Optimal Control of a Problem

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1 Model Description

Model description will be added here.

1.1 Question 1

$x_1 = x$, $x_2 = V_x$, $x_3 = y$, $x_4 = V_y$, $x_5 = r$, $x_6 = \psi$

$u_1 = f_{Fx}$, $u_2 = f_{Rx}$, $u_3 = \delta$

$\dot{x}_1 = x_2$

$\dot{x}_2 = \frac{1}{m}[u_1 cos(u_3) - f_{Fy} sin(u_3) + u_2] + x_4 x_5$

$\dot{x}_3 = x_4$

$\dot{x}_4 = \frac{1}{m}[u_1 sin(u_3) + f_{Fy} cos(u_3) + f_{Ry}] - x_2 x_5$

$\dot{x}_5 = \frac{1}{I_z}[f_{Fy} cos(u_3) + u_1 sin(u_3)] l_F + \frac{1}{I_z} f_{Ry} l_R$

$\dot{x}_6 = x_5$

$s_{Fy} = \frac{V sin\beta cos\delta - V cos\beta sin\delta + r l_F cos\delta}{V cos\beta cos\delta + V sin\beta sin\delta + r l_F sin\delta}$

$s_{Ry} = \frac{V sin\beta - r l_R}{V cos\beta}$

$s_{Fy} = \frac{V_y cos\delta - V_x sin\delta + r l_F cos\delta}{V_x cos\delta + V_y sin\delta + r l_F sin\delta}$

$s_{Ry} = \frac{V_y - r l_R}{V_x}$

$s_{Fy} = \frac{x_4 cos u_3 - x_2 sin u_3 + x_5 l_F cos u_3}{x_2 cos u_3 + x_4 sin u_3 + x_5 l_F sin u_3}$

$s_{Ry} = \frac{x_4 - x_5 l_R}{x_2}$

$f_{Fy} = \sqrt{(\mu f_{Fz})^2 - u_1^2} sin(C arctan(B s_{Fy}))$

$f_{Ry} = \sqrt{(\mu f_{Rz})^2 - u_2^2} sin(C arctan(B s_{Ry}))$

$min(J) = \frac{1}{2} y^2(t_f) + \int_0^{t_f} (1) \mathrm{d}t.$

Initial conditions:

$x_1(0) = 0$, $x_2(0) = 55$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $x_6 = 0$

Final conditions:

$x_1(0) = 0$, $x_2(0) = 55$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $x_6 = 0$

$H = 1 + \lambda_1\{x_2\} + \lambda_2\{\frac{1}{m}[u_1 cos(u_3) - f_{Fy} sin(u_3) + u_2] + x_4 x_5\} + \lambda_3\{x_4\} + \lambda_4\{\frac{1}{m}[u_1 sin(u_3) + f_{Fy} cos(u_3) + f_{Ry}] - x_2 x_5\} + \lambda_5\{\frac{1}{I_z}[f_{Fy} cos(u_3) + u_1 sin(u_3)] l_F + \frac{1}{I_z} f_{Ry} l_R\} + \lambda_6\{x_5\}$

2 Analytical Solutions

1. Let's formulate this problem as an optimal control problem.

   Define the state equations :

   $\dot{X} = V_x \cdot cos(\psi) - V_y \cdot sin(\psi)$

   $\dot{Y} = V_x \cdot sin(\psi) + V_y \cdot cos(\psi)$

   $\dot{V_x} = \frac{1}{m} \cdot (f_{F_x} \cdot cos(\delta) - f_{F_y} \cdot sin(\delta) + f_{R_x}) - V \cdot r$

   $\dot{V_y} = \frac{1}{m} \cdot (f_{F_x} \cdot sin(\delta) + f_{F_y} \cdot cos(\delta) + f_{R_y}) - V_x \cdot r$

   $\dot{r} = \frac{1}{I_z} \cdot (f_{F_y} \cdot cos(\delta) + f_{F_x} \cdot sin(\delta)) \cdot l_F - f_{R_y} \cdot l_R$

   $\dot{\psi} = r$

   Define the inputs :

   f_{F_x}, f_{R_x}, δ

   Define the performance measure :

   $J = \alpha \cdot \phi(x(t_f), t_f) + (1 - \alpha) \cdot \int_0^{t_f} L(x, u, t)dt$

   $J = \alpha \cdot \ y(t_f)^2 + t_f = \alpha \cdot \ y(t_f)^2 + (1 - \alpha) \cdot \int_0^{t_f} 1dt$

   The boundary conditions are :

   $\beta(0) = 0$

   $V_x(0) = 55 kmperhour$

   $V_y(0) = 0$

   $\beta(t_f) \simeq -90 degrees$

$\dot{\psi}(t_f) = 0$

The equations given are the following :

$\beta = arctan(\frac{V_y}{V_x} = arctan(\frac{\dot{X}}{\dot{Y}}) - \psi$

$f_{\star y} = f_{\star y}^{max} \cdot sin(C \cdot arctan(B \cdot s_{\star y})) \quad \star = F, R$

$s_{F_y} = \frac{V \cdot sin(\beta - \delta) + r \cdot l_f \cdot cos(\delta)}{V \cdot cos(\beta - \delta) + r \cdot l_f \cdot sin(\delta)}$

$s_{R_y} = \frac{V \cdot sin(\beta) - r \cdot l_R}{V \cdot cos(\beta)}$

$f_{\star y}^{max} = \sqrt{(\mu f_{\star z})^2 - f_{\star x}^2}, \quad \star = F, R$

$f_{F_z} = m \cdot g \cdot \frac{l_R}{l_F + l_R}$

$f_{R_z} = m \cdot g \cdot \frac{l_F}{l_F + l_R}$

$|f_{\star x}| \leq \mu \cdot f_{\star z} = f_{\star x}^{max}, \quad \star = F, R$

2. Assuming the rear-wheel driven (RWD) vehicle, let us derive the necessary conditions for this optimization problem.

For a rear-drive vehicle, $f_{F_x} = 0$ when driving and $f_{F_x} < 0$ when braking, while f_{R_x} can take both positive (when driving) and negative (when braking) values.
Ideally, $f_{F_x} < 0$ if and only if $f_{R_x} < 0$.

Define the Hamiltonian for the optimal-control problem :

$H(x, \lambda, u, t) = L(x, u, t) + \lambda^T \cdot f(x, u, t) + \mu^T \cdot C(x, u, t)$

$C(x, u, t)$ represents the inputs constraints.

$C_1(x, u, t) = -f_{F_x} \cdot f_{R_x} \leq 0$

$C_2(x, u, t) = (\sqrt{\delta^2} - \delta_{max}) \leq 0$

$C_3(x,u,t) = (\sqrt{f_{R_x}^2} - f_{R_x}^{max}) \leq 0$

$C_4(x,u,t) = (\sqrt{f_{F_x}^2} - f_{F_x}^{max}) \leq 0$

$H(x(t),u(t),t) = 1 + \lambda_x \cdot V_x + \lambda_y \cdot V_y +$
$\lambda_{V_x} \cdot \frac{1}{m} \cdot [(f_{F_x} \cdot cos(\delta) - f_{F_y} \cdot sin(\delta) + f_{R_x}) - V \cdot r] +$
$\lambda_{V_y} \cdot [\frac{1}{m} \cdot (f_{F_x} \cdot sin(\delta) + f_{F_y} \cdot cos(\delta) + f_{R_y}) - V_x \cdot r] +$
$\lambda_r \cdot [\frac{1}{I_z} \cdot (f_{F_y} \cdot cos(\delta) + f_{F_x} \cdot sin(\delta)) \cdot l_F - f_{R_y} \cdot l_R] +$
$\lambda_\psi \cdot r + \mu_1 \cdot (-f_{F_x} \cdot f_{R_x}) + \mu_2 \cdot (\sqrt{\delta^2} - \delta_{max}) +$
$\mu_3 \cdot (\sqrt{f_{R_x}^2 - f_{R_x}^{max}}) + \mu_4 \cdot (\sqrt{f_{F_x}^2 - f_{F_x}^{max}})$

The co-state equations are the following :

$$\dot{\lambda_x} = -\frac{\partial H}{\partial x} = 0 \quad (1)$$

$\dot{\lambda_y} = -\frac{\partial H}{\partial y} = 0$

$\dot{\lambda_{V_x}} = -\frac{\partial H}{\partial V_x} = \lambda_{V_y} \cdot r$

$\dot{\lambda_{V_y}} = -\frac{\partial H}{\partial V_y} = -\lambda_{V_x} \cdot r$

$\lambda_r = -\frac{\partial H}{\partial r} = \dot{-}\lambda_{V_x} \cdot V_y + \lambda_{V_y} \cdot V_x$

$\dot{\lambda_\psi} = -\frac{\partial H}{\partial \psi} = 0$

3. Let us give the boundary conditions for the states $r(t_f) = 0$

The transversality condition for the Hamiltonian is :

$\frac{\partial \phi}{\partial t}(t_f, x_f) + H(t_f, x_f) = 0$

that is $H(t_f, x_f) = -1$

The transversality conditions for the co-states are :

$\frac{\partial \phi}{\partial X}(t_f, x_f) - \lambda(t_f) = 0$

that is :

$\frac{\partial \phi}{\partial x}(t_f, x_f) = \lambda_x(t_f) = 0$

$\frac{\partial \phi}{\partial y}(t_f, x_f) = \lambda_y(t_f) = \alpha \cdot 2 \cdot y(t_f)$

$\frac{\partial \phi}{\partial V_x}(t_f, x_f) = \lambda_{V_x}(t_f) = 0$

$\frac{\partial \phi}{\partial V_y}(t_f, x_f) = \lambda_V(t_f) = 0$

$\frac{\partial \phi}{\partial r}(t_f, x_f) = \lambda_r(t_f) = 0$

$\frac{\partial \phi}{\partial \psi}(t_f, x_f) = \lambda_\psi(t_f) = 0$

4. Let us determine the optimal control strategy :

Looking for a singular control,

$\frac{\partial H}{\partial u} = 0$ which is equivalent to three equations

$\frac{\partial H}{\partial f_{F_x}} = 0$ that is

$\frac{\lambda_{V_x}}{m} \cdot cos(\delta) + \frac{\lambda_{V_y}}{m} \cdot sin(\delta) + \frac{\lambda_r}{I_z} \cdot sin(\delta) - \mu_1 \cdot f_{R_x} - \mu_4 \cdot \frac{f_{F_x}}{\sqrt{f_{F_x}^2}} = 0$

$\frac{\partial H}{\partial f_{R_x}} = 0$ that is

$\frac{\lambda_{V_x}}{m} - \mu 1 \cdot f_{F_x} - \mu_3 \cdot \frac{f_{R_x}}{\sqrt{f_{R_x}^2}} = 0$

$\frac{\partial H}{\partial \delta} = 0$ that is

$-\frac{\lambda_{V_x}}{m} \cdot f_{F_x} \cdot sin(\delta) - \frac{\lambda_{V_x}}{m} \cdot f_{F_y} \cdot cos(\delta) + \frac{\lambda_{V_y} m}{.} f_{F_x} \cdot cos(\delta) - \frac{\lambda_{V_y} m}{.} f_{F_y} \cdot sin(\delta) - \frac{\lambda_r}{I_z} \cdot f_{R_y} \cdot sin(\delta) + \frac{\lambda_r}{I_z} \cdot f_{F_x} \cdot cos(\delta) - \mu_2 \cdot \frac{\delta}{\sqrt{\delta^2}} = 0$

This is not solvable analytically. We tried finding u for higher time derivatives of $H_u(x, u, \lambda, \mu, t)$, using the formula :

$$(-1)^m \cdot \frac{\partial}{\partial u} \cdot (\frac{d^{2m}}{dt^{2m}} \cdot H_u(x, u, \lambda, \mu, t)) \geq 0 \ ,$$

but the equations became very complex and the control still was not appearing explicitly.
Therefore we wrote the equations into Mathematica, however this led to even more complex equations. This justifies the fact that we are using GPOPS to try and solve the problem, because the analytical approach is simply too complicated.