3×3行列の逆行列の計算

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δは行列式とよばれる

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} \leftarrow \text{行列式} \quad = aek + bfg + cdh - afh - bdk - ceg =: \underline{\delta \neq 0} \text{ と仮定}$$

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & k & 0 & 0 & 1 \end{array}\right] \qquad \left.\begin{array}{l} \underline{a \neq 0} \\ \underline{ae-bd \neq 0} \end{array}\right\} \text{と仮定}$$

$$\begin{aligned} & a\delta + (af-cd)(ah-bg) \\ &= a(aek + \cancel{bfg} + \cancel{cdh} - \cancel{afh} - bdk - ceg) \\ &\quad + \cancel{aafh} - \cancel{abfg} - \cancel{acdh} + bcdg \\ &= aeak - bdak - aecg + bdcg \\ &= ae(ak-cg) - bd(ak-cg) \\ &= (ae-bd)(ak-cg) \end{aligned}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ ad & ae & af & 0 & a & 0 \\ ag & ah & ak & 0 & 0 & a \end{array}\right] \leftarrow a\text{をかけた}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & ae-bd & af-cd & -d & a & 0 \\ 0 & ah-bg & ak-cg & -g & 0 & a \end{array}\right] \begin{array}{l} \leftarrow (\text{第1行}) \times d \text{をひいた} \\ \leftarrow (\text{第1行}) \times g \text{をひいた} \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & ae-bd & af-cd & -d & a & 0 \\ 0 & (ae-bd)(ah-bg) & (ae-bd)(ak-cg) & -(ae-bd)g & 0 & a(ae-bd) \end{array}\right] \leftarrow ae-bd \text{ をかけた}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & ae-bd & af-cd & -d & a & 0 \\ 0 & 0 & \alpha & \beta & -a(ah-bg) & a(ae-bd) \end{array}\right] \leftarrow (\text{第2行}) \times (ah-bg) \text{ をひいた}$$

$$\begin{aligned} \alpha &= (ae-bd)(ak-cg) - (af-cd)(ah-bg) \\ &= a^2ek - aceg - abdk + \cancel{bcdg} - a^2fh + abfg + acdh - \cancel{bcdg} = a\delta \end{aligned}$$

$$\beta = -(ae-bd)g + d(ah-bg) = -aeg + \cancel{bdg} + adh - \cancel{bdg} = a(dh-eg)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} a\delta & b\delta & c\delta & \delta & 0 & 0 \\ 0 & (ae-bd)\delta & (af-cd)\delta & -d\delta & a\delta & 0 \\ 0 & 0 & \delta & dh-eg & -(ah-bg) & ae-bd \end{array}\right] \begin{array}{l} \leftarrow \delta\text{をかけた} \\ \\ \leftarrow a\text{でわった} \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} a\delta & b\delta & 0 & \delta - c(dh-eg) & c(ah-bg) & -c(ae-bd) \\ 0 & (ae-bd)\delta & 0 & -d\delta - (af-cd)(dh-eg) & a\delta + (af-cd)(ah-bg) & -(af-cd)(ae-bd) \\ 0 & 0 & \delta & dh-eg & -(ah-bg) & (ae-bd) \end{array}\right] \text{うまくひいた}$$

$$\begin{aligned} & -d\delta - (af-cd)(dh-eg) = -d(aek + bfg + \cancel{cdh} - \cancel{afh} - bdk - \cancel{ceg}) - \cancel{dafh} + \cancel{dcdh} + aefg - \cancel{dceg} \\ &= -aedk - bdfg + bddk + aefg = ae(fg-dk) - bd(fg-dk) = (ae-bd)(fg-dk) \end{aligned}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} a\delta & b\delta & 0 & \delta - c(dh-eg) & c(ah-bg) & -c(ae-bd) \\ 0 & \delta & 0 & -(dk-fg) & ak-cg & -(af-cd) \\ 0 & 0 & \delta & dh-eg & -(ah-bg) & ae-bd \end{array}\right] \leftarrow ae-bd \text{ でわった}$$

つづき

$$\longrightarrow \left[\begin{array}{ccc|ccc} a\delta & b\delta & 0 & \delta-c(dh-eg) & c(ah-bg) & -c(ae-bd) \\ 0 & \delta & 0 & -(dk-fg) & ak-cg & -(af-cd) \\ 0 & 0 & \delta & dh-eg & -(ah-bg) & ae-bd \end{array}\right]$$

(第2行)×b を(第1行)からひく

$$\longrightarrow \left[\begin{array}{ccc|ccc} a\delta & 0 & 0 & \delta-c(dh-eg)+b(dk-fg) & c(ah-bg)-b(ak-cg) & -c(ae-bd)+b(af-cd) \\ 0 & \delta & 0 & & \text{上と同じ} & \\ 0 & 0 & \delta & & & \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} a\delta & 0 & 0 & a(ek-fh) & -a(bk-ch) & a(bf-ce) \\ 0 & \delta & 0 & & \text{上と同じ} & \\ 0 & 0 & \delta & & & \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} \delta & 0 & 0 & ek-fh & -(bk-ch) & bf-ce \\ 0 & \delta & 0 & -(dk-fg) & ak-cg & -(af-cd) \\ 0 & 0 & \delta & dh-eg & -(ah-bg) & ae-bd \end{array}\right] \leftarrow a\text{でわった}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \text{上を}\delta\text{でわったもの} \\ 0 & 0 & 1 & \end{array}\right] \leftarrow \text{全体を}\delta\text{でわった.}$$

ゆえに,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}^{-1} = \frac{1}{\delta}\begin{bmatrix} ek-fh & -(bk-ch) & bf-ce \\ -(dk-fg) & ak-cg & -(af-cd) \\ dh-eg & -(ah-bg) & ae-bd \end{bmatrix}$$

2×2の行列式

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

$$= \frac{1}{\delta}\begin{bmatrix} \begin{vmatrix} e & f \\ h & k \end{vmatrix} & -\begin{vmatrix} b & c \\ h & k \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & k \end{vmatrix} & \begin{vmatrix} a & c \\ g & k \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{\delta}\begin{bmatrix} \begin{vmatrix} e & f \\ h & k \end{vmatrix} & -\begin{vmatrix} d & f \\ g & k \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & k \end{vmatrix} & \begin{vmatrix} a & c \\ g & k \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^{T} \leftarrow \text{転置.}$$

3×3の行列の逆行列のまとめ

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \frac{1}{\delta} \begin{bmatrix} \begin{vmatrix} e & f \\ h & k \end{vmatrix} & -\begin{vmatrix} d & f \\ g & k \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & k \end{vmatrix} & \begin{vmatrix} a & c \\ g & k \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^T$$

← この行列の成分を行列の余因子と呼ぶ
minor

$$\delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = aek + bfg + cdh - afh - bdk - ceg.$$

← これを行列の行列式と呼ぶ
determinant

逆行列を行列式と余因子で表わす公式が存在する!
分母　　分子

2×2行列の行列式の復習

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$$

← これの成分は $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ の余因子

分母は行列式

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$\boxed{\text{注意}}$ 行列 (matrix) と 行列式 (determinant) は全然異なるものである.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$　　$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ ← スカラーになる