Fundamental Theorem of Calculus

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1 Introduction

In the other handouts, you may have learned how to compute derivatives and interpret integrals. In this handout, we go through how one would actually compute these integrals mathematically, by treating the mathematical operation as reverse-differentiation.

2 Definition & Intuition

2.1 Formal Definition

In mathematics, many operations have inverses: addition has subtraction, multiplication has division, and the dual for calculus is that derivatives and integrals both undo each other. This is the basis for the **Fundamental Theorem of Calculus**.

There are two parts to the **Fundamental Theorem of Calculus (FToC)**.

Part 1: The derivative of an integral with an upper bound x is the integrand evaluated at that point x:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\mathrm{d}x = f(x)$$

Part 2: The integral from endpoints a to b is the same as the indefinite integral evaluated at b minus the same thing evaluated at a:

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

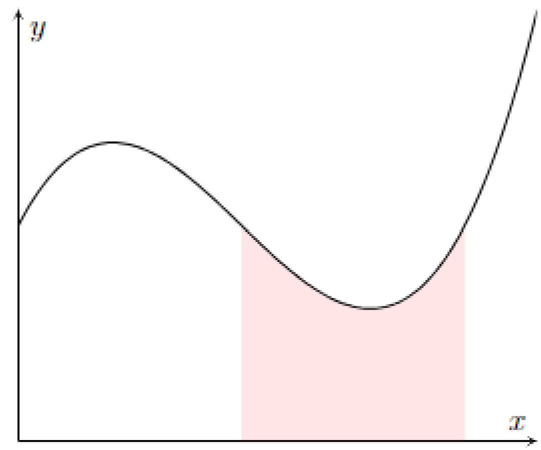
where $F(x) = \int f(x)\mathrm{d}x + C$

2.2 Gaining Intuition

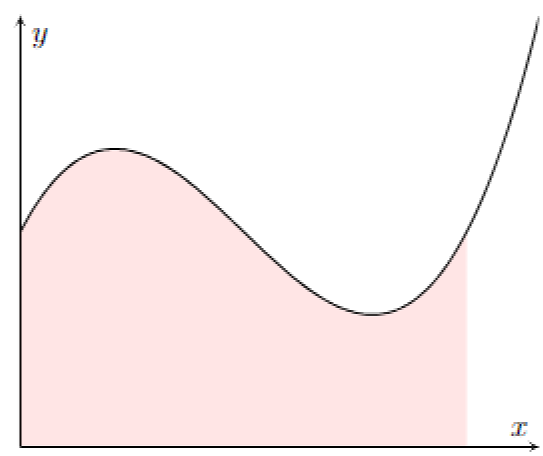
Now, why do these two parts work? Let's first talk about Part 2.

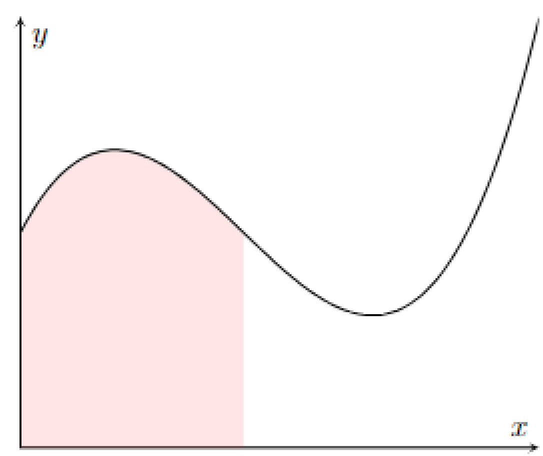
$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

In the "Intuition Behind Integrals" handout, we said that we could interpret integrals as areas under curves.



The area between this region can be computed by the difference of the following two areas:



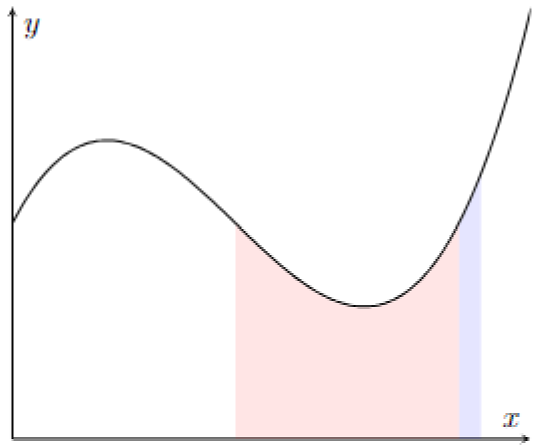


And both of these areas are represented by $F(b)$ and $F(a)$ respectively. Notice that we're only talking about the right endpoints here, and the left endpoint plays no role here, so it is free to be whatever number it wants to be.

Next, what about the first part of the FToC?

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\mathrm{d}x = f(x)$$

Referencing the "Intuition Behind Integrals" handout again, a derivative is just a fraction of infinitesimals. The denominator of the LHS is the familiar infinitesimal change $\mathrm{d}x$, while the numerator is the infinitesimal change in the integral caused by the infinitesimal change in x. Here, x plays the right endpoint of the integral, and hence it's the infinitesimal change in area.



Here, the red region represents the existing area, while the blue region represents the infinitesimal change in the area (exaggerated so it is visible). If we take the blue area and divide it by $\mathrm{d}x$, i.e. computing $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\mathrm{d}x$, then we would get $f(x)$ because the area of the blue area is just $f(x) \times \mathrm{d}x$.

Another way to look at this statement is to notice that integrals are just fancy sums of infinitesimally small stuff. When we change x by $\mathrm{d}x$, we're changing the integral by adding one more infinitesimal thing - and the thing we add is the integrand, $f(x)\mathrm{d}x$. Hence

$$\frac{\mathrm{d}\int_a^x f(x)\mathrm{d}x}{\mathrm{d}x} = \frac{f(x)\mathrm{d}x}{\mathrm{d}x} = f(x)$$

2.3 Why the $+C$?

In the text above, you might have noticed that we defined $F(x)$, the indefinite integral of $f(x)$ as

$$F(x) = \int f(x)\mathrm{d}x + C$$

But it doesn't crop up in definite integrals - why do we even need that $+C$ term?

To explain this, we need to know that integrals are adding changeg, and they only add changes - they cannot tell you what they are adding to, which you have to add by yourself.

An analogy of this is a child playing with a tap which is filling a bucket with water below. Let's say the child keeps varying the flow rate at the tap over time, say, $\frac{\mathrm{d}V}{\mathrm{d}t} = f(t)$ as a function of time. To find the volume of water collected in the bucket, we integrate the function $f(t)$ over t. However, this does not tell us anything about the absolute amount of water in the bucket! Specifically, we do not know if there was already water in the bucket to start with - the integral only tells us how much the volume changed after some time.

Here, the analogy to the initial volume of water in the bucket is the integration constant $+C$.

3 Examples & Applications

3.1 Integrals as Reverse Derivatives

With the FToC defined in place, let's put both parts together!

From Part 2 of FToC, we know

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

Putting this into Part 1 of FToC, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\mathrm{d}x = f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x) - F(a)) = f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$$

since $F(a)$ is just a constant, whose derivative is just zero. Note that by definition $F(x) = \int f(x)\mathrm{d}x + C$.

Now that we've proven that derivatives and integrals undo each other, we can compute integrals with this in mind.

3.2 Computing Integrals

To differentiate, it is straightforward and one simply follows the rules and pray for no mathematical mistakes. However, not all integrals can be easily computed. Some are so hard that mathematicians even name them "non-elementary" and call it a day! However, there exist many tools to help us compute integrals as you shall see in the other handouts.

In this handout, we shall use FToC to compute simple integrals.

Let's say you want to compute

$$\int_0^2 2x\mathrm{d}x$$

Even if you have never taken an integral before, you probably did take a derivative before - and hopefully notice that the derivative of x^2 is precisely $2x$.

Then, by the FToC,

$$\int_0^2 2x\mathrm{d}x = F(2) - F(0) = 4$$

where $F(x) = x^2$

This example is simple, but its principle is crucial and must be kept in the back of your head at all times.

Another example would be the integral

$$\int \sec^2(x)dx$$

If you didn't know about the FToC, you'd need to take some non-straightforward way to compute the integral - but, if you just realize that the derivative of $\tan(x)$ is precisely $\sec^2(x)$, then you can directly say

$$\int \sec^2(x)dx = \tan(x) + C$$

4 Problems & Practices

4.1 Problem 1

Problem 1: Use FToC to find the derivative of the functions

(a)

$$g(x) = \int_6^x \frac{1}{1+t^3}\mathrm{d}t$$

(b)

$$h(t) = \int_9^{\sin(t)} \sqrt{x^2+4}\mathrm{d}x$$

4.2 Problem 2

Problem 2:

(a) Take the derivative of the function

$$\ln(\tan x)$$

(b) Hence, find evaluate the indefinite integral

$$\int \frac{1}{\sin x}\mathrm{d}x$$

4.3 Problem 3

Problem 3:

(a) Take the derivative of the function

$$f(x) = \ln\frac{x-a}{x+a}$$

(b) Hence, find evaluate the indefinite integral

$$\int \frac{1}{x^2 - a^2} \mathrm{d}x$$

4.4 Problem 4

Problem 4: Evaluate the indefinite integral

$$\int \frac{\sin x \sec^2 x - \cos x \tan x}{\sin^2 x}$$

4.5 Problem 5

Problem 5: Evaluate the indefinite integral

$$\int \left(\frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1 - x^2}} \right) \mathrm{d}x$$

5 Solutions

5.1 Problem 1

a)

$$g'(x) = \frac{1}{1 + x^3}$$

b)

$$h'(t) = \sqrt{t^2 + 4} \cdot \cos(t)$$

5.2 Problem 2

a)

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \tan x = \frac{1}{\tan x} \sec^2 x = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x}$$

b)

$$\int \frac{1}{\sin x} dx = \ln \tan \frac{x}{2}$$

5.3 Problem 3

a)

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{x + a}{x - a} \cdot \frac{(x + a) - (x - a)}{(x + a)^2} = \frac{2a}{x^2 - a^2}$$

b)

$$\int \frac{1}{x^2 - a^2} \mathrm{d}x = \frac{1}{2a} \ln \frac{x - a}{x + a}$$

5.4 Problem 4

$$\int \frac{\sin x \sec^2 x - \cos x \tan x}{\sin^2 x} = \frac{\tan x}{\sin x} = \frac{1}{\cos x}$$

5.5 Problem 5

$$\int \left(\frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1-x^2}} \right) \mathrm{d}x = \ln x \sin^{-1} x$$

6 Conclusion

Having the Fundamental Theorem of Calculus in the back of your head can help you evaluate some integrals instantly - congratulations on making it through this handout!