Telescopes

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1 Introduction

In order to understand how telescopes work, foundation in optics is necessary. While finding the magnification, interference, the reason behind chromatic aberration, optics comes to a rescue. So, buckle up, this is gonna be a bumpy ride!

2 Geometrical Optics

2.1 Light

Light is an electromagnetic wave which propagates with speed $\frac{c}{n}$ where n is equal to refractive index of a medium. Light propagates in the direction of the energy flow and the light rays are perpendicular to a wavefronts in a homogenous and isotropic medium. Which explains that the light rays emitted from a point source is propagating in radial direction and that the light rays of plane wave can be represented by a single light ray.

Kepler's first law says that all planets orbit in ellipses with the sun at one focus of the ellipse, which can be generalized to the fact that all objects in closed orbits orbit their central bodies in ellipses with the central body at a focus of the ellipse. Open orbits are either hyperbolas or parabolas, and also have the central body at a focus.

Kepler's first law is important for understanding the orbital behavior of objects, as well as providing the basis for the patched conics technique, which involves combining the paths of conic orbits to create more advanced orbital maneuvers.

elliptical.jpg

Figure 1: A diagram of the closed orbital behavior of a body (Source: BBC)

orbits.png

Figure 2: A diagram of the open orbital behavior of a body (Source: Wikipedia)

2.2 Derivation

From the definition of an ellipse and the Pythagorean theorem, using r as the hypotenuse, $rsin\theta$ as the leg to the focus, and $2ae + rcos\theta$ as the other leg, we can write:

$$r'^2 = r^2 sin^2\theta + (2ae + rcos\theta)^2$$

which can be through expansion and use of the Pythagorean trigonometric identity $sin^2\theta + cos^2\theta = 1$rewritten as

$$r'^2 = r^2 = 4ae(ae + rcos\theta)$$

Using another form of the the ellipse definition that $r + r' = 2a$, substituting in for r' and solving for r:

$$r = \frac{a(1-e^2)}{1+ecos\theta}$$

which is the path of an ellipse, and can be used to map out an orbital path for a bound ellipse. An orbit is bound (elliptical) if the energy overall is negative. The overall orbital energy being negative is another way of rephrasing that the gravitational potential energy is greater than the kinetic energy. For a parabola, where the overall energy is 0 (meaning the gravitational potential perfectly balances the kinetic energy) and the orbital eccentricity is 1,

$$r = \frac{2p}{1 + cos\theta}$$

For a hyperbola, which have positive orbital energies (meaning the gravitational potential is less than the kinetic energy) and eccentricities greater than 1,

$$r = \frac{a(e^2 - 1)}{1 + ecos\theta}$$

This can also be rewritten as

$$r = \frac{L^2/\mu^2}{GM(1 + ecos\theta)}$$

where L is the angular momentum and μ is the reduced mass.

3 Wave optics

3.1 Law and Importance

Kepler's second law says that planets sweep out equal areas in equal times. This is a restatement of the conservation of momentum, only one that Kepler came up with before the conservation of angular momentum!

The area of an entire ellipse is $ab\pi$, so with the period, the area that is swept out over a certain period of time can be easily calculated. This is useful for deriving various important orbital relations, and since the conservation of angular momentum is a crucial basis for orbital mechanics equations.

second.png

Figure 3: An illustration of Kepler's second law (Source: Labster Theory)

Example 2.1. (USAAAO Round One 2022)

usaaao 3.jpg

Solution: From Kepler's second law, equal areas are swept out in equal times, regardless of where in the orbit that area is, and using Δt as the amount of time we are looking for:

$$\frac{\Delta t}{T} = \frac{A_{movement}}{A_{total}}$$

using the total area formula and the area that is swept out as half the total area $\frac{\pi ab}{2}$ plus a triangular area $2 \times \frac{abe}{2}$ that is formed by the semiminor axis and the lines connecting the star to points A and B,

$$\frac{t}{T} = \frac{2 \times \frac{abe}{2} + \frac{\pi ab}{2}}{\pi ab}$$

$$\frac{\Delta t}{T} = \frac{abe + \frac{\pi ab}{2}}{\pi ab}$$

Dividing through,

$$\Delta t = (\frac{e}{\pi} + 0.5)T$$

which is answer choice B.

3.2 Derivation

Warning: This section uses significant amounts of calculus! If you want to attempt reading this section and you do not have a solid calculus background, please check out our Calculus Primer handouts!

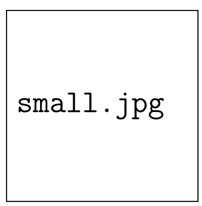

Figure 4: Diagram of a small angular area (Source: Caroll and Ostlie)

From Figure 4,

$$dA = dr(rd\theta)\bar{r}drd\theta$$

Integrating from the principal focus (the focus that the central body is at) to r,

$$dA = \frac{1}{2}r^2 d\theta$$

Taking the time derivative

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

The orbital velocity can be expressed in terms of the radial and tangential velocities,

$$v = v_r + v_t = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{t}\hat{\theta}$$

Solving for v_t and using substituting in,

$$\frac{dA}{dt} = \frac{1}{2}rv_t$$

As v_t is the tangential velocity,

$$rv_t = |r \times v| = \frac{L}{\mu}$$

thus,

$$\frac{dA}{dt} = \frac{L}{2\mu}$$

4 Telescopes

4.1 Law and Importance

Kepler's third law states that **ONLY in the solar system**

$$P^2 \propto a^3$$

where P is the period in **years** and a is the semi-major axis in **AU**. This expression is a specific version of Kepler's third law, known as Newton's version of Kepler's third law, that only applies to our solar system, because the proportionality constant in the solar system evaluates to 1. This is not a coincidence; this reduction comes from the fundamental definition of our units. Newton's version of Kepler's third law is

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}$$

where all values are in SI units. A common way of reducing this is using the approximation that if M_1 is the mass of a star and M_2 is a planet, $M_1 \approx M_1 + M_2$. Kepler's third law is an incredibly important basis for many relations and formulas in orbital mechanics, and all of astronomy in general.

Example 3.1.(USAAAO Round One 2022)

usaaao 17.jpg

Solution: From the graph, there is 4.2 years between the primary and secondary eclipse, which are when the stars eclipse each other. The deeper eclipse is the primary; though this is irrelevant for the problem. The two eclipses occur when the secondary (smaller) star is at periapsis or apoapsis, and thus the orbital period is double this time difference or 8.4 years. Using Kepler's third law,

$$(8.4yrs)^2 = \frac{4\pi^2(14.8AU)^3}{G(M_1 + M_2)}$$

using the appropriate unit conversions to SI and solving for the mass, this results in 46 solar masses, or answer choice E.

Example 3.2. (USAAAO Round One 2020) Planet Nine is a hypothetical planet in the outer Solar System, with a semimajor axis between 400 and 800 AU. Which of the following is a possible orbital period for Planet Nine?
A. 71.1 years
B. 600 years
C. 1,500 years
D. 15,000 years
E. 360,000 years

Solution: Use Kepler's third law for the solar system, where

$$P^2 \propto a^3$$

and they are equal if P is in years and a is in AU. Plugging in for 400 AU and 800 AU, the only choice greater than the period for 400 AU and less than the period for 800 AU is 15,000 years, or answer choice D.

Example 2.3. Does Kepler's third law in the solar system work for Earth?

Solution: Yes, as in the solar system,

$$P^2 \propto a^3$$

and they are equal if P is in years and a is in AU. Plugging in for one year and one AU, we see that it works and the proportionality constant must by 1!

4.2 Derivation

There are two derivations in this section: a simpler version that does not involve calculus and is independent of Kepler's second law, and another more rigorous one based on the second law. The simpler proof involves first writing the centripetal force equal gravitational force:

$$m\frac{v^2}{r} = G\frac{mM}{r^2}$$

$$v^2 = \frac{GM}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

Substituting into using the physics identity based on the distance (the circumference of a circle) divided by the velocity is the period

$$P = \frac{2r\pi}{v}$$

and cross-multiplying

$$v = \frac{2r\pi}{P}$$

$$\sqrt{\frac{GM}{r}} = \frac{2r\pi}{P}$$

Rearranging and solving for P^2

$$P^2 = \frac{4pi^2a^3}{GM}$$

Which is the version of Kepler's third law for central body mass much larger than the orbiting body mass. This version makes clear why the coeffecient is equal to one, as 1 AU is the orbital radius of the Earth, and 1 year is the orbital period of the Earth. For the second derivation, starting with the integrated version of Kepler's second law,

$$A = \frac{L}{2\mu}P$$

substituting $A = ab\pi$

$$P^2 = \frac{4pi^2}{G(m_1 + m_2)}a^3$$

Then using Kepler's first law forms

$$r = \frac{L^2/\mu^2}{GM(1 + ecos\theta)}$$

$$r = \frac{a(1 - e^2}{1 + ecos\theta}$$

Setting them equal to each other and solving for L

$$L = \mu\sqrt{GMa(1 - e^2)}$$

From the geometry of an ellipse

$$b^2 = a^2(1 - e^2)$$

Plugging both of those into the equation derived from Kepler's second law

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)}a^3$$

which is the full, more accurate version of Kepler's third law.

5 CCD

5.1 Ellipses

All closed orbits are ellipses. An ellipse is defined as the locus of points where the distances from two foci sum to a constant.

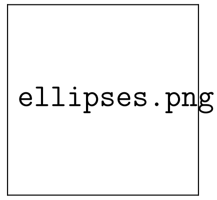

Figure 5: Ellipse diagram (Source: Hyperphysics)

From the diagram, we can define points F_1 and F_2 as the two foci, b as the semi-minor axis, and a as the semimajor axis. Alternatively, it can be defined as the points a given summed distance from a focus and a directrix, or a specific line. The eccentricity of an ellipse, a measure of how "circular" it is, ranges from 0 to 1 and can be defined as

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

or through

$$b^2 = a^2(1 - e^2)$$

which is often more useful. A circle has an eccentricity of 0. Most planets have extremely low eccentricities. Eccentricity can also be expressed in terms of the focus-directrix distance. In this ellipse, we can call the point in the orbit closest to where the central body is (F_1 in this case) the perihelion, perigee, perilune or periapsis, and the farthest point the aphelion, apogee, apolune, or apoapsis, depending upon what body it is orbiting. -helion is the Sun, -gee is the Earth, -lune is the Moon, and -apsis is general. An elliptical orbit can be modeled using the following equation:

$$\frac{(x + ea)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

The periapsis of the orbit is at distance

$$a(1 - e)$$

and the apoapsis is at distance

$$a(1 + e)$$

A circular orbit, a special form of an elliptical orbit, can be modeled using

$$r = \frac{L^2}{GMm^2}$$

where L is the angular momentum, G is the gravitational constant, M is the mass of the central body, and m is the mass of the orbiting body.

Example 4.1. (USAAAO Round One 2022) The orbit of some planet to its star has an eccentricity of 0.086. What is the ratio of the planet's closest distance to its star to the farthest on its orbit?
(a) 0.842
(b) 0.188
(c) 1.188

(d) 0.158
(e) None of the above

Solution: Using the formulae for the apoapsis and periapsis of an orbit, this ratio is equal to

$$\frac{a(1-e)}{a(1+e)}$$

which equals

$$\frac{1-e}{1+e}$$

Plugging in for the given eccentricity,

$$\frac{1-0.086}{1+0.086}$$

This equals 0.842, or answer choice A.

5.2 Parabolas and Hyperbolas

Parabolas and hyperbolas are the two forms of open orbits. Both can also be defined using a focus-directrix definition, and do possess an eccentricity. A parabola has an eccentricity of 1, and a hyperbola has an eccentricity greater than 1. Ellipses, hyperbolas, and parabolas fall into a class of functions known as conic sections. A parabolic orbit can be modeled using

$$y^2 = 2p(0.5p - x)$$

where p is the distance of closest approach, and x and y are coordinate distances relative to the central star the difference from the center can also be expressed using the equation

$$\beta = \frac{RL^2}{(GMm^2)\sqrt{1-e^2}}$$

Where R is the orbital distance, L is the angular momentum, G is the gravitational constant, M is the central body's mass, m is the orbiting body's mass, and e is the eccentricity.

parab.png

Figure 6: A parabolic orbit (Source: Physics Stack Exchange)

Example 4.2. If an orbit has an eccentricity of 2, what shape is it, and it it open or closed?

Solution: The orbit is a hyperbola, as the eccentricity is greater than 1. Hyperbolas are open orbits (the orbiting body never returns to any part of the orbit), so this is an open orbit.

5.3 Orbital Elements

While all orbits are conic sections, more information is needed to identify them in space relative to Earth, and many of those parameters can help better model orbits. Beyond the two most basic orbital parameters, the eccentricity and the semimajor axis length, the inclination i, the longitude of the ascending node Ω, the argument of periapsis ω, and the true anomaly θ are also important.

The inclination is a measure of the tilt of the orbital plane relative to the ecliptic or another reference plane. The longitude of the ascending node refers to the angle from the vernal equinox (the zero right ascension, zero declination point) of the point where the orbital plane intersects with the ecliptic, moving from above the ecliptic to below the ecliptic.

The argument of periapsis is the angle from the line of the center of the orbital plane to the ascending node to the line along the major axis towards the direction of the periapsis. The true anomaly is the angle from the celestial body to the segment along the major axis to the periapsis, and is a way of parameterizing where in the orbit the body is.

There are a series of alternatives choice of angular parameters. One of the most common is the mean anomaly M, which is another useful way to express the position of the body in its orbit. P is the period of the orbit. With τ as the reference time where the object is at periapsis,

$$M = \frac{2\pi}{P}(t - \tau)$$

This equation is useful as the mean anomaly directly represents the angle for Kepler's second law.

`elements.png`

Figure 7: A diagram showing the orbital elements (Source: Wikipedia)

6 Conclusion

Kepler's laws and orbital geometry are tremendously important as they underpin all of orbital mechanics, and now you can use them too! Almost all celestial mechanics relies on Kepler's laws as a crucial foundation, and the celestial mechanics section on the USAAAO especially. Understanding this material will set you up perfectly for success!