus-directrix distance. In this ellipse, we can call the point in the orbit closest to where the central body is (F_1 in this case) the perihelion, perigee, perilune or periapsis, and the farthest point the aphelion, apogee, apolune, or apoapsis, depending upon what body it is orbiting. -helion is the Sun, -gee is the Earth, -lune is the Moon, and -apsis is general. An elliptical orbit can be modeled using the following equation:

$$\frac{(x + ea)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

The periapsis of the orbit is at distance

$$a(1 - e)$$

and the apoapsis is at distance

$$a(1 + e)$$

A circular orbit, a special form of an elliptical orbit, can be modeled using

$$r = \frac{L^2}{GMm^2}$$

where L is the angular momentum, G is the gravitational constant, M is the mass of the central body, and m is the mass of the orbiting body.

Example 4.1: (USAAAO Round One 2022) The orbit of some planet to its star has an eccentricity of 0.086. What is the ratio of the planet's closest distance to its star to the farthest on its orbit?
(a) 0.842
(b) 0.188
(c) 1.188
(d) 0.158
(e) None of the above

Solution: Using the formulae for the apoapsis and periapsis of an orbit, this ratio is equal to

$$\frac{a(1-e)}{a(1+e)}$$

which equals

$$\frac{1-e}{1+e}$$

Plugging in for the given eccentricity,

$$\frac{1-0.086}{1+0.086}$$

This equals 0.842, or **answer choice A**.

5.2 Parabolas and Hyperbolas

Parabolas and hyperbolas are the two forms of open orbits. Both can also be defined using a focus-directrix definition, and do possess an eccentricity. A parabola has an eccentricity of 1, and a hyperbola has an eccentricity greater than 1. Ellipses, hyperbolas, and parabolas fall into a class of functions known as conic sections. A parabolic orbit can be modeled using

$$y^2 = 2p(0.5p - x)$$

where p is the distance of closest approach, and x and y are coordinate distances relative to the central star the difference from the center can also be expressed using the equation

$$\beta = \frac{RL^2}{(GMm^2)\sqrt{1-e^2}}$$

Where R is the orbital distance, L is the angular momentum, G is the gravitational constant, M is the central body's mass, m is the orbiting body's mass, and e is the eccentricity.

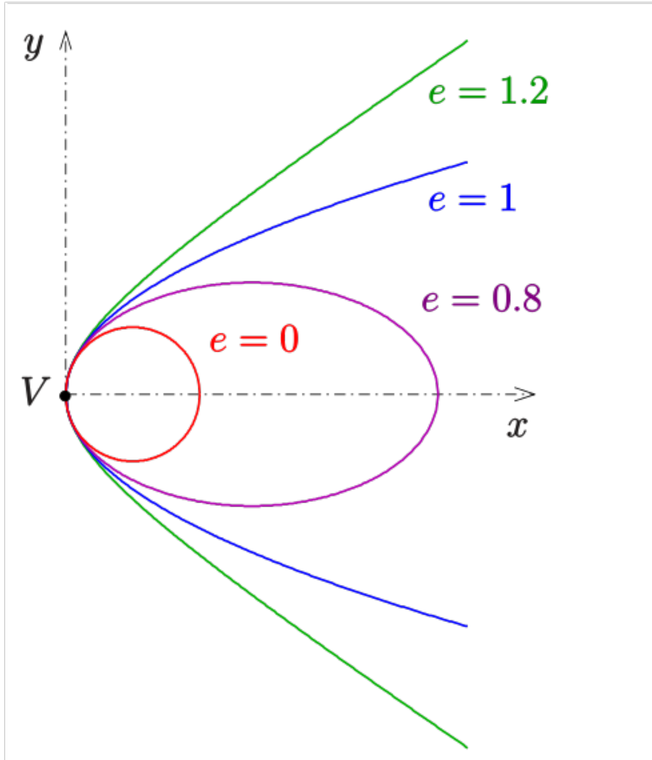


Figure 6: A parabolic orbit (Source: Physics Stack Exchange)

Example 4.2. If an orbit has an eccentricity of 2, what shape is it, and it it open or closed?

Solution: The **orbit is a hyperbola**, as the eccentricity is greater than 1. Hyperbolas are open orbits (the orbiting body never returns to any part of the orbit), so **this is an open orbit**.

5.3 Orbital Elements

While all orbits are conic sections, more information is needed to identify them in space relative to Earth, and many of those parameters can help better model orbits. Beyond the two most basic orbital parameters (the eccentricity and the semimajor axis length) the inclination i, the longitude of the ascending node Ω, the argument of periapsis ω, and the true anomaly θ are also important.

The inclination is a measure of the tilt of the orbital plane relative to the ecliptic or another reference plane. The longitude of the ascending node refers to the angle from the vernal equinox (the zero right ascension, zero declination point) of the point where the orbital plane intersects with the ecliptic, moving from above the ecliptic to below the ecliptic.

The argument of periapsis is the angle from the line of the center of the orbital plane to the ascending node to the line along the major axis towards the direction of the periapsis. The true anomaly is the angle from the celestial body to the segment along the major axis to the periapsis, and is a way of parameterizing where in the orbit the body is.

There are a series of alternatives choice of angular parameters. One of the most common is the mean anomaly M, which is another useful way to express the position of the body in its orbit. P is the period of the orbit. With τ as the reference time where the object is at periapsis,

$$M = \frac{2\pi}{P}(t - \tau)$$

This equation is useful as the mean anomaly directly represents the angle for Kepler's second law.

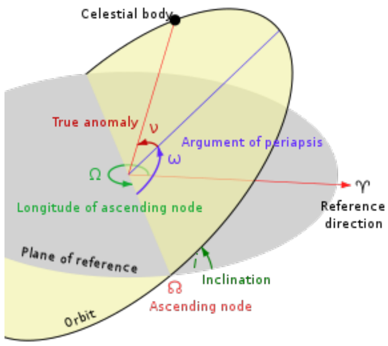


Figure 7: A diagram showing the orbital elements (Source: Wikipedia)

6 Conclusion

Kepler's laws and orbital geometry are tremendously important as they underpin all of orbital mechanics, and now you can use them too! Almost all celestial mechanics questions, especially those in USAAAO, rely on Kepler's laws as a crucial foundation. Understanding this material will set you up for success!