MATH1241 Problem Set Solutions - Algebra

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Contents

Chapter 6

Vector Spaces

6.5 Problem 5

Denote the ijth entry of M by $[M]_{ij}$ where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. For any matrices $A, B \in M_{mn}(\mathbb{C})$ and scalars $\lambda \in \mathbb{C}$.

For axiom 1, because $\mathbb{C}$ is closed under addition, we have

$$[A]_{ij} + [B]_{ij} \in \mathbb{C}, \quad \text{for all} \quad i, j.$$

Hence, $A + B \in M_{mn}(\mathbb{C})$, which shows that axiom 1 is satisfied.

For axiom 3, because $\mathbb{C}$ is commutative, we have

$$[A]_{ij} + [B]_{ij} = [B]_{ij} + [A]_{ij}, \quad \text{for all} \quad i, j.$$

Hence, $A + B = B + A$, which shows that axiom 3 is satisfied.

For axiom 6, because $\mathbb{C}$ is closed under scalar multiplication, we have

$$\lambda [A]_{ij} \in \mathbb{C}, \quad \text{for all} \quad i, j.$$

Hence, $\lambda A \in M_{mn}(\mathbb{C})$, which shows that axiom 6 is satisfied.

For axiom 10, because of the distributive law in $\mathbb{C}$, we have

$$\lambda([A]_{ij} + [B]_{ij}) = \lambda [A]_{ij} + \lambda [B]_{ij}, \quad \text{for all} \quad i, j.$$

Hence, $\lambda(A + B) = \lambda A + \lambda B$, which shows that axiom 10 is satisfied.

6.7 Problem 7

This system is not a vector space.

6.27 Problem 27

a)

Let W' be the intersection of $\{W_k : 1 \leqslant k \leqslant m + 1\}$. We prove this by induction.

For $m = 1$, we have $W = W_1$, which is a subspace of V.

Suppose that for $m > 1$, W is a subspace of V. Then, for $m + 1$, since $W \leqslant V$ and $W_{m+1} \leqslant V$, we have $\mathbf{0} \in W$ and $\mathbf{0} \in W_{m+1}$, and hence, $\mathbf{0} \in W'$. For any vectors $\mathbf{u}, \mathbf{v} \in W'$ and scalars $\lambda, \mu \in \mathbb{F}$, since $W' = W \cap W_{m+1}$, $\mathbf{u}$ and $\mathbf{v}$ must be in both W and W_{m+1}. Also, since W and W_{m+1} are subspaces of V, they are closed under addition and multiplication by scalars from $\mathbb{F}$. Thus, $\lambda\mathbf{u} + \mu\mathbf{v}$ must be in both W and W_{m+1}, and hence in W'. By the alternative Subspace Theorem, W' is a subspace of V.

Therefore, by induction, W is a subspace of V.

b)

Suppose that W is not the set of finite linear combinations of vectors from S. Then, $\exists \mathbf{x} \in W$ such that $\mathbf{x} \notin \text{span}(S)$. However, for any $V_i \leqslant V$ and $V_i \supseteq S$, we have $\text{span}(S) \leqslant V_i$, implying that $\mathbf{x} \notin V_i$, and hence $\mathbf{x} \notin W$, which is a contradiction. Therefore, W is the set of finite linear combinations of vectors from S.

6.36 Problem 36

This problem is equivalent to proving that $\text{span}(S)$ is a subspace of V over field $\mathbb{F}$.

Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\} \subseteq V$.

The zero vector of V is in $\text{span}(S)$ because $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$.

For any vectors $\mathbf{u} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n, \mathbf{v} = \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n \in \text{span}(S)$ where $\lambda_1, ..., \lambda_n$, $\mu_1, ..., \mu_n \in \mathbb{F}$ and any scalar $\lambda \in \mathbb{F}$, we have

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n) + (\mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n) \\ &= (\lambda_1 + \mu_1)\mathbf{v}_1 + \cdots + (\lambda_n + \mu_n)\mathbf{v}_n,\end{aligned}$$

where $(\lambda_1 + \mu_1), ..., (\lambda_n + \mu_n) \in \mathbb{F}$. Thus, $(\mathbf{u} + \mathbf{v}) \in \text{span}(S)$, which implies that $\text{span}(S)$ is closed under addition.

Also, since

$$\begin{aligned}\lambda\mathbf{v} &= \lambda(\mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n) \\ &= (\lambda\mu_1)\mathbf{v}_1 + \cdots + (\lambda\mu_n)\mathbf{v}_1,\end{aligned}$$

where $(\lambda\mu_1), ..., (\lambda\mu_n) \in \mathbb{F}$, we have $(\lambda\mathbf{v}) \in \text{span}(S)$, which implies that $\text{span}(S)$ is also closed under multiplication by a scalar.

Therefore, by the Subspace Theorem, $\text{span}(S)$ is a subspace of V, and hence, the original statement is proved.

6.37 Problem 37

We prove this by induction.

For $n = 1$, $\sum_{k=1}^{1} \lambda_k\mathbf{v}_k = \lambda_1\mathbf{v}_1$.

Suppose that for $n > 1$, $\sum_{k=1}^{n} \lambda_k\mathbf{v}_k = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n$ holds regardless of the order. By the closures under addition and multiplication by a scalar, $\sum_{k=1}^{n} \lambda_k\mathbf{v}_k$ is in the vector space. Hence, for $n + 1$, by the commutative law of addition, we have $\sum_{k=1}^{n+1} \lambda_k\mathbf{v}_k = \sum_{k=1}^{n} \lambda_k\mathbf{v}_k + \lambda_{n+1}\mathbf{v}_{n+1} = \lambda_{n+1}\mathbf{v}_{n+1} + \sum_{k=1}^{n} \lambda_k\mathbf{v}_k$ regardless of the order.

Therefore, by induction, we proved that we do not need to use brackets when writing down linear combinations.