

```
[5]      1  n = 5
      2  s = [sympy.Symbol("s%i"; commutative=false) for i
```

$\Rightarrow \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$

```
[6]      1  A = [j ≤ i+1 ? s[i] : oftype(s[i], 0) for i in 1:
```

$\Rightarrow \begin{bmatrix} s_1 & s_1 & 0 & 0 \\ s_2 & s_2 & s_2 & 0 \\ s_3 & s_3 & s_3 & s_3 \\ s_4 & s_4 & s_4 & s_4 \end{bmatrix}$

```
[7]      1  x1 = s
```

$\Rightarrow \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$

```
      1  x2 = A*x1
```

$\Rightarrow \begin{bmatrix} s_1 s_2 + s_1^2 \\ s_2 s_1 + s_2 s_3 + s_2^2 \\ s_3 s_1 + s_3 s_2 + s_3 s_4 + s_3^2 \\ s_4 s_1 + s_4 s_2 + s_4 s_3 + s_4^2 \end{bmatrix}$

```
[9]      1  x3 = A*x2 .|> expand
```

$\Rightarrow \begin{bmatrix} s_1 s_2 s_1 + s_1 s_2 s_3 + s_1 s_2^2 + s_1^2 s_2 + s_1^3 \\ s_2 s_1 s_2 + s_2 s_1^2 + s_2 s_3 s_1 + s_2 s_3 s_2 + s_2 s_3 s_4 + s_2 s_3^2 + s_2^2 s_1 + s_2^2 s_3 + s_2^3 \\ s_3 s_1 s_2 + s_3 s_1^2 + s_3 s_2 s_1 + s_3 s_2 s_3 + s_3 s_2^2 + s_3 s_4 s_1 + s_3 s_4 s_2 + s_3 s_4 s_3 + s_3 s_4^2 + s_3^2 s_1 + s_3^2 s_2 + s_3^2 s_4 + s_3^3 \\ s_4 s_1 s_2 + s_4 s_1^2 + s_4 s_2 s_1 + s_4 s_2 s_3 + s_4 s_2^2 + s_4 s_3 s_1 + s_4 s_3 s_2 + s_4 s_3 s_4 + s_4 s_3^2 + s_4^2 s_1 + s_4^2 s_2 + s_4^2 s_3 + s_4^3 \end{bmatrix}$

$$S = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \left\{ \begin{array}{cccc} s_1 & H & | & | & | \\ s_2 & | & H & | & | \\ s_3 & | & | & H & | \\ s_4 & | & | & | & H \end{array} \right.$$

$$S = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}, \quad A = \begin{bmatrix} s_1 & s_1 & 0 & 0 \\ s_2 & s_2 & s_2 & 0 \\ s_3 & s_3 & s_3 & s_3 \\ s_4 & s_4 & s_4 & s_4 \end{bmatrix}$$

$s_i$ 's are non-commutative variables.

$$AS \parallel \begin{bmatrix} s_1 s_2 + s_1^2 \\ s_2 s_1 + s_2 s_3 + s_2^2 \\ s_3 s_1 + s_3 s_2 + s_3 s_4 + s_3^2 \\ s_4 s_1 + s_4 s_2 + s_4 s_3 + s_4^2 \end{bmatrix} \left\{ \begin{array}{cccc} s_1 s_2 & H H | | & s_1^2 & H | | | \\ s_2 s_1 & H H | | & s_2 s_3 & | H H | & s_2^2 & | H | | \\ s_3 s_1 & H H | & s_3 s_2 & | H H | & s_3 s_4 & | | H H & s_3^2 & | | H | \\ s_4 s_1 & H | | H & s_4 s_2 & | H H & s_4 s_3 & | | H H & s_4^2 & | | | H \end{array} \right.$$

$$A^2 S \parallel$$

$$\left[ \begin{array}{c} s_1 s_2 s_1 + s_1 s_2 s_3 + s_1 s_2^2 + s_1^2 s_2 + s_1^3 \\ s_2 s_1 s_2 + s_2 s_1^2 + s_2 s_3 s_1 + s_2 s_3 s_2 + s_2 s_3 s_4 + s_2 s_3^2 + s_2^2 s_1 + s_2^2 s_3 + s_2^3 \\ s_3 s_1 s_2 + s_3 s_1^2 + s_3 s_2 s_1 + s_3 s_2 s_3 + s_3 s_2^2 + s_3 s_4 s_1 + s_3 s_4 s_2 + s_3 s_4 s_3 + s_3 s_4^2 + s_3^2 s_1 + s_3^2 s_2 + s_3^2 s_4 + s_3^3 \\ s_4 s_1 s_2 + s_4 s_1^2 + s_4 s_2 s_1 + s_4 s_2 s_3 + s_4 s_2^2 + s_4 s_3 s_1 + s_4 s_3 s_2 + s_4 s_3 s_4 + s_4 s_3^2 + s_4^2 s_1 + s_4^2 s_2 + s_4^2 s_3 + s_4^3 \end{array} \right]$$

$$\left\{ \begin{array}{cccccc} s_4 s_1 s_2 & H H H & s_4 s_1^2 & H | | H & s_4 s_2 s_1 & H H H & s_4 s_2 s_3 & | H H H & s_4 s_2^2 & | H H H \\ s_4 s_3 s_1 & H H H & s_4 s_3 s_2 & | H H H & s_4 s_3 s_4 & | | H H & s_4 s_3^2 & | | H H & s_4^2 s_1 & H | | H \\ s_4^2 s_2 & | H H H & s_4^2 s_3 & | | H H & s_4^3 & | | | H \end{array} \right.$$