MATH1141 Problem Set Solutions - Calculus

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Contents

Chapter 2

Limits

2.10   Problem 10

d)

Method 1

$$\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3} = \lim_{x \to 3} \frac{\left(\frac{1}{x} - \frac{1}{3}\right) \cdot 3x}{(x - 3) \cdot 3x}$$

$$= \lim_{x \to 3} \frac{3 - x}{3x(x - 3)}$$

$$= -\lim_{x \to 3} \frac{1}{3x}$$

$$= -\frac{1}{9}.$$

Method 2

$$\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3} \overset{\text{L'H}}{=} \lim_{x \to 3} \frac{\left(\frac{1}{x} - \frac{1}{3}\right)'}{(x - 3)'}$$

$$= \lim_{x \to 3} \frac{-\frac{1}{x^2}}{1}$$

$$= -\frac{1}{9}.$$

2.11   Problem 11

a)

$$\lim_{x \to 2^-} \frac{|x - 2|}{x - 2} = -\lim_{x \to 2^-} \frac{x - 2}{x - 2} = -1.$$

b)

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{x-2}{x-2} = 1.$$

c)

No. Because the left- and right-hand limit are not equal.

2.12 Problem 12

b)

$$\lim_{x \to 0^-} \frac{4}{x} = -\infty$$
$$\lim_{x \to 0^+} \frac{4}{x} = \infty.$$

Since the left- and right-hand limit are not equal, the limit does not exist.

2.13 Problem 13

a)

For all $x \neq 0$, we have

$$-|x| \leq x \sin \frac{1}{x} \leq |x|.$$

Since

$$\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0,$$

by the Pinching Theorem, we obtain

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

b)

For all $x \neq 0$, we have

$$-x^2 \leq x^2 \sin \frac{1}{2x} \leq x^2.$$

Since

$$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0,$$

by the Pinching Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{2x} = 0.$$

2.14   Problem 14

a)

$$|CB| = \theta$$
$$|CA| = \sin\theta$$
$$|DB| = \tan\theta.$$

b)

According to the graph, for $0 < \theta < \pi/2$, we have

$$\text{Area}(OAC) \leq \text{Area}(OBC) \leq \text{Area}(OBD),$$

which implies that

$$\frac{1}{2}\sin\theta\cos\theta \leq \pi \cdot \frac{\theta}{2\pi} \leq \frac{1}{2}\tan\theta,$$

that is

$$\sin\theta\cos\theta \leq \theta \leq \tan\theta.$$

c)

Applying the result of part b, for $0 < \theta < \pi/2$,

$$\cos\theta = \frac{\sin\theta\cos\theta}{\sin\theta} \leq \frac{\theta}{\sin\theta} \leq \frac{\tan\theta}{\sin\theta} = \frac{1}{\cos\theta}.$$

Since

$$\lim_{\theta\to0^+}\cos\theta = \lim_{\theta\to0^+}\frac{1}{\cos\theta} = 1,$$

by the Pinching Theorem, we obtain

$$\lim_{\theta\to0^+}\frac{\theta}{\sin\theta} = 1.$$

d)

Analogous to part c), by using the result of part b and the Pinching Theorem, we obtain

$$\lim_{\theta\to0^-}\frac{\sin\theta}{\theta} = \lim_{\theta\to0^+}\frac{\sin\theta}{\theta} = 1,$$

which implies that

$$\lim_{\theta\to0}\frac{\sin\theta}{\theta} = 1.$$

2.15   Problem 15

$\cos(1/x)$ oscillates between -1 and 1 as x goes to 0. Hence, the limit of $\cos(1/x)$ as $x \to 0$ does not exist.

Chapter 3

Properties Of Continuous Functions

3.2   Problem 2

a)

$$\lim_{x \to 0^-} \exp(2x) = 1 = \lim_{x \to 0^+} \cos x,$$

which implies that f is continuous on $\mathbb{R}$.

3.5   Problem 5

Noticing that

$$f(-3) = -9 < 0$$
$$f(-2) = 5 > 0,$$

since f is continuous on $[-3, -2]$, by the Intermediate Value Theorem, there exists some $c \in (-3, -2)$ such that $f(c) = 0$.

Same for f on $[0, 1]$ and $[1, 2]$.

3.6   Problem 6

Let

$$f(x) = e^x - 2\cos x.$$

Noticing that

$$f(0) = -1 < 0$$
$$f(\pi) = e^\pi + 2 > 0,$$

since f is continuous on $[0, \pi]$, by the Intermediate Value Theorem, there exists some $c \in (0, \pi)$ such that $f(c) = 0$, that is, there is at least one positive real solution for $e^x = 2\cos x$.

3.9 Problem 9

b)

Yes. Because f is continuous on $[2, 4]$, by the max-Min Theorem, f has both a maximum and a minimum value on $[2, 4]$.

3.10 Problem 10

a)

$$f(x) = \sin \frac{100}{x^2 - 1}$$

b)

$$f(x) = -e^{-x^2}$$

Chapter 4

Differentiable Functions

4.4 Problem 4

a)

f is differentiable everywhere except at $x = 0$. It is continuous everywhere.

b)

f is differentiable and continuous everywhere.

c)

f is differentiable and continuous everywhere except at $x = -2$.

4.6 Problem 6

For $x \neq 0$, it is obvious that f is continuous. Since

$$\lim_{x \to 0} f(x) = 0,$$

f is also continuous at $x = 0$. Hence, f is continuous everywhere.

It is easy to see that f is differentiable for $x \neq 0$, so we only need to prove that f is differentiable at $x = 0$.

Noticing that

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$
$$= \lim_{h \to 0} h \sin \frac{1}{h}$$
$$= 0,$$

along with the fact that f is continuous everywhere, we can conclude that f is also differentiable everywhere.

The derivative of f with respective of x is given by

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Since the limit of f' as $x \to 0$ does not exist, f' is not continuous at $x = 0$.

4.9 Problem 9

b)

Take the derivative of both sides of the equation with respect to x, we have

$$2x - \frac{1}{2}(xy)^{-1/2}\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Rearranging the equation, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x\sqrt{xy} - y}{x - 4y\sqrt{xy}}.$$

4.11 Problem 11

Taking the derivative of both sides with respect to x, we have

$$3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 3 + 3\frac{\mathrm{d}y}{\mathrm{d}x},$$

which implies that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2 - 1}{y^2 - 1}, \quad y \neq \pm 1.$$

The slope is then given by

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1, y=2} = 0.$$

Hence, the tangent line is

$$y = 2.$$

4.12 Problem 12

a)

To guarantee that f is continuous at 0, we need to ensure that the left limit and right limits at 0 are equal. Specifically,

$$\lim_{x \to 0^-} ax + b = \lim_{x \to 0^+} \sin x = f(0),$$

which implies that $b = 0$.

f differentiable at 0 means that f is continuous at 0 and the left and right derivatives are equal. In this case,

$$a = \cos 0 = 1,$$
$$b = 0.$$

b)

Similar to question a, for (i), we have $b = 1$; for (ii), $a = 2, b = 1$.

4.14 Problem 14

For this problem, I combined the answers into one.

Noticing that 8.01 is close to 8, we can apply the linear approximation to $f(8.01)$. To be more specific,

$$f(8.01) \approx f(8) + f'(8)(8.01 - 8) = \frac{2401}{1200},$$

which is slightly better than the approximation by calculating $\sqrt[3]{8}$.

4.16 Problem 16

Denote the base by a, the height by b and the area under the ladder by $S_\triangle$. From the description, we know

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 1 \tag{4.16.1}$$

$$a^2 + b^2 = 5 \tag{4.16.2}$$

$$S_\triangle = \frac{1}{2}ab.$$

Differentiate both sides of (4.16.2) with respect to a, we have

$$\frac{\mathrm{d}b}{\mathrm{d}a} = -\frac{a}{b}. \tag{4.16.3}$$

Hence,

$$
\begin{aligned}
\left.\frac{\mathrm{d}S_\triangle}{\mathrm{d}t}\right|_{b=4} &= \frac{1}{2}\left.\left(\frac{\mathrm{d}a}{\mathrm{d}t}b + a\frac{\mathrm{d}b}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)\right|_{b=4} \\
&= \frac{1}{2}\left.\left(b - a \cdot \frac{a}{b}\right)\right|_{a=3,b=4} \\
&= \frac{7}{8} \ m^2/s
\end{aligned}
$$

where in the second step, we plugged in (4.16.1) and (4.16.3), and used the Pythagorean theorem for a.

Chapter 5

The Mean Value Theorem And Applications

5.3 Problem 3

Noticing that f is continuous on $[1,3]$ and differentiable on $(1,3)$, according to the Mean Value Theorem, there exists at least one point $c \in (1,3)$ such that

$$\begin{aligned}
f'(c) &= \frac{f(3) - f(1)}{3 - 1} \\
&= \frac{\cos 1 - \cos 1}{2} \\
&= 0,
\end{aligned}$$

which implies that f' has a zero on the interval $[1,3]$.

5.4 Problem 4

b)

Let

$$f(t) = -\ln(1 - t) - \frac{t}{1 - t}, \quad t \in [0, x].$$

Since f is continuous on $[0, x]$ and differentiable on $(0, x)$, by the Mean Value Theorem, there exists some $c \in (0, x)$ such that

$$\begin{aligned}
\frac{f(x)}{x} &= \frac{f(x) - f(0)}{x - 0} \\
&= f'(c) \\
&= -\frac{c}{(c - 1)^2} \\
&< 0,
\end{aligned}$$

which implies that

$$-\ln(1 - x) < x/(1 - x)$$

for $x \in (0, 1)$.

c)

Let
$$f(t) = e^t - t - 1, \quad t \in [0, x].$$

Since f is continuous on $[0, x]$ and differentiable on $(0, x)$, by the Mean Value Theorem, there exists some $c \in (0, x)$ such that

$$\frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0}$$
$$= f'(c)$$
$$= e^c - 1$$
$$> 0,$$

which implies that
$$1 + x < e^x$$

for $x > 0$.

5.7   Problem 7

a)

Let
$$f(x) = \sqrt{x}, \quad x \le 0.$$

By the Mean Value Theorem, there exists some $c \in (16, 17)$ such that

$$\sqrt{17} - \sqrt{16} = \frac{f(17) - f(16)}{17 - 16}$$
$$= f'(c)$$
$$= \frac{1}{2\sqrt{c}}$$
$$< \frac{1}{8}.$$

c)

Let
$$f(x) = \frac{1}{x}, \quad x \ne 0.$$

By the Mean Value Theorem, there exists some $c \in (1000, 1002)$ such that

$$\frac{1}{1000} - \frac{1}{1002} = f(1000) - f(1002)$$
$$= -2 \cdot \frac{f(1002) - f(1000)}{1002 - 1000}$$
$$= -2f'(c)$$
$$= \frac{2}{x^2}$$
$$< 2 \times 10^{-6}.$$

12

5.10   Problem 10

e)

Because f is continuous on the closed interval $[0, 3]$, its maximum and minimum values must occur at a critical point on $[0, 3]$.

The critical points of f on $[0, 3]$ are $x = 0$, $x = 1$, $x = 3/2$, $x = 2$ and $x = 3$. Substituting these values into $f(x)$, we obtain the maximum value $f(0) = f(3) = 2$, and the minimum value $f(1) = f(2) = 0$.

5.17   Problem 17

$$p'(x) = 3(x - 3)(x - 5) \begin{cases} > 0, & x < 3 \text{ or } x > 5 \\ < 0, & 3 < x < 5 \end{cases}.$$

Hence, p is continuous on $\mathbb{R}$, monotonically increasing on $(-\infty, 3)$ and $(5, \infty)$ and monotonically decreasing on $(3, 5)$. Also, since $\lim_{x \to -\infty} p(x) < 0$ and $p(3) = 3 > 0$, p has exactly one zero on the interval $(-\infty, 3)$ (This can be easily proved by contradiction using Rolle's Theorem). For the same reason, p has exactly one zero when $x \in (3, 5)$, and one zero when $x \in (5, \infty)$. Therefore, p has three zeroes on $\mathbb{R}$.

5.19   Problem 19

a)

$$\left| \int_0^3 2t - t^2 \, dt \right| = \left| \left(t^2 - \frac{1}{3}t^3 \right) \Big|_{t=0}^{t=3} \right| = 0.$$

b)

From question a, we know that at some point $t_0 > 0$, the velocity of the particle became 0, that is

$$2t_0 - t_0^2 = 0.$$

Solving for t_0, we have $t_0 = 2$. Hence, the total distance travelled is given by

$$2 \left| \int_0^2 2t - t^2 \, dt \right| = \frac{8}{3}.$$

5.21   Problem 21

b)

It is obvious that the limit goes to ∞, as exponential grows faster than polynomial. Alternatively, we can apply L'Hopital's Rule three times to see that the result diverges to infinity.

c)

$$\lim_{x \to -\infty} \frac{e^5}{3x^2} = 0.$$

5.24 Problem 24

Because $\lim_{x \to \infty} (4 + \cos x)/(2 - \cos x)$ does not exist.

5.25 Problem 25

It is easy to see that
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1 = f(0),$$
which implies that f is continuous at 0. Also, noticing that
$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{2h + 1 - 1}{h} = 2$$
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{e^{2h} - 1}{h} = 2,$$

that is, the left and right derivatives of f at 0 are equal, we can conclude that f is differentiable at 0.

5.27 Problem 27

a)

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$$
$$\stackrel{\text{L'H}}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$
$$= - \lim_{x \to 0^+} x$$
$$= 0.$$

b)

$$\lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} x \cdot \lim_{x \to 0^+} x \ln x$$
$$= 0 \cdot 0$$
$$= 0.$$

c)

Firstly, f must be continuous at 0, which means
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

must hold. Hence, $b = 0$.

In addition, the left and right derivatives of f at 0 must equal, that is,
$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h}.$$

14

To be more specific,
$$\lim_{h \to 0^-} \frac{ah + b - b}{h} = a = \lim_{h \to 0^+} \frac{h^2 \ln h - b}{h},$$
which implies that
$$a = \lim_{h \to 0^+} h \ln h = 0.$$

Putting together, we have
$$a = b = 0.$$

Chapter 6

Inverse Functions

6.1 Problem 1

a)

$$(f \circ g)(x) = \sqrt{1 + \left(\sqrt{x^2 - 1}\right)^2} = x, \quad x \geq 1$$
$$(g \circ f)(x) = \sqrt{\left(\sqrt{1 + x^2}\right)^2 - 1} = x, \quad x \geq 0.$$

b)

The domain of $f \circ g$ and $g \circ f$ is $[1, \infty)$.

6.2 Problem 2

b)

$$g^{-1}(x) = -\sqrt{x - 1}, \quad x \in [1, \infty),$$
$$\mathrm{Dom}(g^{-1}) = [-1, \infty),$$
$$\mathrm{Range}(g^{-1}) = (-\infty, 0],$$
$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))} = -\frac{1}{2\sqrt{x - 1}}.$$

6.4 Problem 4

a)

$$f'(x) = 4 - \sin x > 0$$

shows that f is increasing on $\mathbb{R}$ and hence one-to-one. Therefore, f has an inverse function.

b)

Noticing that

$$f^{-1}(2\pi) = \frac{\pi}{2},$$

we obtain

$$g'(2\pi) = \frac{1}{f'((f^{-1})(2\pi))} = \frac{1}{f'(\pi/2)} = \frac{1}{3}.$$

6.5 Problem 5

a)

The derivative of f with respect to x is given by

$$f'(x) = 3(x+1)(x-1),$$

which implies that f is increasing on $(-\infty, -1]$ and $[1, \infty)$ and decreasing on $[-1, 1]$. Also, since f is continuous on $\mathbb{R}$, f is not one-to-one.

b)

The restriction of f to $(-\infty, -1]$ has an inverse with domain $(-\infty, 3]$, the restriction of f to $[-1, 1]$ has an inverse with domain $[-1, 3]$, and the restriction of f to $[1, \infty)$ has an inverse with domain $[-1, \infty)$.

6.7 Problem 7

b)

The inverse function is given by

$$f^{-1}(x) = x^{1/17} - 1, \quad x \in \mathbb{R}.$$

The domain of f^{-1} is $\mathbb{R}$. f is continuous on $\mathbb{R}$ and it is differentiable everywhere except at $x = 0$.

c)

To analyze f, we need to get rid of the modulus sign, which gives

$$f(x) = \begin{cases} x^2 + x, & x > 0 \text{ or } x < -1 \\ -x^2 - x, & -1 \leq x \leq 0 \end{cases}.$$

By taking the derivative of f, it is easy to see that f is increasing on $[-1, -1/2]$ and $[0, \infty)$, and decreasing on $(\infty, -1]$ and $[-1/2, 0]$. Hence, f is one-to-one on each of the four intervals.

6.8 Problem 8

b)

$$\cos(\cos^{-1}(2/5)) = 2/5.$$

c)

$$\cos^{-1}(\cos(-\pi/3)) = \pi/3.$$

f)

$$\sin(\tan^{-1}(3/5)) = 3/\sqrt{34}.$$

6.10   Problem 10

b)

$$y = \tan^{-1} x \implies \tan y = x$$
$$\implies \frac{1}{\cos^2 y}\frac{dy}{dx} = 1$$
$$\implies \frac{dy}{dx} = \cos^2 y = \frac{1}{1+x^2}.$$

6.13   Problem 13

a)

When $x \neq 0$,

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+1/x^2}\cdot\left(-\frac{1}{x^2}\right) \\ &= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\ &= 0. \end{aligned}$$

b)

From question a, we have $f'(x) = 0$ for $x \neq 0$, which means that $f(x)$ is constant on either $(-\infty, 0)$ or $(0, \infty)$. Plugging in $x \pm 1$, we obtain that $f(x) = \pi/2$ on $(0, \infty)$ and $f(x) = -\pi/2$ on $(-\infty, 0)$.

c)

This means that when $x \neq 0$, $\tan^{-1}(x)$ and $\tan^{-1}(1/x)$ are complementary.

6.18   Problem 18

Let the distance be x, then the angle θ that we want to maximize can be expressed by

$$\theta = \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{6}{x}\right) = f(x).$$

Taking the derivative of f with respect to x, we have

$$f'(x) = \frac{96 - 2x^2}{(x^2 + 64)(x^2 + 36)}.$$

We can see that $f'(x)$ is greater than zero when $0 \leq x < 4\sqrt{3}$ and is smaller than zero when $x > 4\sqrt{3}$, indicating that $f(x)$ reaches its maximum at $4\sqrt{3}$. Hence, the distance should be $4\sqrt{3}$ metres.

Chapter 7

Curve Sketching

7.4 Problem 4

b)

The vertical asymptote is $x = -1$. Noticing that

$$\lim_{x \to \infty} \big(f(x) - (x-1)\big) = \frac{1}{x+1} = 0,$$

we can identify the oblique asymptote being $y = x - 1$.

7.5 Problem 5

b)

Noticing that

$$y = \frac{x-1}{x-2} = 1 + \frac{1}{x-2}$$

is obtained by translating $y = 1/x$ to the right by two units, then up by one unit, hence all its features can be derived using $y = 1/x$.

7.7 Problem 7

c)

$$y = x^{2/3}.$$

7.9 Problem 9

a)

Taking the derivative of y with respect to x, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}y/\mathrm{d}t}$$
$$= \frac{-1/(t-1)^2}{1/(t+1)^2}$$
$$= -\frac{(t+1)^2}{(t-1)^2}.$$

The slope of the tangent line at $t = 2$ is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=2} = -9,$$

which implies that the slope of the normal at $t = 2$ is $1/9$. Hence, the normal is

$$y = \frac{1}{9}x + \frac{52}{27}.$$

b)

Expressing t in terms of x and y respectively, we have

$$t = \frac{x}{1-x} = \frac{y}{y-1}, \quad x, y \neq 1.$$

Rearranging the equation, we have

$$x + y - 2xy = 0, \quad x, y \neq 1.$$

Taking the derivative of both sides with respect to x and plugging in the values of x and y when $t = 2$, we obtain

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=2} = \left.\frac{2y-1}{1-2x}\right|_{t=2} = -9.$$

Hence, the gradient of the normal is $1/9$.

7.12 Problem 12

a)

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$
$$= \mathbf{a} + (\mathbf{b}-\mathbf{a})t.$$

Hence,

$$\mathbf{p}(0) = \mathbf{a}$$
$$\mathbf{p}(1) = \mathbf{b}.$$

b)

$$\mathbf{q}(t) = (1-t)\mathbf{b} + t\mathbf{c}$$
$$= \mathbf{b} + (\mathbf{c} - \mathbf{b})t$$
$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Eliminating t, we obtain the Cartesian form

$$x + y - 4 = 0.$$

$$\mathbf{q}(1/2) = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}.$$

c)

$$\mathbf{r}(t) = (1-t)\mathbf{p}(t) + t\mathbf{q}(t)$$
$$= (1-t)(\mathbf{a} + (\mathbf{b} - \mathbf{a})t) + t(\mathbf{b} + (\mathbf{c} - \mathbf{b})t)$$
$$= (t-1)^2\mathbf{a} + 2t(1-t)\mathbf{b} + t^2\mathbf{c}.$$

Equating the coefficients, we obtain

$$p_0(t) = (t-1)^2$$
$$p_1(t) = 2t(1-t)$$
$$p_2(t) = t^2.$$

7.16   Problem 16

a)

$$x = r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta$$
$$y = r\sin\theta = 6\sin^2\theta = 3 - 3\cos 2\theta.$$

Hence,
$$x^2 + (y-3)^3 = (3\sin\theta)^2 + (3\cos\theta)^2 = 9,$$

which is a circle centered at $(0,3)$ with radius 3.

7.17   Problem 17

d)

$$x = r\cos\theta = 2|\cos\theta|\cos\theta = \begin{cases} 1 + \cos 2\theta, & \theta \in [0, \pi/2] \cup [3\pi/2, 2\pi] \\ -1 - \cos 2\theta, & \theta \in [\pi/2, 3\pi/2] \end{cases},$$

$$y = r\sin\theta = 2|\cos\theta|\sin\theta = \begin{cases} \sin 2\theta, & \theta \in [0, \pi/2] \cup [3\pi/2, 2\pi] \\ -\sin 2\theta, & \theta \in [\pi/2, 3\pi/2] \end{cases}.$$

This is equivalent to

$$(x - 1)^2 + y^2 = 1, \quad x \in [0, 2]$$
$$(x + 1)^2 + y^2 = 1, \quad x \in [-2, 0].$$

Therefore, the graph is two unit circles centered at $(1, 0)$ and $(-1, 0)$.

7.18 Problem 18

Since

$$x = r \cos \theta = \frac{a \cos \theta}{\theta},$$

$x \to \infty$ is equivalent to $\theta \to 0$. Hence,

$$\lim_{x \to \infty} (y - 1) = \lim_{\theta \to 0} (r \sin \theta - 1)$$
$$= \lim_{\theta \to 0} \left(\frac{a \sin \theta}{\theta} - 1 \right)$$
$$= 0,$$

which implies that $y = a$ is a horizontal asymptote to the spiral.

Chapter 8

Integration

8.1 Problem 1

a)

i)

$$\underline{S}_{\mathcal{P}_n}(f) = \overline{S}_{\mathcal{P}_n} = 1.$$

ii)

$$\underline{S}_{\mathcal{P}_n}(f) = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=0}^{n-1} k = \frac{n-1}{2n},$$

$$\overline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n+1}{2n}.$$

iii)

$$\underline{S}_{\mathcal{P}_n}(f) = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 = \frac{(n-1)(2n-1)}{6n^2},$$

$$\overline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{(n+1)(2n+1)}{6n^2}.$$

iv)

$$\underline{S}_{\mathcal{P}_n}(f) = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^4} \sum_{k=0}^{n-1} k^3 = \frac{(n-1)^2}{4n^2},$$

$$\overline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{(n+1)^2}{4n^2}.$$

v)

$$\underline{S}_{\mathcal{P}_n}(f) = 0,$$
$$S_{\mathcal{P}_n}(f) = 1.$$

2)

Beware that $v)$ is not Riemann integrable.

8.4 Problem 4

Solving the following equations for x,

$$y = x$$
$$y = x^2 - 2,$$

we obtain the start and end of the integral, namely, $x = -1$ and $x = 2$. Hence, the area is given by

$$\int_{-1}^{2} x - (x^2 - 2) = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x\right]_{-1}^{2} = \frac{9}{2}.$$

8.5 Problem 5

b)

$$\int_{-4}^{2} |x| \, dx = \int_{-4}^{0} -x \, dx + \int_{0}^{2} x \, dx$$
$$= \left[-\frac{1}{2}x^2\right]_{-4}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{2}$$
$$= 10.$$

8.7 Problem 7

$F(x) = -1/x$ is not differentiable at $x = 0$, hence the Second Fundamental Theorem of Calculus does not apply.

8.11 Problem 11

Skipped.

8.13 Problem 13

$$\frac{d}{dx} \int_{x}^{4} (5 - 4t)^5 \, dt = -\frac{d}{dx} \int_{4}^{x} (5 - 4t)^5 \, dt$$
$$= -(5 - 4t)^5.$$

24

8.15 Problem 15

d)

$$\int_{-a}^{a} x^2 \sqrt{a^3 - x^3}\, \mathrm{d}x = \left[-\frac{2}{9}(a^3 - x^3)^{3/2} \right]_{-a}^{a}$$
$$= \frac{4\sqrt{2}a^{9/2}}{9}.$$

8.16 Problem 16

a)

Let $u = \sqrt{x} \geq 0$, then

$$\mathrm{d}u = \frac{1}{2\sqrt{x}}\, \mathrm{d}x,$$

which implies that

$$\mathrm{d}x = 2u\, \mathrm{d}u.$$

Hence,

$$\int \frac{\mathrm{d}x}{1 + \sqrt{x}} = \int \frac{2u}{1 + u}\, \mathrm{d}u$$
$$= \int \left(2 - \frac{2}{1+u} \right) \mathrm{d}u$$
$$= 2u - 2\ln(1 + u) + C$$
$$= 2\sqrt{x} - 2\ln(1 + \sqrt{x}) + C$$

where C is constant. In the third step, we implicitly removed the absolute value symbol because $1 + u > 0$.

b)

Let $u = 5x - 1$, then

$$\mathrm{d}u = 5\, \mathrm{d}x,$$

which implies that

$$\mathrm{d}x = \frac{1}{5}\, \mathrm{d}u.$$

Hence,

$$\int x(5x - 1)^{19}\, \mathrm{d}x = \int \frac{(u + 1)u^{19}}{25}\, \mathrm{d}u$$
$$= \frac{1}{25} \int (u^{20} + u^{19})\, \mathrm{d}u$$
$$= \frac{1}{25} \left(\frac{u^{21}}{21} + \frac{u^{20}}{20} \right) + C$$
$$= \frac{1}{25} \left(\frac{(5x - 1)^{21}}{21} + \frac{(5x - 1)^{20}}{20} \right) + C$$

where C is constant.

d)

Let $u = \sqrt{x} \geq 0$, then

$$\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x,$$

which implies that

$$\mathrm{d}x = 2u\,\mathrm{d}u.$$

Hence,

$$
\begin{aligned}
\int_0^4 \frac{\mathrm{d}x}{5 + \sqrt{x}}\,\mathrm{d}x &= \int_0^2 \frac{2u}{5 + u}\,\mathrm{d}u \\
&= \int_0^2 \left(2 - \frac{10}{5 + u}\right)\mathrm{d}u \\
&= \left[2u - 10\ln(5 + u)\right]_0^2 \\
&= 4 + 10\ln\frac{5}{7}.
\end{aligned}
$$

8.18 Problem 18

e)

$$
\begin{aligned}
\int_1^e x^7 \ln x\,\mathrm{d}x &= \left[\frac{x^8}{8}\ln x\right]_1^e - \int_1^e \frac{x^8}{8}\frac{1}{x}\,\mathrm{d}x \\
&= \left[\frac{x^8}{8}\ln x - \frac{e^8}{64}\right]_1^e \\
&= \frac{7e^8 + 1}{64}.
\end{aligned}
$$

g)

Recall integration by parts

$$uv = \int u\,\mathrm{d}v + \int v\,\mathrm{d}u.$$

Identifying $u = e^x$ and $v' = \cos x$, we have

$$\int e^x \cos x\,\mathrm{d}x = e^x \sin x - \int e^x \sin x\,\mathrm{d}x. \qquad (8.18.1)$$

Identifying $u' = e^x$ and $v = \cos x$, we have

$$\int e^x \cos x\,\mathrm{d}x = e^x \cos x + \int e^x \sin x\,\mathrm{d}x. \qquad (8.18.2)$$

Hence, $(8.18.1) + (8.18.2)$ gives

$$2\int e^x \cos x\,\mathrm{d}x = e^x \sin x + e^x \cos x + C,$$

which implies that

$$\int e^x \cos x\,\mathrm{d}x = \frac{e^2}{2}(\sin x + \cos x) + C$$

where C is constant.

8.19 Problem 19

a)

$$\int_0^\infty e^{-5x}\,\mathrm{d}x = \lim_{R\to\infty} \int_0^R e^{-5x}\,\mathrm{d}x$$

$$= \lim_{R\to\infty} \left[-\frac{1}{5}e^{-5x} \right]_0^R$$

$$= \lim_{R\to\infty} \left(-\frac{1}{5}e^{-5R} + \frac{1}{5} \right)$$

$$= \frac{1}{5}.$$

c)

Noticing that the antiderivative is some multiple of arctan plus constant, the improper integral diverges.

d)

Noticing that

$$\int_0^\infty x^3 e^{-x^4}\,\mathrm{d}x = \lim_{R\to\infty} \int_0^R x^3 e^{-x^4}\,\mathrm{d}x$$

$$= \lim_{R\to\infty} \left[-\frac{1}{4}e^{-x^4} \right]_0^R,$$

which is integrable, since $x^3 e^{-x^4}$ is odd,

$$\int_{-\infty}^\infty x^3 e^{-x^4}\,\mathrm{d}x = 0.$$

f)

Let $u = \ln x \geq 1$ (because $x \geq e$), then

$$\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x,$$

which implies that

$$\mathrm{d}x = e^u\,\mathrm{d}u.$$

Hence,

$$\int_e^\infty \frac{\mathrm{d}x}{x\ln x} = \lim_{R\to\infty} \int_1^R \frac{\mathrm{d}u}{u}$$

$$= \lim_{R\to\infty} \left[\ln u \right]_1^R$$

$$= \lim_{R\to\infty} \ln R,$$

which implies that the integral diverges.

8.22　Problem 22

a)

Because $0 \le \dfrac{1}{\sqrt{1+x^4}} < \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \dfrac{\mathrm{d}x}{x^2}$ converges, by the comparison test, $\displaystyle\int_1^\infty \dfrac{\mathrm{d}x}{\sqrt{1+x^4}}$ converges.

b)

Because $\dfrac{1}{\sqrt[3]{x^2-x}} > \dfrac{1}{\sqrt[3]{x^2}} = \dfrac{1}{x^{2/3}}$ for all $x \ge 2$ and $\displaystyle\int_2^\infty \dfrac{\mathrm{d}x}{x^{2/3}}$ diverges, by the comparison test,

$\displaystyle\int_2^\infty \dfrac{1}{\sqrt[3]{x^2-x}}$ diverges.

8.28　Problem 28

a)

$$\pi(10) = 4,$$
$$\pi(20) = 8,$$
$$\pi(3.14159) = 2.$$

b)

The average number of prime numbers up to x.

c)

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}(x) = \frac{1}{\ln x}$$
$$\mathrm{Li}(2) = 0.$$

d)

Since Li is continuous on $[2, 10^6]$ and differentiable on $(2, 10^6)$, there exists some $c \in (2, 10^6)$ such that
$$\frac{\mathrm{Li}(10^6) - \mathrm{Li}(2)}{10^6 - 2} = \mathrm{Li}'(c) = \frac{1}{\ln c},$$
which implies that
$$\mathrm{Li}(10^6) = \frac{10^6 - 2}{\ln c} + \mathrm{Li}(2) > \frac{10^6 - 2}{6\ln 10}.$$

e)

$$\pi(10^6)/10^6 \gtrapprox 0.078498.$$

Chapter 9

The Logarithmic And Exponential Functions

9.7   Problem 7

a)

$\ln\left(1 + \dfrac{1}{t}\right)$ represents the area under the curve $y = \dfrac{1}{x}$ from 1 to $1 + \dfrac{1}{t}$, which is greater than or equal to its lower Riemann sum $\dfrac{1}{1+t}$ and smaller than or equal to its upper Riemann sum $\dfrac{1}{t}$.

b)

$$\lim_{t \to \infty} \ln\left(1 + \frac{1}{t}\right)^t = \lim_{t \to \infty} t \ln\left(1 + \frac{1}{t}\right)$$

$$= \lim_{t \to \infty} \frac{\ln(1 + 1/t)}{1/t}$$

$$\overset{\text{L'H}}{=} \lim_{t \to \infty} \frac{-\frac{1}{t^2} \frac{1}{1+1/t}}{-\frac{1}{t^2}}$$

$$= \lim_{t \to \infty} \frac{1}{1 + 1/t}$$

$$= 1.$$

Hence,

$$\lim_{t \to \infty} \left(1 + \frac{1}{t}\right)^t = \lim_{t \to \infty} \exp\left\{ \ln\left(1 + \frac{1}{t}\right)^t \right\}$$

$$= \exp\left\{ \lim_{t \to \infty} \ln\left(1 + \frac{1}{t}\right)^t \right\}$$

$$= e.$$

9.9 Problem 9

a)

$$\lim_{x \to \infty} \frac{\ln x}{x^a} \stackrel{\text{L'H}}{=} \lim_{x \to \infty} \frac{1/x}{ax^{a-1}}$$
$$= \lim_{x \to \infty} \frac{1}{ax^a}$$
$$= 0.$$

Chapter 10

The Hyperbolic Functions

10.1   Problem 1

b)

For any $x > 0$, we have

$$
\begin{aligned}
\ln(\sinh x) &= \ln \frac{e^x - e^{-x}}{2} \\
&< \ln \frac{e^x}{2} \\
&= \ln e^x - \ln 2 \\
&= x - \ln 2.
\end{aligned}
$$

10.6   Problem 6

a)

Substituting $x = \cosh t$, $y = \sinh t$ into $x^2 - y^2 = 1$, we have

$$
\begin{aligned}
\cosh^2 t - \sinh^2 t &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\
&= \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} \\
&= 1.
\end{aligned}
$$

Therefore, for every real number t, the point $(\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$.

b)

We can prove the equation either by adding up the two areas split at $x = 1$ or by subtracting the sector-like area from the large triangle from 0 to $\cosh t$. Here we use the latter approach, which directly gives

$$
A(t) = \frac{1}{2} \cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, \mathrm{d}x.
$$

c)

$$A'(t) = \frac{1}{2}(\cosh^2 t + \sinh^t) - \left(\sqrt{\cosh^2 t - 1}\right)\sinh t$$
$$= \frac{1}{2}(\cosh^2 t + \sinh^2 t) - \sinh^2 t$$
$$= \frac{1}{2}\cosh^2 t - \frac{1}{2}\sinh^2 t$$
$$= \frac{1}{2}.$$

Hence,

$$A(t) = \int A'(t)\,\mathrm{d}t = \frac{t}{2} + C,$$

where C is constant.

Since when $t = 0$, the area is zero, we have

$$C = A(0) = 0.$$

Therefore,

$$A(t) = \frac{t}{2}.$$

10.10   Problem 10

b)

$$y = \tanh x \iff y = \frac{\sinh x}{\cosh x}$$
$$\iff y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\iff y = \frac{e^{2x} - 1}{e^{2x} + 1}$$
$$\iff e^{2x} = \frac{1 + y}{1 - y}$$
$$\iff \tan^{-1} y = x = \frac{1}{2}\ln\left(\frac{1 + y}{1 - y}\right).$$

Hence, we obtain

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right).$$

10.12   Problem 12

a)

Let $u = 2x$, then

$$\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u.$$

Hence,

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+4x^2}} &= \int \frac{\mathrm{d}u}{2\sqrt{1+u^2}} \\
&= \frac{1}{2}\sinh^{-1}u + C \\
&= \frac{1}{2}\sinh^{-1}(2x) + C,
\end{aligned}
$$

where C is constant.