MATH1141 Tutorial Solutions - Algebra

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Chapter 1

Problem 31

a)

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

b)

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Problem 34

b)

Parametric vector form:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

From the parametric vector form, we have the following equations

$$\begin{aligned} x_1 &= 1 + 4\lambda \\ x_2 &= 2 - 5\lambda \\ x_3 &= -3 + 6\lambda. \end{aligned}$$

Expressing λ in terms of x, y and z respectively, we obtain the Cartesian form

$$\frac{x_1 - 1}{4} = \frac{x_2 - 2}{-5} = \frac{x_3 + 3}{6}.$$

c)

Parametric form:

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}.$$

Similar to question b, the Cartesian form is given by

$$\frac{x_1 - 1}{5} = -x_2 - 1 = \frac{x_3 - 1}{2}.$$

Problem 36

a)

True. Because

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 4 \\ 12 \end{pmatrix}.$$

b)

False. The line in parametric vector form can be written as

$$2x - 3y - 9 = 0.$$

By comparing the tangents of the two lines, it is easy to see that they are not parallel.

c)

True. Let

$$\frac{x + 10}{5} = y - 7 = \frac{z + 3}{4} = \mu, \quad \mu \in \mathbb{R}.$$

Then, the parametric vector form is given by

$$\mathbf{x} = \begin{pmatrix} -10 \\ 7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}.$$

Because $2 \cdot (5\ 1\ 4)^{\mathrm{T}} = (10\ 2\ 8)^{\mathrm{T}}$, the two lines are parallel.

d)

True. The parametric vector form of the latter line is given by

$$\mathbf{x} = \begin{pmatrix} -10 \\ -5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}, \quad \mu \in \mathbb{R}.$$

Similar to question c, since $2 \cdot (5\ 0\ -2)^{\mathrm{T}} = (10\ 0\ -4)^{\mathrm{T}}$, the two lines are parallel.

Problem 39

a)

Picking two non-parallel vectors parallel to the plane

$$\mathbf{u} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$
$$\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix},$$

the parametric vector form can be given by

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \mathbf{u} + \mu \mathbf{v} = \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix}$$

where $\lambda, \mu \in \mathbb{R}$.

b)

Similar to question a, by picking

$$\mathbf{u} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$
$$\mathbf{v} = \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -14 \\ 5 \end{pmatrix},$$

we obtain one possible parametric vector form as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -14 \\ 5 \end{pmatrix}$$

where $\lambda, \mu \in \mathbb{R}$.

Problem 40

a)

A plane through the origin parallel to $(1\ 2\ 3)^{\mathrm{T}}$ and $(-2\ 3\ 4)^{\mathrm{T}}$.

b)

Noticing that $-2 \cdot (-2\ 1\ 3\ 2)^{\mathrm{T}} = (4\ -2\ -6\ -4)^{\mathrm{T}}$, it is a line through the point $(3, 1, 2, 4)$ parallel to $(-2\ 1\ 3\ 2)^{\mathrm{T}}$.

c)

Noticing that $-3 \cdot (3\ 2\ 1\ 2)^{\mathrm{T}} = (-9\ -6\ -3\ -6)^{\mathrm{T}}$, it is a line through the origin parallel to $(3\ 2\ 1\ 2)^{\mathrm{T}}$.

d)

A plane through the point $(1, 2, 3)$ parallel to $(4 \ -1 \ 2)^{\mathrm{T}}$ and $(8 \ 2 \ 4)^{\mathrm{T}}$.

Problem 41

a)

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

b)

Picking two non-parallel vectors parallel to the plane

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$$
$$\mathbf{v} = \begin{pmatrix} 6 \\ 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix},$$

the parametric vector form is given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

d)

Dividing 12 on both sides of the equation, we have

$$\frac{x_1}{3} - \frac{x_2}{4} + \frac{x_3}{2} = 1.$$

Hence, the plane is through the point $(3, 4, 2)$.

Let $x_2 = \lambda$ and $x_3 = \mu$ where $\lambda, \mu \in \mathbb{R}$, then the parametric vector form can be written as

$$\mathbf{x} = \begin{pmatrix} 3 + \frac{3}{4}\lambda - \frac{3}{2}\mu \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}.$$

f)

Let

$$\frac{x_1 - 5}{7} = \frac{x_2 + 6}{2} = \frac{x_3 - 2}{-3} = \frac{x_4 + 1}{-5} = \mu, \quad \mu \in \mathbb{R},$$

The parametric vector form of the second line can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 + 7\mu \\ -6 + 2\mu \\ 2 - 3\mu \\ -1 - 5\mu \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 2 \\ -3 \\ -5 \end{pmatrix}.$$

Hence, the parametric vector form of the plane is given by

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 0 \\ -4 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 7 \\ 2 \\ -3 \\ -5 \end{pmatrix}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

Problem 44

b)

Method 1

$$\begin{aligned} 0 &= 9x + 4y - z \\ &= 9(-1 + 2\lambda) + 4(2 - 3\lambda) - (3 + 4\lambda) \\ &= -4 + 2\lambda \end{aligned}$$

where $\lambda \in \mathbb{R}$. Solving for λ, we have $\lambda = 2$. Plugging into the parametric vector form of the line, we obtain the point of the intersection $(3, -4, 11)$.

Method 2

The Cartesian form of the plane is given by

$$\mathbf{p} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 9\lambda_1 + 4\lambda_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Hence, the intersection can be found by solving the following equation

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \mathbf{x} = \mathbf{p} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix},$$

which gives the same result as Method 1.

Chapter 2

Problem 1

a)

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)\\ &= \arccos\left(\frac{6}{6\sqrt{2}}\right)\\ &= \frac{\pi}{4}.\end{aligned}$$

Problem 2

b)

$$\cos\angle CAB = \frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|\overrightarrow{AC}||\overrightarrow{AB}|} = \frac{\begin{pmatrix}3\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix}}{\left|\begin{pmatrix}3\\-1\\1\end{pmatrix}\right|\left|\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right|} = -\frac{5\sqrt{33}}{33},$$

$$\cos\angle ABC = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\2\end{pmatrix}}{\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\right|\left|\begin{pmatrix}4\\-2\\2\end{pmatrix}\right|} = \frac{2\sqrt{2}}{3},$$

$$\cos\angle ACB = \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|} = \frac{\begin{pmatrix}-3\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\-2\end{pmatrix}}{\left|\begin{pmatrix}-3\\1\\-1\end{pmatrix}\right|\left|\begin{pmatrix}-4\\2\\-2\end{pmatrix}\right|} = \frac{4\sqrt{66}}{33}.$$

Problem 3

$$\theta = \arccos \frac{\overrightarrow{OF} \cdot \overrightarrow{AG}}{|\overrightarrow{OF}||\overrightarrow{AG}|} = \arccos \frac{1}{3}.$$

Problem 4

a)

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \mathbf{b} \cdot \mathbf{a}.$$

b)

$$\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda a_1 b_1 + \lambda a_2 b_2 + \lambda a_3 b_3 = \lambda(a_1 b_1 + a_2 b_2 + a_3 b_3) = \lambda(\mathbf{a} \cdot \mathbf{b}).$$

Problem 6

Let $ABCD$ be a square, then

$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{BD} &= (\overrightarrow{BC} + \overrightarrow{AB}) \cdot (\overrightarrow{BC} - \overrightarrow{AB}) \\ &= |\overrightarrow{BC}|^2 - |\overrightarrow{AB}|^2 \\ &= 0. \end{aligned}$$

Q.E.D.

Problem 7

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0.$$

Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

Method 1

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{a} &= \mathbf{u}_1 \cdot (\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3) \\ &= \lambda_1 |\mathbf{u}_1|^2 + \lambda_2 \mathbf{u}_1 \cdot \mathbf{u}_2 + \lambda_3 \mathbf{u}_1 \cdot \mathbf{u}_3 \\ &= \lambda_1 \end{aligned}$$

where the last step used the fact that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set. Hence,

$$\lambda_1 = \mathbf{u}_1 \cdot \mathbf{a} = \frac{\sqrt{2}}{2}.$$

Similarly, we have

$$\lambda_2 = -3, \qquad \lambda_3 = \frac{3\sqrt{2}}{2}.$$

Method 2

Alternatively, we can use the Gaussian elimination. The augmented matrix can be reduced to $[\mathbf{I}, \mathbf{b}]$ where $\mathbf{I}$ is the identity matrix and $\mathbf{b} = (\sqrt{2}/2 \;\; -3 \;\; 3\sqrt{2}/2)^{\mathrm{T}}$. Therefore,

$$\lambda_1 = \frac{\sqrt{2}}{2}, \qquad \lambda_2 = -3, \qquad \lambda_3 = \frac{3\sqrt{2}}{2}.$$

This is not ideal as we discarded the information that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

Problem 9

a)

Let

$$\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

By the definition of the vector projection, we have

$$\mathrm{proj}_{\mathbf{v}}\mathbf{b} = \left(\frac{\mathbf{b} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Problem 10

a)

Denote point $(-2, 1, 5)$ by P, point $(1, 2, -5)$ by A and the projection of P on $\mathbf{x}$ by B. Then, we have

$$\overrightarrow{AP} = \begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix}.$$

By the definition of the vector projection, we can compute the projection of $\overrightarrow{AP}$ on $\mathbf{x}$ by

$$\overrightarrow{AB} = \left(\frac{\overrightarrow{AP} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \begin{pmatrix} -6 \\ -3 \\ 4 \end{pmatrix}$$

where $\mathbf{v} = (6 \; 3 \; 4)^{\mathrm{T}}$. Hence, the shortest distance between point P and line $\mathbf{x}$ is given by

$$|\overrightarrow{PB}| = |\overrightarrow{AB} - \overrightarrow{AP}| = \left| \begin{pmatrix} -6 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix} \right| = 7.$$

b)

The parametric vector form of the line is given by

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

Using the same setup as question a, we have

$$\overrightarrow{AP} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix},$$
$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix},$$
$$|\overrightarrow{PB}| = 3.$$

problem 15

$$\mathbf{n} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 6 \end{pmatrix}.$$

Problem 17

b)

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \overrightarrow{AC} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

Hence, one possible normal can be computed by

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}.$$

The area of the parallelogram is $|\overrightarrow{AB} \times \overrightarrow{AC}| = 2\sqrt{2}$.

Problem 18

a)

The area can be computed by

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left| \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{2}.$$

Problem 19

a)

$$\cos \angle DEF = \frac{\overrightarrow{ED} \cdot \overrightarrow{EF}}{|\overrightarrow{ED}||\overrightarrow{EF}|} = \frac{\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\left|\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right|} = -\frac{2\sqrt{2}}{3}.$$

b)

$$S_{\triangle DEF} = \frac{1}{2}|\overrightarrow{ED} \times \overrightarrow{EF}| = \frac{\sqrt{2}}{2}.$$

Problem 24

This can be proved by simply expanding the equation.

Problem 26

We can prove the coplanarity by showing the triple product of DA, DB and DC is zero. This is indeed the case:

$$\overrightarrow{DA} \cdot (\overrightarrow{DB} \times \overrightarrow{DC}) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \left(\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} \right) = 0.$$

Problem 27

a)

Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Point-normal form:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right) = 0.$$

Cartesian form:

$$x_1 - x_2 - 2x_3 = 3.$$

Let $x_2 = \lambda_1$ and $x_3 = \lambda_2$ where $\lambda_1, \lambda_2 \in \mathbb{R}$, the parametric form can be written as

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

b)

One possible normal to the plane is given by

$$\mathbf{n} = \frac{1}{5} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Parametric form:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Point-normal form:

$$\begin{pmatrix}-1\\1\\-1\end{pmatrix}\cdot\left(\mathbf{x}-\begin{pmatrix}1\\2\\-2\end{pmatrix}\right)=0.$$

Cartesian form:

$$-x_1+x_2-x_3=\begin{pmatrix}-1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=3.$$

c)

This problem is equivalent to finding a plane through point $(1,2,-2)$ parallel to $\begin{pmatrix}-2\\-1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\1\\3\end{pmatrix}$.

Hence follows the same steps in question b.

d)

This problem is equivalent to finding a plane through point $(-1,0,0)$ parallel to $\begin{pmatrix}1\\2\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-4\end{pmatrix}$.

Hence follows the same steps in question b.

Problem 30

a)

Denote point $(2,6,-5)$ by P, point $(1,2,3)$ by A and the normal to the plane by $\mathbf{n}$. The shortest distance is the length of the projection of $\overrightarrow{AP}$ on $\mathbf{n}$, given by

$$\left|\mathrm{proj}_{\mathbf{n}}\overrightarrow{AP}\right|=\left|\left(\frac{\overrightarrow{AP}\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n}\right|=3.$$

b)

From the given Cartesian form, we can easily find a point on the plane, say $(0,0,5)$, denoted by A, and the normal $\mathbf{n}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$. We further denote the point $(1,4,1)$ by P. The shortest distance is the length of the projection of $\overrightarrow{AP}$ on $\mathbf{n}$, given by

$$\left|\mathrm{proj}_{\mathbf{n}}\overrightarrow{AP}\right|=\left|\left(\frac{\overrightarrow{AP}\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n}\right|=\sqrt{6}.$$

Problem 31

a)

$$\overrightarrow{AB}=\begin{pmatrix}-1\\-1\\2\end{pmatrix},\qquad\overrightarrow{AC}=\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

Hence, one possible parametric form can be

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

b)

$$\mathbf{n} = -\frac{1}{3} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

c)

By the definition, one possible point-normal form can be

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right) = 0.$$

d)

The shortest distance is the length of the projection of $\overrightarrow{AQ}$ on $\mathbf{n}$, given by

$$\left|\mathrm{proj}_{\mathbf{n}} \overrightarrow{AQ}\right| = \left| \left(\frac{\overrightarrow{AQ} \cdot \mathbf{n}}{|\mathbf{n}|^2} \right) \mathbf{n} \right| = \frac{8\sqrt{3}}{3}.$$

Chapter 3

Problem 5

$$\begin{aligned}
3z &= 6+9i \\
z^2 &= (2+3i)^2 = -5+12i \\
z+2w &= (2+3i)+2(-1+2i) = 7i \\
z(w+3) &= -2+10i \\
\frac{z}{w} &= \frac{2+3i}{-1+2i} = \frac{(2+3i)(-1-2i)}{(-1+2i)(-1-2i)} = \frac{4-7i}{5} \\
\frac{w}{z} &= \frac{5}{4-7i} = \frac{4+7i}{13}.
\end{aligned}$$

Problem 6

a)

$$\frac{1+i}{1+2i} = \frac{(1+i)(1-2i)}{(1+2i)(1-2i)} = \frac{3-i}{5}.$$

b)

$$\frac{2-i}{3+i} - \frac{3-i}{2+i} = \frac{(2-i)(2+i)-(3+i)(3-i)}{(3+i)(2+i)} = -\frac{1-i}{2}.$$

Problem 8

b)

$$z = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm \sqrt{2}i$$

c)

$$z = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i.$$

e)

$$z^4 + 5z^2 + 4 = (z^2+1)(z^2+4),$$

which has roots $z = \pm i, \pm 2i$.

Problem 11

$$\begin{aligned}
\left(\frac{a+bi}{a-bi}\right)^2 - \left(\frac{a-bi}{a+bi}\right)^2 &= \left(\frac{a+bi}{a-bi} + \frac{a-bi}{a+bi}\right)\left(\frac{a+bi}{a-bi} - \frac{a-bi}{a+bi}\right) \\
&= \frac{2(a^2-b^2)}{a^2+b^2} \cdot \frac{4abi}{a^2+b^2} \\
&= \frac{8abi(a^2-b^2)}{(a^2+b^2)^2}.
\end{aligned}$$

Problem 12

$$\mathrm{Re}(-1+i) = -1; \qquad \mathrm{Im}(-1+i) = 1; \qquad \overline{-1+i} = -1-i.$$

$$\mathrm{Re}(2+3i) = 2; \qquad \mathrm{Im}(2+3i) = 3; \qquad \overline{2+3i} = 2-3i.$$

$$\mathrm{Re}\left(\frac{2-i}{1+i}\right) = \mathrm{Re}\left(\frac{1-3i}{2}\right) = \frac{1}{2}; \qquad \mathrm{Re}\left(\frac{2-i}{1+i}\right) = -\frac{3}{2}; \qquad \overline{\frac{2-i}{1+i}} = \frac{1+3i}{2}.$$

$$\mathrm{Re}\left(\frac{1}{(i+1)^2}\right) = \mathrm{Re}\left(\frac{-i}{2}\right) = 0; \qquad \mathrm{Re}\left(\frac{1}{(i+1)^2}\right) = -\frac{1}{2}; \qquad \overline{\frac{1}{(i+1)^2}} = \frac{i}{2}.$$

Problem 13

$$\begin{aligned}
z^2 &= (1+2i)^2 = -3+4i, \\
\frac{\overline{z}}{w} &= \frac{1-2i}{3-4i} = \frac{11}{25} - \frac{2}{25}i.
\end{aligned}$$

Problem 14

Given

$$\begin{aligned}
2z + 3w &= 1 + 12i \\
\overline{z} - \overline{w} &= 3 - i,
\end{aligned} \tag{3.14.1}$$

taking conjugate of both sides of the the second equation, we have

$$z - w = 3 + i. \tag{3.14.2}$$

Solving (3.14.1) and (3.14.2), we obtain

$$\begin{aligned}
z &= 2 + 3i \\
w &= -1 + 2i.
\end{aligned}$$

Problem 17

a)

It can be easily verified by taking the conjugate of both sides of the equation.

b)

Since $3 - 2i$ is a root of the real polynomial, the conjugate complex $3 + 2i$ is also a root. Hence,

$$\begin{aligned} ax^2 + bx + c &= (x - (3 - 2i))(x - (3 + 2i)) \\ &= x^2 - 6x + 13. \end{aligned}$$

c)

Yes.

Problem 18

a)

$$|z| = 6\sqrt{2}; \qquad \operatorname{Arg}(z) = \frac{\pi}{4}; \qquad z = 6\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

b)

$$|z| = 4; \qquad \operatorname{Arg}(z) = \pi; \qquad z = 4(\cos\pi + i\sin\pi).$$

c)

$$|z| = 2; \qquad \operatorname{Arg}(z) = -\frac{\pi}{6}; \qquad z = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right).$$

d)

$$|z| = 1; \qquad \operatorname{Arg}(z) = -\frac{3\pi}{4}; \qquad z = \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}.$$

e)

$$|z| = \sqrt{58}; \quad \operatorname{Arg}(z) = \pi - \arctan\frac{3}{7}; \quad z = \sqrt{58}\left(\cos\left(\pi - \arctan\frac{3}{7}\right) + i\sin\left(\pi - \arctan\frac{3}{7}\right)\right).$$

Problem 21

a)

$$a + bi = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i.$$

b)

$$a + bi = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\frac{3\sqrt{3}}{2} - \frac{1}{2}i.$$

c)

$$a + bi = 3\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i.$$

d)

$$a + bi = 3\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = -\frac{3\sqrt{3}}{2} - \frac{1}{2}i.$$

e)

$$a + bi = 3\left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right) = \frac{3}{2}\left(\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}i\right)$$

where we applied the half angle formula.

Problem 22

a)

Let $z = a + bi$, then $\overline{z} = a - bi$. Hence,

$$z\overline{z} = a^2 + b^2 = |z|^2.$$

Multiplying z^{-1} on both sides of this equation, we obtain

$$\overline{z} = z^{-1}.$$

b)

$$|z| = \sqrt{a^2 + b^2} = |\overline{z}|.$$

c)

$$\overline{z} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta))$$

where we took advantage of the parity of sin and cos.

Problem 26

Skipped.

Problem 27

$$w^6 = (2e^{i\pi/3})^6 = 64e^{2i\pi} = 64.$$

The Cartesian form of w is given by

$$w = 1 + \sqrt{3}i.$$

Hence,

$$z - w = (1 - i) - (1 + \sqrt{3}i) = -(1 + \sqrt{3})i.$$

$$\frac{w}{\overline{z}} = \frac{1 + \sqrt{3}i}{1 + i} = \frac{1 + \sqrt{3}}{2} + \frac{\sqrt{3} - 1}{2}i.$$

Problem 31

Keep in mind that the principal argument is in $(-\pi, \pi]$:

$$\begin{aligned}
\operatorname{Arg}(z) &= \frac{\pi}{3} \\
\operatorname{Arg}(w) &= \frac{\pi}{4} \\
\operatorname{Arg}(zw) &= \operatorname{Arg}(z) + \operatorname{Arg}(w) + 2k\pi = \frac{7\pi}{12}.
\end{aligned}$$

We can easily evaluate zw by

$$zw = |z||w|e^{i\operatorname{Arg}(zw)} = 2\sqrt{2}e^{7i\pi/12}.$$

This should be equal to its Cartesian form that is given by

$$zw = (1+\sqrt{3}i)(1+i) = (1-\sqrt{3}) + (1+\sqrt{3})i.$$

Equating the coefficients, we have

$$\begin{aligned}
2\sqrt{2}\cos\frac{7\pi}{12} &= 1-\sqrt{3} \\
2\sqrt{2}\sin\frac{7\pi}{12} &= 1+\sqrt{3},
\end{aligned}$$

which implies that

$$\begin{aligned}
\cos\frac{7\pi}{12} &= \frac{1-\sqrt{3}}{2\sqrt{2}} \\
\sin\frac{7\pi}{12} &= \frac{1+\sqrt{3}}{2\sqrt{2}}.
\end{aligned}$$

Problem 33

a)

$$\begin{aligned}
(\sqrt{3}+i)^5 &= (2e^{i\pi/6})^5 \\
&= 32e^{i5\pi/6} \\
&= 32(\cos(5\pi/6) + i\sin(5\pi/6)) \\
&= -16\sqrt{3} + 16i.
\end{aligned}$$

Problem 34

a)

Let $z = a + bi$ where $a, b \in \mathbb{R}$ and $a, b \neq 0$, we have

$$z^2 = (a+bi)^2 = 21 - 20i.$$

Solving for a, b and substituting back into the expression of z, we obtain $z = 5 - 2i$ or $z = -5 + 2i$.

37

a)

Using the same steps in question 34(a) to find the roots of $-3+4i$, we have $z' = \pm(1+2i)$. Hence, the roots of $z^2 - 3z + (3-i) = 0$ are given by

$$\begin{aligned} z &= \frac{3 \pm \sqrt{-3+4i}}{2} \\ &= 2+i \text{ or } 1-i. \end{aligned}$$

Problem 40

This problem is equivalent to finding the modulus r and the principal argument θ that satisfies

$$(re^{i\theta})^5 = 16e^{-i\pi/3+2k\pi}, \quad k \in \mathbb{N}.$$

Solving for r and θ, we obtain

$$\begin{aligned} r &= 2 \\ \theta &= -\frac{13\pi}{15}, -\frac{7\pi}{15}, -\frac{\pi}{15}, \frac{\pi}{3}, \frac{11\pi}{15}. \end{aligned}$$

Problem 42

$$\omega + \omega^2 + \cdots + \omega^n = \frac{\omega(1-\omega^n)}{1-\omega} = 0.$$

Q.E.D.

Problem 50

$$\begin{aligned} \cos 3\theta + i \sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= 4\cos^3\theta - 3\cos\theta + i(3\sin\theta\cos^2\theta - \sin^3\theta). \end{aligned}$$

Equating the real parts, we obtain

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Problem 54

Skipped.

Problem 66

Noticing that $p(2) = 0$, by the remainder and factor theorem, $z-2$ must be a factor of $p(z)$. Using long division, we can factorize $p(z)$ as

$$p(z) = (z-2)(2z-5)(z+3).$$

Problem 70

This problem is equivalent to finding the roots of $z^4 + 4 = 0$.

Let $z = re^{i\theta}$, then

$$r^4 e^{i4\theta} = z^4 = -4 = 4e^{i(\pi+2k\pi)}, \quad k \in \mathbb{N}.$$

Equating the coefficients, we obtain the four roots. They are

$$z = \sqrt{2}e^{\pm i\pi/4},\ \sqrt{2}e^{\pm i3\pi/4} = \pm 1 \pm i.$$

Hence,

$$\begin{aligned} p(z) &= (x - (1+i))(x - (1-i))(x - (-1+i))(x - (-1-i)) \\ &= (x^2 - 2x + 2)(x^2 + 2x + 2). \end{aligned}$$

Problem 71

Similar to problem 70,

$$r^4 e^{i4\theta} = -i = e^{i(-\pi/4+2k\pi)}, \quad k \in \mathbb{N}.$$

Equating the coefficients and solving for r and θ, we can factorize $p(z)$ into

$$p(z) = (z - e^{-i5\pi/8})(z - e^{-i\pi/8})(z - e^{i3\pi/8})(z - e^{i7\pi/8}).$$

Problem 72

a)

Similar to problem 70,

$$r^6 e^{i6\theta} = -1 = e^{i(\pi+2k\pi)}, \quad k \in \mathbb{N}.$$

Equating the coefficients and solving for r and θ, we obtain the six roots. They are

$$z = e^{\pm i\pi/6},\ e^{\pm i\pi/2},\ e^{\pm i5\pi/6}.$$

b)

Skipped.

c)

Using the result we obtained in question a, we have

$$z^6 + 1 = (z - e^{i\pi/6})(z - e^{-i\pi/6})(z - e^{i\pi/2})(z - e^{-i\pi/2})(z - e^{i5\pi/6})(z - e^{-i5\pi/6}).$$

d)

$$\begin{aligned} z^6 + 1 &= (z - e^{i\pi/6})(z - e^{-i\pi/6})(z - e^{i\pi/2})(z - e^{-i\pi/2})(z - e^{i5\pi/6})(z - e^{-i5\pi/6}) \\ &= (z^2 - \sqrt{3}z + 1)(z^2 + 1)(z^2 + \sqrt{3}z + 1). \end{aligned}$$

Problem 74

a)

Similar to problem 70,

$$r^5 e^{i5\theta} = 1 = e^{i2k\pi}, \quad k \in \mathbb{N}.$$

Equating the coefficients and solving for r and θ, we obtain the five roots. They are

$$z = 1,\ e^{\pm i2\pi/5},\ e^{\pm 4\pi/5}.$$

Hence,

$$p(z) = (z - e^{i2\pi/5})(z - e^{-i2\pi/5})(z - e^{i4\pi/5})(z - e^{-i4\pi/5}).$$

b)

$$\begin{aligned}
p(z) &= (z - (\cos(2\pi/5) + i\sin(2\pi/5)))(z - (\cos(2\pi/5) - i\sin(2\pi/5))) \\
&\quad (z - (\cos(4\pi/5) + i\sin(4\pi/5)))(z - (\cos(4\pi/5) - i\sin(4\pi/5))) \\
&= (z^2 - 2\cos(2\pi/5)z + 1)(z^2 - 2\cos(4\pi/5)z + 1).
\end{aligned}$$

c)

Divide the equation $p(z) = 0$ by z^2, we have

$$\begin{aligned}
0 &= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 \\
&= \left(z^2 + 2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) - 1 = 0 \\
&= \left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right) - 1 = 0 \\
&= x^2 + x - 1 = 0
\end{aligned}$$

where $x = z + 1/z$.

d)

The results can be easily verified by solving the quadratic equations in terms of x and z.

Problem 76

For real polynomial $f(z)$, since $1 + i$ is a root of $f(z) = 0$, the conjugate $1 - i$ is also a root. Hence, a quadratic factor of $f(z)$ is

$$(z - (1 + i))(z - (1 - i)) = z^2 - 2z + 2.$$

Using the long division, we can factorize $f(z)$ as

$$f(z) = (z^2 - 2z + 2)(z - \sqrt[3]{5})(z^2 + \sqrt[3]{5}z + (\sqrt[3]{5})^2).$$

Therefore, the five roots of $f(z) = 0$ are

$$z = 1 \pm i,\ \sqrt[3]{5},\ \frac{\sqrt[3]{5}}{2}(-1 \pm i\sqrt{3}).$$

Chapter 4

Problem 1

a)

Skipped.

Problem 2

a)

$$\begin{pmatrix} 3 & 2 & 6 \\ 9 & 6 & 36 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 6 \\ 0 & 0 & 18 \end{pmatrix}.$$

No solution.

b)

$$\begin{pmatrix} 3 & 2 & 6 \\ 9 & 4 & 36 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 6 \\ 0 & -2 & 18 \end{pmatrix}.$$

Unique solution.

c)

$$\begin{pmatrix} 1 & -5 & 5 \\ 6 & -30 & 30 \end{pmatrix} \to \begin{pmatrix} 1 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

An infinite number of solutions.

Problem 3

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - (a_{21}a_{12})/a_{11} & b_2 - (a_{21}b_1)/a_{11} \end{pmatrix}.$$

a)

$$a_{22} - (a_{21}a_{12})/a_{11} \neq 0,$$

which implies that

$$a_{11}a_{22} \neq a_{12}a_{21}.$$

b)

$$\begin{aligned} a_{22} - (a_{21}a_{12})/a_{11} &= 0 \\ b_2 - (a_{21}b_1)/a_{11} &\neq 0, \end{aligned}$$

which implies that

$$\begin{aligned} a_{11}a_{22} &= a_{12}a_{21} \\ a_{11}b_2 &\neq a_{21}b_1. \end{aligned}$$

c)

$$\begin{aligned} a_{22} - (a_{21}a_{12})/a_{11} &= 0 \\ b_2 - (a_{21}b_1)/a_{11} &= 0, \end{aligned}$$

which implies that

$$\begin{aligned} a_{11}a_{22} &= a_{12}a_{21} \\ a_{11}b_2 &= a_{21}b_1. \end{aligned}$$

Problem 5

a)

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 5 & 8 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & 2 \end{pmatrix}.$$

Letting $x_3 = \lambda$ and performing back substitution, we obtain the solution set $\left\{ \begin{pmatrix} 1+\lambda \\ 2-2\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$. The solution is the line of intersection of the two planes.

b)

$$\begin{pmatrix} 4 & 5 & -2 & 16 \\ 8 & 10 & -4 & 20 \end{pmatrix} \to \begin{pmatrix} 4 & 5 & -2 & 16 \\ 0 & 0 & 0 & -12 \end{pmatrix}.$$

The solution set is $\varnothing$. The two planes are parallel.

c)

$$\begin{pmatrix} 4 & 5 & -2 & 16 \\ 8 & 10 & -4 & 32 \end{pmatrix} \to \begin{pmatrix} 4 & 5 & -2 & 16 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $x_2 = \lambda$, $x_3 = \mu$, then we obtain the solution set $\left\{ \begin{pmatrix} 4 - 5\lambda/4 + \mu/2 \\ \lambda \\ \mu \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$. The two planes are the same.

Problem 10

a)

$$\begin{pmatrix}1 & 4 & 2 & 3\\ 2 & 6 & 3 & 0\\ 4 & -2 & 4 & 4\end{pmatrix} \xrightarrow[]{\substack{\mathrm{R}_2=\mathrm{R}_2-2\mathrm{R}_1\\ \mathrm{R}_3=\mathrm{R}_3-2\mathrm{R}_1}} \begin{pmatrix}1 & 4 & 2 & 3\\ 0 & -2 & -1 & -6\\ 0 & -18 & -4 & -8\end{pmatrix}$$

$$\xrightarrow{\mathrm{R}_3=\mathrm{R}_3-9\mathrm{R}_2} \begin{pmatrix}1 & 4 & 2 & 3\\ 0 & -2 & -1 & -6\\ 0 & 0 & 5 & 46\end{pmatrix}.$$

b)

Similar to question a.

Problem 12

a) Yes.

b) Yes.

c) No.

d) Yes.

e) Yes.

f) Yes.

g) Yes.

h) No.

Problem 13

b)

$$\begin{pmatrix}3 & 2 & 1 & 1 & 10\\ 0 & 4 & 2 & -4 & 8\\ 0 & 0 & -7 & 14 & 14\end{pmatrix} \xrightarrow[]{\substack{\mathrm{R}_2=\mathrm{R}_2/2\\ \mathrm{R}_3=\mathrm{R}_3/7}} \begin{pmatrix}3 & 2 & 1 & 1 & 10\\ 0 & 2 & 1 & -2 & 4\\ 0 & 0 & -1 & 2 & 2\end{pmatrix}.$$

Letting $x_4 = \lambda$ and performing back substitution, we obtain

$$\mathbf{x} = \begin{pmatrix}2\\3\\-2\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\2\\1\end{pmatrix}, \quad \lambda \in \mathbb{R},$$

which is a line through point $(2, 3, -2, 0)$ parallel to $\begin{pmatrix}-1\\0\\2\\1\end{pmatrix}$.

Problem 14

e)

The augmented matrix is given by

$$\begin{pmatrix} 1 & 2 & 4 & 10 \\ -3 & 3 & 15 & 15 \\ -2 & -1 & 1 & -5 \end{pmatrix}.$$

Using Gaussian elimination, we have

$$\begin{pmatrix} 1 & 2 & 4 & 10 \\ -3 & 3 & 15 & 15 \\ -2 & -1 & 1 & -5 \end{pmatrix} \xrightarrow[]{\substack{\mathrm{R}_2 = \mathrm{R}_2 + 3\mathrm{R}_1 \\ \mathrm{R}_3 = \mathrm{R}_3 + 2\mathrm{R}_1}} \begin{pmatrix} 1 & 2 & 4 & 10 \\ 0 & 9 & 27 & 45 \\ 0 & 3 & 9 & 15 \end{pmatrix} \xrightarrow[]{\substack{\mathrm{R}_2 = \mathrm{R}_2/9 \\ \mathrm{R}_3 = \mathrm{R}_3 - \mathrm{R}_1/3}} \begin{pmatrix} 1 & 2 & 4 & 10 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Letting $x_3 = \lambda$ and performing back substitution, we obtain

$$\mathbf{x} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

f)

The augmented matrix is given by

$$\begin{pmatrix} 1 & -4 & -5 & -6 \\ 2 & -1 & -1 & 2 \\ 3 & 9 & 12 & 30 \end{pmatrix}.$$

Using Gaussian elimination, we have

$$\begin{aligned} \begin{pmatrix} 1 & -4 & -5 & -6 \\ 2 & -1 & -1 & 2 \\ 3 & 9 & 12 & 30 \end{pmatrix} &\xrightarrow[]{\substack{\mathrm{R}_2 = \mathrm{R}_2 - \mathrm{R}_1 \\ \mathrm{R}_3 = \mathrm{R}_3 - 3\mathrm{R}_1}} \begin{pmatrix} 1 & -4 & -5 & -6 \\ 0 & 7 & 9 & 10 \\ 0 & 21 & 27 & 48 \end{pmatrix} \\ &\xrightarrow[]{\mathrm{R}_3 = \mathrm{R}_3 - 3\mathrm{R}_2} \begin{pmatrix} 1 & -4 & -5 & -6 \\ 0 & 7 & 9 & 10 \\ 0 & 0 & 0 & 18 \end{pmatrix} \end{aligned}$$

where we performed row operations simultaneously. Hence, there is no solution.

h)

The augmented matrix is given by

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 1 \\ 3 & 2 & 0 & -2 & 3 \end{pmatrix}.$$

Using Gaussian elimination, we have

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 1 \\ 3 & 2 & 0 & -2 & 3 \end{pmatrix} \xrightarrow[]{\mathrm{R}_3 = \mathrm{R}_3 - 3\mathrm{R}_1 + 4\mathrm{R}_2} \begin{pmatrix} 1 & 2 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -5 \end{pmatrix}.$$

Letting $x_4 = \lambda$ and performing back substitution, we obtain

$$\mathbf{x} = \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Problem 15

b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & 5 & 6 & 2 \\ 0 & 0 & 1 & 7 & 3 \end{pmatrix} \xrightarrow{\mathrm{R}_2 = -\mathrm{R}_2} \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & -5 & -6 & -2 \\ 0 & 0 & 1 & 7 & 3 \end{pmatrix}$$

$$\xrightarrow[]{\substack{\mathrm{R}_1 = \mathrm{R}_1 - 2\mathrm{R}_2 - 13\mathrm{R}_3 \\ \mathrm{R}_2 = \mathrm{R}_2 + 5\mathrm{R}_3}} \begin{pmatrix} 1 & 0 & 0 & -75 & -34 \\ 0 & 1 & 0 & 29 & 13 \\ 0 & 0 & 1 & 7 & 3 \end{pmatrix}$$

where we performed row operations simultaneously.

Letting $x_4 = \lambda$, we obtain

$$\mathbf{x} = \begin{pmatrix} -34 \\ 13 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 75 \\ -29 \\ -7 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

which is a line through point $(-34, 13, 3, 0)$ parallel to $\begin{pmatrix} 75 \\ -29 \\ -7 \\ 1 \end{pmatrix}$ in $\mathbb{R}^4$.

Problem 16

a) Unique solution.

b) No solution.

c) Infinitely many solutions.

d) Infinitely many solutions.

e) Unique solution. Note that the last row gives no information.

Problem 17

Performing Gaussian elimination, we have

$$\begin{pmatrix} 1 & 1 & k & 2 \\ 3 & 4 & 2 & k \\ 2 & 3 & -1 & 1 \end{pmatrix} \xrightarrow[]{\substack{\mathrm{R}_2 = \mathrm{R}_2 - 3\mathrm{R}_1 \\ \mathrm{R}_3 = \mathrm{R}_3 - 2\mathrm{R}_1}} \begin{pmatrix} 1 & 1 & k & 2 \\ 0 & 1 & 2-3k & k-6 \\ 0 & 1 & -1-2k & -3 \end{pmatrix}$$

$$\xrightarrow{\mathrm{R}_3 = \mathrm{R}_3 - \mathrm{R}_2} \begin{pmatrix} 1 & 1 & k & 2 \\ 0 & 1 & 2-3k & k-6 \\ 0 & 0 & -3+k & 3-k \end{pmatrix}.$$

a)

A unique solution indicates $-3 + k \neq 0$, i.e. $k \neq 3$.

b)

No solution implies that $-3 + k \neq 0$ whereas $3 - k = 0$. Apparently, there is no such k.

c)

Infinitely many solutions indicates that $-3 + k = 0$ and $3 - k = 0$, namely, $k = 3$.

Problem 18

Performing Gaussian elimination, we have

$$\begin{pmatrix} 1 & 2 & \lambda & 1 \\ -1 & \lambda & -1 & 0 \\ \lambda & -4 & \lambda & -1 \end{pmatrix} \xrightarrow[]{\substack{\mathrm{R}_2 = \mathrm{R}_2 + \mathrm{R}_1 \\ \mathrm{R}_3 = \mathrm{R}_3 - \lambda\mathrm{R}_1}} \begin{pmatrix} 1 & 2 & \lambda & 1 \\ 0 & \lambda+2 & \lambda-1 & 1 \\ 0 & -2\lambda-4 & \lambda-\lambda^2 & -1-\lambda \end{pmatrix}$$

$$\xrightarrow{\mathrm{R}_3 = \mathrm{R}_3 + 2\mathrm{R}_2} \begin{pmatrix} 1 & 2 & \lambda & 1 \\ 0 & \lambda+2 & \lambda-1 & 1 \\ 0 & 0 & -(\lambda-1)(\lambda-2) & -\lambda+1 \end{pmatrix}.$$

a)

No solution indicates either

$$\begin{aligned} \lambda + 2 &= 0 \\ \lambda - 2 &\neq 0 \\ -\lambda + 1 + \lambda - 2 &\neq 0 \end{aligned}$$

or

$$\begin{aligned} -(\lambda - 1)(\lambda - 2) &= 0 \\ -\lambda + 1 &\neq 0. \end{aligned}$$

Solving for λ, we obtain

$$\lambda = \pm 2.$$

b)

In this case, infinitely many solutions indicates that

$$-(\lambda-1)(\lambda-2) = -\lambda+1 = 0,$$

which implies that

$$\lambda = 1.$$

c)

Unique solution indicates that

$$(\lambda+2)(-(\lambda-1)(\lambda-2)) \neq 0,$$

that is

$$\lambda \neq 1 \text{ and } \lambda \neq \pm 2.$$

In fact, we can first find the λ's that satisfy the condition for unique solution. Then substitute the excluded λ's and check the number of solutions.

Problem 21

The answer is obviously no as long as we see *cddc* out of the five routes, because this gives a zero row in the 5 by 6 augmented matrix, resulting in infinitely many solutions.

Problem 22

a)

$$\begin{pmatrix} 1 & -2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 \\ -2 & 3 & -2 & b_3 \end{pmatrix} \xrightarrow{\mathrm{R}_3 = 2\mathrm{R}_1 + \mathrm{R}_2} \begin{pmatrix} 1 & -2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 \\ 0 & 0 & 1 & 2b_1+b_2+b_3 \end{pmatrix}.$$

Performing back substitution, we obtain

$$\begin{aligned} x_1 &= 7b_1 + 5b_2 + 3b_3 \\ x_2 &= 6b_1 + 4b_2 + 3b_3 \\ x_3 &= 2b_1 + b_2 + b_3. \end{aligned}$$

Problem 24

b)

$$\begin{pmatrix} 1 & 1 & 3 & -1 & b_1 \\ 2 & -1 & 0 & 2 & b_2 \\ 1 & -2 & -3 & 3 & b_3 \\ 0 & 3 & 6 & -4 & b_4 \end{pmatrix} \xrightarrow[\mathrm{R}_3 = \mathrm{R}_1 - \mathrm{R}_3]{\mathrm{R}_2 = \mathrm{R}_2 - 2\mathrm{R}_1} \begin{pmatrix} 1 & 1 & 3 & -1 & b_1 \\ 0 & -3 & -6 & 4 & b_2 - 2b_1 \\ 0 & -3 & -6 & 4 & b_3 - b_1 \\ 0 & -3 & -6 & 4 & -b_4 \end{pmatrix}.$$

To ensure that the system has a solution, we should have

$$b_2 - 2b_1 = b_3 - b_1 = -b_4.$$

Problem 31

This problem is equivalent to solving the linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix}$. Using Gaussian elimination, we obtain

$$\mathbf{x} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}.$$

Hence, the answer is yes, and

$$\begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}.$$

Problem 33

To find the intersection, we have

$$\begin{aligned} 1 + \lambda_1 + 3\lambda_2 &= 2\mu \\ 4\lambda_1 + \lambda_2 &= 18 - 3\mu \\ 4 + \lambda_1 - 2\lambda_2 &= 1 + \mu. \end{aligned}$$

Hence, this problem is equivalent to solving the linear system given by the augmented matrix

$$\begin{pmatrix} 1 & 3 & -2 & -1 \\ 4 & 1 & 3 & 18 \\ 1 & -2 & -1 & -3 \end{pmatrix}.$$

Using Gaussian elimination, the matrix can be reduced to

$$\begin{pmatrix} 1 & 3 & -2 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & -3 \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \mu \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix},$$

and the intersection is point $(6, 9, 4)$.

Problem 34

Plugging

$$\begin{aligned} x_1 &= 6 - 2\lambda_1 + \lambda_2 \\ x_2 &= -2 + \lambda_1 + \lambda_2 \\ x_3 &= 3 + 3\lambda_1 - \lambda_2 \end{aligned}$$

into the first, we have

$$\lambda_1 = 3\lambda_2.$$

Substituting back into the second plane, we obtain the intersection

$$\mathbf{x} = \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 8 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Problem 38

Let $p(x) = ax^2 + bx + c$. This problem is equivalent to solving the linear equations given by the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 4 & 2 & 1 & 7 \\ 9 & 3 & 1 & 13 \end{pmatrix}.$$

Solving for the coefficients and substituting back into the expression of $p(x)$, we obtain

$$p(x) = 2x^2 - 4x + 7.$$