1. $u_{tt}-a^2\Delta_2 u=0$ 先分离 t

$U(\rho,\varphi)\cdot T''(t)-a^2\Delta_2 U(\rho,\varphi)\cdot T(t)=0$

$\therefore \dfrac{T''}{a^2T}=\dfrac{\Delta_2 U}{U}$

由于左侧为 t 的函数，右侧为 ρ,φ 函数，等式成立

则两边必须都为常数，记为 $-k^2$

$\therefore T''+a^2k^2T=0$ (2) $\Delta_2 U+k^2U=0$ (3)

其中(3)式为二维亥姆霍兹方程，在平面极坐标系下：

$U_{\rho\rho}+\frac{1}{\rho}U_\rho+\frac{1}{\rho^2}U_{\varphi\varphi}+k^2U=0$

令 $U(\rho,\varphi)=R(\rho)\phi(\varphi)$ 代入

$R''(\rho)\phi(\varphi)+\frac{1}{\rho}R'(\rho)\cdot\phi(\varphi)+\frac{1}{\rho^2}R(\rho)\cdot\phi''(\varphi)+k^2R(\rho)\phi(\varphi)=0$

$\therefore \dfrac{\rho^2R''}{R}+\dfrac{\rho R'}{R}+k^2\rho^2=-\dfrac{\phi''}{\phi}$

与上述讨论相同，两侧必为常数，记为 m^2

$\therefore \phi''+m^2\phi=0$ (4) $\rho^2R''+\rho R'+(k^2\rho^2-m^2)R=0$ (5)

令 $x=k\rho$ 代入(5) 得贝塞尔方程

$x^2R''+xR'+(x^2-m^2)R=0$

由周期性条件，方程(4)解变为 $\phi_m=A_m\cos m\varphi+B_m\sin m\varphi$

由解在 $\rho\to0$ 有限，方程(2)解为 $T_m=C_m\cos k_m at+D_m\sin k_m at$

5. 3. 先令 $A=\dfrac{h^2}{8\pi\mu}$，$B=Ze^2$

方程在球坐标系下定态的表达式为：

$A\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)}{r^2\sin\theta}+\frac{\frac{\partial^2u}{\partial\varphi^2}}{r^2\sin^2\theta}\right]+B\cdot\frac{u}{r}+Eu=0$

令 $u(r,\theta,\varphi)=R(r)Y(\theta,\varphi)$ 代入

得 $\frac{AY}{r^2}\cdot\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{AR}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{AR}{r^2\sin^2\theta}\frac{\partial^2Y}{\partial\varphi^2}+\left(\frac{B}{r}+E\right)RY=0$

移项，$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{r^2}{A}\left(\frac{B}{r}+E\right)=\frac{-1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)-\frac{\frac{\partial^2Y}{\partial\varphi^2}}{Y\sin^2\varphi}$

同样的讨论，上式两端必须为一常数，记为 $l(l+1)$

$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{B}{A}r+\frac{E}{A}r^2-l(l+1)\right]=R$

即 $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{8\pi^2\mu}{h^2}\left(\frac{Ze^2}{r}+E\right)-\frac{l(l+1)}{r^2}\right]R=0$ (1)

Y 满足：$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2Y}{\partial\varphi^2}+l(l+1)Y=0$ (2)

令 $Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$：则 $\Phi''+m^2\Phi=0$，$\Phi=A_m\cos m\varphi+B_m\sin m\varphi$

Θ 则满足缔合勒让德方程：$(1-x^2)\frac{d^2\Theta}{d\theta^2}-2x\frac{d\Theta}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]\Theta=0$，其中 $x=\cos\theta$

3. $y''-2xy'+(\lambda-1)y=0$

$p(x)=-2x$，$q(x)=\lambda-1$，$x_0=0$ 是方程(1)常点，

设 $y=\sum_{n=1}^{\infty}a_nx^n$，则 $(\lambda-1)y=\sum_{k=0}^{\infty}(\lambda-1)a_kx^k$

$-2xy'=\sum_{n=1}^{\infty}-2na_nx^n=\sum_{k=1}^{\infty}-2ka_kx^k$

$y''=\sum_{n=1}^{\infty}n(n-1)a_nx^{n-2}=\sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k$

代入原方程，$a_{k+2}=\dfrac{2k+1-\lambda}{(k+1)(k+2)}a_k$

$\therefore a_2=\frac{1-\lambda}{2}a_0$ $a_3=\frac{3-\lambda}{6}a_1$

$a_4=\frac{5-\lambda}{3\times4}a_2=\frac{(1-\lambda)(5-\lambda)}{4\times3\times2}a_0$

$a_5=\frac{7-\lambda}{5\times4}=\frac{(3-\lambda)(7-\lambda)}{5\times4\times3\times2}a_1$

$\therefore a_{2k}=\dfrac{(1-\lambda)(5-\lambda)\cdots(4k-3-\lambda)}{(2k)!}a_0$

$a_{2k+1}=\dfrac{(3-\lambda)(7-\lambda)\cdots(4k-1-\lambda)}{(2k+1)!}a_1$

$\therefore y(x)=a_0y_0(x)+a_1y_1(x)$

$y_0(x)=1+\frac{(1-\lambda)}{2!}x^2+\frac{(1-\lambda)(5-\lambda)}{4!}x^4+\cdots+\frac{(1-\lambda)(5-\lambda)\cdots(4k-3-\lambda)}{(2k)!}x^{2k}$

$y_1(x)=x+\frac{3-\lambda}{3!}x^3+\frac{(3-\lambda)(7-\lambda)}{5!}x^5+\cdots+\frac{(3-\lambda)(7-\lambda)\cdots(4k-1-\lambda)}{(2k+1)!}x^{2k+1}$

收敛半径均是无限大

当 $\lambda=4k-3$ 时，$y_0(x)$ 退化为多项式

当 $\lambda=4k-1$ 时，$y_1(x)$ 退化为多项式

使多项式最高幂项为 $(2x)^n$ 形式，记为 $H_n(x)$

取 $k=1$，$\lambda=4k-1=3$ $y_0(x)=1$，记为 $H_0(x)=1$

取 $k=1$，$\lambda=4k-3=1$ $y_0(x)=x$ 记为 $H_1(x)=2y_1(x)=2x$

取 $k=2$，有 $\lambda=4k-3=5$，$y_0(x)=1-2x^2$，记为 $H_3(x)=(2x)^2-2$

取 $k=2$，$\lambda=4k-1=7$，$y_1(x)=x-\frac{2}{3}x^2$，记 $H_4(x)=-2^2\cdot3y_3(x)=(2x)^2-12x$

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4. $R_0(x)=1+\frac{-l(l+1)}{2!}x^2+\cdots+\frac{(2k-2-l)(2k-4-l)\cdots(-l)(l+1)\cdots(l+2k-1)}{(2k)!}x^{2k}+\cdots$ $R_1(x)=x+\frac{(1-l)(l+2)}{3!}x^3+\frac{(2k-1-l)(2k-3-l)\cdots(1-l)(l+2)\cdots(l+2k)}{(2k+1)!}x^{2k+1}$

将 $l=3$ 代入。 解：由于 $x_0=0$ 是方程的常点 令 $y=\sum_{n=0}^{\infty}a_nx^n$，$6y=6\sum_{n=0}^{\infty}a_nx^n$ $-6xy'=\sum_{k=1}^{\infty}-6ka_kx^k$

$y''=\sum_{n=1}^{\infty}n(n-1)a_nx^{n-2}=\sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k$ $-x^2y''=\sum_{n=1}^{\infty}n(n-1)a_nx^n=\sum_{k=1}^{\infty}k(k-1)a_kx^k$

代入原方程后，得 $(k+1)(k+2)a_{k+2}-k(k-1)a_k-6ka_k+6a_k=0$

即 $a_{k+2}=\frac{(k-1)(k+6)}{(k+1)(k+2)}a_k=\frac{[(k+2)-3][(k+2)+4]}{(k+1)(k+2)}a_k$ 将系数递推式代入

$a_2=-\frac{6}{2!}a_0$ $a_3=0\cdot a_1=0$ $a_4=\frac{-8}{3\times4}a_2=\frac{6\times8}{4!}a_0$ $a_5=\gamma a_3=0$ $a_{2k+1}=\gamma' a_3=0$

$a_{2n}=\frac{(2n+4)(2n+2)\cdots6\cdot1\cdot(-1)\cdots(2n-5)(2n-3)}{(2n)!}a_0$ $\therefore y(x)=a_0y_0(x)+a_1y_1(x)=a_0y_0(x)+x$

$y_0(x)=1+\frac{(-1)\cdot6}{2!}x^2+\frac{(-1)\cdot1\cdot6\cdot8}{4!}x^4+\cdots+\frac{(2n-3)\cdots(-1)\cdot1\cdots(2n+4)}{(2n)!}x^{2n}+\cdots$ $R_1(x)=x+\frac{-2\cdot5}{3!}x^3$

$l=3$ 阶勒让德方程 $R_0(x)=1+\frac{-3\cdot4}{2!}x^2+\cdots+\frac{(2n-5)(2n-7)\cdots(-1)(-3)\cdot4\cdot6\cdots(2n+2)}{(2n)!}x^{2n}+\cdots$

$\frac{d^2R_0}{dx^2}=-3\cdot4\frac{(-1)(-3)\cdot4\cdot6\cdot5}{3!}x^2+\frac{(2n-5)\cdots(-3)\cdot4\cdot6\cdots(2n-1)}{(2n-1)!}x^{2n-2}+\cdots=-3\cdot4[1+\frac{-1\cdot6}{2!}x^2+\cdots+\frac{(2n-3)\cdots(-1)\cdot1\cdot6\cdot8\cdots(2n+4)}{(2n)!}x^{2n}+\cdots]$

$=-3\times4\,y_0(x)$

$\frac{d^2R_1}{dx^2}=-2\times5\,y_1(x)$

因此，本题的解确实是3阶勒让德方程解的二阶导数

5. $(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+\lambda(\alpha+\beta+\lambda+1)y=0$ 在 $x_0=0$ 邻域上

设 $y=\sum_{n=0}^{\infty}a_nx^n$ $\lambda(\alpha+\beta+\lambda+1)y=\sum_{k=0}^{\infty}(\lambda+\alpha+\beta+1)a_kx^k$ $(\beta-\alpha)y'=\sum_{n=1}^{\infty}(\beta-\alpha)na_nx^{n-1}=\sum_{k=0}^{\infty}(\beta-\alpha)\cdot(k+1)a_{k+1}x^k$

$-(\alpha+\beta+2)xy'=\sum_{n=1}^{\infty}-(\alpha+\beta+2)ka_kx^k$

$y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k$ $-x^2y''=\sum_{n=2}^{\infty}-n(n-1)a_nx^n=\sum_{k=2}^{\infty}-k(k-1)a_kx^k$

代入雅可比方程：$(k+1)(k+2)a_{k+2}+(\beta-\alpha)(k+1)a_{k+1}+[\lambda(\alpha+\beta+\lambda+1)-(\alpha+\beta+2)k-k(k-1)]a_k=0$

即 $a_{k+2}=\frac{\alpha-\beta}{k+2}a_{k+1}+\frac{(k-\lambda)(k+\alpha+\beta+\lambda+1)}{(k+1)(k+2)}a_k$

$\therefore a_{k+2}=\frac{\alpha-\beta}{k+2}a_{k+1}-\frac{\lambda(\alpha+\beta+\lambda+1)+(\alpha+\beta+2)k-k(k-1)}{(k+1)(k+2)}a_k$