Differentiation Rules

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Table of Contents

1 Trigonometric Derivatives **2**

2 Power Rule **3**

3 Product Rule **3**

4 Quotient Rule **4**

5 Multiple Derivatives **6**

1 Trigonometric Derivatives

Trigonometric derivatives are mostly formulaic, and require an understanding of trig derivative rules in order to solve differentiation problems involving trig functions. Here are the derivatives of common trigonometric functions: The derivative of the sine function, denoted as $\sin(x)$, is the cosine function, denoted as $\cos(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$$

The derivative of the cosine function, denoted as $\cos(x)$, is the negative sine function, denoted as $-\sin(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$$

The derivative of the tangent function, denoted as $\tan(x)$, is the secant squared function, denoted as $\sec^2(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$$

The derivative of the cotangent function, denoted as $\cot(x)$, is the negative cosecant squared function, denoted as $-\csc^2(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = -\csc^2(x)$$

The derivative of the secant function, denoted as $\sec(x)$, is the secant function itself multiplied by the tangent function, denoted as $\sec(x)\tan(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \sec(x)\tan(x)$$

The derivative of the cosecant function, denoted as $\csc(x)$, is the negative cosecant function itself multiplied by the cotangent function, denoted as $-\csc(x)\cot(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc(x) = -\csc(x)\cot(x)$$

Example 1: Consider the trigonometric function $f(x) = -sin(x) - cos(x)$ Apply trig differentiation rules to compute the derivative.

Solution: Step 1: Using the rule $\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x)$, the derivative of $-\sin(x)$ is $-\cos(x)$. Step 2: Using the rule $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x)$, the derivative of $-\cos(x)$ is $\sin(x)$. Now, we have found the derivatives of the individual terms:

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\sin(x)) = -\cos(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\cos(x)) = \sin(x)$$

So, the derivative of $f(x) = -\sin(x) - \cos(x)$ is the sum of the derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\sin(x) - \cos(x)) = -\cos(x) + \sin(x)$$

2 Power Rule

The power rule is a fundamental rule that allows us to differentiate functions of the form $f(x) = x^n$, where n is a constant exponent.

The **Power Rule** states that if $f(x) = x^n$, then the derivative of $f(x)$ with respect to x is given by:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$$

In this equation, $\frac{\mathrm{d}}{\mathrm{d}x}$ represents the derivative operator, and nx^{n-1} is the result of applying the power rule. It shows that when we differentiate a term with an exponent n, we bring down the exponent as the new coefficient and reduce the exponent by 1. For example, if we have $g(x) = x^3$, the power rule tells us that the derivative of $g(x)$ is $g'(x) = 3x^{3-1} = 3x^2$ So, the derivative of $g(x) = x^3$ is $g'(x) = 3x^2$.

Example 2: Consider the function $g(x) = x^{-2}$. Apply the power rule to calculate the derivative.

Solution: Using the power rule, we find: $g'(x) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$. Therefore, the derivative of $g(x) = x^{-2}$ is $g'(x) = \frac{-2}{x^3}$.

3 Product Rule

The product rule allows us to find the derivative of a product of two functions and is a valuable tool that allows us to find the derivative of more complex expressions.

The **Product Rule** states that if we have two functions, $f(x)$ and $g(x)$, then the derivative of their product, $h(x) = f(x) \cdot g(x)$, is given by:

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

In this equation, $f'(x)$ represents the derivative of $f(x)$, and $g'(x)$ represents the derivative of $g(x)$. The product rule tells us that to find the derivative of a product, we take the derivative of the first function and multiply it by the second function, then add the product of the first function and the derivative of the second function. For example, let's consider $f(x) = x^2$ and $g(x) = \sin(x)$. Applying the product rule, we find:

$$h'(x) = (x^2)' \cdot \sin(x) + x^2 \cdot (\sin(x))'$$

Simplifying further:

$$h'(x) = 2x \cdot \sin(x) + x^2 \cdot \cos(x)$$

Therefore, the derivative of $h(x) = x^2 \cdot \sin(x)$ is $h'(x) = 2x \cdot \sin(x) + x^2 \cdot \cos(x)$.

I use a little mnemonic to remember the product rule that goes something like this: "First d-second $\cdot$ second d-first", the first and second representing $f(x)$ and $g(x)$, and the d prefix representing which function must be differentiated

Example 3: Consider the function $g(x) = (2x + 6)(\sin(x))$. Apply the product rule to calculate the derivative.

Solution: Recall the product rule formula:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

where $f(x)$ and $g(x)$ are two differentiable functions. Step 1: identify the functions $f(x)$ and $g(x)$:

$$f(x) = 2x + 6$$

$$g(x) = \sin(x)$$

Step 2: Find the derivatives $f'(x)$ and $g'(x)$:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(2x + 6) = 2$$

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x)$$

Step 3: Apply the product rule:

$$g'(x) = f'(x)g(x) + f(x)g'(x) = 2\sin(x) + (2x + 6)\cos(x)$$

The derivative of $g(x) = (2x + 6)\sin(x)$ with respect to x is $g'(x) = 2\sin(x) + (2x + 6)\cos(x)$.

4 Quotient Rule

The **Quotient Rule** states that if we have two functions, $f(x)$ and $g(x)$, then the derivative of their quotient, $h(x) = \frac{f(x)}{g(x)}$, is given by:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

In this equation, $f'(x)$ represents the derivative of $f(x)$, and $g'(x)$ represents the derivative of $g(x)$.

The quotient rule tells us that to find the derivative of a quotient, we take the derivative of the numerator and multiply it by the denominator, then subtract the product of the numerator and the derivative of the denominator, all divided by the square of the denominator. For example, let's consider $f(x) = \sin(x)$ and $g(x) = x^2$. Applying the quotient rule, we find:

$$h'(x) = \frac{(\sin(x))' \cdot x^2 - \sin(x) \cdot (x^2)'}{(x^2)^2}$$

Simplifying further:

$$h'(x) = \frac{\cos(x) \cdot x^2 - \sin(x) \cdot 2x}{x^4}$$

Therefore, the derivative of $h(x) = \frac{\sin(x)}{x^2}$ is $h'(x) = \frac{\cos(x) \cdot x^2 - \sin(x) \cdot 2x}{x^4}$.

The quotient rule is particularly useful when dealing with functions that cannot be easily differentiated using other basic rules such as the power rule or the product rule.

> **Example 4:** Consider the function $h(x) = \frac{x^3}{\cos(x)}$. Apply the quotient rule to calculate the derivative.

Solution: To find the derivative $h'(x)$, we'll use the quotient rule:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

First, we'll identify $f(x)$ and $g(x)$ and plug them into the quotient rule formula:

$$f(x) = x^3$$
$$g(x) = \cos(x)$$
$$h'(x) = \frac{(x^3)' \cos(x) - x^3 (\cos(x))'}{(\cos(x))^2}$$

Now, let's calculate the derivatives of the individual functions:

$$(x^3)' = 3x^2$$
$$\cos(x))' = -\sin(x)$$

Substituting these derivatives into the quotient rule:

$$h'(x) = \frac{3x^2 \cdot \cos(x) - x^3 \cdot (-\sin(x))}{(\cos(x))^2}$$

Simplify further:

$$h'(x) = \frac{3x^2 \cos(x) + x^3 \sin(x)}{\cos^2(x)}$$

So, the derivative of $h(x) = \frac{x^3}{\cos(x)}$ is $h'(x) = \frac{3x^2 \cos(x) + x^3 \sin(x)}{\cos^2(x)}$.

5 Multiple Derivatives

The double derivative, also known as the second derivative, is a fundamental idea in calculus that represents the rate of change of a function's derivative. In other words, it tells us how the slope of a function is changing as we move along the x-axis. The double derivative provides information about the curvature and concavity of a function.

Let's consider a function $f(x)$. The first derivative of $f(x)$ with respect to x is denoted as $f'(x)$, and the second derivative is denoted as $f''(x)$. Mathematically, the **Second Derivative** is the derivative of the first derivative:

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right) = \frac{d^2}{\mathrm{d}x^2}$$

Multiple differentiation can keep occurring like so:

$$f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\right) = \frac{d^3}{\mathrm{d}x^3}$$

Which represents the third derivative, that is the rate of change of the rate of change of the rate of change.

Now, let's look at an example to illustrate the concept: Let's consider the function

$$f(x) = 3x^3 + 2x^2 - 5x + 1$$

Step 1: Find the first derivative $f'(x)$:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(3x^3 + 2x^2 - 5x + 1)$$

$$f'(x) = 9x^2 + 4x - 5$$

Step 2: Find the second derivative $f''(x)$:

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(9x^2 + 4x - 5)$$

$$f''(x) = 18x + 4$$

So, the second derivative of the function $f(x) = 3x^3 + 2x^2 - 5x + 1$ is $f''(x) = 18x + 4$. In this example, the second derivative $f''(x)$ gives us information about how the slope of the original function $f(x)$ is changing at different points on the graph.

A positive value of $f''(x)$ indicates the function is **concave up** ($\cup$), while a negative value indicates the function is **concave dow**n ($\cap$).

Example 5: Compute the third derivative of the following function: $f(x) = \sin(x)$.

Solution:
Step 1: Evaluate the first derivative:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x)$$

Step 2: Evaluate the second derivative:

$$f''(x) = \frac{d^2}{\mathrm{d}x^2}(\sin(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x)$$

Step 3: Evaluate the third derivative:

$$f'''(x) = \frac{d^3}{\mathrm{d}x^3}(\sin(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(-\sin(x)) = -\cos(x)$$

$$f'''(x) = -\cos(x)$$