2. $f(x)=a_0+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi x}{\frac{T}{2}}+b_k\sin\frac{k\pi x}{\frac{T}{2}}\right)=a_0+\sum_{k=1}^{\infty}\left(a_k\cos\frac{2k\pi x}{T}+b_k\sin\frac{2k\pi x}{T}\right)$

$a_0=\frac{1}{T}\int_0^T f(x)dx=\frac{1}{T}\int_0^T\frac{x}{3}dx=\frac{T}{6}$

$a_k=\frac{2}{T}\int_0^T f(x)\cos\frac{2k\pi x}{T}dx=\frac{2}{T}\int_0^T\frac{x}{3}\cos\frac{2k\pi x}{T}dx=\frac{2}{3T}\left(\frac{Tx}{2k\pi}\sin\frac{2k\pi x}{T}\Big|_0^T-\frac{T}{2k\pi}\int_0^T\sin\frac{2k\pi x}{T}dx\right.$

$=\frac{2}{3T}\frac{T^2}{4k^2\pi^2}\cos\frac{2k\pi x}{T}\Big|_0^T=0$

$b_k=\frac{2}{T}\int_0^T f(x)\sin\frac{2k\pi x}{T}dx=\frac{2}{T}\int_0^T\frac{x}{3}\sin\frac{2k\pi x}{T}dx$

$=\frac{2}{3T}\left[x\frac{-T}{2k\pi}\cos\frac{2k\pi x}{T}\Big|_0^T+\frac{T}{2k\pi}\int_0^T\cos\frac{2k\pi x}{T}dx\right.$

$\therefore f(x)=\frac{T}{6}-\frac{T}{3\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{2k\pi x}{T}$

3. $f(x)=a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$

$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x+x^2)dx=\frac{1}{2\pi}\left(\frac{x^2}{2}+\frac{x^3}{3}\right)\Big|_{-\pi}^{\pi}=\frac{\pi^2}{3}$

$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos kx\,dx=\frac{1}{\pi}\left(\int_{-\pi}^{\pi}x\cos kx\,dx+\int_{-\pi}^{\pi}x^2\cos kx\,dx\right)$

由于第一项奇函数，第二项偶函数 $\therefore a_k=\frac{2}{\pi}\int_0^{\pi}x^2\cos kx\,dx$

$=\frac{2}{\pi}\left(\frac{x^2}{k}\sin kx\Big|_0^{\pi}-\frac{2}{k}\int_0^{\pi}x\sin kx\,dx\right)=\frac{2}{\pi}\left[\frac{2}{k^2}x\cos kx\Big|_0^{\pi}-\frac{2}{k^2}\int_0^{\pi}\cos kx\,dx\right)=\frac{4(-1)^k}{k^2}$

$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin kx\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin kx\,dx=\frac{2}{\pi}\left(\frac{-x\cos kx}{k}\Big|_0^{\pi}+\frac{1}{k}\int_0^{\pi}\cos kx\,dx\right)=\frac{2(-1)^{k+1}}{k}$

$f(x)=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kx+2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin kx}{k}$

由于函数 $f(x)=x+x^2$ 只在 $(-\pi,\pi)$ 成立，$\because f(-\pi)\neq f(\pi)$ $x=\pi$ 是 $f(x)$ 第一类间断点、

$\therefore\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos k\pi+2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin k\pi=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{2}[f(\pi-0)+f(\pi+0)]=\pi^2$

$\therefore 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$

4.(2) $f(x)$ 是偶函数 $f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$

$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-a^2}{1-2a\cos x+a^2}dx=\frac{1-a^2}{2\pi}\oint_{|x|=1}\left[1-a\left(z+\frac{1}{z}\right)+a^2\right]^{-1}\frac{dz}{iz}$

$=\frac{1-a^2}{2\pi}i\oint_{|x|=1}\frac{dz}{az^2-(1+a^2)z+a}=\frac{1-a^2}{2\pi}i\oint_{|x|=1}\frac{dz}{(az-1)(z-a)}=\frac{1-a^2}{2\pi}i\cdot 2\pi i\lim_{x\to a}\frac{1}{az-1}=1$

$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(1-a^2)\cos nx}{1-2a\cos x+a^2}dx=\frac{1-a^2}{\pi}\oint_{|x|=1}\frac{z^x+z^{-x}}{2}\left[1-a\left(z+\frac{1}{z}\right)+a^2\right]^{-1}\frac{dz}{iz}$

$$=\frac{1-a^2}{2\pi i}\oint_{|z|=1}\frac{(z^{2n}+1)dz}{(az-1)(z-a)}=\frac{1-a^2}{2\pi i}\cdot 2\pi i\left[\lim_{z\to a}\frac{z^{2n}+1}{z^n(az-1)}+\frac{1}{(n-1)}\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\frac{z^{2n}+1}{(az-1)(z-a)}\right]$$

$$=(a^2-1)\left[\frac{a^{2n}+1}{a^n(a^2-1)}+\frac{1}{n-1}\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\frac{z^{2n}+1}{a^2-1}\left(\frac{1}{z-a}-\frac{1}{z-\frac{1}{a}}\right)\right]$$

$$=\frac{a^{2n}+1}{a^n}+\frac{1}{(n-1)!}\lim_{z\to 0}C_{n-1}^{0}(z^{2n}+1)\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{z-a}-\frac{1}{z-\frac{1}{a}}\right)$$

$$=\frac{a^{2n}+1}{a^n}+\frac{1}{(n-1)!}\lim_{z\to 0}(z^{2n}+1)(-1)^{n-1}(n-1)!\left[\frac{1}{(z-a)^n}-\frac{1}{(z-\frac{1}{a})^n}\right]$$

$$=\frac{a^{2n}+1}{a^n}+(-1)^{n-1}\left[\frac{1}{(-a)^n}-\frac{1}{(-a^{-1})^n}\right]=\frac{a^{2n}+1}{a^n}+\left[a^n-\frac{1}{a^n}\right]=2a^n$$

4.(4) $(-\pi,\pi)$上，$f(x)=\cosh ax$（a非整数）

由于$f(x)$为偶函数 $\therefore f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$

$$a_0=\frac{1}{\pi}\int_0^{\pi}\cosh ax\,dx=\frac{1}{a\pi}\sinh ax\Big|_0^{\pi}=\frac{\sinh a\pi}{a\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\cosh ax\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}[\cos(n+i\alpha)x+\cos(n-i\alpha)x]dx$$

$$=\frac{1}{\pi}\left[\frac{\sin(n+i\alpha)x}{n+i\alpha}+\frac{\sin(n-i\alpha)x}{n-i\alpha}\right]\Big|_0^{\pi}=\frac{(-1)^n\sin i\alpha\pi}{\pi}\left(\frac{1}{n+i\alpha}-\frac{1}{n-i\alpha}\right)=\frac{(-1)^n 2a\sinh a\pi}{\pi(n^2+a^2)}$$

$$\therefore f(x)=\frac{\sinh a\pi}{a\pi}+\frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2+a^2}$$

5.(2) $f(x)$奇延拓后，得$f(x)=\sum_{n=1}^{\infty}b_n\sin nx$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}x^3\sin nx\,dx=\frac{2}{\pi}\left(x^3\frac{-\cos nx}{n}\Big|_0^{\pi}+\frac{3}{n}\int_0^{\pi}x^2\cos nx\,dx\right)$$

$$=\frac{2}{\pi}\left\{\frac{-(-1)^n\pi^3}{n}+\frac{3}{n}\left[x^2\frac{\sin nx}{n}\Big|_0^{\pi}-\frac{2}{n}\int_0^{\pi}x\sin nx\,dx\right]\right\}=\frac{2}{\pi}\left[-\frac{(-1)^n\pi^3}{n}+\frac{6(-1)^n\pi}{n^3}\right]=(-1)^n\left(-\frac{2\pi^2}{n}+\frac{12}{n^3}\right)$$

$$f(x)=\sum_{n=1}^{\infty}(-1)^n\left(-\frac{2\pi^2}{n}+\frac{12}{n^3}\right)\sin nx,\ x\in(0,\pi)$$

6.(1) 把$(0,l)$上函数偶延拓，得$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$

$a_0=\frac{1}{l}\int_0^l f(x)dx=\frac{1}{l}\int_0^{\frac{l}{2}}\cos\frac{\pi x}{l}dx=\frac{1}{\pi}$

$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx=\frac{2}{l}\int_0^{\frac{l}{2}}\cos\frac{\pi x}{l}\cos\frac{n\pi x}{l}dx=\frac{1}{l}\int_0^{\frac{l}{2}}[\cos\frac{(n+1)\pi x}{l}+\cos\frac{(n-1)\pi x}{l}]dx$

当$n=1$时，$a_1=\frac{1}{l}\int_0^{\frac{l}{2}}(\cos\frac{2\pi x}{l}+1)dx=\frac{1}{2}$

当$n\geqslant 2$时，$a_n=\frac{1}{l}[\frac{-\sin\frac{n+1}{2}\pi}{(n+1)\pi}l+\frac{\sin\frac{n-1}{2}\pi}{(n-1)\pi}l]=\frac{-2}{\pi(n^2-1)}\cos\frac{n\pi}{2}=\begin{cases}\frac{-2(-1)^k}{\pi(4k^2-1)}, & n=2k\\ 0, & n=2k+1\end{cases}$

$f(x)=\frac{1}{\pi}+\frac{1}{2}\cos\frac{\pi x}{l}-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}\cos\frac{2k\pi x}{l}, x\in(0,l)$

1. (1) 实数形式傅里叶变换，$f(t)=\int_0^{\infty}A(w)\cos wt\,dw+\int_0^{\infty}B(w)\sin wt\,dw$

$A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos wt\,dt=\frac{1}{\pi}\int_0^T kt\cos wt\,dt=\frac{k}{\pi}[t\frac{\sin wt}{w}\Big|_0^T-\int_0^T\frac{\sin wt}{w}dt]$

$=\frac{k}{\pi}[T\frac{\sin wT}{w}+\frac{\cos wT-1}{w^2}]=\frac{k}{\pi w^2}(\cos wT+wT\sin T-1)$

$B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin wt\,dt=\frac{k}{\pi}(t\frac{-\cos wt}{w}\Big|_0^T+\int_0^T\frac{\cos wt}{w}dt)$

$=\frac{k}{\pi}(-\frac{T\cos wT}{w}+\frac{\sin wT}{w^2})=\frac{k}{\pi w^2}(\sin wT-wT\cos wT)$

(2) 复数形式傅里叶变换 $f(t)=\int_{-\infty}^{\infty}F(w)e^{iwt}dw$

$F(w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)e^{iwt}dt=\frac{1}{2\pi}\int_0^T kte^{-iwt}dt=\frac{k}{2\pi}(t\frac{e^{-iwt}}{-iw}\Big|_0^T+\int_0^T\frac{e^{-iwt}}{iw}dt)$

$=\frac{k}{2\pi}[\frac{iT}{w}e^{-iwT}+\frac{1}{w^2}(e^{-iwT}-1)]=\frac{k}{2\pi w^2}(iwTe^{-iwT}+e^{-iwT}-1)$

3. $\because f(x)$是奇函数 $\therefore f(t)=\int_0^{\infty}B(w)\sin wt\,dw$

$B(w)=\frac{2}{\pi}\int_0^{\infty}f(t)\sin wt\,dt=\frac{2}{\pi}\int_0^T h\sin wt\,dt=\frac{2h}{\pi w}(1-\cos wT)$

$\therefore f(t)=\frac{2h}{\pi}\int_0^{\infty}\frac{1-\cos wT}{w}\sin wt\,dw$

5. 将$f(x)$奇延拓，令$F(x)=\begin{cases}f(x), & x>0\\ -f(-x), & x<0\end{cases}$ 正弦积分

~~F(x)~~ $B(w)=\frac{2}{\pi}\int_0^{\infty}e^{-\lambda x}\sin wx\,dx=\frac{2}{\pi}[\frac{e^{-\lambda x}}{-\lambda}\sin wx\Big|_0^{\infty}+\frac{w}{\lambda}\int_0^{\infty}e^{-\lambda x}\cos wx\,dx]$

$=\frac{2w}{\pi\lambda}[\frac{e^{-\lambda x}}{-\lambda}\cos wx\Big|_0^{\infty}-\frac{w}{\lambda}\int_0^{\infty}e^{-\lambda x}\sin wx\,dx]=\frac{2w}{\pi\lambda^2}-(\frac{w}{\lambda})^2B(w)$

解得$B(w)=\frac{2w}{\pi(\lambda^2+w^2)}$

$\therefore f(x)=\int_0^{\infty}\frac{2w\sin wx}{\pi(\lambda^2+w^2)}dw, 0<x<\infty$