Stellar Interiors

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1 Introduction

Understanding stellar interiors better is absolutely crucial to developing a better understanding of stars and how they evolve, and thus the universe as a whole!

2 Hydrostatic Equilibrium

Hydrostatic equilibrium refers to the condition that stellar interiors must meet, namely that the inward force of gravity is balanced by the outward force of radiation pressure, preventing the star from expanding outwards infinitely or collapsing inwards on itself. Next, we will derive the hydrostatic equilibrium expression (warning: requires calculus! See the calculus fundamentals handout!)

Consider a cylinder of mass $\mathrm{d}m$ a radius r away from the center of the star with area A and height $\mathrm{d}r$. Assume only gravity and pressure are present, we can write from $F = ma$ and $a = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$, with F_g as the force of gravity, and $F_{P,top}$ as the pressure on the top, and $F_{P,bottom}$ as the pressure on the bottom:

$$\mathrm{d}m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = F_g + F_{P,top} + F_{P,bottom}$$

replacing $F_{P,top} + F_{P,bottom}$ with the difference in pressure $-\mathrm{d}F_p$

$$\mathrm{d}m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = F_g - \mathrm{d}F_p = -G\frac{M_r \mathrm{d}m}{r^2} - A\mathrm{d}P$$

with ρ as the density, $\mathrm{d}m = \rho A \mathrm{d}r$ yielding

$$\rho A\mathrm{d}r\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -G\frac{M_r \rho A \mathrm{d}r}{r^2} - A\mathrm{d}P$$

$$\rho \mathrm{d}r\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -G\frac{M_r \rho \mathrm{d}r}{r^2} - \mathrm{d}P$$

Setting this equal to zero as the acceleration must be zero, and substituting in g as the gravitational acceleration of the star from $g = \frac{GM}{r^2}$ where M is the mass internal to the radius$G$ is the gravitational constant and r is the radius,

$$0 = g\rho \mathrm{d}r - \mathrm{d}P$$

dividing by $\mathrm{d}r$ and rearranging,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$$

which is the condition for hydrostatic equilibrium. This demonstrate that a pressure gradient is capable of supporting the star, and reveals that the pressure is larger in the center.

Example 2.1: (USAAAO Round One 2022) Jupiter's deep atmosphere is very warm due to convection leading to an adiabatic temperature profile that increases with increasing pressure. Assuming (for simplicity) that this outer layer of Jupiter has a temperature of 500

K, perform a back-of-the-envelope estimate of the characteristic thickness (or e-folding scale) of the envelope of Jupiter (you may find that this is independent of pressure level). You may further use that the specific gas constant in Jupiter's atmosphere is 3600 $Jkg^{-1}K^{-1}$.
(a) 20 km (b) 73 km (c) 568 km (d) 3,120 km (e) 10,233 km

Solution: From hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$$

multiplying through by dr and approximating dr as H and dP as P,

$$P = \rho g H$$

Solving for H,

$$H = \frac{P}{\rho g}$$

From the ideal gas law,

$$PV = nRT$$

using $\rho = n/V$

$$\rho = P/RT$$

plugging in,

$$H = \frac{PRT}{Pg} = \frac{RT}{g}$$

Plugging in R, T, and g for Jupiter yield H = **73 km, or answer B**.

More derivations result in a series of equations that are known as the equations of stellar structure, and serve as the basis for understanding stellar interiors precisely in a quantifiable manner. These equations cannot be directly solved typically, and instead are used for modeling applications. With ϵ as the energy efficiency, L as the luminosity, T as the temperature, a as the radiation constant, $\overline{\kappa}$ as the Rosseland mean opacity (a measure of how transparent the star is), c as the speed of light, γ as the adiabatic index of the gas, m_h as the mass of the proton, k as Boltzmann's constant, μ as the average molecular weight,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G\frac{M_r \rho}{r^2}$$

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2 \rho$$

$$\frac{\mathrm{d}L_r}{\mathrm{d}r} = 4\pi r^2 \rho \epsilon$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{-3\overline{\kappa}L_r}{4acT^3 4\pi r^2}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -(1 - \frac{1}{\gamma})\frac{\mu m_h G M_r}{k r^2}$$

where the first temperature gradient equation is used for radiation dominated heat transfer, and the second one is used for adiabatic convection. For stars with not extremely small mass, the first equation is used except when the following equation is met, and then the second equation is used:

$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln T} < \frac{\gamma}{\gamma - 1}$$

When these equations are no longer met, a star dies, and then transitions into a stellar remnant. This either happens when stellar nucleosynthesis stops and a star collapses in on itself, or when it blows itself apart due to radiation pressure. The condition for the second death is known as the **Eddington luminosity**, the upper limit to a star's luminosity. Determining this involves using the expression of the pressure gradient near the surface from the luminosity gradient as:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{-\overline{\kappa}\rho L}{c4\pi r^2}$$

and since hydrostatic equilibrium says that

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$$

we can set these equal to each other and solve for the luminosity

$$L_{Ed} = \frac{4\pi G c M}{\overline{\kappa}}$$

3 Stellar Nucleosynthesis

The ultimate source of most of the energy emitted in the sun is extremely high temperature nuclear fusion happening near the center of the sun. While there is some energy provided by the gravitational collapse and differentiation of the sun, this is extremely minimal.

Nuclear fusion, or stellar nucleosynthesis, where two atoms are combined to form more massive elements, is the initial cause of the energy release. This process requires extremely high energies, and thus temperatures and particle speeds, in order to allow the atoms to overcome the extremely strong electric repulsion between the protons of two atoms that occur when they are forced close together enough for the strong force to take over and nuclear fusion to occur.

This requires temperatures over 10 million kelvin. In fact, even at these extremely high temperatures, this repulsion, known as the **Coloumb barrier**, is too strong to overcome frequently enough to lead to the amount of energy that we see being produced. *Quantum mechanical tunneling* is required in order for protons to overcome this barrier more frequently than would classically be expected, leading to higher rates of stellar nucleosynthesis and thus higher rates of energy production.

The basis of quantum mechanical tunneling is the *Heisenberg uncertainty principle*, which states that the product of the uncertainty in the position of a particle and its momentum must be greater than or equal to one-half the reduced form of Planck's constant, which is most commonly approximated as that product being equal to the reduced form of Planck's constant. This is on the order of 10^{-34} in SI units, rendering this quantum mechanical effect negligible on the scale of everyday life, and even largely negligible on a chemical scale. It is, however, tremendously important for this energy generation processes, and thus for the sun and life as we know it to exist.

The basis of this energy release comes from the fact that when two protons or hydrogen nuclei are combined, their combined mass is higher than that of one helium nuclei or alpha particle. The difference in mass is converted into energy according to Einstein's famous equation, $E = mc^2$. The main way this occurs in stars is through the **proton-proton chain**.

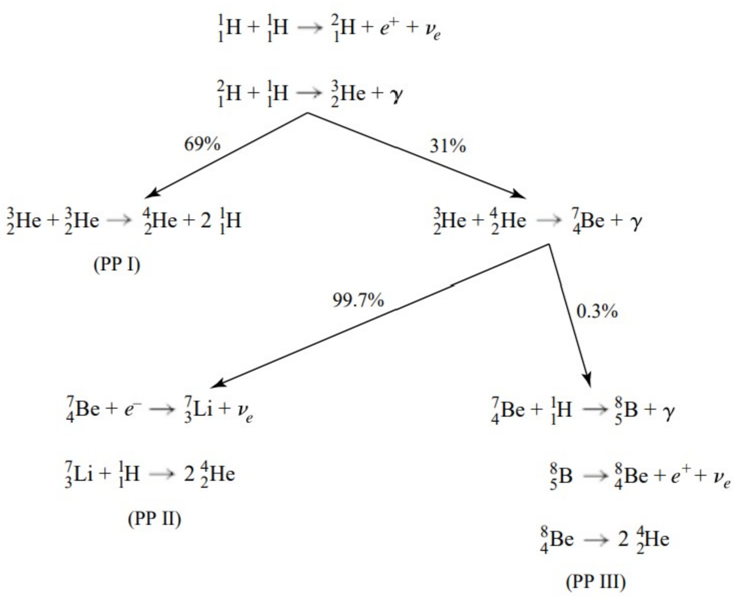

Figure 1: The pathways of the proton-proton or PP chain (Source: Caroll and Ostlie)

A second pathway that can generate the same effects occurs at much hotter temperatures and uses carbon as a catalyst, and is known as the **CNO cycle**.

$$
\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{13}_{7}\mathrm{N} + \gamma \\
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + e^{+} + \nu_e \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + \gamma \\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + \gamma \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + e^{+} + \nu_e \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}.
\end{aligned}
$$

Figure 2: The most common pathway of the CNO cycle (Source: Caroll and Ostlie)

After that happens, higher-mass stars can also fuse helium into carbon through the **triple alpha process**, and often also start hydrogen fusion in a thin layer around the core.

$$
\begin{aligned}
{}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} &\rightleftharpoons {}^{8}_{4}\mathrm{Be} \\
{}^{8}_{4}\mathrm{Be} + {}^{4}_{2}\mathrm{He} &\rightarrow {}^{12}_{6}\mathrm{C} + \gamma.
\end{aligned}
$$

Figure 3: The Triple Alpha Processes (Source: Caroll and Ostlie)

After that, carbon and oxygen are fused, and fusion continues until iron is made. Up until iron, fusion reactions are *exergonic* (energy-producing) as the mass of the product is higher than that of the reactants. However, at iron, the fusion process becomes *endogonic*, and energy is absorbed. This depletes energy, and leads to a buildup of iron "ash" in the core, and eventually a core-collapse supernova.

Most stars never get hot enough to fuse iron. By the time that would have happened in a large star, a series of layers of various elements have been built up in layers, with outwards-inwards the layers being nonburning hydrogen, hydrogen fusion, helium fusion, carbon fusion, oxygen fusion, neon fusion, magnesium fusion, silicon fusion, and then the central iron "ash".

4 Stellar Layers

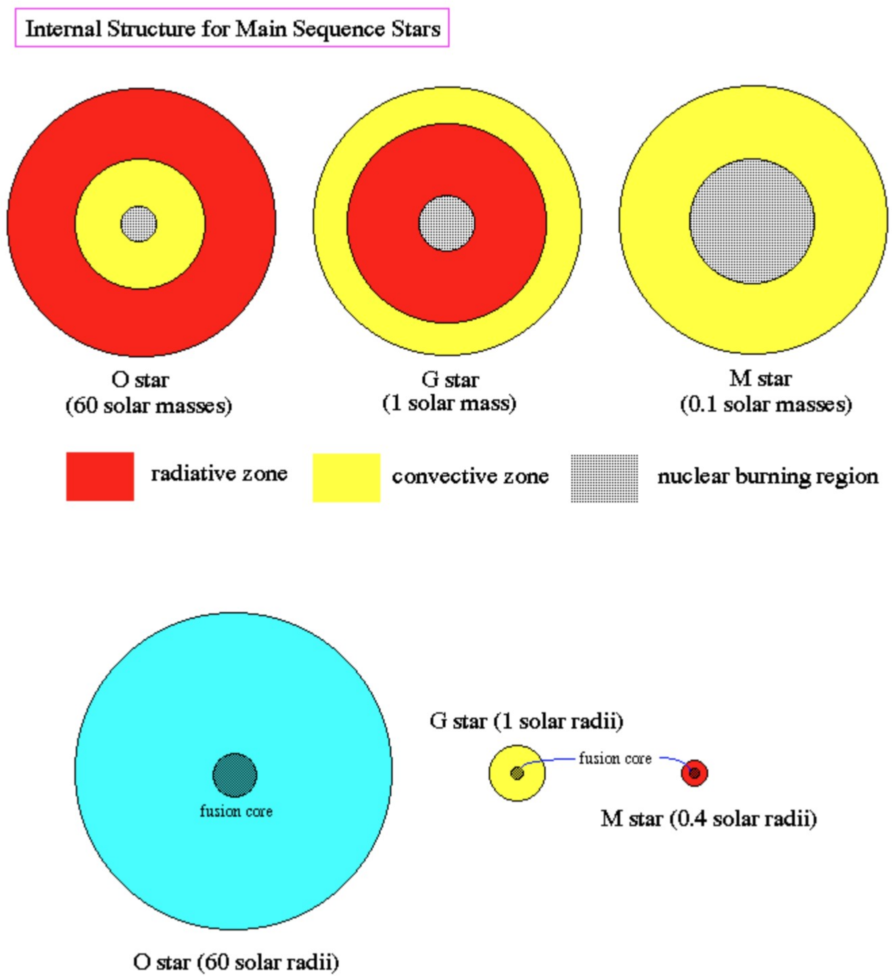

Figure 4: Layers of different-sized main sequence stars (Source: University of Florida)

There are several different types of layers for the star, for different size stars. For extremely small stars, they only have a convective zone. For medium size stars, they have a central radiative zone, and then an outer convective zone. For even larger stars, they have a central convective zone, and then an outwards radiative zone. As a proportion of the total star, the core is larger relative to the rest of the star.

5 Conclusion

With our newfound knowledge of stellar interiors, we are ready to learn more about stars and the universe!