MATH1241 Problem Set Solutions - Algebra

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Contents

Chapter 6

Vector Spaces

6.5 Problem 5

Denote the ijth entry of M by $[M]_{ij}$ where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. For any matrices $A, B \in M_{mn}(\mathbb{C})$ and scalars $\lambda \in \mathbb{C}$.

For axiom 1, because $\mathbb{C}$ is closed under addition, we have

$$[A]_{ij} + [B]_{ij} \in \mathbb{C}, \quad \text{for all} \quad i, j.$$

Hence, $A + B \in M_{mn}(\mathbb{C})$, which shows that axiom 1 is satisfied.

For axiom 3, because $\mathbb{C}$ is commutative, we have

$$[A]_{ij} + [B]_{ij} = [B]_{ij} + [A]_{ij}, \quad \text{for all} \quad i, j.$$

Hence, $A + B = B + A$, which shows that axiom 3 is satisfied.

For axiom 6, because $\mathbb{C}$ is closed under scalar multiplication, we have

$$\lambda [A]_{ij} \in \mathbb{C}, \quad \text{for all} \quad i, j.$$

Hence, $\lambda A \in M_{mn}(\mathbb{C})$, which shows that axiom 6 is satisfied.

For axiom 10, because of the distributive law in $\mathbb{C}$, we have

$$\lambda([A]_{ij} + [B]_{ij}) = \lambda [A]_{ij} + \lambda [B]_{ij}, \quad \text{for all} \quad i, j.$$

Hence, $\lambda(A + B) = \lambda A + \lambda B$, which shows that axiom 10 is satisfied.

6.7 Problem 7

This system is not a vector space.

6.27 Problem 27

a)

Let W' be the intersection of $\{W_k : 1 \leqslant k \leqslant m + 1\}$. We prove this by induction.

For $m = 1$, we have $W = W_1$, which is a subspace of V.

Suppose that for $m > 1$, W is a subspace of V. Then, for $m + 1$, since $W \leqslant V$ and $W_{m+1} \leqslant V$, we have $\mathbf{0} \in W$ and $\mathbf{0} \in W_{m+1}$, and hence, $\mathbf{0} \in W'$. For any vectors $\mathbf{u}, \mathbf{v} \in W'$ and scalars $\lambda, \mu \in \mathbb{F}$, since $W' = W \cap W_{m+1}$, $\mathbf{u}$ and $\mathbf{v}$ must be in both W and W_{m+1}. Also, since W and W_{m+1} are subspaces of V, they are closed under addition and multiplication by scalars from $\mathbb{F}$. Thus, $\lambda\mathbf{u} + \mu\mathbf{v}$ must be in both W and W_{m+1}, and hence in W'. By the alternative Subspace Theorem, W' is a subspace of V.

Therefore, by induction, W is a subspace of V.

b)

Suppose that W is not the set of finite linear combinations of vectors from S. Then, $\exists \mathbf{x} \in W$ such that $\mathbf{x} \notin \mathrm{span}(S)$. However, for any $V_i \leqslant V$ and $V_i \supseteq S$, we have $\mathrm{span}(S) \leqslant V_i$, implying that $\mathbf{x} \notin V_i$, and hence $\mathbf{x} \notin W$, which is a contradiction. Therefore, W is the set of finite linear combinations of vectors from S.