Quantum Painlevé tau-functions

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Plan

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1.1. quantum algebra

1.2. Bäcklund transformations (Weyl group action)

1.3. quantum τ-functions

2. $A^{(1)}_{n-1}$-case ($n \geqq 3$)

2.1. quantum algebra

2.2. Weyl group action

2.3. Hirota-Miwa equation

General theory for the quantum and q-difference version of τ-functions for any symmetrizable GCM

Quantum Algebra (1) Definiton

Consider the associative algebra generated by

- dependent variables: f_i
- parameter variables: $\alpha_i^\vee$
- τ-variables: τ_i

with the relations

- q-Serre relations of f_i.
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$ and f_j.
- τ_i commutes with τ_j and f_j.
- $\tau_i \alpha_j^\vee \tau_i^{-1} = \alpha_j^\vee + \delta_{ij}$. $\quad (\tau_i = \exp(\partial/\partial\alpha_i^\vee))$

Quantum Alegebra (2) q-Serre relations

$[a_{ij}]_{i,j\in I}$: GCM with $d_i a_{ij} = d_j a_{ji}$, $d_i \in \mathbb{Z}_{>0}$.

q: an inderminate.

$$q_i := q^{d_i}, \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}},$$

$$[n]_q! := [1]_q[2]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

q-Serre relations: if $i, j \in I$ and $i \neq j$, then

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_q f_i^{1-a_{ij}-k} f_j f_i^k = 0.$$

Quantum Algebra (3) Relations

- $$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j),$$
- $\alpha_i^\vee \alpha_j^\vee = \alpha_j^\vee \alpha_i^\vee$, $\alpha_i^\vee f_j = f_j \alpha_i^\vee$,
- $\tau_i \tau_j = \tau_j \tau_i$, $\tau_i f_j = f_j \tau_i$,
- $\tau_i \alpha_j^\vee \tau^{-1} = \alpha_j^\vee + \delta_{ij}$

τ_i's are the exponential of the canonical conjugate variables of the parameter variables $\alpha_i^\vee$.

Weyl group action (1)

Weyl group: $W = \langle s_i \mid i \in I \rangle$.

- $s_i^2 = 1$,
- $a_{ij}a_{ji} = 0 \implies s_i s_j = s_j s_i$,
- $a_{ij}a_{ji} = 1 \implies s_i s_j s_i = s_j s_i s_j$,
- $a_{ij}a_{ji} = 2 \implies (s_i s_j)^2 = (s_j s_i)^2$,
- $a_{ij}a_{ji} = 3 \implies (s_i s_j)^3 = (s_j s_i)^3$.

$[A, B]_q := AB - qBA$.

$(\mathrm{ad}_q\, f_i)(x) := [f_i, x]_{q_i^{\langle \alpha_i^\vee, \beta \rangle}}$, where β = the weight of x.

Then $(\mathrm{ad}_q\, f_i)^{k+1}(f_j) = [f_i, (\mathrm{ad}_q\, f_i)^k(f_j)]_{q^{2k+a_{ij}}}$.

Weyl group action (2)

Weyl group action (Bäcklund transformations):

- $s_i(f_i) := f_i,$
- $s_i(f_j) := \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee - k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q f_i)^k (f_j) f_i^{-k} \quad (i \neq j)$
- $s_i(\alpha_j^\vee) := \alpha_j^\vee - a_{ji}\alpha_i^\vee,$
- $s_i(\tau_i) := f_i \tau_i \prod_{j \in I} \tau_j^{-a_{ij}} = f_i \tau_i^{-1} \prod_{j \neq i} \tau_j^{-a_{ij}},$
- $s_i(\tau_j) := \tau_j \quad (i \neq j).$

The action of s_i is an algebra automorphism.

Proof. We can define the algebra automorphism $\tilde{s}_i$ by

$$\tilde{s}_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji}\alpha_i^\vee,$$
$$\tilde{s}_i(\tau_i) = \tau_i \prod_{j\in I} \tau_j^{-a_{ij}}, \qquad \tilde{s}_i(\tau_j) = \tau_j \quad (i \neq j),$$
$$\tilde{s}_i(f_j) = f_j.$$

Then we obtain, for $x = f_j, \alpha_j^\vee, \tau_i$,

$$s_i(x) = f_i^{\alpha_i^\vee} \tilde{s}_i(x) f_i^{-\alpha_i^\vee}.$$

This is an algebra automorphism. □

Useful formulas

$$f_i^{\lambda} f_j f_i^{-\lambda} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\lambda-k)} \begin{bmatrix} \lambda \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, f_i)^k (f_j) f_i^{-k} \qquad (i \neq j).$$

$s_i(f_j) = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee}$.

If $a_{ij} = -1$, then

$$\begin{aligned} s_i(f_j) &= q_i^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_{q_i} (f_i f_j - q_i^{-1} f_j f_i) f_i^{-1} \\ &= [1 - \alpha_i^\vee]_{q_i} f_j + [\alpha_i^\vee]_{q_i} f_i f_j f_i^{-1}. \end{aligned}$$

Therefore

$$s_i(f_j) f_i = [1 - \alpha_i^\vee]_{q_i} f_j f_i + [\alpha_i^\vee]_{q_i} f_i f_j.$$

Quantum τ-functions (1) Definition

Fundamental weights: $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$.

Weight lattice: $P := \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$, $P_+ := \sum_{i \in I} \mathbb{Z}_{\geqq 0}\Lambda_i$.

simple roots: $\alpha_j := \sum_{i \in I} a_{ij}\Lambda_i$

Weyl group action on P: $s_i(\Lambda_j) = \Lambda_j - \delta_{ij}\alpha_i$.

τ-monomial: $\tau^\mu := \prod_{i \in I} \tau_i^{\mu_i} \quad (\mu = \sum_{i \in I} \mu_i \Lambda_i \in P_+)$

(lattice) quantum τ-functions:

$$\tau(\lambda) := w(\tau^\mu) \qquad \text{for} \quad \lambda = w(\mu) \in WP_+.$$

Quantum τ-functions (2) Regularity

Regularity Theorem: All quantum τ-functions $\tau(\lambda)$ ($\lambda \in WP_+$) are (non-commutative) polynomials in the dependent variables f_i. $\square$

Main theorem of `arXiv:1206.3419`.

Proof of the regularity theorem

$\rho := \sum_{i\in I} \Lambda_i$, $w \circ \lambda := w(\lambda + \rho) - \rho$ $(\lambda \in P, w \in W)$.

Assume $\lambda, \mu \in P_+$ and $w \in W$.

$L(\mu)$: highest weight simple module.
$M(w \circ \lambda)$: Verma module with highest weight $w \circ \lambda$.
$M(w \circ \lambda) \subset M(\lambda)$.

Translation functor: $T_\lambda^\mu(M(\lambda \circ \lambda)) \subset M(w \circ \lambda) \otimes L(\mu)$.

Sketch of the proof: $T_\lambda^\mu(M(w \circ \lambda)) \cong M(w \circ (\lambda + \mu))$ implies the regularity theorem. □

Non-trivial relation between the theory of quantum τ-functions and representation theory!

$A^{(1)}_{n-1}$-case $\quad (n \geqq 3)$

$$i, j \in \mathbb{Z}/n\mathbb{Z}$$

$$a_{ij} = \begin{cases} 2 & (i = j) \\ -1 & (i - j = \pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$d_i = 1, \qquad q_i = q.$$

Quantum algebra

Consider the associative algebra generated by

- dependent variables: f_i
- parameter variables: $\alpha_i^\vee$
- τ-variables: τ_i $\qquad\qquad$ $(i \in \mathbb{Z}/n\mathbb{Z})$

with the defining relations

- $f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0.$
- $f_i f_j = f_j f_i \quad (j \neq i \pm 1).$
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$ and f_j.
- $\tau_i = \exp(\partial/\partial\alpha_i^\vee).$

Weyl group action

Using the useful formulas above, we obtain the following formulas:

- $s_i(f_{i\pm1}) = [1-\alpha_i^\vee]_q f_{i\pm1} + [\alpha_i^\vee]_q f_i f_{i\pm1} f_i^{-1}$,
  $s_i(f_j) = f_j \quad (j \neq i \pm 1)$.
- $s_i(\alpha_i^\vee) = -\alpha_i^\vee, \quad s_i(\alpha_{i\pm1}^\vee) = \alpha_{i\pm1}^\vee + \alpha_i^\vee$,
  $s_i(\alpha_j^\vee) = \alpha_j^\vee \quad (j \neq i, i \pm 1)$.
- $s_i(\tau_i) = f_i \dfrac{\tau_{i-1}\tau_{i+1}}{\tau_i}, \quad s_i(\tau_j) = \tau_j \quad (i \neq j)$.

Extension of the weight lattice

$Q^\vee := \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i^\vee \oplus \mathbb{Z}\delta^\vee, \quad P := \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i \oplus \mathbb{Z}\Lambda_0.$

Dual bases: $\varepsilon_i^\vee, \delta^\vee \longleftrightarrow \varepsilon_i, \Lambda_0$.

Assume $\varepsilon_i^\vee = \varepsilon_{i+n}^\vee + \delta^\vee$ and $\varepsilon_{i+n} = \varepsilon_i$.

$\alpha_i^\vee := \varepsilon_i^\vee - \varepsilon_{i+1}^\vee, \quad \alpha_i := \varepsilon_i - \varepsilon_{i+1}.$

$\Lambda_i = \Lambda_0 + \varepsilon_1 + \cdots + \varepsilon_i \quad (i \in \mathbb{Z}_{\geqq 0}).$

Then $P = \bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\varpi_n, \quad \varpi_n = \varepsilon_1 + \cdots + \varepsilon_n$

$P_+ := \sum_{i=0}^{n-1} \mathbb{Z}_{\geqq 0}\Lambda_i + \mathbb{Z}\varpi_n.$

Assume $\Lambda_{i+n} = \Lambda_i + \varpi_n \quad (i \in \mathbb{Z}).$

Extended Weyl group actions on P and $Q^\vee$

$W = \langle s_0, s_1, \dots, s_{n-1} \rangle, \quad s_{i+n} = s_i, \quad \pi(s_i) := s_{i+1},$

$\widetilde{W} = \langle \pi \rangle \ltimes W$: the extended Weyl group.

Assume $\lambda \in P$ and $\beta^\vee \in Q^\vee$.

$s_i(\lambda) := \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i, \quad \alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}.$

$s_i(\beta^\vee) := \beta^\vee - \langle \beta^\vee, \alpha_i \rangle \alpha_i^\vee$

$\pi(\Lambda_i) := \Lambda_{i+1}, \quad \pi(\varpi_n) := \varpi_n.$

$\pi(\varepsilon_i^\vee) := \varepsilon_{i+1}^\vee, \quad \pi(\delta^\vee) := \delta^\vee.$

Translations

$$T_i := s_{i-1} \cdots s_2 s_1 \pi s_{n-1} s_{n-2} \cdots s_i \in \widetilde{W} \quad (i = 1, \dots, n).$$

Assme $\nu = \sum_{i=1}^{n} \nu_i \varepsilon_i \in \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$.

$$T^\nu := \prod_{i=0}^{n-1} T_i^{\nu_i}.$$

Then

$$T^\nu(\varepsilon_i^\vee) = \varepsilon_i^\vee - \nu_i \delta^\vee \quad T^\nu(\delta^\vee) = \delta^\vee,$$

$$T^\nu(\alpha_i^\vee) = \alpha_i^\vee - (\nu_i - \nu_{i+1})\delta^\vee,$$

$$T^\nu(\varepsilon_i) = \varepsilon_i, \quad T^\nu(\Lambda_0) = \Lambda_0 + \nu,$$

$$T^\nu(\Lambda_i) = \Lambda_i + \nu.$$

Hirota-Miwa equation (1)

$$\begin{aligned} \Lambda_i &= \Lambda_{i-1} + \varepsilon_i, & \Lambda_{i+1} &= \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1}, \\ s_i(\Lambda_i) &= \Lambda_{i-1} + \varepsilon_{i+1}, & s_{i+1}(\Lambda_{i+1}) &= \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2}, \\ s_{i+1}s_i(\Lambda_i) &= \Lambda_{i-1} + \varepsilon_{i+2}, & s_i s_{i+1}(\Lambda_{i+1}) &= \Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}. \end{aligned}$$

$$\begin{aligned} \tau_i &= \tau(\Lambda_{i-1} + \varepsilon_i), & \tau_{i+1} &= \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1}), \\ s_i(\tau_i) &= \tau(\Lambda_{i-1} + \varepsilon_{i+1}), & s_{i+1}(\tau_{i+1}) &= \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2}), \\ s_{i+1}s_i(\tau_i) &= \tau(\Lambda_{i-1} + \varepsilon_{i+2}), & s_i s_{i+1}(\tau_{i+1}) &= \tau(\Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}). \end{aligned}$$

Lemma:

$$\begin{aligned} &[\alpha_{i+1}^\vee]_q \tau_i \, s_i s_{i+1}(\tau_{i+1}) + [\alpha_i^\vee]_q s_{i+1} s_i(\tau_i) \tau_{i+1} \\ &\qquad = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q s_i(\tau_i) s_{i+1}(\tau_{i+1}). \end{aligned}$$

Hirota-Miwa equation (2) Proof of Lemma

Warning: τ_i does not commute with $s_i s_{i+1}(\tau_{i+1})$.

$$\tau_i[\alpha_i^\vee]_q = [\alpha_i^\vee + 1]_q \tau_i, \quad \tau_i[1 - \alpha_i^\vee]_q = -[\alpha_i^\vee]_q \tau_i.$$

Proof of Lemma:

$$\begin{aligned}
&\tau_i \, s_i s_{i+1}(\tau_i) \\
&= \tau_i \, s_i \left(f_{i+1} \frac{\tau_i \tau_{i+2}}{\tau_{i+1}} \right) \\
&= \tau_i \, ([1 - \alpha_i^\vee]_q f_{i+1} + [\alpha_i^\vee]_q f_i f_{i+1} f_i^{-1}) f_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i} \frac{\tau_{i+2}}{\tau_{i+1}}, \\
&= (-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1}) \tau_{i-1} \tau_{i+2}.
\end{aligned}$$

Hirota-Miwa equation (3) Proof of Lemma

Thus

$$\tau_i\, s_i s_{i+1}(\tau_i) = (-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1})\tau_{i-1}\tau_{i+2}.$$

Similarly we obtain

$$s_{i+1}s_i(\tau_i)\tau_{i+1} = ([1-\alpha_{i+1}^\vee]_q f_i f_{i+1} + [\alpha_{i+1}^\vee]_q f_{i+1} f_i)\tau_{i-1}\tau_{i+2}$$
$$s_i(\tau_i)s_{i+1}(\tau_{i+1}) = f_i f_{i+1}\tau_{i-1}\tau_{i+2}.$$

q-numbers identity (or addition formula of $\sin$):

$$[\alpha_i^\vee + 1]_q[\alpha_{i+1}^\vee]_q + [\alpha_i^\vee]_q[1-\alpha_{i+1}^\vee]_q = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q.$$

The lemma follows from these calculations. $\square$

Hirota-Miwa equation (4)

Apply T^ν to the formula of Lemma. Then we obtain

Theorem: The quantum τ-functions of type $A^{(1)}{}_{n-1}$ satisfy the quantum q-Hirota-Miwa equations:

$$
\begin{aligned}
&[\varepsilon_i^\vee(\nu) - \varepsilon_{i+1}^\vee(\nu)]_q \quad \tau_i(\nu + \varepsilon_{i+2})\, \tau_i(\nu + \varepsilon_i + \varepsilon_{i+1}) \\
+\, &[\varepsilon_{i+1}^\vee(\nu) - \varepsilon_{i+2}^\vee(\nu)]_q\, \tau_i(\nu + \varepsilon_i) \quad \tau_i(\nu + \varepsilon_{i+1} + \varepsilon_{i+2}) \\
+\, &[\varepsilon_{i+2}^\vee(\nu) - \varepsilon_i^\vee(\nu)]_q \quad \tau_i(\nu + \varepsilon_{i+1})\, \tau_i(\nu + \varepsilon_{i+2} + \varepsilon_i) = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_i(\nu) &:= \tau(\Lambda_{i-1} + \nu), \\
\varepsilon_i^\vee(\nu) &:= T^\nu(\varepsilon_i^\vee) = \varepsilon_i^\vee - \nu_i \delta^\vee. \quad \square
\end{aligned}
$$