# Second Order Differential Equations

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# 1 Introduction

Second-order linear differential equations are of the following form.

**Second order Linear Differential Equations:**

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = d(x). \tag{1.1}$$

In this handout, we study **homogeneous equations**, where $d(x) = 0$. We require constant coefficients (i.e. $a(x), b(x), c(x)$ being constants). Several fundamental concepts in physics, such as Newton's equation and the Euler-Lagrange equations, are second-order linear differential equations. For example, the equation of motion for a system consisting of small oscillations in one dimension is $mx''(t) + kx(t) = 0$ which also has constant coefficients and is homogeneous. Thus, it is very important to know how to solve them.

# 2 The Solution

Consider an equation of the form 1.1 with constant coefficients and $d(x) = 0$. Let us write $a(x), b(x)$, and $c(x)$ as $a, b$, and $c$ respectively. We reduce 1.1 into a simpler one by dividing both sides by $a$ to get

$$y''(x) + \frac{b}{a}y'(x) + \frac{c}{a}y(x) = 0. \tag{2.1}$$

For simplicity, rename $b/a$ as $p$ and $c/a$ as $q$. Thus, we need to solve

$$y''(x) + py'(x) + qy(x) = 0. \tag{2.2}$$

We use an **ansatz**, which is essentially a fancy glorified educated guess to solve the problem. Suppose that $y(x) = e^{cx}$ is a solution to the differential equation. Then, we have $y''(x) = c^2e^{cx}$ and $y'(x) = ce^{cx}$. Plugging this into 2.2 gives

$$c^2e^{cx} + p \cdot ce^{cx} + qe^{cx} = e^{cx}(c^2 + pc + q) = 0.$$

We can divide both sides by $e^{cx}$ to get

$$c^2 + pc + q = 0.$$

This is called the **characteristic equation**. We can find $c$ by factoring when possible and using the quadratic equation when it is not. The method from here depends on what type of roots we get and how many there are.

## 2.1 Two Real Roots

If there are two real roots, say, $c_1, c_2$, then the general solution to the differential equation is $y(x) = Ae^{c_1x} + Be^{c_2x}$ where $A, B$ can be any constants. We know from the ansatz that $e^{c_1x}$ and $e^{c_2x}$ work and it turns out that any linear combination of these terms is also a solution.

### 2.2 One Real Root

When there is one real root (i.e. the characteristic equation is of the form $(c-k)(c-k)$), then the general solution is of the form $y(x) = Ae^{kx} + Bxe^{kx}$.

### 2.3 Two Complex Roots

In this case, we get two complex roots $z_1$ and $z_2$ and plug them in the same way as if we had gotten two real roots to get a general solution of y(x)=$Ae^{z_1x} + Be^{z_2x}$. From here, we can use Euler's formula,

$$e^{ix} = \cos(x) + i\sin(x),$$

to simplify.

## 3 Examples

We compute one example of each type of solution.

**Example 1:** Solve the equation

$$2y''(x) + 16y'(x) + 30y(x) = 0.$$

**Solution:** We first divide both sides by 2 to get

$$y''(x) + 8y'(x) + 15y(x) = 0.$$

The characteristic equation we get is $(c^2 + 8c + 15) = 0$. We can factor the left hand side to get $(c+5)(c+3) = 0$. This implies that the solutions to the characteristic equation are $c = -5$ and $c = -3$. Thus, the general solution to the differential equation is

$$y(x) = Ae^{-5x} + Be^{-3x}.$$

**Example 2:** Solve the equation

$$y''(x) - 8y'(x) + 16y(x) = 0.$$

**Solution:** The characteristic equation is $(c^2 - 8c + 16) = 0$. We can factor the left hand side to get $(c-4)(c-4) = 0$ which implies that there is one real solution, namely $c = 4$. Thus, the general solution to the differential equation is

$$y(x) = Ae^{4x} + Bxe^{4x}.$$

**Example 3:** Solve the equation

$$y''(x) + 4y'(x) + 5y(x) = 0.$$

**Solution:** The characteristic equation is $(c^2 + 4x + 5) = 0$. We see that the discriminant of this equation is $4^2 - 4 \cdot 1 \cdot 5 < 0$ so the equation has two imaginary roots. Using the quadratic formula, we see that the roots of the equation are $z_1 = -2 + i$ and $z_2 = -2 - i$. So, the general solution to the differential equation is

$$y(x) = Ae^{(-2+i)x} + Be^{(-2-i)x}.$$

We can factor out $e^{-2x}$ to get

$$y(x) = e^{-2x}(Ae^{ix} + B^{-ix}).$$

Applying Euler's formula to the terms in parenthesis gives

$$Ae^{ix} + B^{-ix} = A(\cos(x) + i\sin(x)) + B(\cos(-x) + i\sin(-x)).$$

We can use the trigonometric identities $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ to get

$$Ae^{ix} + B^{-ix} = A(\cos(x) + i\sin(x)) + B(\cos(x) - i\sin(x)) = (A + B)(\cos(x)) + i(A - B)(\sin(x)).$$

We can see that $A + B$ and $A - B$ are also constants so we can substitute $c_1$ and $c_2$ for them respectively to get the general solution

$$y(x) = e^{-2x}(c_1\cos(x) + ic_2\sin(x)).$$

We leave the following example as an interesting, real-world occurring, exercise.

**Example 4:** Solve the differential equation for springs mentioned in the introduction. After you have done this, solve the differential equation for when damping is present: $mx''(t) + cx'(t) + kx(t) = 0$.