単拡大定理の例

㊟ $\mathbb{Q}(\omega\sqrt[3]{7}) \subsetneq \mathbb{Q}(\omega, \sqrt[3]{7})$ に注意

$\boxed{1}$ $\mathbb{Q}(\omega, \sqrt[3]{7}) = \mathbb{Q}(\omega + \sqrt[3]{7})$. ここで $\omega = e^{2\pi i/3} = \dfrac{-1+\sqrt{-3}}{2}$,

$\omega + \sqrt[3]{7}$ の $\mathbb{Q}$ 上での最小多項式は $x^6+3x^5+6x^4-7x^3-15x^2+24x+64$.

https://www.wolframalpha.com/input/?i=%28x-%28w%2Ba%29%29%28x-%28w%5E2%2Ba%29%29%28x-%28w%2Bwa%29%29%28x-%28w%5E2%2Bwa%29%29%28x-%28w%2Bw%5E2a%29%29%28x-%28w%5E2%2Bw%5E2a%29%29+where+%7Ba%3D7%5E%281%2F3%29%2C+w%3D%28-1%2B%E2%88%9A%28-3%29%29%2F2%7D&lang=ja

$\downarrow$

Input interpretation

$(x-(w+a))(x-(w^2+a))(x-(w+wa))(x-(w^2+wa))(x-(w+w^2a))(x-(w^2+w^2a))$ where $a=\sqrt[3]{7}$, $w=\frac{1}{2}(-1+\sqrt{-3})$

Result

$$\left(x+\frac{1}{2}(1-i\sqrt{3})-\sqrt[3]{7}\right)\left(x-\frac{1}{2}\sqrt[3]{7}(-1+i\sqrt{3})+\frac{1}{2}(1-i\sqrt{3})\right)$$
$$\left(x-\frac{1}{4}(-1+i\sqrt{3})^2-\sqrt[3]{7}\right)\left(x-\frac{1}{4}(-1+i\sqrt{3})^2-\frac{1}{2}\sqrt[3]{7}(-1+i\sqrt{3})\right)$$
$$\left(x-\frac{1}{4}\sqrt[3]{7}(-1+i\sqrt{3})^2+\frac{1}{2}(1-i\sqrt{3})\right)\left(x-\frac{1}{4}\sqrt[3]{7}(-1+i\sqrt{3})^2-\frac{1}{4}(-1+i\sqrt{3})^2\right)$$

Expanded form

$x^6+3x^5+6x^4-7x^3-15x^2+24x+64$

https://www.wolframalpha.com/input/?i=Is+64+%2B+24+x+-+15+x%5E2+-+7+x%5E3+%2B+6+x%5E4+%2B+3+x%5E5+%2B+x%5E6+irreducible%3F&lang=ja

$\downarrow$

Is 64 + 24 x - 15 x^2 - 7 x^3 + 6 x^4 + 3 x^5 + x^6 irreducible?

Input

IrreduciblePolynomialQ$[64+24x-15x^2-7x^3+6x^4+3x^5+x^6]$

Result

True

つづく

$$\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2},\quad \alpha = \sqrt[3]{7},\quad \theta = \omega + \alpha$$ とおく.

$$F_\omega(x) = x^2 + x + 1,\quad F_\alpha(x) = x^3 - 7,$$

$$G(x) = F_\omega(\theta - x) = (x-\theta)^2 + (\theta - x) + 1 = x^2 - (2\theta+1)x + \theta^2 + \theta + 1$$ とおく.

$$\begin{array}{r}
x + (2\theta+1) \\
x^2-(2\theta+1)x+\theta^2+\theta+1 \;\big)\; x^3 \qquad\qquad -7 \\
x^3 - (2\theta+1)x^2 + (\theta^2+\theta+1)x \\
\hline
(2\theta+1)x^2 - (\theta^2+\theta+1)x - 7 \\
(2\theta+1)x^2 - (2\theta+1)^2 x + (2\theta+1)(\theta^2+\theta+1) \\
\hline
3\theta(\theta+1)\,x - ((2\theta+1)(\theta^2+\theta+1)+7)
\end{array}$$

$$\gcd(G(x), F_\alpha(x)) = x - \sqrt[3]{7} = x - \frac{(2\theta+1)(\theta^2+\theta+1)+7}{3\theta(\theta+1)}$$

$$\sqrt[3]{7} = \frac{(2\theta+1)(\theta^2+\theta+1)+7}{3\theta(\theta+1)}$$

https://www.wolframalpha.com/input/?i=%28%282t%2B1%29%28t%5E2%2Bt%2B1%29%2B7%29%2F%283t%28t%2B1%29%29+where+%7Bt%3D%28-1%2B%E2%88%9A%28-3%29%29%2F2%2B7%5E%281%2F3%29%7D&lang=ja

Input interpretation

$$\boxed{\frac{(2t+1)(t^2+t+1)+7}{3t(t+1)} \text{ where } t = \frac{1}{2}\left(-1+\sqrt{-3}\right)+\sqrt[3]{7}}$$

Result

$$\frac{7+\left(1+2\left(\sqrt[3]{7}+\frac{1}{2}\left(-1+i\sqrt{3}\right)\right)\right)\left(1+\sqrt[3]{7}+\frac{1}{2}\left(-1+i\sqrt{3}\right)+\left(\sqrt[3]{7}+\frac{1}{2}\left(-1+i\sqrt{3}\right)\right)^2\right)}{3\left(\sqrt[3]{7}+\frac{1}{2}\left(-1+i\sqrt{3}\right)\right)\left(1+\sqrt[3]{7}+\frac{1}{2}\left(-1+i\sqrt{3}\right)\right)}$$

Alternate forms

$$\boxed{\sqrt[3]{7}}$$

等しい

$\theta = \omega + \sqrt[3]{7}$, $\omega = e^{2\pi i/3}$ のとき,

$$\frac{(2\theta+1)(\theta^2+\theta+1)+7}{3\theta(\theta+1)} = \sqrt[3]{7}$$

$\boxed{2}$ $\mathbb{Q}(\omega, \sqrt[3]{7}) = \mathbb{Q}(\sqrt{-3}, \sqrt[3]{7}) = \mathbb{Q}(\sqrt{-3} + \sqrt[3]{7})$ $\left(\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}\right)$ である.

$\sqrt{-3} + \sqrt[3]{7}$ の $\mathbb{Q}$ 上での最小多項式は $x^6 + 9x^4 - 14x^3 + 27x^2 + 126x + 76$

https://www.wolframalpha.com/input/?i=%28x-%28v%2Ba%29%29%28x-%28-v%2Ba%29%29%28x-%28v%2Bwa%29%29%28x-%28-v%2Bwa%29%29%28x-%28v%2Bw%5E2a%29%29%28x-%28-v%2Bw%5E2a%29%29%20where%20%7Ba%3D7%5E%281%2F3%29%2C%20v%3D%E2%88%9A%28-3%29%2C%20w%3D%28-1%2Bv%29%2F2%7D

↓

Input interpretation

$(x-(v+a))(x-(-v+a))(x-(v+wa))(x-(-v+wa))\left(x-\left(v+w^2 a\right)\right)\left(x-\left(-v+w^2 a\right)\right)$ where $a = \sqrt[3]{7}$, $v = \sqrt{-3}$, $w = \frac{1}{2}(-1+v)$

Result

$$\left(x-\sqrt[3]{7}-i\sqrt{3}\right)\left(x-\sqrt[3]{7}+i\sqrt{3}\right)\left(x-\frac{1}{2}\sqrt[3]{7}\left(-1+i\sqrt{3}\right)-i\sqrt{3}\right)\left(x-\frac{1}{2}\sqrt[3]{7}\left(-1+i\sqrt{3}\right)+i\sqrt{3}\right)\left(x-\frac{1}{4}\sqrt[3]{7}\left(-1+i\sqrt{3}\right)^2-i\sqrt{3}\right)\left(x-\frac{1}{4}\sqrt[3]{7}\left(-1+i\sqrt{3}\right)^2+i\sqrt{3}\right)$$

Expanded form

$x^6 + 9x^4 - 14x^3 + 27x^2 + 126x + 76$

https://www.wolframalpha.com/input/?i=Is%2076%20%2B%20126%20x%20%2B%2027%20x%5E2%20-%2014%20x%5E3%20%2B%209%20x%5E4%20%2B%20x%5E6%20irreducible%3F

↓

Is 76 + 126 x + 27 x^2 - 14 x^3 + 9 x^4 + x^6 irreducible?

Input

IrreduciblePolynomialQ$[76 + 126x + 27x^2 - 14x^3 + 9x^4 + x^6]$

Result

True

つづく

$\alpha = \sqrt[3]{7}$, $\nu = \sqrt{-3}$, $\eta = \nu + \alpha$ とおく。

$F_\nu(x) = x^2 + 3$, $F_\alpha(x) = x^3 - 7$.

$H(x) = F_\nu(\eta - x) = (x-\eta)^2 + 3 = x^2 - 2\eta x + \eta^2 + 3$ とおく。

$$\gcd(H(x), F_\alpha(x)) = x - \sqrt[3]{7}$$

$$= x - \frac{2\eta(\eta^2+3)+7}{3(\eta^2-1)}$$

$$\sqrt[3]{7} = \frac{2\eta(\eta^2+3)+7}{3(\eta^2-1)}.$$

$$\begin{array}{r l}
 & x + 2\eta \\
x^2 - 2\eta x + \eta^2 + 3 \,\big)\!\! & x^3 \qquad\qquad\qquad -7 \\
 & x^3 - 2\eta x^2 + (\eta^2+3)x \\
\hline
 & 2\eta x^2 - (\eta^2+3)x - 7 \\
 & 2\eta x^2 - 4\eta^2 x + 2\eta(\eta^2+3) \\
\hline
 & 3(\eta^2-1)x - (2\eta(\eta^2+3)+7)
\end{array}$$

https://www.wolframalpha.com/input/?i=%282t%28t%5E2%2B3%29%2B7%29%2F%283%28t%5E2-1%29%29+where+%7Bt%3D%E2%88%9A%28-3%29%2B7%5E%281%2F3%29%7D&lang=ja

Input interpretation

$$\boxed{\frac{2t(t^2+3)+7}{3(t^2-1)} \text{ where } t = \sqrt{-3} + \sqrt[3]{7}}$$

Result

$$\frac{7 + 2\left(\sqrt[3]{7} + i\sqrt{3}\right)\left(3 + \left(\sqrt[3]{7} + i\sqrt{3}\right)^2\right)}{3\left(-1 + \left(\sqrt[3]{7} + i\sqrt{3}\right)^2\right)}$$

Alternate forms

$$\boxed{\sqrt[3]{7}}$$

等しい

$\eta = \sqrt{-3} + \sqrt[3]{7}$ のとき、

$$\frac{2\eta(\eta^2+3)+7}{3(\eta^2-1)} = \sqrt[3]{7}$$

$\boxed{3}$ $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$ である． $\alpha=\sqrt{2}$, $\beta=\sqrt{3}$, $\theta=\alpha+\beta$ とおく．　　特別に易しい場合

$F_\alpha(x)=x^2-2$, $F_\beta(x)=x^2-3$.　$G(x)=F_\alpha(\theta-x)=x^2-2\theta x+\theta^2-2$ とおく，

$\gcd(G(x),F_\beta(x)) = x-\sqrt{3}$.

$$x^2-2\theta x+\theta^2-2 \,\big)\, x^2 - 3 \;=\; 1,\quad x^2-3-(x^2-2\theta x+\theta^2-2) = 2\theta x-(\theta^2+1)$$

$$F_\beta(x) = G(x) + 2\theta\left(x-\frac{\theta^2+1}{2\theta}\right)$$

$$\gcd(G(x),F_\beta(x)) = x-\frac{\theta^2+1}{2\theta}$$

ゆえに，$\sqrt{3} = \dfrac{\theta^2+1}{2\theta}$.

(<u>確認</u> $\dfrac{1}{\theta} = \dfrac{1}{\sqrt{3}+\sqrt{2}} = \sqrt{3}-\sqrt{2}$, $\dfrac{\theta^2+1}{2\theta} = \dfrac{1}{2}\left(\theta+\dfrac{1}{\theta}\right) = \dfrac{1}{2}(\sqrt{3}+\sqrt{2}+\sqrt{3}-\sqrt{2}) = \sqrt{3}$)

$\theta=\sqrt{2}+\sqrt{3}$ の $\mathbb{Q}$ 上での最小多項式：

$$F_\theta(x) = (x-(\sqrt{2}+\sqrt{3}))(x-(-\sqrt{2}+\sqrt{3}))(x-(\sqrt{2}-\sqrt{3}))(x-(-\sqrt{2}-\sqrt{3}))$$

$$= ((x+\sqrt{3})^2-2)((x-\sqrt{3})^2-2) = (x^2+2\sqrt{3}x+1)(x^2-2\sqrt{3}x+1)$$

$$= (x^2+1)^2-(2\sqrt{3}x)^2 = x^4+2x^2+1-12x = x^4-10x+1.$$

$\square$

$\boxed{4}$ $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}) = \mathbb{Q}(\sqrt[3]{2}+\sqrt[3]{3})$ である. $\alpha = \sqrt[3]{2}$, $\beta = \sqrt[3]{3}$, $\theta = \alpha + \beta$ とおく.

$F_\alpha(x) = x^3 - 2$, $F_\beta(x) = x^3 - 3$ とおく.

$G(x) = -F_\alpha(\theta - x) = (x-\theta)^3 + 2 = x^3 - 3\theta x^2 + 3\theta^2 x - \theta^3 + 2$ とおく.

$\gcd(G(x), F_\beta(x)) = x - \beta$ となることを示せる(共通根は β だけ). ← 自分で示せ.

$H(x) = F_\beta(x) - G(x) = 3\theta x^2 - 3\theta^2 x + \theta^3 - 5$ とおく.

$$
\begin{array}{r|l}
 & x - 2\theta \qquad\qquad 3\theta G(x) \\
\hline
3\theta x^2 - 3\theta^2 x + \theta^3 - 5 & 3\theta x^3 - 9\theta^2 x^2 + 9\theta^3 x - 3\theta^4 + 6\theta \\
H(x) & 3\theta x^3 - 3\theta^2 x^2 + (\theta^3 - 5)x \\
\hline
 & -6\theta^2 x^2 + (8\theta^3 + 5)x - 3\theta^4 + 6\theta \\
 & -6\theta^2 x^2 + 6\theta^3 x - 2\theta^4 + 10\theta \\
\hline
 & (2\theta^3 + 5)x - (\theta^4 + 4\theta)
\end{array}
$$

```
In [23]: f = x^3 - a
         g = x^3 - p
         dispallresults(z, x, f, g)
```

0.034458 seconds (430.12 k allocations: 27.470 MiB, 32.49% gc time)

$F_\alpha(x) = x^3 - a$

$F_\beta(x) = x^3 - p$

$R_{\alpha+\beta}(z) = z^9 + (-3a - 3p)z^6 + (3a^2 - 21ap + 3p^2)z^3 - a^3 - 3a^2 p - 3ap^2 - p^3$

$R_{\alpha+\beta}(\alpha+\beta) = 0$

$\beta_{\mathrm{plus}}(z) = \dfrac{z^4 + (-a + 2p)z}{2z^3 + a + p}$

$\beta_{\mathrm{plus}}(\alpha+\beta) = \beta$ is true

$R_{\alpha\beta}(z) = z^9 - 3apz^6 + 3a^2p^2z^3 - a^3p^3$

$R_{\alpha\beta}(\alpha\beta) = 0$

$\beta_{\mathrm{mult}}(z) = -1$

$\beta_{\mathrm{mult}}(\alpha\beta) = \beta$ is false

1-subresultant $= (6z^4 + (3a + 3p)z)x - 3z^5 + (3a - 6p)z^2$

root of 1-subresultant $= \dfrac{z^4 + (-a + 2p)z}{2z^3 + a + p}$

$R(x) = 3\theta G(x) - (x - 2\theta)H(x) = (2\theta^3 + 5)x - (\theta^4 + 4\theta)$ とおく.

$\therefore\ x - \beta = \gcd(G(x), H(x)) = \dfrac{R(x)}{2\theta^3 + 5} = x - \dfrac{\theta(\theta^3 + 4)}{2\theta^3 + 5}$.

$a = 2$, $p = 3$

$\therefore\ \boxed{\dfrac{\theta(\theta^3 + 4)}{2\theta^3 + 5} = \beta.}$

これより, $\mathbb{Q}(\theta) = \mathbb{Q}(\alpha, \beta)$ であることがわかる.

$\dfrac{\theta(\theta^3+4)}{2\theta^3+5}=\beta$ の確認：

$$(\text{分母})=2\theta^3+5=6\alpha^2\beta+6\alpha\beta^2+\overbrace{15}^{=2(2+3)+5}$$

$$(\text{分母})\times\beta=6\alpha^2\beta^2+18\alpha+15\beta$$

$$(\text{分子})=\theta(\theta^3+4)=(\alpha+\beta)(3\alpha^2\beta+3\alpha\beta^2+\overset{2+3+4}{9})$$

$$=6\beta+3\alpha^2\beta^2+9\alpha+3\alpha^2\beta^2+9\alpha+9\beta$$

$$=6\alpha^2\beta^2+18\alpha+15\beta=(\text{分母})\times\beta.$$

$$\therefore\ \frac{\theta(\theta^3+4)}{2\theta^3+5}=\beta.$$

$[\mathbb{Q}(\underset{\sqrt[3]{2}}{\alpha},\underset{\sqrt[3]{3}}{\beta}):\mathbb{Q}]=9$ を別に示せるので，θ の $\mathbb{Q}$ 上での最小多項式は 9 次になる．

$$\theta^3=\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3=3\alpha^2\beta+3\alpha\beta^2+5=3\alpha\beta\theta+5.$$

$3\alpha\beta\theta=\theta^3-5$ の両辺を 3 乗すると $\underbrace{162}_{3^3\cdot 2\cdot 3=2\cdot 3^4=2\cdot 81=162}\theta^3=\theta^9-15\theta^6+75\theta^3-125.$

すなわち，$\boxed{\theta^9-15\theta^6-87\theta^3-125=0.}$

ゆえに，θ の $\mathbb{Q}$ 上での最小多項式は $F_\theta(x)=x^9-15x^6-87x^3-125.$ □

$\boxed{\text{終結式との関係}}$ (この話については理解できなくてもよい.)

多項式 $f(x)=\sum_{i=0}^{m} a_i x^i$ と $g(x)=\sum_{j=0}^{n} b_j x^j$ に対して, $(m+n)\times(m+n)$ の行列式で

$$\mathrm{res}_x(f,g)=\begin{vmatrix} a_m & \cdots & a_1 & a_0 & & \\ & \ddots & & & \ddots & \\ & & a_m & \cdots & a_1 & a_0 \\ b_n & \cdots & b_1 & b_0 & & \\ & \ddots & & & \ddots & \\ & & b_n & \cdots & b_1 & b_0 \end{vmatrix}\begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} m \end{matrix}$$

resultant

と定義される $\mathrm{res}_x(f,g)$ を f と g の (Sylvesterの) 終結式と呼ぶ.

$\boxed{\text{例}}$ $f(x)=ax^2+bx+c$

$g(x)=px^2+qx+r$

のとき,

$$\mathrm{res}_x(f,g)=\begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ p & q & r & 0 \\ 0 & p & q & r \end{vmatrix}$$

$f(x)=a_m\prod_{i=1}^{m}(x-\alpha_i)$, $g(x)=b_n\prod_{j=1}^{n}(x-\beta_j)$ のとき,

$$(*)\qquad \mathrm{res}_x(f,g)=a_m^n b_n^m \prod_{i=1}^{m}\prod_{j=1}^{n}(\alpha_i-\beta_j).$$

← 証明は佐武一郎『線型代数学』の pp.70−74 にある.

これより, z の多項式 $h(z)$ を $h(z)=\mathrm{res}_x(f(z-x),g(x))$ と定めると,

$h(\alpha_1+\beta_1)=0$ が成立する.

(*) をみとめれば証明は易しいので挑戦してみよ.

$\deg h(z)=mn$ となることも示せる.

これらの結果は $\alpha_1+\beta_1$ の最小多項式を求めるために役に立つ.

$f(x), g(x)$ の係数は不定元であると仮定する. ← 簡単のための仮定

$\alpha_1+\beta_1$ の有理式で β_1 になるものは $f(z-x)$ と $g(x)$ の 1次の部分終結式の根として得られる. $f(x)$ と $g(x)$ の k次の部分終結式は次のように定義される.

$$S_k^x(f,g) := \begin{vmatrix} a_m & \cdots\cdots & a_{1+k-(n-k-1)} & x^{n-k-1}f(x) \\ & \ddots & \vdots & \vdots \\ & a_m \cdots & a_{1+k} & f(x) \\ b_n & \cdots\cdots & b_{1+k-(m-k-1)} & x^{m-k-1}g(x) \\ & \ddots & \vdots & \vdots \\ & b_n \cdots & b_{1+k} & g(x) \end{vmatrix}$$

← $(m+n-2k)\times(m+n-2k)$ の行列式

(これは x について k次以下になる)

← 注 $S_0^x(f,g)=\mathrm{res}_x(f,g)$ となることを示せる. 自分で示してみよ.

たとえば $f(x)=ax^2+bx+c$, $g(x)=px^3+qx^2+rx+s$ のとき,

$$S_1^x(f,g) = \begin{vmatrix} a & b & xf(x) \\ & a & f(x) \\ p & q & g(x) \end{vmatrix} = \begin{vmatrix} a & b & ax^3+bx^2+cx \\ 0 & a & ax^2+bx+c \\ p & q & px^3+qx^2+rx+s \end{vmatrix} = \begin{vmatrix} a & b & cx \\ & a & bx+c \\ p & q & rx+s \end{vmatrix}.$$

x^2倍してひく　x^3倍してひく

$S_1^x(f(z-x), g(x))$ は z の多項式を係数とする x の1次式になり, その根として得られる z の有理函数に $\alpha_1+\beta_1$ を代入すると値は β_1 になる

https://nbviewer.org/github/genkuroki/public/blob/main/0023/AbstractAlgebra.jl%20subresultant%20examples.ipynb

In [13]:
```
f = x^2 - a
g = x^3 - p
dispallresults(z, x, f, g)
subresultant(f(z-x), g, 1) |> rootdeg1 |> rhs -> dispeq("\\text{root of 1-subresultant}", rhs)
```

```
0.057067 seconds (201.86 k allocations: 12.256 MiB, 28.52% gc time, 83.09% compilation time)
```

$$F_\alpha(x) = x^2 - a$$

$$F_\beta(x) = x^3 - p$$

$$R_{\alpha+\beta}(z) = z^6 - 3az^4 - 2pz^3 + 3a^2z^2 - 6apz - a^3 + p^2$$

$$R_{\alpha+\beta}(\alpha+\beta) = 0$$

$$\boxed{\beta_{\text{plus}}(z) = \frac{2z^3 - 2az + p}{3z^2 + a}}$$

$\beta_{\text{plus}}(\alpha+\beta) = \beta$ is true

$$R_{\alpha\beta}(z) = z^6 - a^3p^2$$

$$R_{\alpha\beta}(\alpha\beta) = 0$$

$$\beta_{\text{mult}}(z) = \frac{ap}{z^2}$$

$\beta_{\text{mult}}(\alpha\beta) = \beta$ is true

$$\boxed{\text{root of 1-subresultant} = \frac{2z^3 - 2az + p}{3z^2 + a}}$$

等しい

$F_\alpha(\alpha) = 0$, i.e. $\alpha^2 = a$,
$F_\beta(\beta) = 0$, i.e. $\beta^2 = p$ とすると,
$R_{\alpha+\beta}(\alpha+\beta) = 0$ かつ
$\beta_{\text{plus}}(\alpha+\beta) = \beta$ となる.

特に $a=2$, $b=3$ のとき,

$$\beta_{\text{plus}}(z) = \frac{2z^3 - 4z + 3}{3z^2 + 2} = \frac{2z(z^2-2)+3}{3z^2+2}$$

問題5-4の場合

$f(z-x) = x^2 - 2zx + z^2 - a$ より

$$S_1^x(f(z-x), g(x)) = \begin{vmatrix} 1 & -2z & (z^2-a)x \\ 0 & 1 & -2zx + z^2 - a \\ 1 & 0 & -p \end{vmatrix} = (3z^2+a)x - (2z^3 - 2az + p)$$