Cosmology

Faraz Ahmed

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1 Introduction

Building on observations, cosmology helps us to understand and predict everything about the universe, including its evolution, age, shape, and more!

2 Cosmological Principle

The cosmological principle states that at large distances, our universe is homogeneous and isotropic. What it means is that, the universe appears to be the same from every point. Crudely, a **homogenous** universe is one where moving to a different point wont affect your view and **isotropy** means that seeing in a a different direction wont affect your view. To help you understand what it means for a surface to be perfectly homogeneous and isotropic (aka maximally symmetric), think of an infinite flat plane. This is an example of a flat maximally symmetric surface. Can you think of a curved maximally symmetric surface?

3 Scale factor

Consider lattice points on a screen. You observe these points to be separated with each other by some specified distance χ. Now, if you zoom out of the screen, the distance between each point is reduced. If you zoom in, you'll observe that the distance between each point has increased. In both of these instances, the distance between points is scaled by some ratio (also called scale factor). Our universe expands in a similar way. Out of convention, we say that the present-day physical distance is equal to the coordinate distance between the points, or that the current value of scale factor is 1. As we expand or contract, the physical distance is related to the comoving distance as

$$R = a(t)\chi$$

Where $a(t)$ is the scale factor for that instant, χ is the comoving distance, and R is the physical distance.

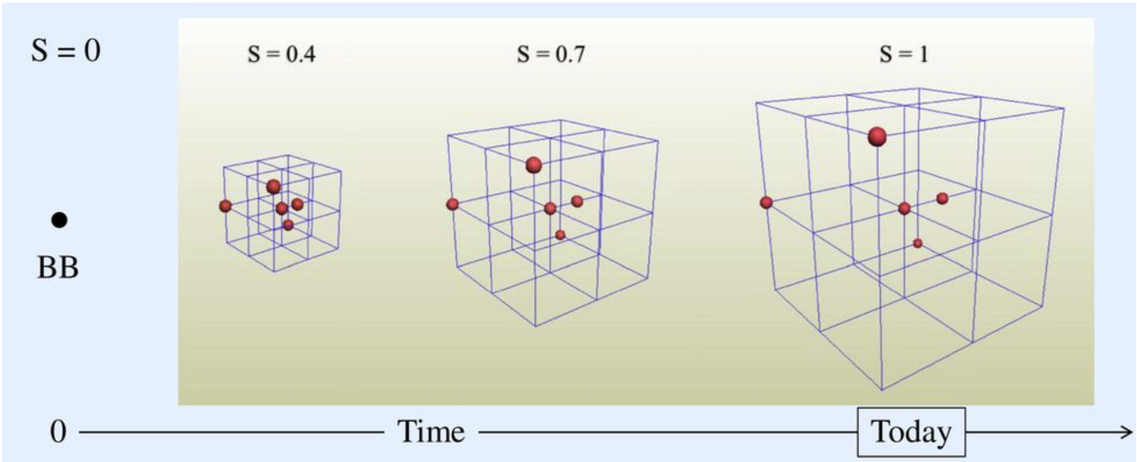

Figure 1: Yo, Mr. White! The universe is expanding! (S is the scale factor)

3.1 Revisiting redshift

Consider a light source emitting photons of wavelength λ. As the photons travel towards us, the universe also expands. This expansion also results in the stretching of the wavelength of light. As a result, when light reaches us, it has the wavelength,

$$\lambda = \lambda_o$$

How would you relate it with the emmitted wavelength? Remember that the ratio of wavelengths and scale factor remains constant because that's the "comoving wavelength", so we have,

$$\lambda_o = \frac{\lambda_e}{a(t)}$$

and finally, we end up with the familiar looking expression,

$$\frac{\lambda_o}{\lambda_e} = \frac{1}{a(t)} = 1 + z$$

where z is the redshift you encountered in the previous chapter. This is powerful because we haven't considered what happens between these two instances, the scale factor has allowed us to elegantly relate the two wavelengths without carrying out any long calculations such as integrating over the path.

3.2 Velocities in an expanding universe

We would like to calculate the velocity of an object resulting from the expansion of the universe. From the previous part,

$$R = a\chi$$

Taking the time-derivative,

$$\dot{R} = \dot{a}\chi + a\frac{d\chi}{dt}$$

Due to how it's defined, χ is a constant. So we get,

$$\dot{R} = \frac{\dot{a}}{a} \cdot a(t)\chi$$

Where I've multiplied and divided by $a(t)$ on the RHS to get a nicer looking expression.

$$\dot{R} = \frac{\dot{a}}{a} \cdot R$$

Does it look familiar? Of course it does! This is the general form of Hubble's law ($v = H_0 d$), which is valid for present-day calculations. We define,

$$\frac{\dot{a}}{a} = H$$

where H is the Hubble parameter for the universe[1].

[1] Current value, $H_0 = 73 \pm 1$ km/s/Mpc

4 Densities in an expanding universe

We'll be mostly dealing with matter density, radiation density, and dark matter and energy densities (boo!). There are lots more but our universe is/was mostly dominated by these.
We represent these quantities as: ρ_M, matter density, ρ_{rad}, radiation density, ρ_{Λ_M}, dark matter density, and ρ_Λ, dark energy density.

4.1 Matter density

As its name suggests, matter density is simply the density of matter in the universe.
So, intuitively,

$$\rho_M \propto \frac{1}{R^3} \propto \frac{1}{a^3}$$

So, in terms of current matter density, at a particular moment in time, we can write

$$\rho_{t,M} = \frac{\rho_{0,M}}{a(t)^3} = \rho_{0,M} \cdot (1+z)^3$$

Or to put it in words, the matter density falls off by the third power of the scale factor.

4.2 Radiation density

Radiation density is the energy density of photons in the universe. Which is given by,

$$\rho_{rad} = \frac{nhc}{\lambda}$$

Where λ is the wavelength (which we previously showed scales with a), h is Planck's constant, c is the speed of light, and n is the number density of photons in the universe, which as you might've guessed, falls off just like the matter density. So we can write,

$$\rho_{rad} \propto \frac{1}{a^4}$$

And similarly,

$$\rho_{t,rad} = \frac{\rho_{0,rad}}{a(t)^4} = \rho_{0,rad} \cdot (1+z)^4$$

4.3 Spooky densities

Crudely, dark energy is the energy of empty space, and dark matter is well, matter we can't see and therefore can only detect gravitationally and the spooky part is that it doesn't interact with radiation! Explaining dark matter and energy in detail is beyond the scope of this guide so you're expected to just take these results as is (sorry!). You can look into WIMPs and MACHOs for more info.
Let's talk about their scaling relations,

$$\rho_\Lambda = \text{constant}$$

Since dark matter is just a form of matter,

$$\rho_{\Lambda_M} \propto \frac{1}{a^3}$$

$$\rho_{t,\Lambda_M} = \frac{\rho_{0,\Lambda_M}}{a^3} = \rho_{0,\Lambda_M} \cdot (1+z)^3$$

5 Distances in an expanding universe

Because we can measure them, we define two distances in the universe: D_{ang}, known as the angular diameter, and D_L, known as the luminosity distance.

$$D_{ang} = \frac{d}{\theta} = R \cdot a(t)$$

D_L is defined as,

$$D_L = \sqrt{\frac{L}{4\pi F}} = \frac{R}{a(t)}$$

Hence, these distances are usually related as[2],

$$D_{ang} \cdot (1+z) = \frac{D_L}{(1+z)} = R$$

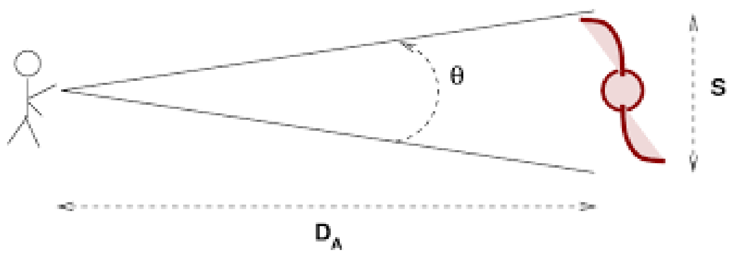

Figure 2: Dang! (Source: CERN)

6 Temperature in an expanding universe

This relation is something that you'd want to derive on your own but I've given the proof here for completeness.
We know,

$$T^4 \propto \rho_{rad} \propto \frac{1}{a^4}$$

Hence,

$$T \propto \frac{1}{a}$$

And we can then write,

$$T_t = \frac{T_0}{a(t)} = T_0 \cdot (1+z)$$

What it also means is that our universe cools down as it expands. How cool!

[2]No proof. Sorry, not sorry. This is cosmology for high-schoolers.

Example 6.1: (IOAA 2008)
The average temperature of the Cosmic Microwave Background is currently $T = 2.73$ K. It yields the origin of CMB to be at redshift $z_{\text{CMB}} = 1100$. The current densities of the Dark Energy, Dark Matter, and Normal Matter components of the Universe as a whole are, $\rho_{DE} = 7.1 \cdot 10^{-30}$ g/cm^3 , $\rho_{DM} = 2.4 \cdot 10^{-30}$ g/cm^3 , and $\rho_{NM} = 0.5 \cdot 10^{-30}$ g/cm^3, respectively. What is the ratio between the density of Dark Matter to the density of Dark Energy at the time CMB was emitted, if we assume that the dark energy is the energy of empty space?

Solution: We know that dark energy density is constant and dark matter density is given by the relation,

$$\rho_{t,\Lambda_M} = \rho_{0,\Lambda_M} \cdot (1+z)^3$$

So the ratio in question is just,

$$\left(\frac{\rho_{\Lambda_M}}{\rho_\Lambda}\right)_{\text{CMB}} = \left(\frac{\rho_{\Lambda_M}}{\rho_\Lambda}\right)_0 \cdot (1+z_{\text{CMB}})^3$$

Which on calculating gives us,

$$\left(\frac{\rho_{\Lambda_M}}{\rho_\Lambda}\right)_{\text{CMB}} \approx 4.5 \cdot 10^8 \;\square$$

7 Friedmann's Equation

I could write the equation directly or give you a bogus Newtonian derivation, you know what, I'll give you a bogus derivation at least you'll remember how to derive in the likely case that you forget the formula in-contest. Alright, roll your sleeves up and consider a particle at the edge of a sphere of radius R. We can write the energy of this particle as,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

where M is the mass of our sphere and v is the velocity of our particle. Let's rewrite this by writing $M = \rho V$ and the expression for velocity we got from subsection 3.2,

$$E = \frac{1}{2}m \cdot \left(\frac{\dot{a}}{a}\right)^2 R^2 - \frac{4\pi G\rho m R^2}{3}$$

We will divide both sides by $\frac{mR^2}{2}$

$$\frac{2E}{mR^2} = \left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho$$

It turns out[3] that we can write the LHS as $\frac{-k}{a^2}$ which is also called the curvature term. And finally, we arrive at Friedmann's equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

We can also expand the density into its constituents and write the equation equivalently as,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_M + \rho_{rad} + \rho_\Lambda) - \frac{k}{a^2}$$

Note that,

$$H = \left(\frac{\dot{a}}{a}\right)$$

where H is the Hubble Parameter as discussed in subsection 3.2

We can also write the densities as a function of initial densities, which gives us,

$$H^2 = \frac{8\pi G}{3}(\rho_{M,0}(1+z)^3 + \rho_{rad}(1+z)^4 + \rho_\Lambda) - \frac{k}{a^2}$$

7.1 Shape of the universe

We said in the previous part that $-\frac{k}{a^2}$ is known as the curvature and rightfully so, because,

$$\begin{aligned} k < 0 &\implies \text{open universe} \\ k = 0 &\implies \text{flat universe} \\ k > 0 &\implies \text{closed universe} \end{aligned}$$

As usual, no proofs.
For the special case of $k = 0$, we define the corresponding density as ρ_{crit}:

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

7.2 Parameters

We define the density parameter Ω as,

$$\Omega = \frac{\rho}{\rho_{crit}}$$

Let's rewrite the Friedmann's equation in terms of Ω

$$H^2 = \frac{8\pi G}{3}\rho_{crit} \cdot \frac{\rho}{\rho_{crit}} - \frac{k}{a^2}$$

$$H^2 = H^2\Omega - \frac{k}{a^2}$$

[3] from General Relativity

Because we like our equations nice and clean, we define a new dummy variable,

$$\Omega_k = -\frac{k}{H^2 a^2}$$

Let's substitute things to get,

$$1 = \Omega + \Omega_k$$

Expand stuff to get,

$$1 = \Omega_M + \Omega_{rad} + \Omega_\Lambda + \Omega_k$$

This is interesting to us because it tells us that any point in time, the sum of the density parameters is just 1.

$$(\Omega_M + \Omega_{rad} + \Omega_\Lambda + \Omega_k)_0 = (\Omega_M + \Omega_{rad} + \Omega_\Lambda + \Omega_k)_t = 1$$

Or,

$$\Omega_k = 1 - \Omega$$

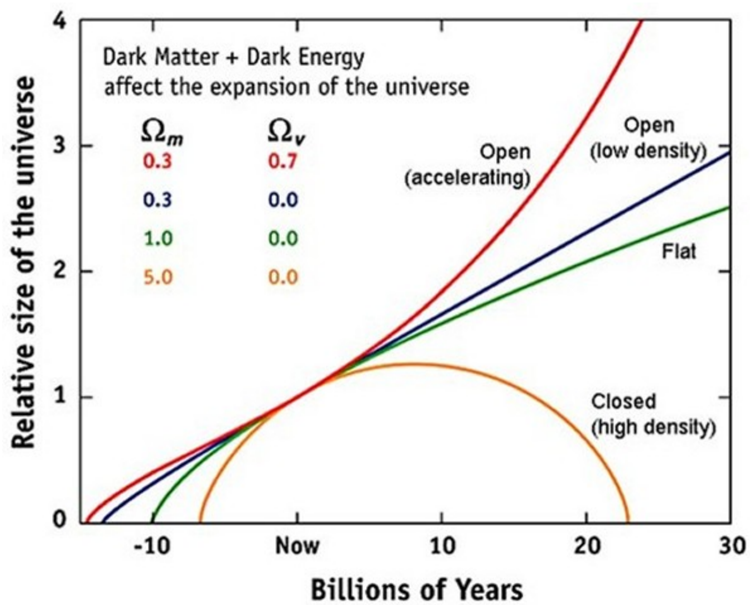

Figure 3: Cosmological society and its future (Source: WMAP-NASA)

Let's do an example problem to make sure you're getting the hang of things.

Example 7.1: (Folklore)
Consider a flat universe with only matter and dark energy. For this universe, write $\Omega_{M,t}$ as a function of $\Omega_{M,0}$ and z

Solution: Because this is a flat universe, we have,

$$\Omega_k = 0$$

and so we can write,

$$\Omega_{M,t} + \Omega_{\Lambda,t} = \Omega_{M,0} + \Omega_{\Lambda,0} = 1$$

we can write $\Omega_{\Lambda,t}$ as,

$$\Omega_{\Lambda,t} = \Omega_{\Lambda,0} \cdot \frac{H_0^2}{H^2}$$

and H^2 as,

$$H^2 = \frac{8\pi G}{3}(\rho_{M,t} + \rho_{\Lambda,t})$$

now we write the densities in terms of initial densities,

$$H^2 = \frac{8\pi G}{3}(\rho_{M,0}(1+z)^3 + \rho_{\Lambda,0})$$

get the Ω in your line,

$$H^2 = H_0^2(\Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0})$$

we substitute this in to get,

$$\Omega_{\Lambda,t} = \Omega_{\Lambda,0} \cdot \frac{1}{\Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0}}$$

and,

$$\Omega_{\Lambda,t} = 1 - \Omega_{M,t} \ ; \ \Omega_{\Lambda,0} = 1 - \Omega_{M,0}$$

Finally,

$$\Omega_{M,t} = \frac{\Omega_{M,0}(1+z)^3}{\Omega_{M,0}(1+z)^3 + 1 - \Omega_{M,0}} \ \square$$

8 Model Universes

In what follows, the universe will be flat. To obtain relations for the scale factor with time, we need to solve the Friedmann equation for some simple cases, that is, cases where a certain type of density is dominant.

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t} \cdot \frac{1}{a}\right) = \sqrt{\frac{8\pi G}{3} \cdot \frac{\rho_0}{a^{3\gamma}}}$$

$$\int a^{\frac{3\gamma}{2}-1}\mathrm{d}a = Ct$$

8.1 Radiation-dominated universe

For radiation,

$$\gamma = \frac{4}{3}$$

and solving the integral gives us,

$$a \propto t^{\frac{1}{2}}$$

8.2 Matter-dominated universe

For matter,

$$\gamma = 1$$

and solving the integral gives us,

$$a \propto t^{\frac{2}{3}}$$

Since, radiation falls off faster than matter, it's fair to assume that a once radiation-dominated universe would turn into a matter-dominated one.

8.3 Dark-energy-dominated universe

For dark-energy,

$$\gamma = 0$$

and solving the integral gives us,

$$a \propto e^{Ct}$$

Which is exponential expansion!
Turns out that this model very accurately models our universe. Our universe is flat, was dominated by radiation after the big bang, then became matter-dominated and is now currently dominated by dark energy.

9 Age of the universe

We know $\frac{\dot{a}}{a}$ for the current day, so we just need to substitute for a and $\dot{a}$, which will allow us to get the age of the universe (started from the bottom now we here?)
I intend to leave the *math* as an exercise for you so I'll just spam the results on your screen. NOW.

Dark-energy-dominated	$t_0 = \frac{1}{H_0}$; also known as Hubble time
Radiation-dominated	$t_0 = \frac{1}{2H_0}$
Matter-dominated	$t_0 = \frac{2}{3H_0}$

Example 9.1: (IOAA 2008)
Consider a type Ia supernova in a distant galaxy which has a luminosity of $5.8 \cdot 10^9 L_\odot$ at its maximum light. Suppose you observe this supernova using your telescope and find that its brightness is $1.6 \cdot 10^{-7}$ times the brightness of Vega. The redshift of its host galaxy is known to be $z = 0.05$. Calculate the distance of this galaxy (in pc) and also the Hubble time. Given $m_\odot = -26.72$

Solution: We know that Vega is our zero-point, so we'll find the brightness of Vega in terms of the Sun.

$$-26.72 - 0 = -2.5 \log\left(\frac{F_\odot}{F_{vega}}\right)$$

Solving, we get

$$F_{vega} \approx 2.09 \cdot 10^{-11} F_\odot$$

Let's substitute Vega's brightness to find the brightness of the supernova in terms of sun's brightness.

$$F_{SN} \approx 3.34 \cdot 10^{-18} F_{\odot}$$

We know the formula for flux,

$$\frac{L_{SN}}{4\pi d_{SN}^2} \approx 3.34 \cdot 10^{-18} \frac{L_{\odot}}{4\pi d_{\odot}^2}$$

we can now solve for d_{SN}.

$$d_{SN} \approx 202 \text{ Mpc}$$

Now, we need to find $\frac{1}{H_0}$, which is not a hard task per se but it does require some unpleasant calculations.

$$v = cz = H_0 \cdot d$$

$$t = \frac{d}{cz} \approx 13.2 \cdot 10^9 \text{ yrs } \square$$

10 Conclusion

You're now set to tackle most problems on Cosmology that appear in the AOs or learn more Cosmology from Intro textbooks by authors like Barbara Ryden or Andrew Liddle.
Cosmology is my favourite part of Astronomy Olympiads and I hope I was able to present it in an attractive and interesting manner.