1.(1) $\frac{e^z}{1+z}$ (i) $\because \lim_{x\to -1} \frac{e^z}{1+z}=\infty$ $\therefore z_0=-1$ 是函数奇点,

又 $\because \lim_{x\to -1}[(1+z)\frac{e^z}{1+z}]=\lim_{x\to -1} e^z=e^{-1}$ $\therefore z_0=-1$ 是一阶零点,留数是 $\frac{1}{e}$, $Resf(-1)=\frac{1}{e}$

(ii) $\because \lim_{x\to\infty} \frac{e^z}{1+z}=\infty$, $z_0=\infty$ 是奇点,由留数定理

$Resf(\infty)=-Resf(-1)=-\frac{1}{e}$

(2) $\frac{z}{(z-1)(z-2)^2}$ (i) 单奇点 $z_0=1$ $Resf(1)=\lim_{z\to 1}\frac{z}{(z-2)^2}=1$

(ii) 二阶奇点 $z_0=2$ $Resf(2)=\lim_{z\to 2}\frac{d}{dz}(\frac{z}{z-1})=\lim_{z\to 2}[\frac{1}{z-1}-\frac{z}{(z-1)^2}]=-1$

(3) $\frac{e^z}{z^2+a^2}$ (i) 单奇点 $z_0=i\alpha$ $Resf(i\alpha)=\lim_{z\to i\alpha}(\frac{e^z}{z+i\alpha})=\frac{e^{i\alpha}}{2i\alpha}$

(ii) 单奇点 $z_0=-i\alpha$, $Resf(-i\alpha)=\lim_{z\to -i\alpha}\frac{e^z}{z-i\alpha}=-\frac{e^{-i\alpha}}{2i\alpha}$

(iii) 本性奇点 $z_0=\infty$ $Resf(\infty)=\frac{e^{-i\alpha}-e^{i\alpha}}{2i\alpha}=-\frac{\sin\alpha}{\alpha}$

(5) $\frac{ze^z}{(z-\alpha)^3}$ 三阶奇点 $z_0=\alpha$, $Resf(\alpha)=\lim_{z\to\alpha}\frac{1}{2!}\frac{d^2}{dz^2}(ze^z)=(1+\frac{\alpha}{2})e^\alpha$

(ii) 本性奇点 $z_0=\infty$, $Resf(\infty)=-Resf(\alpha)=-(1+\frac{\alpha}{2})e^\alpha$

(9) $e^{\frac{1}{1-z}}$ 本性奇点 $z_0=1$,要求 $f(z)=e^{\frac{1}{1-z}}$ 的留数,必须把 $f(z)$ 罗朗展开 $f(z)=1-\frac{1}{z}-\frac{1}{2z^2}-\frac{1}{6z^3}+\frac{1}{24z^4}+\cdots$

$\therefore Resf(1)=-1$

(10) $\frac{1}{1+z^{2n}}$ 令原式中 $1+z^{2n}=0$, $z^{2n}=-1$

$z^n=\pm i=e^{i(2k+1)\pi/2}$

所以 $z_0=e^{i(2k+1)\pi/2n}$, $k=0,1,2\cdots 2n-1$ 为函数 $f(z)$ 的单奇点

$Resf(z_0)=\lim_{z\to z_0}[(z-e^{i(2k+1)\pi/2n})/(1+z^{2n})]$ 用洛必达法则

$=\lim_{z\to z_0}[\frac{1}{2nz^{2n-1}}]=\frac{1}{2n}e^{-i\frac{(2n-1)(2k+1)}{2n}\pi}$

$=\frac{1}{2n}\cdot\frac{e^{i(2k+1)\pi/2n}}{e^{i(2k+1)\pi}}=-\frac{1}{2n}e^{i(2k+1)\pi/2n}$

2. (1) $\oint_L \frac{dz}{(z^2+1)(z-1)^2}$，$L: x^2+y^2-2x-2y=0$ 是以(1,1)为圆心、$\sqrt{2}$为半径的圆

被积函数 $f(z)=\frac{1}{(z^2+1)(z-1)^2}$ 有单奇点 $z_0=\pm i$ 和二阶奇点 $z_0=1$，

其中 $z_0=-i$ 不在积分回路之内，只有极点 $z_0=i$ 和 $z_0=1$ 在积分回路内，

它们的留数分别为 $Resf(i)=\lim_{z\to i}\frac{1}{(z+i)(1-z)^2}=\frac{1}{4}$

$Resf(1)=\lim_{z\to 1}\frac{d}{dz}\left(\frac{1}{1+z^2}\right)=\lim_{z\to 1}\frac{-2z}{(1+z^2)^2}=-\frac{1}{2}$

应用留数定理：$\oint_L \frac{dz}{(z^2+1)(z-1)^2}=2\pi i[Resf(i)+Resf(1)]=2\pi i\left(\frac{1}{4}-\frac{1}{2}\right)=-\frac{\pi}{2}i$

(2) $\oint_{|z|=2}\frac{z\,dz}{\frac{1}{2}-\sin^2 z}$

$f(z)=\frac{2z}{\cos 2z}$，奇点为 $e^{i2z}+e^{-i2z}=0$，$z=\frac{(2k+1)\pi}{4}$，$k=0,\pm1,\pm2\cdots$

其中只有 $z_0=\pm\frac{\pi}{4}$ 在积分回路内

$Resf(-\frac{\pi}{4})=\lim_{z\to-\frac{\pi}{4}}\frac{2z(z+\frac{\pi}{4})}{\cos 2z}=\lim_{z\to-\frac{\pi}{4}}\frac{4z+\frac{\pi}{2}}{-2\sin 2z}=-\frac{\pi}{4}$

$Resf(\frac{\pi}{4})=\lim_{z\to\frac{\pi}{4}}\frac{2z(z-\frac{\pi}{4})}{\cos 2z}=\lim_{z\to\frac{\pi}{4}}\frac{4z-\frac{\pi}{2}}{-2\sin 2z}=-\frac{\pi}{4}$

$\therefore \oint_{|z|=2}\frac{z\,dz}{\frac{1}{2}-\sin^2 z}=2\pi i[Resf(\frac{\pi}{4})+Resf(-\frac{\pi}{4})]=-\pi^2 i$

3. 设 $g(z)=\frac{f(z)}{z-\alpha}$，只有1个单奇点 $z_0=\alpha$，

$Resf(\alpha)=\lim_{z\to\alpha}\left[\frac{f(z)}{z-\alpha}(z-\alpha)\right]=f(\alpha)$

$\therefore \oint_L \frac{f(z)}{z-\alpha}dz=2\pi i\,Resf(\alpha)=2\pi i f(\alpha)$

$\frac{1}{2\pi i}\oint_L \frac{f(z)}{z-\alpha}dz=f(\alpha)$，正是柯西公式

1.(4) $\int_0^{2a} \frac{\sin^2 x}{a+b\cos x}dx \quad (a>b>0)$

变换后原式 $= \oint_{|z|=1} \frac{[(z^2-1)/2iz]^2 \cdot dz/iz}{a+b(z^2+1)/2z} = -\oint_{|z|=1} \frac{(z^2-1)^2 dz}{4iz^3[a+\frac{b}{2z}(z^2+1)]}$

$= -\frac{1}{2bi}\oint_{|z|=1} \frac{(z^2-1)^2 dz}{z^2(z^2+\frac{2az}{b}+1)} = -\frac{1}{2bi}\oint_{|z|=1} \frac{(z^2-1)^2 dz}{z^2[z+\frac{1}{b}(a+\sqrt{a^2-b^2})][z+\frac{1}{b}(a-\sqrt{a^2-b^2})]}$

$= -\frac{1}{2bi}\oint_{|z|=1} f(z)dz$

(5) ~~$\int_0^{\pi} \frac{a\,dx}{a^2+\sin^2 x} \ (a>0)$~~ 上式被积函数中，二阶奇点 $z_0=0$，单奇点

$z_0 = -\frac{1}{b}(a\pm\sqrt{a^2-b^2})$ ~~在单位圆外（即 $|z_0|>1$ 舍~~

其中 $z_0 = -\frac{1}{b}(a+\sqrt{a^2-b^2})$ 在单位圆外（即 $|z_0|>1$，亦即 $a+\sqrt{a^2-b^2}>b$）

其余的奇点在单位圆内，留数分别是 $\mathrm{Res}f(0) = \lim_{z\to 0}\frac{d}{dz}\left[\frac{(z^2-1)^2}{z^2+\frac{2a}{b}z+1}\right] = -\frac{2a}{b}$

$\mathrm{Res}f(-\frac{a-\sqrt{a^2-b^2}}{b})$

$= \lim_{z\to\frac{-a+\sqrt{a^2-b^2}}{b}}\left[\frac{(z^2-1)^2}{z^2[z+\frac{1}{b}(a+\sqrt{a^2-b^2})]}\right] = \frac{[\frac{(\sqrt{a^2-b^2}-a)^2}{b^2}-1]^2}{(\frac{\sqrt{a^2-b^2}-a}{b})^2(\frac{\sqrt{a^2-b^2}-a}{b}+\frac{\sqrt{a^2-b^2}+a}{b})}$

$= \frac{(2a^2-2b^2-2a\sqrt{a^2-b^2})^2}{2b(2a^2-b^2-2a\sqrt{a^2-b^2})\sqrt{a^2-b^2}} = \frac{2\sqrt{a^2-b^2}}{b}$

$\therefore I = 2\pi i(-\frac{1}{2bi})(\frac{2\sqrt{a^2-b^2}}{b}-\frac{2a}{b}) = \frac{2\pi(a-\sqrt{a^2-b^2})}{b^2}$

(5) 令 $\int_0^{\pi}\frac{a\,dx}{a^2+\sin^2 x} = I$，$I = \frac{1}{2}\int_0^{\pi}\frac{a\,dx}{a^2+\sin^2 x} + \frac{1}{2}\int_0^{\pi}\frac{a\,dy}{a^2+\sin^2 y} \to$ 令 $y=x-\pi$

原式 $= \frac{1}{2}\int_0^{\pi}\frac{a\,dx}{a^2+\sin^2 x} + \frac{1}{2}\int_{\pi}^{2\pi}\frac{a\,dx}{a^2+\sin^2 x} = \frac{a}{2}\int_0^{\pi}\frac{dx}{a^2+\sin^2 x}$

$= \frac{a}{2}\oint_{|z|=1}\frac{dz}{iz[a^2+(z+\frac{1}{z})^2/(2i)^2]} = \frac{a}{2}\oint_{|z|=1}\frac{dz}{iz(a+\frac{z+\frac{1}{z}}{2})(a-\frac{z+\frac{1}{z}}{2})}$

$= -\frac{2a}{i}\oint_{|z|=1}\frac{z\,dz}{(z^2+2az-1)(z^2-2az-1)} = \frac{-2a}{i}\oint_{|z|=1}\frac{z\,dz}{(z+a+\sqrt{a^2+1})(z+a-\sqrt{a^2+1})(z-a+\sqrt{a^2+1})(z-a-\sqrt{a^2+1})}$

$= -\frac{2a}{i}\oint_{|z|=1}f(z)dz$，$f(z)$ 在单位圆内有单奇点 $z_0=-a+\sqrt{a^2+1}$ 及 $z_0=a-\sqrt{a^2+1}$

$\mathrm{Res}f(-a+\sqrt{a^2+1}) = \frac{-a+\sqrt{a^2+1}}{2\sqrt{a^2+1}\cdot 2\cdot(-a+\sqrt{a^2+1})(-2a)} = \frac{-1}{8a\sqrt{a^2+1}}$ $\quad \mathrm{Res}f(a-\sqrt{a^2+1}) = \frac{-1}{8a\sqrt{a^2+1}}$

$\therefore \int_0^{\pi}\frac{a\,dx}{a^2+\sin^2 x} = \frac{2a}{i}2\pi i\frac{1}{4a\sqrt{a^2+1}} = \frac{\pi}{\sqrt{a^2+1}}$

(7) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos^2 x}$ 因为是偶函数，原式 $= \frac{1}{4}\int_0^{2\pi} \frac{dx}{1+\cos^2 x} = \frac{1}{4}\oint_{|z|=1} \frac{\frac{dz}{iz}}{1+\left[\frac{z^2+1}{2z}\right]^2}$

$= \frac{1}{i}\oint_{|z|=1} \frac{z\,dz}{z^4+6z^2+1} = \frac{1}{i}\oint_{|z|=1} \frac{z\,dz}{(z^2+3+2\sqrt{2})(z^2+3-2\sqrt{2})}$

$= \frac{1}{i}\oint_{|z|=1} \frac{z\,dz}{(z^2+3+2\sqrt{2})(z+\sqrt{3-2\sqrt{2}}\,i)(z-\sqrt{3-2\sqrt{2}}\,i)}$

被积函数在积分回路之内的奇点有 $z_0=(\sqrt{2}-1)i$ 和 $z_0=(1-\sqrt{2})i$

$\text{Res}\, f((\sqrt{2}-1)i) = \lim_{z\to z_0} \frac{z}{(z^2+3+2\sqrt{2})(z+(\sqrt{2}-1)i)} = \frac{1}{8\sqrt{2}}$

$\text{Res}\, f((1-\sqrt{2})i) = \lim_{z\to z_0} \frac{z}{(z^2+3+2\sqrt{2})(z-(\sqrt{2}-1)i)} = \frac{1}{8\sqrt{2}}$

$\therefore I = 2\pi i \cdot \frac{1}{i} \cdot \frac{1}{4\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$

(8) $\int_0^{2\pi} \cos^{2n} x\,dx = \oint_{|z|=1} \left(\frac{z^2+1}{2z}\right)^{2n} \frac{dz}{iz} = \frac{1}{2^{2n} i}\oint_{|z|=1} \frac{(1+z^2)^{2n}\,dz}{z^{2n+1}}$

被积函数有 $2n+1$ 阶奇点 $z=0$，且 $\text{Res}\, f(0) = \frac{1}{(2n)!}\lim_{z\to 0} \frac{d^{2n}}{dz^{2n}}(1+z^2)^{2n}$

根据二项式公式知 $(1+z^2)^{2n} = \cdots + \frac{(2n)!\, z^{2k}}{(2n-k)!\,k!} + \cdots$

还要对 z 微分 $2n$ 次，故凡是 $2k<2n$ 的 z^{2k} 项，微分 $2n$ 次后都为 0

而 $2k>2n$ 项中，微分 $2n$ 次仍有 z 存在，在 $z\to 0$ 后都为 0，只有 $2k=2n$ 项有值

即 $\text{Res}\, f(0) = \frac{1}{(2n)!}\lim_{z\to 0} \frac{d^{2n}}{dz^{2n}}\left(\frac{(2n)!\, z^{2n}}{(2n-n)!\,n!}\right) = \frac{(2n)!}{(n!)^2}$

$\therefore I = \frac{1}{2^{2n} i} \cdot 2\pi i \cdot \frac{(2n)!}{(n!)^2} = \frac{2\pi \cdot 2^n n! \cdot (2n-1)!!}{2^n \cdot n! \cdot 2^n \cdot n!} = \frac{2\pi \cdot (2n-1)!!}{(2n)!!}$

2.(4) $\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4}$ 其中被积函数

$f(z) = \frac{1}{z^4+a^4} = \frac{1}{[z-\frac{\sqrt{2}}{2}a(1-i)][z+\frac{\sqrt{2}}{2}a(1-i)][z-\frac{\sqrt{2}}{2}a(1+i)][z+\frac{\sqrt{2}}{2}a(1+i)]}$

设 $a>0$，它在上半平面有单奇点 $z_0=\frac{\sqrt{2}}{2}a(i-1)$ 和 $z_0=\frac{\sqrt{2}}{2}a(i+1)$，其留数为

$\text{Res}\, f(\frac{\sqrt{2}}{2}a(i-1)) = \lim_{z\to z_0} \frac{1}{[z-\frac{\sqrt{2}}{2}a(1-i)](z^2-a^2 i)} = \frac{1}{2\sqrt{2}a^3(1+i)}$

$\text{Res}\, f(\frac{\sqrt{2}}{2}a(1+i)) = \lim_{z\to z_0} \frac{1}{(z^2+a^2 i)(z+\frac{\sqrt{2}}{2}a(1+i))} = \frac{1}{2\sqrt{2}a^3(i-1)}$

$\therefore I = 2\pi i \cdot \frac{1}{2} \cdot \frac{1}{2\sqrt{2}a^3}\left(\frac{1}{i+1} + \frac{1}{i-1}\right) = \frac{\pi}{2\sqrt{2}a^3}$

年 月 日 第 页

(6) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^2}dx$

被积函数 $f(z)=\frac{z^2}{(z^2+a^2)^2} = \frac{z^2}{(z+ai)^2(z-ai)^2}$ 在上半平面有一个二阶奇点 $z_0=ai$,

且 $\mathrm{Res}f(ai)=\lim_{z\to ai}\frac{d}{dz}\left(\frac{z^2}{(z+ai)^2}\right) = \lim_{z\to ai}\left[\frac{2z}{(z+ai)^2}-\frac{2z^2}{(z+ai)^3}\right]$

$=\frac{2ai}{(2ai)^2}-\frac{2(ai)^2}{(2ai)^3} = -\frac{i}{4a}$

$\therefore I = 2\pi i\cdot\frac{1}{2}\left(-\frac{i}{4a}\right)=\frac{\pi}{4a}$

3.(1) $\int_0^\infty \frac{\cos mx}{1+x^4}dx \quad (m>0)$

$\because F(z)\cdot e^{imz} = \frac{e^{imz}}{1+z^4} = \frac{e^{imz}}{[z-\frac{\sqrt{2}}{2}(1-i)][z+\frac{\sqrt{2}}{2}(1-i)][z-\frac{\sqrt{2}}{2}(1+i)][z+\frac{\sqrt{2}}{2}(1+i)]}$

在上半平面有两个单奇点 $z_0=\frac{\sqrt{2}}{2}(i-1)$, $z_0=\frac{\sqrt{2}}{2}(i+1)$, 其留数为

$\mathrm{Res}f(z_0)=\lim_{z\to\frac{\sqrt{2}}{2}(i-1)}\left[\frac{e^{imz}}{(z-\frac{\sqrt{2}}{2}(1-i))(z^2-i)}\right]=\frac{e^{im[\frac{\sqrt{2}}{2}(1-i)]}}{-2i(i-1)\sqrt{2}}=\frac{e^{-im\frac{\sqrt{2}}{2}(1-i)}}{2\sqrt{2}(i+1)}$

$\mathrm{Res}f(z_0)=\lim_{z\to\frac{\sqrt{2}}{2}(i+1)}\left[\frac{e^{imz}}{(z^2+i)(z+\frac{\sqrt{2}}{2}(1+i))}\right]=\frac{e^{im[\frac{\sqrt{2}}{2}(1+i)]}}{2i\sqrt{2}(1+i)}=\frac{e^{im[\frac{\sqrt{2}}{2}(1+i)]}}{2\sqrt{2}(i-1)}$

$\therefore I=\pi i\left[\frac{e^{im[\frac{\sqrt{2}}{2}(1-i)]}}{2\sqrt{2}(i-1)}+\frac{e^{im[\frac{\sqrt{2}}{2}(1+i)]}}{2\sqrt{2}(i+1)}\right]=\pi i\frac{(i-1)e^{-im[\frac{\sqrt{2}}{2}(1-i)]}-(i+1)e^{im[\frac{\sqrt{2}}{2}(1+i)]}}{4\sqrt{2}}$

$=\frac{2e^{-\frac{m}{\sqrt{2}}}[-i\cos\frac{m}{\sqrt{2}}-i\sin\frac{m}{\sqrt{2}}]}{4\sqrt{2}}\pi i = \frac{\sqrt{2}\pi e^{-\frac{m}{\sqrt{2}}}(\cos\frac{m}{\sqrt{2}}+\sin\frac{m}{\sqrt{2}})}{4}$

(2) $\int_0^\infty \frac{\sin mx}{x(x^2+a^2)}dx \ (m>0,a>0) = \frac{1}{2i}\int_0^\infty\frac{e^{imx}}{x(x^2+a^2)}dx = I$

考虑积分 $\oint_l \frac{e^{imz}dz}{z(z^2+a^2)}=\int_{C_R}\frac{e^{imz}dz}{z(z^2+a^2)}+\int_{C_\varepsilon}\frac{e^{imz}dz}{z(z^2+a^2)}+\left(\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R}\right)\frac{e^{imx}dx}{x(x^2+a^2)}$

l 内有单奇点 $i\alpha$, 留数是 $\frac{-e^{-m\alpha}}{2a^2}$. 所以原式左端 $=2\pi i\frac{-e^{-m\alpha}}{2a^2}=\frac{-\pi e^{-m\alpha}}{a^2}i$

又在(1)式两端令 $\varepsilon\to0$, $R\to\infty$, 则右端第一项依约当引理为0 右端最后两项 $=2iI$,

于是 $-\frac{\pi e^{-m\alpha}}{a^2}=\lim_{\varepsilon\to0}\int_{C_\varepsilon}\frac{e^{imz}dz}{z(z^2+a^2)}+2iI$.

$\lim_{\varepsilon\to0}\int_{C_\varepsilon}\frac{e^{imz}dz}{z(z^2+a^2)}=\lim_{\varepsilon\to0}\int_{C_\varepsilon}\left(\frac{1}{a^2z}+\text{解析部分}P(z)\right)dz=\lim_{\varepsilon\to0}\int_{-\pi}^{0}\frac{i\varepsilon e^{i\varphi}}{a^2\varepsilon e^{i\varphi}}d\varphi=\frac{-i\pi}{a^2}$

$\therefore 2iI=\frac{i\pi}{a^2}-\frac{i\pi}{a^2}e^{-ma}$, 即 $I=(1-e^{-ma})\frac{\pi}{2a^2}$

17) $\int_0^{\infty}\frac{\sin^2 x}{x^2}dx = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{e^{ix}\sin x}{x^2}dx = I$

考虑 $\oint_C \frac{e^{iz}\sin z\,dz}{z^2} = [\int_{C_R}+\int_{C}]\frac{e^{iz}\sin z}{z^2}dz + (\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R})\frac{e^{ix}\sin x}{x^2}dx$

如图4-3，内无奇点，所以上式左端为0，令 $\varepsilon\to 0$，$R\to\infty$，右端第一项为

$\int_{C_R}\frac{e^{iz}(e^{iz}-e^{-iz})dz}{2iz^2} = \frac{1}{2i}\int_{C_R}(\frac{e^{2iz}}{z^2}-\frac{1}{z^2})dz$，~~其中~~第一项依约当引理 $\to 0$，第二项

在圆上一致趋于0，所以 $\lim_{R\to\infty}\int_{C_R}=0$

$\therefore 2iI = \lim_{\varepsilon\to 0} -\int_{C_\varepsilon}\frac{e^{iz}\sin z}{z^2}dz = \lim_{\varepsilon\to 0}\int_{C_\varepsilon}-(\frac{1}{z}+\text{解析部分}P(z))dz = \int_{\pi}^{0}-\frac{e^{i\varepsilon e^{i\varphi}}}{\varepsilon e^{i\varphi}}d\varphi = i\pi$

所以 $I=\frac{\pi}{2}$ 也即原式 $=\frac{\pi}{2}$

18) $\int_{-\infty}^{\infty}\frac{e^{imx}}{x-i\alpha}dx$，$\int_{-\infty}^{\infty}\frac{e^{imx}}{x+i\alpha}dx$ $(m>0, \operatorname{Re}\alpha>0)$

上半平面 $\frac{e^{imz}}{z-i\alpha}$ 有单奇点 $i\alpha$，$\frac{e^{imz}}{z+i\alpha}$ 在上半平面无奇点

$\therefore \int_{-\infty}^{\infty}\frac{e^{imx}}{x-i\alpha}dx = 2\pi i(\lim_{x\to i\alpha}e^{imx}) = 2\pi i e^{-m\alpha}$，$\int_{-\infty}^{\infty}\frac{e^{imx}}{x+i\alpha}dx = 0$

~~所以~~