Integration by Parts

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1 Introduction

Sometimes we need to find integrals of functions which are products of two different functions. For example, take your time and think how one can integrate $\int x\cos x\mathrm{d}x$. Or, try solving $\int x\sec x^2$. As you can see, these are integrals of product of two completely different functions, being x and $\cos x$, x and $\sec x^2$. In this chapter, you will learn how to integrate these type of integrals.

2 Integration by parts

When we need to find integral of product of two functions, let's say $f(x)$ and $g(x)$: $f(x)g(x)\mathrm{d}x$, we can substitute $f(x)$ with u and $g(x)\mathrm{d}x$ with $\mathrm{d}v$. Then, by utilizing the formula for the derivative of a product written in the differential notation

$$\mathrm{d}(uv) = u\mathrm{d}v + v\mathrm{d}u,$$

$$u\mathrm{d}v = \mathrm{d}(uv) - v\mathrm{d}u$$

$$\int u\mathrm{d}v = \int \mathrm{d}(uv) - \int v\mathrm{d}u$$

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

we get an equation named **Integration by Parts** which can help us solve integrals.

It comes to a rescue when we need integrate the product of two different functions if calculating $\int v\mathrm{d}u$ is easier than calculating $\int u\mathrm{d}v$. For example, let's look at the following problem:

Example 1: Solve $\int x\ln(x)\mathrm{d}x$

Solution: $f(x) = \ln(x)$ $\quad g(x)\mathrm{d}x = x\mathrm{d}x$ $\quad u = \ln(x)$ $\quad \mathrm{d}v = x\mathrm{d}x$ $\quad \mathrm{d}u = \frac{1}{x}\mathrm{d}x$ and $v = \frac{x^2}{2}$. Then, we can say that:

$$\int x\ln(x)\mathrm{d}x = \frac{x^2}{2}\ln(x) - \int \frac{x}{2}\mathrm{d}x = \frac{x^2}{2}(\ln(x) - \frac{1}{2})$$

The tricky part in applying integration by parts is how to select u and $\mathrm{d}v$ so that finding the value will be easy. When $u\mathrm{d}v$ is long, this is especially tricky as finding the proper u and $\mathrm{d}v$ by trial and error takes a lot of time. But with lots of practice, one can find proper u and $\mathrm{d}v$ easily and faster. To select proper u and $\mathrm{d}v$ one should keep in mind before applying the rule that we need to be able to calculate the integral of $\mathrm{d}v$ and we need to be able to integrate $v\mathrm{d}u$. If it is known that one of this will fail, we can select another values for u and $\mathrm{d}v$ and avoid wasting time on u and $\mathrm{d}v$ which will not give a result. It is also possible that the integral maybe solved more easily or faster by u substitution. So, usually testing integration methods which would likely give a result before doing integration by parts is advised. Sometimes, one can apply integration by parts several times in succession. For example, let's look at the following problem.

Example 2: Find $\int x^n e^x \mathrm{d}x$: $u = x^n$, $dv = e^x \mathrm{d}x$. $\mathrm{d}u = nx^{n-1}$, $v = e^x$. Then, we find that:

$$\int x^n e^x \mathrm{d}x = x^n e^x - n \int e^x x^{n-1}.$$

Now, we can see that x^n became x^{n-1} which means that if we continue applying integration by parts rule, the term x^n will be x^0 eventually. So, we can write the equation above as:

$$x^n e^x \mathrm{d}x = x^n e^x - nx^{n-1}e^{n-1} + n(n-1)x^{n-2}e^{n-2} - n(n-1)(n-2)x^{n-3}e^{n-3}...$$

Note that, we continue this, till the power of x and e goes to zero.

Example 3: Another cool example is the reduction of $A_n = \int sin^n x \mathrm{d}x$: $u = sin^{n-1}x$ $dv = sinx\mathrm{d}x$ $\mathrm{d}u = (n-1)sin^{n-2}x \cos x \mathrm{d}x$ $v = -\cos x$

Therefore, $A_n = -sin^{n-1}x \cos x + (n-1)\int \cos x^2 sin^{n-2}\mathrm{d}x$. Now, substituting $\cos x^2$ with $1 - \cos x^2$, we get:

$$A_n = -sin^{n-1}x \cos x + (n-1)\int sin^{n-2}x\mathrm{d}x - (n-1)\int sin^n x \mathrm{d}x$$

$$A_n = -sin^{n-1}x \cos x + (n-1)A_{n-2} - (n-1)A_n$$

$$nA_n = -sin^{n-1}x \cos x + (n-1)A_{n-2}$$

Here, you can see that index of A became smaller, which means if we continue, we can lower the index till it becomes zero or one. If A is odd, then we will use $A_1 = -\cos x$. Otherwise, we will use $A_0 = x$. We could solve the problem by utilizing half angle formulas, but when value of n is big, this becomes cumbersome.

3 Problems

1. Prove:

(a)

$$\int \ln(x)^n \mathrm{d}x = xln(x)^n - n\int \ln(x)^{n-1}\mathrm{d}x$$

(b)

$$\int \ln(x)^5 \mathrm{d}x$$

2. Integrate

$$\int \frac{x^3}{(x^2+5)^2}\mathrm{d}x$$

3. Integrate

$$x^7\sqrt{5+3x^4}\mathrm{d}x$$

4 Solutions

1. (a)

$$\int \ln(x)^n \mathrm{d}x = \int u dv$$
$$u = \ln(x)^n$$
$$dv = \mathrm{d}x$$

$$\mathrm{d}u = \frac{n}{x}\ln(x)^{n-1}\mathrm{d}x$$
$$v = x$$

So,

$$\int u dv = uv - \int v\mathrm{d}u = x\ln(x)^n - n\int \ln(x)^{n-1}\mathrm{d}x$$

(b) To do this, we just need to put the value of n in the formula above:

$$\int \ln(x)^5 \mathrm{d}x = x\ln(x)^5 - 5\int l(x)^4 \mathrm{d}x$$

2. Let's seperate factors x^2 and $\frac{x}{(x^2+5)^2}\mathrm{d}x$. Then, we can name

$$u = x^2$$
$$dv = \frac{x}{(x^2+5)^2}\mathrm{d}x$$
$$\mathrm{d}u = 2x\mathrm{d}x$$
$$v = (-\frac{1}{2})(x^2+5)^{-1} = \frac{-1}{2(x^2+5)}.$$

Therefore,

$$\int \frac{x^3}{(x+5)^2} = -\frac{x^2}{2(x^2+5)} + \int \frac{x\mathrm{d}x}{(x^2+5)}$$

By using "u" substitution,

$$\int \frac{x^3}{(x+5)^2} = -\frac{x^2}{2(x^2+5)} + \frac{1}{2}ln|x^2+5| + C.$$

3.

$$x^7\sqrt{5+3x^4}\mathrm{d}x$$
$$u = x^4$$

$$dv = x^3\sqrt{5+3x^4}\mathrm{d}x$$
$$\mathrm{d}u = 4x^3$$

$$v = \frac{1}{12}\frac{(5+3x^4)^{1.5}}{3/2} = \frac{1}{18}(5+3x^4)^{1.5}$$

Therefore,

$$x^7\sqrt{5+3x^4}\mathrm{d}x = x^4\frac{1}{18}(5+3x^4)^{1.5} - \int \frac{1}{18}(5+3x^4)^{1.5}(4x^3)\mathrm{d}x$$

$$x^7\sqrt{5+3x^4}\mathrm{d}x = x^4\frac{1}{18}(5+3x^4)^{1.5} - \frac{2}{9}\int (5+3x^4)^{1.5}(x^3)\mathrm{d}x$$

We can solve this by using "u" substitution:

$$x^7\sqrt{5+3x^4}\mathrm{d}x = x^4\frac{1}{18}(5+3x^4)^{1.5} - \frac{2}{9}\frac{1}{12}\frac{(5+3x^4)^{2.5}}{2.5} + C$$

$$x^7\sqrt{5+3x^4}\mathrm{d}x = x^4\frac{1}{18}(5+3x^4)^{1.5} - \frac{1}{135}(5+3x^4)^{2.5} + C$$