# Implicit Differentiation

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# 1 Introduction

Up to this point, we've differentiated functions by isolating for $y$ and then taking the derivative of the other side. But it's not always easy nor beneficial to solve directly for $y$. This is where **implicit differentiation** comes in.

**Definition:** The idea of implicit differentiation is to differentiate both sides initially and use the chain rule. This works because differentiation is an operation like addition or subtraction are. If we add 4 to both sides of an equation, the equation is still true. Likewise, if we differentiate both sides of an equation, the equation is still true.

# 2 Examples

**Example 1:** Let's find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given

$$x^2 + y^2 = 1$$

**Solution:**

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(1)$$
$$2x + 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y}$$

In the third line, we used the chain rule to differentiate $y^2$; $y$ is a function of $x$ so we must multiply $2y$ by $\frac{\mathrm{d}y}{\mathrm{d}x}$. Note that you may see $\frac{\mathrm{d}y}{\mathrm{d}x}$ written as $y'$. Both mean the same thing.

**Example 2:** Find $y'$ given $x^2 + xy + +y^3 = 0$.

**Solution:** We have to be careful to use both the chain rule and the product rule here, since the middle term is $xy$, which is a product of two functions.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + xy + y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(0)$$

$$2x + (x \cdot y' + 1 \cdot y) + 3y^2y' = 0$$

Now, we solve for $y'$:

$$y'(x + 3y^2) = -2x - y$$
$$y' = \frac{-2x - y}{x + 3y^2}$$

# 3 Practice

## 3.1 Problems

**Problem 1**: Given $2xy + x^3 - 3y^2 = 5$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

**Problem 2**: Given $x^3y + y^2 - x^2 = 5$, find the slope of the tangent line at the point (2,1).

**Problem 3**: If $\frac{1}{x} + \frac{1}{y} = \frac{1}{12}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

## 3.2 Answers

**Answer 1:** $\frac{-2y-3x^2}{2x-6y}$

**Answer 2:** $-\frac{4}{5}$

**Answer 3:** $-\frac{y^2}{x^2}$

You can find many more practice problems online (Khan Academy, Paul's Online Math Notes, Organic Chemistry Tutor, etc).