1. $\begin{cases} m\ddot{u}_{2n} = \beta(u_{2n+1}-u_{2n}) - \beta(u_{2n}-u_{2n-1}) \\ m\ddot{u}_{2n+1} = \beta(u_{2n+2}+u_{2n}) - 2\beta u_{2n+1} \end{cases}$ 取 $u_{2n} = Ae^{i[\omega t-\frac{2\pi}{\lambda}(2na)]}$ $u_{2n+1} = Be^{i[\omega t-\frac{2\pi}{\lambda}(2n+1)a]}$

化简得 $\begin{cases} (2\beta - m\omega^2)A - 2\beta\cos\frac{2\pi a}{\lambda}B = 0 \\ 2\beta\cos(\frac{2\pi a}{\lambda})A - (2\beta - M\omega^2)B = 0 \end{cases}$ 有唯一解 $\begin{vmatrix} 2\beta - m\omega^2 & -2\beta\cos(\frac{2\pi a}{\lambda}) \\ 2\beta\cos\frac{2\pi a}{\lambda} & -(2\beta - M\omega^2) \end{vmatrix} = 0$

$$\omega^2 = \frac{m+M}{mM}\beta\left[1\pm\sqrt{1-\frac{4mM}{(m+M)^2}\sin^2(\frac{2\pi a}{\lambda})}\right]$$ $k = \frac{l}{N}\cdot\frac{2\pi}{a} = \frac{2\pi}{\lambda}$, $l = 1\cdots N$

∴ 振动模式共 2N 个, $m=M$ 时 $\omega^2 = \frac{2\beta}{m}(1\pm\cos ka) = \dfrac{\frac{4\beta}{m}\cos^2\frac{ka}{2}}{\frac{4\beta}{m}\sin^2\frac{ka}{2}}$

又∵ $\cos^2\frac{ka}{2} = \sin^2(\frac{ka+\pi}{2})$ $N>>1$ 忽略 N 影响

与一维单原子链一一对应

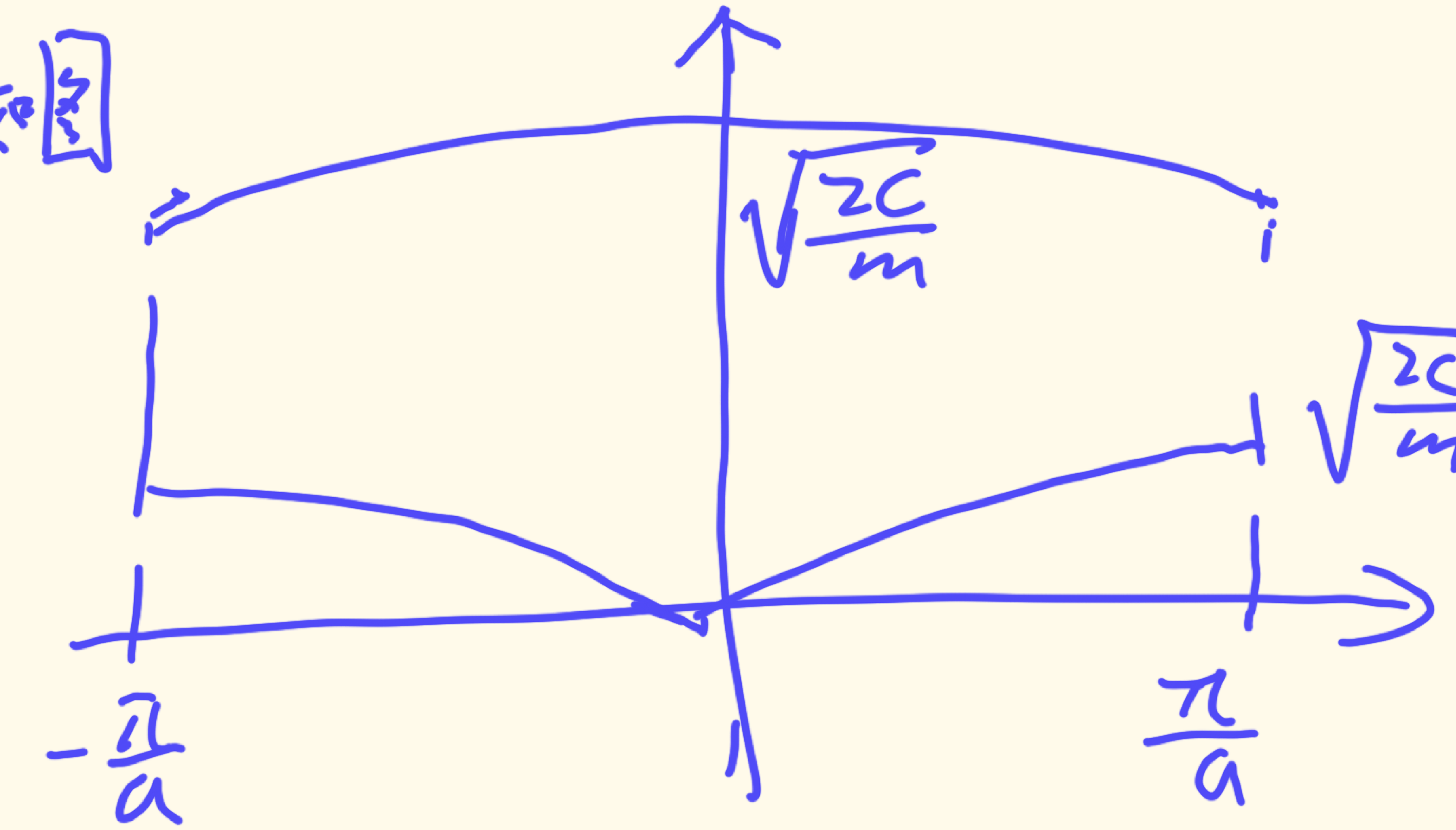
2. 运动方程为 $\begin{cases} m\ddot{x}_{2n} = 10c(x_{2n+1}-x_{2n}) - c(x_{2n}-x_{2n-1}) \\ m\ddot{x}_{2n+1} = c(x_{2n+2}-x_{2n+1}) - 10c(x_{2n+1}-x_{2n}) \end{cases}$

格波解为 $\begin{cases} x_{2n} = Ae^{-i(\omega t - 2n\frac{\varepsilon n}{2})} \\ x_{2n+1} = Be^{-i(\omega t-(2n+1)\frac{\varepsilon n}{2})} \end{cases}$

代入运动方程 $\begin{cases} -m\omega^2 A = c(e^{-i\frac{na}{2}} + 10e^{i\frac{na}{2}})B - 11cA \\ -m\omega^2 B = c(10e^{-i\frac{na}{2}} + e^{i\frac{na}{2}})A - 11cB \end{cases}$

有解条件
$$\begin{vmatrix} m\omega^2-11C & Ce^{\frac{iqa}{2}}+10Ce^{-\frac{iqa}{2}} \\ 10Ce^{\frac{iqa}{2}}+Ce^{-\frac{iqa}{2}} & m\omega^2-11C \end{vmatrix}=0$$

色散关系如图



$$\omega_{\pm}^2=\frac{11C\pm C\sqrt{20\cos qa+101}}{m}$$

3. (1) $U(r)=-\frac{\alpha q^2}{r}+\frac{\beta}{r^n}$ 则 $\beta=\frac{\alpha q^2}{n}r_0^{n-1}$

$$\left.\frac{d^2U_r}{dr^2}\right|_{r=r_0}=\left.-\frac{2\alpha q^2}{r^3}+\frac{n(n+1)\beta}{r^{n+2}}\right|_{r=r_0}=(n-1)\frac{\alpha q^3}{r_0^3}$$

$$\therefore f=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{1}{\mu}\frac{d^2U_r}{dr^2}}=\frac{1}{2\pi}\sqrt{(n-1)\frac{m+M}{mM}\frac{\alpha q^2}{r_0^3}}=1.32\times10^8\,Hz$$

(2) $\lambda=\frac{c}{\nu}=2.27m\gg 61\,\mu m$