Related Rates

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1 Introduction

Related rates is a fascinating branch of calculus that deals with the study of how different quantities change in relation to each other. It is used to solve real-world problems where multiple variables are interdependent and changing with respect to time. By examining the rate of change of one quantity, we can determine how other related quantities change simultaneously.

In related rates problems, we analyze the relationships between variables using derivatives. The process typically involves identifying relevant equations, differentiating them with respect to time (using the chain rule), and then solving for the desired rate of change. Practice is key to mastering related rates problem-solving, and I highly encouraged to draw diagrams and label variables to effectively understand the problem.

1.1 Key Steps

Read the following steps carefully and identify the changing variables and the given rates of change.

1. **Identify the Variables:** Begin by identifying all the variables involved in the problem. Assign meaningful names to each quantity and denote their respective rates of change with respect to time.

2. **Establish an Equation:** Next, establish an equation that relates the variables. This equation usually stems from the problem's context or geometry.

3. **Differentiate with Respect to Time:** Differentiate the equation with respect to time (t) to obtain a new equation that relates the rates of change of the variables.

4. **Plug in Known Values:** Substitute the known values of the variables and their rates into the derived equation.

5. **Solve for the Desired Rate:** Solve for the unknown rate of change, which is usually the rate you are asked to find in the problem.

> **Example 1:** Consider a streetlight mounted on top of a pole and a person walking away from the pole. The person's height is 1.8 meters, and their walking speed is 1.5 m/s. When the person is 10 meters away from the pole, the person's shadow is 6 meters long. At what rate is the tip of the person's shadow moving concerning time when they are 10 meters away from the pole?

Solution: Given:
- Let h be the height of the person (constant) = 1.8 meters.
- Let x be the distance between the person and the pole, changing with time (in meters).
- Let s be the length of the person's shadow, changing with time (in meters).

From the given information, we have $h = 1.8$ m and $\frac{\mathrm{d}x}{\mathrm{d}t} = -1.5$ m/s (negative since the person is walking away). Using similar triangles, we can relate the variables x and s:

$$\frac{s}{x} = \frac{h}{10}$$

Now, differentiate both sides of the equation with respect to time t using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{x}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{h}{10}\right)$$

To find $\frac{\mathrm{d}s}{\mathrm{d}t}$ (the rate at which the tip of the person's shadow is moving concerning time), we need to solve for it:

$$\frac{1}{x}\frac{\mathrm{d}s}{\mathrm{d}t} - \frac{s}{x^2}\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

Plugging in the given values:

$$\frac{1}{10}\frac{\mathrm{d}s}{\mathrm{d}t} - \frac{6}{100}(-1.5) = 0$$

Solving for $\frac{\mathrm{d}s}{\mathrm{d}t}$:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{6}{10} \times 1.5 = 0.9 \text{ m/s}$$

So, the tip of the person's shadow is moving away from the pole at a rate of 0.9 m/s when they are 10 meters away from the pole.

2 Real Application

Example 2: The 1945 Trinity test was an implosion-type plutonium bomb with an estimated 20 Kiloton explosive yield. It created a fireball of super-heated gas and plasma up to 100,000,000°C, hotter than the core of the sun. 700 microseconds after the detonation of a 1 Megaton thermonuclear warhead, the nuclear fireball is expanding at a rate of 330,000 m^3/s. How fast is the surface area of the fireball expanding when the radius is 160 meters?

Solution:
Given:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 330,000$$
$$r = 160$$

The equations that relate the variables are the Spherical Volume formula and the Spherical Surface Area formula:

$$V = \frac{4}{3}\pi r^3$$
$$A = 4\pi r^2$$

Differentiating the equations implicitly with respect to time t:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \frac{\mathrm{d}r}{\mathrm{d}t}$$

Now, substitute the known values into the Volume equation and solve for $\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{330000}{4\pi(160)^2}$$

Finally, find the value of $\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 1.026$$

Now, substitute the known values into the Surface Area equation and solve for $\frac{\mathrm{d}A}{\mathrm{d}t}$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi(160)(1.026)$$

Finally, find the value of $\frac{\mathrm{d}A}{\mathrm{d}t}$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 1.33 \times 10^9$$

So, the surface area of the fireball is expanding at a rate of 1.33×10^9 m^2/s