Optimization

Leon Lian

August 2023

Table of Contents

1 Introduction

Optimization is a fundamental concept in calculus that plays a crucial role in physics. It involves finding the best or optimal solution to a problem, given certain constraints or conditions.

2 Understanding Optimization Problems

2.1 Definition of Optimization

Optimization is the process of maximizing or minimizing a function's value, subject to a set of constraints. In mathematical terms, an optimization problem involves finding the values of variables that optimize the objective function while satisfying the given constraints. The objective function represents the quantity to be optimized, and the constraints define the conditions that the solution must adhere to.

2.2 Objective Function and Constraints

The objective function is a mathematical representation of the quantity that you want to optimize. It can be expressed as a function of one or more variables. The goal is to find the values of these variables that maximize or minimize the objective function.

Constraints, on the other hand, are conditions or limitations that the solution must satisfy. These constraints can be in the form of equations or inequalities and may impose restrictions on the values of the variables.

2.3 Examples

To better understand optimization problems, let's consider some examples:

Maximizing the area of a rectangle with a fixed perimeter.

In this example, suppose you have a fixed length of fencing material and you want to construct a rectangular enclosure. The goal is to find the dimensions of the rectangle that maximize its area. By maximizing the area, you can make the most efficient use of the available fencing material.

Minimizing the cost of producing a certain quantity of goods.

In this example, consider a manufacturing company that produces a certain quantity of goods. The cost of production depends on various factors, such as raw material costs, labor costs, and overhead expenses. The objective is to determine the optimal production quantity that minimizes the overall cost of production. This optimization problem involves minimizing the cost function (the objective function) while satisfying production constraints, such as resource limitations or demand requirements.

Maximizing the profit from selling a product with limited resources.

3 Techniques for Solving Optimization Problems

3.1 Critical Points

In optimization, critical points are the values of the variables where the objective function's derivative is either zero or undefined. These points are important because they can correspond to maximum or minimum values of the objective function.

3.2 The First Derivative Test

The first derivative test is a technique used to determine whether a critical point corresponds to a **relative maximum**, **relative minimum**, or neither. The first derivative indicates if the function is increasing, decreasing, or neither. If the first derivative is positive, the function is increasing, and if it is negative, the function is decreasing.

We take the first derivative of the objective function and set that function equal to zero or undefined, then solve for the x values will help us find the critical point. Once we find the x values which make the derivative 0, we check the interval to the left and to the right of the critical point. We can check the interval to the left and right of the critical points by plugging values to into the derivative. If the derivative is going from positive to negative, the critical point is a max, and if the derivative is going from negative to positive, the critical point is a min. If we are looking for the **absolute maximum**, we find the maximum of all the relative maximum points we calculated, and vice versa.

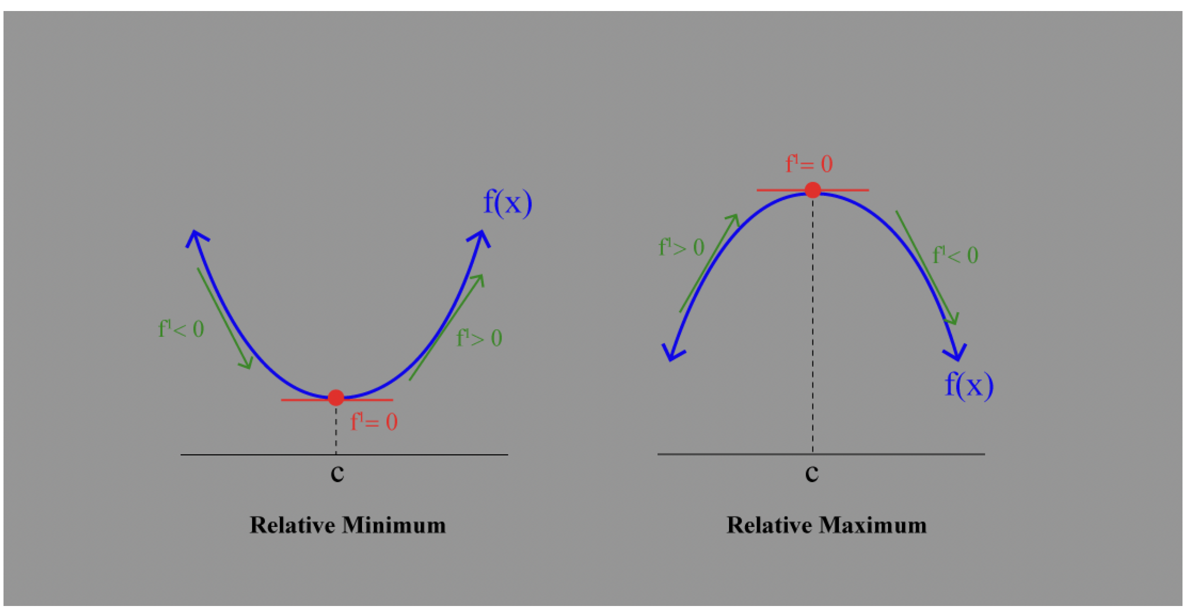

Figure 1: Relative min and max (source: Ximera)

3.3 The Second Derivative Test

The second derivative test is another method for identifying the nature of critical points. The second derivative indicates the concavity of a function. If the second derivative is positive over a certain interval, then the function is concave up of that interval, and if it is negative, then the function is concave down. If the second derivative is zero, we will not have to worry about the x value because it will be an inflection point, and inflection points can never be a maximum or minimum, since concavity changes at inflection points.

We check for critical points by setting the first derivative function equal to zero. Instead of checking the intervals to the left and right of the critical point, we can check if the second derivative is positive or negative at that point, to determine if that point is a relative max or min. We then do the same process, where we find the maximum over all relative maximums or vice versa to find the absolute max or min.

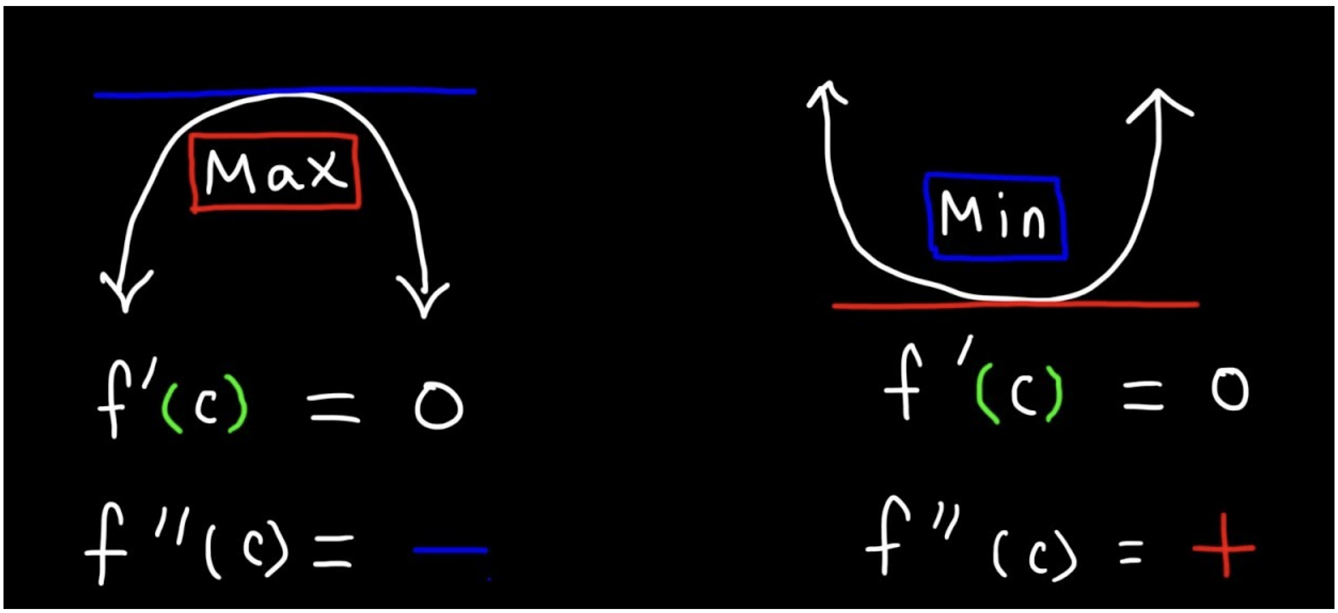

Figure 2: Second Derivative Test (source: Organic Chemistry Tutor)

4 Practice Problems

To reinforce your understanding of optimization, here are some practice problems that you can solve:

4.1 Problem 1

> A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 100ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area? (Source: Openstax)

Solution: Let's assume the length of the rock wall is L and the width of the garden, which will be enclosed by wire fencing, is W. The total length of wire fencing used is given by:

$$2W + L = 100$$

To maximize the area of the garden, we need to find the dimensions that maximize $A = LW$. By solving the equation $2W + L = 100$ for L, we get:

$$L = 100 - 2W$$

Substituting this into the area equation, we have:

$$A = (100 - 2W)W = 100W - 2W^2$$

To find the maximum area, we take the derivative of A with respect to W and set it equal to zero:

$$\frac{dA}{dW} = 100 - 4W = 0$$

Solving this equation gives us $W = 25$.

Taking the second derivative of A with respect to W, we have:

$$\frac{d^2A}{dW^2} = -4$$

Since the second derivative is negative, this indicates a concave-down function, which means that there exists a maximum at $W = 25$.

Substituting $W = 25$ back into $2W + L = 100$, we find $L = 50$.

Therefore, the dimensions that create a garden of maximum area are:

$$W = 25\,\text{ft} \quad \text{and} \quad L = 50\,\text{ft}$$

Substituting these values back into the area equation, we calculate the maximum area:

$$A = LW = 25\,\text{ft} \times 50\,\text{ft} = 1250\,\text{square feet}$$

4.2 Problem 2

> Which point on the graph of y = x is closest to the point (5, 0)?

Solution: To find the point on the graph of $y = x$ that is closest to the point $(5, 0)$, we can use calculus and the concept of minimizing the distance between two points.

Let's denote the point on the graph of $y = x$ that is closest to $(5, 0)$ as (x, x). The distance between these two points can be expressed as:

$$d = \sqrt{(x-5)^2 + (x-0)^2}$$

To find the minimum distance, we need to minimize the function d. We can do this by finding the critical points of d by taking its derivative and setting it equal to zero:

$$\frac{d}{dx} = \frac{2(x-5)+2(x-0)}{2\sqrt{(x-5)^2+(x-0)^2}} = 0$$

Simplifying the equation, we have:

$$2(x-5)+2x = 0$$

Solving for x, we find:

$$4x - 10 = 0 \implies x = \frac{10}{4} = \frac{5}{2}$$

Thus, the critical point occurs at $x = \frac{5}{2}$. To determine whether this point corresponds to a minimum or maximum, we can perform the first derivative test. We evaluate the first derivative of d at a value slightly less than $\frac{5}{2}$ and a value slightly greater than $\frac{5}{2}$:

$$\left.\frac{d}{dx}\right|_{x=\frac{5}{2}^-} = \frac{2(\frac{5}{2}-5)+2(\frac{5}{2}-0)}{2\sqrt{(\frac{5}{2}-5)^2+(\frac{5}{2}-0)^2}} < 0$$

$$\left.\frac{d}{dx}\right|_{x=\frac{5}{2}^+} = \frac{2(\frac{5}{2}-5)+2(\frac{5}{2}-0)}{2\sqrt{(\frac{5}{2}-5)^2+(\frac{5}{2}-0)^2}} > 0$$

Since the derivative changes sign from negative to positive at $x = \frac{5}{2}$, this indicates that $x = \frac{5}{2}$ corresponds to a minimum point. Since $\frac{5}{2}$ is the only relative minimum, it is the absolute maximum.

Therefore, the point on the graph of $y = x$ closest to $(5, 0)$ is $\left(\frac{5}{2}, \frac{5}{2}\right)$.

4.3 Problem 3.

> An open box is to be made from a 16in by 30in piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

Solution: Let's assume that the side length of the squares to be cut out from each corner is x (in inches).

After cutting out the squares and bending up the sides, the length of the resulting box will be $30 - 2x$ inches, the width will be $16 - 2x$ inches, and the height (the side length of the squares) will be x inches.

The volume V of the box can be calculated by multiplying the length, width, and height:

$$V = (30-2x)(16-2x)x$$

Simplifying the expression, we have:

$$V = 4x^3 - 92x^2 + 480x$$

To find the size of the squares that maximizes the volume, we need to find the critical points of V. We can do this by taking the derivative of V with respect to x and setting it equal to zero:

$$\frac{dV}{dx} = 12x^2 - 184x + 480 = 0$$

Solving this quadratic equation for x, we can use factoring, completing the square, or the quadratic formula.

In this case, factoring the quadratic equation gives us:

$$4(3x - 10)(x - 12) = 0$$

This gives us two critical points: $x = \frac{10}{3}$ and $x = 12$. We reject $x = 12$ because $0 \leq x \leq 8$. This happens because if x exceeds 8, then we are unable to cut squares out of the corners, because two squares that share the shorter edge will exceed the value of 16.

To determine which value of x gives the maximum volume, we can evaluate the second derivative of V:

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 24x - 184$$

Substituting $x = \frac{10}{3}$ into the second derivative, we find:

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right|_{x=\frac{10}{3}} = 24\left(\frac{10}{3}\right) - 184 = -8 < 0$$

Since the second derivative is negative for $x = \frac{10}{3}$ this confirms that $x = \frac{10}{3}$ gives the maximum volume as it is the only relative max we have.

Therefore, to obtain a box with the largest volume, the size of the squares to be cut out from each corner should be $\frac{10}{3}$ inches, resulting in a maximum volume of 725.926 in^3.