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5. 所求定解问题 $\begin{cases} \Delta u = 0 \ (0<\rho<\rho_0,\ 0<z<L) \\ u_\rho|_{\rho=\rho_0}=0,\ u|_{\rho\to 0}=\text{有限值} \\ u|_{z=0}=u_0\rho^2,\ u|_{z=L}=u_1 \end{cases}$

由边界条件可知，问题与 φ 无关，所以设方程解 $u(\rho,z)=R(\rho)Z(z)$

代入拉普拉斯方程分离变量得 $Z''(z)-\mu Z(z)=0$

$R''(\rho)+\frac{1}{\rho}R'(\rho)+\mu R(\rho)=0$

代入齐次的侧面边界条件：$R'(\rho_0)=0$，$R(0)=$有限值

求解本征值问题 $\begin{cases} R''(\rho)+\frac{1}{\rho}R'(\rho)+\mu R(\rho)=0 \\ R'(\rho_0)=0,\quad R(0)\to\text{有限值} \end{cases}$ 可得 $\begin{cases} \mu_n=\left(\frac{x_n^{(1)}}{\rho_0}\right)^2,\ n=0,1,2,3\cdots \\ R_n(\rho)=J_0\left(\frac{x_n^{(1)}\rho}{\rho_0}\right) \end{cases}$

其中，$x_n^{(1)}$ 为方程 $J_0'(x)=0$ 第n个根，也是方程 $J_1(x)=0$ 的第n个根

对于方程 $Z_n''(z)-\left(\frac{x_n^{(1)}}{\rho_0}\right)^2 Z_n(z)=0$，当 $n=0$ 时，$Z_0''(z)=0$，$Z_0(z)=A_0+B_0z$，$n\neq 0$，$Z_n(z)=A_n\cosh\left(\frac{x_n^{(1)}z}{\rho_0}\right)$ $+B_n\sinh\left(\frac{x_n^{(1)}z}{\rho_0}\right)$

所以原问题的通解变为 $u(\rho,z)=A_0+B_0z+\sum_{n=1}^{\infty}\left[A_n\cosh\left(\frac{x_n^{(1)}z}{\rho_0}\right)+B_n\sinh\left(\frac{x_n^{(1)}z}{\rho_0}\right)\right]J_0\left(\frac{x_n^{(1)}\rho}{\rho_0}\right)$

代入底面边界条件，$u|_{z=0}=u_0\rho^2$，$u|_{z=L}=u_1$　$u|_{z=0}=A_0+\sum_{n=1}^{\infty}A_nJ_0\left(\frac{x_n^{(1)}\rho}{\rho_0}\right)=u_0\rho^2$

则 $A_0=\frac{1}{[N_0^{(0)}]^2}\int_0^{\rho_0}u_0\rho^2J_0\left(\frac{x_0^{(1)}\rho}{\rho_0}\right)\rho\,d\rho=\frac{2u_0}{\rho_0^2}\cdot\frac{\rho_0^4}{4}=\frac{u_0\rho_0^2}{2}$，$A_n=\frac{1}{[N_n^{(0)}]^2}\int_0^{\rho_0}\rho^2J_0\left(\frac{x_n^{(1)}\rho}{\rho_0}\right)\rho\,d\rho=\frac{2u_0}{\rho_0^2[J_0(x_n^{(1)})]^2}\cdot\frac{2\rho_0^4J_0(x_n^{(1)})}{(x_n^{(1)})^2}=\frac{4u_0\rho_0^2}{[J_0(x_n^{(1)})](x_n^{(1)})^2}$

$u|_{z=L}=A_0+B_0L+\sum_{n=1}^{\infty}\left[A_n\cosh\left(\frac{x_n^{(1)}L}{\rho_0}\right)+B_n\sinh\left(\frac{x_n^{(0)}L}{\rho_0}\right)\right]J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)=u_1$

$A_0+B_0L=\frac{u_1}{[N_0^{(0)}]^2}\int_0^{\rho_0}J_0\left(\frac{x_0^{(1)}\rho}{\rho_0}\right)\rho\,d\rho=\frac{2u_1}{\rho_0^2}\cdot\frac{\rho_0^2}{2}=u_1$，所以 $B_0=\frac{u_1-\frac{u_0\rho_0^2}{2}}{L}$

$A_n\cosh\left(\frac{x_n^{(1)}L}{\rho_0}\right)+B_n\sinh\left(\frac{x_n^{(0)}L}{\rho_0}\right)=\frac{u_1}{[N_n^{(0)}]^2}\int_0^{\rho_0}J_0\left(\frac{x_n^{(1)}\rho}{\rho_0}\right)\rho\,d\rho=0$，$B_n=-A_n\frac{\cosh\left(\frac{x_n^{(1)}L}{\rho_0}\right)}{\sinh\left(\frac{x_n^{(1)}L}{\rho_0}\right)}=-\frac{4u_0\rho_0^2}{[J_0(x_n^{(1)})](x_n^{(1)})^2}\cdot\frac{\cosh\left(\frac{x_n^{(1)}L}{\rho_0}\right)}{\sinh\left(\frac{x_n^{(1)}L}{\rho_0}\right)}$

所以原问题解为 $u(\rho,z)=\frac{u_0\rho_0^2}{2}+\frac{u_1-u_0\rho_0^2/2}{L}z+\sum_{n=1}^{\infty}\frac{4u_0\rho_0^2}{[J_0(x_n^{(1)})](x_n^{(1)})^2\sinh(x_n^{(1)}L/\rho_0)}\sinh\frac{x_n^{(1)}(L-z)}{\rho_0}J_0\left(\frac{x_n^{(1)}\rho}{\rho_0}\right)$

7. 取以圆形膜中心为原点的极坐标系，则定解问题 $u_{tt}-a^2\left(u_{\rho\rho}+\frac{1}{\rho}u_\rho\right)=0$

边界条件：$u|_{\rho=0}$ 有限　$u|_{\rho=\rho_0}=0$，初始条件 $\begin{cases} u|_{t=0}=\left(\frac{1-\rho^2}{\rho_0^2}\right)u_0 \\ u_t|_{t=0}=0 \end{cases}$

设 $u=u(\rho)T(t)$ 分离变量得 $\begin{cases} T''+\lambda^2a^2T=0 \\ u''+\frac{1}{\rho}u'+\lambda^2u=0 \end{cases}$ $\longrightarrow$ 方程有限解为 $u=J_0(\lambda\rho)$，令 $x_n^{(0)}$ 为 $J_0(x)$ 第n个零点 $(n=1,2,3\cdots)$

则 $\lambda_n^{(0)}=\frac{x_n^{(0)}}{\rho_0}$，$v_n=J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)$，$T_n=A_n\cos\frac{x_n^{(0)}}{\rho_0}t+B_n\sin\frac{x_n^{(0)}}{\rho_0}t$

$\therefore u=\sum_{n=1}^{\infty}\left(A_n\cos\frac{x_n^{(0)}}{\rho_0}at+B_n\sin\frac{x_n^{(0)}}{\rho_0}at\right)J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)$，由初始条件 $A_n=\frac{2u_0}{\rho_0^2[J_1(x_n^{(0)})]^2}\int_0^{R}\left(\frac{1-\rho^2}{\rho_0^2}\right)J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)\rho\,d\rho$

$=\frac{2u_0}{\rho_0^2[J_1(x_n^{(0)})]^2}\left[\frac{\rho_0^2}{x_n^{(0)}}J_1(x_n^{(0)})-\frac{\rho_0^2}{x_n^{(0)}}J_1(x_n^{(0)})+\frac{4\rho_0^2}{[x_n^{(0)}]^3}J_1(x_n^{(0)})\right]$

$=\frac{8u_0}{[x_n^{(0)}]^3J_1(x_n^{(0)})}$

于是 $u=8u_0\sum_{n=1}^{\infty}\frac{1}{[x_n^{(0)}]^3J_1(x_n^{(0)})}J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)\cos\frac{x_n^{(0)}at}{\rho_0}$

11. 所求定解问题 $\begin{cases} u_{tt}-a^2\Delta u=0 \ (0<\rho<\rho_0,\ 0<z<L) \\ u_\rho|_{\rho=\rho_0}=0,\ u|_{\rho\to 0}=\text{有限} \\ u|_{z=0}=0,\ u|_{z=L}=0 \\ u|_{t=0}=0,\ u_t|_{t=0}=u_0\rho^2z \end{cases}$

因此，问题与 φ 无关，设原方程解为 $u(\rho,z,t)=R(\rho)Z(z)T(t)$

代入波动方程，分离变量之后 $R''(\rho)+\frac{1}{\rho}R'(\rho)+(k^2+\mu)R(\rho)=0$，$Z''(z)-\mu Z(z)=0$，$T''(t)+k^2a^2T(t)=0$

将齐次侧面 $R'(\rho_0)=0$，$R(0)=$有限值

代入齐次边界条件，$Z(0)=0$，$Z(L)=0$，求解本征值 $\begin{cases} Z''(z)-\mu Z(z)=0 \\ Z(0)=0,\ Z(L)=0 \end{cases}$ 可得 $\begin{cases} -\mu_p=\left(\frac{p\pi}{L}\right)^2,\ p=1,2,3\cdots \\ Z_p(z)=\sin\frac{p\pi z}{L} \end{cases}$

求本征值 $\begin{cases} R''(\rho)+\frac{1}{\rho}R'(\rho)+\mu'R(\rho)=0 \\ R'(\rho_0)=0\quad R(0)=\text{有限值} \end{cases}$，其中 $\mu'=k^2+\mu=k^2-\frac{p^2\pi^2}{L}$ 可得 $\begin{cases} \mu_n'=\left[\frac{x_n^{(1)}}{\rho_0}\right]^2=\left[\frac{x_n^{(1)}}{\rho_0}\right]^2,\ n=1,2,3\cdots \\ R(\rho)=J_0\left[\frac{x_n^{(1)}\rho}{\rho_0}\right] \end{cases}$，其中 $x_n^{(1)}$ 为 $J_0'(x)=0$ 第n个根

解关于t的方程：$T_{np}''(t)+k_{np}^2a^2T_{np}(t)=0$，其中 $k_{np}^2=\left[\frac{x_n^{(1)}}{\rho_0}\right]^2+\frac{p^2\pi^2}{L}$，$T_{np}(t)=A_{np}\cos(k_{np}at)+B_{np}\sin(k_{np}at)$

由初始条件，$u|_{t=0}=0$，可得 $A_{np}=0$，$T_{np}(t)=B_{np}\sin(k_{np}at)$，代入初始条件，$u_t|_{t=0}=u_0\rho^2z$

原问题通解 $u(\rho,t)=\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}B_{np}J_0\left[\frac{x_n^{(1)}\rho}{\rho_0}\right]\sin\frac{p\pi z}{L}\sin(k_{np}at)$

$B_{np}k_{np}a=\frac{u_0}{[N_n^{(0)}]^2}\int_0^{\rho_0}\rho^2J_0\left[\frac{x_n^{(1)}\rho}{\rho_0}\right]\rho\,d\rho\cdot\frac{2}{L}\int_0^{L}z\sin\frac{p\pi z}{L}dz$，其中 $[N_n^{(0)}]^2=\frac{1}{2}\rho_0^2J_0[x_n^{(1)}]^2$

$B_{np}=\frac{(-1)^{p+1}8u_0L\rho_0^2}{ap\pi[x_n^{(1)}]^2J_0[x_n^{(1)}]}\cdot\frac{1}{\sqrt{(x_n^{(1)}/\rho_0)^2+(p\pi/L)^2}}$

所以，原方程解为 $u(\rho,t)=\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}8u_0L\rho_0^2}{ap\pi[x_n^{(1)}]^2J_0[x_n^{(1)}]}\cdot\frac{1}{\sqrt{\left(\frac{x_n^{(1)}}{\rho_0}\right)^2+\left(\frac{p\pi}{L}\right)^2}}J_0\left(\frac{x_n^{(1)}\rho}{\rho_0}\right)\sin\frac{p\pi z}{L}\sin\left(\sqrt{\left(\frac{x_n^{(1)}}{\rho_0}\right)^2+\left(\frac{p\pi}{L}\right)^2}at\right)$

13. $\begin{cases} u_t - a^2\Delta u = 0 \\ u|_{\rho=\rho_0} = u_1 z^2,\ u|_{\rho=0} \text{为有限值} \\ u|_{z=0} = 0,\ \frac{\partial u}{\partial z}|_{z=L} = 2Lu_1 \\ u|_{t=0} = u_0 \end{cases}$ 其中 u_0、u_1 为任意常数

如果令 $u = u_1 z^2 + f$，对 f 有

$\begin{cases} f_t - a^2\Delta f = 2a^2 u_1 & (1) \\ f|_{\rho=\rho_0} = 0,\ f|_{\rho=0} \text{为有限值} & (2) \\ f|_{z=0} = 0,\ \frac{\partial f}{\partial z}|_{z=L} = 0 & (3) \\ f|_{t=0} = u_0 - u_1 z^2 & (4) \end{cases}$

由此，我们求解非齐次方程时，可以用傅里叶级数直接解，也可用冲量定理先将非齐次方程化为齐次方程之后求解

(4)式中如果令 $u = 2u_1 L z + f$，(1)(3)式不变，(4)式变为 $f|_{t=0} = u_0 - 2u_1 L z$，(2)式变为 $f|_{\rho=\rho_1} = u_1 z^2 - 2u_1 L z$，$f_{\rho\to 0}$ 有限值

运用冲量定理法前提是初始条件均为0

将 f 分解为 $f = w + g$

$\begin{cases} w_t - a^2\Delta w = 2a^2 u_1 \\ w|_{\rho=\rho_0} = 0,\ w|_{\rho=0} \text{为有限值} \\ w|_{z=0} = 0,\ \frac{\partial w}{\partial z}|_{z=L} = 0 \\ w|_{t=0} = 0 \end{cases}$

$\begin{cases} g_t - a^2\Delta g = 0 \\ g|_{\rho=\rho_0} = 0,\ g|_{\rho=0} \text{为有限值} \\ g|_{z=0} = 0,\ \frac{\partial g}{\partial z}|_{z=L} = 0 \\ g|_{t=0} = u_0 - u_1 z^2 \end{cases}$

类比冲量定理解齐次方程

可得如下等式：$f = \int_0^t v\,dt + g$

其中 $v(z,t,\tau)d\tau$ 描述很短时间区间 $(\tau, \tau+d\tau)$ 上瞬时热源产生的温度分布

将持续作用的热源看作许多时间上相继的瞬时"热源" $c\rho ds \cdot 2a^2 u_1 \delta(t-\tau)d\tau$ 的叠加。每个瞬时热源在 $(\tau, \tau+d\tau)$ 内，提供热量为 $c\rho ds \cdot 2a^2 u_1 d\tau$，所产生温度分布为 $v(z,t,\tau)d\tau$ 每个瞬时热源在 $(\tau,\tau+d\tau)$ 内，即边界和初始条件温度均为零的有源运输问题的解。瞬时热源的持续累积效果 $\boxed{w = \int_0^t v(z,t,\tau)\,d\tau}$

对于 g 的无源运输问题，直接用傅里叶—贝塞尔级数展开

$\begin{cases} v_t - a^2\Delta v = 2a^2 u_1 \delta(t-\tau) \\ v|_{\rho=\rho_0} = 0,\ v|_{\rho=0} = \text{有限值} \\ v|_{z=0} = 0,\ \frac{\partial v}{\partial z}|_{z=L} = 0 \\ v|_{t=0} = 0 \end{cases}$

在 $t=0$ 时刻，"瞬时"热源尚未作用。τ 至 $\tau+d\tau$ 时刻有热源

此时 $v(t=\tau+d\tau)$ 有增加，即

$c\rho(v|_{t=\tau+d\tau} - v|_{t=\tau-0})dsd\tau = c\rho \cdot 2a^2 u_1 dsd\tau$

由此可得 $c\rho v|_{t=\tau+d\tau} = 2a^2 u_1$　又 $\because d\tau \to 0$，则 $\tau + d\tau$ 可记作 τ，因此关于 v 的方程组

$\begin{cases} v_t - a^2\Delta v = 0 \\ v|_{\rho=\rho_0} = 0 \\ v|_{z=0} = 0 \\ v|_{t=\tau} = 2a^2 u_1 \end{cases}$ 且 $\begin{cases} v|_{\rho=0} \text{为有限值} \\ \frac{\partial v}{\partial z}|_{z=L} = 0 \end{cases}$

而由 w 和 v 关系式，可以求解关于 v 的方程组

由于边界条件齐次，v 与 φ 无关，柱坐标系下 v 的通解为

$\left[\begin{Bmatrix}\cos\nu z \\ \sin\nu z\end{Bmatrix} J_0(\sqrt{\mu}\rho) + \begin{Bmatrix}1 \\ z\end{Bmatrix} J_0(\sqrt{\mu}\rho) + \begin{Bmatrix}\cos\nu z \\ \sin\nu z\end{Bmatrix}\begin{Bmatrix}1 \\ \ln\rho\end{Bmatrix}\right] e^{-k^2 a^2 (t-\tau)}$

由于边界条件 z 和 ρ 都是齐次的，可得

其中 $k^2 \neq 0$，由于 $v(\rho=0)$ 为有限值，则舍去其中的 $\ln\rho$ 项，

$v = \sum_{n=1}^{\infty}\left[C_n\cos\nu z + D_n\sin\nu z + E_n J_0(\sqrt{\mu}\rho) + F_n z J_0(\sqrt{\mu}\rho) + G_n\cos\nu z J_0(\sqrt{\mu}\rho) + H_n\sin\nu z J_0(\sqrt{\mu}\rho)\right] e^{-k^2 a^2(t-\tau)}$

$C_n = D_n = E_n = F_n = G_n = 0$，本征值 $\nu^2 = (p+\frac{1}{2})^2\pi^2/L^2$，$p$ 为非负整数，$k^2 = \nu^2 + \mu$，$\mu_n^{(0)} = (x_n^{(0)}/\rho_0)^2$

$x_n^{(0)}$ 为零阶贝塞尔函数 $J_0(x)$ 第 n 个零点，通解化为 $v = \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} A_{np} J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)\sin\frac{(p+\frac{1}{2})\pi}{L}z\, e^{-\left[\left(\frac{x_n^{(0)}}{\rho_0}\right)^2 + \frac{(p+\frac{1}{2})^2\pi^2}{L^2}\right]a^2(t-\tau)}$

代入初始条件 $v|_{t=\tau} = 2a^2 u_1$，则有 $\sum_{n=1}^{\infty}\sum_{p=0}^{\infty} A_{np} J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)\sin\frac{(p+\frac{1}{2})\pi}{L}z = 2a^2 u_1$，将 u_1 展开为傅里叶级数，就可得到 A_{np} 的解

$A_{np} = \frac{2u_1}{x_n^{(0)}J_1(x_n^{(0)})}\frac{4a^2}{(p+1)\pi} = \frac{8a^2 u_1}{(p+1)\pi x_n^{(0)} J_1(x_n^{(0)})}$，$w = \int_0^t v(z,t,\tau)\,d\tau = \sum_{n=1}^{\infty}\sum_{p}\frac{A_{np}}{k^2 a^2}\left(e^{-k^2 a^2 t} - 1\right) J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)$

解得 $u(\rho,z,t) = u_1 z^2 + \sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\frac{4}{(p+\frac{1}{2})\pi x_n^{(0)}}\left[u_0 + (-1)^{p+1}\frac{2u_1 L^2}{(p+\frac{1}{2})\pi} + \frac{2u_1 L^2}{(p+\frac{1}{2})^2\pi^2}\right]\cdot\exp\left\{-\left[\left(\frac{x_n^{(0)}}{\rho_0}\right)^2 + \frac{(p+\frac{1}{2})^2\pi^2}{L^2}\right]a^2 t\right\} J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)\sin\frac{(p+\frac{1}{2})\pi}{L}z + \sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\frac{8u_1 a^2}{(p+\frac{1}{2})\pi x_n^{(0)} J_1(x_n^{(0)})}\frac{1}{\left(\frac{x_n^{(0)}}{\rho_0}\right)^2 + \frac{(p+\frac{1}{2})^2\pi^2}{L^2}}\cdot\left\{1 - e^{-\left[\left(\frac{x_n^{(0)}}{\rho_0}\right)^2 + \frac{(p+\frac{1}{2})^2\pi^2}{L^2}\right]a^2 t}\right\} J_0\left(\frac{x_n^{(0)}\rho}{\rho_0}\right)\sin\frac{(p+\frac{1}{2})\pi z}{L}$