Trignometric Substitution

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July 2023

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Integration Rules and Examples Your Name February 1, 2024

1 Introduction

We will explore fundamental integration rules used in calculus, and cover essential concepts such as constant rule, exponential rule, power rule, and trigonometric rules. Each rule is illustrated with practical examples, helping readers gain confidence in integrating various functions. This document serves as a valuable reference for understanding and applying integration techniques.

2 When to Use Trig Substitution

Trig substitution is particularly useful when dealing with integrals containing the following types of expressions:

- Quadratic expressions under a square root, e.g., $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{x^2 + a^2}$
- Quadratic expressions with the form $a^2 - x^2$, $a^2 + x^2$, $x^2 - a^2$
- Rational expressions with quadratic denominators

3 Common Trigonometric Substitutions

The choice of trigonometric substitution depends on the form of the integral. Here are the common substitutions:

- For expressions of the form $\sqrt{a^2 - x^2}$, use $x = a \sin\theta$
- For expressions of the form $\sqrt{x^2 - a^2}$, use $x = a \cos\theta$
- For expressions of the form $\sqrt{x^2 + a^2}$, use $x = a \tan\theta$
- For expressions of the form $a^2 - x^2$, use $x = a \sin\theta$
- For expressions of the form $a^2 + x^2$, use $x = a \tan\theta$
- For expressions of the form $x^2 - a^2$, use $x = a \sec\theta$
- For rational expressions with quadratic denominators, use appropriate trigonometric substitutions based on the form.

4 Examples

Let's see some examples of how to use trig substitution to solve integrals:

4.1 Example 1

Evaluate the integral: $\int \frac{1}{\sqrt{4-x^2}}\,dx$

Solution: Let $x = 2\sin\theta$, then $dx = 2\cos\theta d\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{\sqrt{4-(2\sin\theta)^2}} \cdot 2\cos\theta\,d\theta \\
&= \int \frac{1}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\,d\theta \\
&= \int \frac{1}{\sqrt{4\cos^2\theta}} \cdot 2\cos\theta\,d\theta \\
&= \int \frac{1}{2}\,d\theta \\
&= \frac{\theta}{2} + C \\
&= \frac{\arcsin\left(\frac{x}{2}\right)}{2} + C
\end{aligned}$$

4.2 Example 2

Evaluate the integral: $\int \frac{1}{\sqrt{x^2-1}}\,dx$

Solution: Let $x = \sec\theta$, then $dx = \sec\theta\tan\theta\,d\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-1}}\,dx &= \int \frac{1}{\sqrt{\sec^2\theta-1}} \cdot \sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{\sqrt{\tan^2\theta}} \cdot \sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{|\tan\theta|} \cdot \sec\theta\tan\theta\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln|x + \sqrt{x^2-1}| + C
\end{aligned}$$

4.3 Example 3

Evaluate the integral: $\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$

Solution: Let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{1-x^2}}\, dx &= \int \frac{1}{(\sin\theta)^2\sqrt{1-(\sin\theta)^2}} \cdot \cos\theta\, d\theta \\
&= \int \frac{1}{\sin^2\theta\cos\theta}\, d\theta \\
&= \int \frac{1}{\sin^2\theta} \cdot \frac{1}{\cos\theta}\, d\theta \\
&= \int \csc^2\theta\, d\theta \\
&= -\cot\theta + C \\
&= -\cot(\sin^{-1} x) + C
\end{aligned}$$

4.4 Example 4

Evaluate the integral: $\int \frac{1}{\sqrt{x^2+9}}\, dx$

Solution: Let $x = 3\tan\theta$, then $dx = 3\sec^2\theta\, d\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+9}}\, dx &= \int \frac{1}{\sqrt{(3\tan\theta)^2+9}} \cdot 3\sec^2\theta\, d\theta \\
&= \int \frac{1}{\sqrt{9\tan^2\theta+9}} \cdot 3\sec^2\theta\, d\theta \\
&= \int \frac{1}{\sqrt{9(\tan^2\theta+1)}} \cdot 3\sec^2\theta\, d\theta \\
&= \int \frac{1}{\sqrt{9\sec^2\theta}} \cdot 3\sec^2\theta\, d\theta \\
&= \int 1\, d\theta \\
&= \theta + C \\
&= \arctan\left(\frac{x}{3}\right) + C
\end{aligned}$$