

$\sqrt{10}$ の近似値 $\sqrt{10}$ を小数点以下第2桁まで求めよ.

$$f(x) = \sqrt{9+x} \text{ とおく,}$$

$$f(0) = 3, f'(x) = \frac{1}{2}(9+x)^{-\frac{1}{2}}, f'(0) = \frac{1}{6}, f''(x) = -\frac{1}{4}(9+x)^{-\frac{3}{2}}.$$

$$f'(x_1) = f'(0) + \int_0^{x_1} f''(x_2) dx_2.$$

$$f(x) = f(0) + \int_0^x f'(x_1) dx_1 = f(0) + f'(0)x + \int_0^x \left(\int_0^{x_1} f''(x_2) dx_2 \right) dx_1 = 3 + \frac{1}{6}x + \int_0^x \left(\int_0^{x_1} f''(x_2) dx_2 \right) dx_1.$$

$$0 \leq x_2 \leq 1 \text{ のとき } \begin{cases} f''(x_2) = -\frac{1}{4}(9+x)^{-\frac{3}{2}} \geq -\frac{1}{4}9^{-\frac{3}{2}} = -\frac{1}{4} \cdot \frac{1}{3^3} = -\frac{1}{108} \\ f''(x_2) = -\frac{1}{4}(9+x)^{-\frac{3}{2}} \leq -\frac{1}{4}10^{-\frac{3}{2}} = -\frac{1}{4} \frac{1}{10\sqrt{10}} < -\frac{1}{4} \frac{1}{10 \cdot 4} = -\frac{1}{160} \end{cases}$$

$$x=1 \text{ のとき } \int_0^x \left(\int_0^{x_1} c dx_2 \right) dx_1 = c \frac{x^2}{2} \text{ を使うと,}$$

$$\sqrt{10} = f(1) \geq 3 + \frac{1}{6} - \frac{1}{2} \frac{1}{108} > 3.16666666 \dots - 0.005 = 3.16166666 \dots > 3.161,$$

$$\sqrt{10} = f(1) < 3 + \frac{1}{6} - \frac{1}{2} \frac{1}{160} = 3.16666666 \dots - 0.003125 = 3.16354166 \dots < 3.164. \quad \square$$

注意 $\sqrt{10} = 3.16227766 \dots$.

Newton 法

$$a > 0 \text{ とする. } y = x^2 - a \text{ の } x = x_0 \text{ での接線は } y = 2x_0(x - x_0) + x_0^2 - a \text{ である.}$$

$$\text{その零点は } x = \frac{a - x_0^2}{2x_0} + x_0 = \frac{x_0 + a/x_0}{2}.$$

$$x_{n+1} = \frac{x_n + a/x_n}{2} \text{ とおくと, } x_n \rightarrow \sqrt{a}.$$

$$a=10, x_0=3 \text{ のとき,}$$

$$x_1 = \frac{3 + \frac{10}{3}}{2} = \frac{19}{6} = 3.16666666 \dots, x_1^2 = \frac{361}{36} > 10$$

$$x_2 = \frac{\frac{19}{6} + \frac{10 \times 6}{19}}{2} = \frac{721}{288} = 3.16228070 \dots, x_2^2 = \frac{519841}{51984} > 10$$

$$3.162^2 = 9.998244 < 10.$$

$$\text{ゆえに, } 3.162 < \sqrt{10} < 3.1623$$

$$(*) \ x_n = \frac{l}{k}, l^2 = 10k^2 + 1 \text{ のとき, } x_{n+1} = \frac{\frac{l}{k} + \frac{10k}{l}}{2} = \frac{l^2 + 10k^2}{2kl} \text{ である.}$$

$$(l^2 + 10k^2) = (20k^2 + 1)^2, \quad 10(2kl)^2 + 1 = 10(4k^2(10l^2 + 1)) + 1 = 400k^2 + 40k^2 + 1 = (20k+1)^2$$

$$\text{よって } (l^2 + 10k^2)^2 = 10(2kl)^2 + 1.$$