Integration Rules

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Table of Contents

1 Introduction

We will explore the fundamental integration rules used in calculus, and cover essential concepts such as constant rule, exponential rule, power rule, and trigonometric rules. Each rule is illustrated with practical examples, helping readers gain confidence in integrating various functions. This document serves as a valuable reference for understanding and applying integration techniques.

2 Integration Rules

2.1 Constant Rule

The constant rule states that the integral of a constant a with respect to x is the constant times x plus the constant of integration C:

$$\int a \, \mathrm{d}x = ax + C$$

Example

Evaluate the integral: $\int 4 \, \mathrm{d}x$

Solution

$$\int 4 \, \mathrm{d}x = 4x + C$$

2.2 Exponential Rule

The exponential rule states that the integral of e^x with respect to x is e^x plus the constant of integration C:

$$\int e^x \, \mathrm{d}x = e^x + C$$

2.3 Power Rule

The power rule states that the integral of x^n with respect to x (where n is any real number except -1) is $\frac{x^{n+1}}{n+1}$ plus the constant of integration C:

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \quad \text{where } n \neq -1$$

Example

Evaluate the integral: $\int x^2 \, \mathrm{d}x$

Solution

$$\int x^2 \, \mathrm{d}x = \frac{x^3}{3} + C$$

2.4 Multiplication by Constant

The multiplication by constant rule states that if we have a constant c multiplied by a function $f(x)$, then the integral of this product is equal to c times the integral of $f(x)$ with respect to x, plus the constant of integration C:

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x$$

Example

Evaluate the integral: $\int 5x^2\,\mathrm{d}x$

Solution

$$\int 5x^2\,\mathrm{d}x = 5\int x^2\,\mathrm{d}x = 5\cdot\frac{x^3}{3} + C = \frac{5x^3}{3} + C$$

2.5 Sum Rule

The sum rule states that the integral of the sum of two functions $f(x)$ and $g(x)$ with respect to x is equal to the sum of their individual integrals, each with respect to x:

$$\int (f(x) + g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

Example

Evaluate the integral: $\int (2x^2 + 3x)\,\mathrm{d}x$

Solution

$$\int (2x^2 + 3x)\,\mathrm{d}x = \int 2x^2\,\mathrm{d}x + \int 3x\,\mathrm{d}x = \frac{2x^3}{3} + \frac{3x^2}{2} + C$$

2.6 Difference Rule

The difference rule states that the integral of the difference of two functions $f(x)$ and $g(x)$ with respect to x is equal to the difference of their individual integrals, each with respect to x:

$$\int (f(x) - g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x$$

Example

Evaluate the integral: $\int (5x^2 - 2x)\,\mathrm{d}x$

Solution

$$\int (5x^2 - 2x)\,\mathrm{d}x = \int 5x^2\,\mathrm{d}x - \int 2x\,\mathrm{d}x = \frac{5x^3}{3} - x^2 + C$$

2.7 Reciprocal Rule

The reciprocal rule, also known as the "Natural Logarithm Rule," states that the integral of $\frac{1}{x}$ with respect to x is the natural logarithm of the absolute value of x plus the constant of integration C:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

Example

Evaluate the integral: $\int \frac{1}{2x}\,\mathrm{d}x$

Solution

$$\int \frac{1}{2x}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{x}\,\mathrm{d}x = \frac{1}{2}\ln|x| + C$$

2.8 Trigonometric Rules (in radians)

The trigonometric rules involve integrals of trigonometric functions. Here are some important ones (where C is the constant of integration):

$$\int \sin(x)\,\mathrm{d}x = -\cos(x) + C$$

$$\int \sec^2(x)\,\mathrm{d}x = \tan(x) + C$$

$$\int \sec(x)\tan(x)\,\mathrm{d}x = \sec(x) + C$$

$$\int \cos(x)\,\mathrm{d}x = \sin(x) + C$$

$$\int \csc^2(x)\,\mathrm{d}x = -\cot(x) + C$$

$$\int \csc(x)\cot(x)\,\mathrm{d}x = -\csc(x) + C$$

2.9 Integrals of Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin(x) + C$$

$$\int \left(-\frac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x = \arccos(x) + C$$

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x) + C$$

$$\int \left(-\frac{1}{|x|\sqrt{x^2-1}}\right) \mathrm{d}x = \operatorname{arccsc}(x) + C$$
$$\int \frac{1}{|x|\sqrt{x^2-1}} \mathrm{d}x = \operatorname{arcsec}(x) + C$$
$$\int \left(-\frac{1}{1+x^2}\right) \mathrm{d}x = \operatorname{arccot}(x) + C$$

3 Examples

3.1 Example 1

Evaluate the integral: $\int (4x^6 - 2x^3 + 7x - 4)\,\mathrm{d}x$

Solution: To integrate each term, we use the power rule for integration.

$$\int (4x^6 - 2x^3 + 7x - 4)\,\mathrm{d}x = \frac{4}{7}x^7 - \frac{1}{2}x^4 + \frac{7}{2}x^2 - 4x + C$$

3.2 Example 2

Evaluate the integral: $\int (z^7 - 48z^{11} - 5z^{16})\,\mathrm{d}z$

Solution: To integrate each term, we use the power rule for integration.

$$\int (z^7 - 48z^{11} - 5z^{16})\,\mathrm{d}z = \frac{1}{8}z^8 - 4z^{12} - \frac{5}{17}z^{17} + C$$

3.3 Example 3

Evaluate the integral: $\int (10t^{-3} + 12t^{-9} + 4t^3)\,\mathrm{d}t$

Solution: To integrate each term, we use the power rule for integration.

$$\begin{aligned}\int (10t^{-3} + 12t^{-9} + 4t^3)\,\mathrm{d}t &= -\frac{10}{2t^2} - \frac{12}{8t^8} + \frac{4}{4}t^4 + C \\ &= -\frac{5}{t^2} - \frac{3}{2t^8} + t^4 + C\end{aligned}$$

3.4 Example 4

Evaluate the integral: $\int (t^2 - 1)(4 + 3t)\,\mathrm{d}t$

Solution: We use the distributive property and then apply the power rule for integration.

$$\begin{aligned}\int (t^2 - 1)(4 + 3t)\,\mathrm{d}t &= \int (3t^3 + 4t^2 - 3t - 4)\,\mathrm{d}t \\ &= \frac{3}{4}t^4 + \frac{4}{3}t^3 - \frac{3}{2}t^2 - 4t + C\end{aligned}$$

3.5 Example 5

Evaluate the integral: $\int (\sin(x) + 10(\csc(x))^2) \, \mathrm{d}x$

Solution: For the first term, we use the integral of $\sin(x)$, which is $-\cos(x)$. For the second term, we use the integral of $\csc(x)^2$, which is $-\cot(x)$.

$$\int (\sin(x) + 10(\csc(x))^2) \, \mathrm{d}x = -\cos(x) + 10(-\cot(x)) + C$$

3.6 Example 6

Evaluate the integral: $\int (2\cos(w) - \sec(w)\tan(w)) \, dw$

Solution: For the first term, we use the integral of $\cos(w)$, which is $\sin(w)$. For the second term, we use the integral of $\sec(w)\tan(w)$, which is $\sec(w)$.

$$\int (2\cos(w) - \sec(w)\tan(w)) \, dw = 2\sin(w) - \sec(w) + C$$

3.7 Example 7

Evaluate the integral: $\int \left(t^3 - \frac{e^{-t}-4}{e^{-t}}\right) dw$

Solution: First, simplify the expression inside the integral.

$$\begin{aligned} \int \left(t^3 - \frac{e^{-t}-4}{e^{-t}}\right) dw &= \int \left(t^3 - \frac{e^{-t}}{e^{-t}} + \frac{4}{e^{-t}}\right) dw \\ &= \int \left(t^3 - 1 + 4e^t\right) dw \end{aligned}$$

Now, integrate each term separately. The integral of t^3 is $\frac{t^4}{4}$. The integral of 1 with respect to t is t, and the integral of $4e^t$ with respect to t is $4e^t$.

$$\int \left(t^3 - 1 + 4e^t\right) dw = \frac{t^4}{4} - t + 4e^t + C$$

3.8 Example 8

Evaluate the integral: $\int \left(\frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}}\right), \mathrm{d}x$

Solution: Split the integral into two separate integrals and evaluate them separately.

$$\int \left(\frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}}\right) \mathrm{d}x = \int \frac{1}{1+x^2} \mathrm{d}x + \int \frac{12}{\sqrt{1-x^2}} \mathrm{d}x$$

For the first integral, we recognize it as the integral of $\arctan(x)$.

$$\int \frac{1}{1+x^2} \mathrm{d}x = \arctan(x)$$

For the second integral, we recognize it as the integral of $\arcsin(x)$.

$$\int \frac{12}{\sqrt{1-x^2}} \mathrm{d}x = 12 \arcsin(x)$$

Now, we combine the two results.

$$\int \left(\frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}} \right) \mathrm{d}x = \arctan(x) + 12 \arcsin(x) + C$$

3.9 Example 9

Evaluate the integral: $\int \left(\sqrt{x^7} - 7\sqrt[6]{x^5} + 17\sqrt[3]{x^{10}} \right) \mathrm{d}x$

Solution: Get rid of the roots by making the exponents fractional.

$$\int \left(\sqrt{x^7} - 7\sqrt[6]{x^5} + 17\sqrt[3]{x^{10}} \right) \mathrm{d}x = \int \left(x^{\frac{7}{2}} - 7x^{\frac{5}{6}} + 17x^{\frac{10}{3}} \right)$$

Apply the power rule for each term to integrate.

$$\int \left(x^{\frac{7}{2}} - 7x^{\frac{5}{6}} + 17x^{\frac{10}{3}} \right) \mathrm{d}x = \frac{2}{9}x^{\frac{9}{2}} - \frac{42}{11}x^{\frac{11}{6}} + \frac{51}{13}x^{\frac{13}{3}} + C$$