MATH1241 Problem Set Solutions - Algebra

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Contents

Chapter 6

Vector Spaces

6.1 Problem 1

For $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in S$, $(-1)\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin S$, which implies that S is not closed under scalar multiplication. Hence, S is not a vector space.

6.2 Problem 2

For $\mathbf{x} = \begin{pmatrix} 8 \\ 0 \\ 2 \end{pmatrix} \in S$, $(-1)\mathbf{x} = \begin{pmatrix} -8 \\ 0 \\ -2 \end{pmatrix} \notin S$ because $2x_1 + 3x_2^3 - 4x_3^2 = -32 \neq 0$, which implies that S is not closed under scalar multiplication. Hence, S is not a vector space.

6.3 Problem 3

a)

For example, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

b)

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \in S$, but $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \notin S$ because $(3-2) \times 1 \neq 0$, which implies that S is not closed under vector addition.

6.4 Problem 4

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ and $\lambda, \mu \in \mathbb{C}$.

For axiom 1, because $\mathbb{C}$ is closed under addition, we have

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{pmatrix} \in \mathbb{C}^n,$$

which shows that axiom 1 is satisfied.

For axiom 2, because addition in $\mathbb{C}$ is associative, we have

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= \left(\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right) + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} u_1+v_1+w_1 \\ \vdots \\ u_n+v_n+w_n \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right) \\ &= \mathbf{u}+(\mathbf{v}+\mathbf{w}),\end{aligned}$$

which shows that axiom 2 is satisfied.

For axiom 6, because $\mathbb{C}$ is closed under scalar multiplication, we have

$$\lambda\mathbf{v} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \in \mathbb{C}^n,$$

which shows that axiom 6 is satisfied.

For axiom 9, because of the distributive law in $\mathbb{C}$, we have

$$\begin{aligned}(\lambda+\mu)\mathbf{v} &= (\lambda+\mu) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} (\lambda+\mu)v_1 \\ \vdots \\ (\lambda+\mu)v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1+\mu v_1 \\ \vdots \\ \lambda v_n+\mu v_n \end{pmatrix} \\ &= \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} + \begin{pmatrix} \mu v_1 \\ \vdots \\ \mu v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ &= \lambda\mathbf{v}+\mu\mathbf{v},\end{aligned}$$

which shows that axiom 9 is satisfied.

6.5 Problem 5

Suppose $A, B \in M_{mn}(\mathbb{C})$ and $\lambda \in \mathbb{C}$.

For axiom 1, because $\mathbb{C}$ is closed under addition, we have

$$a_{ij}+b_{ij} \in \mathbb{C}, \quad 1 \le i \le m \text{ and } 1 \le j \le n.$$

Hence, $A+B \in M_{mn}(\mathbb{C})$, which shows that axiom 1 is satisfied.

For axiom 3, because $\mathbb{C}$ is commutative, we have

$$a_{ij}+b_{ij} = b_{ij}+a_{ij}, \quad 1 \le i \le m \text{ and } 1 \le j \le n.$$

Hence, $A+B=B+A$, which shows that axiom 3 is satisfied.

For axiom 6, because $\mathbb{C}$ is closed under scalar multiplication, we have

$$\lambda a_{ij} \in \mathbb{C}, \quad 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

Hence, $\lambda A \in M_{mn}(\mathbb{C})$, which shows that axiom 6 is satisfied.

For axiom 10, because of the distributive law in $\mathbb{C}$, we have

$$\lambda(a_{ij}+b_{ij}) = \lambda a_{ij} + \lambda b_{ij}, \quad 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

Hence, $\lambda(A+B) = \lambda A + \lambda B$, which shows that axiom 10 is satisfied.

6.6 Problem 6

It is easy to see that $(\mathbb{C}^n, +, *, \mathbb{R})$ is a vector space because $\mathbb{R}$ is a subfield of $\mathbb{C}$ and $(\mathbb{C}^n, +, *, \mathbb{C})$ is a vector space.

For $(\mathbb{R}^n, +, *, \mathbb{C})$, let $\mathbf{x} \in \mathbb{R}^n$ with all entries being 1, then $i\mathbf{x}$ is a vector with entries being i and is not in $\mathbb{R}^n$. Hence, the system is not closed under scalar multiplication, which implies that it is not a vector space.

6.7 Problem 7

This system is not a vector space.