L'Hôpital's Rule

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Table of Contents

1 Introduction

L'Hôpital's Rule is an important technique that can be used to make our lives easier when dealing with certain forms of tricky limits – those that evaluate to indeterminate form.

2 Identifying Indeterminate Form

2.1 Indeterminate Form

Indeterminate form applies when evaluating a limit leads to an expression that cannot be truly evaluated. While indeterminate form is stated singularly, there are actually many different kinds of limits that satisfy this categorization.

2.1.1 Quotient Forms

Quotient forms are the most common type of indeterminate form. These include the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

2.1.2 Other Forms

- **Product Form**: characterized by $0 \cdot \infty$ (e.g. $\lim_{x \to 0^+} x \ln x$)
- **Difference Form**: characterized by $\infty - \infty$ (e.g. $\lim_{x \to 0^+} \frac{1}{x^2} - \frac{1}{\tan x}$)
- **Power Form**: characterized by ∞^0, 0^0, and 1^∞ (e.g. $\lim_{x \to \infty} x^{\frac{1}{x}}$)

2.2 Getting to Quotient Form

To be useful for L'Hôpital's Rule, it is necessary to get a limit from whatever form it is in to begin with into some sort of quotient. As there is no hard-and-fast rule to do this, some examples will be presented in Section 3.

3 Applying L'Hôpital's Rule

Applying L'Hôpital's Rule is relatively straightforward once the limit has been identified and manipulated to be in quotient form. The rule is defined with a limit in the following form:

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

If the fraction $\frac{f(x)}{g(x)}$ evaluates to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the limit can be properly evaluated by replacing the quotient with

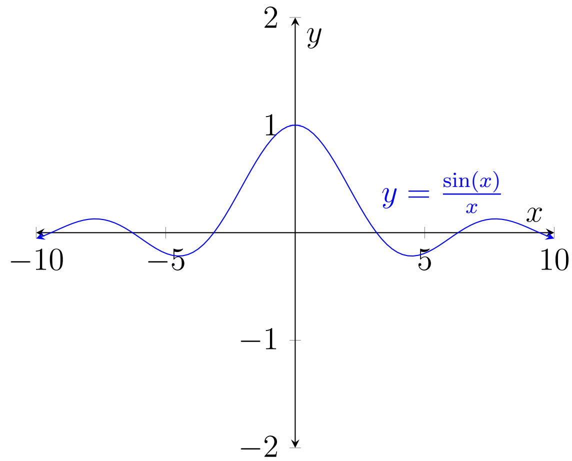
Definition of L'Hôpital's Rule

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

If this results in indeterminate form yet again, L'Hôpital's Rule can be used repetitively.

3.1 Example: $\lim_{x \to 0} \frac{\sin(x)}{x}$

A common and straightforward example to demonstrate a use of L'Hôpital's Rule can be found in $\frac{\sin(x)}{x}$.



As can be seen from the graph, $\lim_{x \to 0} \frac{\sin(x)}{x}$ is clearly 1. However, when evaluating the limit algebraically, it can be seen that both $\sin(x)$ and x evaluate to 0, leaving the indeterminate form $\frac{0}{0}$. Armed with the knowledge of L'Hôpital's Rule, we can set $f(x)$ to $\sin(x)$ and $g(x)$ to x. Evaluating $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ leaves us with a limit $\lim_{x \to 0} \frac{\cos(x)}{1} = 1$, agreeing with the graphical approach. This example represents an extremely common and invaluable use case of L'Hôpital's Rule.

3.2 Example: $\lim_{x \to \infty} \frac{3x+5}{2x+1}$

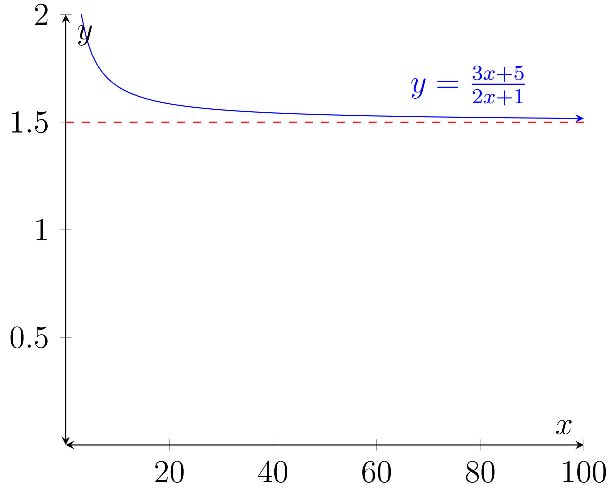
Another relatively common example is having to evaluate a limit as x approaches ∞. In this case, we have the limit

$$\lim_{x \to \infty} \frac{3x + 5}{2x + 1}$$

Applying L'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{3}{2}$$

There is nothing left to plug in, and the limit therefore evaluates to $\frac{3}{2} = 1.5$.
Let's take a look at the graph to see this visually:



3.3 Example: $\lim_{x \to 0^+} x \ln x$

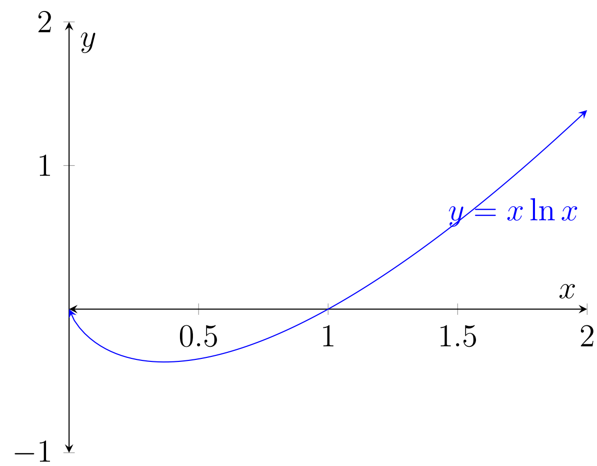
An example for the form $0 \cdot \infty$ can be found in $\lim_{x \to 0^+} x \ln x$. We can rearrange this into a quotient by dividing by $\frac{1}{x}$ instead of multiplying by x:

$$\lim_{x \to 0^+} \frac{\ln x}{1/x}$$

The derivative of the numerator is $\frac{1}{x}$ and that of the denominator is $-\frac{1}{x^2}$. Applyling L'Hôpital's Rule, the limit becomes

$$\lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$
$$\lim_{x \to 0^+} -x$$
$$= 0$$

Therefore, it can be seen the limit evaluates to 0. Let's confirm graphically:



3.4 Example: $\lim_{x\to 0^+} \frac{1}{x^2} - \frac{1}{\tan x}$

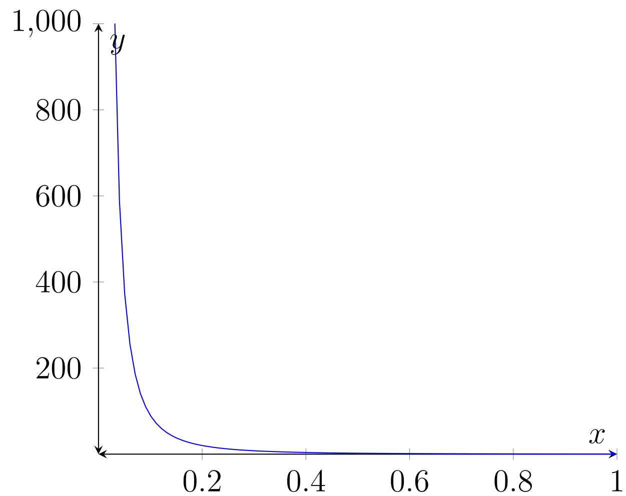
In this case we will be evaluating a limit of the indeterminate form $\infty - \infty$: $\lim_{x\to 0^+} \frac{1}{x^2} - \frac{1}{\tan x}$. As x goes to 0 both denominators do as well, and since the numerator is nonzero both terms approach ∞, requiring another method – L'Hôpital's Rule. The limit must first be converted to a quotient, which can be done by multiplying each term by the other term's denominator:

$$\lim_{x\to 0^+} \frac{\tan(x) - x^2}{x^2 \tan(x)}$$

Taking the necessary derivatives:

$$\lim_{x\to 0^+} \frac{\sec^2(x) - 2x}{x^2 \sec^2(x) + 2x\tan(x)}$$

As x approaches 0, $\sec^2 x$ approaches 1, and $2x$ approaches 0, leaving the numerator to approach 1 as a whole. The denominator, however, approaches 0, leaving the final limit to approach ∞. Again, lets confirm this graphically:



3.5 Example: $\lim_{x\to\infty} x^{\frac{1}{x}}$

This is kind of limit is a bit more complex to evaluate. When we initially attempt to evaluate it algebraically, we get ∞^0, which has no definite solution. To apply L'Hôpital's Rule, we must first create a quotient. This can be done by taking the natural log:

$$\ln x^{\frac{1}{x}} = \frac{\ln x}{x}$$

However, we have now modified the original limit. We can use a simple characterstic of continuous functions

$$\lim_{x\to a} \ln f(x) = \ln(\lim_{x\to a} f(x))$$

In this case $f(x)$ is the original expression, $x^{\frac{1}{x}}$. The previous equation then evaluates to

$$\lim_{x\to\infty} \frac{\ln x}{x} = \ln(\lim_{x\to\infty} x^{\frac{1}{x}})$$

Applying L'Hôpital's Rule:

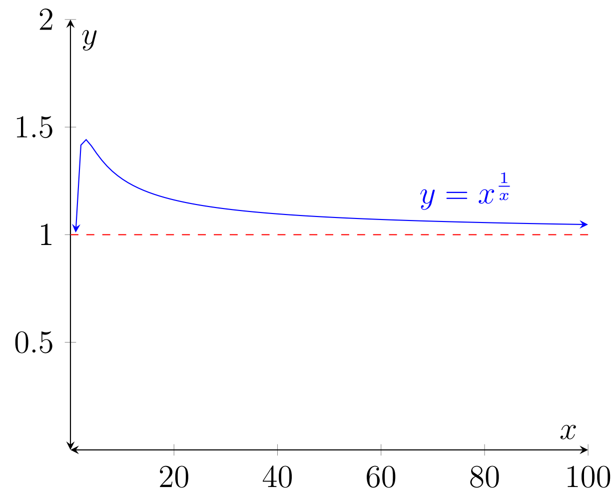
$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Using the previous equation:

$$\ln(\lim_{x\to\infty} x^{\frac{1}{x}}) = 0$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = 1$$

Let's confirm this graphically:



3.6 Proof

We can begin with a simple quotient limit:

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

If the limit evaluated algebraically leads to the form $\frac{0}{0}$ then $f(a) = g(a) = 0$.

$$= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

The numerator and denominator can now be seen to be the difference quotients $f'(x)$ and $g'(x)$. This leaves the final result when $\lim_{x\to a} \frac{f(x)}{g(x)}$ evaluates to indeterminate form to be:

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

3.7 Shortcomings

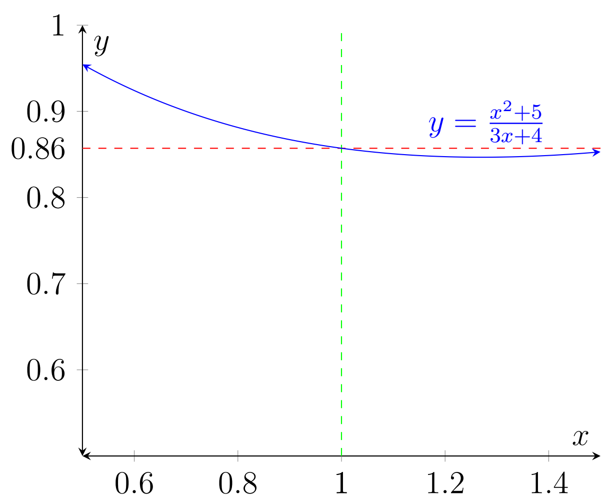
Let's take an example where the result is not indeterminate form to prove that L'Hôpital's Rule works exclusively when indeterminate form is encountered.

$$\lim_{x\to 1} \frac{x^2+5}{3x+4}$$

When evaluating algebraically, the result comes out to $\frac{6}{7}$. Let's see what happens if we try to apply L'Hôpital's Rule.

$$\lim_{x\to 1} \frac{2x}{3}$$

When attempting to evaluate this, we get a different result of $\frac{2}{3}$, which is clearly incorrect if it is continuous at $x = 1$. This becomes even more apparent when looking at the graph:



4 Conclusion

L'Hôpital's Rule is an extremely useful tool which allows to solve many more limits than would've been possible without it. It's simplicity further contributes to it being a useful part of our calculus arsenal.

5 Afterword

Now that you know all there is to know about L'Hôpital's Rule, please enjoy this comic I made sophomore year:

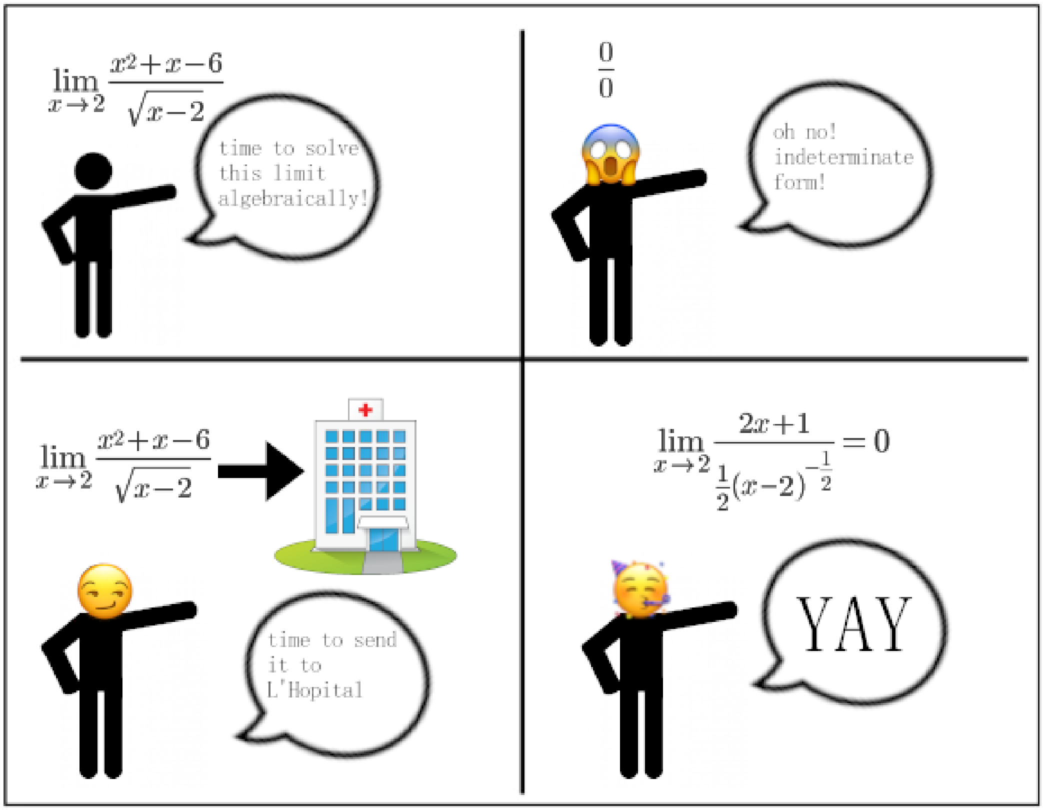


Figure 1: Cool guy saved by awesome application of L'Hôpital's Rule.