MATH1241 Problem Set Solutions - Algebra

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Contents

Chapter 6

Vector Spaces

6.4 Problem 4

Closure under Addition

For any vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n$, because $\mathbb{C}$ is closed under addition, we have

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \in \mathbb{C}^n.$$

Thus, $\mathbb{C}^n$ is closed under addition.

Associative Law of Addition

For any vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n$, because $\mathbb{C}$ is associative, we have

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \left(\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right) + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 + w_1 \\ \vdots \\ u_n + v_n + w_n \end{pmatrix} = \begin{pmatrix} u_1 + (v_1 + w_1) \\ \vdots \\ u_n + (v_n + w_n) \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right) = \mathbf{u} + (\mathbf{v} + \mathbf{w}). \end{aligned}$$

Thus, $\mathbb{C}^n$ is associative.

Closure under Multiplication by a Scalar

For any vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n$ and any scalar $\lambda \in \mathbb{C}$, since $\mathbb{C}$ is closed under multiplication by a scalar, we have

$$\lambda \mathbf{v} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \in \mathbb{C}^n.$$

Thus, $\mathbb{C}^n$ is closed under multiplication by a scalar.

Scalar Distributive Law

For any vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n$ and any scalar $\lambda, \mu \in \mathbb{C}$, due to the scalar distributive law of $\mathbb{C}$, we have

$$(\lambda + \mu)\mathbf{v} = (\lambda + \mu) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} (\lambda + \mu) v_1 \\ \vdots \\ (\lambda + \mu) v_n \end{pmatrix} \in \mathbb{C}^n.$$

Thus, the scalar distributive law holds for $\mathbb{C}^n$.

6.5 Problem 5

Denote the ijth entry of M by $[M]_{ij}$ where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. For any matrices $A, B \in M_{mn}(\mathbb{C})$ and scalars $\lambda \in \mathbb{C}$.

For axiom 1, because $\mathbb{C}$ is closed under addition, we have

$$[A]_{ij} + [B]_{ij} \in \mathbb{C}, \quad \text{for all} \quad i, j.$$

Hence, $A + B \in M_{mn}(\mathbb{C})$, which shows that axiom 1 is satisfied.

For axiom 3, because $\mathbb{C}$ is commutative, we have

$$[A]_{ij} + [B]_{ij} = [B]_{ij} + [A]_{ij}, \quad \text{for all} \quad i, j.$$

Hence, $A + B = B + A$, which shows that axiom 3 is satisfied.

For axiom 6, because $\mathbb{C}$ is closed under scalar multiplication, we have

$$\lambda [A]_{ij} \in \mathbb{C}, \quad \text{for all} \quad i, j.$$

Hence, $\lambda A \in M_{mn}(\mathbb{C})$, which shows that axiom 6 is satisfied.

For axiom 10, because of the distributive law in $\mathbb{C}$, we have

$$\lambda([A]_{ij} + [B]_{ij}) = \lambda [A]_{ij} + \lambda [B]_{ij}, \quad \text{for all} \quad i, j.$$

Hence, $\lambda(A + B) = \lambda A + \lambda B$, which shows that axiom 10 is satisfied.

6.6 Problem 6

It is easy to prove that $(\mathbb{C}^n, +, *, \mathbb{R})$ is a vector space because $\mathbb{R}$ is a subfield of $\mathbb{C}$. It is also easy to see that $(\mathbb{R}^n, +, *, \mathbb{C})$ is not a vector space because, for example, the closure under multiplication by a scalar does not hold.

6.7 Problem 7

This system is not a vector space.

6.8 Problem 8

a)

$$2\mathbf{v} = (1+1)\mathbf{v} = 1\mathbf{v} + 1\mathbf{v} = \mathbf{v} + \mathbf{v}.$$

b)

This can be proved by induction.

6.9 Problem 9

Multiplication of the Zero Vector

$$\lambda\mathbf{0} + \mathbf{0} = \lambda\mathbf{0} = \lambda(\mathbf{0} + \mathbf{0}) = \lambda\mathbf{0} + \lambda\mathbf{0}.$$

By the cancellation property, we obtain $\lambda\mathbf{0} = \mathbf{0}$.

Zero Products

If $\lambda = 0$, by the property of multiplication of the zero vector, we have $\lambda\mathbf{v} = \mathbf{0}$. If $\lambda \neq 0$, then $\lambda^{-1} \neq 0$, and hence,

$$\mathbf{v} = (\lambda^{-1}\lambda)\mathbf{v} = \lambda^{-1}(\lambda\mathbf{v}) = \lambda^{-1}\mathbf{0} = \mathbf{0}.$$

Cancellation Property

If $\lambda\mathbf{v} = \mu\mathbf{v}$, then $(\lambda - \mu)\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$, by the property of zero products, we obtain $\lambda - \mu = 0$, that is, $\lambda = \mu$.

6.23 Problem 23

No, because the zero polynomial of $\mathbb{P}_3$ is not in S.

6.27 Problem 27

a)

Let W' be the intersection of $\{W_k : 1 \leqslant k \leqslant m+1\}$. We prove this by induction.

For $m = 1$, we have $W = W_1$, which is a subspace of V.

Suppose that for $m > 1$, W is a subspace of V. Then, for $m+1$, since $W \leqslant V$ and $W_{m+1} \leqslant V$, we have $\mathbf{0} \in W$ and $\mathbf{0} \in W_{m+1}$, and hence, $\mathbf{0} \in W'$. For any vectors $\mathbf{u}, \mathbf{v} \in W'$ and scalars $\lambda, \mu \in \mathbb{F}$, since $W' = W \cap W_{m+1}$, $\mathbf{u}$ and $\mathbf{v}$ must be in both W and W_{m+1}. Also, since W and W_{m+1} are subspaces of V, they are closed under addition and multiplication by scalars from $\mathbb{F}$. Thus, $\lambda\mathbf{u} + \mu\mathbf{v}$ must be in both W and W_{m+1}, and hence in W'. By the alternative Subspace Theorem, W' is a subspace of V.

Therefore, by induction, W is a subspace of V.

b)

Suppose that W is not the set of finite linear combinations of vectors from S. Then, $\exists \mathbf{x} \in W$ such that $\mathbf{x} \notin \text{span}(S)$. However, for any $V_i \leqslant V$ and $V_i \supseteq S$, we have $\text{span}(S) \leqslant V_i$, implying that $\mathbf{x} \notin V_i$, and hence $\mathbf{x} \notin W$, which is a contradiction. Therefore, W is the set of finite linear combinations of vectors from S.

6.36 Problem 36

This problem is equivalent to proving that $\text{span}(S)$ is a subspace of V over field $\mathbb{F}$.

Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\} \subseteq V$.

The zero vector of V is in $\text{span}(S)$ because $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$.

For any vectors $\mathbf{u} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n, \mathbf{v} = \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n \in \text{span}(S)$ where $\lambda_1, ..., \lambda_n, \mu_1, ..., \mu_n \in \mathbb{F}$ and any scalar $\lambda \in \mathbb{F}$, we have

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n) + (\mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n) \\ &= (\lambda_1 + \mu_1)\mathbf{v}_1 + \cdots + (\lambda_n + \mu_n)\mathbf{v}_n,\end{aligned}$$

where $(\lambda_1 + \mu_1), ..., (\lambda_n + \mu_n) \in \mathbb{F}$. Thus, $(\mathbf{u} + \mathbf{v}) \in \text{span}(S)$, which implies that $\text{span}(S)$ is closed under addition.

Also, since

$$\begin{aligned}\lambda\mathbf{v} &= \lambda(\mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n) \\ &= (\lambda\mu_1)\mathbf{v}_1 + \cdots + (\lambda\mu_n)\mathbf{v}_1,\end{aligned}$$

where $(\lambda\mu_1), ..., (\lambda\mu_n) \in \mathbb{F}$, we have $(\lambda\mathbf{v}) \in \text{span}(S)$, which implies that $\text{span}(S)$ is also closed under multiplication by a scalar.

Therefore, by the Subspace Theorem, $\text{span}(S)$ is a subspace of V, and hence, the original statement is proved.

6.37 Problem 37

We prove this by induction.

For $n = 1$, $\sum_{k=1}^{1} \lambda_k\mathbf{v}_k = \lambda_1\mathbf{v}_1$.

Suppose that for $n > 1$, $\sum_{k=1}^{n} \lambda_k\mathbf{v}_k = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n$ holds regardless of the order. By the closures under addition and multiplication by a scalar, $\sum_{k=1}^{n} \lambda_k\mathbf{v}_k$ is in the vector space. Hence, for $n+1$, by the commutative law of addition, we have $\sum_{k=1}^{n+1} \lambda_k\mathbf{v}_k = \sum_{k=1}^{n} \lambda_k\mathbf{v}_k + \lambda_{n+1}\mathbf{v}_{n+1} = \lambda_{n+1}\mathbf{v}_{n+1} + \sum_{k=1}^{n} \lambda_k\mathbf{v}_k$ regardless of the order.

Therefore, by induction, we proved that we do not need to use brackets when writing down linear combinations.

6.46 Problem 46

For

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_m\mathbf{v}_m = \mathbf{0},$$

multiplying $\dfrac{\mathbf{v}_i}{\|\mathbf{v}_i\|^2}$ on both sides of the equation, since S is orthogonal, we have

$$\lambda_i = \lambda_i \mathbf{v}_i \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|^2} = (\lambda_1 \mathbf{v}_1 + \cdots + \lambda_m \mathbf{v}_m) \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|^2} = \mathbf{0} \cdot \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|^2} = 0, \quad 1 \leqslant i \leqslant m,$$

which implies that $\lambda_1, ..., \lambda_m$ are all zero. Hence, S is a linearly independent set.

6.52 Problem 52

Because by the Rank-nullity Theorem, the rank cannot exceed the number of columns.

6.59 Problem 59

Performing Gaussian Elimination on the matrix whose columns are the vector representations of the polynomials, we have

$$\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & -1 & -4 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, S is a linearly dependent spanning set for $\mathbb{P}_2$. A subset of S which is a basis for $\mathbb{P}_2$ can be $\{p_1, p_2\}$.

6.60 Problem 60

Since $\mathbf{w} \in \operatorname{span}(S)$, $\mathbf{w}$ is some linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Hence, by the definition of linear dependence, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}\}$ is a linearly dependent set.

6.61 Problem 61

Here gives a rather informal proof. For any subspace V of $\mathbb{R}^4$, we must have $0 \leqslant \dim V \leqslant 4$. However, the given subspaces have already covered all the subspaces of dimensions from 0 to 4. Hence, the given subspaces are the only subspaces of $\mathbb{R}^4$.

6.62 Problem 62

$\det(A) = 3$ gives that the columns of A are linearly independent, and hence form a basis for $\mathbb{R}^4$.

Finding the coordinate vector of $\mathbf{v}$ is equivalent to solving $A\mathbf{x} = \mathbf{v}$ for $\mathbf{x}$. Performing Gaussian elimination on the augmented matrix $[A|\mathbf{v}]$, we have

$$\begin{pmatrix} 1 & 2 & -1 & 1 & -2 \\ 3 & 2 & 0 & -2 & -6 \\ 0 & 1 & -1 & 1 & -4 \\ 5 & 3 & 0 & -1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 1 & -2 \\ 0 & -4 & 3 & -5 & 0 \\ 0 & 0 & 1 & 1 & 16 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Using back substitution, we obtain

$$[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix} -2 \\ 4 \\ 12 \\ 4 \end{pmatrix}.$$

6.63 Problem 63

$$\mathbf{v} = A\begin{pmatrix}1\\6\\-1\\4\end{pmatrix} = \begin{pmatrix}1 & 2 & -1 & 1\\3 & 2 & 0 & -2\\0 & 1 & -1 & 1\\5 & 3 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\6\\-1\\4\end{pmatrix} = \begin{pmatrix}18\\7\\11\\19\end{pmatrix}.$$

6.64 Problem 64

$$\mathbf{v} = \begin{pmatrix}1 & 3 & 2\\2 & 7 & 4\\-2 & -5 & 9\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\10\end{pmatrix}.$$

6.65 Problem 65

a)

Solving

$$\mathbf{v} = \begin{pmatrix}1 & 1 & -2\\0 & 1 & 0\\1 & 1 & -1\end{pmatrix}[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix}1\\2\\3\end{pmatrix},$$

for $[\mathbf{v}]_{\mathscr{B}}$, we obtain

$$[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix}3\\2\\2\end{pmatrix}.$$

b)

Solving

$$\mathbf{v} = \begin{pmatrix}1 & 1 & -2\\0 & 1 & 0\\1 & 1 & -1\end{pmatrix}[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},$$

for $[\mathbf{v}]_{\mathscr{B}}$, we obtain

$$[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix}-a_1 - a_2 + 2a_3\\a_2\\-a_1 + a_3\end{pmatrix}.$$

6.66 Problem 66

a)

$$\mathbf{v} = B[\mathbf{v}]_B = \begin{pmatrix}1 & 3 & -2\\-1 & 4 & 3\\2 & 6 & -3\end{pmatrix}\begin{pmatrix}3\\1\\-3\end{pmatrix} = \begin{pmatrix}12\\-8\\21\end{pmatrix}.$$

b)

Solving

$$\mathbf{v} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 4 & 3 \\ 2 & 6 & -3 \end{pmatrix} [\mathbf{w}]_B = \begin{pmatrix} 7 \\ -3 \\ 11 \end{pmatrix}$$

for $[\mathbf{w}]_B$, we obtain

$$[\mathbf{w}]_w = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}.$$

6.67 Problem 67

a)

It is easy to see that $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = 1$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_3 \cdot \mathbf{v}_1 = 0$. Hence, S is an orthonormal set of vectors in $\mathbb{R}^3$.

b)

Let A be a 3 by 3 matrix whose columns are the vectors in S. Then, S is linearly independent because $\det(A) = 1/6 \neq 0$. Also, since $|S| = 3$, we have that S is a basis for $\mathbb{R}^3$.

c)

Let

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R}.$$

Since S is an orthonormal set, we have

$$\begin{aligned} x_1 &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) \cdot \mathbf{v}_1 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_1 = -2\sqrt{2}, \\ x_2 &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) \cdot \mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_2 = 2\sqrt{3}, \\ x_3 &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) \cdot \mathbf{v}_3 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_3 = \sqrt{6}. \end{aligned}$$

Hence,

$$\left[\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}\right]_S = \begin{pmatrix} -2\sqrt{2} \\ 2\sqrt{3} \\ \sqrt{6} \end{pmatrix}.$$

Problem 68

Suppose that

$$x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n = \mathbf{0}, \quad x_1, ..., x_n \in \mathbb{R}. \tag{*}$$

Since S is an orthonormal set, for any $x_j \in \{x_1, ..., x_n\}$, we have

$$x_j = (x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n) \cdot \mathbf{u}_j = \mathbf{0} \cdot \mathbf{u}_j = 0.$$

Hence, (*) holds only when $x_1 = \cdots = x_n = 0$, which implies that S is linearly independent. Also, since $|S| = n$, we have that S is a basis for $\mathbb{R}^n$. Further,

$$[\mathbf{v}]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_j = \mathbf{u}_j \cdot \mathbf{v}.$$