Spherical Trigonometry & Celestial Coordinates

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1 Introduction

Astronomical reference systems have been continually developed for thousands of years. Ancient astronomers considered objects in the night sky to be fixed on a spherical "shell" around the Earth. Amazingly, this geocentric reference system is still used today – after all, we make most observations from the Earth. Since the distances to stars are very large, the Earth's revolution has a negligible effect on celestial coordinates and we can assume that stars, from our perspective, are fixed on a sphere of infinite radius.

> **Example 1.1:** (USAAAO Local Exam, 2019)
> Eltanin, the brightest star in Draco, has the approximate coordinates RA: 17h 56m, Dec: +51.5°. Given that at the observers location, the latitude is +50° and the local sidereal time is 14:00, how far above the horizon will Eltanin appear? Round your answer to the nearest degree.

In order to answer this question, we must be able to answer a few essential questions:

- What do terms like "RA" and "Dec" mean?
- How can we figure out the apparent position of a star based on local time and its fixed coordinates?
- This might require some trig. How will we do that on a sphere?

The answers to the above questions will be enumerated in subsequent sections, and this example will be revisited.

2 Earth (Lat-Long)

Before we understand how astronomers define points on the **celestial sphere**, we should begin with something more down-to-Earth: how geographers define a point on Earth!

Longitude is a measure of how far East-West you are and its lines run North-South. It is measured in degrees, ranging from -180° (East) to +180° (West). A line that connects the North Pole to the South Pole is known as a *meridian.* Longitude is defined by how far away you are from a specific meridian, known as the **Prime Meridian**, which passes through **Greenwich, England**.

Latitude is a measure of how far North-South you are and its lines run East-West (lat is fat). Like longitude, latitude is measured in degrees, ranging from +90° (North Pole) to -90° (South Pole). At 0° lies the **equator**.

Degrees can also be divided into smaller units. There are 60 **minutes** (') in a degree, and 60 **seconds** (") in a minute. For other coordinate systems, we will see *hours* are used instead of degrees. Since there are 360° degrees in a circle and 24hrs in a day, each hour is $\frac{360}{24} = 15°$.

3 Horizontal (Alt-Az)

Perhaps the most intuitive celestial coordinate system is built in reference to the observer.

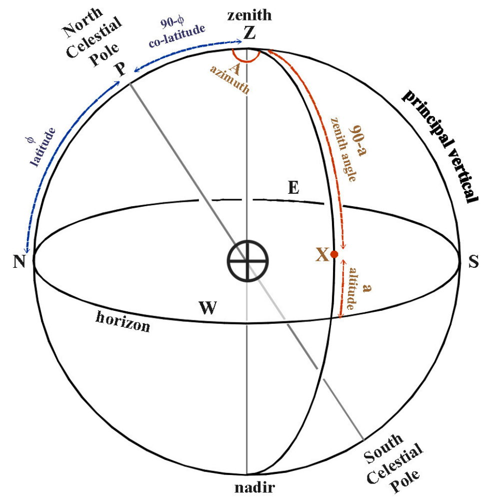

Figure 1: Horizontal System (Source: uhcl.edu)

- The point directly overhead the observer is known as the **zenith** (Z). Its complement is the **nadir**, the point directly *below* the observer on the celestial sphere. For obvious reasons, the observer can not see their nadir.
- a is the **altitude** (h), measured $0 - 90^\circ$ from the horizon up to the zenith (or $0 - -90^\circ$ for objects below the horizon). It's a measure of how high in the sky an object appears.
- **X** is the position of the object in question.
- **A** is the **azimuth** (A_z), measured 00-24H (hours) clockwise starting from the *principal vertical*, the great circle passing through the zenith that contains the North and South Pole.
- When an observer is at the equator, the North and South Pole are both on the horizon. For every degree latitude that the observer travels, the observer's horizon will shift by the same amount. Therefore, if you wanted to figure out the altitude of the North pole for someone at, say 69° N, it would just be 69°! This is why the diagram below represents the altitude of P as *latitude*. Another measurement, the **co-latitude**, expresses the angular distance from P to Z. By definition, the latitude and co-latitude add up to 90°.

4 Equatorial (RA-Dec)

4.1 HA: The Hour Angle

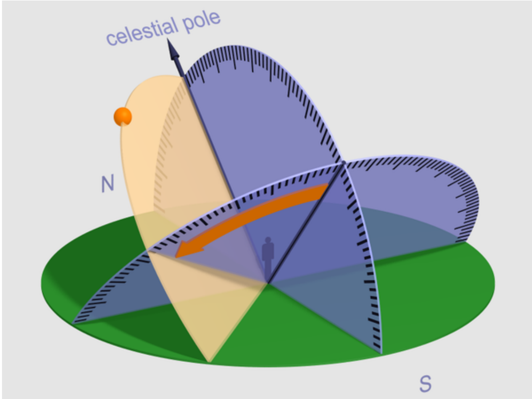

Figure 2: The Hour Angle (Source: Wikipedia)

The **local hour angle** (H, denoted by the orange arrow in Figure 2) is the angle, measured 00-24H clockwise, between the meridian plane of the observer and the great circle connecting the object's position to the celestial poles. You can think of it as how long after noon (the meridian plane of the observer, directly overhead) would it take for the object to reach that position (the great circle connecting the object's position to the celestial poles). Notice how the hour angle varies linearly with time – the Earth's rotation causes the great circle containing the object to appear to rotate, which makes H increase 15° per hour.

4.2 Right Ascension and Declination

With H defined, we are a bit closer to a standardized celestial coordinate system independent of time. However, H still varies and h is not standardized at all. Right ascension and declination remedy these problems:

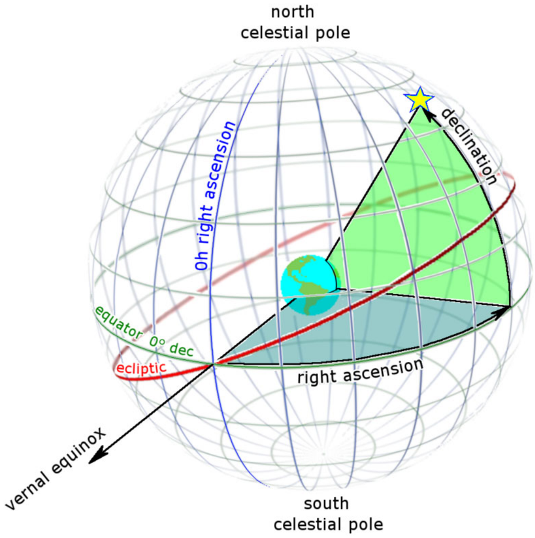

Figure 3: Equatorial Coordinates (Source: Sky and Telescope)

- The **ecliptic** is the great circle projecting the plane of Earth's orbit in the sky. It is situated at a tilt of **23.5°** from the celestial equator.
- The celestial analog to latitude, **declination** (δ), is the angle, measured from $-90°$ at the South Celestial Pole to $90°$ at the North Celestial Pole, between the celestial equator and the object.
- The **vernal equinox** (Υ) is an intersection point between the ecliptic and the celestial equator. This intersection point rotates at exactly the same rate as the rest of the celestial sphere, and therefore can be used to create a fixed longitude-equivalent coordinate.
- The celestial analog to longitude, **right ascension** (α), is the angle measured 00-24H *counterclockwise* from Υ.

One important relation to note between H and α is that

$$\mathrm{ST} = H + \alpha. \tag{1}$$

ST is the **Sidereal Time**, or the hour angle of Υ. One sidereal day is the time between two successive stellar crossings of the same side of the meridian. Similarly, one *Solar Day* is the time between two successive crossings of the Sun (from noon one day to noon the next day). Due to Earth's orbit, the Sun will not be in the same position after one full rotation. As a result, a sidereal day is approximately 4 minutes less than a *Solar Day*.

5 Spherical Trigonometry

5.1 Spherical Coordinates

But what *are* spherical coordinates? And why use them? Let's say we want to figure out the apparent position of a star in the night sky given its right ascension and declination. To do this, we must work with angles on a sphere – for this reason, knowing how spherical coordinates work will help you immensely when solving celestial coordinate problems. Figure 4 shows a basic outline of spherical coordinates. As opposed to rectangular coordinates, which only require two pieces of information about a point to determine its location, spherical coordinates require three. This lecture will not walk you through the derivations of the various relationships found in spherical triangles, but you can learn them in the Further Reading section.

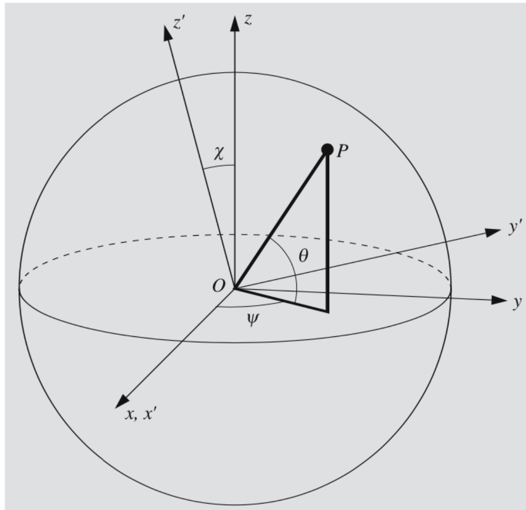

Figure 4: Spherical Coordinates

- ψ is the angle denoting Point P's projected angular location along the xy plane.
- θ is the angle denoting P's distance from the xy plane.
- $\overline{OP}$ is P's distance from the origin.

You may notice that there is an additional set of rectangular coordinate axes. These are utilized in the derivations of spherical triangle laws, which will be skipped for now, but it might intuitively help to think of the standard xyz axes as analogous to the horizontal system and the $x'y'z'$ axes as analogous to the equatorial system.

Before the following sections, it is necessary to define what a *spherical triangle* is:

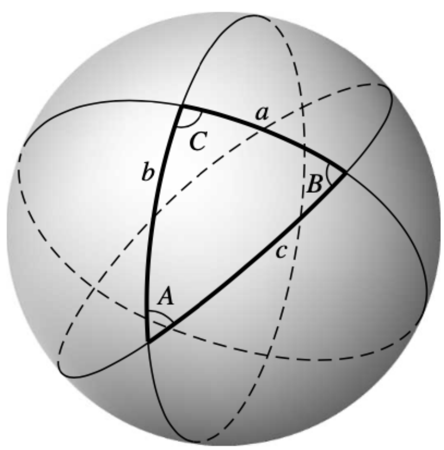

Figure 5: Spherical Triangle (Source: Carroll)

a, b, and c refer to the corresponding arcs' central angles. Angles $A + B + C > 180°$.

5.2 The Spherical Law of Sines

A bit like the planar law of sines, the spherical law of sines states that

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{2}$$

5.3 The Spherical Law of Cosines

The spherical law of cosines states that

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{3}$$

and that

$$\cos B \sin a = -\cos A \sin b \cos c + \cos b \sin c. \tag{4}$$

5.4 The Astronomical Triangle

When converting between the horizontal and equatorial coordinate systems (and vice versa), it is extremely helpful to build what is commonly known as the *astronomical* or *nautical* triangle. Building a spherical triangle connecting the zenith (Z), north celestial pole (P), and object location (S) on the celestial sphere, we can now use the spherical triangle laws outlined above to convert between coordinate systems. How fun!

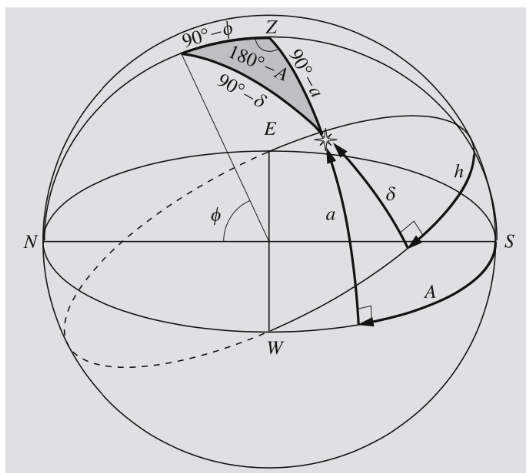

Figure 6: Astronomical Triangle (Source: Karttunen)

Figure 6 outlines the astronomical triangle. Substituting the components of the triangle with components of the spherical coordinate system, and then substituting those values into Eqs. 2, 3, and 4 yield the following relationships:

$$\cos H \cos \delta = \cos A_z \cos h \sin \phi + \sin h \cos \phi, \tag{5}$$

$$\sin H \cos \delta = \sin A_z \cos h, \tag{6}$$

$$\sin \delta = -\cos A_z \cos h \cos \phi + \sin h \sin \phi. \tag{7}$$

Note that H and h actually correspond to the hour angle and altitude, respectively, even though this is not reflected in Figure 6.

As a bit of an exercise, think: which aforementioned astronomical coordinate corresponds to the angle between $\overline{PZ}$ and $\overline{PS}$, where P is the North Celestial Pole and S is the location of the stellar object?

5.5 Example Problem Revisited

Now, we indeed have the tools to solve the example problem given at the beginning of this document. Consider an astronomical triangle between the North Celestial Pole, P; the Zenith, Z; and Eltanin, E. For the exterior angles,

$$\begin{aligned} \overline{PZ} &= 90 - \phi \\ \overline{PE} &= 90 - \delta \\ \overline{ZE} &= 90 - h \end{aligned} \tag{8}$$

And, for the interior angles, all we need to figure out is $\angle EPZ$ so that we can use the Spherical Law of Cosines to solve for its opposite arc, which from Eq. $10 = 90 - h$. By doing this, we can solve for h, the altitude A.K.A. how far above the horizon Eltanin will appear.

$$\angle EPZ = H \tag{9}$$

Take a moment now to visually verify that these above relationships are correct. Visualization is essential to solving celestial coordinate problems. Now, we can use the Spherical Law of Cosines, substituting in the celestial coordinate relationships, to solve for h, the altitude.

$$\cos(90^\circ - h) = \cos(90^\circ - \delta)\cos(90^\circ - \phi) + \sin(90^\circ - \delta)\sin(90^\circ - \phi)\cos H \tag{10}$$

$$\sin h = \sin\delta\sin\phi + \cos\delta\cos\phi\cos H \tag{11}$$

From what is given in the problem, $\delta = +51.5^\circ$ and $\phi = +50^\circ$. To solve for H, we can use α and Eq. 1 to state that $H = 14 - 17.93 = -3.93$, or -59°. Finally, solving for h with Eq. 11 yields an answer of $\boxed{54^\circ}$.

6 Additional Concepts

6.1 Upper and Lower Culmination

It is often of navigational importance to note when celestial objects *transit*, or move across, the meridian of the observer. Each object on the celestial sphere appears to move in a minor circe path parallel to the celestial equator. Thus, each object will cross the meridian twice. An object is said to be in *upper culmination* when it crosses the meridian towards the west. Here, the object reaches its maximum altitude. When the object crosses the meridian (and often the *anti-meridian*) towards the east, it is said to be in *lower culmination*. Here, the object reaches its minimum altitude. We can develop formulae to determine the altitudes at these culminations by observing that $H = 0^h$ in upper culmination and $H = 12^h$ in lower culmination. Reusing Eq. 11 and substituting in $H = 0^h$,

$$\sin h_u = \cos\delta\cos\phi + \sin\delta\sin\phi = \cos(\phi - \delta) = \sin(90^\circ - \phi + \delta). \tag{12}$$

Recall that since $\sin\theta = \sin 180^\circ - \theta$, Eq. 12 has two solutions. The desired solution depends on the cardinal direction of upper culmination.

$$h_u = \begin{cases} 90^\circ - \phi + \theta, & \text{if culmination is south of zenith,} \\ 90^\circ + \phi - \theta, & \text{if culmination is north of zenith.} \end{cases} \tag{13}$$

6.2 Rising and Setting Times

When an object rises and sets, its altitude is zero (neglecting atmospheric refraction). From Eq. 1, we can determine ST given right ascension and hour angle. Assuming right ascension as given, we can again use Eq. 11 to solve for the hour angles corresponding to rising and setting times by isolating $\cos H$:

$$\cos H = \frac{\sin h}{\cos\delta\cos\phi} - \tan\delta\tan\phi. \tag{14}$$

Substituting in $h = 0$ yields,

$$\cos H = -\tan\delta\tan\phi. \tag{15}$$

The two hour angles given from Eq. 15, when substituted into Eq. 1, give the object's rising and setting times in sidereal time.

6.3 Horizons

One somewhat prominent subtopic corresponding to the geographical coordinate system is the distinction between different types of horizons.

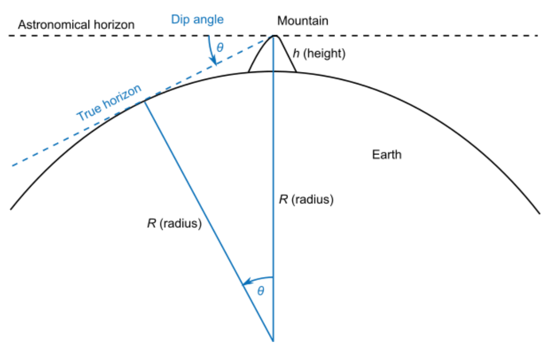

Figure 7: Horizons (Source: mezzacotta)

- The *astronomical horizon* is simply the line perpendicular to the vertical at a given location on Earth.
- The *geometric* or *true* horizon passes through eye level and is tangent to Earth.
- The *dip angle* is the angle of inclination of the true horizon relative to the astronomical horizon.

Using a little bit of trigonometry, we can uncover relationships between θ, R, and h:

$$\theta = \arccos\left(\frac{R}{R+h}\right). \tag{16}$$

As a concrete example, from Eq. 16, it is trivial to then find, neglecting atmospheric refraction and optical obstacles, the maximum distance allowed for an observer to stand in order to be able to view a building of height h.

7 Practice Problems

Problem 1: What are the hour angles of the four cardinal points and the zenith for an observer located in the northern hemisphere? What about an observer in the southern hemisphere?

Problem 2: At what latitudes is the Sun circumpolar at least once a year?

Problem 3: What can the maximum and minimum altitudes of the Sun be, in a city at latitude ϕ?

Problem 4: A star passes through the zenith at 0^h10^m of sidereal time, whereas its altitude on the horizon is $78°12'$ at 9^h2^m of sidereal time. Calculate the latitude of the observer.

Problem 5: What is the angle formed by the ecliptic and the horizon for an observer at the north pole?

Problem 6: A star crosses the south meridian at an altitude of 85° and the north meridian at 45°. Find the declination of the star and the latitude of the observer.

Problem 7: The ecliptic coordinate system centers its coordinates on the ecliptic plane, which is oblique at 23.5° to the celestial equator. It is defined by two coordinates: ecliptic latitude β, the angular distance $[-90°, 90°]$ from the ecliptic; and ecliptic longitude (λ), measured counterclockwise from the vernal equinox. Based on the preceding information, solve for β in terms of δ and α.

8 Further Reading

<u>**Books**</u>

- **Karttunen** – *Fundamental Astronomy*, Chapter 2. Really helpful derivation of spherical trigonometry laws.
- **Vorontsov** – *Astronomical Problems*, Chapters 2-4. Does not derive the spherical trigonometry laws, but derives their application to celestial coordinates very well. Also contains many, many cool problems to give plenty of practice.
- **Carroll & Ostlie** – *An Introduction to Modern Astrophysics*, Chapter 1. Admittedly not the best at explaining this topic, but contains a lot of qualitative information which might help your intuition.
- **Salvati** – *Fundamentals of Astronomy: A guide for Olympiads*, Chapters 1-3. Great diagrams for the celestial sphere. Teaches in a different way than Karttunen and Vorontsov by not mentioning the concept of a spherical triangle.

All of the above titles can be found online except Salvati.

Websites

- Dr. Fiona Vincent's Lecture Notes from the University of St. Andrews. Great visuals and concise explanations.
- Wikipedia. Uses rotation matrices to derive the transformation equations, but individual coordinate system pages are very helpful. Includes nifty animations too.