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3. 定解问题 $\begin{cases} \Delta u = 0 \\ u|_{r=r_1} = u_0 \\ u|_{r=r_2} = u_1\cos^2\theta \end{cases}$ 级数解：$u(r,\theta) = \sum\limits_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta)$

边界条件 $\begin{cases} \sum\limits_{l=0}^{\infty}\left(A_l r_1^l + B_l r_1^{-(l+1)}\right)P_l(\cos\theta) = u_0 \\ \sum\limits_{l=0}^{\infty}\left(A_l r_2^l + B_l r_2^{-(l+1)}\right)P_l(\cos\theta) = u_1\cos^2\theta \end{cases}$

对比系数 $\begin{cases} A_0 + B_0 r_1^{-1} = u_0 \\ A_2 r_1^2 + B_2 r_1^{3} = 0 \end{cases}$ $\begin{cases} A_0 + B_0 r_2^{-1} = \frac{u_1}{3} \\ A_2 r_2^2 + B_2 r_2^{3} = \frac{2}{3}u_1 \end{cases}$

又 $\because u_1\cos^2\theta = u_1\frac{1+\cos 2\theta}{2} = u_1\left[\frac{2}{3}\cdot\frac{1}{4}(3\cos 2\theta + 1) + \frac{1}{3}\right] = u_1\left(\frac{2}{3}P_2 + \frac{1}{3}\right)$

$\therefore A_0 = \frac{\frac{1}{3}u_1 r_2 - u_0 r_1}{r_2 - r_1}$ $A_2 = \frac{\frac{1}{3}u_1 r_2^3}{r_2^5 - r_1^5}$ $B_0 = \frac{\left(u_0 - \frac{u_1}{3}\right)r_1 r_2}{r_2 - r_1}$ $B_2 = \frac{\frac{1}{3}u_1 r_1^5 r_2^3}{r_2^5 - r_1^5}$

4. 由题意得定解问题 $\begin{cases} \Delta u = 0 \ (0 \le \theta < \frac{\pi}{2}) \\ u|_{r=r_0} = u_0 \\ u|_{r=0} \text{ 有限值} \\ u|_{\theta=\frac{\pi}{2}} = 0 \end{cases}$ 将 θ 延拓到 $(\frac{\pi}{2}, \pi]$，设问题的解 $u = \sum\limits_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

$r=0$ 处为有限值 则可约去 $\frac{1}{r^{l+1}}$，$u = \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

由 $r=r_0$ 处边界条件可知 $A_l r_0^l$ 即为 u_0 的勒让德展开式的系数。又由于 $\theta \in [0, \frac{\pi}{2})$

$A_l = \frac{2l+1}{2r_0^l}\int_0^1 u_0 P_l(x)dx = \frac{2l+1}{2r_0^l}u_0\int_0^1 P_l(x)dx = \frac{2(l+1)u_0}{2r_0^l}\cdot\frac{1}{2^l l!}\int_0^1\frac{d^l}{dx^l}(x^2-1)^l dx$

$= \frac{(2l+1)u_0}{2^{l+1}r_0^l l!}\left[\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]\Big|_0^1 = \frac{-(2l+1)u_0}{2^{l+1}r_0^l l!}\left[\frac{d^{l-1}}{dx^{l-1}}\sum\limits_{k=0}^{l}C_l^k x^{2k}(-1)^{l-k}\right]\Big|_{x=0} = \begin{cases} 0 & (l\text{为偶数}, l\neq 0) \\ \frac{(-1)^{l}(2l)!(2l+1)u_0}{2^{2l+2}(l+1)!\,l!\,r_0^l} & (l\text{为奇数}) \end{cases}$

当 $l=0$，$u|_{\theta=\frac{\pi}{2}} = 0$ 可知，u 中 A_0 为 0，$A_l = \frac{(-1)^k(4k+3)(2k-1)!!\,u_0}{(2k+2)!!\cdot r_0^{2k+1}}$

半球电势为 $u = \begin{cases} u_0\sum\limits_{k=1}^{\infty}(-1)^k(4k+3)\frac{(2k-1)!!}{(2k+2)!!}\left(\frac{r}{r_0}\right)^{2k+1}P_{2k+1}(\cos\theta) \\ u_0, \ \theta = \frac{\pi}{2} \end{cases}$

7. 取太阳光直射方向为极轴，温度分布与 φ 无关，仅上半球受照射，方程及边界条件为 $\begin{cases} \Delta u = 0 \ (r < r_0) \\ u|_{r=0} = \text{有限值} \\ \left(\frac{\partial u}{\partial r} + Hu\right)\Big|_{r=r_0} = \begin{cases} q_0\cos\theta & (0 \le \theta < \frac{\pi}{2}) \\ 0 & (\frac{\pi}{2} < \theta \le \pi) \end{cases} \end{cases}$ 其中 $H = \frac{h}{k}$，h 为热交换系数，k 为热传导系数

由于 $r=0$ 处 u 为有限值，设 $u = \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ 则 $H\sum\limits_{l=0}^{\infty} A_l r_0^l P_l(\cos\theta) + \sum\limits_{l=1}^{\infty} A_l l r^{l-1} P_l(\cos\theta) = \begin{cases} q_0\cos\theta & (0 \le \theta < \frac{\pi}{2}) \\ 0 & (\frac{\pi}{2} < \theta \le \pi) \end{cases}$

将等号右边用勒让德多项式展开，令 $\cos\theta = x$，$f_0 = \frac{1}{2}\int_0^1 q_0 x\,dx = \frac{q_0}{4}$，$f_1 = \frac{3}{2}\int_0^1 q_0 x^2 dx = \frac{q_0}{2}$

$\therefore f_l = \frac{1}{2l+1}\int_0^1 q_0 x P_l(x)dx = \frac{q_0}{2l+1}\int_0^1 x\frac{d^l(x^2-1)^l}{dx^l}dx = \frac{q_0}{2l+1}\left\{\left[x\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]\Big|_0^1 - \int_0^1\frac{d^{l-1}(x^2-1)^l}{dx^{l-1}}dx\right\}$

$= \frac{q_0}{2l+1}\left[-\frac{d^{l-2}(x^2-1)^l}{dx^{l-2}}\right]\Big|_0^1 = \frac{-q_0}{2l+1}\left[\frac{d^{l-2}}{dx^{l-2}}\sum\limits_{k=0}^{l}C_l^k x^{2k}(-1)^{l-k}\right]\Big|_{x=0} = \begin{cases} 0, & l = 2n+1 \\ \frac{(-1)^{n+1}(4n+1)(2n-2)!}{2(2n-2)!!(2n+2)!!}; & l = 2n \text{ 且 } n \neq 0 \end{cases}$

比较系数后，$A_0 = \frac{q_0}{4H}$，$A_1 = \frac{1}{Hr_0+1}\cdot\frac{q_0}{2}$ $A_{2n+1} = 0$

$A_{2n} = \frac{(-1)^{n+1}q_0}{Hr_0 - 2n}\cdot\frac{4n+1}{2}\frac{(2n-2)!}{(2n-2)!!(2k+2)!!}\cdot\frac{1}{r_0^{2n-1}}$

综上，温度分布 $u = \frac{q_0}{4H} + \frac{1}{Hr_0+1}\frac{q_0 r}{2}P_1(\cos\theta) + \sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}q_0}{(Hr_0+2k)r_0^{2k-1}}\frac{(3k+1)(2k-2)!}{2(2k-2)!!(2k+2)!!}r^{2k}P_{2k}(\cos\theta)$

9. 将坐标置于环且过环心的轴为极轴，以环心为极点，对于环内 $\begin{cases}\Delta u_1=0\\ u_1|_{r=0}\ 为有限值\\ u_1|_{\theta=0}=u_1|_{\theta=\pi}=\frac{q}{\sqrt{r^2+r_0^2}}\end{cases}$

由$u_1|_{r=0}$为有限值，所以设$u_1=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$，由$\theta$方向边界条件$\sum_{l=0}^{\infty}A_l r^l P_l(1)=\frac{q}{\sqrt{r_0^2+r^2}}$

即$\sum_{l=0}^{\infty}A_l r^l=\frac{q}{r_0}\frac{1}{\sqrt{1+(\frac{r}{r_0})^2}}=\frac{q}{r_0}[1-\frac{1}{2}(\frac{r}{r_0})^2+\frac{1\times3}{2\times4}(\frac{r}{r_0})^4+\cdots+(-1)^k\frac{(2k-1)!!}{(2k)!!}(\frac{r}{r_0})^{2k}+\cdots]$

其中$A_0=\frac{q}{r_0}$，$A_{2k+1}=0$ $\therefore A_{2k}=(-1)^k\frac{(2k-1)!!}{(2k)!!}\frac{q}{r_0^{2k+1}}$

$$u_1=\frac{q}{r_0}+\frac{q}{r_0}\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}(\frac{r}{r_0})^{2k}P_{2k}(\cos\theta)$$

对环外：$\begin{cases}\Delta u_2=0\\ u_2|_{r=\infty}=有限值\\ u_2|_{\theta=0}=u_2|_{\theta=\pi}=\frac{q}{\sqrt{r^2+r_0^2}}\end{cases}$ 由$u_2|_{r=\infty}$为有限值，设$u_2=\sum_{l=0}^{\infty}B_l\frac{1}{r^{l+1}}P_l(\cos\theta)$

由θ方向边界条件，$\sum_{l=0}^{\infty}B_l\frac{1}{r^{l+1}}P_l(\cos\theta)=\frac{q}{\sqrt{r^2+r_0^2}}=\frac{q}{r\sqrt{1+(\frac{r_0}{r})^2}}$

$=\frac{q}{r}[1-\frac{1}{2}(\frac{r_0}{r})^2+\frac{1\cdot3}{2\cdot4}(\frac{r_0}{r})^4+\cdots+(-1)^k\frac{(2k-1)!!}{(2k)!!}(\frac{r_0}{r})^{2k}+\cdots]$

其中$B_0=q$，$B_{2k+1}=0$，$B_{2k}=(-1)^k\frac{(2k-1)!!}{(2k)!!}q r_0^{2k}$，$u_2=\frac{q}{r}+\frac{q}{r_0}\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}(\frac{r_0}{r})^{2k+1}P_{2k}(\cos\theta)$

2. 内：$\begin{cases}\Delta u=0\ (0<r<r_0)\\ u|_{r=r_0}=4\sin^2\theta(\cos\varphi\sin\varphi+\frac{1}{2}),\ u|_{r\to0}为有限值\end{cases}$

拉普拉斯方程在球坐标系通解 $u(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}(C_l r^l+\frac{D_l}{r^{l+1}})P_l^m(\cos\theta)(A_l^m\cos m\varphi+B_l^m\sin m\varphi)$

由$u|_{r\to0}$为有限值，得$D_l=0$，$u=\sum_{l=0}^{\infty}\sum_{m=0}^{l}r^l P_l^m(\cos\theta)(A_l^m\cos m\varphi+B_l^m\sin m\varphi)$

代入边界条件 $u|_{r=r_0}=4\sin^2\theta(\cos\varphi\sin\varphi+\frac{1}{2})=\frac{4}{3}P_0(\cos\theta)-\frac{4}{3}P_2(\cos\theta)+\frac{2}{3}P_2^2(\cos\theta)\sin2\varphi$

比较系数可得$A_0^0=\frac{4}{3}$，$A_2^0 r_0^2=\frac{4}{3}$，$B_2^2 r_0^2=\frac{2}{3}$

所以，在球内 $u(r,\theta,\varphi)=\frac{4}{3}P_0^0(\cos\theta)-\frac{4}{3}(\frac{r}{r_0})^2P_2^0(\cos\theta)+\frac{2}{3}(\frac{r}{r_0})^2P_2^2(\cos\theta)\sin2\varphi$

外：$\begin{cases}\Delta u=0\ (r>r_0)\\ u|_{r=r_0}=4\sin^2\theta(\cos\varphi\sin\varphi+\frac{1}{2}),\ u|_{r\to\infty}为有限值\end{cases}$

将$u|_{r\to\infty}\to$有限值，代入通解，可以发现$C_l=0$

所以有$u(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{r^{l+1}}P_l^m(\cos\theta)(A_l^m\cos m\varphi+B_l^m\sin m\varphi)$

代入 $u|_{r=r_0}=4\sin^2\theta(\cos\varphi\sin\varphi+\frac{1}{2})=\frac{4}{3}P_0(\cos\theta)-\frac{4}{3}P_2(\cos\theta)+\frac{2}{3}P_2^2(\cos\theta)\sin2\varphi$

比较系数可得$\frac{A_0^0}{r_0}=\frac{4}{3}$，$\frac{A_2^0}{r_0}=\frac{4}{3}$ $\frac{B_2^2}{r_0^3}=\frac{2}{3}$

~~此~~ 所以，在球外 $u(r,\theta,\varphi)=\frac{4}{3}\frac{r_0}{r}P_0^0(\cos\theta)-\frac{4}{3}(\frac{r_0}{r})^3P_2^0(\cos\theta)+\frac{2}{3}(\frac{r_0}{r})^3P_2^2(\cos\theta)\sin2\varphi$