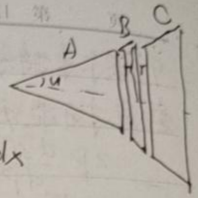
$$) = \begin{pmatrix} a_1 & a_2 & a_3 \\ & b_1 & b_2 \end{pmatrix} \begin{pmatrix} 1 \\ & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ & b_1 & b_2 \end{pmatrix}$$

2. 设想在圆锥杆上截一小段B，C段对B的拉力为Y，



合力为 $Y\frac{\partial}{\partial x}\cdot\pi r^2 u dx$

B段质量为$\rho\pi r^2 dx$ $\rho\pi r^2 dx\cdot u = Y\frac{\partial}{\partial x}\pi r^2 u dx$

$\pi x^2 \rho dx u = Y\frac{\partial}{\partial x}\cdot(\pi x^2 u)dx$

$$x^2 u = \frac{Y}{\rho}\cdot\frac{\partial}{\partial x}(x^2 u)$$

$$\therefore a^2 = \frac{Y}{\rho}$$

$$\therefore u = a^2\cdot\frac{1}{x^2}\frac{\partial}{\partial x} = -\frac{a^2}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)$$

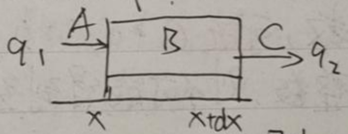
3. 弦所受力除T_1、T_2外，还要受到阻力F的作用

$T_1 \approx T_2$ $\left(T_2 u_x|_{x+dx} - T_1 u_x|_x\right) - R u_1 dx = u_u \rho dx$

$$T u_{xx} - R u_t = \rho u_{tt}$$

$$u_{tt} - a^2 u_{xx} + \frac{R}{\rho}u_t = 0 \quad \left(a^2 = \frac{T}{\rho}\right)$$

4.



(1) 任取一小体积B，在Δt时间内，流入热量为$q_1 s\Delta t$，流出为$q_2 s\Delta t$

净流入为$Q=(q_1-q_2)s\Delta t$，由于$q_1=-k\frac{\partial u}{\partial x}\Big|_x$，$q_2=-k\frac{\partial u}{\partial x}\Big|_{x+\Delta x}$

$Q=\left(k\frac{\partial u}{\partial x}\Big|_{x+\Delta x}-k\frac{\partial u}{\partial x}\Big|_x\right)s\Delta t$ 设升高温度du，比热容为C，密度ρ

$$C\rho\, dx\, du = (q_1-q_2)dt = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)dx\,dt$$

即 $C\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)$

(3) 任取一小体积 dV，在x方向上 $q_1=-Ku_x|_x$，$q_2=-Ku_x|_{x+dx}$

$Q_1=(K\frac{\partial u}{\partial x}|_{x+dx}-K\frac{\partial u}{\partial x}|_x)dydz\Delta t=\frac{\partial}{\partial x}(K\frac{\partial u}{\partial x})dxdydz\Delta t$

通过y的则为 $Q_2=\frac{\partial}{\partial y}(K\frac{\partial u}{\partial y})dydxdz\Delta t$

通过z的则为 $Q_3=\frac{\partial}{\partial z}(K\frac{\partial u}{\partial z})dzdydx\Delta t$

$Q_总=Q_1+Q_2+Q_3$ ，$dV=dxdydz$

$C\rho\, du\,dxdydz=[\frac{\partial}{\partial x}(K\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(K\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(K\frac{\partial u}{\partial z})]dxdydz$

即 $C\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}(K\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(K\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(K\frac{\partial u}{\partial z})$

5、设浇灌后初始水化热密度为 Q_0，则t时刻为 $\int_{Q_0}^{Q}\frac{dQ}{Q}=\int_0^t-\beta dt$

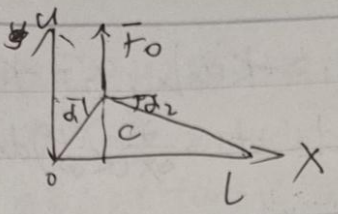
$\ln Q=-\beta t+\ln Q_0$

$Q=Q_0e^{-\beta t}$ ，单位体积内为 $\beta Q_0e^{-\beta t}$

将4的方程代入

$C\rho u_t-[\frac{\partial}{\partial x}(Ku_x)+\frac{\partial}{\partial x}(Ku_y)+\frac{\partial}{\partial z}(Ku_z)]=Q_0\beta e^{-\beta t}$

1.



弦初始 $u|_{t=0}$ $\begin{cases} \frac{c}{h}x \ (0\le x\le h) \\ \frac{c}{l-h}(l-x)\ (h\le x\le l)\end{cases}$

由小振动 $\sin\alpha_1 = \tan\alpha_1 = \frac{c}{h}$

$\sin\alpha_2 = \tan\alpha_2 = \frac{c}{l-h}$

$\cos\alpha_1 \approx \cos\alpha_2 \approx 1$，$dS \approx dx$

$F = T_1\sin\alpha_1 + T_2\sin\alpha_2$

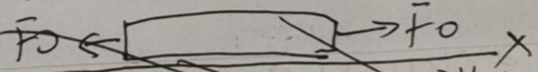
$= T\left(\frac{c}{h} + \frac{c}{l-h}\right)$

$C = \frac{F_0 h(l-h)}{Tl}$ 代入初始位移

$u|_{t=0} = \begin{cases} \frac{F_0(l-h)}{T_0 l}x \ (0\le x\le h) \\ \frac{F_0 h}{T_0 l}(l-x)\ (h\le x\le l)\end{cases}$

初始速度 $u|_{t=0}=0$

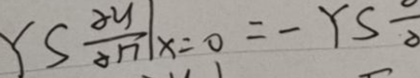
2.



$YS\frac{\partial u}{\partial n}\Big|_{x=0} = -YS\frac{\partial u}{\partial x}\Big|_{x=0} = -F_0$

$\therefore YS\frac{\partial u}{\partial x}\Big|_{x=0} = F_0$

2.

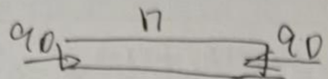


$YS\frac{\partial u}{\partial n}\Big|_{x=0} = -YS\frac{\partial u}{\partial x}\Big|_{x=0} = -F_0$

$\therefore YS\frac{\partial u}{\partial x}\Big|_{x=0} = F_0$

$YS\frac{\partial u}{\partial n}\Big|_{x=l} = YS\frac{\partial u}{\partial x}\Big|_{x=l} = F_0$

3.



在边界上有 $-K\frac{\partial u}{\partial n}\Big|_{x} = q_0$

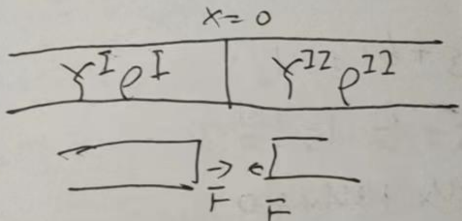
在 $x=l$ 端，$-K\frac{\partial u}{\partial n}\Big|_{x=l} = -K\frac{\partial u}{\partial x}\Big|_{x=l} = -q_0$

即 $K\frac{\partial u}{\partial n}\Big|_{x=0} = q_0$

在 $x=0$ 端，$-K\frac{\partial u}{\partial n}\Big|_{x=0} = K\frac{\partial u}{\partial x}\Big|_{x=0} = -q_0$

$\therefore K\frac{\partial u}{\partial x}\Big|_{x=0} = -q$

6.



由于 $x=0$ 处位移连续，$u^{I}\big|_{x=0} = u^{II}\big|_{x=0}$ (1)

$F_1 = Y^{I} S\frac{\partial u^{I}}{\partial n}\Big|_{x-0} = Y^{I} S\frac{\partial u^{I}}{\partial x}\Big|_{x-0}$

$F_2 = Y^{II} S\frac{\partial u^{II}}{\partial n}\Big|_{x+0} = Y^{II} S\frac{\partial u^{II}}{\partial x}\Big|_{x+0}$

由于 $F_1\ F_2$ 互为反作用力

$Y^{I} S\frac{\partial u^{I}}{\partial x}\Big|_{x-0} = Y^{II} S\frac{\partial u^{II}}{\partial x}\Big|_{x+0}$

1.(1) $au_{xx}+2au_{xy}+au_{yy}+bu_x+cu_y+u=0$

由于 $a_{12}^2-a_{11}a_{22}=a^2-a\cdot a=0$

所以该方程为抛物型特征方程，令 $\xi=y-x,\eta=x$

设 $u_{\eta\eta}=-\frac{1}{A_{22}}(B_1u_\xi+B_2u_\eta+Cu+F)$

$A_{22}=a_{11}\eta_x^2+2a_{12}\eta_x\eta_y+a_{22}\eta_y^2=a+2a\cdot0+a\cdot0=a$

$B_1=a_{11}\xi_{xx}+2a_{12}\xi_{xy}+a_{22}\xi_{yy}+b_1\xi_x+b_2\xi_y=c-b$

$B_2=a_{11}\eta_{xx}+2a_{12}\eta_{xy}+a_{22}\eta_{yy}+b_1\eta_x+b_2\eta_y=b$

$C=1\quad F=0$

$\therefore u_{\eta\eta}=-\frac{1}{a}[(b-c)u_\xi+bu_\xi+u]$

即 $u_{\eta\eta}+\frac{c-b}{a}u_\xi+\frac{b}{a}u_\xi+\frac{u}{a}=0$

(3) $u_{xx}+4u_{xy}+5u_{yy}+u_x+2u_y=0$

$a_{12}^2-a_{11}a_{22}=-1<0$，该方程为椭圆型

特征方程 $\frac{dy}{dx}=\frac{2\pm\sqrt{2^2-5}}{1}=2\pm i$

特征线为 $(2+i)x-y=C_1$ 和 $(2-i)x-y=C_2$

令 $\xi=(2+i)x-y$，$\eta=(2-i)x-y$，

为使计算方便，$\alpha=\frac{\xi+\eta}{2}=2x-y$，$\beta=\frac{\xi-\eta}{2i}=x$

$u_{\alpha\alpha}+u_{\beta\beta}=-\frac{1}{A_{12}}[(B_1+B_2)u_\alpha+i(B_2-B_1)u_\beta+2Cu+F]$

$A_{12}=a_{11}\xi_x\eta_x+a_{12}(\xi_x\eta_y+\xi_y\eta_x)+a_{22}\xi_y\eta_y=2$

$B_1=a_{11}\xi_{xx}+2a_{12}\xi_{xy}+a_{22}\xi_{yy}+b_1\xi_x+b_2\xi_y=i$

$B_2=a_{11}\eta_{xx}+2a_{12}\eta_{xy}+a_{22}\eta_{yy}+b_1\eta_x+b_2\eta_y=-i$

$C=0,F=0$

$\therefore u_{\alpha\alpha}+u_{\beta\beta}=-\frac{1}{2}[i(-2i)u_\beta]$

即 $u_{\alpha\alpha}+u_{\beta\beta}+u_\beta=0$，$u_{\xi\xi}+u_{\eta\eta}+u_\eta=0$

(5) $u_{xx}+xu_{yy}=0$

$a_{12}^2-a_{11}a_{22}=-x$ $\quad \frac{dy}{dx}=-\sqrt{x}$

① $x<0$，方程为双曲型

特征线 $y+\frac{2}{3}(-x)^{\frac{3}{2}}=C_1$ 和 $y-\frac{2}{3}(-x)^{\frac{3}{2}}=C_2$

令 $\xi=\frac{3}{2}y+(-x)^{\frac{3}{2}}$，$\eta=\frac{3}{2}y-(-x)^{\frac{3}{2}}$

$u_{\xi\eta}=-\frac{1}{2A_{12}}(B_1u_\xi+B_2u_\eta+Cu+F)$

$A_{12}=-\frac{3}{2}\sqrt{-x}\cdot\frac{3}{2}\sqrt{-x}+x\cdot\frac{3}{2}\cdot\frac{3}{2}=\frac{9}{2}x$

$B_1=\frac{3}{4}\cdot\frac{1}{\sqrt{-x}}$

$B_2=\frac{3}{4}\cdot\frac{1}{\sqrt{-x}}$

$u_{\xi\eta}=-\frac{1}{9x}\cdot\frac{3}{4\sqrt{-x}}(u_\xi-u_\eta)$

$=\frac{1}{12(-x)^{\frac{3}{2}}}(u_\xi-u_\eta)$

$\xi-\eta=2(-x)^{\frac{3}{2}}$ 代入

$u_{\xi\eta}=\frac{u_\xi-u_\eta}{6(\xi-\eta)}=0$

② $x>0$，方程为椭圆型 $\quad \frac{dy}{dx}=\pm\sqrt{x}\,i$

特征线 $\frac{3}{2}y+ix^{\frac{3}{2}}=C_1$，$\frac{3}{2}y-ix^{\frac{3}{2}}=C_2$

令 $\xi=\frac{3}{2}y$，$\eta=-x^{\frac{3}{2}}$

则 $u_y=u_\xi\xi_y=\frac{3}{2}u_\xi$ $\quad u_{yy}=\frac{9}{4}u_{\xi\xi}$

$u_x=u_\eta(-\frac{3}{2}x^{\frac{1}{2}})=-\frac{3}{2}u_\eta x^{\frac{1}{2}}$ $\quad u_{xx}=u_{\eta\eta}\cdot\frac{x}{4}-u_\eta\frac{1}{4\sqrt{x}}$

方程为 $\frac{9}{4}xu_{\eta\eta}+\frac{9}{4}xu_{\xi\xi}-u_\eta\frac{3}{4\sqrt{x}}=0$

$u_{\xi\xi}+u_{\eta\eta}-\frac{1}{3x^{\frac{3}{2}}}u_\eta=0$

即 $u_{\xi\xi}+u_{\eta\eta}+\frac{u_\eta}{3\eta}=0$

2.(3) $u_{xx}=\frac{1}{a^2}u_y+\alpha u+\beta u_x$

作函数变换 $u=ve^{\lambda x+\mu y}$，并以 u_x,u_y,u_{xx} 及 u 代入原方程约去公共因子 $e^{\lambda x+\mu y}$ 得

$$v_{xx}+(2\lambda-\beta)v_x-\frac{v_y}{a^2}+(\lambda^2-\frac{\mu}{a^2}-\alpha-\lambda\beta)v=0$$

消去 v_x，$\Rightarrow 2\lambda=\beta$，消去 v $\Rightarrow \lambda^2=\frac{\mu}{a^2}+\alpha+\lambda\beta$

即 $\mu=-a^2(\alpha+\frac{\beta^2}{4})$，即 $u=ve^{\frac{\beta}{2}x-a^2(\alpha+\frac{\beta^2}{4})y}$

即该常微分方程化为 $v_{xx}-\frac{v_y}{a^2}=0$

(4) $u_{xy}+3u_x+4u_y+2u=0$

作函数变换 $u=ve^{\lambda x+\mu y}$，代入并约去公共因子

$$v_{xy}+(\mu+3)v_x+(\lambda+4)v_y+(\lambda\mu+3\lambda+4\mu+2)v=0$$

令 $\lambda=-4$，$\mu=-3$，即 $u=ve^{-4x-3y}$

则原方程化为 $v_{xy}-10v=0$

1. 无限长弦的自由振动，设弦初始位移 $\varphi(x)$，初始速度 $-a\varphi'(x)$

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\ -\infty<x<+\infty \\ u|_{t=0}=\varphi(x) \\ u_t|_{t=0}=-a\varphi'(x) \end{cases}$$

这是个一维无限空间问题，由达朗伯公式

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\varphi(\xi)d\xi$$

将初始位移和速度代入得

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}[-a\varphi'(\xi)]d\xi$$

$$=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]-\frac{1}{2}\int_{x-at}^{x+at}\varphi'(\xi)d\xi$$

$$=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]-\frac{1}{2}\varphi(x+at)+\frac{1}{2}\varphi(x-at)$$

$$=\varphi(x-at)$$

波只朝一个方向传播，是一列行波

2.

(1) 电压：$V_{tt}-a^2V_{xx}=0$ 其中 $a^2=\frac{1}{LC}$

初始 $$\begin{cases} V|_{t=0}=A\cos kx=\varphi(x) \\ V_t|_{t=0}=-\frac{1}{C}\sqrt{\frac{C}{L}}Ak\sin(-kx)=aAk\sin kx=\varphi(x) \end{cases}$$

$=(-\frac{1}{C}\cdot j|_{t=0})$

应用达朗伯公式：

$$V(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\varphi(\xi)d\xi$$

$$=\frac{1}{2}[A\cos k(x+at)+A\cos k(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}aA\sin k\xi d\xi$$

$$=\frac{1}{2}[A\cos k(x+at)+A\cos k(x-at)]+\frac{A}{2}[-\cos k(x+at)+\cos k(x-at)]$$

$$=\frac{1}{2}[A\cos k(x+at)$$

$$=A\cos k(x-at)$$

(2) 传输线方程：$j_{tt}-a^2 j_{xx}=0$，式中 $a^2=\frac{1}{LC}$

初始 $\begin{cases} j|_{t=0}=\sqrt{\frac{C}{L}}A\cos kx=\varphi(x) \\ j_t|_{t=0}=-\frac{1}{L}v_x|_{t=0}=\frac{Ak}{L}\sin kx=\varphi(x) \end{cases}$

应用一维无界空间的达朗伯公式

$$j(x,t)=\frac{1}{2}\left[\sqrt{\frac{C}{L}}A\cos k(x+at)+\sqrt{\frac{C}{L}}A\cos k(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\frac{Ak}{L}\sin k\xi d\xi$$

$$=\frac{A}{2}\sqrt{\frac{C}{L}}[\cos k(x+at)+\cos k(x-at)]+\frac{\sqrt{LC}}{2L}A[-\cos k(x+at)+\cos k(x-at)]$$

$$=\sqrt{\frac{C}{L}}A\cos k(x-at)$$

4. $\begin{cases} u_{tt}-a^2u_{xx}=0 \quad (a^2=\frac{T}{\rho}) \\ u|_{t=0}=0 \\ u_t|_{t=0}=\frac{I}{\rho}\delta(\xi-x_0)=\frac{I}{\rho}H'(\xi-x_0) \end{cases}$

$$u(x,t)=\frac{1}{2a}\int_{x-at}^{x+at}\left(\frac{I}{\rho}\right)\delta(x-x_0)d\xi$$

$$=\frac{1}{2a\rho}\int_{x-at}^{x+at}H'(\xi-x_0)d(\xi-x_0)$$

$$=\frac{1}{2a\sqrt{\rho}}H(\xi-x_0)\Big|_{x-at}^{x+at}$$

$$=\frac{1}{2\sqrt{\rho T}}[H(x-x_0+at)-H(x-x_0-at)]$$

5. 已知细圆锥杆振动方程

$$u_{tt} - a^2 \cdot \frac{1}{x^2} \frac{\partial}{\partial x}(x^2 u_x) = 0$$

令 $u(x,t) = \frac{v(x,t)}{x}$

$$\frac{\partial u}{\partial x} = \frac{1}{x} \qquad \frac{\partial v}{\partial x} = -\frac{v}{x^2}$$

$$\frac{\partial}{\partial x}(x^2 u_x) = \frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x} - v\right) = x\frac{\partial^2 v}{\partial x^2}$$

$$u_{tt} = \frac{1}{x} v_{tt}$$

代入原方程得

$$\frac{v_{tt}}{x} - \frac{a^2}{x^2} \cdot x\frac{\partial^2 v}{\partial x^2} = 0$$

即 $v_{tt} - a^2 v_{xx} = 0$

$v(x,t)$ 通解为 $f_1(x-at) + f_2(x+at)$

$$\therefore u(x,t) = \frac{v(x,t)}{x} = \frac{1}{x}[f_1(x-at) + f_2(x+at)]$$