Kinematics

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1 Introduction

A central topic in physics is understanding the motion of a particle. We would like to know the position, velocity, speed, acceleration, jerk, and other quantities relating to the motion of a particle. In several cases, the acceleration of a particle is zero, such as when the net force is zero. In these situations, it is very convenient to use the kinematic equations which we derive in this handout.

2 Preliminaries

We start with particle motion. The position of a particle, $x(t)$ is its position in space. We commonly write it as a function of time, t. In three dimensions, x can be a vector with 3 components. From this, we can obtain two more standard definitions.

Definition 2.1: The velocity of a particle, $v(t)$ is equal to $\frac{\mathrm{d}}{\mathrm{d}t}x(t)$.

Definition 2.2: The acceleration of a particle, $a(t)$ is equal to $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)$.

We say that a particle is "speeding up" if its velocity and acceleration vectors have the same sign and that a particle is "slowing down" otherwise. From these definitions, it is clear that we can go "backwards" via the fundamental theorem of calculus.

Theorem 2.1: Given initial position $x_0 = x(0)$,

$$x(t) = \int_0^t v(t)\,\mathrm{d}t + x_0.$$

Proof. We have by the fundamental theorem of calculus

$$\int_0^t v(t)\,\mathrm{d}t + x_0 = \int_0^t \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t + x_0 = \int_0^t \mathrm{d}x + v_0 = x(t) - x(0) + x_0 = x(t).$$

□

The above theorem shows how to get the position from velocity.

Example 2.1: A particle is at position $x = 5$ at $t = 0$ and its velocity function can be modeled as $v(t) = t^3$. We can find the position function using the prior theorem as follows:

$$x(t) = \int_0^t v(t)\,\mathrm{d}t + x_0 = \int_0^t t^3\,\mathrm{d}t + 5 = \frac{t^4}{4} + 5.$$

We can similarly obtain an expression for velocity in terms of acceleration.

Theorem 2.2: Given initial position $v_0 = v(0)$,

$$v(t) = \int_0^t a(t)\,\mathrm{d}t + v_0.$$

Proof. We have by the fundamental theorem of calculus

$$\int_0^t a(t)\,\mathrm{d}t + v_0 = \int_0^t \frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t + v_0 = \int_0^t dv + v_0 = v(t) - v(0) + v_0 = v(t).$$

□

3 Derivation of the Kinematic Equations

Our goal in this handout is to derive the 4 standard kinematic equations when assuming constant acceleration. We start with the assumption

$$a = c.$$

Rewriting this gives

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = c$$

and integrating once gives

$$v(t) = ct + v(0).$$

This is the first kinematic equation. It tells us that when a particle starts from rest (there is no initial velocity when starting from rest), the velocity of the particle is proportional to the time. We can write the constant acceleration c, as a to obtain the following canonical equation.

Kinematic Equation 1 For a particle undergoing constant acceleration a,

$$v(t) = at + v_0.$$

We can now rewrite this as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ct + v(0)$$

and integrating once gives

$$x(t) = \frac{c}{2}t^2 + v(0)t + x(0).$$

Understanding that a is constant, we can write c as a to get the canonical first kinematic equation that tells us that when a particle is undergoing constant acceleration, its position is proportional to the time squared when it starts from rest at position $x = 0$.

Kinematic Equation 2 For a particle undergoing constant acceleration a,

$$x(t) = \frac{1}{2}at^2 + v_0 t + x(0).$$

From the previous two equations, we can easily derive the third kinematic equation.

Kinematic Equation 3 For a particle undergoing constant acceleration,

$$x(t) - x(0) = \left(\frac{v_0 + v(t)}{2}\right) t.$$

Proof. We can use Theorem 2.1 to get

$$\left(\frac{v_0 + v(t)}{2}\right) t = \left(\frac{at + v_0 + v_0)}{2}\right) t = \frac{at^2 + 2v_0 t}{2} = \frac{1}{2}at^2 + v_0 t.$$

We know that this is equal to $x(t) - x(0)$ by Theorem 2.2. □

The final kinematic equation relates the quantities of position, velocity, and acceleration without mentioning time.

Kinematic Equation 4 For a particle undergoing constant acceleration,

$$v(t)^2 - v(0)^2 = 2a(x(t) - x(0)).$$

Proof. We can use Theorem 2.1 to isolate an expression for a as

$$a = \frac{v(t) - v(0)}{t}.$$

We can now use Theorem 2.3 from above to simplify the right-hand side as

$$2a(x(t) - x(0)) = 2 \cdot \frac{v(t) - v(0)}{t} \cdot (t)\frac{v(t) + v(0)}{2}$$

and cancelling out the $2t$ on the numerator and denominator gives

$$2a(x(t) - x(0)) = (v(t) - v(0))(v(t) + v(0)) = v(t)^2 - v(0)^2$$

as desired. □

Remark 3.1: The theorems above are only valid when a particle is going under constant acceleration and be false otherwise.

Remark 3.2: The $t = 0$ does not have to be the absolute start time. It may be more convenient in some problems to label $t = 0$ after some event occurs that causes the particle under consideration to have constant acceleration.

Remark 3.3: All of the theorems can be derived by the fact that acceleration is the second derivative of position and then the constant acceleration twice. If you ever forget some of the equations, just remember the previous fact and then you can solve any kinematics problem via Theorem 2.1 and Theorem 2.2 (although there may be one additional step).

4 Examples

Example 1: A bug is originally moving to the right at a constant velocity of $2\frac{m}{s}$. After 10 seconds, a kid begins to push the bug and the bug begins to accelerate at $10\frac{m}{s^2}$ from here. What is the position of the bug after 50 seconds?

Solution: Let us consider the motion of the bug into two parts, before and after the kid intervened. Before the kid intervened, we had $x(t) = 2t \implies x(10) = 20m$. Now, for when the kid started pushing we have

$$x(t) = \frac{1}{2}10t^2 + 2t = 5t^2 + 2t \implies x(40) = 8080m.$$

Thus after 50 seconds, the bug is at position $8080 + 20 = 8100m$.

Example 2: A boy who is running at a constant velocity begins tp accelerate at a constant rate of $0.50\frac{m}{s^2}$. After running for $20m$, his speed is $50\frac{m}{s}$. What was speed before accelerating?

Solution: We can use Theorem 2.4 to obtain

$$v(0)^2 = v(t)^2 - 2a(x(t) - x(0)) = 50^2 - 2(0.50)(20) = 2480\frac{m^2}{s^2} \implies v(0) = \sqrt{2480} \approx 49.8\frac{m}{s}.$$