Here, we consider an example of a space deforming from a sphere to a torus. For $R > r > 0$, a point (x, y, z) in a torus is represented as

$$
\begin{aligned}
x(\theta, \eta) &= R\cos\theta + r\cos\eta\cos\theta \\
y(\theta, \eta) &= R\sin\theta + r\cos\eta\sin\theta \\
z(\eta) &= r\sin\eta
\end{aligned}
$$

where $\theta \in [-\pi, \pi]$ and $\eta \in [-\pi, \pi]$. For $R, r > 0$, we consider the following space:

$$
T_{R,r} := \left\{ (x(\theta,\eta), y(\theta,\eta), z(\eta)) \;\middle|\; \begin{array}{l} \theta \in [-\pi, \pi], \\ \eta \in [-\pi + \arccos(\min\{1, R/r\}), \pi - \arccos(\min\{1, R/r\})] \end{array} \right\}
$$

Let $X_{R,r}$ denote a 500-point set of uniform sampling from $T_{R,r}$. If $R > r$, then the space $T_{R,r}$ is a torus. On the other hand, if $R < r$, then the space $T_{R,r}$ is homotopy equivalent to a sphere S^2, that is, topological properties of $T_{R,r}$ change before and after $R = r$ (Figure 6.5). We will characterize this topological change by the kernel PCA for the persistence diagrams.

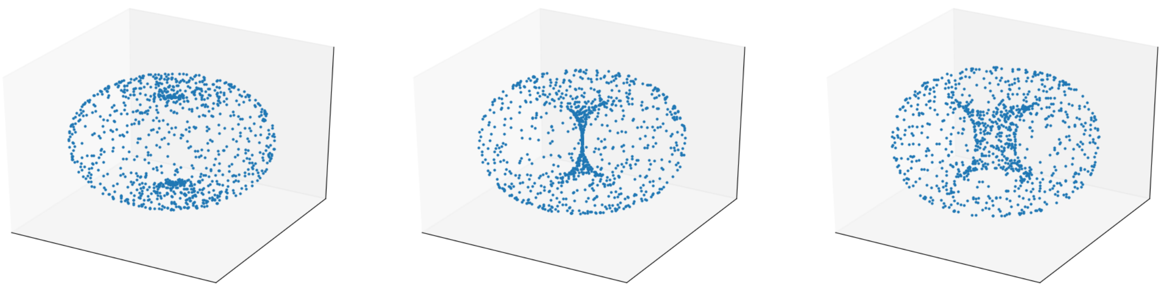

Figure 6.5: Sampling from $T_{R,r}$ for $R < r$ (left, sphere), $R = r$ (middle), and $R > r$ (right, torus).

Let $R_\ell := r(1/2 + \ell/40)$ $(\ell = 0, \ldots, 40)$ and X_ℓ denote $X_{R_\ell,r}$. Then, a topological change of $\{X_\ell\}_{\ell=1}^{40}$ is considered to happen at $\ell = 20$. We apply the kernel PCA to $\{D_1(\mathbb{B}(X_\ell))\}_{\ell=1}^{40}$ with different positive definite kernels, and Figure 6.6 are the results.