2. $u|_{t=0} = bx(l-x)/l^2$

$$\begin{cases} u_t - a^2 u_{xx} = 0 \quad (a^2 = \frac{k}{c\rho})\ (0 \le x \le l) \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = bx(l-x)/l^2 \end{cases}$$

设 $u = X(x)T(t)$，代入

$$\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda$$

$$\therefore \begin{cases} T' + a^2\lambda T = 0 \\ X'' + \lambda X = 0,\ X(0) = X(l) = 0 \end{cases}$$

解得 $X(x) = \sum (A_n' \cos\sqrt{\lambda}x + B_n' \sin\sqrt{\lambda}x)$

又 $\because X(0) = 0 \quad \therefore A_n' = 0 \quad X(l) = 0 \quad \therefore \sin\sqrt{\lambda}l = 0 \quad \sqrt{\lambda}l = n\pi$

$\lambda = \frac{n^2\pi^2}{l^2} \quad X(x) = \sum B_n' \sin\frac{n\pi x}{l}$

又 $\because T_n' + \frac{n^2\pi^2a^2}{l^2}T_n = 0 \quad \therefore T_n = C_n e^{-\frac{n^2\pi^2a^2}{l^2}t}$

$\therefore u(x,t) = \sum\limits_{n=1}^{\infty} B_n' C_n e^{-\frac{n^2\pi^2a^2}{l^2}t} \sin\frac{n\pi x}{l} = \sum\limits_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2a^2t}{l^2}} \sin\frac{n\pi}{l}x$

又 $\because u(x,0) = \frac{bx(l-x)}{l^2}$

$\therefore B_n = \frac{2}{l}\int_0^l \frac{bx(l-x)}{l^2} \sin\frac{n\pi x}{l}dx$

$= \frac{2b}{l^3}\int_0^l x(l-x)\sin\frac{n\pi x}{l}dx$

$= \frac{2b}{l^3}, l \cdot \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{l}x\Big|_0^l - \frac{l^2}{n\pi}x\cos\frac{n\pi x}{l}\Big|_0^l -$

$-\frac{xl^2}{n^2\pi^2}\left(2\sin\frac{n\pi x}{l} - \frac{n\pi x}{l}\cos\frac{n\pi x}{l}\right)\Big|_0^l + \frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l}\Big|_0^l$

$= \frac{2b}{l^3}\left\{-\frac{l^3}{n\pi}(-1)^n + \frac{l^3}{n\pi}(-1)^n + \frac{2l^3}{n^2\pi^2}[(-1)^n - 1]\right\} = \begin{cases} \frac{8b}{\pi^3(2k+1)^3}, & n = 2k+1 \\ 0, & n = 2k \end{cases}$

$\therefore u(x,t) = \sum\limits_{k=0}^{n} B_k e^{-\frac{(2k+1)^2}{l^2}\pi^2a^2t} \sin\frac{(2k+1)\pi x}{l}$

$= \frac{8b}{\pi^3}\sum\limits_{k=0}^{\infty}\frac{1}{(2k+1)^2} e^{-\frac{(2k+1)^2\pi^2a^2t}{l^2}} \cdot \sin\frac{(2k+1)\pi x}{l}$

3. (i) 若为平面锤，
$$\begin{cases} u_{tt}-a^2u_{xx}=0 \ (0<x<l) \\ u|_{x=0}=u|_{x=l}=0 \\ u|_{t=0}=0 \\ u_t|_{t=0}=\begin{cases} 0 \ (0<x<x_0-\delta,\ x_0+\delta<x<l) \\ v_0 \ (x_0-\delta<x<x_0+\delta) \end{cases} \end{cases}$$

根据边界条件，可知本征函数 $\sin\frac{n\pi x}{l}$，弦的一般振动

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi at}{l}+B_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

代入 $u(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}=0$

$$u_t(x,0)=\sum_{n=1}^{\infty}\frac{n\pi a}{l}B_n\sin\frac{n\pi x}{l}=\begin{cases}0,\ (0<x<x_0-\delta,\ x_0+\delta<x<l) \\ v_0 \ (x_0-\delta<x<x_0+\delta)\end{cases}$$

$\therefore A_n=\frac{2}{l}\int_0^l 0\cdot\sin\frac{n\pi x}{l}dx=0$

$$B_n=\frac{2}{n\pi a}\int_{x_0-\delta}^{x_0+\delta}v_0\sin\frac{n\pi x}{l}dx=\frac{-2}{n\pi a}\frac{v_0 l}{n\pi}\cos\frac{n\pi x}{l}\Big|_{x_0-\delta}^{x_0+\delta}$$
$$=\frac{2v_0 l}{n^2\pi^2 a}\left[\cos\frac{n\pi}{l}(x_0-\delta)-\cos\frac{n\pi}{l}(x_0+\delta)\right]=\frac{4v_0 l}{n^2\pi^2 a}\sin\frac{n\pi x_0}{l}\sin\frac{n\pi\delta}{l}$$

$$\therefore u(x,t)=\frac{4v_0 l}{a\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x_0}{l}\sin\frac{n\pi\delta}{l}\sin\frac{n\pi at}{l}\sin\frac{n\pi x}{l}$$

(ii) 若为余弦式锤
$$\begin{cases} u_{tt}-a^2u_{xx}=0 \ (0<x<l) \\ u(0,t)=u(l,t)=0 \\ u(x,0)=0 \\ u_t(x,0)=\begin{cases}0,\ 0<x<x_0-\delta,\ x_0+\delta<x<l \\ v_0\cos\frac{x-x_0}{2\delta}\pi \ (x_0-\delta<x<x_0+\delta)\end{cases} \end{cases}$$

根据边界条件可知本征函数 $\sin\frac{n\pi x}{l}$，一般解可表示为

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi at}{l}+B_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

代入 $u(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}=0$

$$u_t(x,0)=\sum_{n=1}^{\infty}\frac{n\pi a}{l}B_n\sin\frac{n\pi x}{l}=\begin{cases}0 \ (0<x<x_0-\delta,\ x_0+\delta<x<l) \\ v_0\cos\frac{x-x_0}{2\delta}\pi \ (x_0-\delta<x<x_0+\delta)\end{cases}$$

从上列二式可得 $A_n=0$，$B_n=\frac{2}{n\pi a}\int_{x_0-\delta}^{x_0+\delta} V_0\cos\frac{x-x_0}{2\delta}\pi\sin\frac{n\pi x}{l}dx$

$$=\frac{2V_0}{n\pi a}\int_{x_0-\delta}^{x_0+\delta}\left(\cos\frac{x_0\pi}{2\delta}\cos\frac{\pi x}{2\delta}+\sin\frac{x_0\pi}{2\delta}\sin\frac{\pi x}{2\delta}\right)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2V_0}{n\pi a}\int_{x_0-\delta}^{x_0+\delta}\left[\cos\frac{x_0\pi}{2\delta}\left(\sin\frac{l\pi-2\delta n\pi}{2\delta l}x-\sin\frac{l\pi-2\delta n\pi}{2\delta l}x\right)\right.$$

$$\cancel{=\frac{V_0}{n\pi a}}\left.+\sin\frac{x_0\pi}{2\delta}\left(\cos\frac{l\pi-2\delta n\pi}{2\delta l}x-\cos\frac{l\pi+2\delta n\pi}{2\delta l}x\right)\right]dx$$

$$=\frac{V_0}{n\pi a}\left[\cos\frac{x_0\pi}{2\delta}\left(\frac{2l\delta}{l\pi-2\delta n\pi}\cos\frac{l\pi-2\delta n\pi}{2l\delta}x-\frac{2l\delta}{l\pi+2\delta n\pi}\cos\frac{l\pi+2\delta n\pi}{2l\delta}x\right)\Bigg|_{x_0-\delta}^{x_0+\delta}\right.$$

$$\left.+\sin\frac{x_0\pi}{2\delta}\left(\frac{2l\delta}{l\pi-2\delta n\pi}\sin\frac{l\pi-2\delta n\pi}{2l\delta}x-\frac{2l\delta}{l\pi+2\delta n\pi}\sin\frac{l\pi+2\delta n\pi}{2l\delta}x\right)\Bigg|_{x_0-\delta}^{x_0+\delta}\right]$$

$$=\frac{V_0}{n\pi a}\left\{\frac{2l\delta}{l\pi-2\delta n\pi}\left[\cos\frac{2n\pi\cdot x_0-l\pi-2\delta n\pi}{2l}-\cos\frac{2n\pi x_0+l\pi-2\delta n\pi}{2l}\right]\right.$$

$$-\frac{2l\delta}{l\pi-2\delta n\pi}\left[\cos\frac{-2n\pi x_0-l\pi-2\delta n\pi}{2l}-\cos\frac{-2n\pi x_0+l\pi+2\delta n\pi}{2l}\right]$$

$$=\frac{4V_0}{n\pi a}\left(\frac{1}{1-\frac{2\delta n}{l}}\sin\frac{n\pi x_0}{l}\cos\frac{n\pi\delta}{l}+\frac{1}{1+\frac{2\delta n}{l}}\sin\frac{n\pi x_0}{l}\cos\frac{n\pi\delta}{l}\right)$$

$$=\frac{8V_0\delta}{n\pi^2 a}\cdot\frac{1}{1-\left(\frac{2\delta n}{l}\right)^2}\sin\frac{n\pi x_0}{l}\cos\frac{n\pi\delta}{l}$$

$$\therefore u(x,t)=\frac{8V_0\delta}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{1-\left(\frac{2\delta n}{l}\right)^2}\sin\frac{n\pi x_0}{l}\cos\frac{n\pi\delta}{l}\sin\frac{n\pi at}{l}\sin\frac{n\pi x}{l}$$

4. 长为 l 的均匀杆，两端受压长度缩为 $l(1-2e)$，放手后自由振动

$$\begin{cases}u_{tt}-a^2u_{xx}\quad(0<x<l)\\ u_x|_{x=0}=u_x|_{x=l}=0\\ u|_{t=0}=2e\left(\frac{l}{2}-x\right)\\ u_t|_{t=0}=0\end{cases}$$

由于是第二类边界条件，所以要用 $\cos\frac{n\pi x}{l}$ 作本征函数

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi at}{l}+B_n\sin\frac{n\pi at}{l}\right)\cos\frac{n\pi x}{l}$$

$$\because \frac{\partial u}{\partial t}\Big|_{t=0}\sum_{n=1}^{\infty}B_n\frac{n\pi a}{l}\cos\frac{n\pi at}{l}\cos\frac{n\pi x}{l}\Big|_{t=0}=0$$

$\therefore B_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi a t}{l} \cos\frac{n\pi x}{l}$$

$$\because u(x,0) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} = 2\varepsilon\left(\frac{l}{2}-x\right)$$

$$\therefore A_0 = \frac{1}{l}\int_0^l 2\varepsilon\left(\frac{l}{2}-x\right)dx = \varepsilon\int_0^l dx - \frac{2\varepsilon}{l}\int_0^l x\,dx$$

$$= \varepsilon x\Big|_0^l - \frac{2\varepsilon}{l}\frac{x^2}{2}\Big|_0^l = \varepsilon l - \varepsilon l = 0$$

$$A_n = \frac{2}{l}\int_0^l 2\varepsilon\left(\frac{l}{2}-x\right)dx = \varepsilon\int_0^l dx - \frac{2\varepsilon}{l}\int_0^l x\,dx$$

$$= \frac{4\varepsilon}{l}\int_0^l \frac{l}{2}\cos\frac{n\pi x}{l}dx - \frac{4\varepsilon}{l}\int_0^l x\cos\frac{n\pi x}{l}dx$$

$$= 2\varepsilon\left[\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\Big|_0^l - \frac{4\varepsilon}{l}\left(\frac{l}{n\pi}\right)^2\left(\frac{n\pi x}{l}\sin\frac{n\pi x}{l} + \cos\frac{n\pi x}{l}\right)\Big|_0^l$$

$$= -\frac{4\varepsilon l}{n^2\pi^2}(\cos n\pi - 1) = \frac{4\varepsilon l}{n^2\pi^2}[1-(-1)^n]$$

$$= \begin{cases} \dfrac{8\varepsilon l}{\pi^2(2k+1)^2}, & n=2k+1 \\ 0, & n=2k \end{cases}$$

$$\therefore u(x,t) = \frac{8\varepsilon l}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\frac{(2k+1)\pi a t}{l}\cos\frac{(2k+1)\pi x}{l}$$

8. 设中子浓度为u，扩散系数为D，单位体积中子产生数为n，$n=\beta u$

$$\frac{\partial u}{\partial t} = D\Delta u + \beta u$$

临界厚度时，$\frac{\partial u}{\partial t}=0$，原式成为 $D\Delta u + \beta u = 0$

$$u = A_1 e^{i\sqrt{\frac{\beta}{D}}x} + A_2 e^{-i\sqrt{\frac{\beta}{D}}x}$$

~~x=0时~~ $u = A_1 e^{i\sqrt{\frac{\beta}{D}}t} + A_2 e^{i\sqrt{\frac{\beta}{D}}t} = A_1 + A_2$

x=0或x=l时，$u = A_1 e$

$$\begin{cases} 1 - e^{i\sqrt{\frac{\beta}{D}}t} = 1 - \cos\sqrt{\frac{\beta}{D}}t - i\sin\sqrt{\frac{\beta}{D}}l = 0 \\ 1 - e^{-i\sqrt{\frac{\beta}{D}}t} = 1 - \cos\sqrt{\frac{\beta}{D}}l + i\sin\sqrt{\frac{\beta}{D}}l = 0 \end{cases}$$

$$\Rightarrow \sqrt{\frac{\beta}{D}}l = \pi$$

$\therefore$ 临界厚度 $L = \sqrt{\frac{D}{\beta}}\pi = \frac{a\pi}{\sqrt{\beta}}$

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9. $u_t - a^2 u_{xx} = 0$ $\begin{cases} u(0,t)=N_0,\ u(x,0)=0 \\ u(l,t)=N_0 \end{cases}$

令 $u = W + N_0$ 则 $W = u - N_0$

$W_t - a^2 W_{xx} = 0$ $\begin{cases} W(0,t)=0,\ W(x,0)=-N_0 \\ W(l,t)=0 \end{cases}$

令 $W(x,t) = X(x)T(t)$，$\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda$

即 $\begin{cases} T' + a^2\lambda T = 0 \\ X'' + \lambda X = 0,\ X(0)=X(l)=0 \end{cases}$

由(2)得 $X_n(x) = A_n \cos\sqrt{\lambda}x + B_n \sin\sqrt{\lambda}x$

将 $X(0)=0$ 代入，得 $A_n = 0$ $X_n(x) = B_n \sin\sqrt{\lambda}x$

又 $X(l)=0$ ∴ $\sin\sqrt{\lambda}x = 0$ $\lambda = \frac{n^2\pi^2}{l^2}$

∴ $X(x) = \sum B_n \sin\frac{n\pi}{l}x$

∴ $T_n = C_n e^{-\frac{n^2\pi^2 a^2 t}{l^2}}$

∴ $W(x,t) = \sum_{n=1}^{\infty} X_n T_n = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2 a^2 t}{l^2}} \sin\frac{n\pi x}{l}$

为确定 B_n，$W(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = -N_0$

∴ $B_n = \frac{2}{l}\int_0^l -N_0 \sin\frac{n\pi\xi}{l} d\xi = \frac{-2N_0}{l}\cdot\frac{l}{n\pi}\left[-\cos\frac{n\pi\xi}{l}\right]\Big|_0^l$

$= \frac{2N_0}{n\pi}(\cos n\pi - 1) = \frac{2N_0}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} -\frac{4N_0}{\pi(2k+1)}, & n=2k+1 \\ 0, & n=2k \end{cases}$

∴ $W(x,t) = \sum_{k=0}^{\infty} \frac{4N_0}{-\pi(2k+1)} \cdot e^{-\frac{(2k+1)^2\pi^2 a^2 t}{l^2}} \cdot \sin\frac{(2k+1)\pi x}{l}$

∴ $u(x,t) = N_0 + W = N_0 - \frac{4N_0}{\pi}\sum_{k=0}^{\infty} \frac{e^{-\frac{(2k+1)^2\pi^2 a^2 t}{l^2}}}{2k+1} \cdot \sin\frac{(2k+1)\pi x}{l}$

考虑 $e^{-\frac{(2k+1)^2\pi^2 a^2 t}{l^2}}$ 在 $k=1$ 比 $k=0$ 有极剧减小

$u(x,t) \approx N_0 - \frac{4N_0}{\pi} e^{-\frac{\pi^2 a^2 t}{l^2}} \sin\frac{\pi x}{l}$

11. 定解问题 $\begin{cases} \cancel{u_t - a^2 u_{xx} = 0} \quad \Delta u = 0 \ (0<x<a,\ 0<y<b) \\ u|_{x=0} = Ay(b-y) \quad u|_{x=a} = 0 \\ u|_{y=0} = B\sin\frac{\pi x}{a},\ u|_{y=b} = 0 \end{cases}$

设 $u(x,y) = V(x,y) + W(x,y)$，V, W分别满足条件

$\begin{cases} \Delta V = 0 \ (0<x<a,\ 0<y<b) \\ V|_{x=0} = 0 \quad V|_{x=a} = 0 \qquad W同理 \\ V|_{y=0} = B\sin\frac{\pi x}{a},\ V|_{y=b} = 0 \end{cases}$

由分离变量法，通解为 $V(x,y) = \sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi y}{a} + B_n\sinh\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a}$

将 $V|_{y=0} = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{a} = B\sin\frac{\pi x}{a}$ 可得 $A_1 = B \quad A_n = 0 \ (n>1)$

代入边界 $V|_{y=b} = 0$ 得 $B_1 = \frac{B}{\tanh\frac{\pi b}{a}}$，$B_n = 0 \ (n>1)$

所以 $V(x,y) = \left(B\cosh\frac{\pi y}{a} - \frac{B}{\tanh\frac{\pi b}{a}}\sinh\frac{\pi y}{a}\right)\sin\frac{\pi x}{a} = \frac{B\sinh\frac{\pi}{a}(b-y)\sin\frac{\pi x}{a}}{\sinh\frac{\pi b}{a}}$

通解 $W(x,y) = \sum_{n=1}^{\infty}\left(C_n\cosh\frac{n\pi x}{b} + D_n\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}$

将 $W|_{x=0} = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi y}{b} = Ay(b-y)$

$C_n = \frac{2}{b}\int_0^b Ay(b-y)\sin\frac{n\pi y}{b}dy = \frac{4Ab^2[1-(-1)^n]}{\pi^3 n^3}$

$W|_{x=a} = \sum_{n=1}^{\infty}\left(C_n\cosh\frac{n\pi a}{b} + D_n\sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi}{b}y = 0$

$D_n = -\frac{4Ab^2[1-(-1)^n]\cosh\frac{n\pi a}{b}}{\pi^3 n^3 \sinh\frac{n\pi a}{b}}$

$\therefore W(x,y) = \frac{4Ab^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\left(\cosh\frac{n\pi x}{b} - \frac{\cosh\frac{n\pi a}{b}}{\sinh\frac{n\pi a}{b}}\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}$

$= \frac{4Ab^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{\sinh\frac{n\pi(a-x)}{b}}{\sinh\frac{n\pi a}{b}}\sin\frac{n\pi y}{b}$

当 $n=2k$，$W=0$，$n=2k+1$，$W = \frac{8Ab^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\frac{\sinh\frac{(2k+1)\pi(a-x)}{b}}{\sinh\frac{(2k+1)\pi a}{b}}\sin\frac{(2k+1)\pi y}{b}$

$\therefore u(x,y) = \frac{B\sinh\frac{\pi}{a}(b-y)}{\sinh\frac{\pi b}{a}}\sin\frac{\pi x}{a} + \sum_{n=1}^{\infty}\frac{4Ab^2[1-(-1)^n]}{n^3\pi^3}\frac{\sinh\frac{n\pi(a-x)}{b}}{\sinh\frac{n\pi a}{b}}\sin\frac{(2k+1)\pi y}{b}$

$= \frac{B\sinh\frac{\pi}{a}(b-y)}{\sinh\frac{\pi b}{a}}\sin\frac{\pi x}{a} + \frac{8Ab^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\frac{\sinh\frac{n\pi(a-x)}{b}}{\sinh\frac{(2k+1)\pi a}{b}}\sin\frac{(2k+1)\pi y}{b}$

14. 矩形膜，$l_1 l_2$为边长，求本征振动 $\begin{cases} u - a^2\Delta u = 0 \ (0<x<l_1,\ 0<y<l_2) \\ u|_{x=0} = u|_{x=l_1} = 0 \\ u|_{y=0} = u|_{y=l_1} = 0 \end{cases}$

解得

令 $u(x,y,t) = X(x)Y(y)T(t)$ 代入泛定方程得 $\frac{T''}{a^2T} = \frac{X''Y+XY''}{XY} = -\lambda$

$\therefore \begin{cases} T''+\lambda a^2 T = 0 \\ X''Y+XY''+\lambda XY = 0 \end{cases}$

$\therefore T(t) = A\cos\sqrt{\lambda}at + B\sin\sqrt{\lambda}at$

对空间变数 x, y 进一步分离，可得 $-\frac{Y''}{Y} = \frac{X''}{X}+\lambda$

令 $\frac{X''}{X} = -\lambda_1$，即 $X''+\lambda_1 X = 0$，由边界条件 $X(0)=X(l_1)=0$

可知本征函数 $\sin\frac{m\pi x}{l_1}$，$\lambda_1 = \frac{m^2\pi^2}{l_1^2}$ $(m=1,2,3\cdots)$

同理我们令 $\frac{Y''}{Y} = -\lambda_2$，本征函数 $\sin\frac{n\pi y}{l_2}$，$\lambda_2 = \frac{n^2\pi^2}{l_2^2}$ $(n=1,2,3\cdots)$

$\therefore \lambda = \lambda_1+\lambda_2 = \frac{m^2\pi^2}{l_1^2}+\frac{n^2\pi^2}{l_2^2} = w_{m,n}^2$ $\quad w_{m,n} = \sqrt{\frac{m^2\pi^2}{l_1^2}+\frac{n^2\pi^2}{l_2^2}}$

本征振动模式 $u_{m,n}(A_{m,n}\cos w_{m,n}at + B_{m,n}\sin w_{m,n}at)\sin\frac{m\pi}{l_1}\sin\frac{n\pi}{l_2}y$

18.

由题意定解问题 $\begin{cases} C_{01}+\sum_{m=1}^{\infty}(A_{1m}\cos m\varphi+B_{1m}\sin m\varphi)a^m = -E_0\cos\varphi+\sum_{m=1}^{\infty}(A_{2m}\cos m\varphi+B_{2m}\sin m\varphi)a^{-m} \\ \varepsilon\sum_{m=1}^{\infty}(A_{1m}\cos m\varphi+B_{1m}\sin m\varphi)ma^{m-1} = -E_0\cos\varphi + \sum_{m=1}^{\infty}(A_{2m}\cos m\varphi+B_{2m}\sin m\varphi)(-m)a^{-(m+1)} \end{cases}$

比较 $\cos m\varphi$, $\sin m\varphi$ 系数可得

$\begin{cases} C_{01}=0 \\ A_{11}a^m = -E_0 a + A_{21}a^{-m} \\ A_{1m}a^m = A_{2m}a^{-m}(m\neq 1) \\ B_{1m}a^m = B_{2m}a^{-m} \end{cases}$ $\quad \begin{cases} \varepsilon A_{11} = -E_0 - A_{21}a^{-2} \\ \varepsilon A_{1m}a_m^{m-1} = -A_{2m}a^{-(m+1)}m \ (m\neq 1) \\ \varepsilon B_{1m}a^{m-1}m = -B_{2m}a^{-(m+1)}m \end{cases}$

解得
$$\begin{cases} C_{01}=0 \\ A_{11}=\dfrac{-2E_0}{1+\varepsilon} \\ A_{21}=\dfrac{\varepsilon-1}{\varepsilon+1}a^2E_0 \\ A_{1m}=A_{2m}=0\ (m\neq1) \\ B_{1m}=B_{2m}=0 \end{cases}$$

$$\begin{cases} \Delta u_1=0\ (0\le\rho<a) \\ \Delta u_2=0\ (\rho>a) \\ u_1(\varphi+2\pi)=u_1(\varphi) \quad u_2(\varphi+2\pi)=u_2(\varphi) \\ u_1|_{\rho=0}=C \quad u_2|_{\rho=\infty}=-E_0\rho\cos\varphi \\ u_1|_{\rho=a}=u_2|_{\rho=a} \\ \varepsilon\dfrac{\partial u_1}{\partial\rho}\Big|_{\rho=a}=\dfrac{\partial u_2}{\partial\rho}\Big|_{\rho=a} \end{cases}$$

球内电势 u_1 在 $\rho=0$ 处有限

设 $u_1(\rho,\varphi)=C_{01}+\sum_{m=1}^{\infty}(A_{1m}\cos m\varphi+B_{1m}\sin m\varphi)\rho^m$

对球外，$u_2(\rho,\varphi)=C_{02}+D_{02}\ln\rho+\sum_{m=1}^{\infty}(A_{2m}\cos m\varphi+B_{2m}\sin m\varphi)(C_{2m}\rho^m+D_{2m}\rho^{-m})$

在 $\rho=\infty$ 处，$u_2|_{\rho=\infty}=-E_0\rho\cos\varphi$，比较系数

$C_{02}=0$，$D_{02}=0$，$C_{2m}=0\ (m\neq1)$ $\quad A_{21}C_{21}=-E_0$

代入以上系数 $u_2(\rho\varphi)=-E_0\rho\cos\varphi+\sum_{m=1}^{\infty}(A_{2m}\cos m\varphi+B_{2m}\sin m\varphi)\rho^{-m}$

将衔接条件代入，可以求得

柱内电势 $u_1(\rho,\varphi)=\dfrac{-2E_0}{1+\varepsilon}\rho\cos\varphi$

柱外电势 $u_2(\rho,\varphi)=-(\rho-\dfrac{\varepsilon-1}{\varepsilon+1}\dfrac{a^2}{\rho})E_0\cos\varphi$

柱内场强 $E_1=\dfrac{-\partial u_1}{\partial x}=-\dfrac{\partial}{\partial x}\left(\dfrac{-2E_0x}{1+\varepsilon}\right)=\dfrac{2E_0}{1+\varepsilon}$ 不变

说明柱内为匀强电场

柱内极化强度 $P_1=(\varepsilon-1)\varepsilon_0E_1=2\varepsilon_0\dfrac{\varepsilon-1}{\varepsilon+1}E_0$

柱内束缚电荷面密度 $=P_n=2\varepsilon_0\dfrac{\varepsilon-1}{\varepsilon+1}E_0\cos\varphi$

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19. 由题意得定解问题为 $\begin{cases} \Delta u=0 \ (0<\rho<a) \\ [ku_\rho+Hu]|_{\rho=a}=\begin{cases} q\sin\varphi (0<\varphi<\pi) \\ 0 \ (\pi<\varphi<2\pi) \end{cases} \end{cases}$

设问题的解 $u(\rho,\varphi)=C_0+D_0\ln\rho+\sum_{m=1}^{\infty}(A_m\cos m\varphi+B_m\sin m\varphi)(C_m\rho^m+D_m\rho^{-m})$

由 $\rho=0$ 处 u 为有限值，得 $D_0=D_m=0$，则

$u(\rho,\varphi)=C_0+\sum_{m=1}^{\infty}(A_m\cos m\varphi+B_m\sin m\varphi)\rho^m$

代入另一边界条件得 $[HC_0+H\sum_{m=1}^{\infty}(A_m\cos m\varphi+B_m\sin m\varphi)m\rho^{m-1}]|_{\rho=a}$

$=\begin{cases} q\sin\varphi (0<\varphi<\pi) \\ 0 \ (\pi<\varphi<2\pi) \end{cases}$

即 $HC_0+\sum_{m=1}^{\infty}(Ha^m+kma^{m-1})(A_m\cos m\varphi+B\sin m\varphi)=\begin{cases} q\sin\rho (0<\varphi<\pi) \\ 0 \ (\pi<\varphi<2\pi) \end{cases}$

进行傅里叶展开，$C_0=\frac{1}{H}\frac{1}{2\pi}\int_0^{\pi}q\sin\varphi=\frac{q}{\pi H}$

$A_m=\frac{1}{Ha^m+kma^{m-1}}\cdot\frac{1}{\pi}\int_0^{\pi}q\sin\varphi\cos\varphi d\varphi$

$=\frac{1}{Ha^m+kma^{m-1}}\cdot\frac{q}{\pi}\int_0^{\pi}\frac{\sin(m+1)\varphi+\sin(1-m)\varphi}{2}d\varphi$

$=\frac{q}{2(Ha^m+kma^{m-1})\pi}\left[\frac{-1-(-1)^{m+1}}{1+m}+\frac{1-(-1)^{m-1}}{1-m}\right]=\begin{cases} 0, m=2l+1, \ l=0,1,2 \\ \frac{q}{Ha^{2k}+2kla^{2l-1}}\cdot\frac{1}{1-4l^2}, m=2l \end{cases}$

$B_m=\frac{1}{(Ha^m+kma^{m-1})}\cdot\frac{1}{\pi}\int_0^{\pi}q\sin\varphi\sin m\varphi d\varphi$

若 $m\neq1$，则 $\sin\varphi$ 与 $\sin m\varphi$ 正交，$B_m=0$

若 $m=0,1$ 则 $B_m=\frac{q}{\pi(Ha^m+kma^{m-1})}\int_0^{\pi}\sin^2\varphi d\varphi=\frac{q}{2(k+Ha)}$

综上所述

$u(\rho,\varphi)=\frac{q}{H\pi}+\frac{1}{k+Ha}\cdot\frac{q}{2}\rho\sin\rho+\sum_{l=1}^{\infty}\frac{2q}{\pi a^{2l-1}(2lk+Ha)(1-4l^2)}\cdot\rho^{2l}\cos 2l\varphi$

26. 由题意得定解问题 $\begin{cases} u_t - a^2 u_{xx} = \beta u \\ u_x|_{x=0} = 0, u|_{x=l} = 0 \\ u|_{t=0} = u_0 x(x-l) \end{cases}$

令 $u(x,t) = X(x)T(t)$，代入泛定方程得 $XT' - a^2 X''T = \beta XT$

$$\therefore \frac{T'}{a^2 T} = \frac{X''}{X} + \frac{\beta}{a^2} = -\lambda \quad (\lambda > 0)$$

解得 $X(x) = A\cos\sqrt{\lambda + \frac{\beta}{a^2}}x + B\sin\sqrt{\lambda + \frac{\beta}{a^2}}x$，$T(t) = Ce^{-\lambda a^2 t}$

代入 $X'(0) = 0$ $X(l) = 0$，得 $B = 0$，$\sqrt{\lambda + \frac{\beta}{a^2}}\,l = \frac{2n+1}{2}\pi$，$n = 0, 1, 2 \cdots$

$$X(x) = A_n \cos\frac{2n+1}{2l}\pi x$$

所以 $u(x,t) = X(x)T(t) = \sum_{n=0}^{\infty} e^{-\lambda a^2 t} \cos(n+\frac{1}{2})\frac{\pi x}{l}$，$\lambda = \frac{\frac{n(n+1)}{2}}{l} - \frac{\beta}{a^2}$

代入初始条件 $u(x,0) = \sum_{n=0}^{\infty} A_n \cos(n+\frac{1}{2})\frac{\pi x}{l} = u_0 x(x-l)$

将 $u_0 x(x-l)$ 用 $\cos(n+\frac{1}{2})\frac{\pi x}{l}$ 展开

$$A_n = \frac{2}{l}\int_0^l u_0 x(x-l)\cos\frac{(n+\frac{1}{2})\pi x}{l}\,dx$$

经分部积分之后

$$A_n = \frac{2u_0 l^2}{(n+\frac{1}{2})^3\pi^3}\left[(n+\tfrac{1}{2})\pi + 2(-1)^{n+1}\right]$$

$$\therefore u(x,t) = \sum_{n=0}^{\infty} \frac{2u_0 l^2}{(n+\frac{1}{2})^3\pi^3}\left[(n+\tfrac{1}{2})\pi + 2(-1)^{n+1}\right] e^{\left[\frac{\beta-(n+\frac{1}{2})^2\pi^2a^2}{l}\right]} \cos\frac{(n+\frac{1}{2})\pi x}{l}$$

$$\begin{cases} u_t(x,t) = a^2 u_{xx}(x,t) \\ u_x\big|_{x=0} = -\dfrac{B}{k}\sin wt, \quad u\big|_{x\to+\infty} \neq \infty \end{cases}$$

长时间后忽略初始条件 $U(x,t) = \bar{u}(x,w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{iwt}dt$

即方程为
$$\begin{cases} \bar{u}_{xx}(x,w) + \dfrac{iw}{a^2}\bar{u}(x,w) = 0 \quad (x>0) \\ \bar{u}_x\big|_{x=0} = \dfrac{B}{k}\sqrt{2\pi}\,\delta(w'-w) \quad \bar{u}\big|_{x\to+\infty} \neq \infty \end{cases}$$

$$\bar{u}_{xx}(x,w) - \frac{w}{a^2}e^{-i\frac{\pi}{2}}\bar{u}(x,w) = 0$$

$$\bar{u}(x,w) = C_1 e^{e^{i\frac{\pi}{4}}\frac{\sqrt{w}}{a}x} + C_2 e^{-e^{-i\frac{\pi}{4}}\frac{\sqrt{w}}{a}x} = C_1 e^{(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)\frac{\sqrt{w}}{a}x} + C_2 e^{(\frac{-\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)\frac{\sqrt{w}}{a}x}$$

由于 $\bar{u}\big|_{x\to\infty} \neq \infty$ $\therefore C_1 = 0$ 将 $x=0$ 代入得

$$\bar{u}(x,\cancel{t}) = \left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}i}{2}\right)\frac{Ba}{k\sqrt{w}}e^{(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})\frac{\sqrt{w_0}}{a}x - iw_0t}$$

$$= \frac{aB}{k\sqrt{w}}e^{(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})\frac{\sqrt{w_0}}{a}x - iw_0t + \frac{5\pi}{4}i}$$

取虚部 $u(x,t) = \frac{aB}{k\sqrt{w}}e^{-\frac{x}{a}\sqrt{\frac{w}{2}}}\sin\left(wt - \frac{x}{a}\sqrt{\frac{w}{2}} - \frac{\pi}{4}\right)$

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