Intuition Behind Integrals

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1 Introduction

Integrals are often seen in higher physics, and might look daunting to the uninformed. However, they are a crucial in developing theories that accurately model the world - let's take a look at how to get familiar with them!

2 Definition & Intuition

When you first learn integration in high school, you might have merely been introduced to it as a mathematical operator - something that inputs a function, and spits out another function. However, that severely undermines the ideas that lie beneath it!

Take a look at the integration symbol:

$$\int$$

It looks like a weird "S", and the reason for that is because it stands for "sum".

Sometimes, you may see numbers on the top and bottom of the symbol:

$$\int_a^b$$

This means to add from the lower bound "a" to the upper bound "b". With bounds, the integral is called a **definite integral**, because it has definite endpoints. Without them, it would be called an **indefinite integral** for the same reason.

All of the time, you would see the term "dx" in the integral:

$$\int_a^b \mathrm{d}x$$

The term dx is called a **differential element** - you might have encountered it when learning differentiation, written as

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

The idea is to interpret the derivative operator as a **fraction** (ignore the wailing mathematicians) - the reason to motivate this perspective can be seen from the definition of a derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If you've seen this before, great! If not, even better! Because seeing derivatives as the limit of a fraction is a great way to gain intuition behind this mathematical machinery.

Going back to integrals, what would the dx in the integral mean? If we take dx to mean the same thing as in derivatives, it would mean that it is something that is very very small in the limit - which is precisely what it is! An integral is nothing but a summation of infinitely small things. To a sprinkle of mathematical rigor, the definition of a definite integral is

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{k=0}^{n} f(x_k)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_k = a + k \cdot \Delta x$.

Now that might seem a lot to take in - where did the term $f(x)$ come from, and since when did infinity join the party?! Let's break down each part one by one.

The reason we put $f(x)$ inside the integral is because most of the time we want to add something that varies with x - if x denotes time, this might be something like velocity, for example, where the velocity might not be constant in time but we want to find where something is at the end of the interval. To quantify how velocity would vary with time, we write it as a function of time, specifically $f(x)$.

Now, on the right side of the definition, the word "lim" refers to a **limit**, where we take something to one extreme, in this case to infinity (because infinity is not a number, but a concept). In this case, we sum up infinitely many terms - but what do we sum up?

The summands are the function in question that we want to integrate $f(x)$, multiplied by a finite interval Δx. Again, think back to the kinematics example - if x were to denote time, and $f(x)$ to denote velocity, then we would be adding velocity $\times$ time, which is just displacement. Notice that we are evaluating the velocity at different times x_k, where x_k would slowly increment from $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, all the way up to $x_n = b$.

Now, one problem that might arise is: what about the velocities in between x_0 and x_1? Do we not account for them when computing the total displacement? Well, yes we would: when we take the limit to infinity, the intervals Δx would tend towards zero, hence there would ideally be no gaps between x_0 and x_1 in the limit, and all velocities would be accounted for.

You can play around with the idea over at this Desmos plot - do play around with the sliders and the function!

3 Examples & Applications

3.1 Integrals as Areas

Arguably the most popular application of integrals in math courses is to find areas under certain graphs - the reason integrals can directly correspond to areas is shown in the Desmos plot above: integrals help break down the graph into infinitely many small rectangular chunks, which when added together gives the area. One thing to note is that the areas are **signed**, that is, there are **positive and negative areas**, highlighted as red and green in the Desmos plot linked earlier.

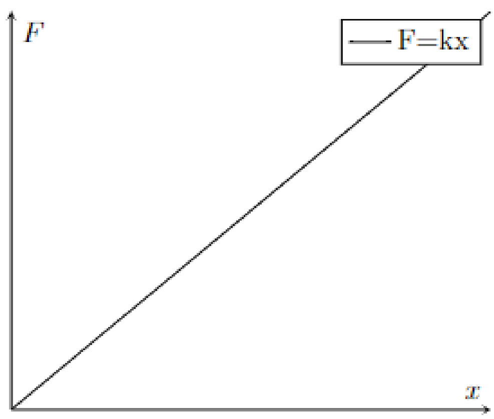
A picture is worth a thousand words, and graphs are undeniably useful in physics as they can give us lots of information, from the behaviour of the function, to its gradient, and the most relevant one: the area enclosed under it.

Areas under curves crop up a lot in physics: from simple scenarios like deriving the formula for the energy of a spring to more advanced cases like Maxwell's equal area construction in thermodynamics.

Let's look at the simpler example of the energy of a spring system. By Hooke's Law, we know that the force exerted by a spring is

$$\vec{F} = -k\vec{x}$$

with the negative sign indicating the opposite direction. Plotting its absolute values on a graph, we see a linear function:



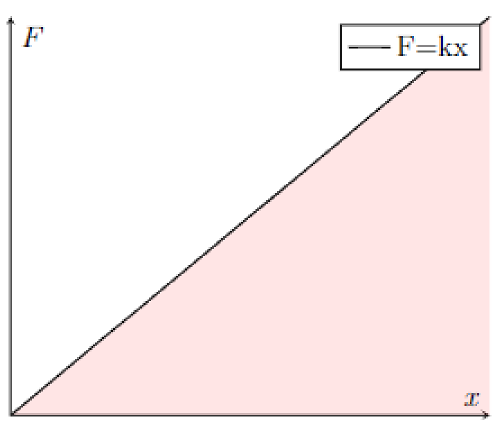
By definition, potential energy of a system at some state is the amount of energy put in to bring the system to that state. The infinitesimal work done when the spring is stretched by a miniscule length dx is

$$dU = -Fdx = kxdx$$

Hence, when the spring is stretched from $x = 0$ to $x = x$, the energy that needs to be put in is

$$U = \int_{x=0}^{x=x} dU = \int_0^x kx \cdot dx$$

Graphically, this corresponds to the area under the linear function.



This gives the result

$$U = \frac{1}{2}kx^2$$

which is the familiar expression for the energy of a spring.

3.2 Adding changes

Another way to view integrals is that they add together infinitely many tiny changes. Kinematics provide a great example of this: by definition, velocity is

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$

(i.e. the infinitesimal change in displacement $\mathrm{d}x$ divided by an infinitesimal interval of time $\mathrm{d}t$).

To find the total displacement, we can add together infinitely many tiny displacements with an integral:

$$x = \int \mathrm{d}x$$

And, with the definition of velocity, we get

$$x = \int v \cdot \mathrm{d}t$$

This is useful because, if we know how velocity varies with time, i.e. $v(t)$, then we can precisely know how much the object has moved.

3.3 Infinitesimal Superposition

One more important place where integrals pop up is **superposition**. (feel free to come back to this later when you've learned electrostatics)

Namely, in EM problems, we often have a charge distribution instead of point charges. As an example, let's consider an infinitely long line of charge with linear charge density λ, i.e. for each segment Δl of the line there is $\Delta Q = \lambda \Delta l$ of charge. Let's say we want to know the electric potential at some point in space with perpendicular distance to the line d - how do we compute that?

For the case of one point charge Q, we know that the electric potential is simply

$$\phi = \frac{kQ}{r}$$

but how do we extend this to an infinitely long line of charge?

We need the **principle of superposition**. The idea says that potentials can be added linearly, i.e. $\phi_{total} = \phi_1 + \phi_2 + ...$

With this, we can break up the infinite line into infinitesimal segments of dx. If we do this, then each segment contributes

$$\mathrm{d}\phi = \frac{k \cdot \mathrm{d}q}{r}$$

Since the line is infinite, we can choose some point on the line as the origin of our coordinate system, which we'll choose to be the point right under the point of concern with perpendicular distance d.

$$\mathrm{d}\phi = \frac{k \cdot \lambda \mathrm{d}x}{\sqrt{x^2 + d^2}}$$

With this, we can write the total electric potential as:

$$\phi = \int \mathrm{d}\phi = \int_{x=-\infty}^{x=+\infty} \frac{k\lambda \mathrm{d}x}{\sqrt{x^2 + d^2}}$$

which one can compute with, say, a table of integrals.

4 Problems & Practices

4.1 Problem 1

Problem 1: An ideal gas is a volume of gas which obeys the equation of state

$$pV = nRT$$

where p is the pressure, V is the volume, n is the number of moles, T is the temperature in Kelvins and R is the molar gas constant. What is the total work done by the gas when it expands isothermally from V_i to V_f?

Hint: Isothermal means that the temperature T is constant, and when an ideal gas expands by dV, it does work $W = pdV$ on its surroundings.

4.2 Problem 2

Problem 2: Derive the expression for the gravitational potential energy of a mass m, if the expression for the gravitational force is

$$\vec{F} = \frac{GMm}{r^2}(-\hat{r})$$

where M and m are masses and $\vec{r}$ is a vector pointing from the center of one mass to the other.

Note that the energy is defined to be zero at $r = \infty$.

Hint: Use the definition of work $dW = -dU = \vec{F} \cdot \vec{dr}$

4.3 Problem 3

Problem 3: Find the resistance of an Ohmic spherical shell with and resistivity ρ, inner radius a and outer radius b.

Hint: the formula for resistance is $R = \frac{\rho L}{A}$, where L is the length of the resistor and A the cross-sectional area. Resistors in series obey the principle of superposition.

4.4 Problem 4

Problem 4: This problem deals with infinitesimals. Convince yourself that the following identity is true:

$$\ddot{x} = \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}$$

4.5 Problem 5

Problem 5: In rotational mechanics, instead of using mass as a quantity for inertia, we use a quantity called the **moment of inertia**, often denoted I.

When a point mass m is spinning about some axis, its moment of inertia is

$$I = mr^2$$

where r is the perpendicular distance to the axis of rotation. And, just like mass in translational mechanics, the moment of inertia is additive.

With this information, derive the moment of inertia of a thin rod of length L and mass m spinning about its center.

4.6 Problem 6

Problem 6: The coordinates that we are familiar with are the xy coordinate system, also known as **rectangular coordinates**.

When we want to find the area of some patch of the grid, we can write an infinitesimal patch of area as

$$dA = dx \cdot dy$$

and integrate over the region of interest.

Another useful coordinate system is called **polar coordinates**. In this system, points in space are denoted with (r, θ), where r is the distance from the origin, and θ the angle from some fixed axis.

Derive the formula for an infinitesimal patch of area dA in polar coordinates.

Hint: the area of a circular sector is $A = \frac{1}{2}r^2\theta$

4.7 Problem 7

Problem 7: When water in a container is rotated, its shape changes and curves at the edges. In this problem we derive the shape of the surface.

Consider a cylindrical container, rotating about its center. We let our origin be at the axis of rotation, at the cylinder's base. Let the density of water be ρ, gravitational acceleration be g, and the angular velocity of the rotation be ω.

(a) Consider an infinitesimal blob of water at the base of the cylinder, and at some distance r from the axis of rotation. Write down the force balance equation for the blob to be in circular motion in the container.

Hint: think pressures!

(b) Derive the pressure gradient, $\frac{\mathrm{d}p}{\mathrm{d}r}$ as a function of the distance r.

(c) From this, derive the gradient of the surface of the water, $\frac{\mathrm{d}y}{\mathrm{d}r}$, if the surface of the water has the shape y as a function of r.

(d) Derive the shape of the water surface from the result of (c).

Note: the result of d) implies that the surface of the liquid has a very well defined focal point - telescopes have been built by rotating liquids like this (**liquid-mirror telescopes**), like the Large Zenith Telescope!

5 Solutions

5.1 Problem 1

$$W = nRT \ln \frac{V_f}{V_i}$$

5.2 Problem 2

$$U = \int \mathrm{d}U = -\int_{-\infty}^{r} \frac{GMm}{r^2} - \mathrm{d}r = -\frac{GMm}{r}$$

5.3 Problem 3

$$R = \int_a^b \frac{\rho \mathrm{d}r}{4\pi r^2} = \frac{\rho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right)$$

5.4 Problem 4

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}$$

5.5 Problem 5

$$I = \int_{-L/2}^{L/2} r^2 \mathrm{d}m = \int_{-L/2}^{L/2} x^2 \frac{M}{L} \mathrm{d}x = \frac{1}{12} ML^2$$

5.6 Problem 6

$$\mathrm{d}A = \frac{1}{2} r^2 \mathrm{d}\theta$$

5.7 Problem 7

a)

$$p(r + \mathrm{d}r)\mathrm{d}A - p(r)\mathrm{d}A = r\omega^2\mathrm{d}m$$

b)

$$p(r + \mathrm{d}r)\mathrm{d}A - p(r)\mathrm{d}A = r\omega^2\rho\mathrm{d}A\mathrm{d}r$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \rho\omega^2 r$$

c)

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\rho g\mathrm{d}y}{\mathrm{d}r}$$

$$\frac{\mathrm{d}y}{\mathrm{d}r} = \frac{\omega^2}{g}r$$

d) Integrating (c) gives

$$y = y_0 + \frac{\omega^2}{2g}r^2$$

6 Conclusion

Being acquainted with calculus-based physics is one of the big stepping stones on your way to becoming a theoretical physicist - congratulations on making it through this handout!