Stellar Flux Relationships

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Table of Contents

1 Introduction

Radiation laws and their uses are integral to comprehending how we see celestial bodies and how to interpret relevant data. In this handout, you will learn many of the formulas that are crucial to answering questions about radiation.

2 Background

2.1 Blackbodies

Blackbodies are theoretical bodies that simply absorb and re-emit all incoming radiation without scattering or reflection. Therefore, they have an **albedo** of 0. While they do not truly exist because of their ideal behavior, they serve as good models for many celestial objects.

2.2 Luminosity

Luminosity is a vague term you may have heard before for something like “brightness”, but let it be vague no longer. Luminosity is simply the power outputted by an object, generally a blackbody. It is often measured in watts or solar luminosities ($L_\odot$).

2.3 Flux

Flux is simply the total radiation (energy per second) passing perpendicularly to a surface, as shown in Figure 1. This surface can be imaginary, such as a hollow sphere with the Sun at its center and the Earth at the edge. This is useful for understanding flux density. This is also useful for luminosity, which is generally used interchangeably with flux and can be considered the flux through a blackbody's surface.

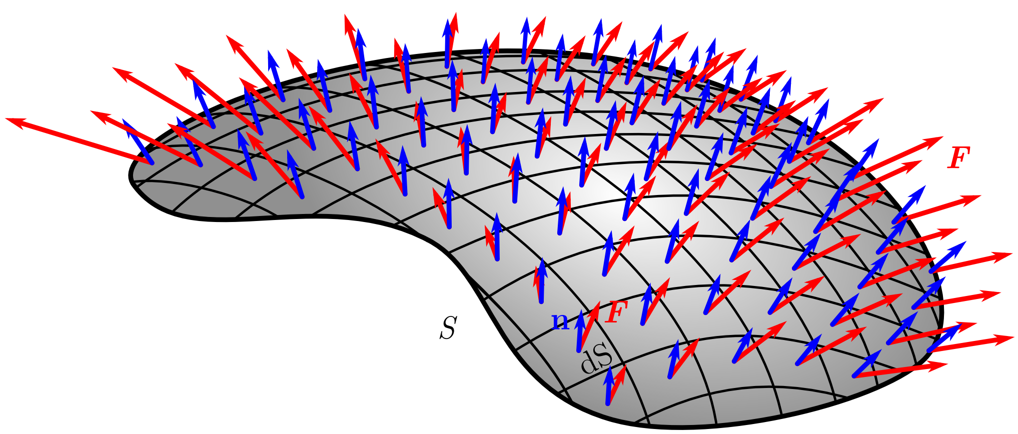

Figure 1: A representation of the perpendicular component of radiation. (Source: LibreTexts)

2.4 Flux Density

Flux density is the amount of radiation per unit area. The aforementioned Earth-Sun sphere can be used to get the flux density at a distance equal to the distance from the Sun to the Earth, and can then be used to get the incident flux from the Sun on Earth. This will be covered quantitatively later in this document.

3 Blackbody Radiation Laws

3.1 Planck's Law (Qualitative)

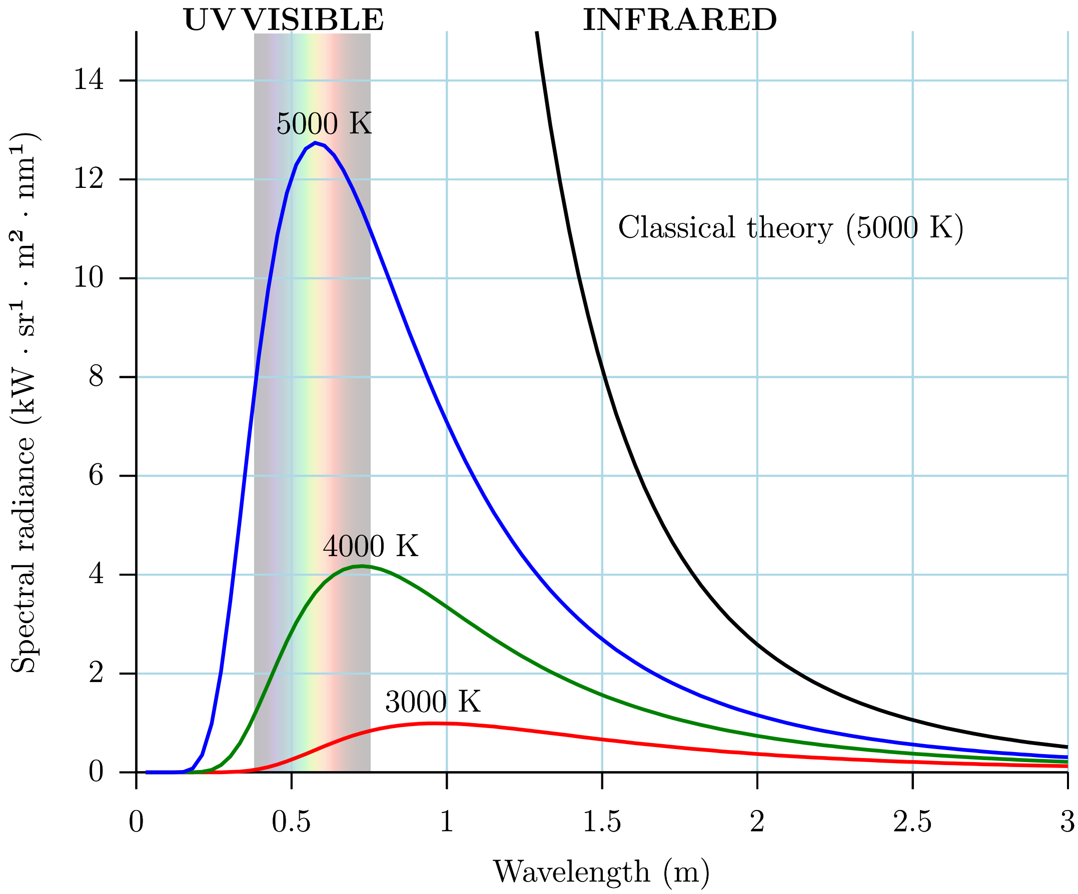

Figure 2: The curves of blackbodies at different temperatures, colored by their peak wavelength.

Planck's Law provides a relation regarding spectral radiance and is useful for the derivation of other laws in blackbody radiation. It is built on the Rayleigh-Jeans Law and accounts for the Ultraviolet Catastrophe.

3.2 Wien's Law

Wien's law creates a simple inverse relationship between the peak wavelength and temperature. It makes it very clear why hotter stars look bluer and cooler stars look redder.

$$T \cdot \lambda_{max} = const$$

Here, the constant is known as Wien's constant, with a value of about $2900\,\mu\text{m} \cdot \text{K}$ and symbol b. The equation then becomes:

$$\lambda_{max} = \frac{b}{T}$$

Example 3.1: Peak Wavelength of the Sun The Sun has a surface temperature of about 5800 K. Calculate the wavelength in which its radiation peaks.

Solution: From Wien's Law:

$$\lambda_{max} = \frac{b}{T}$$
$$\lambda_{max} = \frac{2900\,\mu\text{m}\cdot\text{K}}{5800K}$$
$$\lambda_{max} \approx 500\,\text{nm} \quad \square$$

Figure 2 shows that Sun peaks somewhere in the visible spectrum.

> **Example 3.2: Peak Wavelength of the Human Body** The human body has a surface temperature of about 37 °C, or 310 K. Calculate the wavelength in which a person would "glow" most.

Solution: From Wien's law:

$$\lambda_{max} = \frac{b}{T}$$
$$\lambda_{max} = \frac{2900\,\mu\text{m}\cdot\text{K}}{310\,\text{K}}$$
$$\lambda_{max} \approx 9.35\,\mu\text{m} \quad \square$$

This wavelength is somewhere in the infrared, explaining why we don't see other humans glow unless looking through an infrared camera.

3.3 Stefan-Boltzmann

The **Stefan-Boltzmann law** is quite important in terms of understanding the energy output of a star, and therefore how much energy is incident on other bodies. It can then be used (at least in the Solar System) to calculate equilibrium temperature. It is based on the idea that the flux density F at the surface of a blackbody is equal to:

$$F = \sigma T^4$$

where σ is the Stefan-Boltzmann constant, $5.67 \times 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}$ It can be easily remembered as the numbers in the constant are in the order 5, 6, 7, 8. Since the flux density is given in the previous equation, it is simple to arrive at the luminosity by multiplying the flux density with the surface area of a spherical blackbody:

$$L = 4\pi R^2 F$$
$$L = 4\pi R^2 \sigma T^4$$

> **Example 3.3: Equilibrium Temperature of the Earth** The sun has a surface temperature of about 5800 K, and a radius of about 6.96×10^8 m. 1 AU is approximately 1.496×10^{11} m. Equilibrium temperature is the temperature of an object (particularly a planet) under the assumption that it is a blackbody and that there is no greenhouse effect in place. Find the equilibrium temperature of the Earth.

Solution: From the Stefan-Boltzmann law:

$$L = 4\pi R^2 \sigma T^4$$

$$L = 4\pi(6.96 \times 10^8\,\mathrm{m})^2 \sigma (5800\,\mathrm{K})^4$$

$$L \approx 3.9 \times 10^{26}\,\mathrm{W}$$

This is the net flux (luminosity) outputted by the Sun. To get the flux density at Earth, we simply need to create an imaginary sphere with the Sun at the center with the radius equal to the distance between the Earth and the Sun. Since the Sun's rays will be perpendicular to this imaginary sphere, we need not worry about cross products such as in Figure 1. The flux density will then become the total luminosity divided by the surface area of this imaginary sphere.

$$A = 4\pi r_\oplus^2$$

$$A \approx 2.81 \times 10^{23}\,\mathrm{m}^2$$

$$F = \frac{L}{A} = \frac{3.9 \times 10^{26}\,\mathrm{W}}{2.81 \times 10^{23}\,\mathrm{m}^2} \approx 1389 W/m^2$$

Now that the flux density has been obtained, it can be use to find the incident flux on Earth. Instead of multiplying it by the surface area of the Earth, it must be multiplied by the cross-sectional area. If you're having trouble understanding why, take a look at Figure 1 again to see how the incident angles can be simplified.

$$P_\oplus = F \times A_\oplus = F \times \pi R_\oplus^2$$

$$P_\oplus \approx 1.78 \times 10^{17}\,\mathrm{W}$$

Notice that this is much, *much* smaller than the power output of the Sun. It is finally time to go back to our wonderful Stefan-Boltzmann law to take this incident power/flux and get the temperature of the Earth.

$$P_\oplus = 4\pi R_\odot^2 \sigma T_\oplus^4$$

$$T_\oplus = (\frac{P_\oplus}{4\pi R_\oplus^2 \sigma})^{1/4}$$

$$T_\oplus \approx 280\,\mathrm{K} \quad \square$$

It is recommended you try this example again with generic values and without plugging in values for variables in order to get a simplified equation for equilibrium temperature independent of the planet's radius.

4 Magnitudes

Magnitudes introduce a simple way to say how bright a star appears in a quantitative way that follows a simple scale in which every visible star (including the Sun!) remains at most two digits in magnitude.

4.1 What is Magnitude?

4.1.1 The Pogson Formula

The **Pogson ratio** describes a scale that relates the magnitudes of stars, meaning each star that is 1 magnitude higher is about $100^{1/5} \approx 2.512$ times *less* intense (the magnitude scale works so that brighter stars have lower magnitude, with some going negative). While Pogson did not create a formula per se, the formulation of the scale will be hereafter called the Pogson formula for consistency. The formulation does not necessarily give us the exact magnitude of a star but rather describes a relation between two stars. It is described as

$$m_1 - m_2 = -2.5 \log(\frac{I_1}{I_2})$$

where I is the intensity of the given body.

4.2 Apparent Magnitude

The **apparent magnitude** is the magnitude as seen from the Earth, and generally denoted as m. The flux density can be used to determine the apparent magnitude since the flux density depends on the current location relative to the radiating body. This changes the Pogson formula to

$$m_1 - m_2 = -2.5 \log(\frac{F_1}{F_2})$$

where F is the flux density of the given body.

4.2.1 Calculating Apparent Magnitude

We can use the Pogson formula by relating a known flux density at Earth and the unknown flux density at Earth. This is generally either Vega, since it is a benchmark star with apparent magnitude 0, or the Sun, with it's magnitude of -26.74 and well-known flux density as the solar constant $1362\,\frac{\mathrm{W}}{\mathrm{m}^2}$.

> Example 4.1: **Apparent Magnitude of Sirius** Given either the flux density of Sirius at Earth or the distance to Sirius and the luminosity of Sirius it can be compared to the known Solar constant (flux density of the Sun at Earth), about $1362\,\mathrm{W/m^2}$. Given that the flux density of Sirius at Earth is $F_S \approx 1.18 \times 10^{-7}\,\mathrm{W/m^2}$, what is the apparent magnitude of the star?

Solution: With 4 variables and 3 knowns, values can just be plugged in and the unknown can be solved for. The equation becomes

$$m_S - m_\odot = -2.5 \log(F_S/F_\odot)$$

$$m_S = -2.5\log(\frac{1.18\times10^{-7}\,\mathrm{W/m^2}}{1362\,\mathrm{W/m^2}}) - 26.74$$

$$m_S \approx -1.59 \quad \square$$

This is fairly off from the actual value of -1.44, though this is likely due to both atmospheric and interstellar extinction.

4.3 Absolute Magnitude

The *luminosity* can be used to determine the **absolute magnitude** since it is independent of distance as distance is held constant. This changes the Pogson formula to:

$$M_1 - M_2 = -2.5\log(\frac{L_1}{L_2})$$

Using the fact that luminosity is equal to $4\pi R^2\sigma T^4$, this equation can be split like

$$M_1 - M_2 = -5\log(\frac{R_1}{R_2}) - 10\log(\frac{T_1}{T_2})$$

4.3.1 Calculating Absolute Magnitude

We can use the Pogson formula by relating a star with a known luminosity and the unknown luminosity. This is easy if the luminosity is desired in solar luminosities because we then only need the absolute magnitude of the sun, $M_\odot$, which is 4.83. We can also use the distance modulus as discussed in the next section if the apparent magnitude and distance are given.

4.3.2 Distance Modulus

The distance modulus in a way to relate the distance to a body and the apparent and absolute magnitude of that body. Since we can see from Stefan-Boltzmann's law that the flux *density* is inversely proportional to distance squared.

$$L = 4\pi R^2\sigma T^4$$

$$F = \frac{L}{4\pi r^2}$$

$$F \propto \frac{1}{r^2}$$

where R is the radius of the body and r is the distance to the body. A distance relation can then be created between absolute and apparent magnitude by using Pogson's formula with apparent magnitude and comparing the flux densities by using its relation to distance:

$$m - M = -2.5\log(\frac{(\frac{L}{4\pi d^2})}{(\frac{L}{4\pi(10\,\mathrm{pc})^2})})$$

$$m - M = -2.5\log(\frac{(10\,\mathrm{pc})^2}{d^2})$$

$$m - M = 2.5\log(\frac{d^2}{(10\,\mathrm{pc})^2})$$

$$m - M = 5\log(\frac{d}{(10\,\mathrm{pc})})$$

$$m - M = 5\log(d) - 5$$

These last two are both common ways of representing the **distance modulus**, the relation between the apparent magnitude m and the absolute magnitude M. However we can solve for d and get:

$$d = 10^{\frac{m-M+5}{5}}$$

This is a quick rearragement to arrive at the distance given the absolute and apparent magnitudes of an object.

> **Example 4.2: Distance to Vega** As previously stated, Vega is known to have an apparent magnitude of about 0 (in reality it is about 0.03 and fluctuates.) Given that it has an absolute magnitude of about 0.58, calculate the distance.

Solution: It is then trivial to calculate the distance using the distance modulus solved for d. The equation becomes

$$d = 10^{\frac{(0.03-0.58+5)}{5}}\,\mathrm{pc}$$

$$d = 10^{0.89}\,\mathrm{pc}$$

$$d \approx 7.76\,\mathrm{pc} \quad \square$$

Therefore, Vega is about **7.76 parsecs** away from us.

4.4 Bolometric Magnitude

Bolometric magnitude is the magnitude that takes into account all wavelengths of radiation, not just visible. It can be found with the bolometric luminosity/flux density analogously to absolute and apparent magnitude and is denoted by M_{bol} and m_{bol} respectively.

4.5 Extinction

Extinction is a phenomenon caused by the combination of scattering and absorption, generally by the interstellar medium but also by the atmosphere.

4.5.1 Interstellar Extinction and Optical Thickness

Optical thickness (τ) is a dimensionless constant that represents the opacity of a certain medium to light. It is 0 for a perfect vacuum and approaches ∞ in a perfectly opaque substance. The apparent luminosity of an object (L) is related to the actual flux outputted by it (L_0) by the equation

$$L = L_0 e^{-\tau}$$

It can also be given in magnitudes as A where it can be used to correct for the apparent magnitude of a star (absolute magnitude is considered without extinction):

$$m = m_0 + A$$

A distance modulus with extinction can also be derived which can be helpful in determining various quantities:

$$m - M = 5\log\left(\frac{r}{10\,\mathrm{pc}}\right) + A$$

4.5.2 Atmospheric Extinction

Atmospheric extinction is scattering and absorption caused by our own atmosphere on Earth.

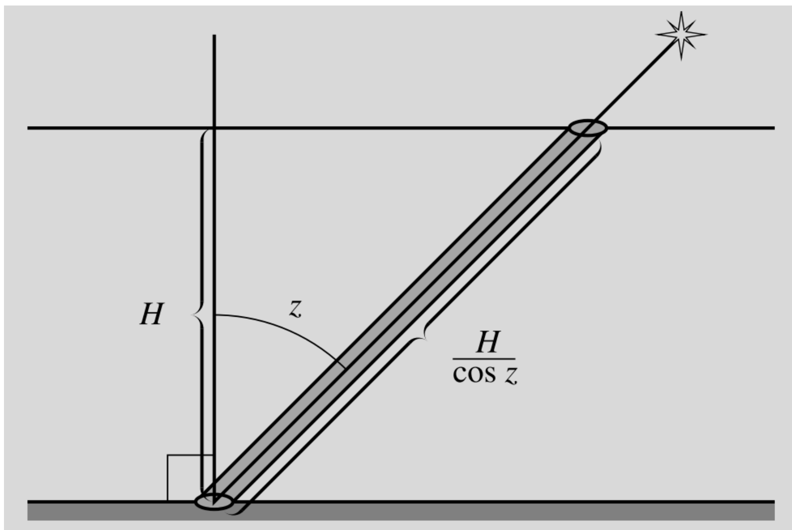

Figure 3: A visualization of the atmospheric planar approximation for atmospheric extinction. (Source: Karttunen's *Fundamental Astronomy*)

Taking a look at Figure 3, it is clear to see that a star at an angular distance z from the zenith will go through $\frac{H}{\cos z} = H \sec z$ times as much atmosphere. Setting H to 1 as the unit for air mass, the total air mass X becomes just $\sec z$, and the magnitude correction becomes kX where k is the extinction coefficient. The final equation to correct for magnitude is

$$m = m_0 + kX$$

4.5.3 Color Index and Color Excess

The color index is a measure of the blueness or redness of a star, and is given by $B - V$, the magnitude in a blue filter minus the magnitude in a visual filter. If it's negative, it is more blue, and if it is positive, then it is more red. Due to interstellar extinction, more blue light is absorbed than red, and the color index can be determined with

$$B - V = M_B - M_V + A_B - A_V$$

$$B - V = (B - V)_0 + E_{B-V}$$

In the last equation, the first term on the righthand side is known as the intrinsic color and the second term is called the *color excess*. The ratio of A_V to the color excess E_{B-V} is usually ≈ 3.0 for stars.

5 Selected Problems

- USAAAO First Round 2023 Q4
- USAAAO First Round 2022 Q8
- USAAAO NAC 2022 Short Questions Q3
- USAAAO NAC 2021 Short Questions Q1
- USAAAO NAC 2021 Long Questions Q1e
- IOAA 2007 Theory Q9
- IOAA 2011 Theory Q8
- IOAA 2007 Theory Q16

6 Conclusion

Now that we know how to see and calculate flux and radiation, we can now more easily understand why we see what we do, and what we can do with that information.