Taylor Series

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1 Introduction

Taylor Series are a remarkable piece of mathematical machinery that every aspiring theoretical physicist should know about! The reason they are so powerful is because their purpose is to transform an awkward function into a nice, polynomial one.

It is also useful in providing the rigor required to say that $\sin(x) \approx \tan(x) \approx x$, a common hand-wavy trick that makes physicists get on mathematicians' nerves.

Let's take a look at what the Taylor Series formally is!

2 Definition & Intuition

Definition: Given an n-time differentiable function $f(x)$, the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called a **Taylor Polynomial**, and its limit as you add infinitely many terms, $S(x)$

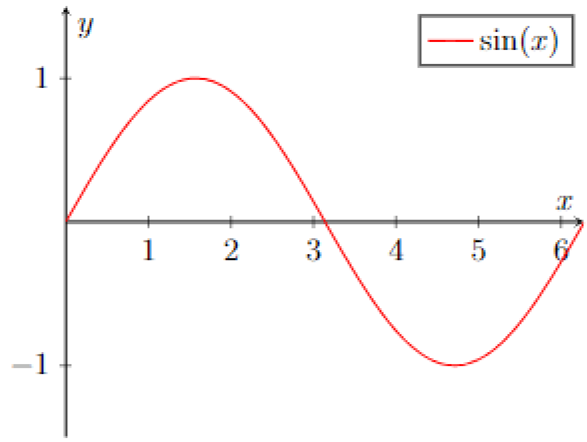
$$S(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the **Taylor Series** of the function $f(x)$ centered around $x = a$.

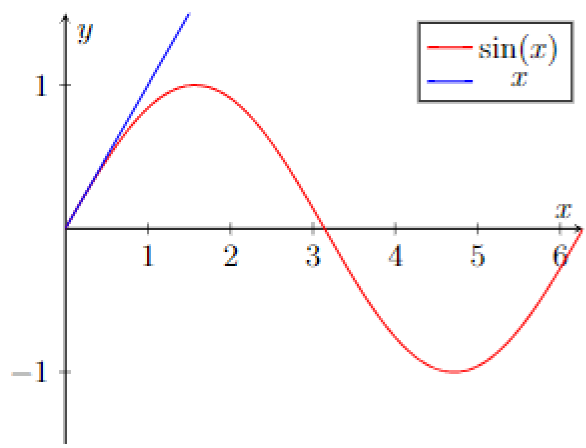
If $S(x)$ converges and is equal to the value of $f(x)$ everywhere, it is said that the **Taylor Expansion** of $f(x)$ is $S(x)$.

When $a = 0$ (i.e. the expansion is centered around the origin) the Taylor Expansion is called the **Maclaurin Expansion**.

To get an intuition of how this works, consider the function $f(x) = \sin(x)$. First, let's only look at values of x near the origin.



Close to the origin, the graph of $\sin(x)$ looks very similar to the linear function $y = x$, doesn't it? Let's plot both of them on the same graph.



Those two match really well near the origin! But, why is that?

The reason for their similarity is because their **first derivatives match** near the origin. When you differentiate $\sin(x)$, you get $\cos(x)$, and at $x = 0$ $\cos(x)$ is equal to 1. Hence the first derivative of $\sin(x)$ is 1 at the origin, which is same for the line $y = x$.

Now, can we improve on this? One thing we know about the sine function is that it is **odd**, that is, $\sin(-x) = -\sin(x)$. Hence, when we model them as a polynomial, the polynomial better be an odd function too - that is to say, $P_n(-x) = -P_n(x)$, which leads us to rule out all even powers of x (x^2, x^4, etc.).

Our next best guess at modelling the function is to add a downward-curving cubic term that matches the **higher derivatives of** $\sin(x)$. Let's call our new curve $y = x - cx^3$ for some constant c, as a guess. We put a negative because we need the line to curve downwards. Differentiating, we get

$$y' = 1 - 3cx^2$$
$$y'' = -6cx$$
$$y''' = -6c$$

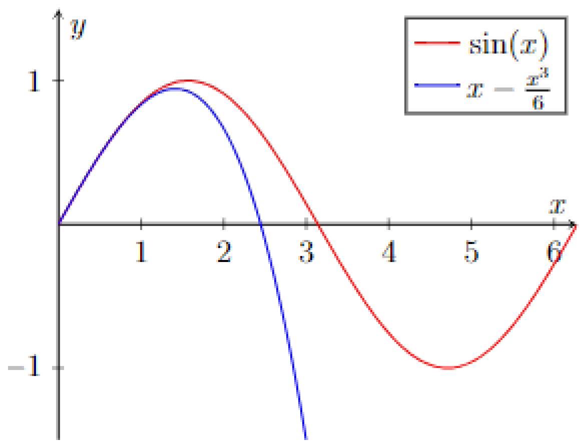
For the function $y = \sin(x)$, when we differentiate and plug in $x = 0$, we get

$$y' = \cos(0) = 1$$

$$y'' = -\sin(0) = 0$$

$$y''' = -\cos(0) = -1$$

For the third derivatives to match, we need $c = \frac{1}{6}$. Plotting this on the graph, we get:



And that's the core idea of Taylor series! When you do this same process over and over, matching higher and higher derivatives, the polynomial approaches the original function better and better. If instead of matching the function close to $x = 0$, you want to model the function around some point $x = a$, then you can just compute the derivatives at a and swap each power of x from x^k to $(x - a)^k$.

Feel free to play around with the idea in this Desmos plot.

Of course, there are some nuances about convergence and some oddly-behaving functions (see the problems below!), but let's leave that for the real-analysis mathematicians for now.

3 Common Taylor Series

3.1 Binomial Expansion

You might have learned this somewhere in school, and might be thinking "hey, this isn't a Taylor Series!" - you're totally right, but since these two ideas have many similar principles, it would be fitting to put them in the same discussion.

Definition: Given a binomial raised to a power, its expanded form can be written in the form

$$(x + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the **binomial coefficient**.

To see why this is true, let's see take a look when $x = 0$. On the RHS, all the terms would vanish except for the $k = 0$ term (because $0^0 = 1$, don't tell the mathematicians), leaving us with $\binom{n}{0}a^n = a^n$, which is exactly the same as the LHS.

Taking the derivatives of both sides, we get

$$n(x+a)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} k x^{k-1}$$

Again, at $x = 0$, the only nonzero term on the RHS is when $k = 1$, i.e. $\binom{n}{1}a^{n-1} = na^{n-1}$, which is also exactly the same as the LHS.

If you do this for all n higher derivatives, you would get that both sides always match, and hence the binomial expansion is true when viewed as a Taylor polynomial.

3.2 Trigonometric Functions

Here are a few classic trigonometric Taylor Expansions of trig functions. Try to convince yourself that they are true!

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

You might be wondering what the Taylor expansion for $\tan(x)$ is, but that's a rabbit hole in and of itself, involving Bernoulli numbers from some magical math major's textbook. To expand $\tan(x)$ in Olympiad problems, just take the first few derivatives and forget about deriving a general expression for the expansion!

3.3 Exponential & Logarithmic Functions

Just as above, try convincing yourself that these equations hold true!

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$\ln\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{1}{k} x^k$$

Careful with the $\ln(x)$ function though! It starts at $k = 1$, because we don't want to be dividing by zero now do we...

You might notice that the form chosen for the Taylor expansion for $\ln(x)$ differs from other sources from the internet, but I find that this form is easier to remember.

3.4 Power Series

As with the Binomial Expansion, this one on the list isn't really a Taylor series either, but has significant overlap with the topic.

Definition: A power series $P(x)$ has the form

$$P(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

where each of the coefficients a_k are constants.

A simple and familiar power series would be the power series of $\frac{1}{1-x}$, where all the constants a_K are just 1 because

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$= 1 + x + x^2 + x^3 + ...$$

This is the generalized form of Taylor series, and can be found in many places of higher math & physics, ranging from generating functions used for combinatoric problems to solving the Schrodinger's equation for a Hydrogen-like atom.

One usually uses the power series to **construct a solution** to an equation, by plugging it into the equation and deducing what the coefficients must be.

An example is the familiar simple harmonic motion equation of motion:

$$y'' = -\omega^2 y$$

There are many ways to solve this, and one of them involves using power series.

Let $y = \sum_{k=0}^{\infty} a_k t^k$ be a solution to the SHM equation. Plugging it in, we get

$$\sum_{k=0}^{\infty} a_k \cdot k \cdot (k-1) \cdot x^{k-2} = -\omega^2 \sum_{k=0}^{\infty} a_k t^k$$

$$\sum_{k=0}^{\infty} a_{k+2} \cdot (k+2) \cdot (k+1) \cdot x^k = -\omega^2 \sum_{k=0}^{\infty} a_k t^k$$

after some reindexing. Since the coefficients of x^k on each side must be identical, we obtain the relation

$$a_{k+2} = -\omega^2 \frac{1}{(k+2)(k+1)} a_k$$

Notice that this relation only talks about coefficients which are 2 indices away, so that means we have 2 families of a: one for odd k and one for even k, both determined by the initial conditions a_1 and a_0 respectively.

The recurrence relation reads

$$a_k = \frac{-\omega^2}{k(k-1)} a_{k-2}$$

$$a_k = \frac{-\omega^2}{k(k-1)} \cdot \frac{-\omega^2}{(k-2)(k-3)} a_{k-4}$$

$$a_k = \frac{-\omega^2}{k(k-1)} \cdot \frac{-\omega^2}{(k-2)(k-3)} \cdot \frac{-\omega^2}{(k-4)(k-5)} \ldots$$

We now split into two cases: k can either be even or odd.

When k is **even**, it can be written as $k = 2n$. We notice that when the recurrence relation is brought down to a_0, there would be n copies of $-\omega^2$ in the numerator and $k!$ in the denominator, that is to say

$$a_{2n} = \frac{(-\omega^2)^n}{(2n)!} a_0$$

When k is **odd**, it can be written as $k = 2n+1$. A similar argument would yield

$$a_{2n+1} = \frac{(-\omega^2)^n}{(2n+1)!} a_1$$

Finally, putting this into the original power series expression for y gives

$$y(t) = \sum_{\text{even}} a_k t^k + \sum_{\text{odd}} a_k t^k$$

$$y(t) = a_0 \sum_{n=0}^{\infty} \frac{(-\omega^2)^n}{(2n)!} t^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-\omega^2)^n}{(2n+1)!} t^{2n+1}$$

$$y(t) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\omega t)^{2n} + \frac{a_1}{\omega} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\omega t)^{2n+1}$$

from which we recognize the two sums as

$$y(t) = a_0 \cos(\omega t) + \frac{a_1}{\omega} \sin(\omega t)$$

4 Applications

4.1 Approximations

Let's go back to the power series of $\frac{1}{1-x}$, where all the terms in the expansion differs by a factor of x. If x is a very small number, eg. 0.01, we would find that

n	x^n	$P_n(0.001)$	$P_n(x)$
0	1	1	1
1	0.1	1.001	$1+x$
2	0.01	1.001001	$1+x+x^2$
3	0.001	1.001001001	$1+x+x^2+x^3$

The values that add onto the Taylor polynomial become smaller and smaller. In physics, one philosophy that one must adhere to is that **nothing is perfect**. As said by Greg in the hit show Steven Universe: "if all porkchops were perfect, we wouldn't have hotdogs!" If all meter rulers were perfect, we wouldn't have vernier calipers, micrometers, etc. The thing is, only up to a certain precision is a number useful for. Beyond that, we can never measure something that precisely, and the digits would mean nothing.

All of this is to say that the expression $\frac{1}{1-x}$ can be approximated as $1 + x$, as that's the useful range of precision we could hope for. (in different scenarios, the range of precision would be different too, but that's up to the physicist to be the judge of what precision is necessary).

Some other famous approximations would include

$$\sin(x) \approx x$$

$$\cos(x) \approx 1 - \frac{x^2}{2}$$

$$e^x \approx 1 + x$$

$$(1 + x)^n \approx 1 + nx$$

all of which only hold true when $|x| << 1$

4.2 SHM angular frequency

We know that the simple harmonic motion equation of motion has the form

$$\ddot{x} = -\omega^2 x$$

where ω is the angular frequency of the system, however the system is setup. What if we want to generalize this and find ω knowing only the potential energy U?

Suppose we have a potential energy function $U(x)$, and the system oscillates about an equilibrium point x_0. You can imagine a mass on a spring oscillating under the influence of gravity - the equilibrium point is somewhere lower than the spring's natural length. At this point

Most forces we deal with have a special property, that is that they are **conservative** and obey the equation $F = -\frac{dU}{dx}$ One can see this from the fact that work, the change in energy, is defined as $W = \vec{F} \cdot \vec{\Delta s}$.

We know that, by Newton's 2nd law, $F = m\ddot{x}$, hence

$$-\frac{dU}{dx} = m\ddot{x}$$

Here comes the magic. If we do a Taylor expansion of the potential energy function about the point x_0, one gets

$$U(x) \approx U(x_0) + U'(x_0)\frac{(x - x_0)}{1!} + U''(x_0)\frac{(x - x_0)^2}{2!}$$

from neglecting the higher order terms by assuming that the oscillations are small.

From here, we get the expression for $\frac{\mathrm{d}U}{\mathrm{d}x}$ as

$$\frac{\mathrm{d}U}{\mathrm{d}x} \approx U'(x_0) + U''(x_0)(x - x_0)$$

$$\frac{\mathrm{d}U}{\mathrm{d}x} \approx U''(x_0)(x - x_0)$$

because by definition of being an equilibrium point, $U'(x_0) = F_{x=x_0} = 0$.

If we let the perturbation from equilibrium $x - x_0$ be ϵ, then $\ddot{x} = \ddot{\epsilon}$. Putting all these into the original equation of motion, we get

$$-\frac{dU}{dx} = m\ddot{x}$$

$$-U''(x_0)\epsilon = m\ddot{\epsilon}$$

$$\ddot{\epsilon} = -\frac{U''(x_0)}{m}\epsilon$$

which is exactly the form of SHM. Directly comparing coefficients, we get

$$\omega = \sqrt{\frac{U''(x_0)}{m}}$$

5 Problems & Practices

5.1 Problem 1

Problem 1: Show that both of the following "famous" definitions of e^x are equivalent:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

(Note: these are the real definitions of e^x - 2.71828... is not just a number!)

5.2 Problem 2

Problem 2: Find the Taylor expansion of $\arctan x$.

5.3 Problem 3

Problem 3: From the Desmos plot earlier in this handout, you played around with the function $\sin(x)$, now let's take a look at the expansion of $\ln\left(\frac{1}{1-x}\right)$

(a) Change the function in the Desmos to the logarithmic function. Does the Taylor expansion match the function well? What if you add more and more terms?

(b) Take a look at the coefficients in the Taylor expansion

$$\ln\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{1}{k} x^k$$

Under what circumstances will the series converge? (hint: think ratios)

(c) Obtain the range of values x on which the infinite series converges. This result is called the **radius of convergence**, and is the range where the Taylor expansion holds. (careful about the inequality boundaries!)

5.4 Problem 4

Problem 4: Prove Euler's Formula,

$$e^{ix} = \cos(x) + i\sin(x)$$

where i is the imaginary number which obeys $i^2 = -1$.

5.5 Problem 5

Problem 5: Suppose there are two electric charges q and $-q$, respectively, at $(0, d)$ and $(0, -d)$ in vacuum. Derive the potential V at (x, y), which is located sufficiently far from the origin, up to the first order with respect to d. Let the potential at infinity be zero, and denote the vacuum permittivity by ϵ_0.

Hint: The potential of one electric charge Q at a distance r is $V = \frac{Q}{4\pi\epsilon_0 r}$, and potentials obey the principle of superposition (i.e. $V_{\text{tot}} = V_1 + V_2$)

5.6 Problem 6

Problem 6: In the presence of a uniform and weak magnetic field B pointing into the page, a particle of charge q and mass m with velocity v is shot to the right toward a point P on the screen which is located at distance L away from the initial position of the particle. The incident particle is moving perpendicularly to the screen.

Derive the distance l between P and the point where the particle actually hits the screen, assuming that the deflection angle of the particle (which indicates the change in the advancing direction) is sufficiently small.

Hint: the particle undergoes circular motion under the centripetal force supplied by the Lorentz force, $F = q\vec{v} \times \vec{B}$.

5.7 Problem 7

Problem 7: You might have heard of the James Webb Space Telescope (JWST) that's been supplying the pretty images we see on social media, but did you know that its position in space is a very special one?

It orbits in space at a special point: the **Lagrange point** (specifically L2), which moves in tandem with the Earth, orbiting along the same line joining the Earth and the Sun. (for a better visualisation, search up some GIFs on Google!)

Derive the distance between the L2 point, at which JWST is currently orbiting around, and the Earth.

Hint: The reason it can achieve this special configuration is because the gravitational force from both the Sun and the Earth perfectly supply the centripetal force needed for JWST to orbit at that particular point.

Bonus: if you're up for a challenge, try your hand at IPhO 2011 Q1!

6 Solutions

6.1 Problem 1

Applying binomial theorem on the first definition and simplifying would give the second definition.

6.2 Problem 2

Differentiate repeatedly to get

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

6.3 Problem 3

a) Diverges at the sides, only converges on some interval

b) When the terms you add get smaller and smaller, i.e.

$$|\frac{T_{n+1}}{T_n}| < 1$$

$$|\frac{\frac{1}{n+1}x^{n+1}}{\frac{1}{n}x^n}| < 1$$

$$|\frac{n}{n+1}x| < 1$$

c) The inequality derived above must hold even when n is infinitely large

$$\lim_{n\to\infty} |\frac{n}{n+1}x| < 1$$

$$|x| < 1$$

$$-1 < x < 1$$

But notice that we can also include $x = -1$ too because

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

Hence

$$-1 \leq x < 1$$

6.4 Problem 4

Use Taylor expansion of e^x and change variables $x \to ix$, then separate real and imaginary parts to get Taylor series $\cos x$ and $\sin x$.

6.5 Problem 5

For some point in space (x, y) the potential is

$$V = \frac{kq}{\sqrt{x^2 + (y-d)^2}} - \frac{kq}{\sqrt{x^2 + (y+d)^2}}$$

To simplify this, we can write

$$\frac{1}{\sqrt{x^2 + (y \pm d)^2}} = \frac{1}{\sqrt{x^2 + y^2}} \left(1 + \frac{\pm 2yd + d^2}{x^2 + y^2}\right)^{-\frac{1}{2}}$$

And using $(1 + x)^n \approx 1 + nx$ for small x we get

$$\frac{1}{\sqrt{x^2 + (y \pm d)^2}} \approx \frac{1}{\sqrt{x^2 + y^2}} \left(1 - \frac{\pm yd + d^2/2}{x^2 + y^2}\right)$$

$$\approx \frac{1}{\sqrt{x^2 + y^2}} \left(1 \mp \frac{yd}{x^2 + y^2}\right)$$

where we discarded the second order term of d. Therefore the potential is

$$V = \frac{kq \cdot 2yd}{(x^2 + y^2)^{\frac{3}{2}}}$$

6.6 Problem 6

Under the influence of a magnetic field a charged particle undergoes circular motion with radius

$$r = \frac{mv}{qB}$$

From geometry,

$$l = r - r\cos\theta \approx r\frac{\theta^2}{2}$$

Also from geometry,

$$\sin\theta = \frac{L}{r} \approx \theta$$

Hence

$$l \approx r\frac{\left(\frac{L}{r}\right)^2}{2} = \frac{L^2}{2r} = \frac{qBL^2}{2mv}$$

6.7 Problem 7

At the Lagrange point L2 force balance gives

$$\frac{GM_S}{(R+d)^2} + \frac{GM_E}{d^2} = (R+d)\omega^2$$

where ω is the angular velocity of the Earth's revolution around the Sun. This can also be expressed with Kepler's 3rd Law:

$$GM_S = w^2R^3$$

Substituting that into the original force balance equation,

$$\frac{GM_S}{R^2}\left(1+\frac{d}{R}\right)^{-2} + \frac{GM_E}{d^2} = R\left(1+\frac{d}{R}\right)\frac{GM_S}{R^3}$$

$$\frac{GM_S}{R^2}\left(1-2\frac{d}{R}\right) + \frac{GM_E}{d^2} = \frac{GM_S}{R^2}\left(1+\frac{d}{R}\right)$$

$$-2\frac{GM_Sd}{R^3} + \frac{GM_E}{d^2} = \frac{GM_Sd}{R^3}$$

$$\frac{M_E}{d^3} = \frac{3M_S}{R^3}$$

$$d = R\left(\frac{M_E}{3M_S}\right)^{\frac{1}{3}}$$

IPhO 2011 Q1 solution can be found here.

7 Conclusion

Understanding Taylor Series and its power in simplifying equations is your first step in becoming a mathematically well-equipped theoretical physicist! Congratulations on making it through this handout!