1. 解：$-\frac{\partial^n\psi}{\partial t^n}$ 表为回路积分如下：

$$\left.\frac{\partial^n\psi}{\partial t^n}\right|_{t=0} = \frac{n!}{2\pi i}\oint_L \frac{e^{2\xi x-\xi^2}}{(\xi-t)^{n+1}}\,d\xi = \frac{n!}{2\pi i}\oint_L \frac{e^{2\xi x-\xi^2}}{\xi^{n+1}}\,d\xi_0$$

将 $\xi = x-z$ 代入

$$\left.\frac{\partial^n\psi}{\partial t^n}\right|_{t=0} = \frac{n!}{2\pi i}\oint_L \frac{e^{x^2-z^2}}{(x-z)^{n+1}}\,d(-z)$$

$$= \frac{n!}{2\pi i}\oint_L \frac{e^{x^2}\cdot e^{-z^2}}{(-1)^n\,(z-x)^{n+1}}\,dz$$

$$= e^{x^2}\frac{n!}{2\pi i}\oint \frac{(-1)^n e^{-z^2}dz}{(z-x)^{n+1}}$$

$$= (-1)^n e^{x^2}\frac{d^n e^{-x^2}}{dx^n}$$ ，得证

2. 表为回路积分

$$\frac{\partial^n\psi}{\partial t^n} = \frac{n!}{2\pi i}\oint_L \frac{e^{-\frac{x\xi}{1-\xi}}/(1-\xi)\,d\xi_n}{(\xi-t)^{n+1}}$$

$$\left.\frac{\partial^n\psi}{\partial t^n}\right|_{t=0} = \frac{n!}{2\pi i}\oint_L \frac{e^{-\frac{x\xi}{1-\xi}}/1-\xi}{\xi^{n+1}(1-\xi)}\,d\xi_n$$

将 $\xi = \frac{z-x}{z}$ 代入上式

$$\left.\frac{\partial^n\psi}{\partial t^n}\right|_{t=0} = \frac{n!}{2\pi i}\oint_L \frac{e^{-x\frac{z-x}{t}}/(1-\frac{z-x}{z})}{(\frac{z-x}{z})^{n+1}(1-\frac{z-x}{z})}\cdot(\frac{x}{z^2})\,dz$$

$$\Rightarrow = \frac{n!}{2\pi i}\oint_L \frac{z^{n+1}\cdot e^{-(z-x)\frac{z}{x}}}{(z-x)^{n+1}}(\frac{x}{z^2})\,dz$$

$$= e^x\frac{n!}{2\pi i}\oint_L \frac{z^n e^{-z}}{(z-x)^{n+1}}\,dz$$

$$= e^x\frac{d^n}{dx^n}(x^n e^{-x})$$

补充题1：$|E| = \frac{\lambda}{2\pi\varepsilon_0 r}$ ~~$(r>a)$~~

用复数表示：$E(z) = E_x + E_y = \frac{\lambda}{2\pi\varepsilon_0 r e^{-i\theta}}$

$$\begin{cases} E_x = -\frac{\partial u}{\partial x} \\ E_y = -\frac{\partial u}{\partial y} \end{cases}$$

$$\frac{dW(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\bar{E}(z) = -\frac{\lambda}{2\pi\varepsilon_0 z}$$

$\therefore W(z) = \frac{-\lambda}{2\pi\varepsilon_0}\ln z + C$ 将 $z = re^{i\theta}$ 代入

$W(z) = (-\frac{\lambda}{2\pi\varepsilon_0}\ln r + C_1) + i(C_2 - \frac{\lambda}{2\pi\varepsilon_0}\theta)$

$u(x,y) = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C_1$

$v(x,y) = C_2 - \frac{\lambda}{2\pi\varepsilon_0}\theta$

$\oint_L \frac{dz}{z-\alpha}$ 则表示从xy平面~~的a高度处的电势到c高度~~的绕a的小单元绕n圈(圈数与L有关)

每一圈的值为 $2\pi i$

补充题2：$\oint_L \frac{dz}{(z-\alpha)^n} = \oint_C R^{-n}e^{-in\varphi}\,\frac{1}{\cancel{Re^{i\varphi}\,id\varphi}}\,d(\alpha+Re^{i\varphi})$

$= \int_0^{2\pi} R^{-n}e^{-in\varphi}\frac{1}{Re^{i\varphi}}\,id\varphi$

$= iR^{-n-1}\int_0^{2\pi} e^{-i(n+1)\varphi}d\varphi$

$= iR^{-n-1}\frac{-1}{i(n+1)}e^{-i(n+1)\varphi}\Big|_0^{2\pi}$

$= 0$

$\cancel{x=(-1)^{\frac{1}{k}}(1+\frac{1}{k})}$

3. (1) $\sum_{i=1}^{\infty}\frac{1}{k}(z-i)^k$

收敛半径 $R=\lim\limits_{k\to\infty}\left|\frac{\frac{1}{k}}{\frac{1}{k+1}}\right|=\lim\limits_{i\to\infty}\left|\frac{k+1}{k}\right|=1$

$\therefore$ 收敛圆为 $|z-i|=1$

(2) $\sum_{i=1}^{\infty}k^{\ln k}(z-2)^k$,

收敛半径 $R=\lim\limits_{k\to\infty}\left|\frac{k^{\ln k}}{(k+1)^{\ln(k+1)}}\right|$

令 $(k+1)^{\ln(k+1)}=(k+1)^{\ln[k(1+\frac{1}{k})]}=(k+1)^{\ln k}\cdot(k+1)^{\ln(1+\frac{1}{k})}$

故 $R=\lim\limits_{k\to\infty}\frac{k^{\ln k}}{(k+1)^{\ln k}}\cdot\frac{1}{(k+1)^{\ln(1+\frac{1}{k})}}$.

$\lim\limits_{k\to\infty}\ln\frac{k^{\ln k}}{(k+1)^{\ln k}}=\lim\limits_{k\to\infty}\frac{\ln(1+\frac{1}{k})}{\frac{1}{\ln k}}$ 利用洛必达法则

$=\lim\limits_{k\to\infty}\frac{\frac{1}{1+\frac{1}{k}}(-\frac{1}{k^2})}{-\frac{1}{(\ln k)^2}\frac{1}{k}}=\lim\limits_{k\to\infty}\frac{(\ln k)^2}{k+1}=\lim\limits_{k\to\infty}\frac{2(\ln k)\frac{1}{k}}{1}=\lim\limits_{k\to\infty}\frac{2\ln k}{k}\to 0$

$\therefore\lim\limits_{k\to\infty}\frac{k^{\ln k}}{(k+1)^{\ln k}}=1$.

同理 $\lim\limits_{k\to\infty}\ln(1+\frac{1}{k})\cdot\ln(k+1)=\lim\limits_{k\to\infty}\frac{\ln(1+\frac{1}{k})}{\frac{1}{\ln(k+1)}}=\frac{\frac{1}{1+\frac{1}{k}}(-\frac{1}{k^2})}{\frac{1}{[\ln(k+1)]^2}\cdot\frac{1}{k+1}}$

$=\lim\limits_{k\to\infty}\frac{[\ln(k+1)]^2}{k}\cdot\lim\limits_{k\to\infty}(1+\frac{1}{k})=\lim\limits_{k\to\infty}\frac{[\ln(k+1)]^2}{k}$ 用洛必达法则

$=\lim\limits_{k\to\infty}[2\ln(k+1)]\frac{1}{k+1}=\lim\limits_{k\to\infty}\frac{2\ln(k+1)}{k+1}=\lim\limits_{k\to\infty}2\cdot\frac{1}{k+1}=0$

$\therefore\lim\limits_{k\to\infty}\ln(1+\frac{1}{k})\ln(k+1)=1$

$\therefore$ 收敛半径为1，收敛圆为 $|z_1-2|=1$

(3) $\sum_{k=1}^{\infty}(\frac{z}{k})^k$

收敛半径：$R=\lim\limits_{k\to\infty}\frac{1}{\sqrt[k]{|a_k|}}=\lim\limits_{k\to\infty}\frac{1}{\sqrt[k]{\frac{1}{k^k}}}$

$=\lim\limits_{k\to\infty}\sqrt[k]{k^k}=\lim\limits_{k\to\infty}k=\infty$

(4) $\sum_{k=1}^{\infty} k!\left(\frac{z}{k}\right)^k$

收敛半径：$R=\lim_{k\to\infty} \frac{k!}{(k+1)!} \cdot \frac{(k+1)^{k+1}}{k^k} = \lim_{k\to\infty}\left[\frac{1}{k+1} \cdot \frac{(k+1)^{k+1}}{k^k}\right] = \lim_{k\to\infty} \frac{(k+1)^k}{k^k}$

$=\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e$，所以收敛圆是$|z|=e$

(5) $\sum_{k=1}^{\infty} k^k (z-3)^k$

收敛半径：$R=\lim_{k\to\infty} \frac{1}{\sqrt[k]{|a_k|}} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k^k}} = \lim_{k\to\infty} \frac{1}{k} = 0$

4. (1) 若$R_1 \le R_2$，则$|z|=R_1$内部$\sum_{k=0}^{\infty} a_k z^k$和$\sum_{k=0}^{\infty} b_k z^k$都绝对收敛，从而$\sum_{k=0}^{\infty}(a_k+b_k)z^k$绝对收敛，$\therefore$收敛半径$\ge \min[R_1, R_2]$

(2) $\sum_{k=0}^{\infty}(a_k-b_k)z_k$　若$R_1 \le R_2$ 则$|z|=R_1$内$\sum_{k=0}^{\infty}-b_k z^k$同样收敛

结论与上题相同，收敛半径$\ge \min[R_1, R_2]$

(3) $\sum_{k=0}^{\infty} a_k b_k z^k$　$R=\lim_{k\to\infty}\left|\frac{a_k b_k}{a_{k+1} b_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}} \cdot \frac{b_k}{b_{k+1}}\right| = R_1 R_2$

(4) $\sum_{k=0}^{\infty} \frac{a_k}{b_k} z^k$　$R=\lim_{k\to\infty}\left|\frac{a_k/b_k}{a_{k+1}/b_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{a_k/a_{k+1}}{b_k/b_{k+1}}\right| = \frac{R_1}{R_2}$

(1) $\operatorname{arctg} z$ 在 $z_0=0$　令 $f(z)=\operatorname{arctg} z$，则

$f(z)=\operatorname{arctg} z$，$f(0)=0$

$f'(z)=\frac{1}{1+z^2}$　$f''(z)=\frac{-2z}{(1+z^2)^2}$　$f''(0)=0$　$f^{(3)}(z)=\frac{6z^2-2}{(1+z^2)^3}$　$f^{(3)}(0)=-2$

$\therefore f(z)=z-\frac{1}{3}z^3+\frac{1}{5}z^5-\frac{1}{7}z^7+\cdots$，$(|z|<1)$

(3) $f(z)=\ln z$，$f(i)=\ln i$　$f'(z)=\frac{1}{z}$　$f'(i)=\frac{1}{i}$

$f''(z)=-\frac{1}{z^2}$，$f''(i)=-\frac{1}{i^2}$　$f^{(3)}(z)=\frac{2}{z^3}$；$f^{(3)}(i)=\frac{2}{i^3}$

$\therefore f(z)=\ln i+\frac{1}{i}(z-i)-\frac{1}{2i^2}(z-i)^3+\frac{1}{3i^3}(z-i)^3+\cdots$

(7) $f(z)=z^{\frac{1}{m}}$ $\quad f(1)=1 \quad f'(z)=\frac{1}{m}z^{\frac{1}{m}-1}$, $f'(1)=\frac{1}{m}$

$f''(z)=\frac{1-m}{m^2}z^{\frac{1}{m}-2}$, $f''(1)=\frac{1-m}{m^2}$ $\quad f^{(3)}(z)=\frac{(1-m)(1-2m)}{m^3}z^{\frac{1}{m}-3}$ $\quad f^{(3)}(1)=\frac{(1-m)(1-2m)}{m^3}$

故其泰勒级数

$$f(z)=1+\frac{1}{m}(z-1)+\frac{1-m}{2!m^2}(z-1)^2+\frac{(1-m)(1-2m)}{3!m^3}(z-1)^3+\cdots$$

(8) $f(z)=\sin^2 z$ $\quad f(0)=0 \quad f'(z)=2\sin z\cos z \quad f'(0)=0$

$f''(z)=2\cos 2z$, $f''(0)=2$ $\quad f^{(3)}(z)=-4\sin 2z$, $f^{(3)}(0)=0$

$f^{(4)}(z)=-8\cos 2z$, $f^{(4)}(0)=-2^3$

故其泰勒级数为 $f(z)=\frac{2}{2!}z^2-\frac{2^3}{4!}z^4+\frac{2^5}{6!}z^6-\cdots=\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2z)^{2k}}{(2k)!}$

1) $z^5e^{\frac{1}{z}}$ 在 $z_0=0$ 由 $e^t=1+t+\frac{t^2}{2!}+\cdots+\frac{t^n}{n!}+\cdots \ (|t|<\infty)$ 知

$$e^{1/z}=1+\frac{1}{z}+\frac{1}{2!z^2}+\cdots+\frac{1}{n!}\left(\frac{1}{z}\right)^n+\cdots \quad (0<|z|)$$

(2) $\frac{1}{z^2(z-1)}$ 在 $z_0=1$ $\quad \because \frac{1}{z^2(z-1)}=\frac{1}{(z-1)[1-(1-z)]^2}$，当 $|t|<1$ 时，

$$\frac{1}{(1-t)^2}=\frac{d}{dt}\frac{1}{1-t}=\frac{d}{dt}\sum_{k=0}^{\infty}t^k=\sum_{k=0}^{\infty}(t^k)'=\sum_{k=1}^{\infty}kt^{k-1}$$

所以，当 $0<|z-1|<1$ 时，有 $\frac{1}{z^2(z-1)}=\frac{1}{z-1}\sum_{k=1}^{\infty}k(1-z)^{k-1}$

$$=\sum_{k=1}^{\infty}(-1)^{k-1}k(z-1)^{k-1}$$

即 $\frac{1}{z^2(z-1)}=\sum_{k=-1}^{\infty}(-1)^{k+1}(k+2)(z-1)^k \quad (0<|z-1|<1)$

(4) $\because |z|>1$，所以 $\left|\frac{1}{z}\right|<1$，则 $\frac{1}{1-z}=\frac{-1}{z(1-\frac{1}{z})}=-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$

$$=-\left(\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

从而可得 $e^{\frac{1}{1-z}}=1-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)+\frac{1}{2!}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)^2-\frac{1}{3!}\left(\frac{1}{z}+\frac{1}{z^2}+\cdots\right)^3$

$+\frac{1}{4!}\left(\frac{1}{z}+\frac{1}{z^2}+\cdots\right)^4-\frac{1}{5!}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)^5+\cdots$

$$=1-\frac{1}{z}-\frac{1}{2z^2}-\frac{1}{6z^3}+\frac{1}{24z^4}-\frac{19}{120z^6}+\cdots \quad (|z|>1)$$

(5) $\frac{1}{(z-2)(z-3)}$ 在 $|z|>3$ ∵ $\frac{1}{(z-2)(z-3)} = \frac{z-2-(z-3)}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}$

$= \frac{1}{z}\frac{1}{1-\frac{3}{z}} - \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \cancel{\sum_{k=0}^{\infty}\frac{3^k}{z^{k+1}}}$

当 $|z|>3$ 时 $\frac{1}{z}\cdot\frac{1}{1-\frac{3}{z}} = \sum_{k=0}^{\infty}\frac{3^k}{z^{k+1}} = \sum_{k=-\infty}^{-1} 3^{-(k+1)}z^k$

同理 $\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} = \sum_{k=-\infty}^{-1} 2^{-(k+1)}z^k$

所以 $\frac{1}{(z-2)(z-3)} = \sum_{k=-\infty}^{-1}[3^{-(k+1)}-2^{-(k+1)}]z^k \ (|z|>3)$

(9) e^z/z 在奇点、由于 $\frac{e^z}{z}$ 在 $z=0$ 解析，可作泰勒展开

$\frac{e^z}{z} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{k!}z^k = \sum_{k=-1}^{\infty}\frac{z^k}{(k+1)!} = \frac{1}{z}+\sum_{k=1}^{\infty}\frac{z^{k-1}}{k!} \ (0<|z|<\infty)$

(11) $\sin\frac{1}{z}$ 在奇点 $\sin\frac{1}{z} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{z}\right)^{2k+1}, (0<|z|<\infty)$

(15) $\frac{1}{z^2(z^2-1)^2}$ 在 $0<|z|<1$，在 $1<|z|<\infty$

(i) 在 $0<|z|<1$ 时，

$\frac{1}{z^2(z^2-1)^2} = \frac{1}{z^2}\cdot\frac{1}{2z}\frac{d}{dz}\left(\frac{1}{1-z^2}\right) = \frac{1}{2z^3}\frac{d}{dz}\sum_{k=0}^{\infty}z^{2k}$

$= \frac{1}{2z^3}\sum_{k=0}^{\infty}2kz^{2k-1} = \sum_{k=-1}^{\infty}(k+2)z^{2k}$

(ii) 在 $1<|z|<\infty$ 时，$\frac{1}{z^2(z^2-1)^2} = \frac{1}{z^6(1-\frac{1}{z^2})^2} = \frac{1}{z^6}\left(\frac{-z^3}{2}\right)\frac{d}{dz}\left(\frac{1}{1-\frac{1}{z^2}}\right)$

$= -\frac{1}{2z^3}\frac{d}{dz}\sum_{k=0}^{\infty}\left(\frac{1}{z^2}\right)^k = -\frac{1}{2z^3}\sum_{k=0}^{\infty}(-2k)\frac{1}{z^{2k+1}}$

$= -\sum_{k=-\infty}^{-1}(k+2)z^{2k}$

36.(1) $f(z)g(z)$ $f(z)=\frac{\phi(z)}{(z-z_0)^m}$，$g(z)=\frac{\psi(z)}{(z-z_0)^n}$，其中 $\phi(z)$ 与 $\psi(z)$ 在 $z=z_0$ 的邻域上解析，且二者均不等于0

因此 $f(z)g(z) = \frac{\phi(z)\psi(z)}{(z-z_0)^{m+n}}$

∴ z_0 是 $f(z)g(z)$ 的 $(m+n)$ 阶极点.

(2) $f(z)/g(z)$

同上题设 $f(z)=\frac{\phi(z)}{(z-z_0)^m}$ $g(z)=\frac{\psi(z)}{(z-z_0)^n}$

$\frac{f(z)}{g(z)}=\frac{\phi(z)}{\psi(z)}\cdot(z-z_0)^{n-m}$

如果 $m>n$, 则 z_0 是 $f(z)/g(z)$ 在 $m-n$ 阶极点

如果 $m<n$, 则 z_0 不是 $f(z)/g(z)$ 的奇点

(3) $f(z)+g(z)$

同上题设 $f(z)=\frac{\phi(z)}{(z-z_0)^m}$, $g(z)=\frac{\psi(z)}{(z-z_0)^n}$

$\therefore f(z)+g(z)=\frac{\phi(z)}{(z-z_0)^m}+\frac{\psi(z)}{(z-z_0)^n}$

$\therefore$ z_0 是 $f(z)+g(z)$ 的极点, 其阶数为 m 和 n 中较大的一个

如 $m=n$, 则极点的阶数可能小于 m