MATH1241 Problem Set Solutions - Algebra

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2022 Term 2

Contents

Chapter 6

Vector Spaces

6.4 Problem 4

Closure under Addition

For any vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n$, because $\mathbb{C}$ is closed under addition, we have

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \in \mathbb{C}^n.$$

Thus, $\mathbb{C}^n$ is closed under addition.

Associative Law of Addition

For any vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n$, because $\mathbb{C}$ is associative, we have

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \left(\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right) + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 + w_1 \\ \vdots \\ u_n + v_n + w_n \end{pmatrix} = \begin{pmatrix} u_1 + (v_1 + w_1) \\ \vdots \\ u_n + (v_n + w_n) \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right) = \mathbf{u} + (\mathbf{v} + \mathbf{w}).
\end{aligned}$$

Thus, $\mathbb{C}^n$ is associative.

Closure under Multiplication by a Scalar

For any vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n$ and any scalar $\lambda \in \mathbb{C}$, since $\mathbb{C}$ is closed under multiplication by a scalar, we have

$$\lambda \mathbf{v} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \in \mathbb{C}^n.$$

Thus, $\mathbb{C}^n$ is closed under multiplication by a scalar.

Scalar Distributive Law

For any vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n$ and any scalar $\lambda, \mu \in \mathbb{C}$, due to the scalar distributive law of $\mathbb{C}$, we have

$$(\lambda + \mu)\mathbf{v} = (\lambda + \mu) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} (\lambda + \mu) v_1 \\ \vdots \\ (\lambda + \mu) v_n \end{pmatrix} \in \mathbb{C}^n.$$

Thus, the scalar distributive law holds for $\mathbb{C}^n$.

6.5 Problem 5

Denote the ijth entry of M by $[M]_{ij}$ where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. For any matrices $A, B \in M_{mn}(\mathbb{C})$ and scalars $\lambda \in \mathbb{C}$.

For axiom 1, because $\mathbb{C}$ is closed under addition, we have

$$[A]_{ij} + [B]_{ij} \in \mathbb{C}, \quad \text{for all} \quad i, j.$$

Hence, $A + B \in M_{mn}(\mathbb{C})$, which shows that axiom 1 is satisfied.

For axiom 3, because $\mathbb{C}$ is commutative, we have

$$[A]_{ij} + [B]_{ij} = [B]_{ij} + [A]_{ij}, \quad \text{for all} \quad i, j.$$

Hence, $A + B = B + A$, which shows that axiom 3 is satisfied.

For axiom 6, because $\mathbb{C}$ is closed under scalar multiplication, we have

$$\lambda [A]_{ij} \in \mathbb{C}, \quad \text{for all} \quad i, j.$$

Hence, $\lambda A \in M_{mn}(\mathbb{C})$, which shows that axiom 6 is satisfied.

For axiom 10, because of the distributive law in $\mathbb{C}$, we have

$$\lambda([A]_{ij} + [B]_{ij}) = \lambda [A]_{ij} + \lambda [B]_{ij}, \quad \text{for all} \quad i, j.$$

Hence, $\lambda(A + B) = \lambda A + \lambda B$, which shows that axiom 10 is satisfied.

6.6 Problem 6

It is easy to prove that $(\mathbb{C}^n, +, *, \mathbb{R})$ is a vector space because $\mathbb{R}$ is a subfield of $\mathbb{C}$. It is also easy to see that $(\mathbb{R}^n, +, *, \mathbb{C})$ is not a vector space because, for example, the closure under multiplication by a scalar does not hold.

6.7 Problem 7

This system is not a vector space.

6.8 Problem 8

a)

$$2\mathbf{v} = (1+1)\mathbf{v} = 1\mathbf{v} + 1\mathbf{v} = \mathbf{v} + \mathbf{v}.$$

b)

This can be proved by induction.

6.9 Problem 9

Multiplication of the Zero Vector

$$\lambda\mathbf{0} + \mathbf{0} = \lambda\mathbf{0} = \lambda(\mathbf{0} + \mathbf{0}) = \lambda\mathbf{0} + \lambda\mathbf{0}.$$

By the cancellation property, we obtain $\lambda\mathbf{0} = \mathbf{0}$.

Zero Products

If $\lambda = 0$, by the property of multiplication of the zero vector, we have $\lambda\mathbf{v} = \mathbf{0}$. If $\lambda \neq 0$, then $\lambda^{-1} \neq 0$, and hence,

$$\mathbf{v} = (\lambda^{-1}\lambda)\mathbf{v} = \lambda^{-1}(\lambda\mathbf{v}) = \lambda^{-1}\mathbf{0} = \mathbf{0}.$$

Cancellation Property

If $\lambda\mathbf{v} = \mu\mathbf{v}$, then $(\lambda - \mu)\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$, by the property of zero products, we obtain $\lambda - \mu = 0$, that is, $\lambda = \mu$.

6.23 Problem 23

No, because the zero polynomial of $\mathbb{P}_3$ is not in S.

6.27 Problem 27

a)

Let W' be the intersection of $\{W_k : 1 \leqslant k \leqslant m+1\}$. We prove this by induction.

For $m = 1$, we have $W = W_1$, which is a subspace of V.

Suppose that for $m > 1$, W is a subspace of V. Then, for $m+1$, since $W \leqslant V$ and $W_{m+1} \leqslant V$, we have $\mathbf{0} \in W$ and $\mathbf{0} \in W_{m+1}$, and hence, $\mathbf{0} \in W'$. For any vectors $\mathbf{u}, \mathbf{v} \in W'$ and scalars $\lambda, \mu \in \mathbb{F}$, since $W' = W \cap W_{m+1}$, $\mathbf{u}$ and $\mathbf{v}$ must be in both W and W_{m+1}. Also, since W and W_{m+1} are subspaces of V, they are closed under addition and multiplication by scalars from $\mathbb{F}$. Thus, $\lambda\mathbf{u} + \mu\mathbf{v}$ must be in both W and W_{m+1}, and hence in W'. By the alternative Subspace Theorem, W' is a subspace of V.

Therefore, by induction, W is a subspace of V.

b)

Suppose that W is not the set of finite linear combinations of vectors from S. Then, $\exists \mathbf{x} \in W$ such that $\mathbf{x} \notin \text{span}(S)$. However, for any $V_i \leqslant V$ and $V_i \supseteq S$, we have $\text{span}(S) \leqslant V_i$, implying that $\mathbf{x} \notin V_i$, and hence $\mathbf{x} \notin W$, which is a contradiction. Therefore, W is the set of finite linear combinations of vectors from S.

6.36 Problem 36

This problem is equivalent to proving that $\text{span}(S)$ is a subspace of V over field $\mathbb{F}$.

Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\} \subseteq V$.

The zero vector of V is in $\text{span}(S)$ because $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$.

For any vectors $\mathbf{u} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n, \mathbf{v} = \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n \in \text{span}(S)$ where $\lambda_1, ..., \lambda_n, \mu_1, ..., \mu_n \in \mathbb{F}$ and any scalar $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n) + (\mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n) \\ &= (\lambda_1 + \mu_1)\mathbf{v}_1 + \cdots + (\lambda_n + \mu_n)\mathbf{v}_n, \end{aligned}$$

where $(\lambda_1 + \mu_1), ..., (\lambda_n + \mu_n) \in \mathbb{F}$. Thus, $(\mathbf{u} + \mathbf{v}) \in \text{span}(S)$, which implies that $\text{span}(S)$ is closed under addition.

Also, since

$$\begin{aligned} \lambda\mathbf{v} &= \lambda(\mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n) \\ &= (\lambda\mu_1)\mathbf{v}_1 + \cdots + (\lambda\mu_n)\mathbf{v}_1, \end{aligned}$$

where $(\lambda\mu_1), ..., (\lambda\mu_n) \in \mathbb{F}$, we have $(\lambda\mathbf{v}) \in \text{span}(S)$, which implies that $\text{span}(S)$ is also closed under multiplication by a scalar.

Therefore, by the Subspace Theorem, $\text{span}(S)$ is a subspace of V, and hence, the original statement is proved.

6.37 Problem 37

We prove this by induction.

For $n = 1$, $\displaystyle\sum_{k=1}^{1} \lambda_k\mathbf{v}_k = \lambda_1\mathbf{v}_1$.

Suppose that for $n > 1$, $\displaystyle\sum_{k=1}^{n} \lambda_k\mathbf{v}_k = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n$ holds regardless of the order. By the closures under addition and multiplication by a scalar, $\displaystyle\sum_{k=1}^{n} \lambda_k\mathbf{v}_k$ is in the vector space. Hence, for $n + 1$, by the commutative law of addition, we have $\displaystyle\sum_{k=1}^{n+1} \lambda_k\mathbf{v}_k = \sum_{k=1}^{n} \lambda_k\mathbf{v}_k + \lambda_{n+1}\mathbf{v}_{n+1} = \lambda_{n+1}\mathbf{v}_{n+1} + \sum_{k=1}^{n} \lambda_k\mathbf{v}_k$ regardless of the order.

Therefore, by induction, we proved that we do not need to use brackets when writing down linear combinations.

6.46 Problem 46

For

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_m\mathbf{v}_m = \mathbf{0},$$

multiplying $\dfrac{\mathbf{v}_i}{\|\mathbf{v}_i\|^2}$ on both sides of the equation, since S is orthogonal, we have

$$\lambda_i = \lambda_i \mathbf{v}_i \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|^2} = (\lambda_1 \mathbf{v}_1 + \cdots + \lambda_m \mathbf{v}_m) \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|^2} = \mathbf{0} \cdot \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|^2} = 0, \quad 1 \leqslant i \leqslant m,$$

which implies that $\lambda_1, ..., \lambda_m$ are all zero. Hence, S is a linearly independent set.

6.52 Problem 52

Because by the Rank-nullity Theorem, the rank cannot exceed the number of columns.

6.59 Problem 59

Performing Gaussian Elimination on the matrix whose columns are the vector representations of the polynomials, we have

$$\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & -1 & -4 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, S is a linearly dependent spanning set for $\mathbb{P}_2$. A subset of S which is a basis for $\mathbb{P}_2$ can be $\{p_1, p_2\}$.

6.60 Problem 60

Since $\mathbf{w} \in \operatorname{span}(S)$, $\mathbf{w}$ is some linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Hence, by the definition of linear dependence, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}\}$ is a linearly dependent set.

6.61 Problem 61

Here gives a rather informal proof. For any subspace V of $\mathbb{R}^4$, we must have $0 \leqslant \dim V \leqslant 4$. However, the given subspaces have already covered all the subspaces of dimensions from 0 to 4. Hence, the given subspaces are the only subspaces of $\mathbb{R}^4$.

6.62 Problem 62

$\det(A) = 3$ gives that the columns of A are linearly independent, and hence form a basis for $\mathbb{R}^4$.

Finding the coordinate vector of $\mathbf{v}$ is equivalent to solving $A\mathbf{x} = \mathbf{v}$ for $\mathbf{x}$. Performing Gaussian elimination on the augmented matrix $[A|\mathbf{v}]$, we have

$$\begin{pmatrix} 1 & 2 & -1 & 1 & -2 \\ 3 & 2 & 0 & -2 & -6 \\ 0 & 1 & -1 & 1 & -4 \\ 5 & 3 & 0 & -1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 1 & -2 \\ 0 & -4 & 3 & -5 & 0 \\ 0 & 0 & 1 & 1 & 16 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Using back substitution, we obtain

$$[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix} -2 \\ 4 \\ 12 \\ 4 \end{pmatrix}.$$

6.63 Problem 63

$$\mathbf{v} = A\begin{pmatrix}1\\6\\-1\\4\end{pmatrix} = \begin{pmatrix}1 & 2 & -1 & 1\\3 & 2 & 0 & -2\\0 & 1 & -1 & 1\\5 & 3 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\6\\-1\\4\end{pmatrix} = \begin{pmatrix}18\\7\\11\\19\end{pmatrix}.$$

6.64 Problem 64

$$\mathbf{v} = \begin{pmatrix}1 & 3 & 2\\2 & 7 & 4\\-2 & -5 & 9\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\10\end{pmatrix}.$$

6.65 Problem 65

a)

Solving

$$\mathbf{v} = \begin{pmatrix}1 & 1 & -2\\0 & 1 & 0\\1 & 1 & -1\end{pmatrix}[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix}1\\2\\3\end{pmatrix},$$

for $[\mathbf{v}]_{\mathscr{B}}$, we obtain

$$[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix}3\\2\\2\end{pmatrix}.$$

b)

Solving

$$\mathbf{v} = \begin{pmatrix}1 & 1 & -2\\0 & 1 & 0\\1 & 1 & -1\end{pmatrix}[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},$$

for $[\mathbf{v}]_{\mathscr{B}}$, we obtain

$$[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix}-a_1 - a_2 + 2a_3\\a_2\\-a_1 + a_3\end{pmatrix}.$$

6.66 Problem 66

a)

$$\mathbf{v} = B[\mathbf{v}]_B = \begin{pmatrix}1 & 3 & -2\\-1 & 4 & 3\\2 & 6 & -3\end{pmatrix}\begin{pmatrix}3\\1\\-3\end{pmatrix} = \begin{pmatrix}12\\-8\\21\end{pmatrix}.$$

b)

Solving

$$\mathbf{v} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 4 & 3 \\ 2 & 6 & -3 \end{pmatrix} [\mathbf{w}]_B = \begin{pmatrix} 7 \\ -3 \\ 11 \end{pmatrix}$$

for $[\mathbf{w}]_B$, we obtain

$$[\mathbf{w}]_B = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}.$$

6.67 Problem 67

a)

It is easy to see that $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = 1$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_3 \cdot \mathbf{v}_1 = 0$. Hence, S is an orthonormal set of vectors in $\mathbb{R}^3$.

b)

Let A be a 3 by 3 matrix whose columns are the vectors in S. Then, S is linearly independent because $\det(A) = 1/6 \neq 0$. Also, since $|S| = 3$, we have that S is a basis for $\mathbb{R}^3$.

c)

Let

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R}.$$

Since S is an orthonormal set, we have

$$\begin{aligned} x_1 &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) \cdot \mathbf{v}_1 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_1 = -2\sqrt{2}, \\ x_2 &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) \cdot \mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_2 = 2\sqrt{3}, \\ x_3 &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) \cdot \mathbf{v}_3 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_3 = \sqrt{6}. \end{aligned}$$

Hence,

$$\left[\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}\right]_S = \begin{pmatrix} -2\sqrt{2} \\ 2\sqrt{3} \\ \sqrt{6} \end{pmatrix}.$$

6.68 Problem 68

Suppose that

$$x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n = \mathbf{0}, \quad x_1, ..., x_n \in \mathbb{R}. \tag{*}$$

Since S is an orthonormal set, for any $x_j \in \{x_1, ..., x_n\}$, we have

$$x_j = (x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n) \cdot \mathbf{u}_j = \mathbf{0} \cdot \mathbf{u}_j = 0.$$

Hence, (*) holds only when $x_1 = \cdots = x_n = 0$, which implies that S is linearly independent. Also, since $|S| = n$, we have that S is a basis for $\mathbb{R}^n$. Further,

$$[\mathbf{v}]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_j = \mathbf{u}_j \cdot \mathbf{v}.$$

6.69 Problem 69

a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

b)

No. Because the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not in S.

6.70 Problem 70

a)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

b)

Yes. This can be easily proved using the Subspace Theorem.

6.71 Problem 71

Suppose that

$$\lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$. Equating the coefficients gives

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= 0, \\ \lambda_3 &= 0, \\ \lambda_4 &= 0. \end{aligned}$$

This is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which has unique solution $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Therefore, the four matrices are linearly independent, and hence form a basis for $M_{22}(\mathbb{R})$.

6.72 Problem 72

The proof is the same as Problem 71. Because $\mathbb{R}$ is a subfield of $\mathbb{C}$, all properties we used for $\mathbb{R}$ hold automatically for $\mathbb{C}$.

6.73 Problem 73

Suppose that

$$\lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$. Equating the cofficients gives

$$\begin{aligned} \lambda_1 + \lambda_4 &= 0, \\ \lambda_2 - i\lambda_3 &= 0, \\ \lambda_2 + i\lambda_3 &= 0, \\ \lambda_1 - \lambda_4 &= 0. \end{aligned}$$

Performing Gaussian elimination on the coefficient matrix, we have

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

where all columns are leading columns. Thus, the system of equations has unique solution $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Therefore, the four matrices are linearly independent, and hence form a basis for $M_{22}(\mathbb{C})$.

6.74 Problem 74

a)

$$[A]_{\mathscr{B}} = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

b)

This problem is equivalent to solving the system of equations

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

Performing Gaussian elimination on the augmented matrix and solving for the unknowns, we obtain

$$[A]_{\mathscr{P}} = \begin{pmatrix} (a_{11}+a_{22})/2 \\ (a_{12}+a_{21})/2 \\ (-a_{12}+a_{21})/(2i) \\ (a_{11}-a_{22}/2) \end{pmatrix}.$$

6.75 Problem 75

a)

$$\begin{pmatrix} -4 & 2 \\ -1 & -3 \end{pmatrix} = -4\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} - 3\begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix}.$$

b)

No. Because $|R| = 3 < 4 = \dim M_{22}(\mathbb{R})$.

6.77 Problem 77

Here gives the proof of axioms 2 and 4, the other two can be proved in a similar manner.

Associative Law of Addition

For any functions $f, g, h \in \mathcal{C}[X]$, we have

$$\begin{aligned} ((f+g)+h)(x) &= (f+g)(x) + h(x) \\ &= f(x) + g(x) + h(x) \\ &= f(x) + (g(x) + h(x)) \\ &= f(x) + (g+h)(x) \\ &= (f + (g+x))(x). \end{aligned}$$

Thus, $\mathcal{C}[X]$ is associative.

Existence of the Zero

The zero function is $f(x) = 0, x \in X$, such that for any function $g \in \mathcal{C}[X]$, we have

$$(f+g)(x) = (g+f)(x) = g(x), \quad x \in X.$$

6.78 Problem 78

This can be easily proved using the Subspace Theorem.

6.79 Problem 79

No. Because the zero function $y(x) = 0$ of $\mathcal{R}[\mathbb{R}]$ is not in S.

6.80 Problem 80

Firstly, we notice that $\mathcal{C}^{(k)}[\mathbb{R}]$ is a subset of the vector space $\mathcal{R}[\mathbb{R}]$. The zero function $f(x) = 0, x \in \mathbb{C}$ is in $\mathcal{C}^{(k)}[\mathbb{R}]$. In addition, for any functions $f, g \in \mathcal{C}^{(k)}[\mathbb{R}]$ and any scalars $\lambda, \mu \in \mathbb{R}$, we have that for $(\lambda f + \mu g)$, the first k derivatives exist and is continuous. Hence, by the alternative Subspace Theorem, $\mathcal{C}^{(k)}[\mathbb{R}]$ is a subspace of $\mathcal{R}[\mathbb{R}]$.

6.82 Problem 82

It is easy to verify that the zero function $f(x) = 0, x \in [-\pi, \pi]$ is in S.

For any functions $f, g \in S$ and any scalars $\lambda, \mu \in \mathbb{R}$, since $\mathcal{R}[[-\pi, \pi]]$ is a vector space, we have $(\lambda f + \mu g) \in \mathcal{R}[[-\pi, \pi]]$, and

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(x+t)((\lambda f + \mu g)(t))dt &= \int_{-\pi}^{\pi} \cos(x+t)(\lambda f + \mu g)(t)dt \\
&= \int_{-\pi}^{\pi} \cos(x+t)(\lambda f(t) + \mu g(t))dt \\
&= \lambda \int_{-\pi}^{\pi} \cos(x+t)f(t)dt + \mu \int_{-\pi}^{\pi} \cos(x+t)g(t)dt \\
&= \lambda 0 + \mu 0 \\
&= 0,
\end{aligned}$$

which implies that $(\lambda f + \mu g) \in S$. Therefore, by the alternative Subspace Theorem, S is a subspace of $\lambda, \mu \in \mathcal{R}[[-\pi, \pi]]$.

6.83 Problem 83

Suppose that

$$\lambda_1 f_1(x) + \cdots + \lambda_n f_n(x) = 0, \tag{*}$$

where $\lambda_1, ..., \lambda_n \in \mathbb{R}$. Multiplying $f_i(x)$ on both sides of (*), where $1 \leqslant i \leqslant n$, we have

$$(\lambda_1 f_1(x) + \cdots + \lambda_n f_n(x))f_i(x) = (0)f_i(x),$$

that is,

$$\lambda_1 f_1(x)f_i(x) + \cdots + \lambda_i f_i^2(x) + \cdots + \lambda_n f_n(x)f_i(x) = 0.$$

Integrating both sides from a to b with respect to x gives

$$\int_a^b (\lambda_1 f_1(x)f_i(x) + \cdots + \lambda_i f_i^2(x) + \cdots + \lambda_n f_n(x)f_i(x))dx = \int_a^b 0\,dx = 0.$$

Noticing that the left-hand side can be further reduced to

$$\lambda_1 \int_a^b f_1(x)f_i(x)dx + \cdots + \lambda_i \int_a^b f_i^2(x)dx + \cdots + \lambda_n \int_a^b f_n(x)f_i(x)dx = \lambda_i,$$

we have $\lambda_i = 0$ for $1 \leqslant i \leqslant n$. Therefore, (*) has unique solution $\lambda_1 = \cdots = \lambda_n = 0$, which implies that S is a linearly independent set.

6.84 Problem 84

This can be easily proved using the Subspace Theorem. I may come back and update this at some point.

6.85 Problem 85

No, because the zero polynomial is not in S.

6.86 Problem 86

This can be easily proved using the Subspace Theorem. I may come back and update this at some point.

6.87 Problem 87

This problem is equivalent to check the solvability of

$$\begin{pmatrix} 1 & -4 & -5 \\ 2 & -1 & -1 \\ 3 & 9 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 30 \end{pmatrix}.$$

Performing Gaussian elimination on the augmented matrix, we have

$$\begin{pmatrix} 1 & -4 & -5 & -6 \\ 2 & -1 & -1 & 2 \\ 3 & 9 & 12 & 30 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -4 & -5 & -6 \\ 0 & 7 & 9 & 14 \\ 0 & 0 & 0 & 6 \end{pmatrix},$$

where the right-most column is a leading column. Therefore, the system of equations has no solution, that is, $p \notin \operatorname{span}(p_1, p_2, p_3)$.

6.88 Problem 88

Let $p(z) = a_0 + a_1 x + a_2 x^2$. This problem is equivalent to finding the conditions for a_0, a_1, a_2 such that the following system of equations has a solution:

$$\begin{pmatrix} 3 & -3 & -6 \\ 2 & -2 & -4 \\ 0 & 5 & 15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix}.$$

Performing Gaussian elimination on the augmented matrix, we have

$$\begin{pmatrix} 3 & -3 & -6 & a_2 \\ 2 & -2 & -4 & a_1 \\ 0 & 5 & 15 & a_0 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & -3 & -6 & a_2 \\ 0 & 1 & 3 & \frac{1}{5}a_0 \\ 0 & 0 & 0 & a_1 - \frac{2}{3}a_2 \end{pmatrix}.$$

Hence, the condition $a_1 - \dfrac{2}{3}a_2 = 0$ must be satisfied so that p can be a linear combination of p_1, p_2 and p_3.

6.89 Problem 89

No for both problems. Because $\operatorname{rank}(A) = 2 < 3 = \dim \mathbb{P}_2$.

6.90 Problem 90

$$\begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & -4 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

where the third column is not leading. Therefore, S is not a linearly independent set.

To find an expression of p_3 in terms of p_1 and p_2, we can treat the row echelon form as an augmented matrix and use back substitution, which gives

$$p_3 = -p_1 + 3p_2.$$

6.92 Problem 92

This problem is equivalent to solving the system of equations $A\mathbf{x} = \mathbf{b}$. The steps have been omitted for simplicity. The answer is $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

6.95 Problem 95

Suppose that

$$x_1 p_1(x) + \cdots + x_n p_n(x) = 0, \tag{*}$$

where $x_1, ..., x_n \in \mathbb{R}$. Multiplying $p_i(x)$ on both sides of (*), where $1 \leqslant i \leqslant n$, we have

$$(x_1 p_1(x) + \cdots + x_n p_n(x)) p_i(x) = (0) p_i(x),$$

that is,

$$x_1 p_1(x) p_i(x) + \cdots + x_i p_i^2(x) + \cdots + x_n p_n(x) p_i(x) = 0.$$

Integrating both sides from a to b with respect to x gives

$$\int_a^b (x_1 p_1(x) p_i(x) + \cdots + x_i p_i^2(x) + \cdots + x_n p_n(x) p_i(x)) dx = \int_a^b 0 \, dx = 0.$$

Noticing that the left-hand side can be further reduced to

$$x_1 \int_a^b p_1(x) p_i(x) dx + \cdots + x_i \int_a^b p_i^2(x) dx + \cdots + x_n \int_a^b p_n(x) p_i(x) dx = x_i,$$

we have $x_i = 0$ for $1 \leqslant i \leqslant n$. Therefore, (*) has unique solution $x_1 = \cdots = x_n = 0$, which implies that S is a linearly independent set. Also, since $|S| = n = \dim \mathbb{P}_{n-1}(\mathbb{R})$, S is a basis for $\mathbb{P}_{n-1}(\mathbb{R})$, and hence, by the definition of coordinate vector,

$$[p]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{where} \quad x = \int_a^b p_i(x) p(x) dx.$$

Chapter 7

Linear Transformations

7.3 Problem 3

a)

$T : \mathbb{C} \to \mathbb{R}$ is a linear map because

$$\begin{aligned} T(\lambda z_1 + \mu z_2) &= \mathrm{Re}(\lambda z_1 + \mu z_2) \\ &= \lambda \mathrm{Re}(z_1) + \mu \mathrm{Re}(z_2) \\ &= \lambda T(z_1) + \mu T(z_2) \end{aligned}$$

for any $z_1, z_2 \in \mathbb{C}$ and any $\lambda, \mu \in \mathbb{R}$.

b)

$T : \mathbb{C} \to \mathbb{R}$ is a linear map because

$$\begin{aligned} T(\lambda z_1 + \mu z_2) &= \mathrm{Im}(\lambda z_1 + \mu z_2) \\ &= \lambda \mathrm{Im}(z_1) + \mu \mathrm{Im}(z_2) \\ &= \lambda T(z_1) + \mu T(z_2) \end{aligned}$$

for any $z_1, z_2 \in \mathbb{C}$ and any $\lambda, \mu \in \mathbb{R}$.

c)

$T : \mathbb{C} \to [0, \infty)$ is not a linear map because $T(-1) \neq -T(1)$.

d)

$T : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ is not a linear map because neither the domain nor the codomain is a vector space.

e)

$T : \mathbb{C} \to \mathbb{C}$ is a linear map because it preserves linear combination.

7.5 Problem 5

If $n = 1$, then $T(\lambda_1 \mathbf{v}_1) = \lambda_1 T(\mathbf{v}_1)$.

Suppose that for $n > 1$,

$$T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n)$$

holds. Then,

$$\begin{aligned} T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_{n+1} \mathbf{v}_{n+1}) &= T((\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) + \lambda_{n+1} \mathbf{v}_{n+1}) \\ &= 1 \cdot T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) + \lambda_{n+1} T(\mathbf{v}_{n+1}) \\ &= \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_{n+1} T(\mathbf{v}_{n+1}). \end{aligned}$$

7.10 Problem 10

If T is a linear map, then

$$\begin{aligned} T\begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix} = T\left(3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}\right) = 3T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + T\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \\ = 3\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \end{pmatrix} \neq \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \end{aligned}$$

Therefore, T is not a linear map.

7.16 Problem 16

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}.$$

Since

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\begin{pmatrix} -1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

we have

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

7.17 Problem 17

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}.$$

Since

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

we have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

7.18 Problem 18

Here gives an intuitive but slightly cumbersome solution.

We notice that $T(\mathbf{p}) = \mathbf{q}$ satisfies

$$\mathbf{p} + \mathbf{q} = \lambda \mathbf{d} \quad \text{for some} \quad \lambda \in \mathbb{R} \tag{*}$$

and

$$\text{proj}_{\mathbf{x}}\mathbf{p} = \text{proj}_{\mathbf{x}}\mathbf{q}. \tag{**}$$

(*) implies that

$$q_i = \lambda d_i - p_i \quad \text{for} \quad 1 \leqslant i \leq n.$$

(**) implies that

$$\left(\frac{\mathbf{p} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\right) \mathbf{d} = \left(\frac{\mathbf{q} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\right) \mathbf{d},$$

that is,

$$\left(\sum_{j=1}^{n} p_j d_j\right) \mathbf{d} = \left(\sum_{j=1}^{n} q_j d_j\right) \mathbf{d} = \left(\sum_{j=1}^{n} (\lambda d_j - p_j) d_j\right) \mathbf{d},$$

where we have cancelled $1/\|\mathbf{d}\|^2$ on both sides. Rearranging the equation, we obtain

$$\lambda = \frac{2\sum_{j=1}^{n} p_j d_j}{\sum_{k=1}^{n} d_k^2} = \frac{2\sum_{j=1}^{n} p_j d_j}{\|\mathbf{d}\|^2}.$$

Plugging this back into (**), we have

$$q_i = \frac{2\sum_{j=1}^{n} p_j d_j d_i}{\|\mathbf{d}\|^2} - p_i = p_i \left(\frac{2d_i^2}{\|\mathbf{d}\|^2} - 1\right) + \sum_{j=1, j\neq i}^{n} p_j \frac{2d_i d_j}{\|\mathbf{d}\|^2}.$$

Hence, the matrix A is in the form of

$$[A]_{ij} = \begin{cases} \dfrac{2d_i^2}{\|\mathbf{d}\|^2} - 1 & \text{if} \quad i = j \\ \dfrac{2d_i d_j}{\|\mathbf{d}\|^2} & \text{if} \quad i \neq j \end{cases}.$$

7.20 Problem 20

This can be generalized to $\mathbb{R}^n$.

T is a linear map because for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and any scalars $\lambda, \mu \in \mathbb{R}$, we have

$$\begin{aligned} T(\lambda\mathbf{u} + \mu\mathbf{v}) &= \frac{(\lambda\mathbf{u} + \mu\mathbf{v}) \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} \\ &= \lambda\frac{\mathbf{u} \cdot \mathbf{b}}{|\mathbf{b}|^2} + \mu\frac{\mathbf{v} \cdot \mathbf{b}}{|\mathbf{b}|^2} \\ &= \lambda T(\mathbf{u}) + \mu T(\mathbf{v}). \end{aligned}$$

Hence, T is a linear map. Further,

$$T(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{\mathbf{b}\mathbf{b}^{\mathrm{T}}}{\mathbf{b}^{\mathrm{T}}\mathbf{b}}\mathbf{a}$$

implies that $A = \dfrac{\mathbf{b}\mathbf{b}^{\mathrm{T}}}{\mathbf{b}^{\mathrm{T}}\mathbf{b}}$.

7.21 Problem 21

S is not a linear map because for any $\mathbf{b} \neq \mathbf{0}$, $S(-\mathbf{b}) = S(\mathbf{b}) \neq -S(\mathbf{b})$.

7.22 Problem 22

$$\begin{aligned}
A_\theta A_\phi &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix} \\
&= A_{\phi+\theta},
\end{aligned}$$

which says that rotating in the plane by angles ϕ and θ consecutively is equivalent to rotating by angle $\phi + \theta$.

7.23 Problem 23

$$R_\alpha(\mathbf{i}) = \begin{pmatrix} \cos\alpha \\ 0 \\ -\sin\alpha \end{pmatrix}, \quad R_\alpha(\mathbf{j}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R_\alpha(\mathbf{k}) = \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}.$$

Hence,

$$A = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix},$$

which is obviously a linear map.

7.27 Problem 27

By the definition of kernel, matrix A can be

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

7.30 Problem 30

a)

For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and any scalars $\lambda, \mu \in \mathbb{R}$, we have

$$T(\lambda\mathbf{u} + \mu\mathbf{v}) = \mathbf{b} \times (\lambda\mathbf{u} + \mu\mathbf{v}) = \lambda(\mathbf{b} \times \mathbf{u}) + \mu(\mathbf{b} \times \mathbf{v}) = \lambda T(\mathbf{u}) + \mu T(\mathbf{v}).$$

Hence, T is a linear map.

b)

$$T(\mathbf{x}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3x_2 + 2x_3 \\ 3x_1 - x_3 \\ -2x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives the system of equations

$$\begin{aligned} -3x_2 + 2x_3 &= 0 \\ 3x_1 - x_3 &= 0 \\ -2x_1 + x_2 &= 0. \end{aligned}$$

Performing Gaussian elimination on the coefficient matrix, we obtain

$$\begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix},$$

which implies that the nullity of T is 1.

7.39 Problem 39

If a) holds, then A is invertible, that is, $\det(A) \neq 0$, which implies that the nullity is 0, and hence b) and c) hold. Then, by the Rank-nullity Theorem for matrix, we have $\operatorname{rank}(A) = n - \operatorname{nullity}(A) = n - 0 = n$. Thus, d) and e) hold, and the n columns of A are linearly independent, which implies that f) holds. Finally, if f) holds, we have that the columns of A are linearly independent, meaning that A is invertible, and hence a) holds.

7.40 Problem 40

The rank of A is r means that after Gaussian elimination, only the top r rows are not all zeroes.

7.41 Problem 41

Because by the Rank-nullity Theorem for matrix, the rank of A is $n-\nu$. Hence, there are $m-(n-\nu) = m - n + \nu$ zero rows after Gaussian elimination.

7.42 Problem 42

In this case, A is a full rank matrix that has no zero rows after Gaussian elimination.

7.43 Problem 43

All the statements are the properties of a linear isomorphism.

7.44 Problem 44

For any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4$ and any scalars $\lambda, \mu \in \mathbb{R}$, we have

$$T(\lambda \mathbf{a} + \mu \mathbf{b}) = \begin{pmatrix} \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 \\ \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \end{pmatrix} = \lambda \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \mu \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \lambda T(\mathbf{a}) + \mu T(\mathbf{b}).$$

Hence, T is a linear map.

7.45 Problem 45

For any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4$ and any scalars $\lambda, \mu \in \mathbb{R}$, we have

$$\begin{aligned} T(\lambda \mathbf{a} + \mu \mathbf{b}) &= \begin{pmatrix} 3(\lambda a_1 + \mu b_1) - 2(\lambda a_4 + \mu b_4) & (\lambda a_4 + \mu b_4) + 2(\lambda a_3 + \mu b_3) \\ -5(\lambda a_2 + \mu b_2) + 3(\lambda a_3 + \mu b_3) & \lambda a_1 + \mu b_1 \end{pmatrix} \\ &= \lambda \begin{pmatrix} 3a_1 - 2a_4 & a_4 + 2a_3 \\ -5a_2 + 3a_3 & a_1 \end{pmatrix} + \mu \begin{pmatrix} 3b_1 - 2b_4 & b_4 + 2b_3 \\ -5b_2 + 3b_3 & b_1 \end{pmatrix} \\ &= \lambda T(\mathbf{a}) + \mu T(\mathbf{b}). \end{aligned}$$

Hence, T is a linear map.

7.46 Problem 46

Yes.

7.48 Problem 48

It is easy to prove that T is a linear map, and we can also find the matrix A by

$$T \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_1 - 3a_2 \\ 2a_3 - 3a_4 \\ a_2 \\ 3a_1 - a_2 + 2a_3 + 4a_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 4 \end{pmatrix}}_{A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

7.51 Problem 51

For any $p_1, p_2 \in \mathbb{P}_3$ and any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned} (T(\lambda_1 p_1 + \lambda_2 p_2))(x) &= \int_0^x (\lambda_1 p_1 + \lambda_2 p_2)(t)dt \\ &= \int_0^x (\lambda_1 p_1)(t)dt + \int_0^x (\lambda_2 p_2)(t)dt \\ &= \lambda_1 \int_0^x p_1(t)dt + \lambda_2 \int_0^x p_2(t)dt \\ &= \lambda_1 (T(p_1))(x) + \lambda_2 (T(p_2))(x), \end{aligned}$$

which implies that $T(\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$. Hence, T is a linear map.

7.52 Problem 52

The zero function of $\mathcal{R}[\mathbb{R}]$ is in V.

For any functions $f_1, f_2 \in V$ and any scalars $\lambda_1, \lambda_2 \in \mathbb{R}$, since $\mathcal{R}[\mathbb{R}]$ is a vector space, it is closed under linear combination, so we have $(\lambda_1 f_1 + \lambda_2 f_2) \in \mathcal{R}[\mathbb{R}]$, and hence, $\int_0^x (\lambda_1 f_1 + \lambda_2 f_2)(t)dt$ also exists for all $x \in \mathbb{R}$, which implies that $(\lambda_1 f_1 + \lambda_2 f_2) \in V$.

Therefore, by the alternative Subspace Theorem, V is a subspace.

For the second part of the problem, for any functions $f_1, f_2 \in V$ and any scalars $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned}
(T(\lambda_1 f_1 + \lambda_2 f_2))(x) &= \int_0^x (\lambda_1 f_1 + \lambda_2 f_2)(t)dt \\
&= \int_0^x (\lambda_1 f_1)(t)dt + \int_0^x (\lambda_2 f_2)(t)dt \\
&= \lambda_1 \int_0^x f_1(t)dt + \lambda_2 \int_0^x f_2(t)dt \\
&= \lambda_1 (T(f_1))(x) + \lambda_2 (T(f_2))(x),
\end{aligned}$$

which implies that $T(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 T(f_1) + \lambda_2 T(f_2)$. Hence, T is a linear map.

7.53 Problem 53

S is not a linear map because $\mathbb{Z}$ is not a vector space. Also, S is not a linear map either because $S(1.2 + 1.4) = S(2.6) = 3 \neq 2 = S(1.2) + S(1.4)$.

7.54 Problem 54

Since the Laplace transform is linear, applying it to both sides of the equation, we have

$$\begin{aligned}
\text{LHS} &= L(y'' + 4y' + 3y)(s) \\
&= L(y'')(s) + 4L(y')(s) + 3L(y)(s) \\
&= \int_0^\infty e^{-st} y''(t)dt + 4\int_0^\infty e^{-st} y'(t)dt + 3y_L(s) \\
&= \left[e^{-st} y'(t)\right]_0^\infty + (s+4)\int_0^\infty e^{-st} y'(t)dt + 3y_L(s) \\
&= -y'(0) + (s+4)\left(\left[e^{-st} y(t)\right]_0^\infty + y_L(s)\right) + 3y_L(s) \\
&= -y'(0) + (s+4)(-y(0)) + (s+3)(s+1)y_L(s) \\
&= -(s+6) + (s+3)(s+1)y_L(s),
\end{aligned}$$

which equals to

$$\text{RHS} = L\left(e^{-3s}\right) = \frac{1}{s+3}.$$

Solving for $y_L(s)$, we obtain

$$y_L(s) = \frac{s^2 + 9s + 19}{(s+3)^2(s+1)}.$$

7.55 Problem 55

a)

For any $p_1(x), p_2(x) \in \mathbb{P}_3(\mathbb{R})$ and any scalars $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned}
T(\lambda_1 p_1(x) + \lambda_2 p_2(x)) &= T((\lambda_1 a_1 + \lambda_2 a_2) + (\lambda_1 b_1 + \lambda_2 b_2)x + (\lambda_1 c_1 + \lambda_2 c_2)x^2 + (\lambda_1 d_1 + \lambda_2 d_2)x^3) \\
&= \begin{pmatrix} (\lambda_1 a_1 + \lambda_2 a_2) - (\lambda_1 b_1 + \lambda_2 b_2) \\ (\lambda_1 c_1 + \lambda_2 c_2) - (\lambda_1 d_1 + \lambda_2 d_2) \end{pmatrix} \\
&= \lambda_1 \begin{pmatrix} a_1 - b_1 \\ c_1 - d_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} a_2 - b_2 \\ c_2 - d_2 \end{pmatrix} \\
&= \lambda_1 T(p_1(x)) + \lambda_2 T(p_2(x)).
\end{aligned}$$

Hence, T is linear.

b)

$$T(p(x)) = T\begin{pmatrix} (-2) - (-2) \\ 3 - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies that $p(x)$ is in the kernel of T.

7.56 Problem 56

a)

$$T\begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix} = (1 - 2(-3)) + (2 - 4)x = 7 - 2x.$$

b)

For any $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} \in \mathbb{R}^4$ and any scalars $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned}
T\left(\lambda_1 \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}\right) &= T\begin{pmatrix} \lambda_1 a_1 + \lambda_2 a_2 \\ \lambda_1 b_1 + \lambda_2 b_2 \\ \lambda_1 c_1 + \lambda_2 c_2 \\ \lambda_1 d_1 + \lambda_2 d_2 \end{pmatrix} \\
&= ((\lambda_1 a_1 + \lambda_2 a_2) - 2(\lambda_1 b_1 + \lambda_2 b_2)) + ((\lambda_1 c_1 + \lambda_2 c_2) + (\lambda_1 d_1 + \lambda_2 d_2))x \\
&= \lambda_1((a_1 - 2b_1) + (c_1 + d_1)x) + \lambda_2((a_2 - 2b_2) + (c_2 + d_2)x) \\
&= \lambda_1 T\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \lambda_2 T\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}.
\end{aligned}$$

Hence, T is a linear transformation.

c)

Let $T\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=0$, we can easily come up with a non-zero vector lying in $\ker(T)$, for example, $\begin{pmatrix}2\\1\\1\\-1\end{pmatrix}$.

7.57 Problem 57

Here gives the solution to $T\begin{pmatrix}i\\2\\-1\end{pmatrix}$. The other one can be obtained with the same manner.

$$T\begin{pmatrix}i\\2\\-1\end{pmatrix}=iT\begin{pmatrix}1\\0\\0\end{pmatrix}+2T\begin{pmatrix}0\\1\\0\end{pmatrix}-T\begin{pmatrix}0\\0\\1\end{pmatrix}=(2+i)+(7-4i)z+2z^2-3iz^3.$$

7.58 Problem 58

a)

For any $p(x),q(x)\in\mathbb{P}_3$ and any scalars $\lambda,\mu\in\mathbb{R}$, we have

$$\begin{aligned}(T(\lambda p+\mu q))(x)&=\int_0^x(\lambda p+\mu q)(t)dt\\&=\int_0^x(\lambda p(t)+\mu q(t))dt\\&=\lambda\int_0^x p(t)dt+\mu\int_0^x q(t)dt\\&=\lambda(T(p))(x)+\mu(T(q))(x).\end{aligned}$$

Hence, $T(\lambda p+\mu q)=\lambda T(p)+\mu T(q)$, which implies that T is a linear map.

b)

Since

$$\begin{aligned}T(\mathbf{e}_1)&=\int_0^x 1dt=x,\\T(\mathbf{e}_2)&=\int_0^x tdt=\frac{1}{2}x,\\T(\mathbf{e}_3)&=\int_0^x t^2dt=\frac{1}{3}x^3,\\T(\mathbf{e}_4)&=\int_0^x t^3dt=\frac{1}{4}x^4,\end{aligned}$$

we have

$$A=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{4}\end{pmatrix}.$$

c)

Noticing that the columns of A are linearly independent, they can be a basis for $\mathrm{im}(T)$.

d)

Since all the columns of A are linearly independent, we have $\ker(T) = \{\mathbf{0}\}$, which implies that the basis for the kernel of T is the empty set.

7.61 Problem 61

Let $B_V = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ and $B_W = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$, then

$$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and hence

$$\left[T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{B_W} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \left[T \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]_{B_W} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

Therefore, by the Matrix Representation Theorem, $A = \begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix}$.

7.62 Problem 62

Noticing that the matrix representing G with respect to the standard bases in both domain and codomain is just A itself, we have

$$G(p) = A \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 0 \end{pmatrix},$$

, which implies that $G(p) = -6 + 4x$.

7.65 Problem 65

Let $p(x) = \sum_{k=0}^{n} a_k x^k$. Since T is a linear map,

$$q(x) = \sum_{k=0}^{n} a_k T(x^k) = \sum_{k=0}^{n} ((k(k-1)a_k) - (3ka_k) + 3a_k)x^k = \sum_{k=0}^{n} (k-1)(k-3)a_k x^k, \quad 0 \leqslant k \leqslant n.$$

To fit it into the matrix representation, let $i = k + 1$, then we end up with a $(n + 1)$ by $(n + 1)$ diagonal matrix with

$$[A]_{ii} = (i - 2)(i - 4).$$

Hence, $\ker(T) = \lambda x + \mu x^3$ where $\lambda, \mu \in \mathbb{R}$ and the nullity of T is 2.

7.66 Problem 66

The proof of the orthonormality of B' is omitted for simplicity.

Let Q be the matrix whose columns are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and Q' be the matrix whose columns are $\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3$. Since

$$\mathbf{a} = Q \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = Q' \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix}$$

and Q' is orthonormal, there exists a matrix $E = Q'^{-1}Q$ such that

$$E \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix}.$$

Hence, T is a linear map, and

$$E = Q'^{-1}Q = \left(Q \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \right)^{-1} Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix}.$$

Finally, solving the equations for $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, we obtain

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{\sqrt{2}}(\mathbf{u}'_1 - \mathbf{u}'_2), \\ \mathbf{u}_2 &= -\mathbf{u}'_3, \\ \mathbf{u}_3 &= \frac{1}{\sqrt{2}}(\mathbf{u}'_1 + \mathbf{u}'_2). \end{aligned}$$

Hence, for any vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in V$, we have

$$\begin{aligned} \mathbf{v} &= v_1\mathbf{u}_1 + v_2\mathbf{u}_2 + v_3\mathbf{u}_3 \\ &= v_1\frac{1}{\sqrt{2}}(\mathbf{u}'_1 - \mathbf{u}'_2) + v_2(-\mathbf{u}'_3) + v_3\frac{1}{\sqrt{2}}(\mathbf{u}'_1 + \mathbf{u}'_2) \\ &= \frac{1}{\sqrt{2}}(v_1 + v_3)\mathbf{u}'_1 + \frac{1}{\sqrt{2}}(-v_1 + v_3)\mathbf{u}'_2 + (-v_2)\mathbf{u}'_3, \end{aligned}$$

which implies that

$$[\mathbf{v}]_{B'} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v_1 + v_3) \\ \frac{1}{\sqrt{2}}(-v_1 + v_3) \\ -v_2 \end{pmatrix}.$$

Since this is equal to $E\mathbf{v}$, the matrix is a matrix representation of the identity map.

7.67 Problem 67

$$\begin{aligned} [(T+S)(\mathbf{v})]_{B_W} &= [T(\mathbf{v}) + S(\mathbf{v})]_{B_W} \\ &= [T(\mathbf{v})]_{B_W} + [S(\mathbf{v})]_{B_W} \\ &= A[\mathbf{v}]_{B_V} + B[\mathbf{v}]_{B_V} \\ &= (A+B)[\mathbf{v}]_{B_V}. \end{aligned}$$

Hence, $(A+B)$ is the matrix representing $T+S$ with respect to the bases B_V and B_W.

7.68 Problem 68

$$\begin{aligned}[(S \circ T)(\mathbf{u})]_{B_W} &= [S(T(\mathbf{u}))]_{B_W} \\ &= [S(A[\mathbf{u}]_{B_U})]_{B_W} \\ &= B(A[\mathbf{u}]_{B_U}) \\ &= (BA)[\mathbf{u}]_{B_U}.\end{aligned}$$

Hence, BA is the matrix which represents the composition function $S \circ T$ with respect to the bases B_U and B_W.

7.69 Problem 69

a)

Suppose that

$$\begin{aligned}\lambda_1 e^x + \lambda_2(x-1)e^x + \lambda_3(x-1)(x-2)e^x &= e^x((\lambda_1 - \lambda_2 + 2\lambda_3) + (\lambda_2 - 3\lambda_3)x + \lambda_3 x^2) \\ &= 0,\end{aligned}$$

then we obtain the system of equations

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since all the columns of the coefficient matrix are leading, we have that the system has only zero solution. Hence, B is linearly independent.

b)

i)

By the Matrix Representation Theorem, the columns of the matrix are the vectors for

$$\begin{aligned} D(e^x) &= e^x, \\ D((x-1)e^x) &= xe^x, \\ D((x-1)(x-2)e^x) &= (x^2 - x - 1)e^x. \end{aligned}$$

Representing these vectors with respect to B, for e^x, we have the system of equations

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solving for x_1, x_2, x_3, we obtain the vector representation for e^x with respect to B, which is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Similarly, the vector representations for xe^x and $(x^2 - x - 1)e^x$ with respect to B are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ respectively.

Therefore, the matrix for D with respect to the ordered basis B of W is

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

ii)

Since D is a linear map, the composition $D \circ D$ is also a linear map whose matrix representation is

$$AA = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

iii)

Because all the columns of the product of AA are leading, the system of equations has unique solution. Hence, the statement holds.

7.70 Problem 70

a)

This problem is omitted as similar proofs have been given in the previous problems.

b)

This problem is equivalent to solving the system of equations

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and we obtain

$$\ker(T) = \lambda \begin{pmatrix} -\frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda \in \mathbb{F}.$$

c)

$\operatorname{rank}(T) = 3$.

d)

No. Because T is linear, it is injective if and only if $\ker(T) = \{\mathbf{0}\}$, whereas the nullity of T is 1.

e)

By the Matrix Representation Theorem, the matrix of T with respect to the standard bases of $M_{22}(\mathbb{F})$ and $\mathbb{F}^3$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix}.$$