Quasisymmetric embedding of discrete subset and its extensibility*

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1. Definitions

Let $\Omega \subset \mathbb{C}$ be a domain, and $f : \Omega \to \mathbb{C}$ be a homeomorphism into $\mathbb{C}$. For $z \in \Omega$ and $r > 0$ such that $S(z,r) = \{w \in \mathbb{C} \mid \ |z-w| = r\} \subset \Omega$, we set

$$L(z,f,r) = \max_{w \in S(z,r)} |f(z) - f(w)|,$$
$$\ell(z,f,r) = \min_{w \in S(z,r)} |f(z) - f(w)|.$$

Then the circular dilatation $H_f(z) \in [1,\infty]$ of f at z is defined by

$$H_f(z) = \limsup_{r \to 0} \frac{L(z,f,r)}{\ell(z,f,r)}.$$

Let $K \geq 1$. Then f is said to be K-*quasiconformal* if $H_f(z) < \infty$ for all points $z \in \Omega$ and satisfies $H_f(z) \leq K$ almost everywhere in Ω. For a quasiconformal mapping $f : \Omega \to \mathbb{C}$, the quantity $K(f) = \text{ess.sup}_\Omega H_f(z)$ is called the maximal dilatation of f.

Next, let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism and $X \subset \mathbb{C}$ be a subset. An injection $f : X \to \mathbb{C}$ is said to be η-quasisymmetric if the following inequality holds for any three points $x, y, z \in X$ $(x \neq z)$;

$$\left| \frac{f(x) - f(y)}{f(x) - f(z)} \right| \leq \eta \left(\left| \frac{x-y}{x-z} \right| \right).$$

These two concepts are closely related by the so-called egg-yolk principle, see [5, Theorem 11.14]. In particular, for homeomorphisms from $\mathbb{C}$ onto itself, the quasiconformality and the quasisymmetry are quantitatively equivalent.

2. Results and The Väisälä problem

In this talk, we give the following:

Theorem A (F. [4, 2016]) Every η-quasisymmetric embedding $f : \mathbb{Z} \to \mathbb{R}$ admits a $K = K(\eta)$-quasiconformal extension $F : \mathbb{C} \to \mathbb{C}$ where $K(\eta)$ is a constant depending only on η.

This result means that the integer set $\mathbb{Z}$ is an example of unbounded discrete subset for which the one dimensional Väisälä problem, stated below, can be solved affirmatively.

Question (The Väisälä 8th problem [8, 1995]) Let $X \subset \mathbb{R}^n$. Can η-quasisymmetric mapping $f : X \to \mathbb{R}^n$ be extended to a K-quasiconformal mapping $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, where $K \geq 1$ is a constant depending only on η and n?

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The Väisälä problem has been studied mainly in the case that $X \subset \mathbb{R}^n$ is connected or bounded, see [1–3, 7, 9]. In [6, Theorem 6.21, 1999], Trotsenko–Väisälä proved that; if $X \subset \mathbb{R}^n$ is not relatively connected, then there is a quasisymmetric embedding $f : X \to \mathbb{R}^n$ which admits no quasiconformal extensions $F : H \to H$ to any Hilbert spaces H. Therefore, the Väisälä problem cannot be solved affirmatively for general subsets even if $n = 1$.

The proof of Theorem A is divided into the following two theorems.

Theorem B For a subset $A \subset \mathbb{R}$, the following conditions are quantitatively equivalent;

1. There exists an η-quasisymmetric bijection $f : \mathbb{Z} \to A$.
2. A can be written as a monotone increasing sequence $A = \{a_n\}_{n\in\mathbb{Z}}$ with $a_n \to \pm\infty$ as $n \to \pm\infty$, and there exists a constant $M \geq 1$ such that the following inequality holds for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$;
$$\frac{1}{M} \leq \frac{a_{n+k} - a_n}{a_n - a_{n-k}} \leq M.$$
3. There exists a K-quasiconformal mapping $F : \mathbb{C} \to \mathbb{C}$, such that $F(\mathbb{Z}) = A$.

Further, if A satisfies the second condition, then there exists a $K = K(M)$-quasiconformal mapping $F : \mathbb{C} \to \mathbb{C}$ such that $F(n) = a_n$ for all $n \in \mathbb{Z}$.

Theorem C For a bijection $f : \mathbb{Z} \to \mathbb{Z}$, the following conditions are quantitatively equivalent;

1. f is η-quasisymmetric.
2. f is M-biLipschitz.
3. f admits an M-biLipschitz extension $F : \mathbb{C} \to \mathbb{C}$.
4. f admits a K-quasiconformal extension $F : \mathbb{C} \to \mathbb{C}$.

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