Substitutions

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1 Introduction

Among the multitude of mathematical tools to conquer integral computations, the u-substitution is one of the most fundamental, as it helps simplify otherwise ugly looking integrals. Let's get familiar with them in this handout!

2 Definition and Examples

2.1 Why u-sub works

Definition: This integration technique is basically applying **reverse chain rule** by the Fundamental Theorem of Calculus. It aims to write integrals into the form

$$\int f(g(x))g'(x)\mathrm{d}x$$

which, when using a substitution of $u = g(x)$ gives:

$$\int f(u)\mathrm{d}u$$

because $g'(u) = \frac{\mathrm{d}u}{\mathrm{d}x}$.

2.2 Steps to perform a u-sub

There are a few steps to performing a u-substitution. As an example, let's say we wanted to evaluate

$$\int_3^5 \sin(1 + 69x)\mathrm{d}x$$

Here's how you could do it using u-sub.

1. **Choose a simplifying transformation:** Here, notice that if we could apply the power rule, that would be wonderful. Hence we choose the substitution

$$u = 1 + 69x$$

2. **Determine the corresponding differential:** We want to change everything from the x world into the u world: although we changed the part inside the parentheses, we have yet to change the differential $\mathrm{d}x$ to $\mathrm{d}u$. To find that, we can compute the ratio $\frac{\mathrm{d}u}{\mathrm{d}x}$ with differentiation:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{d}{\mathrm{d}x}(1 + 69x) = 69$$
$$\mathrm{d}u = 69\mathrm{d}x$$

3. **Change the bounds of integration:** This part only applies if you have definite integrals. For $x = 3$, $u = 1 + 69(3) = 208$. Similarly for $x = 5$, $u = 346$.

4. **Assembling the integral:** Putting all three steps from above together, we get:

$$\int_{x=3}^{x=5} \sin(1+69x)\mathrm{d}x = \int_{u=208}^{u=346} \sin(u) \cdot \left(\frac{1}{69}\mathrm{d}u\right)$$

From here on, it is straightforward to perform the integral because we know the integral of $\sin(x)$, which one can obtain from the Fundamental Theorem of Calculus, or by looking up an integral table/formula sheet.

The reason we have to perform all the steps above is because u-substitution is basically the same thing as changing a graph's axes - if one pictures integrals as areas under curves, then u-substitution is mapping one x-axis into another u-axis. Not only does the scales on the axes change (which implies that $\mathrm{d}x$ changes into $\mathrm{d}u$), but also the bounds of integration too.

Note that you don't always have to perform substitutions of u as a function of x. You could also perform substitutions where x is a function of u, as we will see below.

2.3 Trigonometric Substitutions

As you continue on your calculus journey, you will encounter more and more substitutions. One particular family of substitutions are **trigonometric substitutions**. The reason they are so useful and even deserve to be named "a family of substitutions" is because of the trigonometric identities, which can help simplify an integral greatly.

Example 1: Let's consider the integral:

$$\int \frac{1}{(1+x^2)^{3/2}}\mathrm{d}x$$

At first glance, this seems hard because you can't do reverse-power-rule on this. However, there is a beautiful coincidence:

$$1 + \tan^2 x = \sec^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$

The derivative of $\tan x$ is exactly the same as 1 plus itself squared! This motivates us to perform the substitution $x = \tan u$. This implies that:

$$\mathrm{d}x = \sec^2 u\mathrm{d}u$$

Plugging both of them into the original integral:

$$\int \frac{1}{(1+x^2)^{3/2}}\mathrm{d}x = \int \frac{1}{(1+\tan^2 u)^{3/2}}\sec^2 u\mathrm{d}u = \int \frac{1}{\sec^3 u}\sec^2 u\mathrm{d}u = \int \cos u\mathrm{d}u$$

The original integral then evaluates to

$$\int \frac{1}{(1+x^2)^{3/2}}\mathrm{d}x = \sin u + C = \sin(\arctan x) + C$$

which one can further simplify using trigonometry, but that would be outside the focus of this handout.

The thing about trigonometric substitutions is that whenever you see a certain "form" of function in the integrand, then one should use a certain substitution. The problem then transforms from a polynomial type function into a trigonometric type function.

Here's a list of useful **trigonometric substitutions**:

Form	Substitution	Reason
$\sqrt{a^2+x^2}$	$u = a\tan x$	$1+\tan^2 x = \sec^2 x = (\tan x)'$
$\sqrt{a^2-x^2}$	$u = a\sin x$	$1-\sin^2 x = \cos^2 x = ((\sin x)')^2$
$\sqrt{x^2-a^2}$	$u = a\sec x$	$\sec^2 x - 1 = \tan^2 x = \left(\frac{(\sec x)'}{\sec x}\right)^2$

where $(f(x))' = \frac{d}{\mathrm{d}x}f(x)$ and $(\sec x)' = \sec x\tan x$

2.4 A Note on Substitutions

As seen from above, the list of substitutions is extensive - in there are many types of substitutions out there too, including Weierstrass substitutions, Euler substitutions, and many other tricks that one will discover along their path on the journey through calculus.

The key to substitutions is not memorising all of them, but rather, choosing the correct one which simplifies the problem at hand.

3 Problems & Practices

3.1 Problem 1

Evaluate the integral

$$\int 3x^2(x^3+1)^6\mathrm{d}x$$

3.2 Problem 2

Evaluate the integral

$$\int \frac{(\ln x)^2}{x}\mathrm{d}x$$

3.3 Problem 3

Evaluate the integral

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\mathrm{d}x$$

3.4 Problem 4

Prove the King Property:

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x$$

3.5 Problem 5

Evaluate the integral

$$\int_0^\pi \frac{x\sin x}{(1+\cos^2 x)^2}\mathrm{d}x$$

Hint: use the result from Problem 3, and think about the periodicity of the sinusoidal functions.

3.6 Problem 6

Evaluate the integral

$$\int \frac{1}{x^2(x^4+1)^{3/4}}\mathrm{d}x$$

3.7 Problem 7

Evaluate the integral

$$\int \frac{x^2-2}{(x^4+5x^2+4)\arctan\left(\frac{x^2+2}{x}\right)}\mathrm{d}x$$

3.8 Problem 8

Evaluate the integral

$$\int_0^{\pi/2} \ln(\sin x)\mathrm{d}x$$

Hint: use the result from Problem 3. You might use the identity $\sin 2x = 2\sin x\cos x$

4 Solutions

4.1 Problem 1

$$\frac{(x^3+1)^7}{7} + C$$

4.2 Problem 2

$$\frac{(\ln x)^3}{3} + C$$

4.3 Problem 3

$$\frac{1}{b}\arcsin\frac{b}{a}x + C$$

4.4 Problem 4

Perform the substitution $x = a + b - u$ to get

$$\int_b^a f(a+b-u)(-\mathrm{d}u) = \int_a^b f(a+b-u)\mathrm{d}u$$

Then notice that the integral is definite, and the variable u is a dummy variable and can be named anything - by changing it to x we complete the proof of the King Property.

4.5 Problem 5

$$\frac{\pi^2}{4}$$

4.6 Problem 6

Perform the substitution $u = x^{-4}$ to get

$$\frac{-1}{4}\int \frac{\mathrm{d}u}{(1+u)^{3/4}}$$

which evaluates to

$$-(1+x^{-4})^{1/4} + C$$

4.7 Problem 7

$$\ln\left(\arctan\left(\frac{x^2+2}{x}\right)\right) + C$$

4.8 Problem 8

$$\frac{-\pi \ln 2}{2}$$

5 Conclusion

Congratulations on making it through this handout! The problems were tough, so give yourself a pat on the back if you managed to do them all - you can now compute many different types of integrals!