Minima and Maxima

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1 Introduction

Whether it is finding the highest point of a projectile or calculating the quickest path light can take through two different mediums, knowing how to find the extrema of a function is crucial in physics. In this handout we will cover how to find and classify local and global extrema of functions.

2 Critical Points

Suppose we want to find the maximum/minimum value of $f(x)$ on a certain interval. Then we have the following conditions:

Critical Points: The minimum or maximum values of $f(x)$ on the interval $[a, b]$ can only occur at one of the following:

- Points where $f'(x) = 0$
- Points where $f'(x)$ is undefined
- At $x = a$ or $x = b$. That is, the endpoints of the interval.

Intuitively, this should make sense: at a maximum the function must go up and down a "hill" so we should have some point where the slope of the tangent line is zero (condition 1). Sometimes, we have a sharp corner at the top of the hill, and the derivative is undefined (2). Also, the function can just be constantly increasing in which case the maximum is just an endpoint (3).

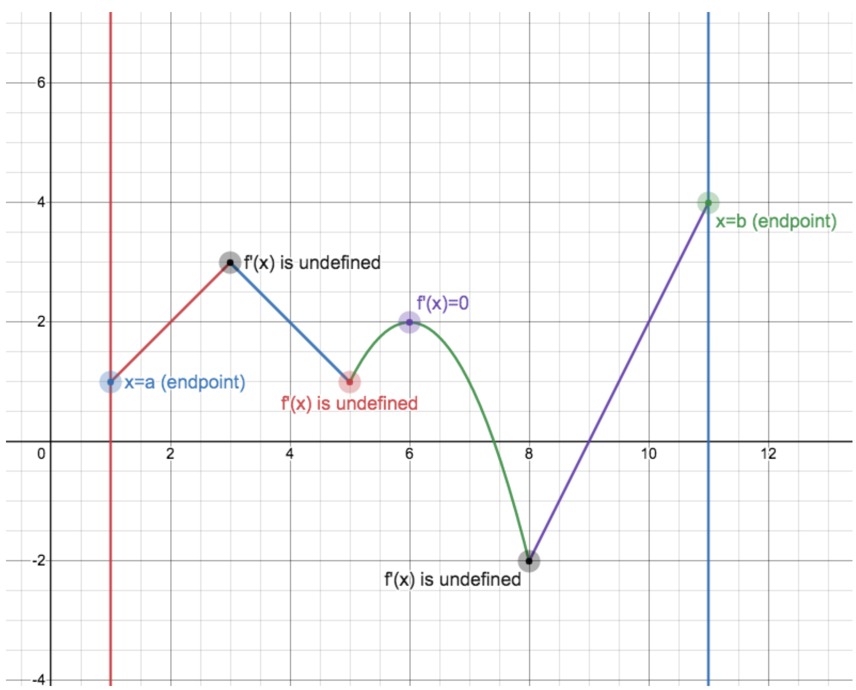

Figure 1: examples in a graph

Notice that these conditions are *not sufficient*. That is, just because an x value satisfies these conditions does not mean it is a minimum/maximum. Notice this is evident in our graph. Clearly, not all of the marked points are minimums or maximums! Let's try a few examples:

Example 1: Find the critical points of $f(x) = x^2 - 6x + 1$ on the interval $[2, 5]$.

Solution. Let's go through the conditions one by one. We see that $f'(x) = 2x - 6$ so $f'(x) = 0$ at $x = 3$. Clearly, $f'(x)$ is always defined so we don't have to worry about condition two. For condition three, simply take the endpoints $x = 2$ and $x = 5$. Hence, the critical points are at $\boxed{x = 2, 3, 5}$.

Example 2: Find the critical points of the following function on the interval $[1, 12]$:

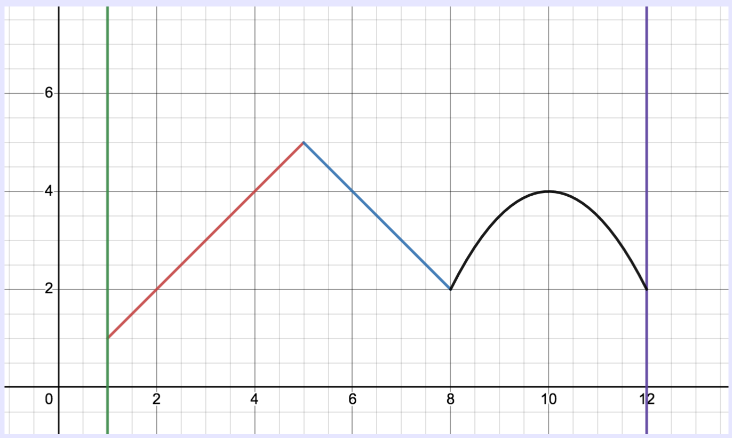

Figure 2: example 2

Solution. We look at each of the conditions:

- For condition one, we have $x = 10$ only.
- For condition two, we have $x = 5, 8$.
- For condition three, we have $x = 1, 12$.

3 Local and Global Extrema

3.1 Definitions

We have a ***local* minimum** of a function at $x = c$ if $f(c) \leq f(x)$ for all x in *some* open interval (a, b).

For example, in example 2 we have a local minimum at $x = 8$ since it is the minimum of $f(x)$ on the interval $(6, 10)$. On the other hand, there is no local minimum at $x = 6$ since there is no open interval such that $x = 6$ is the minimum of $f(x)$.

On the other hand, we have a ***global* minimum** of an interval $[a, b]$ at $x = c$ if $f(c) \leq f(x)$ for all x in $[a, b]$.

For example, for the interval $[7, 10]$ the global minimum is at $x = 8$. Notice global minimums are by definition local minimums. We can define local/global maximums similarly.

3.2 How to Find Global Extrema

Since we know how to find critical points, we can automatically find global extrema. How?

To find the global maximum/minimum in an interval we can evaluate all the critical points and see which one is largest/smallest.

Sometimes, we only have one local minimum/maximum in an interval. Then, this point must automatically be the global minimum/maximum in the interval. Can you see why by sketching an example?

4 Classifying Extrema

We have covered how to identify where potential extrema are. In this section, we find out whether a point is a local maximum or minimum.

4.1 First Derivative Test

First Derivative Test, formal: Let f be a continuous function and suppose c is a real number with $f'(c) = 0$ or $f'(c)$ undefined.

- If there exists an open interval (a, b) containing c such that f is decreasing on (a, c) and increasing on (c, b) then $x = c$ is a local minimum.
- If there exists an open interval (a, b) containing c such that f is increasing on (a, c) and decreasing on (c, b) then $x = c$ is a local maximum.

Here, decreasing and increasing are *not* strict.

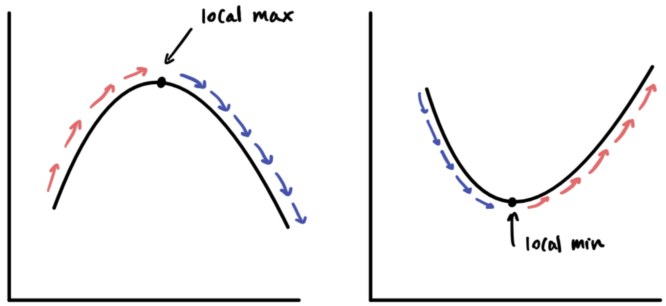

Figure 3: first derivative test, red denotes increasing and blue denotes decreasing

In practice, we just take one point to the left of $x = c$ and one point to the right of $x = c$. If the derivative of the right point is non-positive (≤ 0) and the derivative of the left point is non-negative (≥ 0) we have $x = c$ as a local maximum and vice versa for local minimums.

It is worth noting that the above method does not always work if we take the right and left points too far from $x = c$, so the method is not completely rigorous. In practice, it should work most of the time for simple functions if we take points sufficiently close, though.

Example 3: For the function $f(x) = x^2$ is $x = 0$ a local minimum or maximum?

Solution: First, we know $f'(x) = 2x$. Clearly, $f'(0) = 0$. Take a point to the left of 0, say -1, and we find $f'(-1) = -2 \leq 0$. Take a point to the right of 0, say 2, and we find $f'(2) = 4 \geq 0$ so $x = 0$ is a local minimum by the first derivative test.

4.2 Second Derivative Test

Second Derivative Test: Suppose $f'(c) = 0$. Then,

- If $f''(c) > 0$ then $x = c$ is a local minimum.
- If $f''(c) < 0$ then $x = c$ is a local maximum.

This should make sense intuitively as concave functions (those with $f''(x) > 0$) are shaped like a "U" so we will have a minimum. Conversely, convex functions (those with $f''(x) < 0$) are shaped like an upside down "U" so we will have a maximum. Also, if $f''(c) = 0$ we cannot make any conclusions about c.

Example 4: Is $x = 3$ a local minimum or maximum for the function $f(x) = -x^2 + 6x$?

Solution: We find $f'(x) = -2x + 6$ so $f'(3) = 0$. Also, $f''(x) = -2$ so $f''(3) = -2 < 0$. Hence, we have $x = 3$ is a local maximum by the second derivative test.

5 Practice Problems

Problem 1

When is $f(x) = 2x^3 - 15x^2 + 36x$ maximized between $x = 0$ and $x = 3.4$?

Problem 2

A baseball is thrown straight up into the air with initial velocity v. Find the maximum height of the baseball. Assume air resistance is negligible.

Problem 3

Walt is selling water. If he sells each gallon at a price of x dollars, Walt expects to sell $10 - x^2$ gallons of water. What is the value of x where Walt's profits are maximized? (Note: x can be any non-negative real number)

Problem 4

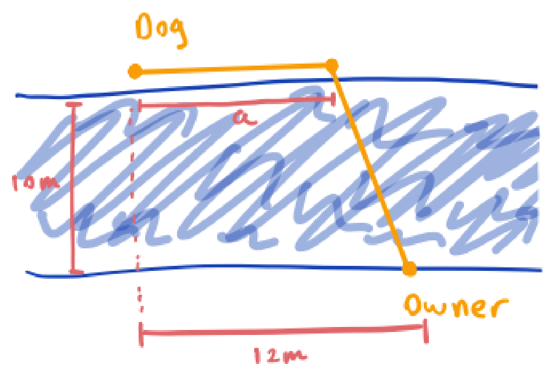

fig 5. problem 4

As shown in the figure, a dog is trying to get back to her owner, who is across the river. The river is 10 meters wide and the dog is 12 meters downriver from her owner. If the dog runs along the shore for a meters at 5 meters per second then swims directly to her owner at 2 meters per second, which value of a minimizes the time spent getting to her owner? (Note: this problem is computationally heavy so the use of a calculator is permitted)

6 Solutions to Practice Problems

Solution 1

Note $f'(x) = 6x^2 - 30x + 36 = 6(x-3)(x-2)$. We find the critical points to be $x = 0, 2, 3, 3.4$ as $f'(x)$ is always defined. Testing all these values, we see that we have the largest value at $\boxed{x = 2}$. Note we can see $x = 2$ is a local maximum by the first or second derivative test.

Solution 2

By a basic formula from kinematics, the y coordinate of the baseball can be written as

$$f(t) = vt - \tfrac{1}{2}gt^2$$

at a given time t, where $g = 9.81$ m/s^2. Taking the derivative, we see $f'(t) = v - gt$ so f is maximized at $t = v/g$. This is because we can check $f'(t) > 0$ for $t < v/g$ and $f'(t) < 0$ for $t > v/g$ so $t = v/g$ is an global maximum by the first derivative test. Hence, the maximum height of the baseball is

$$f\left(\frac{v}{g}\right) = \frac{v^2}{g} - \frac{g}{2} \cdot \frac{v^2}{g^2} = \boxed{\frac{v^2}{2g}}$$

Solution 3

Walts's profits can be calculated by the number of gallons of water he sells times the price of each gallon, which is $f(x) = x(10 - x^2)$. Since $f'(x) = 10 - 3x^2$ we have $x = \pm\sqrt{10/3}$ as critical points. Clearly, $x \geq 0$ so we can discard the negative root.

Since $f''(x) = -6x$ we have $f''(\sqrt{10/3}) < 0$ so $x = \sqrt{10/3}$ is a local maximum by the second derivative test. Note only one local extrema point in the interval $[0, \infty)$ so $\boxed{x = \sqrt{10/3}}$ is value of x where Walt's profits are maximized.

Solution 4

First, note a is bounded by 0 and 12. We calculate the time the dog takes to travel the two segments of her path. For the segment on land, she takes $a/5$ seconds. The length of the segment in water is $\sqrt{(12-a)^2 + 10^2}$ by the Pythagorean theorem, so the time taken is $\frac{1}{2}\sqrt{(12-a)^2 + 10^2}$ seconds. Hence, it suffices to minimize

$$f(a) = \frac{a}{5} + \frac{\sqrt{(12-a)^2 + 10^2}}{2} = \frac{a}{5} + \frac{\sqrt{244 - 24a + a^2}}{2}$$

Taking the derivative using Chain Rule,

$$f'(a) = \frac{1}{5} + \frac{-24 + 2a}{4\sqrt{244 - 24a + a^2}}$$

Setting this equal to zero, we have

$$\begin{aligned} \frac{1}{5} = \frac{24 - 2a}{4\sqrt{244 - 24a + a^2}} &\implies 2\sqrt{244 - 24a + a^2} = 5(12 - a) \\ &\implies 4(244 - 24a + a^2) = 25(12 - a)^2 = 25(144 - 24a + a^2) \\ &\implies a \approx 7.64, 16.36 \end{aligned}$$

In the interval $[0, 12]$ the local extrema is at $\boxed{a = 7.64}$. We now must verify that $a = 7.64$ indeed yields a local minimum, which would finish as there is only one local minimum/maximum. We can plug in $a = 7.7$ and $a = 7.6$ and check $f'(7.7) > 0$ and $f'(7.6) < 0$, which implies the desired result by the first derivative test.