MATH1141 Tutorial Solutions - Calculus

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Chapter 2

Problem 10

d)

Method 1

$$\begin{aligned}\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3} &= \lim_{x\to 3}\frac{\left(\frac{1}{x}-\frac{1}{3}\right)\cdot 3x}{(x-3)\cdot 3x}\\ &= \lim_{x\to 3}\frac{3-x}{3x(x-3)}\\ &= -\lim_{x\to 3}\frac{1}{3x}\\ &= -\frac{1}{9}.\end{aligned}$$

Method 2

$$\begin{aligned}\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3} &\overset{\text{L'H}}{=} \lim_{x\to 3}\frac{\left(\frac{1}{x}-\frac{1}{3}\right)'}{(x-3)'}\\ &= \lim_{x\to 3}\frac{-\frac{1}{x^2}}{1}\\ &= -\frac{1}{9}.\end{aligned}$$

Problem 11

a)

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2} = -\lim_{x\to 2^-}\frac{x-2}{x-2} = -1.$$

b)

$$\lim_{x\to 2^+}\frac{|x-2|}{x-2} = \lim_{x\to 2^-}\frac{x-2}{x-2} = 1.$$

c)

No. Because the left- and right-hand limit are not equal.

Problem 12

b)

$$\lim_{x\to 0^-} \frac{4}{x} = -\infty$$
$$\lim_{x\to 0^+} \frac{4}{x} = \infty.$$

Since the left- and right-hand limit are not equal, the limit does not exist.

Problem 13

a)

For all $x \neq 0$, we have

$$-|x| \leq x \sin\frac{1}{x} \leq |x|.$$

Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0,$$

by the Pinching Theorem, we obtain

$$\lim_{x\to 0} x \sin\frac{1}{x} = 0.$$

b)

For all $x \neq 0$, we have

$$-x^2 \leq x^2 \sin\frac{1}{2x} \leq x^2.$$

Since

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0,$$

by the Pinching Theorem, we obtain

$$\lim_{x\to 0} x^2 \sin\frac{1}{2x} = 0.$$

Problem 14

a)

$$|CB| = \theta$$
$$|CA| = \sin\theta$$
$$|DB| = \tan\theta.$$

b)

According to the graph, for $0 < \theta < \pi/2$, we have

$$\text{Area}(OAC) \leq \text{Area}(OBC) \leq \text{Area}(OBD),$$

which implies that

$$\frac{1}{2}\sin\theta\cos\theta \leq \pi \cdot \frac{\theta}{2\pi} \leq \frac{1}{2}\tan\theta,$$

that is

$$\sin\theta\cos\theta \leq \theta \leq \tan\theta.$$

c)

Applying the result of part b, for $0 < \theta < \pi/2$,

$$\cos\theta = \frac{\sin\theta\cos\theta}{\sin\theta} \leq \frac{\theta}{\sin\theta} \leq \frac{\tan\theta}{\sin\theta} = \frac{1}{\cos\theta}.$$

Since

$$\lim_{\theta\to 0^+}\cos\theta = \lim_{\theta\to 0^+}\frac{1}{\cos\theta} = 1,$$

by the Pinching Theorem, we obtain

$$\lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} = 1.$$

d)

Analogous to part c), by using the result of part b and the Pinching Theorem, we obtain

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1,$$

which implies that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Problem 15

$\cos(1/x)$ oscillates between -1 and 1 as x goes to 0. Hence, the limit of $\cos(1/x)$ as $x \to 0$ does not exist.

Chapter 3

Problem 2

a)

$$\lim_{x \to 0^-} \exp(2x) = 1 = \lim_{x \to 0^+} \cos x,$$

which implies that f is continuous on $\mathbb{R}$.

Problem 5

Noticing that

$$\begin{aligned} f(-3) &= -9 < 0 \\ f(-2) &= 5 > 0, \end{aligned}$$

since f is continuous on $[-3, -2]$, by the Intermediate Value Theorem, there exists some $c \in (-3, -2)$ such that $f(c) = 0$.

Same for f on $[0, 1]$ and $[1, 2]$.

Problem 6

Let

$$f(x) = e^x - 2\cos x.$$

Noticing that

$$\begin{aligned} f(0) &= -1 < 0 \\ f(\pi) &= e^\pi + 2 > 0, \end{aligned}$$

since f is continuous on $[0, \pi]$, by the Intermediate Value Theorem, there exists some $c \in (0, \pi)$ such that $f(c) = 0$, that is, there is at least one positive real solution for $e^x = 2\cos x$.

Problem 9

b)

Yes. Because f is continuous on $[2, 4]$, by the max-Min Theorem, f has both a maximum and a minimum value on $[2, 4]$.

Problem 10

a)

$$f(x) = \sin \frac{100}{x^2 - 1}$$

b)

$$f(x) = -e^{-x^2}$$

Chapter 4

Problem 4

a)

f is differentiable everywhere except at $x = 0$. It is continuous everywhere.

b)

f is differentiable and continuous everywhere.

c)

f is differentiable and continuous everywhere except at $x = -2$.

Problem 6

For $x \neq 0$, it is obvious that f is continuous. Since

$$\lim_{x \to 0} f(x) = 0,$$

f is also continuous at $x = 0$. Hence, f is continuous everywhere.

It is easy to see that f is differentiable for $x \neq 0$, so we only need to prove that f is differentiable at $x = 0$.

Noticing that

$$\begin{aligned} \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} \frac{f(h)}{h} \\ &= \lim_{h \to 0} h \sin \frac{1}{h} \\ &= 0, \end{aligned}$$

along with the fact that f is continuous everywhere, we can conclude that f is also differentiable everywhere.

The derivative of f with respective of x is given by

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} .$$

Since the limit of f' as $x \to 0$ does not exist, f' is not continuous at $x = 0$.

Problem 9

b)

Take the derivative of both sides of the equation with respect to x, we have

$$2x - \frac{1}{2}(xy)^{-1/2}\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Rearranging the equation, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x\sqrt{xy} - y}{x - 4y\sqrt{xy}}.$$

Problem 11

Taking the derivative of both sides with respect to x, we have

$$3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 3 + 3\frac{\mathrm{d}y}{\mathrm{d}x},$$

which implies that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2 - 1}{y^2 - 1}, \quad y \neq \pm 1.$$

The slope is then given by

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1,y=2} = 0.$$

Hence, the tangent line is

$$y = 2.$$

Problem 12

a)

To guarantee that f is continuous at 0, we need to ensure that the left limit and right limits at 0 are equal. Specifically,

$$\lim_{x \to 0^-} ax + b = \lim_{x \to 0^+} \sin x = f(0),$$

which implies that $b = 0$.

f differentiable at 0 means that f is continuous at 0 and the left and right derivatives are equal. In this case,

$$\begin{aligned} a &= \cos 0 = 1, \\ b &= 0. \end{aligned}$$

b)

Similar to question a, for (i), we have $b = 1$; for (ii), $a = 2, b = 1$.

Problem 14

For this problem, I combined the answers into one.

Noticing that 8.01 is close to 8, we can apply the linear approximation to $f(8.01)$. To be more specific,

$$f(8.01) \approx f(8) + f'(8)(8.01 - 8) = \frac{2401}{1200},$$

which is slightly better than the approximation by calculating $\sqrt[3]{8}$.

Problem 16

Denote the base by a, the height by b and the area under the ladder by $S_\triangle$. From the description, we know

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 1 \tag{4.16.1}$$

$$a^2 + b^2 = 5 \tag{4.16.2}$$

$$S_\triangle = \frac{1}{2}ab.$$

Differentiate both sides of (4.16.2) with respect to a, we have

$$\frac{\mathrm{d}b}{\mathrm{d}a} = -\frac{a}{b}. \tag{4.16.3}$$

Hence,

$$\begin{aligned}
\left.\frac{\mathrm{d}S_\triangle}{\mathrm{d}t}\right|_{b=4} &= \frac{1}{2}\left.\left(\frac{\mathrm{d}a}{\mathrm{d}t}b + a\frac{\mathrm{d}b}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)\right|_{b=4} \\
&= \frac{1}{2}\left.\left(b - a \cdot \frac{a}{b}\right)\right|_{a=3,b=4} \\
&= \frac{7}{8}\ m^2/s
\end{aligned}$$

where in the second step, we plugged in (4.16.1) and (4.16.3), and used the Pythagorean theorem for a.

Chapter 5

Problem 3

Noticing that f is continuous on $[1, 3]$ and differentiable on $(1, 3)$, according to the Mean Value Theorem, there exists at least one point $c \in (1, 3)$ such that

$$\begin{aligned} f'(c) &= \frac{f(3) - f(1)}{3 - 1} \\ &= \frac{\cos 1 - \cos 1}{2} \\ &= 0, \end{aligned}$$

which implies that f' has a zero on the interval $[1, 3]$.

Problem 4

b)

Let

$$f(t) = -\ln(1 - t) - \frac{t}{1 - t}, \quad t \in [0, x].$$

Since f is continuous on $[0, x]$ and differentiable on $(0, x)$, by the Mean Value Theorem, there exists some $c \in (0, x)$ such that

$$\begin{aligned} \frac{f(x)}{x} &= \frac{f(x) - f(0)}{x - 0} \\ &= f'(c) \\ &= -\frac{c}{(c - 1)^2} \\ &< 0, \end{aligned}$$

which implies that

$$-\ln(1 - x) < x/(1 - x)$$

for $x \in (0, 1)$.

c)

Let

$$f(t) = e^t - t - 1, \quad t \in [0, x].$$

Since f is continuous on $[0, x]$ and differentiable on $(0, x)$, by the Mean Value Theorem, there exists some $c \in (0, x)$ such that

$$\begin{aligned}\frac{f(x)}{x} &= \frac{f(x) - f(0)}{x - 0} \\ &= f'(c) \\ &= e^c - 1 \\ &> 0,\end{aligned}$$

which implies that

$$1 + x < e^x$$

for $x > 0$.

Problem 7

a)

Let

$$f(x) = \sqrt{x}, \quad x \leq 0.$$

By the Mean Value Theorem, there exists some $c \in (16, 17)$ such that

$$\begin{aligned}\sqrt{17} - \sqrt{16} &= \frac{f(17) - f(16)}{17 - 16} \\ &= f'(c) \\ &= \frac{1}{2\sqrt{c}} \\ &< \frac{1}{8}.\end{aligned}$$

c)

Let

$$f(x) = \frac{1}{x}, \quad x \neq 0.$$

By the Mean Value Theorem, there exists some $c \in (1000, 1002)$ such that

$$\begin{aligned}\frac{1}{1000} - \frac{1}{1002} &= f(1000) - f(1002) \\ &= -2 \cdot \frac{f(1002) - f(1000)}{1002 - 1000} \\ &= -2f'(c) \\ &= \frac{2}{x^2} \\ &< 2 \times 10^{-6}.\end{aligned}$$

Problem 10

e)

Because f is continuous on the closed interval $[0, 3]$, its maximum and minimum values must occur at a critical point on $[0, 3]$.

The critical points of f on $[0,3]$ are $x=0$, $x=1$, $x=3/2$, $x=2$ and $x=3$. Substituting these values into $f(x)$, we obtain the maximum value $f(0)=f(3)=2$, and the minimum value $f(1)=f(2)=0$.

Problem 17

$$p'(x) = 3(x-3)(x-5)\begin{cases} >0, & x<3 \text{ or } x>5 \\ <0, & 3<x<5 \end{cases}.$$

Hence, p is continuous on $\mathbb{R}$, monotonically increasing on $(-\infty,3)$ and $(5,\infty)$ and monotonically decreasing on $(3,5)$. Also, since $\lim_{x\to-\infty} p(x) < 0$ and $p(3)=3>0$, p has exactly one zero on the interval $(-\infty,3)$ (This can be easily proved by contradiction using Rolle's Theorem). For the same reason, p has exactly one zero when $x\in(3,5)$, and one zero when $x\in(5,\infty)$. Therefore, p has three zeroes on $\mathbb{R}$.

Problem 19

a)

$$\left|\int_0^3 2t - t^2 \, \mathrm{d}t\right| = \left|(t^2 - \frac{1}{3}t^3)\Big|_{t=0}^{t=3}\right| = 0.$$

b)

From question a, we know that at some point $t_0 > 0$, the velocity of the particle became 0, that is

$$2t_0 - t_0^2 = 0.$$

Solving for t_0, we have $t_0 = 2$. Hence, the total distance travelled is given by

$$2\left|\int_0^2 2t - t^2 \, \mathrm{d}t\right| = \frac{8}{3}.$$

Problem 21

b)

It is obvious that the limit goes to ∞, as exponential grows faster than polynomial. Alternatively, we can apply L'Hopital's Rule three times to see that the result diverges to infinity.

c)

$$\lim_{x\to-\infty} \frac{e^5}{3x^2} = 0.$$

Problem 24

Because $\lim_{x\to\infty}(4+\cos x)/(2-\cos x)$ does not exist.

Problem 25

It is easy to see that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1 = f(0),$$

which implies that f is continuous at 0. Also, noticing that

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{2h+1-1}{h} = 2 \\
\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{e^{2h}-1}{h} = 2,
\end{aligned}$$

that is, the left and right derivatives of f at 0 are equal, we can conclude that f is differentiable at 0.

Problem 27

a)

$$\begin{aligned}
\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\
&= -\lim_{x\to 0^+} x \\
&= 0.
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{x\to 0^+} x^2\ln x &= \lim_{x\to 0^+} x \cdot \lim_{x\to 0^+} x\ln x \\
&= 0\cdot 0 \\
&= 0.
\end{aligned}$$

c)

Firstly, f must be continuous at 0, which means

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

must hold. Hence, $b=0$.

In addition, the left and right derivatives of f at 0 must equal, that is,

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}.$$

To be more specific,

$$\lim_{h\to 0^-} \frac{ah+b-b}{h} = a = \lim_{h\to 0^+} \frac{h^2\ln h - b}{h},$$

which implies that

$$a = \lim_{h\to 0^+} h\ln h = 0.$$

Putting together, we have

$$a = b = 0.$$

Chapter 6

Problem 1

a)

$$(f \circ g)(x) = \sqrt{1 + \left(\sqrt{x^2 - 1}\right)^2} = x, \quad x \geq 1$$
$$(g \circ f)(x) = \sqrt{\left(\sqrt{1 + x^2}\right)^2 - 1} = x, \quad x \geq 0.$$

b)

The domain of $f \circ g$ and $g \circ f$ is $[1, \infty)$.

Problem 2

b)

$$g^{-1}(x) = -\sqrt{x - 1}, \quad x \in [1, \infty),$$
$$\mathrm{Dom}(g^{-1}) = [-1, \infty),$$
$$\mathrm{Range}(g^{-1}) = (-\infty, 0],$$
$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))} = -\frac{1}{2\sqrt{x - 1}}.$$

Problem 4

a)

$$f'(x) = 4 - \sin x > 0$$

shows that f is increasing on $\mathbb{R}$ and hence one-to-one. Therefore, f has an inverse function.

b)

Noticing that

$$f^{-1}(2\pi) = \frac{\pi}{2},$$

we obtain

$$g'(2\pi) = \frac{1}{f'((f^{-1})(2\pi))} = \frac{1}{f'(\pi/2)} = \frac{1}{3}.$$

Problem 5

a)

The derivative of f with respect to x is given by

$$f'(x) = 3(x+1)(x-1),$$

which implies that f is increasing on $(-\infty, -1]$ and $[1, \infty)$ and decreasing on $[-1, 1]$. Also, since f is continuous on $\mathbb{R}$, f is not one-to-one.

b)

The restriction of f to $(-\infty, -1]$ has an inverse with domain $(-\infty, 3]$, the restriction of f to $[-1, 1]$ has an inverse with domain $[-1, 3]$, and the restriction of f to $[1, \infty)$ has an inverse with domain $[-1, \infty)$.

Problem 7

b)

The inverse function is given by

$$f^{-1}(x) = x^{1/17} - 1, \quad x \in \mathbb{R}.$$

The domain of f^{-1} is $\mathbb{R}$. f is continuous on $\mathbb{R}$ and it is differentiable everywhere except at $x = 0$.

c)

To analyze f, we need to get rid of the modulus sign, which gives

$$f(x) = \begin{cases} x^2 + x, & x > 0 \text{ or } x < -1 \\ -x^2 - x, & -1 \leq x \leq 0 \end{cases} .$$

By taking the derivative of f, it is easy to see that f is increasing on $[-1, -1/2]$ and $[0, \infty)$, and decreasing on $(\infty, -1]$ and $[-1/2, 0]$. Hence, f is one-to-one on each of the four intervals.

Problem 8

b)

$$\cos(\cos^{-1}(2/5)) = 2/5.$$

c)

$$\cos^{-1}(\cos(-\pi/3)) = \pi/3.$$

f)

$$\sin(\tan^{-1}(3/5)) = 3/\sqrt{34}.$$

Problem 10

b)

$$\begin{aligned} y = \tan^{-1} x \implies & \tan y = x \\ \implies & \frac{1}{\cos^2 y} \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \\ \implies & \frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 y = \frac{1}{1+x^2}. \end{aligned}$$

Problem 13

a)

When $x \neq 0$,

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+1/x^2} \cdot \left(-\frac{1}{x^2} \right) \\ &= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\ &= 0. \end{aligned}$$

b)

From question a, we have $f'(x) = 0$ for $x \neq 0$, which means that $f(x)$ is constant on either $(-\infty, 0)$ or $(0, \infty)$. Plugging in $x \pm 1$, we obtain that $f(x) = \pi/2$ on $(0, \infty)$ and $f(x) = -\pi/2$ on $(-\infty, 0)$.

c)

This means that when $x \neq 0$, $\tan^{-1}(x)$ and $\tan^{-1}(1/x)$ are complementary.

Problem 18

Let the distance be x, then the angle θ that we want to maximize can be expressed by

$$\theta = \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{6}{x}\right) = f(x).$$

Taking the derivative of f with respect to x, we have

$$f'(x) = \frac{96 - 2x^2}{(x^2+64)(x^2+36)}.$$

We can see that $f'(x)$ is greater than zero when $0 \leq x < 4\sqrt{3}$ and is smaller than zero when $x > 4\sqrt{3}$, indicating that $f(x)$ reaches its maximum at $4\sqrt{3}$. Hence, the distance should be $4\sqrt{3}$ metres.

Chapter 7

Problem 4

b)

The vertical asymptote is $x = -1$. Noticing that

$$\lim_{x\to\infty} \big(f(x) - (x-1)\big) = \frac{1}{x+1} = 0,$$

we can identify the oblique asymptote being $y = x - 1$.

Problem 5

b)

Noticing that

$$y = \frac{x-1}{x-2} = 1 + \frac{1}{x-2}$$

is obtained by translating $y = 1/x$ to the right by two units, then up by one unit, hence all its features can be derived using $y = 1/x$.

Problem 7

c)

$$y = x^{2/3}.$$

Problem 9

a)

Taking the derivative of y with respect to x, we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}y/\mathrm{d}t} \\ &= \frac{-1/(t-1)^2}{1/(t+1)^2} \\ &= -\frac{(t+1)^2}{(t-1)^2}.\end{aligned}$$

The slope of the tangent line at $t = 2$ is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=2} = -9,$$

which implies that the slope of the normal at $t = 2$ is $1/9$. Hence, the normal is

$$y = \frac{1}{9}x + \frac{52}{27}.$$

b)

Expressing t in terms of x and y respectively, we have

$$t = \frac{x}{1-x} = \frac{y}{y-1}, \quad x, y \neq 1.$$

Rearranging the equation, we have

$$x + y - 2xy = 0, \quad x, y \neq 1.$$

Taking the derivative of both sides with respect to x and plugging in the values of x and y when $t = 2$, we obtain

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=2} = \left.\frac{2y-1}{1-2x}\right|_{t=2} = -9.$$

Hence, the gradient of the normal is $1/9$.

Problem 12

a)

$$\begin{aligned}\mathbf{p}(t) &= (1-t)\mathbf{a} + t\mathbf{b} \\ &= \mathbf{a} + (\mathbf{b} - \mathbf{a})t.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{p}(0) &= \mathbf{a} \\ \mathbf{p}(1) &= \mathbf{b}.\end{aligned}$$

b)

$$\begin{aligned}\mathbf{q}(t) &= (1-t)\mathbf{b} + t\mathbf{c} \\ &= \mathbf{b} + (\mathbf{c} - \mathbf{b})t \\ &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.\end{aligned}$$

Eliminating t, we obtain the Cartesian form

$$x + y - 4 = 0.$$

$$\mathbf{q}(1/2) = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}.$$

c)

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\mathbf{p}(t) + t\mathbf{q}(t)\\ &= (1-t)(\mathbf{a} + (\mathbf{b}-\mathbf{a})t) + t(\mathbf{b} + (\mathbf{c}-\mathbf{b})t)\\ &= (t-1)^2\mathbf{a} + 2t(1-t)\mathbf{b} + t^2\mathbf{c}.\end{aligned}$$

Equating the coefficients, we obtain

$$\begin{aligned}p_0(t) &= (t-1)^2\\ p_1(t) &= 2t(1-t)\\ p_2(t) &= t^2.\end{aligned}$$

Problem 16

a)

$$\begin{aligned}x &= r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta\\ y &= r\sin\theta = 6\sin^2\theta = 3 - 3\cos 2\theta.\end{aligned}$$

Hence,

$$x^2 + (y-3)^3 = (3\sin\theta)^2 + (3\cos\theta)^2 = 9,$$

which is a circle centered at $(0,3)$ with radius 3.

Problem 17

d)

$$x = r\cos\theta = 2|\cos\theta|\cos\theta = \begin{cases}1+\cos 2\theta, & \theta \in [0,\pi/2] \cup [3\pi/2, 2\pi]\\ -1-\cos 2\theta, & \theta \in [\pi/2, 3\pi/2]\end{cases},$$

$$y = r\sin\theta = 2|\cos\theta|\sin\theta = \begin{cases}\sin 2\theta, & \theta \in [0,\pi/2] \cup [3\pi/2, 2\pi]\\ -\sin 2\theta, & \theta \in [\pi/2, 3\pi/2]\end{cases}.$$

This is equivalent to

$$\begin{aligned}(x-1)^2 + y^2 &= 1, \quad x \in [0,2]\\ (x+1)^2 + y^2 &= 1, \quad x \in [-2,0].\end{aligned}$$

Therefore, the graph is two unit circles centered at $(1,0)$ and $(-1,0)$.

Problem 18

Since

$$x = r\cos\theta = \frac{a\cos\theta}{\theta},$$

$x \to \infty$ is equivalent to $\theta \to 0$. Hence,

$$\begin{aligned}
\lim_{x\to\infty}(y-1) &= \lim_{\theta\to 0}(r\sin\theta - 1) \\
&= \lim_{\theta\to 0}\left(\frac{a\sin\theta}{\theta} - 1\right) \\
&= 0,
\end{aligned}$$

which implies that $y = a$ is a horizontal asymptote to the spiral.

Chapter 8

Problem 1

a)

i)

$$\underline{S}_{\mathcal{P}_n}(f) = \overline{S}_{\mathcal{P}_n} = 1.$$

ii)

$$\underline{S}_{\mathcal{P}_n}(f) = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=0}^{n-1} k = \frac{n-1}{2n},$$
$$\overline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n+1}{2n}.$$

iii)

$$\underline{S}_{\mathcal{P}_n}(f) = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 = \frac{(n-1)(2n-1)}{6n^2},$$
$$\overline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{(n+1)(2n+1)}{6n^2}.$$

iv)

$$\underline{S}_{\mathcal{P}_n}(f) = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^4} \sum_{k=0}^{n-1} k^3 = \frac{(n-1)^2}{4n^2},$$
$$\overline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{(n+1)^2}{4n^2}.$$

v)

$$\underline{S}_{\mathcal{P}_n}(f) = 0,$$
$$S_{\mathcal{P}_n}(f) = 1.$$

2)

Beware that v) is not Riemann integrable.

Problem 4

Solving the following equations for x,

$$\begin{aligned} y &= x \\ y &= x^2 - 2, \end{aligned}$$

we obtain the start and end of the integral, namely, $x = -1$ and $x = 2$. Hence, the area is given by

$$\int_{-1}^{2} x - (x^2 - 2) = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x\right]_{-1}^{2} = \frac{9}{2}.$$

Problem 5

b)

$$\begin{aligned} \int_{-4}^{2} |x| \,\mathrm{d}x &= \int_{-4}^{0} -x \,\mathrm{d}x + \int_{0}^{2} x \,\mathrm{d}x \\ &= \left[-\frac{1}{2}x^2\right]_{-4}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{2} \\ &= 10. \end{aligned}$$

Problem 7

$F(x) = -1/x$ is not differentiable at $x = 0$, hence the Second Fundamental Theorem of Calculus does not apply.

Problem 11

Skipped.

Problem 13

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \int_{x}^{4} (5 - 4t)^5 \,\mathrm{d}t &= -\frac{\mathrm{d}}{\mathrm{d}x} \int_{4}^{x} (5 - 4t)^5 \,\mathrm{d}t \\ &= -(5 - 4t)^5. \end{aligned}$$

Problem 15

d)

$$\begin{aligned}\int_{-a}^{a} x^2 \sqrt{a^3 - x^3}\,\mathrm{d}x &= \left[-\frac{2}{9}(a^3 - x^3)^{3/2} \right]_{-a}^{a} \\ &= \frac{4\sqrt{2}a^{9/2}}{9}.\end{aligned}$$