Basic Astrophysics

Logan Reich

February 2023

Table of Contents

1 Introduction **2**

2 Orbital Geometry **2**
2.1 Types of Orbits 2
2.2 Mathematical Description 2
2.3 Orbits in the Solar System 3

3 Gravity **4**
3.1 Newton's Law of Gravitation 4
3.2 Orbital Velocity and Escape Velocity 4

4 Kepler's Laws **5**
4.1 Kepler's First Law 5
4.2 Kepler's Second Law 5
4.3 Kepler's Third Law 6

5 Energy Balance **7**
5.1 Stefan-Boltzmann Law 7
5.2 Wien's Law 7
5.3 Flux 8
5.4 Spectral Band Absorption 8

6 Energy Balance **8**
6.1 Equilibrium Temperature of a Planet 8

7 Conclusion **9**

1 Introduction

Astrophysics is the application of the laws of physics to the study of astronomy. For Earth Science based purposes, this is typically limited to processes happening within the solar system. This handout will cover the very basics of astrophysics that can be used to help explain many Earth Science phenomena.

2 Orbital Geometry

Orbital geometry describes the shape of an object's orbits. These could include planets, asteroids, satellites, etc.

2.1 Types of Orbits

Objects orbit in conic sections: ellipses, hyperbolas, or parabolas. Most bodies that stay in the solar system orbit in ellipses, but interstellar visitors and some long-period comets have effectively parabolic or hyperbolic orbits.

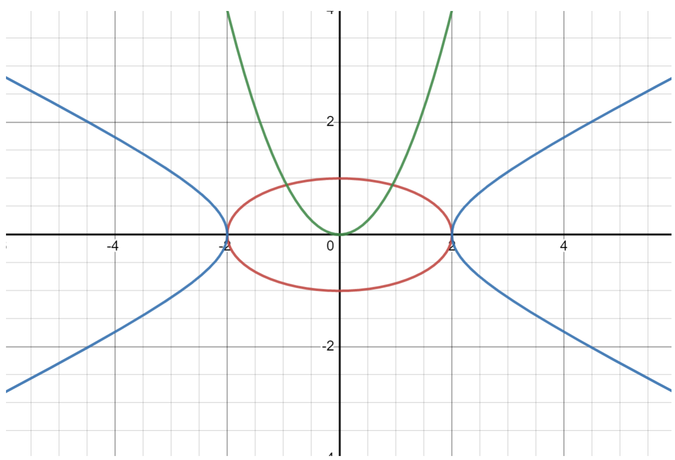

Figure 1: The three shapes of orbit (made using Desmos).

2.2 Mathematical Description

Since all objects that stay in the solar system orbit in ellipses, we will focus on those. Ellipses are defined by any two of their eccentricity e, semi-minor axis b, and semi-major axis a. The semi-minor and semi-major axes are marked below. The eccentricity is a measure of how "squished" a conic section is relative to a circle: an eccentricity of 0 is a perfect circle, while an eccentricity of 0.8 would be a fairly "squashed" circle. Ellipse eccentricities range from 0 to 1, not inclusive. An eccentricity of 1 represents a parabola, while an eccentricity greater than 1 represents a hyperbola and an eccentricity of infinity is that of a line. Eccentricity equals $\sqrt{1-\frac{b^2}{a^2}}$. The farthest distance is the aphelion, apogee, or apoapsis depending on which body it is orbiting, and the nearest is perihelion, perigee, or periapsis. The semimajor axis is the average of these two distances.

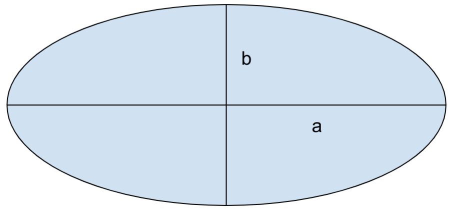

Figure 2: A labeled diagram of an ellipse (Image Credit: Labster).

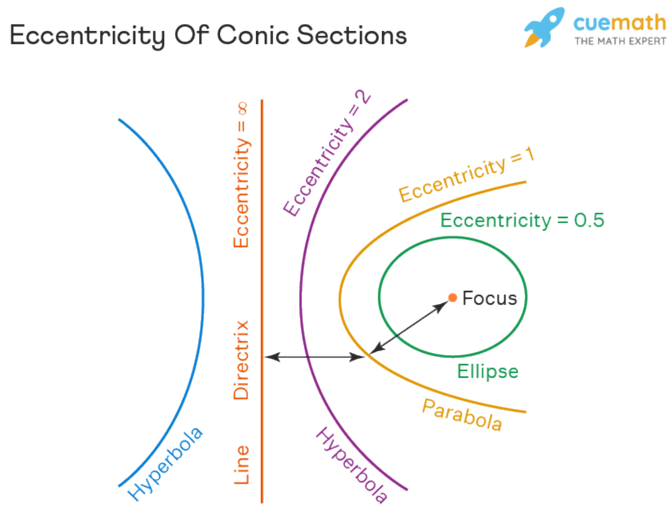

Figure 3: What conic sections of varied eccentricities look like. (Image Credit: Cuemath)

2.3 Orbits in the Solar System

Most planets in the solar system have eccentricities very close to zero, meaning their orbits are almost circular. Of the planets, Mercury has the most eccentric orbit. Dwarf planets (like Pluto) and other objects in the solar system tend to have much higher eccentricities since a lot of them are captured objects that did not form with the rest of the solar system.

Planet	Eccentricity
Mercury	0.2056
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0484
Saturn	0.0542
Uranus	0.0472
Neptune	0.0086
Pluto	0.2488

Table 1: Table of eccentricities for selected bodies in the solar system.

3 Gravity

3.1 Newton's Law of Gravitation

The force that keeps all objects in orbit is gravity. The general form for the force of gravity is:

$$F_g = \frac{Gm_1m_2}{r^2}$$

where G is the universal gravitational constant of $6.673 * 10^{-11} m^3 kg^{-1} s^{-2}$, m_1 is the mass of one of the objects, m_2 is the mass of the other object, and r is the distance between their centers of mass.

Example 3.1. The gravity of the moon causes tides on Earth. What is the force of gravity exerted by the moon on a $1m^3$ parcel of water, if the density of water is $1000kg/m^3$, the mass of the moon is $7.3 * 10^{22} kg$, the distance to the moon is 384,400 km, and the moon's radius is 1,740 km?

Solution: 0.033 N. Plug into the formula, being sure to use SI units!

3.2 Orbital Velocity and Escape Velocity

For a circular orbit, the object in orbit has a constant velocity of:

$$\sqrt{GM/r}$$

where G is the universal gravitational constant of $6.673 * 10^{-11} m^3 kg^{-1} s^{-2}$, M is the mass of the central body (not the orbiting one!), and R is the radius of the orbit (which equals the distance between their centers of mass, not their surfaces). This can be derived by setting the centripetal force equal to the force of gravity,

$$\frac{mv^2}{r} = F_c = F_g = \frac{Gm_1m_2}{r^2}$$

and solving for velocity. There is also a specific threshold for overcoming the force of gravity, known as escape velocity, which is:

$$\sqrt{2GM/r}$$

where G is the universal gravitational constant of $6.673 \cdot 10^{-11} m^3/kg * s^2$, M is the mass of the central body (not the orbiting one!), and R is the initial distance between their centers of mass (not their surfaces). This can be derived by setting the change in kinetic energy of a particle equal to the change in gravitational potential energy (which is the integral of the gravitational force with respect to distance.)

$$E_{grav} = \int \frac{-Gm_1m_2}{r^2}\ dx = \frac{Gm_1m_2}{r}$$

with the negative sign indicating that it is an attractive force. Thus,

$$\frac{v^2}{2} = \frac{Gm_1}{r}$$

and the formula is the expression after solving for velocity.

Example 3.2. A molecule of hydrogen exists high in Earth's atmosphere, at a distance of 9,000 km. If Earth's radius is 6,400 km and its mass is $6 * 10^{24} kg$, and assuming the motion of this particle is effectively the same as it would be in orbit, what is the hydrogen molecule's velocity? What is its escape velocity?

Solution: 9,432 m/s. Plug into the formula, being sure to convert units! For an extra-interesting challenge, compare this to the velocity that the hydrogen atom would have from the temperature at that altitude. Can hydrogen exist high in Earth's atmosphere?

4 Kepler's Laws

Kepler's Laws are the key to understanding basic planetary motion.

4.1 Kepler's First Law

Kepler's First Law states that all planets orbit in ellipses, with the Sun as one of the foci. All moons also orbit in ellipses, with the planet the orbit around at one of the foci. The foci, singular focus, are effectively special points in the geometry of an ellipse that are used in its mathematical definition.

4.2 Kepler's Second Law

Kepler's Second Law states that an orbiting planet sweeps out equal areas in equal times during each part of an ellipse, as illustrated below. This is a consequence of the conservation of angular momentum. We can show that this is true using the conservation of energy in polar coordinates and the conservation of angular momentum, solving for the angle, integrating, and then yielding an expression for the distance in terms of angle that is the polar form of an ellipse, which also shows that the first law is true.

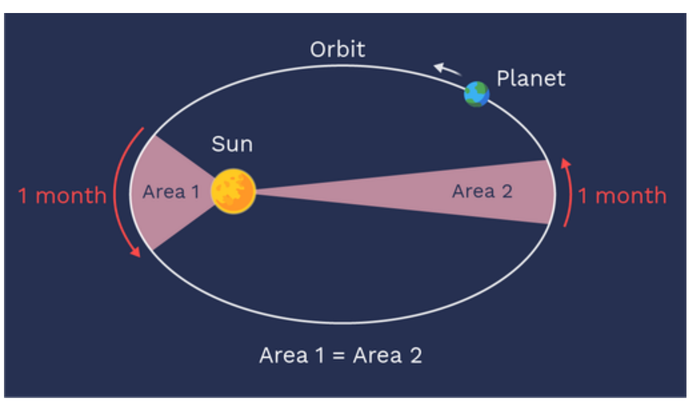

Figure 4: Illustration of Kepler's Second Law (made using Google Drawings).

4.3 Kepler's Third Law

Kepler's Third Law states that the period of an orbiting body P squared is proportional to its semimajor axis a cubed. This form, without the constants, *only applies to objects orbiting the sun, and when P is in Earth years and a is in AU.*[1] An AU is an astronomical unit, and is the distance from the Earth to the Sun.

$$P^2 \propto a^3$$

Newton later extended this to work for any body orbiting another body, where m_1 and m_2 are the masses of the two bodies. Usually, one is much more massive than the other, so the smaller mass can be ignored. This equation uses standard S.I. units.

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}$$

We can derive this by setting centripetal acceleration, $a_c = \omega^2 r^2 = \frac{GM}{r}$ Then we can substitute in $2\pi/P$ for ω, yielding the equation when rearranged.

Example 4.3.a How long is a year on Mercury, which has an orbital radius of 2500 km, assuming the sun has a radius of 696,340 km and a mass of $2 * 10^{30} kg$?

Solution: 88 days. Plug into the formula, being sure to convert units!

Example 4.3.b (USESO Open Round 2022) Mimas is a shepherd moon of Saturn that clears the Cassini Division, a gap in Saturn's rings. Particles on the inner edge of the Cassini Division are known to have a 2:1 orbital resonance with Mimas. If Mimas is 3 planetary radii (R) away from Saturn, estimate the distance from Saturn to the Cassini Division in terms of R.
A. 1.5R
B. 1.9R
C. 2.7R
D. 3.5R
E. 4.8R

Solution: An orbital resonance of 2:1 means that Mimas orbits once for every 2 orbits of the Cassini Division. By Kepler's Third Law we have $P^2 \propto a^3$. Plugging in numbers, we have a = $(\frac{3^3}{2^2})^{1/3}$= 1.89. Hence, the answer is B.

[1] This is the case since when the units are in years and AU, $\frac{4\pi^2}{G(m_1+m_2)} = 1$.

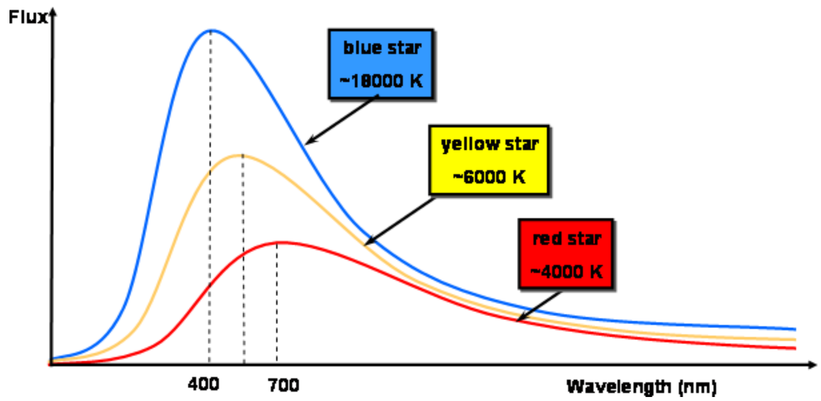

Figure 5: Blackbody Distribution (Source: Centre for Astrophysics and Supercomputing).

5 Energy Balance

In addition to dealing with orbits, astrophysics also deals with how light is transferred, and how solar energy reaches a planet!

5.1 Stefan-Boltzmann Law

The Stefan-Boltzmann law describes the luminosity of an object, or the amount of energy emitted as light per unit time. Be sure to use temperature in Kelvin! $L = A\varepsilon\sigma T^4$ where A is surface area of the object, T is the temperature of the object, ϵ is the emissivity of the object[2], and σ is the Stefan-Boltzmann constant, which equals $5.67 \cdot 10^{-8} W m^{-2} K^{-4}$. When this relationship is applied to a spherical object (like a planet or a star!) it becomes:

$$L = 4\pi\varepsilon\sigma R^2 T^4$$

where R is the radius of the planet or star.

5.2 Wien's Law

Wien's law relates the wavelength of the object's peak (most luminous) emission λ_{max} to temperature of the object T, allowing for the temperature of an object to be determined based on the light it emits fairly easily.

$$\lambda_{max} = \frac{0.0029\ Km}{T}$$

[2]An object's emissivity ε is defined to be its effectiveness in emitting thermal radiation. For blackbodies (perfect absorbers and emitters of radiation, which include all stars), $\varepsilon = 1$ and is often omitted from calculations as it doesn't make a difference. However, for other objects, such as planets (especially those with a greenhouse effect), $\varepsilon \neq 1$ and therefore cannot be ignored. You will see its use later, in section 6.1.

5.3 Flux

Another important aspect of emitted light is that it spreads out. The flux F (how bright something appears, or energy per unit time per unit area) of an object with luminosity L at a distance d is:

$$F = \frac{L}{4\pi d^2}$$

which is known as the inverse square law.

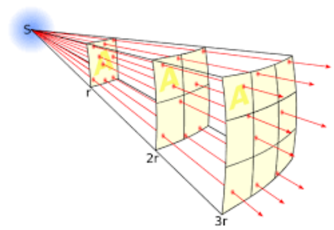

Figure 6: Inverse-Square Law (Source: Wikipedia).

5.4 Spectral Band Absorption

Light can also be absorbed by atoms and molecules in specific wavelengths, as light can hit electrons and cause in increase in the energy of that electron once it is absorbed. However, since electrons can only have certain quantized energy values, only certain wavelengths of light can be absorbed, leading to each atom or molecule having a distinct spectral absorption fingerprint.

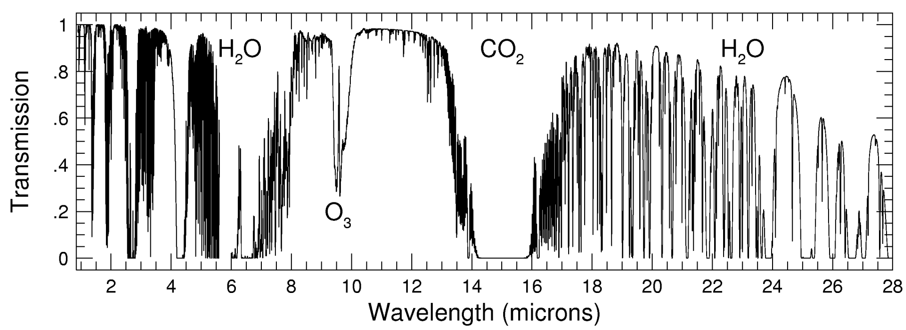

Figure 7: Absorption Spectrum of Earth's Atmosphere (Source: Ohio State Astronomy Department).

6 Energy Balance

6.1 Equilibrium Temperature of a Planet

Just from the inverse-square flux law and the Stefan-Boltzmann law, we are ready to determine the theoretical equilibrium temperature of a planet! First, we start of by setting the absorbed energy from the star equal to the luminosity of the planet, which must be true for it to have a constant (equilibrium temperature):

$$E_{absorbed} = L_{planet}$$

The amount of absorbed energy is the incoming flux times 1-albedo (denoted as α) times the area that is absorbing, so

$$F_{incoming}(1-\alpha)(\pi R_p^2) = L_{planet}$$

where R_p is the radius of the planet.
Using the inverse-square law on the LHS and the Stefan-Boltzmann law on the planet, we obtain

$$\frac{L_{star}(1-\alpha)\pi R_p^2}{4\pi D^2} = 4\pi R_p^2 \varepsilon\sigma T_{eq}^4$$

where D is the distance from the planet to the star and ε is the planet's emissivity.[3]
Solving for T_{eq}, we get

$$T_{eq} = \sqrt[4]{\frac{L_{star}(1-\alpha)}{16\pi\varepsilon\sigma D^2}}.$$

By plugging in $L_{star} = 4\pi R_{star}^2 \sigma T_{star}^4$, we can cancel out more terms to obtain a cleaner formula for T_{eq}:

$$T_{eq} = T_{star}\left(\frac{1-\alpha}{4\varepsilon}\right)^{1/4}\left(\frac{R_{star}}{D}\right)^{1/2}.$$

Interestingly, equilibrium temperature does not depend on the size of the planet!

7 Conclusion

Astrophysical laws are super useful for helping explain many key phenomena in Earth Science!

[3] If, when solving a problem, you are not given emissivity (as is often the case), you can assume $\varepsilon = 1$ and leave it out of your calculations.