

# MATH1141 Tutorial Solutions - Algebra

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# Chapter 1

## Problem 31

a)

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \lambda \in \mathbb{R}.\end{aligned}$$

b)

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.\end{aligned}$$

## Problem 34

b)

Parametric vector form:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

From the parametric vector form, we have the following equations

$$\begin{aligned}x_1 &= 1 + 4\lambda \\ x_2 &= 2 - 5\lambda \\ x_3 &= -3 + 6\lambda.\end{aligned}$$

Expressing  $\lambda$  in terms of  $x$ ,  $y$  and  $z$  respectively, we obtain the Cartesian form

$$\frac{x_1 - 1}{4} = \frac{x_2 - 2}{-5} = \frac{x_3 + 3}{6}.$$

**c)**

Parametric form:

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}.$$

Similar to question b, the Cartesian form is given by

$$\frac{x_1 - 1}{5} = -x_2 - 1 = \frac{x_3 - 1}{2}.$$

## Problem 36

**a)**

True. Because

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 4 \\ 12 \end{pmatrix}.$$

**b)**

False. The line in parametric vector form can be written as

$$2x - 3y - 9 = 0.$$

By comparing the tangents of the two lines, it is easy to see that they are not parallel.

**c)**

True. Let

$$\frac{x + 10}{5} = y - 7 = \frac{z + 3}{4} = \mu, \quad \mu \in \mathbb{R}.$$

Then, the parametric vector form is given by

$$\mathbf{x} = \begin{pmatrix} -10 \\ 7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}.$$

Because  $2 \cdot (5 \ 1 \ 4)^T = (10 \ 2 \ 8)^T$ , the two lines are parallel.

**d)**

True. The parametric vector form of the latter line is given by

$$\mathbf{x} = \begin{pmatrix} -10 \\ -5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}, \quad \mu \in \mathbb{R}.$$

Similar to question c, since  $2 \cdot (5 \ 0 \ -2)^T = (10 \ 0 \ -4)^T$ , the two lines are parallel.

## Problem 39

a)

Picking two non-parallel vectors parallel to the plane

$$\mathbf{u} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$
$$\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix},$$

the parametric vector form can be given by

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \mathbf{u} + \mu \mathbf{v} = \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix}$$

where  $\lambda, \mu \in \mathbb{R}$ .

b)

Similar to question a, by picking

$$\mathbf{u} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$
$$\mathbf{v} = \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -14 \\ 5 \end{pmatrix},$$

we obtain one possible parametric vector form as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -14 \\ 5 \end{pmatrix}$$

where  $\lambda, \mu \in \mathbb{R}$ .

## Problem 40

a)

A plane through the origin parallel to  $(1 \ 2 \ 3)^T$  and  $(-2 \ 3 \ 4)^T$ .

b)

Noticing that  $-2 \cdot (-2 \ 1 \ 3 \ 2)^T = (4 \ -2 \ -6 \ -4)^T$ , it is a line through the point  $(3, 1, 2, 4)$  parallel to  $(-2 \ 1 \ 3 \ 2)^T$ .

c)

Noticing that  $-3 \cdot (3 \ 2 \ 1 \ 2)^T = (-9 \ -6 \ -3 \ -6)^T$ , it is a line through the origin parallel to  $(3 \ 2 \ 1 \ 2)^T$ .

d)

A plane through the point  $(1, 2, 3)$  parallel to  $(4 \ -1 \ 2)^T$  and  $(8 \ 2 \ 4)^T$ .

## Problem 41

a)

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

b)

Picking two non-parallel vectors parallel to the plane

$$\begin{aligned} \mathbf{u} &= \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{v} &= \begin{pmatrix} 6 \\ 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}, \end{aligned}$$

the parametric vector form is given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

d)

Dividing 12 on both sides of the equation, we have

$$\frac{x_1}{3} - \frac{x_2}{4} + \frac{x_3}{2} = 1.$$

Hence, the plane is through the point  $(3, 4, 2)$ .

Let  $x_2 = \lambda$  and  $x_3 = \mu$  where  $\lambda, \mu \in \mathbb{R}$ , then the parametric vector form can be written as

$$\mathbf{x} = \begin{pmatrix} 3 + \frac{3}{4}\lambda - \frac{3}{2}\mu \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}.$$

f)

Let

$$\frac{x_1 - 5}{7} = \frac{x_2 + 6}{2} = \frac{x_3 - 2}{-3} = \frac{x_4 + 1}{-5} = \mu, \quad \mu \in \mathbb{R},$$

The parametric vector form of the second line can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 + 7\mu \\ -6 + 2\mu \\ 2 - 3\mu \\ -1 - 5\mu \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 2 \\ -3 \\ -5 \end{pmatrix}.$$

Hence, the parametric vector form of the plane is given by

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 0 \\ -4 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 7 \\ 2 \\ -3 \\ -5 \end{pmatrix}$$

where  $\lambda_1, \lambda_2 \in \mathbb{R}$ .

## Problem 44

b)

Method 1

$$\begin{aligned} 0 &= 9x + 4y - z \\ &= 9(-1 + 2\lambda) + 4(2 - 3\lambda) - (3 + 4\lambda) \\ &= -4 + 2\lambda \end{aligned}$$

where  $\lambda \in \mathbb{R}$ . Solving for  $\lambda$ , we have  $\lambda = 2$ . Plugging into the parametric vector form of the line, we obtain the point of the intersection  $(3, -4, 11)$ .

Method 2

The Cartesian form of the plane is given by

$$\mathbf{p} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 9\lambda_1 + 4\lambda_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Hence, the intersection can be found by solving the following equation

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \mathbf{x} = \mathbf{p} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix},$$

which gives the same result as Method 1.

# Chapter 2

## Problem 1

a)

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) \\ &= \arccos\left(\frac{6}{6\sqrt{2}}\right) \\ &= \frac{\pi}{4}.\end{aligned}$$

## Problem 2

b)

$$\begin{aligned}\cos \angle CAB &= \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{|\overrightarrow{AC}||\overrightarrow{AB}|} = \frac{\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}{\left| \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right| \left| \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right|} = -\frac{5\sqrt{33}}{33}, \\ \cos \angle ABC &= \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right| \left| \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \right|} = \frac{2\sqrt{2}}{3}, \\ \cos \angle ACB &= \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|} = \frac{\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}}{\left| \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} \right| \left| \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} \right|} = \frac{4\sqrt{66}}{33}.\end{aligned}$$

### Problem 3

$$\theta = \arccos \frac{\overrightarrow{OF} \cdot \overrightarrow{AG}}{|\overrightarrow{OF}| |\overrightarrow{AG}|} = \arccos \frac{1}{3}.$$

### Problem 4

a)

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \mathbf{b} \cdot \mathbf{a}.$$

b)

$$\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda a_1 b_1 + \lambda a_2 b_2 + \lambda a_3 b_3 = \lambda (a_1 b_1 + a_2 b_2 + a_3 b_3) = \lambda (\mathbf{a} \cdot \mathbf{b}).$$

### Problem 6

Let  $ABCD$  be a square, then

$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{BD} &= (\overrightarrow{BC} + \overrightarrow{AB}) \cdot (\overrightarrow{BC} - \overrightarrow{AB}) \\ &= |\overrightarrow{BC}|^2 - |\overrightarrow{AB}|^2 \\ &= 0. \end{aligned}$$

Q.E.D.

### Problem 7

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0.$$

Therefore,  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is an orthonormal set.

Method 1

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{a} &= \mathbf{u}_1 \cdot (\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3) \\ &= \lambda_1 |\mathbf{u}_1|^2 + \lambda_2 \mathbf{u}_1 \cdot \mathbf{u}_2 + \lambda_3 \mathbf{u}_1 \cdot \mathbf{u}_3 \\ &= \lambda_1 \end{aligned}$$

where the last step used the fact that  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is an orthonormal set. Hence,

$$\lambda_1 = \mathbf{u}_1 \cdot \mathbf{a} = \frac{\sqrt{2}}{2}.$$

Similarly, we have

$$\lambda_2 = -3, \quad \lambda_3 = \frac{3\sqrt{2}}{2}.$$



### Method 2

Alternatively, we can use the Gaussian elimination. The augmented matrix can be reduced to  $[\mathbf{I}, \mathbf{b}]$  where  $\mathbf{I}$  is the identity matrix and  $\mathbf{b} = (\sqrt{2}/2 \ -3 \ 3\sqrt{2}/2)^T$ . Therefore,

$$\lambda_1 = \frac{\sqrt{2}}{2}, \quad \lambda_2 = -3, \quad \lambda_3 = \frac{3\sqrt{2}}{2}.$$

This is not ideal as we discarded the information that  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is an orthonormal set.

## Problem 9

a)

Let

$$\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

By the definition of the vector projection, we have

$$\text{proj}_{\mathbf{v}} \mathbf{b} = \left( \frac{\mathbf{b} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} = \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{pmatrix}.$$

## Problem 10

a)

Denote point  $(-2, 1, 5)$  by  $P$ , point  $(1, 2, -5)$  by  $A$  and the projection of  $P$  on  $\mathbf{x}$  by  $B$ . Then, we have

$$\overrightarrow{AP} = \begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix}.$$

By the definition of the vector projection, we can compute the projection of  $\overrightarrow{AP}$  on  $\mathbf{x}$  by

$$\overrightarrow{AB} = \left( \frac{\overrightarrow{AP} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} = \begin{pmatrix} -6 \\ -3 \\ 4 \end{pmatrix}$$

where  $\mathbf{v} = (6 \ 3 \ 4)^T$ . Hence, the shortest distance between point  $P$  and line  $\mathbf{x}$  is given by

$$|\overrightarrow{PB}| = |\overrightarrow{AB} - \overrightarrow{AP}| = \left| \begin{pmatrix} -6 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix} \right| = 7.$$

b)

The parametric vector form of the line is given by

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Using the same setup as question a, we have

$$\begin{aligned}\overrightarrow{AP} &= \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}, \\ \overrightarrow{AB} &= \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \\ |\overrightarrow{PB}| &= 3.\end{aligned}$$

### problem 15

$$\mathbf{n} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 6 \end{pmatrix}.$$

### Problem 17

b)

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

Hence, one possible normal can be computed by

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}.$$

The area of the parallelogram is  $|\overrightarrow{AB} \times \overrightarrow{AC}| = 2\sqrt{2}$ .

### Problem 18

a)

The area can be computed by

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \left| \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{2}.$$

### Problem 19

a)

$$\cos \angle DEF = \frac{\overrightarrow{ED} \cdot \overrightarrow{EF}}{|\overrightarrow{ED}| |\overrightarrow{EF}|} = \frac{\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right|} = -\frac{2\sqrt{2}}{3}.$$

b)

$$S_{\triangle DEF} = \frac{1}{2} |\vec{ED} \times \vec{EF}| = \frac{\sqrt{2}}{2}.$$

## Problem 24

This can be proved by simply expanding the equation.

## Problem 26

We can prove the coplanarity by showing the triple product of  $DA$ ,  $DB$  and  $DC$  is zero. This is indeed the case:

$$\vec{DA} \cdot (\vec{DB} \times \vec{DC}) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \left( \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} \right) = 0.$$

## Problem 27

a)

Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Point-normal form:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \cdot \left( \mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right) = 0.$$

Cartesian form:

$$x_1 - x_2 - 2x_3 = 3.$$

Let  $x_2 = \lambda_1$  and  $x_3 = \lambda_2$  where  $\lambda_1, \lambda_2 \in \mathbb{R}$ , the parametric form can be written as

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

b)

One possible normal to the plane is given by

$$\mathbf{n} = \frac{1}{5} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Parametric form:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Point-normal form:

$$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \left( \mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right) = 0.$$

Cartesian form:

$$-x_1 + x_2 - x_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 3.$$

c)

This problem is equivalent to finding a plane through point  $(1, 2, -2)$  parallel to  $\begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ .

Hence follows the same steps in question b.

d)

This problem is equivalent to finding a plane through point  $(-1, 0, 0)$  parallel to  $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ .

Hence follows the same steps in question b.

## Problem 30

a)

Denote point  $(2, 6, -5)$  by  $P$ , point  $(1, 2, 3)$  by  $A$  and the normal to the plane by  $\mathbf{n}$ . The shortest distance is the length of the projection of  $\overrightarrow{AP}$  on  $\mathbf{n}$ , given by

$$|\text{proj}_{\mathbf{n}} \overrightarrow{AP}| = \left| \left( \frac{\overrightarrow{AP} \cdot \mathbf{n}}{|\mathbf{n}|^2} \right) \mathbf{n} \right| = 3.$$

b)

From the given Cartesian form, we can easily find a point on the plane, say  $(0, 0, 5)$ , denoted by  $A$ , and the normal  $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ . We further denote the point  $(1, 4, 1)$  by  $P$ . The shortest distance is the length of the projection of  $\overrightarrow{AP}$  on  $\mathbf{n}$ , given by

$$|\text{proj}_{\mathbf{n}} \overrightarrow{AP}| = \left| \left( \frac{\overrightarrow{AP} \cdot \mathbf{n}}{|\mathbf{n}|^2} \right) \mathbf{n} \right| = \sqrt{6}.$$

## Problem 31

a)

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Hence, one possible parametric form can be

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

**b)**

$$\mathbf{n} = -\frac{1}{3} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

**c)**

By the definition, one possible point-normal form can be

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \left( \mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right) = 0.$$

**d)**

The shortest distance is the length of the projection of  $\overrightarrow{AQ}$  on  $\mathbf{n}$ , given by

$$|\text{proj}_{\mathbf{n}} \overrightarrow{AQ}| = \left| \left( \frac{\overrightarrow{AQ} \cdot \mathbf{n}}{|\mathbf{n}|^2} \right) \mathbf{n} \right| = \frac{8\sqrt{3}}{3}.$$

# Chapter 3

## Problem 5

$$\begin{aligned}3z &= 6 + 9i \\ z^2 &= (2 + 3i)^2 = -5 + 12i \\ z + 2w &= (2 + 3i) + 2(-1 + 2i) = 7i \\ z(w + 3) &= -2 + 10i \\ \frac{z}{w} &= \frac{2 + 3i}{-1 + 2i} = \frac{(2 + 3i)(-1 - 2i)}{(-1 + 2i)(-1 - 2i)} = \frac{4 - 7i}{5} \\ \frac{w}{z} &= \frac{5}{4 - 7i} = \frac{4 + 7i}{13}.\end{aligned}$$

## Problem 6

a)

$$\frac{1 + i}{1 + 2i} = \frac{(1 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{3 - i}{5}.$$

b)

$$\frac{2 - i}{3 + i} - \frac{3 - i}{2 + i} = \frac{(2 - i)(2 + i) - (3 + i)(3 - i)}{(3 + i)(2 + i)} = -\frac{1 - i}{2}.$$

## Problem 8

b)

$$z = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm \sqrt{2}i$$

c)

$$z = \frac{6 \pm \sqrt{36 - 40}}{2} = 3 \pm i.$$

e)

$$z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4),$$

which has roots  $z = \pm i, \pm 2i$ .

### Problem 11

$$\begin{aligned}\left(\frac{a+bi}{a-bi}\right)^2 - \left(\frac{a-bi}{a+bi}\right)^2 &= \left(\frac{a+bi}{a-bi} + \frac{a-bi}{a+bi}\right)\left(\frac{a+bi}{a-bi} - \frac{a-bi}{a+bi}\right) \\ &= \frac{2(a^2-b^2)}{a^2+b^2} \cdot \frac{4abi}{a^2+b^2} \\ &= \frac{8abi(a^2-b^2)}{(a^2+b^2)^2}.\end{aligned}$$

### Problem 12

$$\operatorname{Re}(-1+i) = -1; \quad \operatorname{Im}(-1+i) = 1; \quad \overline{-1+i} = -1-i.$$

$$\operatorname{Re}(2+3i) = 2; \quad \operatorname{Im}(2+3i) = 3; \quad \overline{2+3i} = 2-3i.$$

$$\operatorname{Re}\left(\frac{2-i}{1+i}\right) = \operatorname{Re}\left(\frac{1-3i}{2}\right) = \frac{1}{2}; \quad \operatorname{Re}\left(\frac{2-i}{1+i}\right) = -\frac{3}{2}; \quad \frac{\overline{2-i}}{1+i} = \frac{1+3i}{2}.$$

$$\operatorname{Re}\left(\frac{1}{(i+1)^2}\right) = \operatorname{Re}\left(\frac{-i}{2}\right) = 0; \quad \operatorname{Re}\left(\frac{1}{(i+1)^2}\right) = -\frac{1}{2}; \quad \frac{\overline{1}}{(i+1)^2} = \frac{i}{2}.$$

### Problem 13

$$\begin{aligned}z^2 &= (1+2i)^2 = -3+4i, \\ \frac{\bar{z}}{w} &= \frac{1-2i}{3-4i} = \frac{11}{25} - \frac{2}{25}i.\end{aligned}$$

### Problem 14

Given

$$2z + 3w = 1 + 12i \tag{3.14.1}$$

$$\bar{z} - \bar{w} = 3 - i,$$

taking conjugate of both sides of the the second equation, we have

$$z - w = 3 + i. \tag{3.14.2}$$

Solving (3.14.1) and (3.14.2), we obtain

$$\begin{aligned}z &= 2 + 3i \\ w &= -1 + 2i.\end{aligned}$$

### Problem 17

a)

It can be easily verified by taking the conjugate of both sides of the equation.

b)

Since  $3 - 2i$  is a root of the real polynomial, the conjugate complex  $3 + 2i$  is also a root. Hence,

$$\begin{aligned} ax^2 + bx + c &= (x - (3 - 2i))(x - (3 + 2i)) \\ &= x^2 - 6x + 13. \end{aligned}$$

c)

Yes.

## Problem 18

a)

$$|z| = 6\sqrt{2}; \quad \text{Arg}(z) = \frac{\pi}{4}; \quad z = 6\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).$$

b)

$$|z| = 4; \quad \text{Arg}(z) = \pi; \quad z = 4(\cos \pi + i \sin \pi).$$

c)

$$|z| = 2; \quad \text{Arg}(z) = -\frac{\pi}{6}; \quad z = 2 \left( \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right).$$

d)

$$|z| = 1; \quad \text{Arg}(z) = -\frac{3\pi}{4}; \quad z = \cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4}.$$

e)

$$|z| = \sqrt{58}; \quad \text{Arg}(z) = \pi - \arctan \frac{3}{7}; \quad z = \sqrt{58} \left( \cos \left( \pi - \arctan \frac{3}{7} \right) + i \sin \left( \pi - \arctan \frac{3}{7} \right) \right).$$

## Problem 21

a)

$$a + bi = 3 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i.$$

b)

$$a + bi = 3 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = -\frac{3\sqrt{3}}{2} - \frac{1}{2}i.$$



c)

$$a + bi = 3 \left( \cos \left( -\frac{2\pi}{3} \right) + i \sin \left( -\frac{2\pi}{3} \right) \right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i.$$

d)

$$a + bi = 3 \left( \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right) = -\frac{3\sqrt{3}}{2} - \frac{1}{2}i.$$

e)

$$a + bi = 3 \left( \cos \left( \frac{\pi}{8} \right) + i \sin \left( \frac{\pi}{8} \right) \right) = \frac{3}{2} \left( \sqrt{2 + \sqrt{2}} + \sqrt{2 - \sqrt{2}}i \right)$$

where we applied the half angle formula.

## Problem 22

a)

Let  $z = a + bi$ , then  $\bar{z} = a - bi$ . Hence,

$$z\bar{z} = a^2 + b^2 = |z|^2.$$

Multiplying  $z^{-1}$  on both sides of this equation, we obtain

$$\bar{z} = z^{-1}.$$

b)

$$|z| = \sqrt{a^2 + b^2} = |\bar{z}|.$$

c)

$$\bar{z} = r(\cos \theta - i \sin \theta) = r(\cos(-\theta) + i \sin(-\theta))$$

where we took advantage of the parity of sin and cos.

## Problem 26

Skipped.

## Problem 27

$$w^6 = (2e^{i\pi/3})^6 = 64e^{2i\pi} = 64.$$

The Cartesian form of  $w$  is given by

$$w = 1 + \sqrt{3}i.$$

Hence,

$$z - w = (1 - i) - (1 + \sqrt{3}i) = -(1 + \sqrt{3})i.$$

$$\frac{w}{\bar{z}} = \frac{1 + \sqrt{3}i}{1 + i} = \frac{1 + \sqrt{3}}{2} + \frac{\sqrt{3} - 1}{2}i.$$

### Problem 31

Keep in mind that the principal argument is in  $(-\pi, \pi]$ :

$$\begin{aligned}\operatorname{Arg}(z) &= \frac{\pi}{3} \\ \operatorname{Arg}(w) &= \frac{\pi}{4} \\ \operatorname{Arg}(zw) &= \operatorname{Arg}(z) + \operatorname{Arg}(w) + 2k\pi = \frac{7\pi}{12}.\end{aligned}$$

We can easily evaluate  $zw$  by

$$zw = |z||w|e^{i\operatorname{Arg}(zw)} = 2\sqrt{2}e^{7i\pi/12}.$$

This should be equal to its Cartesian form that is given by

$$zw = (1 + \sqrt{3}i)(1 + i) = (1 - \sqrt{3}) + (1 + \sqrt{3})i.$$

Equating the coefficients, we have

$$\begin{aligned}2\sqrt{2}\cos\frac{7\pi}{12} &= 1 - \sqrt{3} \\ 2\sqrt{2}\sin\frac{7\pi}{12} &= 1 + \sqrt{3},\end{aligned}$$

which implies that

$$\begin{aligned}\cos\frac{7\pi}{12} &= \frac{1 - \sqrt{3}}{2\sqrt{2}} \\ \sin\frac{7\pi}{12} &= \frac{1 + \sqrt{3}}{2\sqrt{2}}.\end{aligned}$$

### Problem 33

a)

$$\begin{aligned}(\sqrt{3} + i)^5 &= (2e^{i\pi/6})^5 \\ &= 32e^{i5\pi/6} \\ &= 32(\cos(5\pi/6) + i\sin(5\pi/6)) \\ &= -16\sqrt{3} + 16i.\end{aligned}$$

### Problem 34

a)

Let  $z = a + bi$  where  $a, b \in \mathbb{R}$  and  $a, b \neq 0$ , we have

$$z^2 = (a + bi)^2 = 21 - 20i.$$

Solving for  $a, b$  and substituting back into the expression of  $z$ , we obtain  $z = 5 - 2i$  or  $z = -5 + 2i$ .

## 37

a)

Using the same steps in question 34(a) to find the roots of  $-3 + 4i$ , we have  $z' = \pm(1 + 2i)$ . Hence, the roots of  $z^2 - 3z + (3 - i) = 0$  are given by

$$\begin{aligned} z &= \frac{3 \pm \sqrt{-3 + 4i}}{2} \\ &= 2 + i \text{ or } 1 - i. \end{aligned}$$

## Problem 40

This problem is equivalent to finding the modulus  $r$  and the principal argument  $\theta$  that satisfies

$$(re^{i\theta})^5 = 16e^{-i\pi/3+2k\pi}, \quad k \in \mathbb{N}.$$

Solving for  $r$  and  $\theta$ , we obtain

$$\begin{aligned} r &= 2 \\ \theta &= -\frac{13\pi}{15}, -\frac{7\pi}{15}, -\frac{\pi}{15}, \frac{\pi}{3}, \frac{11\pi}{15}. \end{aligned}$$

## Problem 42

$$\omega + \omega^2 + \cdots + \omega^n = \frac{\omega(1 - \omega^n)}{1 - \omega} = 0.$$

Q.E.D.

## Problem 50

$$\begin{aligned} \cos 3\theta + i \sin 3\theta &= (\cos \theta + i \sin \theta)^3 \\ &= \cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 \\ &= 4 \cos^3 \theta - 3 \cos \theta + i(3 \sin \theta \cos^2 \theta - \sin^3 \theta). \end{aligned}$$

Equating the real parts, we obtain

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.$$

## Problem 54

Skipped.

## Problem 66

Noticing that  $p(2) = 0$ , by the remainder and factor theorem,  $z - 2$  must be a factor of  $p(z)$ . Using long division, we can factorize  $p(z)$  as

$$p(z) = (z - 2)(2z - 5)(z + 3).$$

## Problem 70

This problem is equivalent to finding the roots of  $z^4 + 4 = 0$ .

Let  $z = re^{i\theta}$ , then

$$r^4 e^{i4\theta} = z^4 = -4 = 4e^{i(\pi+2k\pi)}, \quad k \in \mathbb{N}.$$

Equating the coefficients, we obtain the four roots. They are

$$z = \sqrt{2}e^{\pm i\pi/4}, \quad \sqrt{2}e^{\pm i3\pi/4} = \pm 1 \pm i.$$

Hence,

$$\begin{aligned} p(z) &= (x - (1 + i))(x - (1 - i))(x - (-1 + i))(x - (-1 - i)) \\ &= (x^2 - 2x + 2)(x^2 + 2x + 2). \end{aligned}$$

## Problem 71

Similar to problem 70,

$$r^4 e^{i4\theta} = -i = e^{i(-\pi/4+2k\pi)}, \quad k \in \mathbb{N}.$$

Equating the coefficients and solving for  $r$  and  $\theta$ , we can factorize  $p(z)$  into

$$p(z) = (z - e^{-i5\pi/8})(z - e^{-i\pi/8})(z - e^{i3\pi/8})(z - e^{i7\pi/8}).$$

## Problem 72

a)

Similar to problem 70,

$$r^6 e^{i6\theta} = -1 = e^{i(\pi+2k\pi)}, \quad k \in \mathbb{N}.$$

Equating the coefficients and solving for  $r$  and  $\theta$ , we obtain the six roots. They are

$$z = e^{\pm i\pi/6}, \quad e^{\pm i\pi/2}, \quad e^{\pm i5\pi/6}.$$

b)

Skipped.

c)

Using the result we obtained in question a, we have

$$z^6 + 1 = (z - e^{i\pi/6})(z - e^{-i\pi/6})(z - e^{i\pi/2})(z - e^{-i\pi/2})(z - e^{i5\pi/6})(z - e^{-i5\pi/6}).$$

d)

$$\begin{aligned} z^6 + 1 &= (z - e^{i\pi/6})(z - e^{-i\pi/6})(z - e^{i\pi/2})(z - e^{-i\pi/2})(z - e^{i5\pi/6})(z - e^{-i5\pi/6}) \\ &= (z^2 - \sqrt{3}z + 1)(z^2 + 1)(z^2 + \sqrt{3}z + 1). \end{aligned}$$

## Problem 74

a)

Similar to problem 70,

$$r^5 e^{i5\theta} = 1 = e^{i2k\pi}, \quad k \in \mathbb{N}.$$

Equating the coefficients and solving for  $r$  and  $\theta$ , we obtain the five roots. They are

$$z = 1, e^{\pm i2\pi/5}, e^{\pm i4\pi/5}.$$

Hence,

$$p(z) = (z - e^{i2\pi/5})(z - e^{-i2\pi/5})(z - e^{i4\pi/5})(z - e^{-i4\pi/5}).$$

b)

$$\begin{aligned} p(z) &= (z - (\cos(2\pi/5) + i \sin(2\pi/5)))(z - (\cos(2\pi/5) - i \sin(2\pi/5))) \\ &\quad (z - (\cos(4\pi/5) + i \sin(4\pi/5)))(z - (\cos(4\pi/5) - i \sin(4\pi/5))) \\ &= (z^2 - 2 \cos(2\pi/5)z + 1)(z^2 - 2 \cos(4\pi/5)z + 1). \end{aligned}$$

c)

Divide the equation  $p(z) = 0$  by  $z^2$ , we have

$$\begin{aligned} 0 &= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 \\ &= \left(z^2 + 2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) - 1 = 0 \\ &= \left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right) - 1 = 0 \\ &= x^2 + x - 1 = 0 \end{aligned}$$

where  $x = z + 1/z$ .

d)

The results can be easily verified by solving the quadratic equations in terms of  $x$  and  $z$ .

## Problem 76

For real polynomial  $f(z)$ , since  $1 + i$  is a root of  $f(z) = 0$ , the conjugate  $1 - i$  is also a root. Hence, a quadratic factor of  $f(z)$  is

$$(z - (1 + i))(z - (1 - i)) = z^2 - 2z + 2.$$

Using the long division, we can factorize  $f(z)$  as

$$f(z) = (z^2 - 2z + 2)(z - \sqrt[3]{5})(z^2 + \sqrt[3]{5}z + (\sqrt[3]{5})^2).$$

Therefore, the five roots of  $f(z) = 0$  are

$$z = 1 \pm i, \sqrt[3]{5}, \frac{\sqrt[3]{5}}{2}(-1 \pm i\sqrt{3}).$$

# Chapter 4

## Problem 1

a)

Skipped.

## Problem 2

a)

$$\begin{pmatrix} 3 & 2 & 6 \\ 9 & 6 & 36 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & 6 \\ 0 & 0 & 18 \end{pmatrix}.$$

No solution.

b)

$$\begin{pmatrix} 3 & 2 & 6 \\ 9 & 4 & 36 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & 6 \\ 0 & -2 & 18 \end{pmatrix}.$$

Unique solution.

c)

$$\begin{pmatrix} 1 & -5 & 5 \\ 6 & -30 & 30 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

An infinite number of solutions.

## Problem 3

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - (a_{21}a_{12})/a_{11} & b_2 - (a_{21}b_1)/a_{11} \end{pmatrix}.$$

a)

$$a_{22} - (a_{21}a_{12})/a_{11} \neq 0,$$

which implies that

$$a_{11}a_{22} \neq a_{12}a_{21}.$$

**b)**

$$\begin{aligned}a_{22} - (a_{21}a_{12})/a_{11} &= 0 \\ b_2 - (a_{21}b_1)/a_{11} &\neq 0,\end{aligned}$$

which implies that

$$\begin{aligned}a_{11}a_{22} &= a_{12}a_{21} \\ a_{11}b_2 &\neq a_{21}b_1.\end{aligned}$$

**c)**

$$\begin{aligned}a_{22} - (a_{21}a_{12})/a_{11} &= 0 \\ b_2 - (a_{21}b_1)/a_{11} &= 0,\end{aligned}$$

which implies that

$$\begin{aligned}a_{11}a_{22} &= a_{12}a_{21} \\ a_{11}b_2 &= a_{21}b_1.\end{aligned}$$

## Problem 5

**a)**

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 5 & 8 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & 2 \end{pmatrix}.$$

Letting  $x_3 = \lambda$  and performing back substitution, we obtain the solution set  $\left\{ \begin{pmatrix} 1 + \lambda \\ 2 - 2\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$ .

The solution is the line of intersection of the two planes.

**b)**

$$\begin{pmatrix} 4 & 5 & -2 & 16 \\ 8 & 10 & -4 & 20 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 5 & -2 & 16 \\ 0 & 0 & 0 & -12 \end{pmatrix}.$$

The solution set is  $\emptyset$ . The two planes are parallel.

**c)**

$$\begin{pmatrix} 4 & 5 & -2 & 16 \\ 8 & 10 & -4 & 32 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 5 & -2 & 16 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let  $x_2 = \lambda$ ,  $x_3 = \mu$ , then we obtain the solution set  $\left\{ \begin{pmatrix} 4 - 5\lambda/4 + \mu/2 \\ \lambda \\ \mu \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$ . The two planes are the same.

## Problem 10

a)

$$\begin{pmatrix} 1 & 4 & 2 & 3 \\ 2 & 6 & 3 & 0 \\ 4 & -2 & 4 & 4 \end{pmatrix} \xrightarrow{\begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}} \begin{pmatrix} 1 & 4 & 2 & 3 \\ 0 & -2 & -1 & -6 \\ 0 & -18 & -4 & -8 \end{pmatrix}$$
$$\xrightarrow{R_3 = R_3 - 9R_2} \begin{pmatrix} 1 & 4 & 2 & 3 \\ 0 & -2 & -1 & -6 \\ 0 & 0 & 5 & 46 \end{pmatrix}.$$

b)

Similar to question a.

## Problem 12

a) Yes.

b) Yes.

c) No.

d) Yes.

e) Yes.

f) Yes.

g) Yes.

h) No.

## Problem 13

b)

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 10 \\ 0 & 4 & 2 & -4 & 8 \\ 0 & 0 & -7 & 14 & 14 \end{pmatrix} \xrightarrow{\begin{matrix} R_2 \longleftrightarrow R_2/2 \\ R_3 \longleftrightarrow R_3/7 \end{matrix}} \begin{pmatrix} 3 & 2 & 1 & 1 & 10 \\ 0 & 2 & 1 & -2 & 4 \\ 0 & 0 & -1 & 2 & 2 \end{pmatrix}.$$

Letting  $x_4 = \lambda$  and performing back substitution, we obtain

$$\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

which is a line through point  $(2, 3, -2, 0)$  parallel to  $\begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ .



## Problem 14

e)

The augmented matrix is given by

$$\begin{pmatrix} 1 & 2 & 4 & 10 \\ -3 & 3 & 15 & 15 \\ -2 & -1 & 1 & -5 \end{pmatrix}.$$

Using Gaussian elimination, we have

$$\begin{pmatrix} 1 & 2 & 4 & 10 \\ -3 & 3 & 15 & 15 \\ -2 & -1 & 1 & -5 \end{pmatrix} \xrightarrow{\substack{R_2 = R_2 + 3R_1 \\ R_3 = R_3 + 2R_1}} \begin{pmatrix} 1 & 2 & 4 & 10 \\ 0 & 9 & 27 & 45 \\ 0 & 3 & 9 & 15 \end{pmatrix} \xrightarrow{\substack{R_2 = R_2/9 \\ R_3 = R_3 - R_1/3}} \begin{pmatrix} 1 & 2 & 4 & 10 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Letting  $x_3 = \lambda$  and performing back substitution, we obtain

$$\mathbf{x} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

f)

The augmented matrix is given by

$$\begin{pmatrix} 1 & -4 & -5 & -6 \\ 2 & -1 & -1 & 2 \\ 3 & 9 & 12 & 30 \end{pmatrix}.$$

Using Gaussian elimination, we have

$$\begin{pmatrix} 1 & -4 & -5 & -6 \\ 2 & -1 & -1 & 2 \\ 3 & 9 & 12 & 30 \end{pmatrix} \xrightarrow{\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - 3R_1}} \begin{pmatrix} 1 & -4 & -5 & -6 \\ 0 & 7 & 9 & 10 \\ 0 & 21 & 27 & 48 \end{pmatrix} \\ \xrightarrow{R_3 = R_3 - 3R_2} \begin{pmatrix} 1 & -4 & -5 & -6 \\ 0 & 7 & 9 & 10 \\ 0 & 0 & 0 & 18 \end{pmatrix}$$

where we performed row operations simultaneously. Hence, there is no solution.

h)

The augmented matrix is given by

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 1 \\ 3 & 2 & 0 & -2 & 3 \end{pmatrix}.$$

Using Gaussian elimination, we have

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 1 \\ 3 & 2 & 0 & -2 & 3 \end{pmatrix} \xrightarrow{R_3 = R_3 - 3R_1 + 4R_2} \begin{pmatrix} 1 & 2 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -5 \end{pmatrix}.$$

Letting  $x_4 = \lambda$  and performing back substitution, we obtain

$$\mathbf{x} = \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

## Problem 15

b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & 5 & 6 & 2 \\ 0 & 0 & 1 & 7 & 3 \end{pmatrix} \xrightarrow{R_2 = -R_2} \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & -5 & -6 & -2 \\ 0 & 0 & 1 & 7 & 3 \end{pmatrix}$$

$$\xrightarrow{\begin{matrix} R_1 = R_1 - 2R_2 - 13R_3 \\ R_2 = R_2 + 5R_3 \end{matrix}} \begin{pmatrix} 1 & 0 & 0 & -75 & -34 \\ 0 & 1 & 0 & 29 & 13 \\ 0 & 0 & 1 & 7 & 3 \end{pmatrix}$$

where we performed row operations simultaneously.

Letting  $x_4 = \lambda$ , we obtain

$$\mathbf{x} = \begin{pmatrix} -34 \\ 13 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 75 \\ -29 \\ -7 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

which is a line through point  $(-34, 13, 3, 0)$  parallel to  $\begin{pmatrix} 75 \\ -29 \\ -7 \\ 1 \end{pmatrix}$  in  $\mathbb{R}^4$ .

## Problem 16

- a) Unique solution.
- b) No solution.
- c) Infinitely many solutions.
- d) Infinitely many solutions.
- e) Unique solution. Note that the last row gives no information.

## Problem 17

Performing Gaussian elimination, we have

$$\begin{pmatrix} 1 & 1 & k & 2 \\ 3 & 4 & 2 & k \\ 2 & 3 & -1 & 1 \end{pmatrix} \xrightarrow{\substack{R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1}} \begin{pmatrix} 1 & 1 & k & 2 \\ 0 & 1 & 2-3k & k-6 \\ 0 & 1 & -1-2k & -3 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 1 & k & 2 \\ 0 & 1 & 2-3k & k-6 \\ 0 & 0 & -3+k & 3-k \end{pmatrix}.$$

a)

A unique solution indicates  $-3+k \neq 0$ , i.e.  $k \neq 3$ .

b)

No solution implies that  $-3+k \neq 0$  whereas  $3-k = 0$ . Apparently, there is no such  $k$ .

c)

Infinitely many solutions indicates that  $-3+k = 0$  and  $3-k = 0$ , namely,  $k = 3$ .

## Problem 18

Performing Gaussian elimination, we have

$$\begin{pmatrix} 1 & 2 & \lambda & 1 \\ -1 & \lambda & -1 & 0 \\ \lambda & -4 & \lambda & -1 \end{pmatrix} \xrightarrow{\substack{R_2 = R_2 + R_1 \\ R_3 = R_3 - \lambda R_1}} \begin{pmatrix} 1 & 2 & \lambda & 1 \\ 0 & \lambda+2 & \lambda-1 & 1 \\ 0 & -2\lambda-4 & \lambda-\lambda^2 & -1-\lambda \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 + 2R_2} \begin{pmatrix} 1 & 2 & \lambda & 1 \\ 0 & \lambda+2 & \lambda-1 & 1 \\ 0 & 0 & -(\lambda-1)(\lambda-2) & -\lambda+1 \end{pmatrix}.$$

a)

No solution indicates either

$$\begin{aligned} \lambda+2 &= 0 \\ \lambda-2 &\neq 0 \\ -\lambda+1+\lambda-2 &\neq 0 \end{aligned}$$

or

$$\begin{aligned} -(\lambda-1)(\lambda-2) &= 0 \\ -\lambda+1 &\neq 0. \end{aligned}$$

Solving for  $\lambda$ , we obtain

$$\lambda = \pm 2.$$

**b)**

In this case, infinitely many solutions indicates that

$$-(\lambda - 1)(\lambda - 2) = -\lambda + 1 = 0,$$

which implies that

$$\lambda = 1.$$

**c)**

Unique solution indicates that

$$(\lambda + 2)(-(\lambda - 1)(\lambda - 2)) \neq 0,$$

that is

$$\lambda \neq 1 \text{ and } \lambda \neq \pm 2.$$

In fact, we can first find the  $\lambda$ 's that satisfy the condition for unique solution. Then substitute the excluded  $\lambda$ 's and check the number of solutions.

## **Problem 21**

The answer is obviously no as long as we see *cddc* out of the five routes, because this gives a zero row in the 5 by 6 augmented matrix, resulting in infinitely many solutions.