# Chapter 2 Probability Distributions

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## 2.1 Binary Variables

### (2.3)

$$\begin{aligned}
\mathbb{E}[\mathbf{x}] &= \sum_x x p(x) \\
&= \sum_x x \mu^x (1-\mu)^{1-x} \\
&= 1 \cdot \mu^1 (1-\mu)^0 + 0 \cdot \mu^0 (1-\mu)^1 \\
&= \mu.
\end{aligned}$$

### (2.4)

$$\begin{aligned}
\mathrm{var}[\mathbf{x}] &= \mathbb{E}[x^2] - \mathbb{E}[x]^2 \\
&= \sum_x x^2 p(x) - \mu^2 \\
&= \sum_x x^2 \mu^x (1-\mu)^{1-x} - \mu^2 \\
&= 1^2 \cdot \mu^1 (1-\mu)^0 + 0^2 \cdot \mu^0 (1-\mu)^1 - \mu^2 \\
&= \mu(1-\mu).
\end{aligned}$$

### (2.7)

$$\begin{aligned}
\mu_{\mathrm{ML}} &= \arg_\mu \left( \frac{\partial}{\partial \mu} \ln p(\mathcal{D}|\mu) = 0 \right) \\
&= \arg_\mu \left( \frac{\partial}{\partial \mu} \sum_{n=1}^{N} \{ x_n \ln \mu + (1-x_n) \ln(1-\mu) \} = 0 \right) \\
&= \arg_\mu \left( \sum_{n=1}^{N} \frac{x_n}{\mu} - \sum_{n=1}^{N} \frac{1-x_n}{1-\mu} = 0 \right) \\
&= \frac{1}{N} \sum_{n=1}^{N} x_n.
\end{aligned}$$

## (2.11) and (2.12)

We prove these two equations through problem 2.3 and 2.4. Firstly, notice that

$$\begin{aligned}\binom{N}{m}+\binom{N}{m-1} &= \frac{N!}{(N-m)!m!}+\frac{N!}{(N-m+1)!(m-1)!}\\ &= \frac{N!(N-m+1)}{(N-m+1)!m!}+\frac{N!m}{(N-m+1)!m!}\\ &= \frac{(N+1)!}{(N+1-m)!m!}\\ &= \binom{N+1}{m}. \qquad (*)\end{aligned}$$

Next, we prove by induction the *binomial theorem* that is given by

$$(1+x)^N=\sum_{m=0}^{N}\binom{N}{m}x^m.$$

When $N=0$, we have

$$\sum_{m=0}^{0}\binom{0}{m}x^m=1=(1+x)^0.$$

If the equation is correct for any integer $N>0$, then for $N+1$, we have

$$\begin{aligned}(1+x)^{N+1} &= (1+x)^N(1+x)\\ &= \sum_{m=0}^{N}\binom{N}{m}x^m(1+x)\\ &= \sum_{m=0}^{N}\binom{N}{m}x^m+\sum_{m=0}^{N}\binom{N}{m}x^{m+1}\\ &= \binom{N}{0}x^0+\sum_{m=1}^{N}\binom{N}{m}x^m+\sum_{m=0}^{N-1}\binom{N}{m}x^{m+1}+\binom{N}{N}x^{N+1}\\ &= \binom{N+1}{0}x^0+\left[\sum_{m=1}^{N}\binom{N}{m}x^m+\sum_{m=1}^{N}\binom{N}{m-1}x^m\right]+\binom{N+1}{N+1}x^{N+1}\\ &= \binom{N+1}{0}x^0+\sum_{m=1}^{N}\binom{N+1}{m}x^m+\binom{N+1}{N+1}x^{N+1}\\ &= \sum_{m=0}^{N+1}\binom{N+1}{m}x^m\end{aligned}$$

where the fifth step used (*). Hence, the binomial theorem holds.

Then, we prove the binomial distribution is normalized. Specifically,

$$\begin{aligned}\sum_{m=0}^{N}\mathrm{Bin}(m|N,\mu) &= \sum_{m=0}^{N}\binom{N}{m}\mu^m(1-\mu)^{N-m}\\ &= (1-\mu)^N\sum_{m=0}^{N}\binom{N}{m}\left(\frac{\mu}{1-\mu}\right)^m\\ &= (1-\mu)^N\left(1+\frac{\mu}{1-\mu}\right)^N\\ &= 1 \qquad (**)\end{aligned}$$

where the second last step used the binomial theorem that we just proved.

Differentiating both sides of (**) with respect to $\mu$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\mu}\sum_{m=0}^{N}\mathrm{Bin}(m|N,\mu) &= \sum_{m=0}^{N}\binom{N}{m}\left(m\mu^{m-1}(1-\mu)^{N-m} - (N-m)\mu^{m}(1-\mu)^{N-m-1}\right)\\
&= \sum_{m=0}^{N}\binom{N}{m}\left(\frac{m}{\mu}\mu^{m}(1-\mu)^{N-m} - \frac{N-m}{1-\mu}\mu^{m}(1-\mu)^{N-m}\right)\\
&= \frac{1}{\mu(1-\mu)}\sum_{m=0}^{N}(m-N\mu)\binom{N}{m}\mu^{m}(1-\mu)^{N-m}\\
&= \frac{1}{\mu(1-\mu)}\sum_{m=0}^{N}(m-N\mu)\mathrm{Bin}(m|N,\mu)\\
&= 0.
\end{aligned}$$

Rearranging the equation, we obtain

$$\begin{aligned}
\mathbb{E}[m] &= \sum_{m=0}^{N} m\mathrm{Bin}(m|N,\mu)\\
&= N\mu\sum_{m=0}^{N}\mathrm{Bin}(m|N,\mu)\\
&= N\mu
\end{aligned}$$

where we used the fact we just proved that the binomial distribution is normalized.

To compute the variance, we further differentiate both sides of the above equation with respect to $\mu$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\mu}\mathbb{E}[m] &= \sum_{m=0}^{N} m\binom{N}{m}\left(m\mu^{m-1}(1-\mu)^{N-m} - (N-m)\mu^{m}(1-\mu)^{N-m}\right)\\
&= \frac{1}{\mu(1-\mu)}\sum_{m=0}^{N}(m^2-mN\mu)\binom{N}{m}u^{m}(1-\mu)^{N-m}\\
&= \frac{1}{\mu(1-\mu)}\sum_{m=0}^{N}(m^2-mN\mu)\mathrm{Bin}(m|N,\mu)\\
&= \frac{1}{\mu(1-\mu)}\left\{\sum_{m=0}^{N}m^2\mathrm{Bin}(m|N,\mu) - N\mu\sum_{m=0}^{N}m\mathrm{Bin}(m|N,\mu)\right\}\\
&= \frac{1}{\mu(1-\mu)}(\mathbb{E}[m^2]-\mathbb{E}[m]^2)\\
&= \frac{1}{\mu(1-\mu)}\mathrm{var}[m]\\
&= N.
\end{aligned}$$

Therefore,

$$\mathrm{var}[m] = N\mu(1-\mu).$$

## (2.14)

From the definition of the gamma function

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,\mathrm{d}u,$$

we have

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty \exp(-x)x^{a-1}\,\mathrm{d}x \int_0^\infty \exp(-y)y^{b-1}\,\mathrm{d}y \\ &= \int_0^\infty \int_0^\infty \exp(-(x+y))x^{a-1}y^{b-1}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Substituting $t = x + y$, we have

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty \int_x^\infty \exp(-t)x^{a-1}(t-x)^{b-1}\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right|\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^\infty \int_x^\infty \exp(-t)x^{a-1}(t-x)^{b-1}\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^\infty \int_0^t \exp(-t)x^{a-1}(t-x)^{b-1}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We further substitute $x = t\mu$, which gives

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty \int_0^1 \exp(-t)(t\mu)^{a-1}(t-t\mu)^{b-1}\left|\frac{\mathrm{d}x}{\mathrm{d}\mu}\right|\mathrm{d}\mu\,\mathrm{d}t \\ &= \int_0^\infty \int_0^1 \exp(-t)(t\mu)^{a-1}(t-t\mu)^{b-1}t\,\mathrm{d}\mu\,\mathrm{d}t \\ &= \int_0^\infty \exp(-t)t^{a+b-1}\,\mathrm{d}t \int_0^1 \mu^{a-1}(1-\mu)^{b-1}\,\mathrm{d}\mu \\ &= \Gamma(a+b)\int_0^1 \mu^{a-1}(1-\mu)^{b-1}\,\mathrm{d}\mu.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 \mathrm{Beta}(\mu|a,b)\,\mathrm{d}\mu &= \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}\,\mathrm{d}\mu \\ &= 1.\end{aligned}$$

## (2.15)

$$\begin{aligned}\mathbb{E}[\mu] &= \int_0^1 \mu\mathrm{Beta}(\mu|a,b)\,\mathrm{d}\mu \\ &= \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a}(1-\mu)^{b-1}\,\mathrm{d}\mu \\ &= \frac{\Gamma(a+b)\Gamma(a+1)}{\Gamma(a+b+1)\Gamma(a)}\int_0^1 \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b)}\mu^{a}(1-\mu)^{b-1}\,\mathrm{d}\mu \\ &= \frac{\Gamma(a+b)\Gamma(a+1)}{\Gamma(a+b+1)\Gamma(a)} \\ &= \frac{a\Gamma(a+b)\Gamma(a)}{(a+b)\Gamma(a+b)\Gamma(a)} \\ &= \frac{a}{a+b}\end{aligned}$$

where the third step used the fact that the gamma distribution is normalized, and the second last step used the property $\Gamma(x+1) = x\Gamma(x)$.

### (2.16)

$$
\begin{aligned}
\operatorname{var}[\mu] &= \mathbb{E}[\mu^2] - \mathbb{E}[\mu]^2 \\
&= \int_0^1 \mu^2 \operatorname{Beta}(\mu|a,b)\,\mathrm{d}\mu - \left(\frac{a}{a+b}\right)^2 \\
&= \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a+1}(1-\mu)^{b-1}\,\mathrm{d}\mu - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{\Gamma(a+b)\Gamma(a+2)}{\Gamma(a+b+2)\Gamma(a)} \int_0^1 \frac{\Gamma(a+b+2)}{\Gamma(a+2)\Gamma(b)} \mu^{a+1}(1-\mu)^{b-1}\,\mathrm{d}\mu - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{\Gamma(a+b)\Gamma(a+2)}{\Gamma(a+b+2)\Gamma(a)} - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{a(a+1)\Gamma(a+b)\Gamma(a)}{(a+b)(a+b+1)\Gamma(a+b)\Gamma(a)} - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{ab}{(a+b)^2(a+b+1)}
\end{aligned}
$$

where, again, in the fifth step, we used the fact that the gamma distribution is normalized, and in the sixth step, we used the property $\Gamma(x+1) = x\Gamma(x)$.

### (2.19)

$$
\begin{aligned}
p(x=1|\mathcal{D}) &= \int_0^1 p(x=1,\mu|\mathcal{D})\,\mathrm{d}\mu \\
&= \int_0^1 p(x=1|\mu,\mathcal{D})p(\mu|\mathcal{D})\,\mathrm{d}\mu \\
&= \int_0^1 p(x=1|\mu)p(\mu|\mathcal{D})\,\mathrm{d}\mu \\
&= \int_0^1 \mu p(\mu|\mathcal{D})\,\mathrm{d}\mu \\
&= \mathbb{E}[\mu|\mathcal{D}],
\end{aligned}
$$

which is the expected value of $\mu$ after observing the dataset $\mathcal{D}$.

The third step omitted $\mathcal{D}$ by the i.i.d assumption such that the probability of $x = 1$ given $\mu$ does not depend on the observed data. In the second last step, we assumed that $x \sim \operatorname{Bern}(\mu)$, and hence, $p(x = 1|\mu) = \mu$.

## 2.2 Multinomial Variables

### (2.29)

$$
p(\mathcal{D}|\boldsymbol{\mu}) = \prod_{n=1}^{N} p(\mathbf{x}_n|\boldsymbol{\mu}) = \prod_{n=1}^{N}\prod_{k=1}^{K} \mu_k^{x_{nk}} = \prod_{k=1}^{K}\left(\prod_{n=1}^{N} \mu_k^{x_{nk}}\right) = \prod_{k=1}^{K} \mu_k^{\sum_n x_{nk}} = \prod_{k=1}^{K} \mu_k^{m_k}.
$$

## 2.3 The Gaussian Distribution

### (2.45)

To see that the matrix $\mathbf{\Sigma}$ can be taken to be symmetric, for precision matrix $\mathbf{\Lambda}$, denote symmetric matrix $\mathbf{\Lambda}^{\mathrm{S}} = (\mathbf{\Lambda} + \mathbf{\Lambda}^{\mathrm{T}})/2$, and anti-symmetric matrix $\mathbf{\Lambda}^{\mathrm{A}} = (\mathbf{\Lambda} - \mathbf{\Lambda}^{\mathrm{T}})/2$, then

$$\begin{aligned}\Delta^2 &= \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{\Lambda}(\mathbf{x}-\boldsymbol{\mu}) \\ &= \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}(\mathbf{\Lambda}^{\mathrm{S}}+\mathbf{\Lambda}^{\mathrm{A}})(\mathbf{x}-\boldsymbol{\mu}) \\ &= \frac{1}{2}\sum_{i=1}^{D}\sum_{j=1}^{D}(x_i-\mu_i)\mathbf{\Lambda}^{\mathrm{S}}_{ij}(x_j-\mu_j) + \frac{1}{2}\sum_{i=1}^{D}\sum_{j=1}^{D}(x_i-\mu_i)\mathbf{\Lambda}^{\mathrm{A}}_{ij}(x_j-\mu_j).\end{aligned}$$

In the last step, for each $\mathbf{\Lambda}^{\mathrm{A}}_{ij}$, there is a corresponding $\mathbf{\Lambda}^{\mathrm{A}}_{ji} = -\mathbf{\Lambda}^{\mathrm{A}}_{ij}$. Hence, the second term vanishes, which implies that the covariance matrix can be chosen to be symmetric.

### (2.46)

To prove that all the eigenvalues of a real symmetric matrix are real, suppose that for

$$\mathbf{\Sigma}\mathbf{u} = \lambda\mathbf{u}, \tag{*}$$

$\lambda = a + bi$ and $\bar{\lambda} = a - bi$. Taking conjugates of (*), we have

$$\mathbf{\Sigma}\bar{\mathbf{u}} = \bar{\lambda}\bar{\mathbf{u}},$$

transposing both sides, we have

$$\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{\Sigma} = \bar{\mathbf{u}}^{\mathrm{T}}\bar{\lambda}. \tag{**}$$

Multiplying $\bar{\mathbf{u}}^{\mathrm{T}}$ to the left of (*) on both sides gives

$$\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{u} = \bar{\mathbf{u}}^{\mathrm{T}}\lambda\mathbf{u}.$$

Similarly, by multiplying $\mathbf{u}$ to the right of (**) on both sides, we obtain

$$\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{u} = \bar{\mathbf{u}}^{\mathrm{T}}\bar{\lambda}\mathbf{u}$$

Therefore,

$$\bar{\mathbf{u}}^{\mathrm{T}}\lambda\mathbf{u} = \bar{\mathbf{u}}^{\mathrm{T}}\bar{\lambda}\mathbf{u}.$$

Since $\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{u} \neq 0$, we must have $\lambda = \bar{\lambda}$, that is, $a + bi = a - bi$. Hence, $b = 0$, which implies that $\lambda$ is real.

For any pair of $\mathbf{u}_i$ and $\mathbf{u}_j$ where $i \neq j$, we have

$$\begin{aligned}\mathbf{u}_i^{\mathrm{T}}\lambda_i\mathbf{u}_j &= (\lambda_i\mathbf{u}_i)^{\mathrm{T}}\mathbf{u}_j \\ &= (\mathbf{\Sigma}\mathbf{u}_i)^{\mathrm{T}}\mathbf{u}_j \\ &= \mathbf{u}_i^{\mathrm{T}}\mathbf{\Sigma}^{\mathrm{T}}\mathbf{u}_j \\ &= \mathbf{u}_i^{\mathrm{T}}\mathbf{\Sigma}\mathbf{u}_j \\ &= \mathbf{u}_i^{\mathrm{T}}\lambda_j\mathbf{u}_j.\end{aligned}$$

Since $\lambda_i \neq \lambda_j$, we obtain $\lambda_i\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_j = \lambda_j\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_j$, which means that $\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_j = 0$, that is, $\mathbf{u}_i \perp \mathbf{u}_j$. Therefore,

$$\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}},$$

indicating that the set of eigenvectors can be chosen to be orthonormal.

## (2.48)

$$\boldsymbol{\Sigma} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}} = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_D\end{bmatrix} \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_D\end{bmatrix} \begin{bmatrix}\mathbf{u}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{u}_D^{\mathrm{T}}\end{bmatrix} = \sum_{i=1}^{D} \lambda_i \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}}.$$

## (2.49)

$$\boldsymbol{\Sigma}^{-1} = (\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}})^{-1} = \mathbf{U}^{-\mathrm{T}}\boldsymbol{\Lambda}^{-1}\mathbf{U}^{-1} = \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^{\mathrm{T}} = \sum_{i=1}^{D} \frac{1}{\lambda_i}\mathbf{u}_i \mathbf{u}_i^{\mathrm{T}}.$$

## (2.60)

According to (2.52)

$$\mathbf{y} = \mathbf{U}(\mathbf{x} - \boldsymbol{\mu}) = \mathbf{U}\mathbf{z},$$

we obtain

$$\mathbf{z} = \mathbf{U}^{\mathrm{T}}\mathbf{y} = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_D\end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_D\end{bmatrix} = \sum_{j=1}^{D} y_j \mathbf{u}_j$$

where $y_j$ is defined by (2.51).

## (2.120)

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\mu}} \ln p(\mathbf{X}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) &= -\frac{1}{2}\frac{\partial}{\partial \boldsymbol{\mu}} \sum_{n=1}^{N} (\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - 2\mathbf{x}_n^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) \\
&= -\sum_{n=1}^{N} (\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \boldsymbol{\Sigma}^{-1}\mathbf{x}_n) \\
&= \sum_{n=1}^{N} \boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \boldsymbol{\mu}).
\end{aligned}$$

## (2.122)

Setting the derivative of likelihood function with respect to $\boldsymbol{\Sigma}^{-1}$ to $\mathbf{0}$, we have

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\Sigma}^{-1}} \ln p(\mathbf{X}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \frac{\partial}{\partial \boldsymbol{\Sigma}^{-1}} \left( -\frac{N}{2}\ln|\boldsymbol{\Sigma}| - \frac{1}{2}\sum_{n=1}^{N} (\mathbf{x}_n - \boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \boldsymbol{\mu}) \right) \\
&= \frac{\partial}{\partial \boldsymbol{\Sigma}^{-1}} \left( -\frac{N}{2}\ln|\boldsymbol{\Sigma}| - \frac{1}{2}\sum_{n=1}^{N} \mathrm{Tr}\left[ (\mathbf{x}_n - \boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \boldsymbol{\mu}) \right] \right) \\
&= \frac{\partial}{\partial \boldsymbol{\Sigma}^{-1}} \left( -\frac{N}{2}\ln|\boldsymbol{\Sigma}| - \frac{1}{2}\sum_{n=1}^{N} \mathrm{Tr}\left[ \boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^{\mathrm{T}} \right] \right) \\
&= \frac{N}{2}\boldsymbol{\Sigma} - \frac{1}{2}\sum_{n=1}^{N} (\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^{\mathrm{T}} \\
&= \mathbf{0},
\end{aligned}$$

which implies that

$$\boldsymbol{\Sigma} = \frac{1}{N}\sum_{n=1}^{N} (\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^{\mathrm{T}}.$$

## (2.123)

This is a multivariate generalization of (1.56).

$$\begin{aligned}\mathbb{E}[\boldsymbol{\mu}_{\mathrm{ML}}] &= \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}\mathbf{x}_n\right] \\ &= \frac{1}{N}\mathbb{E}\left[\sum_{n=1}^{N}\mathbf{x}_n\right] \\ &= \frac{1}{N}N\boldsymbol{\mu} \\ &= \boldsymbol{\mu}\end{aligned}$$

where in the second last step, we took advantage of the i.i.d assumption such that $\mathbb{E}[\mathbf{x}_n] = \boldsymbol{\mu}$ for any $n \in \{1, ..., N\}$.

## (2.124)

This is a multivariate generalization of (1.57), see (1.57) in *Chapter 1 Introduction* for details.

## (2.126)

$$\begin{aligned}\boldsymbol{\mu}_{\mathrm{ML}}^{(N)} &= \frac{1}{N}\sum_{n=1}^{N}\mathbf{x}_n \\ &= \frac{1}{N}\mathbf{x}_N + \frac{1}{N}\sum_{n=1}^{N-1}\mathbf{x}_n \\ &= \frac{1}{N}\mathbf{x}_N + \frac{N-1}{N}\frac{1}{N-1}\sum_{n=1}^{N-1}\mathbf{x}_n \\ &= \frac{1}{N}\mathbf{x}_N + \frac{N-1}{N}\boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)} \\ &= \boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)} + \frac{1}{N}(\mathbf{x}_N - \boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)}).\end{aligned}$$

## (2.135)

The observed value of $z$ is with respect to $x_N$ where we have the estimated $\theta$ based on the previous $N-1$ observations.

## (2.136)

$$\begin{aligned}z &= \frac{\partial}{\partial \mu_{\mathrm{ML}}}\left[-\ln p(x|\mu_{\mathrm{ML}}, \sigma^2)\right] \\ &= \frac{\partial}{\partial \mu_{\mathrm{ML}}}\left[\frac{(x-\mu_{\mathrm{ML}})^2}{2\sigma^2}\right] \\ &= -\frac{1}{\sigma^2}(x - \mu_{\mathrm{ML}}).\end{aligned}$$

Substituting back into (2.135), we have

$$\begin{aligned}\mu_{\mathrm{ML}}^{(N)} &= \mu_{\mathrm{ML}}^{(N-1)} - a_{N-1}\frac{\partial}{\partial \mu_{\mathrm{ML}}^{(N-1)}}\left[-\ln p(x_N|\mu_{\mathrm{ML}}^{(N-1)})\right] \\ &= \mu_{\mathrm{ML}}^{(N-1)} + a_{N-1}\frac{1}{\sigma^2}(x_N - \mu_{\mathrm{ML}}^{(N-1)}).\end{aligned}$$

Comparing with (2.126), we obtain

$$a_N = \frac{\sigma^2}{N}.$$

## (2.141) and (2.142)

Recall

$$p(\mu|\mathbf{x}) \propto p(\mathbf{x}|\mu)p(\mu)$$

where

$$\begin{aligned} p(\mu|\mathbf{x}) &= \mathcal{N}(\mu|\mu_N, \sigma_N^2) \\ p(\mu) &= \mathcal{N}(\mu|\mu_0, \sigma_0^2). \end{aligned}$$

By completing the square in the exponent, specifically, comparing the coefficients of the quadratic and linear term, we have

$$-\frac{1}{2\sigma_N^2} = -\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma^2}\sum_{n=1}^{N} 1 \tag{*}$$

$$\frac{1}{\sigma_N^2}\mu_N = \frac{1}{\sigma_0^2}\mu_0 + \frac{1}{\sigma^2}\sum_{n=1}^{N} x_n. \tag{**}$$

Solving for $\sigma_N$ using (**) and substituting back into (*) for $\mu_N$, we obtain

$$\begin{aligned} \mu_N &= \frac{\sigma^2}{N\sigma_0^2 + \sigma^2}\mu_0 + \frac{N\sigma_0^2}{N\sigma_0^2 + \sigma^2}\mu_{\mathrm{ML}} \\ \frac{1}{\sigma_N^2} &= \frac{1}{\sigma_0^2} + \frac{N}{\sigma^2} \end{aligned}$$

where

$$\mu_{\mathrm{ML}} = \frac{1}{N}\sum_{n=1}^{N} x_n.$$

## (2.146)

To see the gamma distribution is normalized, consider

$$\begin{aligned} \int_0^\infty \mathrm{Gam}(\lambda|a,b)\,\mathrm{d}\lambda &= \int_0^\infty \frac{1}{\Gamma(a)} b^a \lambda^{a-1} \exp(-b\lambda)\,\mathrm{d}\lambda \\ &= \frac{b^a}{\Gamma(a)}\int_0^\infty \lambda^{a-1}\exp(-b\lambda)\,\mathrm{d}\lambda. \end{aligned}$$

Substituting $u = b\lambda$, we have

$$\begin{aligned} \int_0^\infty \mathrm{Gam}(\lambda|a,b)\,\mathrm{d}\lambda &= \frac{b^a}{\Gamma(a)}\int_0^\infty \left(\frac{u}{b}\right)^{a-1} \exp(-u)\left|\frac{\mathrm{d}\lambda}{\mathrm{d}u}\right|\mathrm{d}u \\ &= \frac{b^a}{\Gamma(a)}\int_0^\infty \left(\frac{u}{b}\right)^{a-1} \exp(-u)\frac{1}{b}\,\mathrm{d}u \\ &= \frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}\exp(-u)\,\mathrm{d}u \\ &= \frac{1}{\Gamma(a)}\Gamma(a) \\ &= 1. \end{aligned}$$

**(2.147)**

$$\begin{aligned}
\mathbb{E}[\lambda] &= \int_0^\infty \mathrm{Gam}(\lambda|a,b)\lambda \,\mathrm{d}\lambda \\
&= \int_0^\infty \frac{1}{\Gamma(a)} b^a \lambda^{a-1} \exp(-b\lambda)\lambda \,\mathrm{d}\lambda \\
&= \frac{1}{\Gamma(a)} \int_0^\infty (b\lambda)^a \exp(-b\lambda) \,\mathrm{d}\lambda.
\end{aligned}$$

Substituting $u = b\lambda$, we have

$$\begin{aligned}
\mathrm{E}[\lambda] &= \frac{1}{\Gamma(a)} \int_0^\infty u^a \exp(-u) \left|\frac{\mathrm{d}\lambda}{\mathrm{d}u}\right| \mathrm{d}u \\
&= \frac{a}{b\Gamma(a)} \int_0^\infty u^{a-1} \exp(-u) \,\mathrm{d}u \\
&= \frac{a}{b\Gamma(a)} \Gamma(a) \\
&= \frac{a}{b}.
\end{aligned}$$

**(2.148)**

$$\begin{aligned}
\mathrm{var}[\lambda] &= \mathbb{E}[\lambda^2] - \mathbb{E}[\lambda]^2 \\
&= \int_0^\infty \frac{1}{\Gamma(a)} b^a \lambda^{a-1} \exp(-b\lambda)\lambda^2 \,\mathrm{d}\lambda - \left(\frac{a}{b}\right)^2 \\
&= \frac{1}{b\Gamma(a)} \int_0^\infty (b\lambda)^{a+1} \exp(-b\lambda) \,\mathrm{d}\lambda - \left(\frac{a}{b}\right)^2.
\end{aligned}$$

Substituting $u = b\lambda$, we have

$$\begin{aligned}
\mathrm{var}[\lambda] &= \frac{1}{b\Gamma(a)} \int_0^\infty u^{a+1} \exp(-u) \left|\frac{\mathrm{d}\lambda}{\mathrm{d}u}\right| \mathrm{d}\lambda - \left(\frac{a}{b}\right)^2 \\
&= \frac{1}{b^2\Gamma(a)} \int_0^\infty u^{a+1} \exp(-u) \,\mathrm{d}\lambda - \left(\frac{a}{b}\right)^2 \\
&= \frac{1}{b^2\Gamma(a)} \Gamma(a+2) - \left(\frac{a}{b}\right)^2 \\
&= \frac{a(a+1)\Gamma(a)}{b^2\Gamma(a)} - \left(\frac{a}{b}\right)^2 \\
&= \frac{a}{b^2}
\end{aligned}$$

where the second last step used the property $\Gamma(x+1) = x\Gamma(x)$.

**(2.152)**

$$\begin{aligned}
p(\mathbf{x}|\mu,\lambda) &= \prod_{n=1}^N \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(x_n-\mu)^2\right\} \\
&= \left(\frac{\lambda}{2\pi}\right)^{N/2} \exp\left\{-\frac{\lambda}{2}\sum_{n=1}^N (x_n-\mu)^2\right\} \\
&\propto \left[\lambda^{1/2} \exp\left(-\frac{\lambda\mu^2}{2}\right)\right]^N \exp\left\{\lambda\mu\sum_{n=1}^N x_n - \frac{\lambda}{2}\sum_{n=1}^N x_n^2\right\}.
\end{aligned}$$

## (2.158)

$$
\begin{aligned}
p(x|\mu,a,b) &= \int_0^\infty \mathcal{N}(x|\mu,\tau^{-1})\mathrm{Gam}(\tau|a,b)\,\mathrm{d}\tau \\
&= \int_0^\infty \frac{b^a e^{-b\tau}\tau^{a-1}}{\Gamma(a)}\left(\frac{\tau}{2\pi}\right)^{1/2}\exp\left\{-\frac{\tau}{2}(x-\mu)^2\right\}\mathrm{d}\tau \\
&= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty \tau^{a-1/2}\exp\left\{-b\tau-\frac{\tau}{2}(x-\mu)^2\right\}\mathrm{d}\tau.
\end{aligned}
$$

By making the change of variable $z = \tau[b + (x-\mu)^2/2]$, we have

$$
\begin{aligned}
p(x|\mu,a,b) &= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty z^{a-1/2}\left[b+\frac{(x-\mu)^2}{2}\right]^{-a+1/2}\exp(-z)\left|\frac{\mathrm{d}\tau}{\mathrm{d}z}\right|\mathrm{d}z \\
&= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty z^{a-1/2}\left[b+\frac{(x-\mu)^2}{2}\right]^{-a+1/2}\exp(-z)\left[b+\frac{(x-\mu)^2}{2}\right]^{-1}\mathrm{d}z \\
&= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2}\left[b+\frac{(x-\mu)^2}{2}\right]^{-a-1/2}\int_0^\infty z^{a-1/2}\exp(-z)\,\mathrm{d}z \\
&= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2}\left[b+\frac{(x-\mu)^2}{2}\right]^{-a-1/2}\Gamma(a+1/2).
\end{aligned}
$$

## (2.162)

$$
\begin{aligned}
\mathrm{St}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Lambda},\nu) &= \int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \int_0^\infty \frac{|\eta\boldsymbol{\Lambda}|^{1/2}}{(2\pi)^{D/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\eta\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu})\right\}\frac{1}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^{\nu/2}\eta^{\nu/2-1}\exp\left\{-\frac{\nu\eta}{2}\right\}\mathrm{d}\eta \\
&= \frac{|\Lambda|^{1/2}(\nu/2)^{\nu/2}}{(2\pi)^{D/2}\Gamma(\nu/2)}\int_0^\infty \exp\left\{-\frac{\eta}{2}(\Delta^2+\nu)\right\}\eta^{D/2+\nu/2-1}\,\mathrm{d}\eta
\end{aligned}
$$

where we have defined

$$
\Delta^2 = (\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu}),
$$

and in the last step we used the property that $|\eta\boldsymbol{\Lambda}| = \eta^D|\boldsymbol{\Lambda}|$.

Now, we substitute $z = \eta(\Delta^2+\nu)/2$, which gives

$$
\begin{aligned}
\mathrm{St}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Lambda},\nu) &= \frac{|\Lambda|^{1/2}(\nu/2)^{\nu/2}}{(2\pi)^{D/2}\Gamma(\nu/2)}\int_0^\infty \exp(-z)\left(\frac{2z}{\Delta^2+\nu}\right)^{D/2+\nu/2-1}\left|\frac{\mathrm{d}\eta}{\mathrm{d}z}\right|\mathrm{d}z \\
&= \frac{|\Lambda|^{1/2}(\nu/2)^{\nu/2}}{(2\pi)^{D/2}\Gamma(\nu/2)}\int_0^\infty \exp(-z)\left(\frac{2z}{\Delta^2+\nu}\right)^{D/2+\nu/2-1}\frac{2}{\Delta^2+\nu}\,\mathrm{d}z \\
&= \frac{|\Lambda|^{1/2}(\nu/2)^{\nu/2}}{(2\pi)^{D/2}\Gamma(\nu/2)}\left(\frac{2z}{\Delta^2+\nu}\right)^{D/2+\nu/2}\int_0^\infty \exp(-z)z^{D/2+\nu/2-1}\,\mathrm{d}z \\
&= \frac{|\Lambda|^{1/2}(\nu/2)^{\nu/2}}{(2\pi)^{D/2}\Gamma(\nu/2)}\left(\frac{2z}{\Delta^2+\nu}\right)^{D/2+\nu/2}\Gamma(D/2+\nu/2) \\
&= \frac{\Gamma(D/2+\nu/2)}{\Gamma(\nu/2)}\frac{|\boldsymbol{\Lambda}|^{1/2}}{(\pi\nu)^{D/2}}\left[1+\frac{\Delta^2}{\nu}\right]^{-D/2-\nu/2}.
\end{aligned}
$$

## (2.164)

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}\sim \mathrm{St}(\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)}[\mathbf{x}] &= \int_{\mathbf{x}} \mathbf{x}\mathrm{St}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{x}} \mathbf{x}\bigg(\int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta\bigg)\,\mathrm{d}\mathbf{x} \\
&= \int_0^\infty \bigg(\int_{\mathbf{x}} \mathbf{x}\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\,\mathrm{d}\mathbf{x}\bigg)\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \int_0^\infty \mathbb{E}_{\mathbf{x}\sim\mathcal{N}(\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})}[\mathbf{x}]\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \boldsymbol{\mu}\int_0^\infty \mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \boldsymbol{\mu}
\end{aligned}
$$

where we used the property that the expected value of a Gaussian random variable is its mean, and the fact that the gamma distribution is normalized.

## (2.165)

Firstly, notice that

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}\sim \mathrm{St}(\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] &= \int_{\mathbf{x}} \mathbf{x}\mathbf{x}^{\mathrm{T}}\mathrm{St}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{x}} \mathbf{x}\mathbf{x}^{\mathrm{T}}\bigg(\int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta\bigg)\,\mathrm{d}\mathbf{x} \\
&= \int_0^\infty \bigg(\int_{\mathbf{x}} \mathbf{x}\mathbf{x}^{\mathrm{T}}\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\,\mathrm{d}\mathbf{x}\bigg)\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \int_0^\infty \mathbb{E}_{\mathbf{x}\sim\mathcal{N}(\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})}[\mathbf{x}\mathbf{x}^{\mathrm{T}}]\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \int_0^\infty (\mathrm{cov}_{\mathbf{x}\sim\mathcal{N}(\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})}[\mathbf{x}] + \mathbb{E}_{\mathbf{x}\sim\mathcal{N}(\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})}[\mathbf{x}]\mathbb{E}_{\mathbf{x}\sim\mathcal{N}(\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})}[\mathbf{x}^{\mathrm{T}}])\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \int_0^\infty ((\eta\boldsymbol{\Lambda})^{-1} + \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}})\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}}\int_0^\infty \mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta + \boldsymbol{\Lambda}^{-1}\int_0^\infty \eta^{-1}\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \boldsymbol{\Lambda}^{-1}\int_0^\infty \eta^{-1}\mathrm{Gam}(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \frac{\boldsymbol{\Lambda}^{-1}(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\int_0^\infty \eta^{\nu/2-2}\exp\bigg(-\frac{\nu\eta}{2}\bigg)\,\mathrm{d}\eta.
\end{aligned}
$$

Making the change of variable $z = \nu\eta/2$, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbf{x}\sim\mathrm{St}(\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] &= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \frac{\boldsymbol{\Lambda}^{-1}(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \int_0^\infty \left(\frac{2z}{\nu}\right)^{\nu/2-2} \exp(-z) \left|\frac{\mathrm{d}\eta}{\mathrm{d}z}\right| \mathrm{d}z \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \frac{\boldsymbol{\Lambda}^{-1}(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \int_0^\infty \left(\frac{2z}{\nu}\right)^{\nu/2-2} \exp(-z) \left(\frac{2}{\nu}\right) \mathrm{d}z \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \frac{\boldsymbol{\Lambda}^{-1}}{\Gamma(\nu/2)} \frac{\nu}{2} \int_0^\infty z^{\nu/2-2} \exp(-z) \, \mathrm{d}z \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \frac{\boldsymbol{\Lambda}^{-1}\nu}{2} \frac{\Gamma(\nu/2-1)}{\Gamma(\nu/2)} \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \frac{\boldsymbol{\Lambda}^{-1}\nu}{2} \frac{\Gamma(\nu/2-1)}{(\nu/2-1)\Gamma(\nu/2-1)} \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \frac{\nu}{\nu-2}\boldsymbol{\Lambda}^{-1}
\end{aligned}$$

where in the second last step we used the property $\Gamma(x+1) = x\Gamma(x)$.

Therefore,

$$\begin{aligned}
\mathrm{cov}_{\mathbf{x}\sim\mathrm{St}(\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)}[\mathbf{x}] &= \mathbb{E}_{\mathbf{x}\sim\mathrm{St}(\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] - \mathbb{E}_{\mathbf{x}\sim\mathrm{St}(\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)}[\mathbf{x}]\mathbb{E}_{\mathbf{x}\sim\mathrm{St}(\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)}[\mathbf{x}^{\mathrm{T}}] \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \frac{\nu}{\nu-2}\boldsymbol{\Lambda}^{-1} - \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} \\
&= \frac{\nu}{\nu-2}\boldsymbol{\Lambda}^{-1}.
\end{aligned}$$

## (2.228)

$$\begin{aligned}
\nabla_{\boldsymbol{\eta}} \ln p(\mathbf{X}|\boldsymbol{\eta}) &= \nabla_{\boldsymbol{\eta}} \left( \sum_{n=1}^N \ln h(\mathbf{x}_n) + N \ln g(\boldsymbol{\eta}) + \boldsymbol{\eta}^{\mathrm{T}} \sum_{n=1}^N \mathbf{u}(\mathbf{x}_n) \right) \\
&= N \nabla_{\boldsymbol{\eta}} \ln g(\boldsymbol{\eta}) + \sum_{n=1}^N \mathbf{u}(\mathbf{x}_n) \\
&= \mathbf{0}.
\end{aligned}$$

Rearranging the equation, we obtain

$$-\nabla_{\boldsymbol{\eta}} \ln g(\boldsymbol{\eta}_{\mathrm{ML}}) = \frac{1}{N} \sum_{n=1}^N \mathbf{u}(\mathbf{x}_n).$$