南開大學 作业纸

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1.试推导极坐标中柯西-黎曼方程

解、从 $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$ 将 $\rho=\sqrt{x^2+y^2}$ 和 $\varphi=\operatorname{arctg}\frac{y}{x}$ 代入

$$\begin{cases} \dfrac{\partial \rho}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{\rho} = \cos\varphi \\ \dfrac{\partial \rho}{\partial y} = \dfrac{y}{\rho} = \sin\varphi \end{cases}$$

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = \dfrac{-y}{x^2+y^2} = -\dfrac{\sin\varphi}{\rho} \\ \dfrac{\partial \varphi}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{\cos\varphi}{\rho} \end{cases}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\cdot\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x} = \cos\varphi\cdot\frac{\partial u}{\partial \rho} - \frac{1}{\rho}\sin\varphi\frac{\partial u}{\partial \varphi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \rho}\cdot\frac{\partial \rho}{\partial y} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y} = \sin\varphi\frac{\partial u}{\partial \rho} + \frac{1}{\rho}\cos\varphi\frac{\partial u}{\partial \varphi}$$

同理 $\dfrac{\partial v}{\partial x} = \cos\varphi\dfrac{\partial v}{\partial \rho} - \dfrac{1}{\rho}\sin\varphi\dfrac{\partial v}{\partial \varphi}$ $\quad \dfrac{\partial v}{\partial y} = \sin\varphi\dfrac{\partial v}{\partial \rho} + \dfrac{1}{\rho}\cos\varphi\dfrac{\partial v}{\partial \varphi}$

代入柯西-黎曼方程：

$$\cos\varphi\cdot\frac{\partial u}{\partial \rho} - \frac{1}{\rho}\sin\varphi\frac{\partial u}{\partial \varphi} = \sin\varphi\frac{\partial v}{\partial \rho} + \frac{1}{\rho}\cos\varphi\frac{\partial v}{\partial \varphi} \quad (1)$$

$$\sin\varphi\frac{\partial u}{\partial \rho} + \frac{1}{\rho}\cos\varphi\frac{\partial u}{\partial \varphi} = -\cos\varphi\frac{\partial v}{\partial \rho} + \frac{1}{\rho}\sin\varphi\frac{\partial v}{\partial \varphi} \quad (2)$$

(1) × $\sin\varphi$ − (2) × $\cos\varphi$ 得 $\begin{cases} \dfrac{\partial u}{\partial \rho} = \dfrac{1}{\rho}\dfrac{\partial v}{\partial \varphi} \\ \dfrac{\partial v}{\partial \rho} = -\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi} \end{cases}$

(1) 某个区域上解析函数如为实函数，试证它必为常数

设 $w(z)=u(x,y)+iv(x,y)$ $\because$ 实函数 $\therefore iv(x,y)=0$

由柯西黎曼方程 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0$ $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0$

$\therefore$ u 必为常数 即该解析函数必为常数

2.(2)

$$\begin{cases} \frac{\partial u}{\partial x}=e^x(x\cos y+\cos y-y\sin y)=\frac{\partial v}{\partial y} \\ -\frac{\partial u}{\partial y}=e^x(x\sin y+\sin y+y\cos y)=\frac{\partial v}{\partial x} \end{cases}$$

$$\therefore v=e^x(x\sin y+y\cos y)+C$$

$$f(z)=e^x(x\cos y-y\sin y)+ie^x(x\sin y+y\cos y)+iC$$

$$=xe^x(\cos y+i\sin y)-e^xy(\sin y-i\cos y)+iC$$

$$=xe^x\cdot e^{iy}+iye^xe^{iy}+iC=e^{x+iy}(x+iy)+iC$$

$\because f(0)=iC=0$ $\therefore C=0$

$$\therefore f(z)=ze^z$$

2.(3) $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=\frac{4\sin 2x(e^{2x}-e^{-2y})}{(e^{2y}+e^{-2y}-2\cos 2x)^2}$

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{4\cos 2x(e^{2x}+e^{-2y}-2\cos 2x)-8\sin^2 2x}{(e^{2y}+e^{-2y}-2\cos 2x)^2}$$

$$dv=\frac{4\sin 2x(e^{2y}+e^{-2y})dx+4[\cos x(e^{2y}+e^{-2y})-2]dy}{(e^{2y}+e^{-2y}-2\cos 2x)^2}$$

$$v=-\frac{e^{2y}-e^{-2y}}{e^{2y}+e^{-2y}-2\cos 2x},\ \because f\left(\frac{\pi}{2}\right)=0\ \therefore C=0$$

$$f(z)=u+iv=\frac{2\sin x-i(e^{2y}-e^{-2y})}{e^{2y}+e^{-2y}-2\cos 2x}=\operatorname{ctg} z$$

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2(4) 令:

$v=\frac{1}{\rho}\sin\varphi$

利用 $\frac{\partial u}{\partial \rho}=\frac{1}{\rho}\frac{\partial v}{\partial \varphi}=\frac{1}{\rho^2}\cos\varphi$

$\frac{1}{\rho}\frac{\partial u}{\partial \varphi}=-\frac{\partial v}{\partial \rho}=\frac{1}{\rho^2}\sin\varphi$

$\therefore u=-\frac{1}{\rho}\cos\varphi+C$

$f(z)=\frac{1}{\rho}(-\cos\varphi+i\sin\varphi)+C$

$=\frac{1}{\rho}e^{-i\varphi}+C=-\frac{1}{z}+C$

$f(2)=0 \therefore C=\frac{1}{2}$

$\therefore f(z)=\frac{1}{2}-\frac{1}{z}$

(8) $\frac{\partial u}{\partial x}=3x^2+12xy-3y^2=\frac{\partial v}{\partial y}$

$-\frac{\partial u}{\partial y}=-6x^2+6xy+6y^2=\frac{\partial v}{\partial x}$

$\therefore v=-2x^3+3x^2y+6xy^2-y^3+C$

$f(z)=x^3+6x^2y-3xy^2-2y^3+i(-2x^3+3x^2y+6xy^2-y^3)+iC$

$=(x+iy)^3-2i(x+iy)^3+iC=z^3(1-2i)+iC$

$\because f(0)=0 \therefore C=0 \quad \therefore f(z)=z^3(1-2i)$

(10) $\frac{\partial u}{\partial \rho}=\frac{1}{\rho}\frac{\partial v}{\partial \varphi}=\frac{1}{\rho}$ $\quad \therefore \begin{cases}\frac{\partial v}{\partial \varphi}=1\\ \frac{\partial v}{\partial \rho}=0\end{cases}$ $\quad \therefore v=\varphi+C$

$\frac{1}{\rho}\frac{\partial u}{\partial \varphi}=-\frac{\partial v}{\partial \rho}=0$

$f(z)=\ln\rho+i\varphi+iC=\ln z+iC$ 又 $\because f(1)=0$

$\therefore f(z)=\ln z$

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2.(1) $\frac{\partial u}{\partial \rho}=\frac{1}{\rho}\frac{\partial v}{\partial \varphi}=0 \Rightarrow \begin{cases}\frac{\partial v}{\partial \varphi}=0\\ \frac{\partial v}{\partial \rho}=-\frac{1}{\rho}\end{cases} \quad v=-\ln\rho+C$

$\frac{1}{\rho}\frac{\partial u}{\partial \varphi}=-\frac{\partial v}{\partial \rho}=\frac{1}{\rho}$

$f(z)=\varphi-i\ln\rho+iC=-i\ln z+iC$

$\because f(1)=0 \quad \therefore f(z)=-i\ln z$

1. $f(z)=\frac{1}{(x-2)-i(y+1)}=\frac{x-2}{(x-2)^2+(y+1)^2}+i\frac{-(y+1)}{(x-2)^2+(y+1)^2}$

得 $C_1=\frac{x-2}{(x-2)^2+(y+1)^2}$ 故等温网为在(2,1)与x=2, y=-1相切的圆族

$C_2=\frac{y+1}{(x-2)^2+(y+1)^2}$

3. 令 $u=F(t)\ (t=x^2+y^2)$

(i) $\begin{cases}u_x=2xF'\\ u_y=2yF'\end{cases} \quad u_{xx}=2F'+4x^2F'' \quad u_{yy}=2F'+4y^2F'' \qquad \frac{F''}{F'}=\frac{-1}{x^2+y^2}=-\frac{1}{t} \quad F'=\frac{C_1}{t}$

$\therefore F=C_1\ln(x^2+y^2)+C_2 \quad u=C_1\ln(x^2+y^2)+C_2$

(ii) $u_x=\frac{C_1 2x}{x^2+y^2}=v_y,\ u_y=\frac{2yC_1}{x^2+y^2} \qquad \therefore v=C_1\int^{(y)}\frac{2x}{x^2+y^2}dy-2C_1\operatorname{artg}\frac{y}{x}+C_4(x)$

又 $\because v_x=2C_1\frac{-y}{x^2+y^2}+C_4'(x)=-u_y=\frac{-2yC_1}{x^2+y^2}$

$\therefore C_4'(x)=0 \quad C_4(x)=C_3$

$v=2C_1\operatorname{arctg}\frac{y}{x}+C_3=-iC_1\ln\frac{(x+iy)^2}{x^2+y^2}+C_3$

(iii) $f(z)=C_1\ln(x^2+y^2)+C_2+i[-iC_2\ln\frac{(x+iy)^2}{x^2+y^2}+C_3]$

$=C_1\ln(x^2+y^2)+C_2+C_1\ln\frac{(x+iy)^2}{x^2+y^2}+iC_3$

$=C\ln z^2+C_2+iC_3=2C_1\ln z+C_2+iC_3$