Telescopes

Austin Chen

June 2023

Table of Contents

1 Introduction **2**

2 Basic Angle Formulas **2**

3 Rayleigh Criterion **3**

4 Focal Ratio **5**

5 Magnification **6**

6 Magnitudes **6**

7 Conclusion **8**

1 Introduction

Telescopes are the instruments that allow astronomers to view the universe. In this paper, let's *observe* how they work.

2 Basic Angle Formulas

You should know how to convert degrees into radians and vice-versa:

$$180^\circ = \pi \text{ rad}.$$

And also degrees into arcminutes and arcseconds. There are 60 arcminutes in a degree and 60 arcseconds in an arcminute:

$$1^\circ = 60' = 3600'',$$

where arcminute is denoted by $'$ and arcsecond denoted by $''$.

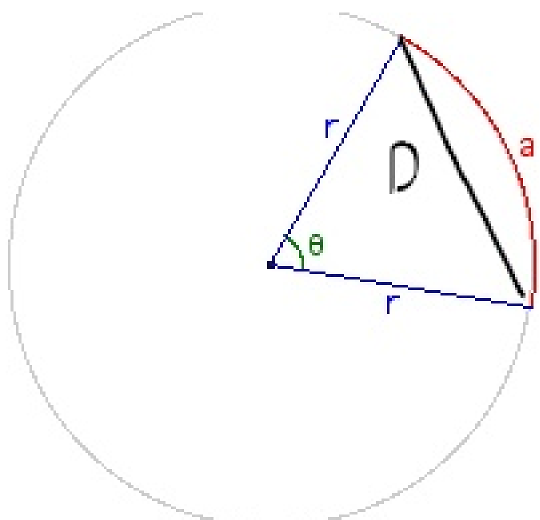


Figure 1

See Fig. 1. By definition, θ in radians is equal to the arc length subtended over the radius of the circle:

$$\theta = \frac{a}{r}$$

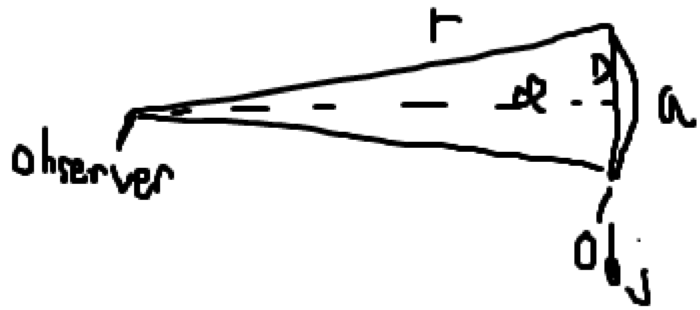


Figure 2

See Fig. 2. D is the length of the line connecting the two line segments of length r, and d is the distance from the center of the circle to the line segment of length D. When $a << r$, $a \approx D$ and $r \approx d$, thus

$$\theta \approx \frac{D}{d}.$$

Now consider viewing an object: You are standing at the center of the circle and the object of length D is placed a distance d away from you. If the object is sufficiently far, to satisfy $a << r$, (almost all cases in Astronomy), then the angular diameter of the object is simply

$$\theta = \frac{D}{d}.$$

Example 2.1. What is the angular diameter of the Sun, in arcminutes? Use that the radius of Sun is 6.96×10^5 km.

Solution: Use that the Sun is 1 AU away.

$$\theta = \frac{2 \cdot 6.96 \times 10^8 \text{ m}}{1.50 \times 10^{11} \text{ m}} = 31.9'$$

About half a degree!

3 Rayleigh Criterion

When viewing an object through a telescope, a diffraction pattern is produced as shown in Fig. 3.

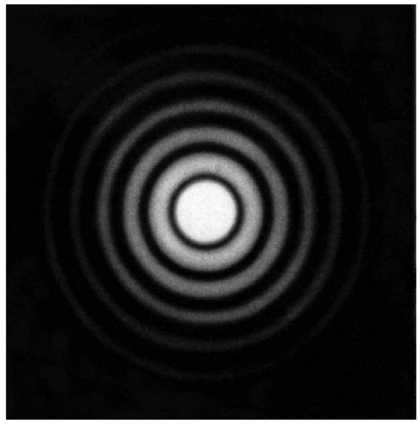
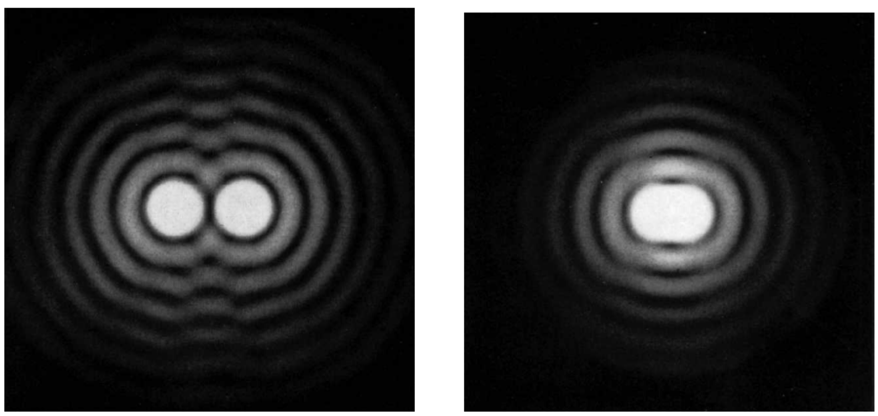


Figure 3: Circular aperture diffraction pattern of a point source.

This phenomenon is closely related to the single-slit diffraction pattern. Now consider when viewing two objects as shown in Fig. 4.



(a) (b)

Figure 4: Diffraction pattern of two point sources. By the Rayleigh criterion: (a) The sources are resolvable. (b) The sources are not resolvable.

The Rayleigh criterion says that two objects are unresolved if the central maximum of one diffraction pattern falls inside the first minimum of the other, and that they are resolved otherwise. This condition is quite arbitrary - you may consider Fig. 4. (b) to be resolvable - but this will be the condition IOAA uses. The solution of the Rayleigh Criterion is

$$\theta = 1.22\frac{\lambda}{D},$$

where λ is the wavelength of light being observed, D is the diameter of the aperture of the telescope, and θ is the minimum angle, in radians, at which if two objects are separated an angle θ in the sky, they are said to be resolvable by this particular telescope. Think about why this formula makes sense: As you increase λ, such as from visible light to radio waves, it becomes harder to distinguish two objects as the diffraction patterns get larger and thus θ increases. As you increase D, it is easier to resolve with a larger aperture as the diffraction patterns get smaller and thus θ decreases. This is one of the **MOST IMPORTANT** formulas in all of IOAA/USAAAO and will be used many times.

Example 3.1. How far would two objects have to be placed on the surface of the moon to be able to be resolved by the human eye? Use that the distance to the moon is 3.82×10^5 km and that the diameter of a human pupil is 3 mm.

Solution: Use that the human eye is observing at visible light with $\lambda = 550$ nm.

$$D = d\theta = 3.82 \times 10^5 \text{ km}(1.22 \cdot \frac{550 \times 10^{-9} \text{ m}}{3 \times 10^{-3} \text{ m}}) = 85.4 \text{ km}.$$

Example 3.2. (USAAAO Round One 2022, Question 6)

The resolution of a space telescope is theoretically limited by diffraction from its primary mirror. In this problem, we will compare the diffraction limit of the Hubble Space Telescope (HST) (primary mirror diameter $d = 2.4\,\text{m}$) and the James Webb Space Telescope (JWST) ($d = 6.5\,\text{m}$). The operating wavelengths for the two telescopes are 500 nm and 10 µm respectively. Calculate the ratio of the diffraction limited angular resolution $\frac{\theta(\text{HST})}{\theta(\text{JWST})}$. Which telescope can resolve smaller angular features if limited only by diffraction?

(a) 0.014, JWST
(b) 0.14, HST
(c) 1.4, JWST
(d) 14, HST
(e) 140, JWST

Solution:

$$\frac{\theta(\text{HST})}{\theta(\text{JWST})} = \frac{1.22\frac{500\times10^{-9} \text{ m}}{2.4 \text{ m}}}{1.22\frac{10\times10^{-6} \text{ m}}{6.5 \text{ m}}} = 0.135.$$

This means that JWST has a larger minimum angular resolution, so HST can resolve smaller angular features. The answer is $\boxed{B}$.

4 Focal Ratio

If the focal length of a telescope is f, and the diameter of its aperture is D, then the focal ratio is

$$F = \frac{f}{D}.$$

If a telescope is said to be "f/n", n denotes its focal ratio; $F = n$.

Example 4.1. (USAAAO Round One 2022, Question 19)

Consider a f/9 telescope with focal length $f = 1.0$ m that operates at visible wavelength $\lambda = 5000$ Å. What is the farthest distance at which an open cluster of radius $R_C = 4.1$ pc can be resolved by this telescope?

(a) 1.2×10^6 pc
(b) 1.5×10^6 pc
(c) 3.0×10^6 pc
(d) 4.2×10^6 pc
(e) 5.8×10^6 pc

Solution: Note: I realize it's been kind of confusing when I've been using D for both the diameter of the aperture of a telescope and the diameter of an object. I'll also use D_o for the diameter of the aperture of a telescope.

$$d = \frac{D}{\theta} = \frac{DD_o}{1.22\lambda} = \frac{Df}{1.22F\lambda} = \frac{2 \cdot 4.1 \text{ pc} \cdot 1.0 \text{ m}}{1.22 \cdot 9 \cdot 5000 \times 10^{-10} \text{ m}} = 1.49 \times 10^6 \text{ pc}.$$

The answer is $\boxed{B}$.

5 Magnification

If the focal length of a telescope's primary or *objective* lens is f_o (same as f used in section 4), and the focal length of its eyepiece is f_e, then the magnification of the telescope is

$$m = \frac{f_o}{f_e}.$$

If the field of view of a telescope's eyepiece is FOV_e, then the field of view of the telescope is

$$FOV = \frac{FOV_e}{m}.$$

6 Magnitudes

If the magnitudes of two objects are m_1, m_2, and the flux of the image of the objects are F_1, F_2, then

$$m_1 - m_2 = -2.5 \log_{10} \frac{F_1}{F_2}.$$

Additionally, the Light-Gathering Power, or how much total light a telescope receives, is proportional to D_o^2.

Example 6.1. (USAAAO NAO, Medium Question 1)
An astronomer used his f/5 telescope with a diameter of 130 mm to observe a binary system.

He is using an eyepiece with a field of view of 45° and a focal length of 25 mm. In this system, star A has a mass of 18.9 solar masses, and an apparent magnitude in the V filter of 9.14. Star B has a mass of 16.2 solar masses, and an apparent magnitude in the V filter of 9.60. The period of the system is 108 days, and the distance between the binary stars and the Solar System is 2.29 kpc. The binary system has an edge-on orbit relative to the Solar System.
(a) What is the field of view of the telescope?
(b) What is the limiting magnitude of the telescope?
(c) What is the angular resolution of the telescope?
(d) What is the angular separation between the stars?
(e) Is the astronomer able to observe both stars as distinct points in the telescope? Answer as YES or NO.
The limiting magnitude for the human eye is 6.0, and the diameter of the pupil is equal to 7.0 mm. Also consider that visible light has a wavelength of 550 nm.

Solution:
(a)

$$FOV = \frac{FOV_e}{m} = \frac{FOV_e f_e}{f_o} = \frac{FOV_e f_e}{FD_o} = \frac{45° \cdot 25 \text{ mm}}{5 \cdot 130 \text{ mm}} = \boxed{1.73°.}$$

(b) This one is bit tricky. You have to use the human eye as reference. Let m_1, m_2 be the limiting magnitudes of the human eye and the telescope. $\frac{F_1}{F_2}$ is the ratio of the flux received by the human eye and the telescope of the limiting magnitude objects. This ratio is going to be less than 1: the telescope can receive less flux at the limit because it has more Light-Gathering Power. This ratio is equal to the inverse ratio of their Light-Gathering Power thus equal to the inverse ratio of the squares of their aperture diameters:

$$m_1 - m_2 = -2.5 \log_{10}(\frac{D_2^2}{D_1^2})$$

$$6.0 - m_2 = -2.5 \log_{10} \frac{(130 \text{ mm})^2}{(7.0 \text{ mm})^2}$$

$$m_2 = \boxed{12.3.}$$

(c)

$$\theta = 1.22 \frac{\lambda}{D_o} = 1.22 \frac{550 \times 10^{-9}}{130 \times 10^{-3}} = \boxed{5.16 \times 10^{-6} \text{ rad}}$$

(d) Using Kepler's third law, you can determine that the separation between the two stars is

$$a = 1.45 \text{ AU}.$$

Then

$$\theta = \frac{a}{d} = \frac{1.45 \cdot 1.50 \times 10^{11} \text{ m}}{2.29 \times 10^3 \cdot 3.09 \times 10^{16} \text{ m}} = \boxed{3.07 \times 10^{-9} \text{ rad.}}$$

(e) The angular separation between the stars is smaller than the limiting angular resolution of the telescope. This means that the telescope cannot resolve the stars and sees them as one point, so $\boxed{NO}$.

7 Conclusion

Understanding telescopes are the first steps to becoming an Astronomer and also increasing your USAAAO scores!