QUASICLASSICAL ASYMPTOTICS OF SOLUTIONS TO THE KZ EQUATIONS

Nicolai Reshetikhin* and Alexander Varchenko”

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Abstract. The quasiclassical asymptotics of the Knizhnik-Zamolodchikov system is studied. Solutions to this system in this limit are related naturally to Bethe vectors in the Gaudin model of spin chains.

Introduction.

The Knizhnik-Zamolodchikov differential equation

$$\frac{\partial F}{\partial z_i}(z) = \frac{1}{\kappa} H_i(z)F(z) , \qquad i = 1, \dots, n ,$$

appeared first as a holonomic system of differential equations on conformal blocks in a WZW model of conformal field theory [KZ]. Here $F(z_1, \dots, z_n)$ is a function with values in the tensor product $V_1 \otimes \cdots \otimes V_n$ of representations of a simple Lie algebra $\mathfrak{g}$, $\kappa = k + g$, where k is the central charge of the model, and g is the dual Coxeter number of the simple Lie algebra $\mathfrak{g}$.

One of the remarkable properties of the KZ system is that the coefficient functions $H_i(z)$ commute and that the form $w = \sum_i H_i(z) dz_i$ is closed:

$$\frac{\partial H_i}{\partial z_j} = \frac{\partial H_j}{\partial z_i} , \qquad [H_i, H_k] = 0 .$$

In this work we study asymptotics of solutions to the KZ equation when κ tends to zero. In this limit, solutions to the KZ equation turn into normalized eigenvectors of commuting linear operators $H_1(z), \dots, H_n(z)$.

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* Department of Mathematics, University of California, Berkeley, California 94720

E-mail address: reshetik@math.berkeley.edu

” Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

E-mail address: varchenko@math.unc.edu

There are integral representations for solutions to the KZ equation,

$$F(z) = \int_{\mathcal{C}} \exp(S(t,z)/\kappa) A(t,z) dt \ ,$$

where $S(t,z)$ is some multivalued scalar function of z and t, $A(z,t)$ is a rational vector valued function, and $\mathcal{C}$ is a cycle on which $S(z,t)$ is a single-valued function of t, $t = (t_1, \ldots, t_k)$ [SV]. If κ tends to zero, the integral is localized at critical points of S with respect to t. It turns out that every nondegenerate critical point $t(z)$ gives an eigenvector $A(t(z),z)$ of $H_1(z), \ldots, H_n(z)$.

The algebraic Bethe Ansatz is a certain construction of eigenvectors for a system of commuting operators. The idea of this construction is to find a vector-valued function of a special form and determine its arguments in such a way that the value of this function will be an eigenvector. The equations which determine these special values of arguments are called the Bethe equations. For more details about the algelbraic Bethe ansatz see [FT].

One of the systems of commuting operators which can be diagonalized by the ABA is the Gaudin model [G] of an inhomogeneous magnetic chain.

It turns out that the functions $A(z,t)$ are exactly those special functions which appear in the ABA for the Gaudin model and that the Bethe equations for the Gaudin model coincide with the equations on critical points of the function $S(z,t)$:

$$\frac{\partial S(z,t)}{\partial t_i} = 0, \qquad i = 1, ..., k.$$

We show that for every nondegenerate critical point $t(z)$:

$$B(A(t(z),z), A(t(z),z)) = \text{const} \cdot \ \text{Hess}_t(S(t(z),z))$$

where B is the Shapovalov form on $V_1 \otimes \cdots \otimes V_n$ and Hess is the Hessian of the function $S(t,z)$ as a function of t at $t = t(z)$. This is an analog of formulae for norms of Bethe vectors in [K,R].

As an example we consider the case of $\mathfrak{g} = sl_2$. We show that the constructed eigenvectors are paiwise orthogonal with respect to the Shapovalov form and form a basis in the space of singular vectors for generic z. We describe asymptotics of this basis when $|z_1| << |z_2| << ... << |z_n|$.

The first part of the paper contains some general facts about asymptoticaly flat sections for families of flat holomorphic bundles. In the second part we recall some basic facts about Knizhnik-Zamolodchikov equations. In section 3 we recall integral representations for solutions to the KZ equations from [SV],[V1]. Section 4 contains the description of asymptoticaly flat sections of the KZ bundles. Section 5 contains a conjecture about the structure of critical points in the case n=2. In section 6 we analise the asymptotic behaviour of critical points. Sections 7,8,9 contain detailed analysis of sl_2 case. In section 10 we study the Bethe basis for generic configuration of points $z_1, ..., z_n$. Section 11 contains some remarks about the relation between asymptoticaly flat sections of the KZ bundle and the Lame functions. In the last section 12 we discuss branching properties of the Bethe vectors in the Gaudin model.

The main result of this work is Theorem (4.10) and its Corollary (4.16), it was found in 1990 and has been the subject of several talks given in 1992-1993.

When this manuscript was completed we learned about the papers [Ba][BF] where the relation between asymptotic solutions to the KZ system and the Bethe vectors has been found.

Similarly quasiclassical assymptotics of q-KZ equations [FR] reproduces Bethe vectors for corresponding spin chains [R2],[TV].

The structure of the Bethe vectors for the Gaudin model is naturally related to the representation theory of affine Lie algebras at the critical level. This subject has been studied in [FFR].

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1. Asymptotically flat sections.

Let $\mathcal{D}$ be a ball in $\mathbb{C}^n$, $\pi : \mathbb{C}^N \times \mathcal{D} \to \mathcal{D}$ the projection. Let

$$\nabla_\kappa = \kappa d - \omega$$

be a *family of flat holomorphic connections* in π, $\kappa \in \mathbb{C}$. Here

$$\omega = H_1 dz_1 + \cdots + H_n dz_n$$

where $\{H_i\}$ are matrix valued functions.

The flatness means that

$$\kappa d\omega + \omega \wedge \omega = 0$$

for all $\kappa \in \mathbb{C}$, or

$$d\omega = 0 \ , \qquad \omega \wedge \omega = 0,$$

which implies

$$\frac{\partial H_i}{\partial z_j} = \frac{\partial H_j}{\partial z_i} \ , \qquad [H_i, H_j] = 0 \ , \tag{1.1}$$

for all i and j.

We are interested in *asymptotically flat sections* F for our family of flat bundles which have the form

$$F = \exp(S/\kappa)(f_0 + \kappa f_1 + \dots) \quad \text{for } \kappa \to 0 \ . \tag{1.2}$$

Here $S(z_1, \dots, z_n)$ is a function, $\{f_j(z_1, \dots, z_n)\}$ are sections of π, and F must be a formal solution to the system of equations

$$\nabla_\kappa F = 0 \ . \tag{1.3}$$

We will call $\exp(S/\kappa) f_0$ *an asymptotically flat section of the first order* if there exists a power series (1.2) which provides an asymptotic solution to (1.3) modulo terms of order κ^2.

We describe asymptotically flat sections in terms of fundamental solutions to the system (1.3). First we arrange them into an *asymptotic fundamental solution to* (1.3) which is a matrix with columns being asymptotically flat sections (1.2) such that its first terms form a basis of sections of π. Assume that

- linear operators $\{H_i(z)\}_{i=1}$ are simultaneously diagonalizable for each $z \in \mathcal{D}$,
- the spectrum of operators $\{H_i(z)\}_{i=1}^n$ is simple for each $z \in \mathcal{D}$ ($\lambda_i^\alpha \neq \lambda_i^\beta$ for some $i = 1, \ldots, n$ if $\alpha \neq \beta$).

Let $\phi(z)$ be the matrix with columns being eigenvectors of $H_i(z)$, $i = 1, \ldots n$, and $\Lambda_i(z)$ the diagonal matrix with eigenvalues of $H_i(z)$ on the corresponding places in the diagonal. We have

$$H_i(z)\phi(z) = \Lambda_i(z)\phi(z) \tag{1.4}$$

for each $z \in \mathcal{D}$.

(1.5) Proposition. *We have the identity*

$$\partial_i \Lambda_j - \partial_j \Lambda_i = 0 \tag{1.6}$$

Proof. We have the following equalities:

$$\begin{aligned}
\partial_i \Lambda_j - \partial_j \Lambda_i &= \partial_j(\phi^{-1} H_i \phi) - \partial_i(\phi^{-1} H_j \phi) \\
&= -\phi^{-1}\partial_j\phi\Lambda_i + \Lambda_i\phi^{-1}\partial_j\phi + \phi^{-1}\partial_i\phi\Lambda_j + \Lambda_j\phi^{-1}\partial_i\phi \\
&= -[\phi^{-1}\partial_j\phi, \Lambda_i] + [\phi^{-1}\partial_i\phi, \Lambda_j] \ .
\end{aligned}$$

Clearly the diagonal part of the right-hand side is zero, which implies equality (1.6).

(1.7) Corollary. *There exists a function $S(z)$ on $\mathcal{D}$ with values in the diagonal $N \times N$ matrices, such that*

$$\Lambda_i(z) = \partial_i S(z). \tag{1.8}$$

(1.9) Corollary. *Let $\{S_\alpha(z)\}$ be diagonal elements of $S(z)$. Then we have*

$$(\phi^{-1}\partial_i\phi)_{\alpha\beta} = N_{\alpha\beta}(\partial_i S_\alpha - \partial_i S_\beta) \ , \qquad \alpha \neq \beta \tag{1.10}$$

for some matrix $N(z)$ which does not depend on i.

This follows from identity (1.6).

The following theorem gives a description of asymptotic fundamental solutions to system (1.3).

(1.11) Theorem. *Let $\Phi(z)$ be the following asymptotic series expansion:*

$$\Phi(z) = \phi(z) \sum_{K\geq 0} \kappa^K \psi^{(K)}(z) e^{\frac{S(z)}{\kappa}} C \ . \tag{1.12}$$

Then:
(1) $\Phi(z)$ is an asymptotic fundamental solution to the system

$$\kappa \frac{\partial \Phi}{\partial z_i} = H_i \Phi \ , \qquad i = 1, \dots N \tag{1.13}$$

if

(i) *C is a constant*
(ii) *$\phi(z)$ is as in (1.4)*
(iii) *functions $\psi^{(K)}(z)$ satisfies the following recursive relations:*

$$\psi^{(K)}_{\alpha\beta} = \sum_\gamma N_{\alpha\gamma} \partial_i S_{\alpha\gamma} \psi^{(K-1)}_{\gamma\beta} + \frac{\partial_i \psi^{(K-1)}_{\alpha\beta}}{\partial_i S_\alpha - \partial_i S_\beta} \ , \qquad \alpha \neq \beta \tag{1.14}$$

$$\begin{aligned} &\partial_i \psi^{(K)}_{\alpha\alpha} + \sum_\gamma N_{\alpha\gamma} \psi^{(K)}_{\gamma\alpha} (\partial_i S_\alpha - \partial_i S_\gamma) = 0 \\ &\psi^{(0)}_{\alpha\beta} = \delta_{\alpha\beta} \end{aligned} \tag{1.15}$$

where $N_{\alpha\beta}$ is the same as in (1.10).

(iv)

$$(\phi^{-1} \partial_i \phi)_{\alpha\alpha} = 0 \quad \textit{for each } \alpha = 1, \dots, N \ . \tag{1.16}$$

(2) There exists a solution to (1.14) and (1.15) which is unique up to

$$\psi^{(K-1)}_{\gamma\beta} \longrightarrow \textit{const} \sum_{S+T=K} \psi^{(S)}_{\alpha\beta} C^{(T)}_\beta ,$$

and the matrix $\phi(z)$ has property (iv). Therefore, there exists an asymptotic fundamental solution to system (1.13) .

Proof. (1) We just have to substitute (1.12) into system (1.13). Comparing coefficients of asymptotic expansion with respect to κ we obtain the recursive relations for $\psi^{(K)}$:

$$(\phi^{-1} \partial_i \phi) \psi^{(K-1)} + \partial_i \psi^{(K-1)} = [\partial_i S, \psi^{(K)}]$$

and the condition

$$H_i \phi = \phi \partial_i S \ .$$

Equations (1.14) and (1.15) are the off-diagonal and diagonal matrix elements of (1.14), respectively. Let us prove that the system consisting of (1.14) and (1.15) has solutions. For this we must show:

$$(\alpha) \qquad \frac{\partial_i \psi^{(K)}_{\alpha\beta} + \sum_\gamma \partial_i S_{\alpha\gamma} N_{\alpha\gamma} \psi^{(K)}_{\alpha\beta}}{\partial_i S_{\alpha\beta}} = \frac{\partial_j \psi^{(K)}_{\alpha\beta} + \sum_\gamma \partial_j S_{\alpha\gamma} N_{\alpha\gamma} \psi^{(K)}_{\alpha\beta}}{\partial_j S_{\alpha\beta}}$$

$$(\beta) \qquad \partial_j \sum_\gamma N_{\alpha\gamma} \psi^{(K)}_{\gamma\alpha} \partial_i S_{\alpha\gamma} = \partial_i \sum_\gamma N_{\alpha\gamma} \psi^{(K)}_{\gamma\alpha} \partial_j S_{\alpha\gamma}$$

where $S_{\alpha\beta} := S_\alpha - S_\beta$. Let us first prove (α).

(1.17) Lemma.

$$\partial_i N_{\alpha\beta} = L_{\alpha\beta}\partial_i S_{\alpha\beta} \tag{1.18}$$

for some $L_{\alpha\beta}$.

Proof. Let us recall the definition of $N_{\alpha\beta}$ and compute $\partial_j N_{\alpha\beta}$:

$$\partial_j N_{\alpha\beta} = \frac{\partial_j(\phi^{-1}\partial_i\phi)_{\alpha\beta}}{\partial_i S_{\alpha\beta}} - \frac{(\phi^{-1}\partial_i\phi)_{\alpha\beta}\partial_i\partial_j S_{\alpha\beta}}{(\partial_i S_{\alpha\beta})^2}$$

where $S_{\alpha\beta} := S_\alpha - S_\beta$. Then we have:

$$\begin{aligned}\partial_j(\phi^{-1}\partial_i\phi)_{\alpha\beta} &= -(\phi^{-1}\partial_j\phi\phi^{-1}\partial_i\phi)_{\alpha\beta} + (\phi^{-1}\partial_i\partial_j\phi)_{\alpha\beta}\\ &= -\sum_\gamma N_{\alpha\gamma}N_{\gamma\beta}\partial_j S_{\alpha\gamma}\partial_i S_{\gamma\beta} + (\phi^{-1}\partial_i(\phi[\partial_j S, N]))_{\alpha\beta}\\ &= -\sum_\gamma N_{\alpha\gamma}N_{\gamma\beta}\partial_j S_{\alpha\gamma}\partial_i S_{\gamma\beta} + ([\partial_i S, N][\partial_j S, N])_{\alpha\beta}\\ &+ [\partial_i\partial_j S, N]_{\alpha\beta} + [\partial_j S, \partial_i N]_{\alpha\beta}\ .\end{aligned}$$

Therefore:

$$\partial_j N_{\alpha\beta} = \frac{\partial_j S_{\alpha\beta}\partial_i N_{\alpha\beta}}{\partial_i S_{\alpha\beta}}$$

which implies (1.6).

Now we can prove (α) by induction. Lemma 1.17 provides a base for induction:

$$\psi^{(1)}_{\alpha\beta} = N_{\alpha\beta}\ .$$

Relation (1.18) means that

$$\partial_i\psi^{(1)} = L_{\alpha\beta}\partial_i S_{\alpha\beta}.$$

Assume (1.14) holds for K. Let us prove that this implies (α) and therefore consistency of (1.14) for $K+1$.

$$
\begin{aligned}
&\partial_i S_{\alpha\beta}\left(\partial_j \psi^{(K)}_{\alpha\beta}+\sum_\gamma \partial_j S_{\alpha\gamma} N_{\alpha\gamma}\psi^{(K)}_{\gamma\beta}\right)-(i\leftrightarrow j)=\\
&=\partial_i S_{\alpha\beta}\left(\partial_j\left(\frac{\sum_\gamma(\phi^{-1}\partial_i\phi)_{\alpha\gamma}\psi^{(K-1)}_{\gamma\beta}+\partial_i\psi^{(K-1)}_{\alpha\beta}}{\partial_i S_{\alpha\beta}}\right)+\sum_\gamma \partial_j S_{\alpha\gamma}N_{\alpha\gamma}\psi^{(K)}_{\gamma\beta}\right)-(i\leftrightarrow j)\\
&=\Bigg(-\sum_{\gamma,\tau}(\phi^{-1}\partial_i\phi)_{\alpha\gamma}(\phi^{-1}\partial_j\phi)_{\gamma\tau}\psi^{(K-1)}_{\tau\beta}+\sum_\gamma(\phi^{-1}\partial_j\partial_i\phi)_{\alpha\gamma}\psi^{(K-1)}_{\gamma\beta}\\
&\quad+\sum_\gamma(\phi^{-1}\partial_i\phi)_{\alpha\gamma}\partial_j\psi^{(K-1)}_{\gamma\beta}+\partial_i\partial_j\psi^{(K-1)}_{\alpha\beta}-\psi^{(K)}_{\alpha\beta}\partial_i\partial_j S_{\alpha\beta}+\sum_\gamma\partial_i S_{\alpha\beta}(\phi^{-1}\partial_j\phi)_{\alpha\gamma}\psi^{(K)}_{\gamma\beta}\Bigg)-(i\leftrightarrow j)\\
&=-\sum_\gamma(\phi^{-1}\partial_j\phi)_{\alpha\gamma}(\phi^{-1}\partial_i\phi)_{\gamma\tau}\psi^{(K-1)}_{\tau\beta}\\
&\quad+\sum_\gamma(\phi^{-1}\partial_i\phi)_{\alpha\gamma}\partial_j\psi^{(K-1)}_{\gamma\beta}+\partial_i S_{\alpha\beta}(\phi^{-1}\partial_j\phi)_{\alpha\gamma}\psi^{(K)}_{\gamma\beta}-(i\leftrightarrow j)\\
&=-\sum_\gamma(\phi^{-1}\partial_j\phi)_{\alpha\gamma}[\sum_\tau(\phi^{-1}\partial_i\phi)_{\gamma\tau}\psi^{(K-1)}_{\tau\beta}+\partial_i\psi^{(K-1)}_{\gamma\beta}]\\
&\quad+\sum_\gamma(\phi^{-1}\partial_i\phi)_{\alpha\gamma}[\sum_\tau(\phi^{-1}\partial_j\phi)_{\gamma\tau}\psi^{(K-1)}_{\tau\beta}+\partial_i\psi^{(K-1)}_{\gamma\beta}]\\
&\quad+\partial_i S_{\alpha\beta}\sum_\gamma(\phi^{-1}\partial_j\phi)_{\alpha\gamma}\psi^{(K)}_{\gamma\beta}-\partial_j S_{\alpha\beta}\sum_\gamma(\phi^{-1}\partial_i\phi)_{\alpha\gamma}\psi^{(K)}_{\alpha\beta}\\
&=-\sum_\gamma(\phi^{-1}\partial_j\phi)_{\alpha\gamma}\partial_i S_{\alpha\beta}\psi^{(K)}_{\gamma\beta}+\sum_\gamma(\phi^{-1}\partial_i\phi)_{\alpha\gamma}\partial_j S_{\gamma\beta}\psi^{(K)}_{\gamma\beta}\\
&\quad+\sum_\gamma\partial_i S_{\alpha\beta}(\phi^{-1}\partial_j\phi)_{\alpha\gamma}\psi^{(K)}_{\gamma\beta}-\sum_\gamma\partial_j S_{\alpha\beta}(\phi^{-1}\partial_i\phi)_{\alpha\gamma}\psi^{(K)}_{\gamma\beta}\\
&=\sum_\gamma N_{\alpha\gamma}\psi^{(K)}_{\gamma\beta}\left(-\partial_j S_{\alpha\gamma}\partial_i S_{\gamma\beta}+\partial_i S_{\alpha\gamma}\partial_j S_{\gamma\beta}+\partial_i S_{\alpha\beta}\partial_j S_{\alpha\gamma}-\partial_j S_{\alpha\beta}\partial_i S_{\alpha\gamma}\right)=0
\end{aligned}
$$

This proves (α).

Now let us prove (β):

$$
\begin{aligned}
&\partial_j((\phi^{-1}\partial_i\phi)\psi^{(K-1)})-\partial_i((\phi^{-1}\partial_j\phi)\psi^{(K-1)})=\\
&=-\phi^{-1}\partial_j\phi\phi^{-1}\partial_i\phi\psi^{(K-1)}+\phi^{-1}\partial_i\phi\partial_j\psi^{(K-1)}\\
&\quad+\phi^{-1}\partial_i\phi\phi^{-1}\partial_j\phi\psi^{(K-1)}-\phi^{-1}\partial_j\phi\partial_i\psi^{(K-1)}\\
&=-[\phi^{-1}\partial_j\phi,\phi^{-1}\partial_i\phi]\psi^{(K-1)}-\phi^{-1}\partial_i\phi\cdot\phi^{-1}\partial_j\phi\psi^{(K-1)}\\
&\quad+\phi^{-1}\partial_i\phi[\partial_j S,\psi^{(K)}]+\phi^{-1}\partial_j\phi\phi^{-1}\partial_i\phi\psi^{(K-1)}-\phi^{-1}\partial_j\phi[\partial_i S,\psi^{(K)}]\\
&=[\partial_i S,N][\partial_j S,\psi^{(K)}]-[\partial_j S,N][\partial_i S,\psi^{(K)}]
\end{aligned}
$$

Diagonal matrix elements of the right-hand side are equal to:

$$\sum_{\gamma} \partial_i S_{\alpha\beta} N_{\alpha\beta} \partial_j S_{\beta\alpha} \psi^{(K)}_{\beta\alpha} - \sum_{\beta} \partial_j S_{\alpha\beta} N_{\alpha\beta} \partial_i S_{\beta\alpha} \psi^{(K)}_{\beta\alpha} = 0$$

which proves (β). And finally, it is clear that if each $\psi^{(S)}_{\alpha\beta}$, $S = 0, \ldots, K$, satisfies relations (1.14) and (1.15) then $\psi^{(P)} = \sum_{S=0}^{\rho} \psi^{(S)}_{\alpha\beta} C^{(P-S)}_{\beta}$ satisfy the same relations. This property reflects the fact that $\psi^{(K)}(z)$ in (1.12) is determined only modulo multiplication from the right by a κ-adic. Theorem 1.11 is proved.

Let V^* be a linear space dual to V and let $H_i^* : V^* \to V^*$ be linear operators dual to H_i. Denote by ∇_i^* the differential operator dual to ∇_i:

$$\nabla_i^* = -\kappa \frac{\partial}{\partial z_i} - H_i^* \tag{1.19}$$

Denote by $\langle \cdot , \cdot \rangle : V^* \otimes V \to \mathbb{C}$ the pairing between V^* and V .

Below we do not assume simplicity of spectrum of H_i.

(1.20) Proposition. *If f is a solution to (1.3) and g is a solution to the dual system:*

$$\nabla_i^* g = 0, \qquad i = 1, \ldots, n, \tag{1.21}$$

then

$$\langle g, f \rangle = \text{const.} \tag{1.22}$$

The proof is obvious.

Consider asymptotic solutions f and g to (1.13) and (1.21), respectively:

$$f(z) = \exp\left(\frac{S(z)}{\kappa}\right)(f_0(z) + \kappa\ f_1(z) + \ldots) \tag{1.23}$$

$$g(z) = \exp\left(\frac{T(z)}{\kappa}\right)(g_0(z) + \kappa\ g_1(z) + \ldots) \tag{1.24}$$

Here $S(z)$ and $T(z)$ are two functions which determine eigenvalues of H_i and H_i^* respectively (see (1.8)).

(1.25) Proposition. *(1) If $S - T \neq$ const, we have*

$$\langle g_0(z), f_0(z) \rangle = 0$$

(2) If $S - T =$ const, we have

$$\langle g_0(z), f_0(z) \rangle = \text{const.}$$

PROOF. Indeed, Proposition 1.20 implies implies

$$\langle g(z), f(z)\rangle = \mathrm{const}(\kappa).$$

On the other hand,

$$\langle g(z), f(z)\rangle = \exp\left(\frac{S(z)-T(z)}{\kappa}\right)(\langle g_0(z), f_0(z)\rangle + ...)$$

This proves (1) and (2).

(1.26) Remark. Assume that V has a nondegenerate bilinear form $(\cdot\ ,\ \cdot) : V\otimes V \to \mathbb{C}$ and the linear operators H_i are symmetric with respect to the form. In this case Proposition 1.20 implies

$$(f, g) = \mathrm{const}(\kappa) \tag{1.27}$$

if f is an asymptotic solution to (1.3) and g is an asymptotic solution to (1.3) in which κ is replaced by $-\kappa$.

Proposition 1.25 then implies that for any two asymptotic solutions f and g to (1.3) having forms (1.23) and (1.24), respectively, we have the following:

i) $(g_0(z), f_0(z)) = 0$ if $S - T \neq$ const,
ii) $(g_0(z), f_0(z)) =$ const if $S - T =$ const.

and, in particular, we have:

$$(f_0(z), f_0(z)) = \mathrm{const}$$

for every asymptotic solution to $\nabla_\kappa f = 0$.

(1.28) Remark. Under assumptions of (1.26) one can easily prove the following generalization of (1.26). For any two asymptotically flat sections of the first order f and g we have statements i) and ii) of (1.26).

2. Knizhnik-Zamolodchikov connections.

Let V be a vector space and let $r_{ij} : \mathbb{C}^* \to \mathrm{End}(V)$, $1 \le i < j \le n$, be functions satisfying the classical Yang-Baxter equation:

$$[r_{ij}(u), r_{ik}(u+v)] + [r_{ij}(u), r_{jk}(v)] + [r_{ik}(u+v), r_{jk}(v)] = 0 \tag{2.1}$$

for all triples $i < j < k$. Assume the following skew-symmetry condition

$$r_{ij}(u) = -r_{ji}(-u)\ . \tag{2.2}$$

Let $d_i : V \to V$, $i = s, \dots, n$, be linear maps with the following property:

$$[d_i + d_j, r_{ij}(u)] = 0\ , \qquad [d_i, d_j] = 0\ . \tag{2.3}$$

Consider

$$H_i(z_1,\dots,z_n)=\sum_{j\neq i} r_{ij}^{V_iV_j}(z_i-z_j)+d_i\ .$$

It is easy to verify that this collection of linear operators provides a family of flat connections on $\mathbb{C}^n\backslash\{\text{diagonals}\}$ which we will call the *KZ family of flat connections* corresponding to $\{r_{ij}(u),d_i\}$.

The *KZ equation* is the equation of flat sections:

$$\nabla_\kappa F=0.$$

Let $\mathfrak{g}$ be a complex simple Lie algebra, and let $(\rho_1,V_1),\dots,(\rho_n,V_n)$ be $\mathfrak{g}$-modules, $\rho_i:\mathfrak{g}\to \mathrm{End}(V_i)$. Define $V=V_1\otimes\cdots\otimes V_n$ and $\Omega\in\mathfrak{g}\otimes\mathfrak{g}$ as the element of $S^2(\mathfrak{g})\overset{\text{inv}}{\hookrightarrow}\mathfrak{g}\otimes\mathfrak{g}$ corresponding to the Killing form. If $\{I_a\}$ is an orthonormal basis in $\mathfrak{g}$, then $\Omega=\sum_a I_a\otimes I_a$.

Consider the linear operators Ω_{ij} acting in V as

$$\Omega_{ij}=\sum_{a=1}^{\dim\mathfrak{g}} 1\otimes\cdots\otimes\rho_i(I_a)\otimes\cdots\otimes\rho_j(I_a)\otimes\cdots\otimes 1$$

for $i<j$ and $\Omega_{ij}=\Omega_{ji}$ for $i>j$.

Define

$$r_{ij}(u)=\frac{\Omega_{ij}}{u}\ .$$

Fix any element d of $\mathfrak{g}$ and let $d_i=1\otimes\cdots\otimes \underset{i}{d}\otimes\cdots\otimes 1$.

It is an easy exercise to check that elements $r_{ij}(u)$ and d_i of $\mathrm{End}(V)$ satisfy conditions (2.1)–(2.3) and, therefore, the linear operators

$$H_i=\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}+d_i$$

determine a family of flat connections in the trivial vector bundle over $\mathbb{C}^n\backslash\{\text{diagonals}\}$ with fiber V.

We consider the case in which all d_i are equal to zero and

$$H_i=\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}. \tag{2.4}$$

Let $e_1,\dots,e_r,\ f_1,\dots,f_r,\ h_1,\dots,h_r$ be the Chevalley generators of $\mathfrak{g}$ such that

$$[e_i,f_j]=\delta_{ij}h_i\ ,$$
$$[h_i,e_j]=a_{ij}e_j\ ,$$
$$[h_i,f_j]=-a_{ij}f_j\ ,$$

where (a_{ij}) is the Cartan matrix.

The map $\tau:\mathfrak{g}\to\mathfrak{g}$, sending $e_1,\dots,e_r,\ f_1,\dots,f_r,\ h_1,\dots,h_r$ to $f_1,\dots,f_r,\ e_1,\dots,e_r,\ h_1,\dots,h_r$, respectively, generate an antiinvolution of $\mathfrak{g}$.

(2.5) Lemma. *τ preserves the Killing form on $\mathfrak{g}$.*

Corollary.

$$\tau\Omega = \sum \tau(I_a) \otimes \tau(I_a) = \Omega \ .$$

Let W be a $\mathfrak{g}$ module with highest weight vector w. The *Shapovalov form* B on W is the symmetric bilinear form defined by the conditions:

$$B(w,w) = 1, \qquad B(xu,v) = B(u,\tau(x)v) \tag{2.6}$$

for all $u, v \in W$ and $x \in \mathfrak{g}$.

(2.7) Lemma. *Let W_1 and W_2 be modules with highest weight, and let B_1 and B_2 be their Shapovalov forms. Then*

$$B_1 \otimes B_2(\Omega(x \otimes y), u \otimes v) = B_1 \otimes B_2(x \otimes y, \Omega(u \otimes v)) \tag{2.8}$$

for all $x \otimes y,\ \ u \otimes v \in W_1 \otimes W_2$.

Assume that $V_1, \dots, V_n$ are $\mathfrak{g}$ modules with highest weight. Let B_i be the Shapovalov form on V_i. Set

$$B = B_i \otimes \cdots \otimes B_n \ . \tag{2.9}$$

B is a symmetric bilinear form on V.

(2.10) Lemma. *The operators $H_1, \dots, H_n$ in (2.4) are symmetric with respect to B.*

If $V_1, \dots, V_n$ are irreducible finite-dimensional $\mathfrak{g}$-modules, or Verma modules with generic highest weights, then the form B is nondegenerate.

3. Integral representation for solutions of the KZ equation.

Let $V_1, ..., V_n$ be $\mathfrak{g}$ modules with highest weight.

Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$, $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$ the simple roots, $(\ \cdot\ ,\ \cdot\ )$ the bilinear form on $\mathfrak{h}^*$ induced by the Killing form, $\Lambda_1, \dots, \Lambda_n$ the highest weights of modules $V_1, \dots, V_n$, respectively.

Set $\Lambda = \Lambda_1 + \cdots + \Lambda_n$. For $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r_{\geq 0}$, set

$$\begin{aligned} V_\lambda &= \{v \in V \mid h_i v = (\Lambda - \sum_{j=1}^r \lambda_j \alpha_j, \alpha_i) v,\ i = 1, \dots, r\} \ , \\ \text{Sing } V_\lambda &= \{v \in V_\lambda \mid e_i v = 0,\ i = 1, \dots, r\} \ . \end{aligned} \tag{3.1}$$

Consider the KZ equation

$$\kappa\ dF = \sum_{i<j} \Omega_{ij}\ F\ \frac{d(z_i - z_j)}{(z_i - z_j)}. \tag{3.2}$$

We will recall a construction of solutions to the KZ equation with values in a given Sing V_λ [SV].

An arbitrary solution of the KZ equation is represented as a linear combination of the constructed solutions since the KZ connection commutes with the $\mathfrak{g}$ action on V .

Let $k = \lambda_1 + \cdots + \lambda_n$. Consider the space $\mathbb{C}^k$ with coordinates $t_1(1), t_1(2), \ldots, t_1(\lambda_1)$, $\ldots$, $t_r(1), t_r(2), \ldots t_r(\lambda_r)$, the space $\mathbb{C}^{n+k}$ with coordinates $t_1(1), t_1(2), \ldots, t_r(\lambda_r)$, z_1, $\ldots, z_n$, and the space $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$. Let $p : \mathbb{C}^{n+k} \to \mathbb{C}^n$ be the natural projection. Set

$$\begin{aligned}\Phi(t,z) = &\prod_{1\le m<\ell\le n} (z_m - z_\ell)^{(\Lambda_m,\Lambda_\ell)/\kappa} \cdot \\ &\cdot \prod_{m,i,j} (z_m - t_i(j))^{-(\Lambda_m,\alpha_i)/\kappa} \prod_{\substack{i<\ell \text{ or} \\ i=\ell \text{ and } j<m}} (t_i(j) - t_\ell(m))^{(\alpha_i,\alpha_\ell)/\kappa} .\end{aligned} \tag{3.3}$$

The function Φ is a multi-valued holomorphic function on $\mathbb{C}^{n+k}$ with singularities at diagonal hyperplanes.

A *monomial of weight* λ is an element $M \in V_\lambda$ of the form

$$M = f_{i_1} \ldots f_{i_{k_i}} v_1 \otimes f_{j_1} \ldots f_{j_{k_2}} v_2 \otimes \cdots \otimes f_{\ell_1} \ldots f_{\ell_{k_n}} v_n \ . \tag{3.4}$$

Here v_i is the highest weight vector of V_i, f's are elements of $\{f_1, \ldots, f_r\}$.

In [SV], for any monomial $M \in V_\lambda$, a differential k-form $\eta(M)$ is constructed. The form $\eta(M)$ is a rational form on $\mathbb{C}^{n+k}$ with poles at diagonal hyperplanes.

Consider the V_λ-valued k-form

$$N = \sum_{M\in V_\lambda} \Phi \cdot \eta(M) \otimes M \ . \tag{3.5}$$

In [SV] it is proved that

(3.6) For every i, the form

$$\left(\kappa \frac{\partial}{\partial z_i} - \sum_{j\ne i} \frac{\Omega_{ij}}{z_i - z_j} \right) N$$

is the sum of differential of a suitable $(k-1)$-form and a form which has zero restriction to fibers of the projection p.

(3.7) For every j, the form $e_j N = \sum \Psi \cdot \eta(M) \otimes e_j M$ is the sum of differential of a suitable $(k-1)$-form and a form which has zero restriction to fibers of p.

(3.8) All forms mentioned in (3.6) and (3.7) have the shape $\sum \Phi\ \omega(M) \otimes M$ where the sum is over all monomials in V, $\{\omega(M)\}$ are suitable rational forms with poles at diagonal hyperplanes.

Now let $z = (z_1, \dots, z_n)$ be a point in $\mathbb{C}^n$ lying in the complement to the diagonals. Let $\gamma(z)$ be a k-cycle lying in $p^{-1}(z) \subset \mathbb{C}^{n+k}$ outside the diagonals. Assume that $\gamma(z)$ continuously depends on z. Then the function

$$F(z) = \int_{\gamma(z)} N \tag{3.9}$$

takes values in Sing V_λ and satisfies the KZ equation. See a more precise description of $\gamma(z)$ in [SV, V].

4. Asymptotic solutions to the KZ equation.

Let $\kappa \to 0$. We use the form N defined in (3.5) and the method of steepest descent to construct asymptotic solutions to the KZ equation (3.2). The function Φ can be written in the form

$$\Phi(t, z) = \exp(S(t, z)/\kappa) \tag{4.1}$$

where

$$\begin{aligned} S(t,z) = & \sum_{1 \le m < \ell \le n} (\Lambda_m, \Lambda_\ell) \ln(z_m - z_\ell) \quad - \sum_{m,i,j} (\Lambda_m, \alpha_i) \ln(z_m - t_i(j)) \\ & + \sum_{\substack{i<\ell \text{ or} \\ i=\ell \text{ and } j<m}} (\alpha_i, \alpha_\ell) \ln(t_i(j) - t_\ell(m)) \end{aligned} \tag{4.2}$$

Its derivatives have the form

$$\begin{aligned} \frac{\partial S}{\partial z_m} &= \sum_{\ell \ne m} (\Lambda_m, \Lambda_\ell) \, \frac{1}{z_m - z_\ell} - \sum_{i,j} (\Lambda_m, \alpha_i) \frac{1}{z_m - t_i(j)} \ , \\ \frac{\partial S}{\partial t_i(j)} &= \sum_{(\ell,m) \ne (i,j)} (\alpha_i, \alpha_\ell) \frac{1}{t_i(j) - t_\ell(m)} - \sum_m (\Lambda_m, \alpha_i) \, \frac{1}{t_i(j) - z_m} \ , \\ \frac{\partial^2 S}{\partial t_\ell(m) t_i(j)} &= (\alpha_i, \alpha_\ell) \, \frac{1}{(t_i(j) - t_\ell(m))^2} \ , \\ \frac{\partial^2 S}{\partial t_i(j)^2} &= - \sum_{(\ell,m) \ne (i,j)} (\alpha_i, \alpha_\ell) \, \frac{1}{(t_i(j) - t_\ell(m))^2} + \sum_m (\Lambda_m, \alpha_i) \, \frac{1}{(t_i(j) - z_m)^2} \ . \end{aligned} \tag{4.3}$$

Set

$$\mathrm{Hess}_t(-S) = \det \left(- \frac{\partial^2 S}{\partial t_i(j) \partial t_\ell(m)} \right) \ . \tag{4.4}$$

For a fixed $z \in \mathbb{C}^n$, consider the equations of t-critical points for S:

$$\frac{\partial S}{\partial t_i(j)} = 0 \ , \qquad i = 1, \ldots, r, \quad j = 1, \ldots, \lambda_r \ . \tag{4.5}$$

Let $t = t(z)$ be a non-degenerate solution to system (4.5) holomorphically depending on z in a neighborhood of a point $z^0 \in \mathbb{C}^n$. Set

$$\overline{S}(z) = S(t(z), z) \ . \tag{4.6}$$

Let $B \subset \mathbb{C}^k$ be a small ball with center at $t(z_0)$. Let $\delta \subset B$ be a k-chain such that

$$\text{Re } S(t(z_0), z_0) > \text{Re } S(t, z_0)|_{\partial\delta} \tag{4.7}$$

where $\partial\delta$ is boundary of δ. This chain is unique in the sense explained in the remark below. For z close to z_0, let $\delta(z) \in p^{-1}(z)$ be the image of the chain δ under the natural isomorphism $p^{-1}(z) \simeq \mathbb{C}^k$. Then for z close to z_0 we have

$$\text{Re } S(t(z), z) > \text{Re } S(t, z)|_{\partial\delta(z)} \tag{4.8}$$

Set

$$F(z) = \kappa^{-\frac{k}{2}} \int_{\delta(z)} N \tag{4.9}$$

where N is given by (3.5). The function F is defined in a neighborhood of z_0.

(4.10) Theorem.

1. Let $\kappa \in \mathbb{R}$. As $\kappa \to +0$, the function F has an asymptotic expansion

$$F(z) = \exp(\overline{S}(z)/\kappa) \sum_{m=0}^{\infty} f_m(z) \kappa^m$$

where $\{f_m\}$ are holomorphic functions of z.

2. The function F gives an asymptotic solution to the KZ equation.

3. The functions $\{f_m\}$ take values in $\text{Sing}V_\lambda$.

Part 1 of the theorem is a direct corollary of the method of steepest descent; see, for example, §11 in [AGV]. Parts 2 and 3 are direct corollaries of (3.6)–(3.8) and (4.7).

Remark. Let

$$B^- = \{t \in B \mid \text{Re } S(t(z_0), z_0) > \text{Re } S(t, z_0)\} \ . \tag{4.11}$$

It is well known that $H_k(B, B^-, \mathbb{Z}) = \mathbb{Z}$ and, moreover, if δ and δ' are two cycles generating the same element in H_k, then $\int_{\delta(z)} N$ and $\int_{\delta'(z)} N$ have the same asymptotic expansions; see §11 in [AGV]. In what follows we assume that δ is a cycle generating H_k.

There exist local coordinates $u_1, \ldots, u_k$ in $\mathbb{C}^k$ centered at $t(z_0)$ such that

$$S(u, z_0) = -u_1^2 - \cdots - u_k^2 + \text{const} \ . \tag{4.12}$$

For such coordinates, δ is a disc

$$\{(u_1, \ldots, u_k) \in \mathbb{R}^k \mid u_1^2 + \cdots + u_k^2 \ \leq \ \varepsilon\} \tag{4.13}$$

where ε is a small positive number.

The first term of asymptotics.

Let $F(z)$ be given by (4.9) where δ is defined by (4.13). Then the first term of asymptotics can be computed explicitly. Namely, let $M \in V_\lambda$ be a monomial. The form $\eta(M)$ can be written as

$$\eta(M) = A_M(t,z) dt_1(1) \wedge \cdots \wedge dt_r(\lambda_r) \tag{4.14}$$

where A_M is a rational function. Then the standard formula of the method of steepest descent states that

$$f_0(z) = \pm(2\pi)^{k/2}(\mathrm{Hess}_t(-S(t(z),z))^{-1/2} \sum_{M\in V_\lambda} A_M(t(z),z)\ M \tag{4.15}$$

where the sign depends on orientation of δ. According to our previous considerations, we have the following corollary.

(4.16) Corollary.

The vector $f_0(z)$ lies in $\mathrm{Sing}V_\lambda$. *For any i, $f_0(z)$ is an eigenvector of H_i with eigenvalue $\frac{\partial S}{\partial z_i}(t(z),z)$. The vector $f_0(z)$ has constant length with respect to the Shapovalov form B defined in (2.9).*

The last statement can be reformulated as follows. Set

$$g(t,z) = \sum_{M\in V_\lambda} A_M(t,z)\ M \tag{4.17}$$

We will call $g(t(z),z)$ *the Bethe vector corresponding to the critical point* $(t(z),z)$. Then

$$B(g(t(z),z), g(t(z),z)) = \mathrm{const}\cdot \mathrm{Hess}_t(S(t(z),z))\ . \tag{4.18}$$

This is a rather surprising relation between critical points of S and the Shapovalov form, cf [K,R]. It is shown in [V] that const $= 1$ for $\mathfrak{g} = sl_2$.

5. Conjectures.

Let us study critical points of the function S in the case $n = 2$. Introduce new variables

$$s_i(\ell) = \frac{t_i(\ell) - z_1}{z_2 - z_1}\ .$$

Obviously,

$$S(t(s,z),z) \ = \ T(s) \ + \ \mathrm{const}\cdot \ln(z_1 - z_2)$$

where S is the function defined by (4.2) and

$$\begin{aligned} T(s) = \quad & -\sum_{i,j}(\ (\Lambda_1,\alpha_i)\ln(-s_i(j)) + (\Lambda_2,\alpha_i)\ln(1-s_i(j)) \\ & + \sum_{\substack{i<\ell \text{ or} \\ i=\ell \text{ and } j<m}} (\alpha_i,\alpha_\ell)\ln(s_i(j) - s_\ell(m)) \end{aligned}$$

Remarkably, equations (4.5) of $t-$critical points become equations of $s-$critical points for $\{\{s_i(\ell)\}_{\ell=1}^{\lambda_i}\}_{i=1}^r$ which do not depend on variables z_1 and z_2 :

$$\frac{(\Lambda_1,\alpha_j)}{s_i(j)}+\frac{(\Lambda_2,\alpha_j)}{s_i(j)-1}=\sum_{(\ell,m)\neq(i,j)}\frac{(\alpha_i,\alpha_\ell)}{s_i(j)-s_\ell(m)}\ . \tag{5.1}$$

Let $g(t,z)$ be the vector defined by (4.17). Set

$$G \ = \ g(t,z)/(z_1-z_2)^k.$$

Substitute $t=t(s,z)$. Then G becomes a function of s,z_1,z_2.

(5.2) Proposition. *The vector G does not depend on z_1,z_2.*

This proposition follows from the explicit form of functions $A_M(t,z)$ in (4.17) (see [SV]).

(5.3) Conjecture. *For generic weights Λ_1 and Λ_2 the following is true:*

(1) *Each solution to (5.1) determines a nondegenerate critical point of the function T.*

(2) *For each solution s^0 to (5.1), we have*

$$B(G(s^0),G(s^0))=\mathrm{Hess}_s(T(s^0)).$$

(3) *The product of symmetric groups $\mathcal{S}=\mathcal{S}_{\lambda_1}\times...\times\mathcal{S}_{\lambda_r}$ naturally acts on the set of critical points of the function T. We conjecture that for each pair of solutions s^0 and s^1 to system (5.1) which belong to different $\mathcal{S}-$orbits we have*

$$B(G(s^0),G(s^1))=0.$$

(4) *The set of vectors $\{G(s^0)\}$, where s^0 runs through all $\mathcal{S}-$orbits of critical points of T, forms a basis in Sing$(V_1\otimes V_2)_\lambda$.*

Similarly, for dominant integer weights Λ_1 and Λ_2 we have

(5.4) Conjecture. *Let $\tilde V_1$ and $\tilde V_2$ be irreducible highest weight modules with highest weights Λ_1 and Λ_2. Then (1)–(4) from Conjecture 5.3 hold in this case also.*

6. Asymptotic properties of the critical points.

Let $1<l<n$ and $v(z)$ be a nondegenerate critical point with the weight $\sum_{i=1}^l\Lambda_i-\sum_{i=1}^r a_i\alpha_i$ of the function $S(t,z_1,\dots,z_l)$. Here we assume as above that $z_i\in\mathbb{C},\ \ z_i\neq z_j$, and to each point z_i we associate a highest weight Verma module with highest weight Λ_i.

Similarly let $u(x)$ be a nondegenerate critical point with weight $\sum_{i=l+1}^n\Lambda_i-\sum_{j=1}^r b_j\alpha_j$ of the function $S(t,x_1,\dots,x_{n-l-1})$. Here we assume that to each point x_i we associate a highest weighit Λ_{i+l}.

Let $(\{s_i(j)\}_{j=1}^{c_i})_{i=1}^r$ be a solution to (5.1) with Λ_1 replaced to $\Lambda_1+\dots+\Lambda_l-\sum_{i=1}^r a_i\alpha_i$ and Λ_2 replaced to $\Lambda_{l+1}+\dots+\Lambda_n-\sum_{j=1}^r b_j\alpha_j$.

Theorem 6.1. *(1) There exists a unique nondegenerate critical point of $S(t, z_1, \dots, z_N)$ with weight $\sum_{i=1}^n \Lambda_i - \sum_{j=1}^r (a_j + b_j + c_j)\alpha_j$ such that asymptotically, when $z_{l+1}, \dots, z_n \to \infty$, $z_1, \dots, z_l, \dots, z_{l+i} - z_{l+1}, \dots, z_M - z_{l+1}$ are finite, it has the form*

$$\{t(z)\} = \{v(z_1, \dots z_l) + \mathcal{O}(\frac{1}{z_{l+1}})\} \cup \{s_i(j) z_{l+1} + \mathcal{O}(1)\} \cup \{u(x_1, \dots, x_{n-l-1}) + z_{l+1} + \mathcal{O}(\frac{1}{z_{l+1}})\}$$

where $x_i = z_{l+i+1} - z_{l+1}$.

Proof. We will use the following lemma.

Lemma. *Let $t(z) = \sum_{n \geq 0} t^{(n)} z^{-n}$. Then*

$$\frac{1}{t(z) - w} = \frac{1}{t^{(0)} - w} + \sum_{m \geq 1} z^{-m} \sum_{1 \leq n \leq m} \frac{(-1)^n}{(t^{(0)} - w)^{n+1}} \sum_{\substack{p_1 + \dots + p_n = m \\ p_i \geq 1}} t^{(p_1)} \dots t^{(p_n)} \tag{6.1}$$

Proof is by straightforward computation.

The proof for an arbitrary simple Lie algebra $\mathfrak{g}$ is absolutely parallel to the one for sl_2. To avoid technicalities we present here the proof for sl_2.

Let $t_\alpha(z)$ be the asymptotic power series of the following type when $z_{l+1}, \dots, z_n \to \infty$ and $x_i = z_i - z_{l+1}$ is finite, $i = l+1, \dots, n$.

$$\begin{aligned} t_\alpha(z) &= v_\alpha(\tilde{z}) + \sum_{p \geq 1} v_\alpha^{(p)}(\tilde{z}, x) z_{l+1}^{-p}, \qquad \alpha = 1, \dots a \\ t_\alpha(z) &= z_{K+1} s_{\alpha - a} + \sum_{p \geq 0} s_{\alpha - a}^{(p)}(\tilde{z}, x) z_{l+1}^{-p}, \qquad \alpha = a+1, \dots, a+c \\ t_\alpha(z) &= z_{l+1} + u_{\alpha - a - c}(x) + \sum_{p \geq 1} u_{\alpha - a - c}^{(p)}(\tilde{z}, x) z_{l+1}^{-p}, \qquad \alpha = a+c+1, \dots, a+c+b \end{aligned} \tag{6.2}$$

where $z = (z_1, \dots, z_n)$, $\tilde{z} = (z_1, \dots z_l)$, $x = (0, x_{l+2}, \dots, x_n)$. Then when $\alpha = 1, \dots, a$ we have the following:

$$\begin{aligned} \sum_{\substack{\beta \neq \alpha \\ \beta = 1}}^{a+b+c} \frac{1}{t_\alpha(z) - t_\beta(z)} &= \sum_{\substack{\beta \neq \alpha \\ \beta = 1}}^{a} \frac{1}{v_\alpha(\tilde{z}) - v_\beta(\tilde{z})} + \\ \sum_{M \geq 1} z_{K+1}^{-M} \Bigg\{ \sum_{1 \leq L \leq M} \frac{(-1)^L}{v_{\alpha\beta}^{L+1}} &\sum_{\substack{p_1 + \dots + p_L = M \\ p_i \geq 1}} v_{\alpha\beta}^{(p_1)} \dots v_{\alpha\beta}^{(p_L)} - \\ - \sum_{1 \leq L \leq M-1} \frac{(-1)^L}{s_\beta^{L+1}} &\sum_{\substack{p_1 + \dots + p_L = M-1 \\ p_i \geq 1}} s_{\alpha\beta}^{(p_1)} \dots s_{\alpha\beta}^{(p_L)} - \\ - \sum_{1 \leq L \leq M-1} (-1)^L &\sum_{\substack{p_1 + \dots + p_L = M-1 \\ p_i \geq 1}} u_{\alpha\beta}^{(p_1)} \dots u_{\alpha\beta}^{(p_L)} \Bigg\} \end{aligned} \tag{6.3}$$

Here $v_{\alpha\beta}^{(p)} = v_\alpha^{(p)} - v_\beta^{(p)}$, $u_{\alpha\beta}^{(p)} = v_\alpha^{(p+1)} - u_\beta^{(p+1)}$ and $s_{\alpha\beta}^{(p)} = v_\alpha^{(p+1)} - s_\beta^{(p+1)}$. We have similar formulas for $\alpha = a+1, \dots, a+c$ and for $\alpha = a+c+1, \dots, a+c+b$, and for expansions of

$$\sum_{i=1}^{n} \frac{\lambda_i}{t_\alpha - z_i} \,. \tag{6.4}$$

Then one can verify that the equality

$$\sum_{i=1}^{n} \frac{\lambda_i}{t_\alpha - z_i} = \sum_{\beta\neq\alpha}^{a+b+c} \frac{2}{t_\alpha - t_\beta} \tag{6.5}$$

for t_α being an asymptotic power series (6.2) is equivalent to the following recursive system of equations:

(6.6)

$$\sum_{\beta=1}^{a} K_{\alpha\beta} v_\beta^{(m)} = \text{certain function of } (v, v^{(1)}, \dots, t^{(m-1)}; s, \dots, s^{(m-1)}; u, \dots, u^{(m-1)})$$

$$\sum_{\beta=c+1}^{a+c} L_{\alpha\beta} s_\beta^{(m)} = \text{certain function of } (v, v^{(1)}, \dots, t^{(m-1)}; s, \dots, s^{(m-1)}; u, \dots, u^{(m-1)})$$

$$\sum_{\beta=a+c+1}^{a+b+c} M_{\alpha\beta} u_\beta^{(m)} = \text{certain function of } (v, v^{(1)}, \dots, t^{(m-1)}; s, \dots, s^{(m-1)}; u, \dots, u^{(m-1)})$$

Here

(6.7)

$$K_{\alpha\beta} = \left\{ \sum_{i=1}^{l} \frac{\lambda_i}{(v_\alpha - z_i)^2} - \sum_{\beta\neq\alpha} \frac{2}{(v_\alpha - v_\beta)^2} \right\} \delta_{\alpha\beta} - (1-\delta_{\alpha\beta}) \frac{2}{(v_\alpha - v_\beta)^2}$$

$$L_{\alpha\beta} = \left\{ \frac{\lambda_1 + \dots + \lambda_l}{s_\alpha^2} + \frac{\lambda_{l+1} + \dots + \lambda_l}{(1-s_\alpha)^2} - \sum_{\beta\neq\alpha} \frac{2}{(s_\alpha - s_\beta)^2} \right\} \delta_{\alpha\beta} - (1-\delta_{\alpha\beta}) \frac{2}{(s_\alpha - s_\beta)^2}$$

$$M_{\alpha\beta} = \left\{ \sum_{i=l+1}^{n} \frac{\lambda_i}{(u_\alpha - z_i)^2} - \sum_{\beta\neq\alpha} \frac{2}{(u_\alpha - u_\beta)^2} \right\} \delta_{\alpha\beta} - (1-\delta_{\alpha\beta}) \frac{2}{(u_\alpha - u_\beta)^2}$$

These matrices are nondegenerate according to our assumptions and to Conjecture 5.3. Therefore, recurrencies determine power series (6.2) uniquely and these series will satisfy equations (6.5).

This finishes the proof of Theorem 6.1 for $\mathfrak{g} = sl_2$. For other simple Lie algebras the proof is absolutely similar.

7. sl_2-case.

Let $\mathfrak{g} = sl_2$. In terms of the standard generators e, f, h we have

$$\Omega = \tfrac{1}{2} h \otimes h + e \otimes f + f \otimes e \ .$$

Let $V_1, \dots, V_n$ be sl_2-modules with highest weights $\Lambda_1, \dots, \Lambda_n$ and highest weight vectors $v_1, \dots, v_n$, respectively. Set $\Lambda = \Lambda_1 + \dots + \Lambda_n$. For $k \in \mathbb{Z}_{\geq 0}$, set

$$\begin{aligned} V \ &= V_1 \otimes \dots \otimes V_n \ , \\ (V)_k &= \{ v \in V \mid hv = (\Lambda - k\alpha, \alpha) v \} \ , \\ \mathrm{Sing}(V)_k &= \{ v \in (V)_k \mid ev = 0 \} \ , \end{aligned}$$

where α is the simple root.

Consider the space $\mathbb{C}^{n+k}$ with coordinates $t_1, \dots, t_k, \ \ z_1, \dots, z_k$. The functions Φ and S, defined in (3.3) and (4.2), now have the form

$$\Phi = \prod_{1 \leq m < \ell \leq n} (z_m - z_\ell)^{(\Lambda_m, \Lambda_\ell)/\kappa} \prod_{m,i} (z_m - t_i)^{-(\Lambda_m, \alpha)/\kappa} \prod_{i<j} (t_i - t_j)^{2/\kappa}$$

(7.1) $$S = \kappa \ln \Phi.$$

A monomial of weight k is an element $M_K \in (V)_k$ of the form

(7.2) $$M_K = f^{k_1} v_1 \otimes \dots \otimes f^{k_n} v_n \ ,$$

$K = (k_1, \dots, k_n), \ \ k_1 + \dots + k_n = k$. For a monomial M_K, the differential form $\eta(M_K)$, used in (3.5) and (3.9) to construct solutions to the KZ equation, has the form

(7.3) $$\begin{aligned} \eta(M_K) &= A_{M_K} dt_1 \wedge \dots \wedge dt_k \ , \\ A_M &= \sum_{\sigma \in S(k, k_1, \dots, k_n)} \prod_{i=1}^{k} \frac{1}{t_i - z_{\sigma(i)}} \ . \end{aligned}$$

The sum is over the set $S(k, k_1, \dots, k_n)$ of maps σ from $\{1, \dots, k\}$ to $\{1, \dots, n\}$ such that for all m the cardinality of $\sigma^{-1}(m)$ is k_m.

Let $t = t(z)$ be a nondegenerate solution to the system of equations

(7.4) $$\frac{\partial S}{\partial t_i} \, (t, z) = 0 \ , \qquad i = 1, \dots, k \ .$$

Then formula (4.15) gives a vector $f_0(z) \in \mathrm{Sing}(V)_k$ with properties indicated in (4.16). The function $\exp(S(t(z), z)/\kappa) f_0(z)$ is an asymptotically flat section of first order.

From now on we assume that $V_1, \dots, V_n$ are Verma modules. In §9 we will show that the eigenvectors constructed by (4.15) form a basis in $\mathrm{Sing}(V)_k$ for generic $\Lambda_1, \dots, \Lambda_n$, and $z_1, \dots, z_n$. We will show that these vectors are pairwise orthogonal with respect to the Shapovalov form. §8 contains a preliminary information.

8. Iterated singular vectors.

Let W_i, $i = 1, 2$, be sl_2 modules. Let $\omega_i \in W_i$ be a singular vector of weight $m_i \in \mathbb{C}$, that is, $e\omega_i = 0$ and $h\omega_i = m_i\omega_i$. For a nonnegative integer ℓ, the vector

$$(\omega_1, \omega_2)_\ell := \left(\prod_{j=1}^{\ell} (m_1 + m_2 + j + 1 - 2\ell) \right) \cdot$$

(8.1) $$\cdot \sum_{p=0}^{\ell} (-1)^p \binom{\ell}{p} \cdot \prod_{j=0}^{p-1} (m_1 - j)^{-1} \cdot \prod_{j=0}^{\ell-p-1} (m_2 - j)^{-1} f^p \omega_1 \otimes f^{\ell-p} \omega_2$$

is a singular vector in $W_1 \otimes W_2$ of weight $m_1 + m_2 - 2\ell$. We assume that all denominators in this formula are not zero.

Let Ω be the Casimir operator acting on $W_1 \otimes W_2$.

(8.2)
$(\omega_2, \omega_2)_\ell$ is an eigenvector of Ω with eigenvalue $\lambda(m_1, m_2; \ell) = \frac{1}{2} m_1 m_2 - \ell(m_1 + m_2) + \ell(\ell - 1)$.

Now let $V_1, \ldots, V_n$ be sl_2 Verma modules with highest weights $\Lambda_1, \ldots, \Lambda_n$ and highest vectors $v_1, \ldots, v_n$, respectively. Set $m_i = (\Lambda_i, \alpha)$.

For any sequence of non-negative integers $I = (i_2, \ldots, i_n)$, $i_2 + \cdots + i_n = k$, set

(8.3) $$v_I = (\ldots((v_1, v_2)_{i_2}, v_3)_{i_3}, \ldots, v_n)_{i_n}$$

v_I is a singular vector in $V = V_1 \otimes \cdots \otimes V_n$ of weight $m_1 + \cdots + m_n - 2k$. We assume that all denominators in v_I are not zero. The vector v_I will be called the *iterated singular vector.*

(8.4) For generic $\Lambda_1, \ldots, \Lambda_n$, the iterated vectors form a basis in $\mathrm{Sing}(V)_k$

It is easy to write explicitly the nondegeneracy conditions.

For any $\ell = 2, \ldots, n$, set

(8.5) $$G_\ell = \sum_{i<\ell} \Omega_{i\ell}$$

(8.6) Lemma. *For any ℓ and any $I = (i_2, \ldots, i_n)$, the iterated vector v_I is an eigenvector of G_ℓ with eigenvalue*

$$\lambda(m_1 + \cdots + m_{\ell-1} - 2(i_2 + \cdots + i_{\ell-1}), m_\ell; i_\ell).$$

The lemma easily follows from the following property of the Casimir operator:

$$[\Omega, 1 \otimes x + x \otimes 1] = 0$$

for any $x \in sl_2$.

(8.7) Corollary. *For generic $\Lambda_1, \dots, \Lambda_n$ we have the following property:*

Let $I = (i_2, \dots, i_n)$, $i_2 + \dots + i_n = k$, and $J = (j_2, \dots, j_n)$, $j_2 + \dots + j_n = k$, be different sequences. Then there exists $\ell \in \{2, \dots, n\}$ such that the eigenvalues of vectors v_I and v_J with respect to G_ℓ are different.

9. Critical points of function Φ.

Let

$$\Phi_2(t) = \prod_{i=1}^{k} t_i^{-m_1/\kappa}(t_i - 1)^{-m_2/\kappa} \cdot \prod_{1 \le i < j \le k} (t_i - t_j)^{2/\kappa} \tag{9.1}$$

where $m_1, m_2, \kappa \in \mathbb{C}$ are parameters. This function is a special case of the function Φ defined in (7.1). Let $S_2 = \kappa \ln \Phi_2$.

Let

$$\lambda_1 = t_1 + \dots + t_k, \qquad \lambda_2 = \sum t_i t_j, \qquad \dots, \qquad \lambda_k = t_1 \cdot \ldots \cdot t_k$$

be the standard symmetric functions. Let

$$\mu_1 = (1 - t_1) + \dots + (1 - t_k), \qquad \mu_2 = \sum (1 - t_i)(1 - t_j), \qquad \cdots$$

$$\mu_k = (1 - t_1) \dots (1 - t_k), \qquad \delta = \prod_{1 \le i < j \le k} (t_i - t_j)^2.$$

(9.2) Theorem, [V], cf. [Sz, 6.7].

If $t = (t_1, \dots, t_k)$ is a critical point of Φ_2, then

$$\lambda_\ell = \binom{k}{\ell} \cdot \prod_{j=1}^{\ell} \frac{(m_1 + j - k)}{(m_1 + m_2 + j + 1 - 2k)} ,$$

$$\mu_\ell = \binom{k}{\ell} \cdot \prod_{j=1}^{\ell} \frac{(m_2 + j - k)}{(m_1 + m_2 + j + 1 - 2k)} ,$$

for all ℓ,

$$\delta = \prod_{j=0}^{k-1} (j+1)^{j+1} \frac{(-m_1 + j)^j (-m_2 + j)^j}{(-m_1 - m_2 + 2k - 2 - j)^{2k-2-j}},$$

$$\mathrm{Hess}(-S_2(t)) = k! \prod_{j=0}^{k-1} \frac{(-m_1 - m_2 + 2k - 2 - j)^3}{(-m_1 + j)(-m_2 + j)} .$$

Let t be a critical point of Φ_2. Consider the vector f_0 given by Theorem (4.10):

$$f_0 = \pm (2\pi)^{k/2} (\mathrm{Hess}(-S_2(t)))^{-\frac{1}{2}} \sum_{\ell=0}^{k} A_{k-\ell,\ell} f^{k-\ell} v_1 \otimes f^\ell v_2 , \tag{9.3}$$

$$A_{k-\ell,\ell} = \sum_{1 \le i_1 < \dots < i_\ell \le k} \frac{1}{(t_{i_1} - 1)} \cdots \frac{1}{(t_{i_\ell} - 1)} \prod_{j \notin (i_1, \dots, i_\ell)} \frac{1}{t_j} .$$

Here v_p, $p = 1, 2$, is the highest weight vector of the Verma module V_p with highest weight $m_p \in \mathbb{C}$.

(9.4) Lemma, [V]. *We have*

$$A_{k-\ell,\ell} = (-1)^\ell \binom{k}{\ell} \cdot \prod_{j=1}^{k} (m_1 + m_2 + j + 1 - 2k) \cdot \prod_{j=0}^{k-\ell-1} (m_1 - j)^{-1} \cdot \prod_{j=0}^{\ell-1} (m_2 - j)^{-1}$$

for all ℓ.

The vector f_0 is a singular vector of weight $m_1 + m_2 - 2k$, it is proportional to the vector $(v_1, v_2)_k$ in (8.1).

Now let Φ be the function defined by (7.1). Set $m_\ell = (\Lambda_\ell, \alpha)$ for all ℓ.

Consider in $\mathbb{C}^k$ the configuration $\mathcal{C}$ of hyperplanes

$$\begin{aligned} &H_{ij} : t_i = t_j \ , \qquad 1 \le i < j \le k \\ &H_i^\ell : t_i = z_\ell \ , \qquad i = 1, \dots, k \ , \ell = 1, \dots, n \ . \end{aligned} \tag{9.5}$$

Let T be the complement in $\mathbb{C}^k$ to the union of all hyperplanes of $\mathcal{C}$. Set

$$T_{\mathbb{R}} = T \cap \mathbb{R}^k \ .$$

(9.6) Theorem, [V]. *Assume that $z_1, \dots, z_n$ are real and pairwise different. Assume that $m_1, \dots, m_n$ are negative. Then all critical points of Φ lie in the union of bounded connected components of $T_{\mathbb{R}}$. Each bounded component contains exactly one critical point. This critical point is nondegenerate.*

Assume that $z_1 < z_2 < \dots < z_n$.

We say that a bounded connected component is *admissible* if it lies in the cone $t_1 < t_2 < \dots < t_k$.

Admissible components are enumerated by sequences of nonnegative integers $I = (i_2, \dots, i_n)$, $i_2 + \dots + i_n = k$. The corresponding component has the form

(9.7)

$$\mathcal{D}_I(z) = \{ t \in \mathbb{R}^k | z_1 < t_1 < \dots < t_{i_2} < z_2 < \dots < z_{n-1} < t_{i_2 + \dots + i_{n-1} + 1} < \dots < t_k < z_n \}$$

cf. (8.3). The critical point of Φ in $\mathcal{D}_I(z)$ is denoted by $t_I(z)$.

Consider the Bethe vector

$$g_I(z) = \sum_{k_1 + \dots + k_n = k} A_{(k_1, \dots, k_n)}(t_I(z), z) f^{k_1} v_1 \otimes \dots \otimes f^{k_n} v_n \tag{9.8}$$

where $A_k(t,z)$ is given by (7.3). $g_I(z)$ is a vector in $\mathrm{Sing}(V)_k$ where V is the tensor product of Verma modules with highest weights $\Lambda_1, \dots, \Lambda_n$.

It is well known that for generic $\Lambda_1, \dots, \Lambda_n$ the dimension of $\mathrm{Sing}(V)_k$ is equal to the number of iterated singular vectors described in §8 and, therefore, it is equal to the number of admissible domains.

Consider the set of vectors $\{g_I(z)\}_{I \in \mathrm{Adm}}$, where I ranges over the set of sequences of nonnegative integers $(i_2, \dots, i_n)$, $i_2 + \dots + i_n = k$. In [V] it is proved that under explicitly written conditions on $\Lambda_1, \dots, \Lambda_n$ the Bethe vectors $\{g_I(z)\}_{I \in \mathrm{Adm}}$, form a basis in $\mathrm{Sing}(V)_k$. Here we'll describe asymptotics of these vectors for $z_1 << z_2 \dots << z_n$.

(9.9) Theorem. *Assume that* $z_j = s^j$, $j = 1, \dots, n$, *and* $s \to +\infty$. *Then for any* $I = (i_2, \dots, i_n)$, $i_2 + \dots + i_n = k$, *we have*

$$g_I(z(s)) = s^{d(I)}(v_I + O(s^{-1}))$$

where v_I *is given by (8.3) and* $d(I)$ *is an integer. Moreover, for any* $\ell = 2, \dots, n$, *the operator* $H_\ell(z) = \sum_{j \neq \ell} \Omega_{j\ell}/(z_j - z_\ell)$ *has the following asymptotics:*

$$H_\ell = s^{-\ell}(\Omega_{1,\ell} + \Omega_{2,\ell} + \dots + \Omega_{\ell-1,\ell} + O(s^{-1})) \ .$$

(9.10) Corollary. *The Bethe vectors* $\{g_I(z)\}_{I \in \mathrm{Adm}}$ *form a basis in* $\mathrm{Sing}(V)_k$ *for generic* $z_1, \dots, z_n$ *and generic* $\Lambda_1, \dots, \Lambda_n$.

(9.11) Corollary. *Let* $\lambda_I^\ell(z)$ *be the eigenvalue of* $g_I(z)$ *with respect to* $H_\ell(z)$, *then*

$$\gamma_I^\ell(z(s)) = \lambda(m_1 + \dots + m_{\ell-1} - 2(i_2 + \dots + i_{\ell-1}), m_\ell; i_\ell) + O(s^{-1})$$

for $\ell > 1$, *see (8.6).*

(9.12) Corollary. *For generic* $\Lambda_1, \dots, \Lambda_n$ *we have the following property:*

Let I *and* J *be different admissible sequences. Then there exists* ℓ *such that the eigenvalues of the Bethe vectors* $g_I(z(s))$ *and* $g_J(z(s))$ *with respect to* $H_\ell(z)$ *are different for* $s >> 1$.

(9.13) Corollary. *Let* I *and* J *be different admissible sequences. Then the Bethe vectors* $g_I(z(s))$ *and* $g_J(z(s))$ *are orthogonal with respect to the Shapabalov form* B.

(9.14) Remark. It is shown in [V] that

$$B(g_I(z(s)), g_I(z(s))) = \mathrm{Hess}_t(S(t_I(z), z)) \ .$$

Proof of the theorem. The statement on H_ℓ is trivial. We prove the statement on g_I. Make a change of variables: $t_j = s^\ell u_j$ if $i_1 + \dots + i_{\ell-1} < j \le i_2 + \dots + i_\ell, \ell = 2, ..., n$ (we assume that $i_1 = 0$) . For any $\ell = 2, \dots, n$, introduce a function

$$S_\ell(u) = -\sum_{j=i_{\ell-1}+1}^{i_\ell} (a_\ell \ln u_j + m_\ell \ln(u_j - 1)) + 2 \sum_{i_{\ell-1}+1 \le i < j \le i_\ell} \ln(u_i - u_j)$$

where $a_\ell = m_1 + \dots + m_{\ell-1} - 2(i_2 + \dots + i_{\ell-1})$. Then we have

$$S(t(u)) = A \ln s + S_2(u) + \dots + S_n(u) + O(s^{-1}) \tag{9.15}$$

for some number A. This formula and the explicit formula for g_I imply the theorem.

Assume that $m_\ell = (\Lambda_\ell, \alpha)$, $\ell = 1, \dots, n$, are positive integers. Let $V_1, \dots, V_n$ be irreducible sl_2 modules with highest weights $\Lambda_1, \dots, \Lambda_n$, respectively.

Apply the construction of Section 4 to this situation. For any $t-$critical point $t = t(z)$ of the function Φ we get a Bethe vector $g(t(z), z) \in \mathrm{Sing}(V)_k$ given by (9.8).

We say that a $t-$critical point $t = t(z)$ is *nontrivial* if $g(t(z), z) \neq 0$, otherwise we will call it *trivial.*

The set of $t-$critical points of Φ is invariant with respect to the group of permutations of coordinates $t_1, \dots, t_k$. Critical points lying in the same orbit give the same eigenvector.

Consider the family $\{g(t(z), z)\}$ of the Bethe vectors where $t(z)$ runs through the set of orbits of critical points.

(9.16) Theorem. *For any z every trivial t-critical point is degenerate and all Bethe vectors $\{g(t(z), z)\}$ are pairwise orthogonal with respect to the Shapovalov form. For generic z, all nontrivial critical points are nondegenerate and the corresponding Bethe vectors form a basis in* $\mathrm{Sing}(V)_k$.

PROOF. Trivial critical points are degenerate by (9.14). All Bethe vectors are pairwise orthogonal by (9.13). Therefore, it suffices to show that there are at least dim $\mathrm{Sing}(V)_k$ orbits of nondegenerate critical points for generic z.

Let $I = (i_2, \dots, i_n)$ be a sequence of nonnegative integers such that $i_1 + \cdots + i_n = k$. We say that I is a *good sequence* if for any $\ell = 2, \dots, n$ we have

$$i_\ell \leq \min(m_1 + \cdots + m_{\ell-1} - 2(i_2 + \cdots + i_{\ell-1}), m_\ell) \; .$$

The number N of good sequences is equal to dimension of $\mathrm{Sing}(V)_k$.

We show that there are at least N orbits of nondegenerate critical points if $z_\ell = s^\ell$, $\ell = 1, \dots, n$, and s tends to $+\infty$.

Let I be a good sequence. Make a change of variables $t_j = s^\ell u_j$ if $i_1 + \cdots + i_{\ell-1} < j \leq i_2 + \cdots + i_\ell$. Then we have formula (9.15).

Critical points of Φ coincide with critical points of $S(t(u))$. The critical point equations for $S(t(u))$ are a deformation of the critical point equations for $S_2(u) + \cdots + S_n(u)$. Critical points of $S_2 + \cdots + S_n$ are described by (9.2).

Let u_I be a critical point of $S_2 + \cdots + S_n$. Let $u_I(s)$ be the critical point of $S(t(u))$ which is the deformation of u_I. Let $t_I(s)$ be the corresponding critical point of Φ and $g_I(t_I(s), z(s))$ the corresponding Bethe vector. Then

$$g(t_I(s), z(s)) = s^{d(I)}(v_I + O(s^{-1}))$$

where $d(I)$ is some integer. This describes asymptotics of $\{g(t(z), z)\}$ and proves the theorem.

10. Trees, asymptotic zones, and Bethe bases.

Let $V_1, \dots, V_n$ be finite-dimensional irreducible sl_2 modules. Consider the family of Bethe vectors $\{g(t(z), z)\} \in \mathrm{Sing}(V)_k$. We proved that for

$$|z_2 - z_1| << |z_3 - z_2| << \cdots << |z_n - z_{n-1}| \tag{10.1}$$

the Bethe vectors form a basis in $\mathrm{Sing}(V)_k$. Moreover, the Bethe vectors can be enumerated by good sequences $\{I\}$ in such a way that for every I,

$$g(t_I(z), z) \sim \mathrm{const}(z) \cdot v_I \tag{10.2}$$

where v_I is the corresponding iterated vector, $\mathrm{const}(z)$ is a scalar function, const $\neq 0$.

These facts have the following generalization. *An n-tree* is a planar tree with n tops, one root, and $n-1$ internal triple vertices. See an example in the figure below.

Figure 1. An n-tree T_0.

The n-tree T_0, shown in the figure, defines an "asymptotic zone" described by (10.1) and a basis $\{v_I\} \in \operatorname{Sing}(V)_k$ where I runs through the good sequences.

Similarly, every n-tree T defines two objects: an asymptotic zone, described by inequalities similar to (10.1), and a distinguished basis $\mathcal{B}_T$ in $\operatorname{Sing}(V)_k$ consisting of iterated vectors where iterations are defined according to the shape of the tree.

(10.3) Theorem.

For any n-tree T, we restrict the Bethe vectors $\{g(t(z), z)\}$ to the asymptotic zone defined by T. Then the Bethe vectors can be enumerated by elements of the basis $\mathcal{B}_T$ in such a way that for every $v \in \mathcal{B}_T$ we have

$$g_v(t(z), z) \sim \operatorname{const}(z) \cdot v$$

where $\operatorname{const}(z)$ *is a scalar function,* $\operatorname{const} \neq 0$.

The proof is the same as for (9.16).

Now consider an arbitrary complex simple Lie algebra $\mathfrak{g}$ instead of sl_2. Assume that Conjecture (5.4) is true for $\mathfrak{g}$.

Let $V_1, \dots, V_n$ be finite-dimensional irreducible $\mathfrak{g}$ modules. Consider the KZ equation with values in Sing $(V)_\lambda$, defined by (3.2). Let $\{g(t(z), z)\} \in \operatorname{Sing}(V_\lambda)$ be the Bethe vectors constructed in Section 4.

For any n-tree T we can construct a distinguished basis $\mathcal{B}_T$ in $\operatorname{Sing}(V)_\lambda$. The basis consists of the iterated vectors where iterations are constructed according to the shape of the tree, and at each step of the iteration we use the eigenvectors discussed in Conjecture (5.4). For example, for the tree shown in the figure and $g = sl_2$, the iteration procedure is given by (8.3) and the eigenvector used in each step of iteration (8.3) is given by (8.1).

(10.4) Theorem. *Assume that Conjecture (5.4) is true for $\mathfrak{g}$. For any n-tree T, we restrict the Bethe vectors $\{g(t(z), z)\}$ to the asymptotic zone defined by T. Then for every $v \in \mathcal{B}_T$ there exists a Bethe vector $g(t(z), z)$ such that $g(t(z), z) \sim$* $\operatorname{const}(z) \cdot v$ *where* $\operatorname{const}(z)$ *is a scalar function,* $\operatorname{const} \neq 0$.

The proof is the same as for (9.16).

11. Bethe vectors and Lame functions.

Consider the following problem [Sz, 6.8].

(11.1) Problem. *Let $A(t)$ and $B(t)$ be given polynomials of degree n and $n-1$, respectively. To determine a polynomial $C(t)$ of degree $n-2$ such that the differential equation*

$$A(t)y''(t) - B(t)y'(t) + C(t)y(t) = 0 \tag{11.2}$$

has a solution which is a polynomial of preassigned degree k.

A polynomial solution $y(t)$ is called a *Lame function.* Hein (1878) proved that, in general, there are exactly $\binom{k+n-2}{k}$ determinations of $C(z)$.

Example. Let $A = (1-t^2)$, $B = (\alpha - \beta + (\alpha+\beta+2)t)$, then $C = k(k+\alpha+\beta+1)$ and the corresponding polynomial solution of degree k, normalized by the condition $y(1) = \binom{k+\alpha}{k}$, is called *the Jacobi polynomial* and is denoted by $P_k^{(\alpha,\beta)}(t)$.

Let $A = (t-z_1)\dots(t-z_n)$,

$$\frac{B(t)}{A(t)} = \frac{m_1}{t-z_1} + \dots + \frac{m_n}{t-z_n} \tag{11.3}$$

(11.4) Theorem, Stieltjes [Sz, §6.8]. *Let A and B be given polynomials of degree n and $n-1$, respectively. Then there exists a polynomial C of degree $n-2$ and a polynomial solution $y = (t-t_1)\dots(t-t_k)$ of (11.2) if and only if $t = (t_1, \dots, t_k)$ is a critical point of the function*

$$\Psi(t_1,\dots,t_k) = \prod_{j=1}^{k}\prod_{i=1}^{n}(t_j - z_i)^{-m_i/\kappa} \prod_{1\le i<j\le k}(t_i - t_j)^{2/\kappa}$$

where κ is an arbitrary nonzero number.

Therefore, having a Lame function $y = (t-t_1)\dots(t-t_k)$ we can define a Bethe vector $g(z) \in \mathrm{Sing}(V)_k$ by formulae (9.8) and (7.3). Here $V = V_1 \otimes \dots \otimes V_n$, V_j is an sl_2 highest weight module with highest weight m_j.

It turns out that if V_j, $j = 1,\dots,n$, is the Verma module with highest weight m_j, then, in general, a Bethe vector defines a Lame function, and, therefore, Problem 11.1 is equivalent to the problem of diagonalization in $\mathrm{Sing}(V)_k$ of the KZ operators

$$H_i(z) = \sum_{j\ne i}\frac{\Omega_{ij}}{z_i - z_j}\ , \qquad i = 1,\dots,n\ .$$

Namely, let $g(z)$ be a Bethe vector given by (9.8) and (7.3). Then the vector

$$\prod_{j=1}^{k}\prod_{i=1}^{n}(t_j - z_i)g(z) = \sum_{k_1+\dots+k_n=k} B_{k_1,\dots,k_n} f^{k_1}v_1 \otimes \dots \otimes f^{k_n}v_n$$

polynomially depends on $t_1, \dots, t_k,\ \ z_1, \dots, z_n$.

For a sequence $L = (\ell_1 \geq \cdots \geq \ell_k)$ of positive integers define a symmetric polynomial p_L by

$$p_L(t_1, \dots, t_k) = \sum t^\alpha$$

where the sum is over all permutations α of the sequence $(\ell_1 - 1, \dots, \ell_k - 1)$. We have

$$B_K(z,t) = \sum_L M_{K,L}(z) \cdot p_L(t)\ .$$

Here $K = (k_1, \dots, k_n),\ \ k_1 + \cdots + k_n = k,\ \ k_j \in \mathbb{Z}_{\geq 0}$. The sum is over $L = (\ell_1 \geq \cdots \geq \ell_k)$, $n \geq \ell_1$. Denote by $\mathcal{K}_{k,n}$ (respectively $\mathcal{L}_{k,n}$) the set of such K (respectively L).

For every K, L, the coefficient $M_{K,L}$ is a polynomial in z depending only on k, n and independent of $m_1, \dots, m_n$. It is easy to see that $\#\mathcal{K}_{k,n} = \#\mathcal{L}_{k,n}$.

(11.5) Theorem. *We have*

$$\det(M_{K,L}(z)) \not\equiv 0.$$

Corollary. *If* $\det(M_{K,L}(z)) \neq 0$ *for some* z*, then the coefficients* $B_K(t,z)$ *of a Bethe vector uniquely determine the symmetric functions* $\{p_L(t)\}$*, the orbit of the critical point* $t = (t_1, \dots, t_k)$*, and the corresponding Lame function, cf. [S].*

Remark. In several examples, we have

$$\det(M_{K,L}(z)) = \text{const} \prod (z_i - z_j)^a$$

for suitable a.

Proof. Let $z_j = s^j,\ \ j = 1, \dots, n$. Then $(M_{K,L}(z(s)))$ is a polynomial in s. Let $d_{K,L} = \deg_s M_{K,L}(z(s)),\ \ d_L = \max_K d_{K,L}$. Let $m_{K,L}$ be the coefficient of s^{d_L} in $M_{K,L}$. We'll show that $\det(m_{K,L}) \neq 0$. This would imply the theorem.

Define the lexicographical order in $\mathcal{L}_{k,n}$ by the rule: $(\ell_1 \geq \cdots \geq \ell_k) > (\ell'_1 \geq \cdots \geq \ell'_k)$ if $\ell_k = \ell'_k, \dots, \ell_{j+1} = \ell'_{j+1},\ \ \ell_j > \ell'_j$ for some $j = 1, \dots, k$.

Define a bijection $\mathcal{L}_{k,n} \to \mathcal{K}_{k,n}$ by the rule $(\ell_1 \geq \cdots \geq \ell_k) \mapsto \sum_{j=1}^k (0, \dots, 0, 1_{\ell_j}, 0, \dots, 0)$. This bijection and the lexicographical order on $\mathcal{L}_{k,n}$ induce an order on $\mathcal{K}_{k,n}$. It is easy to see that the matrix $(m_{K,L})$ is a triangular matrix with nonzero diagonal elements and, therefore, $\det(m_{K,L}) \neq 0$.

An explanation of a connection between the Bethe vectors and the Lame functions is given in the Sklyanin paper [S] in which the tensor product V of sl_2 modules is realized in a suitable space of jets of functions of one variable, the sl_2 action is realized by differential operators, and the KZ operators are realized in terms of differential operators (11.2).

12. Branching of the Bethe vectors, Jordan blocks of the KZ operators.

The Bethe vectors discussed in this work may have branching outside diagonals $z_i = z_j$. Here is an example. Let V be an Sl_2 module with highest weight $m \neq 0$. Consider the Bethe equation corresponding to $(V \otimes V \otimes V)_1$:

$$\frac{1}{t - z_1} + \frac{1}{t - z_2} + \frac{1}{t - z_3} = 0 \ .$$

For $(z_1, z_2, z_3) = (0, 1, s)$, this equation gives $3t^2 - 2(1 + s) + s = 0$. Its discriminant has two roots s_1 and s_2. Therefore, for $s \neq s_1, s_2$ there are two Bethe vectors and they form a basis in the two-dimensional $\mathrm{Sing}(V \otimes V \otimes V)_1$. When s goes around s_j, $j = 1, 2$, the Bethe vectors interchange positions. For $s = s_j$, the KZ operators $H_\ell(z(s_j))$, $\ell = 1, 2, 3$, have a two-dimensional Jordan block. Their single eigenvector is the Bethe vector corresponding to the single critical point.

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Department of Mathematics
University of California
Berkeley, California 94720
and
Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina 27599