Advanced Orbital Mechanics

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1 Introduction

At this point, you should have gone through the Basic Physics and Kepler's Laws handouts (if not, please do those first). In this handout, we'll focus on advanced applications of orbital mechanics, such as how to use conservation of energy and how orbital maneuvers work.

2 Energy

2.1 Basics

The total energy of an object in an elliptical orbit (at all points) is

$$E = -\frac{GMm}{2a},$$

where a is the semi-major axis of the orbit. The proof is trivial and left as an exercise to the reader.

Just kidding, of course! To prove this, we start by equating the energies at the perigee (closest point of orbit) and apogee (farthest point of orbit):

$$E = \frac{mv_1^2}{2} - \frac{GMm}{r_1} = \frac{mv_2^2}{2} - \frac{GMm}{r_2}$$

Now, let's multiply the first equation by r_1^2 and the second equation by r_2^2:

$$Er_1^2 = \frac{mv_1^2 r_1^2}{2} - GMmr_1$$

$$Er_2^2 = \frac{mv_2^2 r_2^2}{2} - GMmr_2$$

Subtracting these equations and using the fact that $v_1 r_1 = v_2 r_2$ (by conservation of angular momentum) yields

$$E(r_1^2 - r_2^2) = -GMm(r_1 - r_2)$$

Isolating for E and using the difference of squares factorization and the fact that $r_1 + r_2 = 2a$ (properties of an ellipse), we finally get

$$E = -GMm\frac{r_1 - r_2}{r_1^2 - r_2^2} = \frac{-GMm}{r_1 + r_2} = \frac{-GMm}{2a}$$

Note that the total energy of an elliptical (or circular) orbit is always negative, meaning the object can't escape. This is why elliptical and circular orbits are called **closed** orbits. For an object in a closed orbit to escape its host star, it must be *given* enough speed such that it has a positive energy.

A **parabolic** orbit is an orbit with an eccentricity of 1. **The total energy of a parabolic orbit is 0**. This means that

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = 0$$

at all points in the orbit, so

$$v = \sqrt{\frac{2GM}{r}}$$

at all points in the orbit. In other words, in a parabolic orbit, the object's velocity is always equal to the **escape velocity**; it can "just barely" escape.

> **Example 2.1:** (2021 USAAAO First Round #15) An interesting phenomena that happens in the Solar System is the capture of comets in the interstellar medium. Assume that a comet with a mass of $7.15 * 10^{16}$ kg is captured by the solar system. The perihelion of this comet's orbit after it is captured is equal to 4.64 AU, and its velocity with respect to the Sun before being captured by the Solar System was very small. Calculate the velocity of the comet at the perihelion.

Solution: This problem seems very daunting at first. We don't even know what type of orbit the comet is following. Or do we? We have to read the problem carefully. We are told the initial velocity is very small, which means that the initial kinetic energy is 0. We also know that the comet was "captured" from far away, so the initial gravitational potential energy is 0. Thus, the total initial energy, and energy at all points in the orbit, is 0, meaning the orbit is parabolic! We can thus use conservation of energy at the perihelion:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = 0$$

$$v = \sqrt{\frac{2GM}{r}}$$

Plug in all the numbers (making sure to convert to SI base units) to get that $v =$**19.6 km/s** □

The total energy of a **hyperbolic** orbit is positive. This means that the object is on a trajectory that will escape its host star.

2.2 Vis-Viva Equation

The **vis-viva equation** is very common in orbital mechanics. It gives the velocity of an object at every point in a closed orbit. It's essentially just a statement of conservation of energy. We start with

$$E_{tot} = -\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Now, if we solve for v^2, we get

$$v^2 = GM(\frac{2}{r} - \frac{1}{a})$$

This is the vis-viva equation.

> **Example 2.2:** (2022 USAAAO First Round #16) In 2025, the Parker Solar Probe will pass just $6.9 * 10^6$ km from the Sun, becoming the closest man-made object to the Sun in history.

It will make five orbits, passing close to the Sun once every 89 days, before the planned end of the mission in 2026. How fast will the Parker Solar Probe be traveling at its closest approach to the Sun?

Solution: We want the velocity, so we know we'll need to the vis-viva equation. We know the r value, but we don't know the semi-major axis a. But we do have the period of orbit, so we can use Kepler's third law and solve for a:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_{sun}}$$

so

$$a = (\frac{GM_{Sun}T^2}{4\pi^2})^{\frac{1}{3}} = 5.8 \cdot 10^7 \text{ km}$$

Now we use the vis-viva equation to find the velocity at the closest approach:

$$v = \sqrt{GM_{sun}(\frac{2}{r} - \frac{1}{a})} = \mathbf{190\ km/s} \ \square$$

2.3 Virial Theorem

In the Basic Physics handout, we introduced the **Virial Theorem**. The theorem states that in a circular orbit, the kinetic energy K and potential energy U are related by

$$K = -\frac{1}{2}U \text{ or } 2K + U = 0$$

Because we know that $E_{tot} = K + U$, we can also write that

$$K + K + U = 0 \rightarrow K + E_{tot} = 0 \rightarrow E_{tot} = -K$$

The Virial Theorem can be used for spherical bodies such as stars and galaxies, in addition to orbits.

Example 2.3: (2020 USAAAO First Round #14) As a consequence of the virial theorem, how does the stellar temperature (T) change if we add more arbitrary energy (E) to the star?

Solution: We use the Viriral Theorem, specifically that $E_{tot} = -K$. The key thing to realize is that E_{tot} is negative, so when we add energy, we are making E_{tot} more positive and thus *decreasing* the magnitude of the total energy. For example, the total energy could go from -500 J to -450 J, which means the kinetic energy would *decrease* from 500 J to 450 J. Because the kinetic energy decreases, the stellar temperature would also **decrease**. $\square$

3 Orbital Maneuvers

In this section, we will study two types of transfers that rockets undergo when moving between orbits. The math here is a bit complicated, but when you're doing these types of problems, the most important thing to keep in mind is that you don't need any advanced formulas; you just need conservation of energy, the velocity in a circular orbit, and occasionally Kepler's laws.

3.1 Hohmann Transfer

In a **Hohmann transfer**, a satellite moves from one circular orbit to another circular orbit using an elliptical transfer orbit.

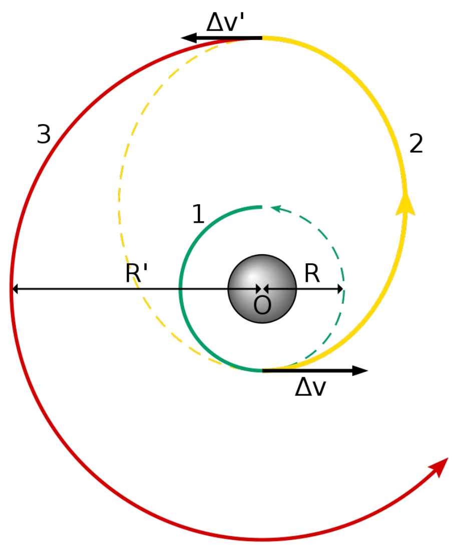

Figure 1: A Hohmann transfer orbit (Source: Wikipedia)

The Hohmann transfer requires two **impulses**, instances where the satellite changes its velocity. The first impulse moves the satellite from the original circular orbit to an elliptical orbit (1 to 2 in the figure). The second impulse occurs when the satellite adjusts its orbit from the elliptical transfer orbit to the second circular orbit (2 to 3 in the diagram).

Let's calculate these impulses. Again, impulses are essentially changes in velocity; we will denote them with Δv. The main equations we will use are the velocity of an object in a circular orbit ($v = \sqrt{\frac{GM}{R}}$) and the vis-viva equation previously derived.

Let's say the satellite goes from a circular orbit of radius r_1 to a circular orbit of radius r_2. The initial velocity of the satellite is

$$v_1 = \sqrt{\frac{GM}{r_1}}$$

Then, the satellite fires its thrusters and it *instantaneously* (at the same point) moves into an elliptical orbit. The semi-major axis of this elliptical orbit (yellow in the diagram) is $a = \frac{r_1+r_2}{2}$. You can visualize this using the diagram; the major axis is just $r_1 + r_2$ so the semi-major axis is half of that. The distance from the satellite to the central object (in the elliptical orbit) is r_1 so we can use the vis-viva equation to get the velocity of the satellite right after it fires its thrusters and goes into an elliptical orbit:

$$v_2 = \sqrt{GM(\frac{2}{r_1} - \frac{2}{r_1 + r_2})} = \sqrt{GM(\frac{2r_2}{r_1(r_1 + r_2)})}$$

We can then calculate the first impulse as

$$\Delta v_1 = v_2 - v_1 = \sqrt{\frac{GM}{r_1}}(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1)$$

Now, the satellite completes half of the elliptical orbit. Its velocity afterwards at the "halfway point" can be found using the vis-viva equation again, except this time, the satellite is r_2 away from the central object:

$$v_3 = \sqrt{GM(\frac{2}{r_2} - \frac{2}{r_1 + r_2})} = \sqrt{GM(\frac{2r_1}{r_2(r_1 + r_2)})}$$

Now, the satellite will fire its thrusters again to move to a circular orbit. It's velocity afterwards will just be the velocity of a satellite in a circular orbit of radius r_2:

$$v_4 = \sqrt{\frac{GM}{r_2}}$$

The second impulse is thus

$$\Delta v_2 = v_4 - v_3 = \sqrt{\frac{GM}{r_2}}(1 - \sqrt{\frac{2r_1}{r_1 + r_2}})$$

The total Δv required for a Hohmann transfer is $\Delta v_1 + \Delta v_2$.

4 Further Reading

If you want to learn more about the Hohmann transfer and other orbital maneuvers, section 14.8 of Roy and Clarke Astronomy: Principles and Practice is really good. You can learn more about transfer time and how to "coordinate" the transfer such that the satellite enters the secondary orbit right when the secondary object is there (e.g. Earth to Mars).

Another important but less well-known maneuver is the **bi-elliptic transfer orbit**. It's very similar to a Hohmann transfer except it consists of two half-elliptical transfer orbits rather than 1. The process of calculating the Δv's is the exact same. You can read more about it here.

5 Practice Problems

Below are some practice problems you can try. The ones marked with a * are challenging because they are either quite complicated or require knowledge outside of this handout.

- **2023 First Round:** 10*, 12, 21*, 26*, 28
- **2022 First Round:** 2, 16, 21*
- **2021 First Round:** 7
- **2020 First Round:** 18, 23

6 Conclusion

Orbital mechanics show up a lot on the USAAAO (both First Round and USAAAO). Memorizing the formulas in this handout isn't enough; you need to know how to apply them, which is why doing practice problems is so important. But once you thoroughly understand the material, you will be very successful on USAAAO orbital mechanics problems.