Chapter 8 Graphical Models

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8.1 Bayesian Networks

(8.15)

$$
\begin{aligned}
\mathbb{E}[x_i] &= \mathbb{E}\left[\sum_{j\in\mathrm{pa}_i} w_{ij}\mathbb{E}[x_j] + b_i\right] \\
&= \sum_{j\in\mathrm{pa}_i} w_{ij}\mathbb{E}[\mathbb{E}[x_j]] + \mathbb{E}[b_i] \\
&= \sum_{j\in\mathrm{pa}_i} w_{ij}\mathbb{E}[x_j] + b_i.
\end{aligned}
$$

(8.16)

$$
\begin{aligned}
\mathrm{var}[x_i, x_j] &= \mathbb{E}[(x_i - \mathbb{E}[x_i])(x_j - \mathbb{E}[x_j])] \\
&= \mathbb{E}\left[(x_i - \mathbb{E}[x_i])\left\{\sum_{k\in\mathrm{pa}_j} w_{jk}x_k + b_j + \sqrt{v_j}\epsilon_j - \left(\sum_{k\in\mathrm{pa}_i} w_{jk}\mathbb{E}[x_k] + b_j\right)\right\}\right] \\
&= \mathbb{E}\left[(x_i - \mathbb{E}[x_i])\left\{\sum_{k\in\mathrm{pa}_j} w_{jk}(x_k - \mathbb{E}[x_k]) + \sqrt{v_j}\epsilon_j\right\}\right] \\
&= \sum_{k\in\mathrm{pa}_j} w_{jk}\mathbb{E}[(x_i - \mathbb{E}[x_i])(x_k - \mathbb{E}[x_k])] + \mathbb{E}[(x_i - \mathbb{E}[x_i])\sqrt{v_j}\epsilon_j] \\
&= \sum_{k\in\mathrm{pa}_j} w_{jk}\mathbb{E}[(x_i - \mathbb{E}[x_i])(x_k - \mathbb{E}[x_k])] + \mathbb{E}\left[\left\{\sum_{l\in\mathrm{pa}_i} w_{il}(x_l - \mathbb{E}[x_l]) + \sqrt{v_i}\epsilon_i\right\}\sqrt{v_j}\epsilon_j\right] \\
&= \sum_{k\in\mathrm{pa}_j} w_{jk}\mathrm{cov}[x_i, x_k] + \sqrt{v_j}\,\mathbb{E}\left[\sum_{l\in\mathrm{pa}_i} w_{il}(x_l - \mathbb{E}[x_l])\right]\mathbb{E}[\epsilon_j] + \sqrt{v_i v_j}\,\mathbb{E}[\epsilon_i\epsilon_j] \\
&= \sum_{k\in\mathrm{pa}_j} w_{jk}\mathrm{cov}[x_i, x_k] + I_{ij}v_j
\end{aligned}
$$

where we assumed that ϵ is independent of x.

8.2 Conditional Independence

N/A

8.3 Markov Random Fields

N/A

8.4 Inference in Graphical Models

(8.52)

$$
\begin{aligned}
p(x_n) &= \sum_{x_1} \cdots \sum_{x_{n-1}} \sum_{x_{n+1}} \cdots \sum_{x_N} p(\mathbf{x}) \\
&= \frac{1}{Z} \sum_{x_1} \cdots \sum_{x_{n-1}} \sum_{x_{n+1}} \cdots \sum_{x_N} \psi_{1,2}(x_1, x_2) \psi_{2,3}(x_2, x_3) \cdots \psi_{N-1,N}(x_{N-1}, x_N) \\
&= \frac{1}{Z} \left[\sum_{x_{n-1}} \psi_{n-1,n}(x_{n-1}, x_n) \cdots \left[\sum_{x_2} \psi_{2,3}(x_2, x_3) \left[\sum_{x_1} \psi_{1,2}(x_1, x_2) \right] \right] \right] \\
&\quad \left[\sum_{x_{n+1}} \psi_{n,n+1}(x_n, x_{n+1}) \cdots \left[\sum_{x_N} \psi_{N-1,N}(x_{N-1}, x_N) \right] \right].
\end{aligned}
$$

(8.58)

$$
\begin{aligned}
p(x_{n-1}, x_n) &= \sum_{x_1} \cdots \sum_{x_{n-2}} \sum_{x_{n+1}} \cdots \sum_{x_N} p(\mathbf{x}) \\
&= \frac{1}{Z} \sum_{x_1} \cdots \sum_{x_{n-2}} \sum_{x_{n+1}} \cdots \sum_{x_N} \psi_{1,2}(x_1, x_2) \psi_{2,3}(x_2, x_3) \cdots \psi_{N-1,N}(x_{N-1}, x_N) \\
&= \frac{1}{Z} \left[\sum_{x_{n-2}} \psi_{n-1,n-2}(x_{n-1}, x_{n-2}) \cdots \left[\sum_{x_2} \psi_{2,3}(x_2, x_3) \left[\sum_{x_1} \psi_{1,2}(x_1, x_2) \right] \right] \right] \psi_{n-1,n}(x_{n-1}, x_n) \\
&\quad \left[\sum_{x_{n+1}} \psi_{n,n+1}(x_n, x_{n+1}) \cdots \left[\sum_{x_N} \psi_{N-1,N}(x_{N-1}, x_N) \right] \right].
\end{aligned}
$$