Exponential Decay and Growth

Arpit Mittal

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1 Introduction

In science, several types of behaviors can be modeled by curves that appear to increase or decrease very fast as well as curves that seem to be bounded above. For example, if you are modeling the amount of people living in a city, and a city can only hold 100 people, then the graph showing the city's population over time may look like a curve that increases very fast but then remains constant at 100. These are examples of exponential growth and decay, which we study in this handout.

2 The Derivatives of Exponentials

Before we can start talking about exponential growth and decay, we need to first discuss the most important topic: the derivative of e^x. What is e, you might wonder?

What is e?

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

It is well known that

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$

This means that the slope of the tangent line to e^x at any given x is equal to e^x. This beautiful result has several proofs and we give one now. We start with the definition of the derivative using a limit (refer to a previous handout if you are not fully comfortable with it or just skip this section and accept on faith that the derivative of e^x is e^x).

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x\left(\frac{e^h-1}{h}\right).$$

Now, define $n = e^h - 1$. This implies that $n+1 = e^h$, which means that $h = \ln(n+1)$. We can also compute

$$\lim_{h\to 0} n(h) = \lim_{h\to 0} e^h - 1 = 0.$$

So, when we substitute back in our new expressions in terms of n, we can replace the variable of the limit with n. We get

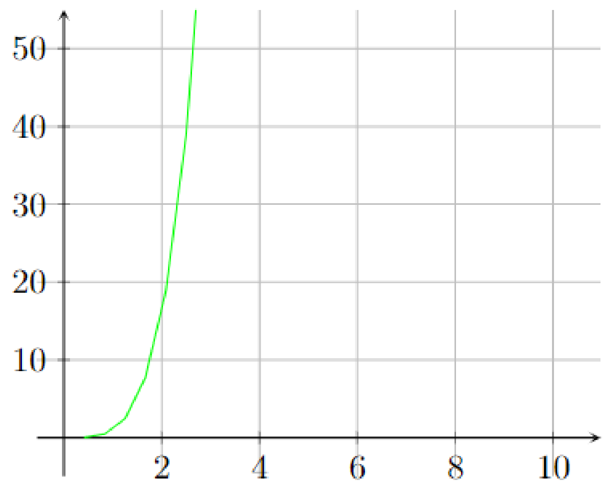
$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x\lim_{n\to 0}\frac{n}{\ln(n+1)} = e^x\lim_{n\to 0}\frac{1}{\frac{1}{n}\ln(n+1)} = e^x\lim_{n\to 0}\frac{1}{\ln((n+1)^{\frac{1}{n}})}.$$

We get the last equality by using the property of logarithms that $\ln(a^b) = b\ln(a)$. We can use the property of the limits of composite functions to put the limit inside of the logarithm and use our original definition for e to finally get

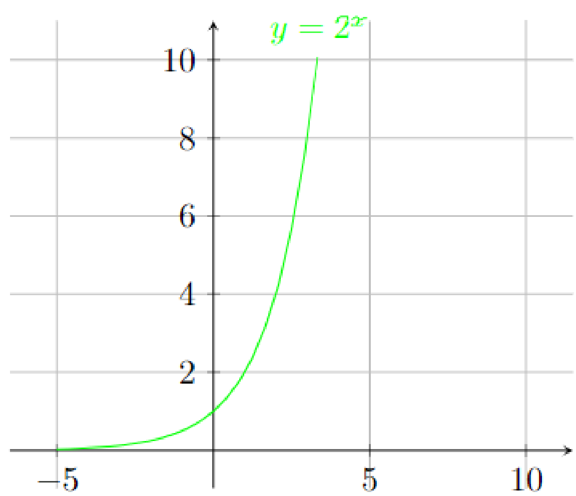
$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x\frac{1}{\ln(\lim_{n\to 0}(n+1)^{\frac{1}{n}})} = e^x\frac{1}{\ln(e)} = e^x.$$

3 Motivation

You have likely heard the phrase "exponential growth" several times before. However, it was probably used incorrectly most of the time. People tend to use the word "exponential" for something that seems to grow very large but this need not always be the case. For example, consider the following function.



This function seems to grow very fast, but it is actually not the graph of an "exponential" function. It is simply the graph of $f(x) = x^4$ which as we previously learned has derivative $f'(x) = 4x^3$ by the power rule. Exponential functions have the form $f(x) = c_1 e^{c_2 x}$. One example of an exponential function is depicted below.



This does not seem to be an exponential function, as one may wonder, where is the e? To see that this is exponential, we use a key property of logarithms by rewriting this as $f(x) = 2^x = e^{\ln(2^x)} = e^{x\ln(2)}$ which fits the criteria outlined above. We can also find the derivative of this function as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}2^x = \frac{\mathrm{d}}{\mathrm{d}x}e^{x\ln(2)} = e^{x\ln(2)} \cdot \ln(2) = 2^x \ln(2)$$

via the chain rule. We can see that $y'(x) = \ln(2)y(x) = ky(x)$. In other words, $y'(x) \propto y(x)$ Now, let us see if this holds in general for exponential functions.

$$\frac{\mathrm{d}}{\mathrm{d}x}c_1 e^{c_2 x} = c_1(e^{c_2 x}c_2) = (c_1 c_2)e^{c_2 x}.$$

We see that $f'(x) = (c_1 \cdot c_2)f(x) = kf(x)$ where k is a constant and is set to $c_1 \cdot c_2$. We have derived that the derivative of an exponential function is proportional to the function itself, or in other words

$$f'(x) \propto f(x).$$

Conceptually, when k is positive, this means that as a function is taking on a larger value, it will start to increase more rapidly. This is known as exponential growth. Sometimes k is negative so $f(x) = -kf(x)$ for positive k. For these functions, as the function takes on smaller values, it starts to decrease slower. This is known as exponential decay. The canonical example of exponential decay is **radioactive decay**.

4 Exponential Growth

Consider we are given that $f'(x) = kf(x)$ where k is a positive constant. From the last section, we can see that $f(x)$ is a function consisting of exponential growth. Our task becomes to derive $f(x)$. The first step is to rewrite this in Leibniz notation, as that makes the following steps clearer.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) = kf(x).$$

Now, let $y = f(x)$. Then, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky.$$

Separating variables gives

$$\frac{\mathrm{d}y}{y} = k\mathrm{d}x.$$

We integrate both sides to get

$$\int \frac{\mathrm{d}y}{y} = \int k\mathrm{d}x.$$

Using the fact that the derivative of $\ln(x)$ is x^{-1} and the power rule for integrals gives

$$\ln(y) = kx + c.$$

Remembering that exponentials are the inverse of logarithms, we can exponentiate each side to get

$$y = e^{kx+c}.$$

We can now use the key rule of exponents that $a^{b+c} = a^b a^c$ to rewrite this as

$$y = e^{kx}e^c$$

and substitute c_2 for k and c_1 for e^c to get this in the recognizable form

$$y = c_1 \cdot e^{c_2 x}$$

which is the expected and appropriate function for this differential equation. Suppose we know the value of $y(0) = a$. Then, we have

$$c_1 \cdot e^0 = a \implies c_1 = a.$$

And, if also given the value of y at $x = b$, $f(b) = z$ then we have

$$a \cdot e^{c_2 b} = z \implies e^{c_2 b} = \frac{z}{a} \implies c_2 b = \ln(\frac{z}{a}) \implies c_2 = \ln(\frac{z}{a})/b = \frac{1}{b}(\ln(z) - \ln(a)).$$

5 Exponential Decay

Consider k being negative (i.e. we are given that $f'(x) = -kf(x)$). We can write this using Leibniz notation as

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -kf(x)$$

and separate variables to get

$$\frac{\mathrm{d}f}{f(x)} = -k\mathrm{d}x$$

and integrate to get

$$\int \frac{\mathrm{d}f}{f(x)} = \int -k\mathrm{d}x.$$

We can compute the integrals to get

$$\ln(f(x)) = -kx + c_3.$$

We exponentiate to undo the logarithm to get

$$f(x) = e^{-kx+c_3} = e^{c_3}e^{-kx}.$$

We can see that e^{c_3} is a constant so we denote this as c_1 to get the final form

$$f(x) = c_1 e^{-kx}.$$

If we know the initial condition, i.e. $f(0) = a$, then we must have

$$c_1 e^{-k(0)} = a \implies c_1 = a.$$

6 Examples

Example 1: A student begins to spread a rumor at her school. Initially, only five students know about the rumor. Each student continuously tells the rumor to two other students in the school. Write a formula modeling the number of students in the school who know the rumor at a given time t.

Solution: By the result of Section 3, we have $c_1 = 5$. And, since if there are $f(t)$ students who know the rumor at any given time we have $f'(t) = 2f(t)$. Thus,

$$f(t) = 5e^{2t}.$$

Example 2: Consider the same example as before except we do not know how many students each student tells the rumor to but we do know that at time $t = 10$, $f(t) = 100$.

Solution: We have that $c_1 = 5$ as before. Now, we can use the results obtained at the end of Section 3 to get $c_2 = \frac{1}{10}(\ln(100) - \ln(5)) \approx 0.3$ so

$$f(t) = 5e^{0.3t}.$$

Example 3: Carbon-14 is always undergoing exponential decay with constant k. Carbon-14 has a half-life of approximately 5,730 years. This means that every 5,730 years the amount of Carbon-14 halves. If we have 50g of Carbon-14 today how much will be left after 100 years?

Solution: First, we derive a formula for half-life. If an isotope's exponential decay is modeled by $f(t) = f(0)e^{-kt}$ and has a half-life of z, then

$$f(a+z) = \frac{f(a)}{2} \implies f(0)e^{-ka-kz} = f(0)e^{-ka}e^{-kz} = f(0)\frac{e^{-ka}}{2} \implies e^{-kz} = \frac{1}{2}$$

so

$$-kz = \ln(0.5) \implies k = -\ln(0.5)/z.$$

Thus, in this example, $z = 5730$ so $k = -\ln(0.5)/5730 \approx 1.21 \cdot 10^{-4}$. Thus,

$$f(t) = 50e^{-1.21 \cdot 10^{-4} \cdot t} \implies f(100) \approx 49.3986g.$$