Chain Rule

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1 Introduction

How do we take the derivative of sin(3x)? We know how to take the derivative of sin(x), so it may seem like sin(3x) would just be cos(3x) - but it's not. Instead, we have to apply the chain rule. The chain rule is a simple yet incredibly important rule in calculus that allows us to take the derivative of more advanced functions more easily.

At its most basic level, the **chain rule** states that if:

$$f(x) = g(h(x))$$

then:

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = g'(h(x))h'(x)$$

In words, this means that the derivative of a function f that equals g of h of x, where h and g are any function, then the derivative of f, f prime of x, equals g prime of h of x, or h(x) plugged into the derivative of the function g times the derivative of the function h evaluated at x. This may initially seem strange or abstract, and makes more sense with an example.

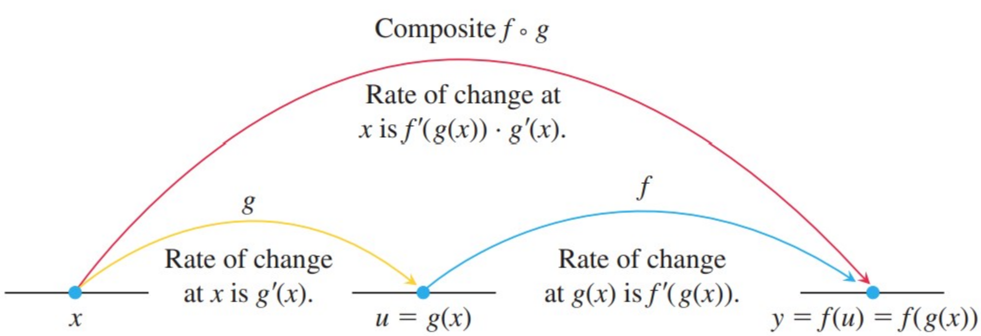

Figure 1: A Depiction of the Chain Rule (Thomas Calculus)

Example 1: What is the derivative of sin(3x)?

Solution: Let $f(x) = \sin(3x) = g(h(x))$ where $g(x) = \sin(x)$ and $h(x) = 3x$ then:

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = g'(h(x))h'(x) = \cos(3x)3 = 3\cos(3x)$$

since the derivative of sin is cos and the derivative of 3x is 3.

This principle can be extended to even more complex functions, such as in the below example.

Example 2: What is the derivative of $\sin(\sin(\sin(x)))$?

Solution: Let $f(x) = \sin(\sin(\sin(x))) = g(h(x)) = g(p(q(x)))$ where $g(x) = \sin(x)$ $h(x) = p(q(x)) = \sin(\sin(x))$ $p(x) = \sin(x)$ $q(x) = \sin(x)$ so:

$$f'(x) = \frac{df(x)}{dx} = g'(h(x))h'(x) = \cos(\sin(\sin(x)))h'(x)$$

$$= \cos(\sin(\sin(x)))p'(q(x))q'(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)$$

And that's the chain rule for you!

Now, time for a quick proof of the chain rule!

We can express the derivative of f(g(x)) as:

$$(f(g(x)))' = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

Noting that g(x) does not equal g(a) for x near a unless the function is a straight, horizontal line in which case we do not need the chain rule:

$$= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a}$$

Since the limit of the product is the product of the limits, and then using the definition of a derivative, this equals our now-familiar $f'(g(x))g'(x)$

Now it's time for some examples! A rocket, like the one shown below, has a position function

$$r(t) = 5t + 0.5\sin(3t^2 + 4t)$$

Figure 2: A Rocket (Wikipedia)

What is its velocity? Ans: $v(t) = 5 + 0.5\cos(3t^2 + 4t)(6t + 4)(6)$

Take the derivative of the following functions.

$$4(5x + 34)^2$$

Ans: $8(5x + 34)5$

$$5(4x^2 + 6x + 38)^2$$

Ans: $10(4x^2 + 6x + 38)(8x + 6)(8)$

$$6(2x + 3)^3$$

Ans: $18(2x + 3)^2(2)$

$$\sin(4x^2 + 3)$$

Ans: $\cos(4x^2 + 3)(8x)(8)$

$$tan(3x + 2)$$

Ans: $\sec^2(3x + 2)(3)$

$$(\frac{4x - 3}{2x + 1})^2$$

Ans: $2(\frac{4x-3}{2x+1})(\frac{(2x+1)(4)-(4x-3)(2)}{(2x+1)^2})$

$$\sin(\cos(4x^2))$$

Ans:$\cos(\cos(4x^2))(-\sin(4x^2))(8x)$

$$\sin(10x)$$

Ans:$\cos(10x)(10)$

$$tan(3x^2 + 4x)$$

Ans:$\sec^2(3x^2 + 4x)(6x + 4)(6)$

$$3e^{3x}$$

Ans: $3e^{3x}3$

$$5e^{16x^2+4x}$$

Ans: $5e^{16x^2+4x}(32x+4)(32)$

$$\sin(5e^{16x^2+4x})$$

Ans: $\cos(5e^{16x^2+4x})5e^{16x^2+4x}(32x+4)(32)$

$$\sin(\cos(\sin(\cos(10x^2+1))))$$

Ans: $\cos(\cos(\sin(\cos(10x^2+1))))(-\sin(\sin(\cos(10x^2+1))))\cos(\cos(10x^2+1))(-\sin(10x^2+1))(20x)(20)$