Quantum Painlevé tau-functions

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Summary

- For any symmetrizable GCM, we can introduce the proper non-commutativity of quantum τ-functions by $\tau_i = \exp(\partial/\partial\alpha_i^\vee)$.

- Quantum q-Hirota-Miwa equations for $A_{n-1}^{(1)}$-case.

- Quantized Lax and Sato-Wilson forms of the extended affine Weyl group action for $A_{n-1}^{(1)}$-case.

- Quantized $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$-action for mutually prime m and n.

- An appropriate quantization of $q\mathbf{P_{IV}}$.

General theory
of the quantum and
q-difference version of
τ-functions
generated by the Weyl group action
for any symmetrizable GCM

We will consider the birational action of the Weyl group (Bäcklund transformations).

Want to construct quantizations of classical τ-functions of Painlevé systems (differrential and q-difference).

Difficulty. How to find the proper non-commutativity of quantized τ-functions?

My Answer.

- parameter variable $\alpha_i^\vee$ ↔ simple coroot.
- classical τ_i ↔ exp(fundamental weight)
- In the situation above, the appropriate definition of quantized τ_i is

$$\tau_i = \exp(\partial/\partial\alpha_i^\vee).$$

Quantum Algebra: Definiton

Consider the associative algebra
(precisely the non-commutative field) generated by

- dependent variables: f_i
- parameter variables: $\alpha_i^\vee$
- τ-variables: τ_i

with the relations

- q-Serre relations of f_i.
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$ and f_j.
- τ_i commutes with τ_j and f_j.
- $\tau_i \alpha_j^\vee \tau_i^{-1} = \alpha_j^\vee + \delta_{ij}.$ $(\tau_i = \exp(\partial/\partial\alpha_i^\vee))$

Quantum Algebra: q-Serre relations

$[a_{ij}]_{i,j \in I}$: GCM with $d_i a_{ij} = d_j a_{ji}$, $d_i \in \mathbb{Z}_{>0}$.

q: an inderminate.

$$q_i := q^{d_i}, \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}},$$

$$[n]_q! := [1]_q[2]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

q-Serre relations: if $i, j \in I$ and $i \neq j$, then

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0.$$

Quantum Algebra: Relations

- $$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j),$$

- $\alpha_i^\vee \alpha_j^\vee = \alpha_j^\vee \alpha_i^\vee, \; \alpha_i^\vee f_j = f_j \alpha_i^\vee,$

- $\tau_i \tau_j = \tau_j \tau_i, \; \tau_i f_j = f_j \tau_i,$

- $\tau_i \alpha_j^\vee \tau^{-1} = \alpha_j^\vee + \delta_{ij}$

τ_i's are the exponentials of the canonical conjugate variables of the parameter variables $\alpha_i^\vee$.

Quantum Algebra: Summary

- $f_i \leftrightarrow$ Chevalley generators of $U_q(\mathfrak{n}_-)$
- $\alpha_i^\vee \leftrightarrow$ simple coroots
- $\tau_i \leftrightarrow$ fundamental weights

- f_i ($i \in I$) satisfy the q-Serre relations.
- $\alpha_i^\vee$ and τ_i commute with f_i.
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$.
- τ_i commutes with τ_j.
- $\tau_i \alpha_j^\vee = (\alpha_j^\vee + \delta_{ij})\tau_i \quad (\tau_i = \exp(\partial/\partial\alpha_i^\vee))$.

Weyl group action

Weyl group: $W = \langle s_i \mid i \in I \rangle$.

- $s_i^2 = 1$,
- $a_{ij}a_{ji} = 0 \implies s_i s_j = s_j s_i$,
- $a_{ij}a_{ji} = 1 \implies s_i s_j s_i = s_j s_i s_j$,
- $a_{ij}a_{ji} = 2 \implies (s_i s_j)^2 = (s_j s_i)^2$,
- $a_{ij}a_{ji} = 3 \implies (s_i s_j)^3 = (s_j s_i)^3$.

$[A, B]_q := AB - qBA$.

$(\mathbf{ad}_q f_i)(x) := [f_i, x]_{q_i^{\langle \alpha_i^\vee, \beta \rangle}}$, where β = the weight of x.

Then $(\mathbf{ad}_q f_i)^{k+1}(f_j) = [f_i, (\mathbf{ad}_q f_i)^k(f_j)]_{q^{2k+a_{ij}}}$.

Weyl group action (Bäcklund transformations):

- $s_i(f_i) := f_i,$
- $s_i(f_j) := \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee - k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, f_i)^k (f_j) f_i^{-k} \quad (i \neq j)$
- $s_i(\alpha_j^\vee) := \alpha_j^\vee - a_{ji}\alpha_i^\vee,$
- $s_i(\tau_i) := f_i \tau_i \prod_{j \in I} \tau_j^{-a_{ij}} = f_i \tau_i^{-1} \prod_{j \neq i} \tau_j^{-a_{ij}},$
- $s_i(\tau_j) := \tau_j \quad (i \neq j).$

Remark.

- $\tau_i = \exp(\partial/\partial\alpha_i^\vee) \leftrightarrow$ the fundamental weight Λ_i
- $\tau_i \prod_{j \in I} \tau_j^{-a_{ij}} \leftrightarrow s_i(\Lambda_i) = \Lambda_i - \alpha_i$
  $(\alpha_i = \sum_{j \in I} a_{ji}\Lambda_j,$ simple root$).$

The action of s_i is an algebra automorphism.

Proof. We can define the algebra automorphism $\tilde{s}_i$ by

$$\tilde{s}_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji}\alpha_i^\vee,$$
$$\tilde{s}_i(\tau_i) = \tau_i \prod_{j \in I} \tau_j^{-a_{ij}}, \qquad \tilde{s}_i(\tau_j) = \tau_j \quad (i \neq j),$$
$$\tilde{s}_i(f_j) = f_j.$$

Then we obtain, for $x = f_j, \alpha_j^\vee, \tau_i,$

$$s_i(x) = f_i^{\alpha_i^\vee} \tilde{s}_i(x) f_i^{-\alpha_i^\vee}.$$

This is an algebra automorphism. □

Useful formulas

$$f_i^\gamma f_j f_i^{-\gamma} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\gamma-k)} \begin{bmatrix} \gamma \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q f_i)^k (f_j) f_i^{-k} \quad (i \neq j).$$

$$s_i(f_j) = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee}.$$

If $a_{ij} = -1$, then

$$s_i(f_j) = q_i^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_{q_i}(f_i f_j - q_i^{-1} f_j f_i) f_i^{-1}$$
$$= [1 - \alpha_i^\vee]_{q_i} f_j + [\alpha_i^\vee]_{q_i} f_i f_j f_i^{-1}.$$

Therefore

$$s_i(f_j) f_i = [1 - \alpha_i^\vee]_{q_i} f_j f_i + [\alpha_i^\vee]_{q_i} f_i f_j.$$

Remark: quantum geometric crystal

Since f_i ($i \in I$) satisfy the Verma relations, for example,

$$f_i^a f_j^{a+b} f_i^b = f_j^b f_i^{a+b} f_j^a \quad \text{if } a_{ij} a_{ji} = 1,$$

we can consider the actions of f_i^γ,

$$e_i(\gamma) : x \mapsto f_i^\gamma x f_i^{-\gamma},$$

as quantum version of a geometric crystal.

For the definition of classical geometric crystal, see Berenstein-Kazhdan `arXiv:math/9912105`, `arXiv:math/0601391`.

Quantum τ-functions: Definition

Fundamental weights: $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$.

Weight lattice: $P := \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$, $P_+ := \sum_{i \in I} \mathbb{Z}_{\geqq 0}\Lambda_i$.

simple roots: $\alpha_j := \sum_{i \in I} a_{ij}\Lambda_i$

Weyl group action on P: $s_i(\Lambda_j) = \Lambda_j - \delta_{ij}\alpha_i$.

τ-monomial: $\tau^\mu := \prod_{i \in I} \tau_i^{\mu_i}$ $\quad (\mu = \sum_{i \in I} \mu_i\Lambda_i \in P_+)$

(lattice) quantum τ-functions:

$$\tau(\lambda) := w(\tau^\mu) \qquad \text{for} \quad \lambda = w(\mu) \in WP_+.$$

Quantum τ-functions: Regularity

Regularity Theorem: All quantum τ-functions $\tau(\lambda)$ ($\lambda \in WP_+$) are (non-commutative) polynomials in the dependent variables f_i. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

Main theorem of `arXiv:1206.3419`.

Proof of the regularity theorem

$\rho := \sum_{i \in I} \Lambda_i$, $w \circ \lambda := w(\lambda + \rho) - \rho$ $(\lambda \in P, w \in W)$.

Assume $\lambda, \mu \in P_+$ and $w \in W$.

$L(\mu)$: highest weight simple module.
$M(w \circ \lambda)$: Verma module with highest weight $w \circ \lambda$.
$M(w \circ \lambda) \subset M(\lambda)$.

Translation functor: $T_\lambda^\mu(M(\lambda \circ \lambda)) \subset M(w \circ \lambda) \otimes L(\mu)$.

Sketch of the proof: $T_\lambda^\mu(M(w \circ \lambda)) \cong M(w \circ (\lambda + \mu))$ implies the regularity theorem. □

Non-trivial relation between the theory of quantum τ-functions and representation theory!

$A_{n-1}^{(1)}$-case $(n \geqq 3)$

$$i, j \in \mathbb{Z}/n\mathbb{Z}$$

$$a_{ij} = \begin{cases} 2 & (i = j) \\ -1 & (i - j = \pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$d_i = 1, \qquad q_i = q.$$

We will show that the quantum lattice τ-functions satisfy the quantum q-Hirota-Miwa equations.

Quantum algebra

Consider the associative algebra generated by

- dependent variables: f_i
- parameter variables: $\alpha_i^\vee$
- τ-variables: τ_i ($i \in \mathbb{Z}/n\mathbb{Z}$)

with the defining relations

- $f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0$.
- $f_i f_j = f_j f_i \quad (j \neq i \pm 1)$.
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$ and f_j.
- $\tau_i = \exp(\partial/\partial\alpha_i^\vee)$.

Weyl group action

Using the useful formulas above, we can show the following formulas:

- $s_i(f_{i\pm1}) = [1 - \alpha_i^\vee]_q f_{i\pm1} + [\alpha_i^\vee]_q f_i f_{i\pm1} f_i^{-1}$,

  $s_i(f_j) = f_j \quad (j \neq i \pm 1)$.

- $s_i(\alpha_i^\vee) = -\alpha_i^\vee, \quad s_i(\alpha_{i\pm1}^\vee) = \alpha_{i\pm1}^\vee + \alpha_i^\vee$,

  $s_i(\alpha_j^\vee) = \alpha_j^\vee \quad (j \neq i, i \pm 1)$.

- $s_i(\tau_i) = f_i \dfrac{\tau_{i-1}\tau_{i+1}}{\tau_i}, \quad s_i(\tau_j) = \tau_j \quad (i \neq j)$.

Extended coroot and weight lattices

$$Q^\vee := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i^\vee \oplus \mathbb{Z}\delta^\vee, \quad P := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i \oplus \mathbb{Z}\Lambda_0.$$

Dual bases: $\varepsilon_i^\vee, \delta^\vee \longleftrightarrow \varepsilon_i, \Lambda_0$.

Assume $\varepsilon_i^\vee = \varepsilon_{i+n}^\vee + \delta^\vee$ and $\varepsilon_{i+n} = \varepsilon_i$.

$$\alpha_i^\vee := \varepsilon_i^\vee - \varepsilon_{i+1}^\vee, \quad \alpha_i := \varepsilon_i - \varepsilon_{i+1}.$$

$$\Lambda_i = \Lambda_0 + \varepsilon_1 + \cdots + \varepsilon_i \quad (i \in \mathbb{Z}_{\geqq 0}).$$

Then $P = \bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\varepsilon_{\text{all}}, \quad \varepsilon_{\text{all}} = \varepsilon_1 + \cdots + \varepsilon_n$

$$P_+ := \sum_{i=0}^{n-1} \mathbb{Z}_{\geqq 0}\Lambda_i + \mathbb{Z}\varepsilon_{\text{all}}.$$

Assume $\Lambda_{i+n} = \Lambda_i + \varepsilon_{\text{all}} \quad (i \in \mathbb{Z})$.

Extended affine Weyl group

$$W := W(A_{n-1}^{(1)}) = \langle s_0, s_1, \ldots, s_{n-1} \rangle, \quad s_{i+n} = s_i.$$

$$\pi(s_i) := s_{i+1},$$

$$\widetilde{W} := \widetilde{W}(A_{n-1}^{(1)}) := \langle \pi \rangle \ltimes W = \langle \pi, s_0, \ldots, s_{n-1} \rangle.$$

(Do not assume $\pi^n = 1$.)

Assume $\lambda \in P$ and $\beta^\vee \in Q^\vee$.

$$s_i(\lambda) := \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i, \quad \alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}.$$

$$s_i(\beta^\vee) := \beta^\vee - \langle \beta^\vee, \alpha_i \rangle \alpha_i^\vee$$

$$\pi(\Lambda_i) := \Lambda_{i+1}, \quad \pi(\varepsilon_{\mathsf{all}}) := \varepsilon_{\mathsf{all}}.$$

$$\pi(\varepsilon_i^\vee) := \varepsilon_{i+1}^\vee, \quad \pi(\delta^\vee) := \delta^\vee.$$

Translation part of $\widetilde{W}$

$T_i := s_{i-1} \cdots s_2 s_1 \pi \, s_{n-1} s_{n-2} \cdots s_i \in \widetilde{W}$ $(i = 1, \ldots, n)$.

Assme $\nu = \sum_{i=1}^{n} \nu_i \varepsilon_i \in \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$.

$T^\nu := \prod_{i=0}^{n-1} T_i^{\nu_i}$.

Then

$T^\nu(\varepsilon_i^\vee) = \varepsilon_i^\vee - \nu_i \delta^\vee \quad T^\nu(\delta^\vee) = \delta^\vee$,

$T^\nu(\alpha_i^\vee) = \alpha_i^\vee - (\nu_i - \nu_{i+1})\delta^\vee$,

$T^\nu(\varepsilon_i) = \varepsilon_i, \quad T^\nu(\Lambda_0) = \Lambda_0 + \nu$,

$T^\nu(\Lambda_i) = \Lambda_i + \nu$.

Hirota-Miwa equation (1)

$$\Lambda_i = \Lambda_{i-1} + \varepsilon_i, \qquad \Lambda_{i+1} = \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1},$$
$$s_i(\Lambda_i) = \Lambda_{i-1} + \varepsilon_{i+1}, \qquad s_{i+1}(\Lambda_{i+1}) = \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2},$$
$$s_{i+1}s_i(\Lambda_i) = \Lambda_{i-1} + \varepsilon_{i+2}, \qquad s_i s_{i+1}(\Lambda_{i+1}) = \Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}.$$

$$\tau_i = \tau(\Lambda_{i-1} + \varepsilon_i), \qquad \tau_{i+1} = \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1}),$$
$$s_i(\tau_i) = \tau(\Lambda_{i-1} + \varepsilon_{i+1}), \qquad s_{i+1}(\tau_{i+1}) = \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2}),$$
$$s_{i+1}s_i(\tau_i) = \tau(\Lambda_{i-1} + \varepsilon_{i+2}), \qquad s_i s_{i+1}(\tau_{i+1}) = \tau(\Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}).$$

Lemma:

$$[\alpha_{i+1}^\vee]_q \tau_i \, s_i s_{i+1}(\tau_{i+1}) + [\alpha_i^\vee]_q s_{i+1} s_i(\tau_i) \tau_{i+1}$$
$$= [\alpha_i^\vee + \alpha_{i+1}^\vee]_q s_i(\tau_i) s_{i+1}(\tau_{i+1}).$$

Warning: τ_i does not commute with $s_i s_{i+1}(\tau_{i+1})$.

$$\tau_i[\alpha_i^\vee]_q = [\alpha_i^\vee + 1]_q \tau_i, \quad \tau_i[1 - \alpha_i^\vee]_q = -[\alpha_i^\vee]_q \tau_i.$$

Proof of Lemma:

$$\tau_i \, s_i s_{i+1}(\tau_i)$$

$$= \tau_i \, s_i \left(f_{i+1} \frac{\tau_i \tau_{i+2}}{\tau_{i+1}} \right)$$

$$= \tau_i \left([1 - \alpha_i^\vee]_q f_{i+1} + [\alpha_i^\vee]_q f_i f_{i+1} f_i^{-1} \right) f_i \frac{\tau_{i-1} \tau_{i+1}}{\tau_i} \frac{\tau_{i+2}}{\tau_{i+1}},$$

$$= \tau_i \left([1 - \alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee]_q f_i f_{i+1} \right) \tau_i^{-1} \tau_{i-1} \tau_{i+2},$$

$$= \left(-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1} \right) \tau_{i-1} \tau_{i+2}.$$

Thus

$$\tau_i \, s_i s_{i+1}(\tau_i) = (-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1}) \tau_{i-1} \tau_{i+2}.$$

Similarly we obtain

$$s_{i+1} s_i(\tau_i) \tau_{i+1} = ([1 - \alpha_{i+1}^\vee]_q f_i f_{i+1} + [\alpha_{i+1}^\vee]_q f_{i+1} f_i) \tau_{i-1} \tau_{i+2}$$

$$s_i(\tau_i) s_{i+1}(\tau_{i+1}) = f_i f_{i+1} \tau_{i-1} \tau_{i+2}.$$

q-numbers identity (or addition formula of **sin**):

$$[\alpha_i^\vee + 1]_q [\alpha_{i+1}^\vee]_q + [\alpha_i^\vee]_q [1 - \alpha_{i+1}^\vee]_q = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q.$$

Therefore

$$[\alpha_{i+1}^\vee]_q \tau_i \, s_i s_{i+1}(\tau_{i+1}) + [\alpha_i^\vee]_q s_{i+1} s_i(\tau_i) \tau_{i+1}$$
$$= [\alpha_i^\vee + \alpha_{i+1}^\vee]_q s_i(\tau_i) s_{i+1}(\tau_{i+1}).$$

In LHS the $f_{i+1} f_i$-terms are canceled. $\qquad\square$

Hirota-Miwa equation (3)

Apply T^ν to the formula of Lemma. Then we obtain

Theorem: The quantum τ-functions of type $A_{n-1}^{(1)}$ satisfy the quantum q-Hirota-Miwa equations:

$$[\varepsilon_i^\vee(\nu) - \varepsilon_{i+1}^\vee(\nu)]_q \ \tau_i(\nu + \varepsilon_{i+2})\,\tau_i(\nu + \varepsilon_i + \varepsilon_{i+1})$$
$$+ [\varepsilon_{i+1}^\vee(\nu) - \varepsilon_{i+2}^\vee(\nu)]_q\,\tau_i(\nu + \varepsilon_i) \ \ \tau_i(\nu + \varepsilon_{i+1} + \varepsilon_{i+2})$$
$$+ [\varepsilon_{i+2}^\vee(\nu) - \varepsilon_i^\vee(\nu)]_q \ \ \tau_i(\nu + \varepsilon_{i+1})\,\tau_i(\nu + \varepsilon_{i+2} + \varepsilon_i) = 0$$

where

$$\tau_i(\nu) := \tau(\Lambda_{i-1} + \nu),$$
$$\varepsilon_i^\vee(\nu) := T^\nu(\varepsilon_i^\vee) = \varepsilon_i^\vee - \nu_i \delta^\vee. \quad \square$$

Lax and Sato-Wilson forms of the affine Weyl group action

The relation between
the $RLL = LLR$ formalism of quantum groups and
the Lax and Sato-Wilson forms of the Painlevé systems
is non-trivial.

Assume that m and n are mutually prime.

Lax form: RLL=LLR

$A_{m-1}^{(1)}$-type R-matrix:

$$R(z) = (q - q^{-1}z) \sum_{i=1}^{m} E_{ii} \otimes E_{ii} + (1 - z) \sum_{i \neq j} E_{ii} \otimes E_{jj}$$
$$+ (q - q^{-1}) \sum_{i<j} (E_{ij} \otimes E_{ji} + z E_{ji} \otimes E_{ij}).$$

Local L-matrices: for $k = 1, \ldots, n$,

$$L_k(z) = \sum_{i=1}^{m} a_{ik} E_{ii} + \sum_{i=1}^{m-1} b_{ik} E_{i,i+1} + z b_{mk} E_{m1}.$$

$$L_k(z)^1 := L_k(z) \otimes 1, \quad L_k(z)^2 := 1 \otimes L_k(z).$$

Fundamental relations:

$$R(z/w)L_k(z)^1 L_k(w)^2 = L_k(w)^2 L_k(z)^1 R(z/w),$$
$$L_k(z)^1 L_l(w)^2 = L_l(w)^2 L_k(z)^1 \quad (k \neq l).$$

Equivalent to the q-commutation relations:

$$a_{ik}b_{ik} = q^{-1}b_{ik}a_{ik}, \quad a_{ik}b_{i+1,k} = qb_{i+1,k}a_{ik},$$
$$a_{ik}a_{jk} = a_{jk}a_{ik}, \quad b_{ik}b_{jk} = b_{jk}b_{ik}, \quad \text{etc.}$$

If $k \neq l$, then a_{ik} and b_{ik} commute with a_{jl} and b_{jl}.

Another form of the bidiagonal matrix $L_k(z)$.

$$a_k := \mathrm{diag}(a_{1k}, \ldots, a_{mk}),$$
$$b_k := \mathrm{diag}(b_{1k}, \ldots, b_{mk}),$$

$$\Lambda(z) := \sum_{i=1}^{m} E_{i,i+1} + zE_{m1} = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ z & & & 0 \end{bmatrix} \quad \text{(shift matrix)}.$$

Then

$$L_k(z) = a_k + b_k\Lambda(z) = \begin{bmatrix} a_{1k} & b_{1k} & & \\ & a_{2k} & \ddots & \\ & & \ddots & b_{m-1,k} \\ zb_{mk} & & & a_{mk} \end{bmatrix}.$$

Lax form: $\widehat{L}(z)$

1. $L(z) := L_1(z) \cdots L_n(z)$, the global L-operator.

$\tilde{a}_i := a_{i1} \cdots a_{in}$.

$\tilde{a} := \mathrm{diag}(\tilde{a}_1, \ldots, \tilde{a}_m)$, the diagonal part of $L(z)$.

2. $\widetilde{L}(z) := L(z)\tilde{a} \leftarrow$ doubling the diagonal part of $L(z)$.

3. $\widehat{L}(z) := \widetilde{C}\widetilde{L}(z)\widetilde{C}^{-1}$.

Here $\widetilde{C} = \mathrm{diag}(\tilde{c}_1, \ldots, \tilde{c}_m)$ is uniquely characterized by

$$\tilde{c}_1 = 1,$$
$$\widehat{L}(z) = \sum_{k=0}^{n-1} \hat{\ell}_k \Lambda(z)^k + \underbrace{\Lambda(z)\Lambda(rz) \cdots \Lambda(r^{n-1}z)}_{\text{highest term}},$$

where $\hat{\ell}_0, \hat{\ell}_1, \ldots, \hat{\ell}_{n-1}$ are diagonal matrices.

$$\widehat{L}(z) = \hat{\ell}_0 + \hat{\ell}_1 \Lambda(z) + \cdots + \Lambda(z)\Lambda(rz)\cdots\Lambda(r^{n-1}z).$$

$$t = \mathrm{diag}(t_1,\ldots,t_n) := \tilde{c}\tilde{a}\tilde{c}^{-1}.$$

Then $\hat{\ell}_0 = t^2$ and $t_i\widehat{L}(z) = \widehat{L}(z)t_i$.

Define $\hat{b}_i$ and $\hat{f}_i$ by

$$\hat{\ell}_1 = \mathrm{diag}(\hat{b}_i)_{i=1}^n = \mathrm{diag}\left((q^{-1}-q)t_it_{i+1}\hat{f}_i\right)_{i=1}^n.$$

$$r\widehat{L}(z) = \widehat{L}(z)r.$$

Assume $t_{i+n} = r^{-1}t_i$ and $\hat{f}_{i+n} = r\hat{f}_i$.

Example ($q\mathrm{P}_{\mathrm{IV}}$ case): $(m, n) = (3, 2)$.

$$\widehat{L}(z) = \begin{bmatrix} t_1^2 & (q^{-1} - q)t_1 t_2 \hat{f}_1 & 1 \\ rz & t_2^2 & (q^{-1} - q)t_2 t_3 \hat{f}_2 \\ rz(q^{-1} - q)t_3 t_4 \hat{f}_3 & z & t_3^2 \end{bmatrix}.$$

Assume $\widetilde{L}(z) = A + B\Lambda(z) + C\Lambda(z)^2$, A, B, C are diagonal, and $C = \mathbf{diag}(c_1, c_2, c_3)$. Then

$$c_i = b_{i1} b_{i+1,2} a_{i+2,1} a_{i+2,2},$$
$$\tilde{c}_1 = 1, \quad \tilde{c}_3 = c_1, \quad \tilde{c}_2 = c_1 c_3, \quad r = c_1 c_3 c_2.$$

$\widetilde{C} = \mathbf{diag}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$, $\widehat{L}(z) = \widetilde{C}\widetilde{L}(z)\widetilde{C}^{-1}$, $r\widehat{L}(z) = \widehat{L}(z)r$.

Lax form: $\widehat{M}(z)$

$T_{z,r} := r^{z \partial/\partial z} : f(z) \mapsto f(rz)$, r-difference operator.

4. $\widehat{M}(z) := \widehat{L}(z) T_{z,r}^n$, matrix coefficient r-difference op.

5. Assume $t_i = q^{-\varepsilon_i^\vee}$ and $r = q^{-\delta^\vee}$.

Then $[\alpha_i^\vee]_q = (t_{i+1}/t_i - t_i/t_{i+1})/(q - q^{-1})$

6. $g_i := (t_i^2 - t_{i+1}^2)/\hat{b}_i = [\alpha_i^\vee]_q/\hat{f}_i$.

$G_i := g_i E_{i+1,\,i} \quad (i = 1, \ldots, n-1)$.

$(G_n(z) := rz^{-1} g_n E_{1n}.)$

Lax form: Weyl group action

Consider the algebra generated by the matrix elements of $\widehat{L}(z)$ (precisely of $\hat{\ell}_0, \ldots, \hat{\ell}_{n-1}$).

7. Algebra automorphism Weyl group action:

$$s_i(\widehat{M}(z)) := G_i \widehat{M}(z) G_i^{-1},$$
$$\pi(\widehat{M}(z)) := (\Lambda(z) T_{z,r}) \widehat{M}(z) (\Lambda(z) T_{z,r})^{-1}$$
$$= \Lambda(z) \widehat{L}(rz) \Lambda(r^n z) T_{z,r}^n.$$

Then

$$s_i(t_i) = t_{i+1}, \quad s_i(t_{i+1}) = t_i,$$
$$s_i(\hat{b}_i) = \hat{b}_i, \quad s_i(\hat{b}_{i\pm1}) = \hat{b}_{i\pm1} \pm (t_i^2 - t_{i+1}^2)/\hat{b}_i.$$

Sato-Wilson form: z-variables

8. Introduce τ_0 and z_i by

   - $\tau_0 = \exp(\partial/\partial\delta^\vee)$: $\tau_0 r = q^{-1}r\tau_0$, $\tau_0 t_j = t_j\tau_0$.
   - $z_i = \exp(\partial/\partial\varepsilon_i^\vee)$: $z_i r = r z_i$, $z_i t_j = q^{-\delta_{ij}}t_j z_i$.
   - τ_0 and z_i commute with τ_0, z_j, $\hat{f}_j$.

9. $D_Z := \mathrm{diag}(z_1,\ldots,z_n)$, $Z(z) := U(z)D_Z$, where

   $$U(z) = E + \sum_{k=1}^{\infty} u_k \Lambda(z)^k,$$

   $u_1, u_2, \ldots$ are diagonal matrices,

   $$\widehat{M}(z) = U(z)t^2 T_{z,r}^n U(z)^{-1}.$$

Then

$$\widehat{M}(z) = Z(z)(qt)^2 T_{z,r}^n Z(z)^{-1}.$$

Sato-Wilson form: Weyl group action

10. The Weyl group action can be extended by

$$s_i(U(z)) = G_i U(z) S_i^g, \quad s_i(D_Z) = (S_i^g)^{-1} D_Z S_i,$$

$$s_i(t) = S_i^{-1} t S_i, \quad s_i(Z(z)) = G_i(z) Z(z) S_i,$$

$$\pi(A(z)) = (\Lambda(z) T_{z,r}) A(z) (\Lambda(z) T_{z,r})^{-1},$$
$$(A(z) = U(z), D_Z, t, Z(z))$$

where $\quad g_i = (t_i^2 - t_{i+1}^2)/\hat{b}_i = [\alpha_i^\vee]_q / \hat{f}_i,$

$$S_i^g := g_i^{-1} E_{i,i+1} - g_i E_{i+1,i} + \sum_{j \neq i, i+1} E_{jj},$$

$$S_i := [\alpha_i^\vee + 1]_q E_{i,i+1} - [\alpha_i^\vee - 1]_q^{-1} E_{i+1,i} + \sum_{j \neq i, i+1} E_{jj}.$$

S^g and S_i are permutation matrices $i \leftrightarrow i + 1$.

11. Assume $z_{j+m} = z_j$, $\tau_j = \tau_{j-1} z_i$,
and $s_i(\tau_0) = \tau_0$ $(i = 1, 2)$. Then, for $i = 1, 2$,

$$s_i(z_i) = \hat{f}_i z_{i+1}, \quad s_i(z_{i+1}) = \hat{f}_i^{-1} z_i, \quad s_i(\tau_i) = \hat{f}_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i},$$

$$\pi(z_i) = z_{i+1}, \quad \pi(\tau_i) = \tau_{i+1}.$$

Because $g_i = [\alpha_i^\vee]_q / \hat{f}_i$ and $z_i \varepsilon_j^\vee = (\varepsilon_j^\vee + \delta_{ij}) z_i$ implies

$$\begin{bmatrix} 0 & g_i \\ -g_i^{-1} & 0 \end{bmatrix}^{-1} \begin{bmatrix} z_i & 0 \\ 0 & z_{i+1} \end{bmatrix} \begin{bmatrix} 0 & [\alpha_i^\vee + 1]_q \\ -[\alpha_i^\vee - 1]_q^{-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} g_i^{-1} z_{i+1} [\alpha_i^\vee + 1]_q & 0 \\ 0 & g_i z_i [\alpha_i^\vee - 1]_q^{-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_i z_{i+1} & 0 \\ 0 & \hat{f}_i^{-1} z_i \end{bmatrix}. \quad \square$$

Quantum $q\mathrm{P_{IV}}$

Both canonically quantized and *q*-difference.

(m, n)-case: $X_k(z)$

Assume that **m** and **n** are mutually prime (gcm = 1).

There exist unique diagonal matrices $\widetilde{C}_1, \ldots, \widetilde{C}_1$ such that $\widetilde{C}_1 = \widetilde{C}_{n+1} = \widetilde{C}$ and, for $k = 1, \ldots, n-1$,

$$X_k(r^{k-1}z) := \widetilde{C}_k L_k(z) \widetilde{C}_{k+1}^{-1} = x_k + \Lambda(r^{k-1}z),$$
$$X_n(r^{n-1}z) := \widetilde{C}_n L_n(z) \tilde{a} \widetilde{C}_1^{-1} = x_n + \Lambda(r^{n-1}z),$$
$$x_k = \mathbf{diag}(x_{1k}, \ldots, x_{m,k}).$$

Then

$$\widehat{L}(z) = X_1(z) X_2(rz) \cdots X_n(r^{n-1}z).$$

Assume $x_{i+m,k} = r^{-1} x_{ik}$ and $x_{i,k+n} = x_{ik}$.

(m, n)-case: q-commutation relations of x_{ik}

Theorem. $x_{ik}x_{jl} = q_{j-i,l-k}^{(m,n)} x_{jl}x_{ik}, \quad q_{\mu\nu}^{(m,n)} \in \{1, q^{\pm 2}\}.$ $\square$

Example. If $(m, n) = (2g + 1, 2)$ and $x_i := x_{i1}$, $y_i := x_{i2}$, then

$$x_i y_i = y_i x_i = t_i^2,$$
$$x_i x_{i+\mu} = q^{(-1)^{\mu-1}2} x_{i+\mu} x_i, \qquad x_i y_{i+\mu} = q^{-(-1)^{\mu-1}2} y_{i+\mu} x_i,$$
$$y_i y_{i+\mu} = q^{(-1)^{\mu-1}2} y_{i+\mu} y_i, \qquad y_i x_{i+\mu} = q^{-(-1)^{\mu-1}2} x_{i+\mu} y_i,$$
$$t_i \text{ commutes with } t_j, x_j, y_j. \quad \square$$

Example. $(m, n) = (3, 5), (5, 3)$. $x_{ik}x_{jl} = q^{(m,n)}_{j-i,l-k}x_{jl}x_{ik}$,
where $q_{\mu+m,\nu} = q_{\mu\nu}$, $q_{\mu,\nu+n} = q_{\mu\nu}$, and

$$
\left[q^{(3,5)}_{\mu\nu} \right] = \begin{bmatrix} 1 & 1 & q^{-2} & q^2 & 1 \\ q^{-2} & q^2 & 1 & q^{-2} & q^2 \\ q^2 & q^{-2} & q^2 & 1 & q^{-2} \end{bmatrix},
$$

$$
\left[q^{(5,3)}_{\mu\nu} \right] = \begin{bmatrix} 1 & q^{-2} & q^2 \\ 1 & q^2 & q^{-2} \\ q^{-2} & 1 & q^2 \\ q^2 & q^{-2} & 1 \\ 1 & q^2 & q^{-2} \end{bmatrix}.
$$

Note that $q^{(5,3)}_{\mu\nu} = q^{(3,5)}_{\nu\mu}$. $\quad\square$

Assume that $0 < \widetilde{m} < n$ and $m\widetilde{m} \equiv 1 \pmod{n}$.
Assume that $0 < \tilde{n} < n$ and $n\tilde{n} \equiv 1 \pmod{m}$.

Theorem. Define $B^{(m,n)}$ and $p_{\mu\nu}^{(m,n)}$ by

$$B^{(m,n)} := \{ (\mu \bmod m, \ \mu \bmod n) \mid 0 \leqq \mu < \widetilde{m}m \}.$$

$$p_{\mu\nu}^{(m,n)} := \begin{cases} q & \text{if } (\mu \bmod m, \ \nu \bmod n) \in B, \\ 1 & \text{if } (\mu \bmod m, \ \nu \bmod n) \notin B. \end{cases}$$

Then

$$q_{\mu\nu}^{(m,n)} = (p_{\mu\nu}/p_{\mu-1,\nu})^2 \in \{1, q^{\pm 2}\}. \qquad \square$$

Cor. (duality) $q_{\mu\nu}^{(m,n)} = q_{\nu\mu}^{(n,m)}$. $\qquad \square$

(m, n)-case: Weyl group action

The action of $\widetilde{W}(A_{m-1}^{(1)})$ on t_i, $\hat{f}_i$, etc. can be extended to the one on x_{ik}.

Using the duality above, we can construct the action of $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$ on x_{ik}.

We shall write $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$ as

$$\widetilde{W}(A_{m-1}^{(1)}) = \langle \pi, s_0, \ldots, s_{m-1} \rangle,$$
$$\widetilde{W}(A_{n-1}^{(1)}) = \langle \varpi, r_0, \ldots, r_{n-1} \rangle.$$

$q\mathbf{P}_{IV}$-case: Weyl group action

Example ($q\mathbf{P}_{IV}$-case) $(m, n) = (3, 2)$. $\quad x_i := x_{i1}, \quad y_i := x_{i2}.$

$$s_i(x_i) = (x_i + y_{i+1})x_{i+1}(y_i + x_{i+1})^{-1},$$
$$s_i(x_{i+1}) = (x_i + y_{i+1})^{-1}x_i(y_i + x_{i+1}),$$
$$s_i(y_i) = (y_i + x_{i+1})y_{i+1}(x_i + y_{i+1})^{-1},$$
$$s_i(y_{i+1}) = (y_i + x_{i+1})^{-1}y_i(x_i + y_{i+1}),$$
$$s_i(x_{i+2}) = x_{i+2}, \quad s_i(y_{i+2}) = y_{i+2},$$
$$s_i(t_i) = t_{i+1}, \quad s_i(t_{i+1}) = t_i, \quad s_i(t_{j+2}) = t_{j+2},$$
$$\pi(x_i) = x_{i+1}, \quad \pi(y_i) = y_{i+1}, \quad \pi(t_i) = t_{i+1}.$$
$$Q_i := y_{i+2}y_{i+1} + y_{i+2}x_i + x_{i+1}x_i,$$
$$r_1(x_i) = r^{-1}Q_{i+1}^{-1}y_iQ_i,$$
$$r_1(y_i) = rQ_{i+1}x_iQ_i^{-1},$$
$$r_1(t_i) = t_i, \quad \varpi(x_i) = y_i, \quad \varpi(y_i) = x_i, \quad \varpi(t_i) = t_i.$$

$q\mathbf{P}_{\mathrm{IV}}$-case: Lax form

$G'_i := \varpi(G_i)$.   Then

$$s_i(X(z)) = G_i X(z) G'^{-1}_i,$$
$$s_i(Y(z)) = G'_i Y(z) G^{-1}_i,$$
$$\pi(X(z)) = \Lambda(z) X(z) \Lambda(z)^{-1}$$

$$r_1(X(z)Y(rz)) = X(z)Y(rz),$$
$$r_1 : x_{i+2}x_{i+1}x_i \leftrightarrow y_{i+3}y_{i+2}y_{i+1},$$
$$\varpi : X(z) \leftrightarrow Y(z).$$

These relations uniquely characterize the quantized birational action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$.

$q\mathbf{P_{IV}}$-case: discrete time evolution

$U_1 := r_1 \varpi \in$ translation part of $\widetilde{W}(A_1^{(1)})$.

The U_1-action is the discrete time evolution of $q\mathbf{P_{IV}}$ and the $\widetilde{W}(A_2^{(1)})$-action is its symmetry.

$a_i := t_i/t_{i+1}$ and $F_i := x_{i+1}x_i/(t_{i+1}t_i)$.

Then
$$F_i F_{i+1} = q^2 F_{i+1} F_i,$$
$$a_i a_j = a_j a_i, \ a_i F_j = F_j a_i,$$
$$F_{i+3} = F_i, \ a_{i+3} = a_i.$$

Discrete time evolution of quantized $q\mathrm{P}_{\mathrm{IV}}$.

$$U_1(F_i) = (1 + q^2 a_{i-1} F_{i-1} + q^2 a_{i-1} a_i F_{i-1} F_i)$$
$$\times a_i a_{i+1} F_{i+1}$$
$$\times (1 + q^2 a_i F_i + q^2 a_i a_{i+1} F_i F_{i+1})^{-1},$$
$$U_1(a_i) = a_i.$$

Classical case:

Kajiwara-Noumi-Yamada `arXiv:nlin/0012063`

$$\overline{F_i} = a_i a_{i+1} F_{i+1} \frac{1 + a_{i-1} F_{i-1} + a_{i-1} a_i F_{i-1} F_i}{1 + a_i F_i + a_i a_{i+1} F_i F_{i+1}},$$
$$\overline{a_i} = a_i.$$

$q\mathrm{P_{IV}}$-case: symmetry

Symmetry of quantum $q\mathrm{P_{IV}}$.

$$s_i(F_i) = F_i,$$
$$s_i(F_{i-1}) = F_{i-1}\frac{a_i + F_i}{1 + a_i F_i}, \quad s_i(F_{i+1}) = \frac{1 + a_i F_i}{a_i + F_i}F_{i+1},$$
$$s_i(a_i) = a_i^{-1}, \quad s_i(a_{i\pm 1}) = a_i a_{i\pm 1}.$$

These formulas coincide with the ones obtained by Koji Hasegawa `arXiv:0703036`, which quantizes Kajiwara-Noumi-Yamada `arXiv:nlin/0012063`.

Summary

- For any symmetrizable GCM, we can introduce the proper non-commutativity of quantum τ-functions by $\tau_i = \exp(\partial/\partial\alpha_i^\vee)$.

- Quantum q-Hirota-Miwa equations for $A_{n-1}^{(1)}$-case.

- Quantized Lax and Sato-Wilson forms of the extended affine Weyl group action for $A_{n-1}^{(1)}$-case

- Quantized $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$-action for mutually prime m and n.

- An appropriate quantization of $q\mathrm{P}_{\mathrm{IV}}$.