# MATH1241 Problem Set Solutions - Algebra

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# Contents

# Chapter 6

# Vector Spaces

## 6.5 Problem 5

For axiom 1, if $\mathbf{u}, \mathbf{v} \in M_{mn}(\mathbb{C})$, then $u_{ij} + v_{ij} \in \mathbb{C}$ because $\mathbb{C}$ is closed under addition. Hence $\mathbf{u} + \mathbf{v} \in M_{mn}(\mathbb{C})$.

For axiom 3, if $\mathbf{u}, \mathbf{v} \in M_{mn}(\mathbb{C})$, then for any pair of entries $u_{ij}$ and $v_{ij}$, we have $u_{ij}+v_{ij} = v_{ij}+u_{ij}$, which implies that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

For axiom 6, if $\mathbf{v} \in M_{mn}(\mathbb{C})$ and $\lambda \in \mathbb{C}$, then for any entry $v_{ij}$, we have $\lambda v_{ij} \in \mathbb{C}$ because $\mathbb{C}$ is closed under multiplication. Hence, $\lambda \mathbf{v} \in M_{mn}(\mathbb{C})$.

For axiom 10, if $\mathbf{u}, \mathbf{v} \in M_{mn}(\mathbb{C})$ and $\lambda \in \mathbb{C}$, then for any pair of entries $u_{ij}$ and $v_{ij}$, we have $\lambda(u_{ij} + v_{ij}) = \lambda u_{ij} + \lambda v_{ij}$, which implies that $\lambda(\mathbf{u} + \mathbf{v}) = \lambda \mathbf{u} + \lambda \mathbf{v}$.

## 6.7 Problem 7

Here gives a rather informal answer by eyeballing.

1. Axiom 1 – Satisfied. Because $\mathbf{R}$ is closed under addition and multiplication.
2. Axiom 2 – Violated. Because of the minus sign in the addition rule.
3. Axiom 3 – Violated. Because of the minus sign in the addition rule.
4. Axiom 4 – Satisfied. The zero is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
5. Axiom 5 – Satisfied.
6. Axiom 6 – Satisfied.
7. Axiom 7 – Violated. 4 is multiplied twice for $\lambda(\mu \mathbf{v})$, while only once for $(\lambda\mu)\mathbf{v}$.
8. Axiom 8 – Violated. Because of the 4 in the multiplication rule.
9. Axiom 9 – Violated. Because of the 4 in the multiplication rule.
10. Axiom 10 – Satisfied.

Hence, the system is not a vector space.

## 6.27 Problem 27

### a)

We prove this by induction.

For $m = 1$, it is obvious that $W = W_1 \leq V$. Suppose that for $m \geq 2$, the intersection of the $m$ subspaces $W' \leq V$. Then, for $m+1$ subspaces, the intersection $W = W' \cap W_{m+1}$. Hence, for $\mathbf{u}, \mathbf{v} \in W$ and $\lambda \in \mathbb{F}$, we have

(1) $\mathbf{0} = 0 \cdot \mathbf{u} \in W$.

(2) $\lambda \mathbf{u} + \mathbf{v} \in W'$ and $\lambda \mathbf{u} + \mathbf{v} \in W_{m+1}$. Thus, $\lambda \mathbf{u} + \mathbf{v} \in W$.

Therefore, by the subspace test, $W \leq V$.

### b)

We prove this by contradiction.

Suppose that $\exists \mathbf{x} \in W$ such that $\mathbf{x} \neq \displaystyle\sum_{i=1}^{n} \lambda_i \mathbf{s}_i, \forall \mathbf{s}_i \in V, \lambda_i \in \mathbb{F}$, that is, $\mathbf{x} \notin \text{span}(S)$. Then, for any subspace $V' \leq V, S \subseteq V'$, we have $\mathbf{x} \neq V' \supseteq W$, which is contradictory to the assumption that $\mathbf{x} \in W$. Hence, $W$ is the set of finite linear combinations of vectors from $S$.