# Quantum Painlevé tau-functions

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December 11, 2018

Conformal field theory, isomonodromy tau-functions
and Painlevé equations, 2018
December 10 - 12, 2018
Room B301, Graduate School of Science, Kobe University

2018-12-11 Version 1.0

https://genkuroki.github.io/documents/20181211QuantumPainleveTau.pdf

# Summary

- For any symmetrizable GCM, we can introduce the proper non-commutativity of quantum $\tau$-functions. $\tau_i = \exp(\partial/\partial\alpha_i^\vee)$.
- Quantum $q$-Hirota-Miwa equations for $A_{n-1}^{(1)}$-case.
- Quantized Lax and Sato-Wilson forms of the extended affine Weyl group action for $A_{n-1}^{(1)}$-case.
- Quantized $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A(1)_{n-1})$-action for mutually prime $m$ and $n$.
- An appropriate quantization of $q\mathrm{P}_{\mathrm{IV}}$.

# General theory of the quantum and $q$-difference version of $\tau$-functions for any symmetrizable GCM

# Quantum Algebra: Definiton

Consider the associative algebra generated by

- dependent variables: $f_i$
- parameter variables: $\alpha_i^\vee$
- $\tau$-variables: $\tau_i$

with the relations

- $q$-Serre relations of $f_i$.
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$ and $f_j$.
- $\tau_i$ commutes with $\tau_j$ and $f_j$.
- $\tau_i \alpha_j^\vee \tau_i^{-1} = \alpha_j^\vee + \delta_{ij}$. $\quad (\tau_i = \exp(\partial/\partial\alpha_i^\vee))$

# Quantum Algebra: $q$-Serre relations

$[a_{ij}]_{i,j\in I}$: GCM with $d_i a_{ij} = d_j a_{ji}$, $d_i \in \mathbb{Z}_{>0}$.

$q$: an inderminate.

$$q_i := q^{d_i}, \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}},$$

$$[n]_q! := [1]_q[2]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

$q$-Serre relations: if $i, j \in I$ and $i \neq j$, then

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_q f_i^{1-a_{ij}-k} f_j f_i^k = 0.$$

# Quantum Algebra: Relations

- $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j)$,
- $\alpha_i^\vee \alpha_j^\vee = \alpha_j^\vee \alpha_i^\vee$, $\alpha_i^\vee f_j = f_j \alpha_i^\vee$,
- $\tau_i \tau_j = \tau_j \tau_i$, $\tau_i f_j = f_j \tau_i$,
- $\tau_i \alpha_j^\vee \tau^{-1} = \alpha_j^\vee + \delta_{ij}$

$\tau_i$'s are the exponential of the canonical conjugate variables of the parameter variables $\alpha_i^\vee$.

# Quantum Algebra: Summary

- $f_i \leftrightarrow$ Chavalley generators of $U_{q,-}$
- $\alpha_i^\vee \leftrightarrow$ simple coroots
- $\tau_i \leftrightarrow$ fundamental weights

- $f_i$ $(i \in I)$ satisfy the $q$-Serre relations.
- $\alpha_i^\vee$ and $\tau_i$ commute with $f_i$.
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$.
- $\tau_i$ commutes with $\tau_j$.
- $\tau_i \alpha_j^\vee = (\alpha_j^\vee + \delta_{ij})\tau_i \quad (\tau_i = \exp(\partial/\partial\alpha_i^\vee))$.

# Weyl group action

Weyl group: $W = \langle s_i \mid i \in I \rangle$.

- $s_i^2 = 1$,
- $a_{ij}a_{ji} = 0 \implies s_i s_j = s_j s_i$,
- $a_{ij}a_{ji} = 1 \implies s_i s_j s_i = s_j s_i s_j$,
- $a_{ij}a_{ji} = 2 \implies (s_i s_j)^2 = (s_j s_i)^2$,
- $a_{ij}a_{ji} = 3 \implies (s_i s_j)^3 = (s_j s_i)^3$.

$[A, B]_q := AB - qBA$.

$(\mathrm{ad}_q f_i)(x) := [f_i, x]_{q_i^{\langle \alpha_i^\vee, \beta \rangle}}$, where $\beta$ = the weight of $x$.

Then $(\mathrm{ad}_q f_i)^{k+1}(f_j) = [f_i, (\mathrm{ad}_q f_i)^k(f_j)]_{q^{2k+a_{ij}}}$.

**Weyl group action** (Bäcklund transformations):

- $s_i(f_i) := f_i,$
- $s_i(f_j) := \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee - k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, f_i)^k (f_j) f_i^{-k} \quad (i \neq j)$
- $s_i(\alpha_j^\vee) := \alpha_j^\vee - a_{ji}\alpha_i^\vee,$
- $s_i(\tau_i) := f_i \tau_i \prod_{j \in I} \tau_j^{-a_{ij}} = f_i \tau_i^{-1} \prod_{j \neq i} \tau_j^{-a_{ij}},$
- $s_i(\tau_j) := \tau_j \quad (i \neq j).$

**Remark.**

- $\tau_i = \exp(\partial/\partial\alpha_i^\vee) \leftrightarrow$ the fundamental weight $\Lambda_i$
- $\tau_i \prod_{j \in I} \tau_j^{-a_{ij}} \leftrightarrow s_i(\Lambda_i) = \Lambda_i - \alpha_i$

# The action of $s_i$ is an algebra automorphism.

**Proof.** We can define the algebra automorphism $\tilde{s}_i$ by

$$\tilde{s}_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji}\alpha_i^\vee,$$
$$\tilde{s}_i(\tau_i) = \tau_i \prod_{j\in I} \tau_j^{-a_{ij}}, \qquad \tilde{s}_i(\tau_j) = \tau_j \quad (i \neq j),$$
$$\tilde{s}_i(f_j) = f_j.$$

Then we obtain, for $x = f_j, \alpha_j^\vee, \tau_i$,

$$s_i(x) = f_i^{\alpha_i^\vee} \tilde{s}_i(x) f_i^{-\alpha_i^\vee}.$$

This is an algebra automorphism. □

# Useful formulas

$$f_i^{\lambda} f_j f_i^{-\lambda} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\lambda-k)} \begin{bmatrix} \lambda \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, f_i)^k (f_j) f_i^{-k} \qquad (i \neq j).$$

$$s_i(f_j) = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee}.$$

If $a_{ij} = -1$, then

$$\begin{aligned} s_i(f_j) &= q_i^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_{q_i} (f_i f_j - q_i^{-1} f_j f_i) f_i^{-1} \\ &= [1 - \alpha_i^\vee]_{q_i} f_j + [\alpha_i^\vee]_{q_i} f_i f_j f_i^{-1}. \end{aligned}$$

Therefore

$$s_i(f_j) f_i = [1 - \alpha_i^\vee]_{q_i} f_j f_i + [\alpha_i^\vee]_{q_i} f_i f_j.$$

# Quantum $\tau$-functions: Definition

**Fundamental weights:** $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$.

**Weight lattice:** $P := \bigoplus_{i\in I} \mathbb{Z}\Lambda_i$, $P_+ := \sum_{i\in I} \mathbb{Z}_{\geqq 0}\Lambda_i$.

**simple roots:** $\alpha_j := \sum_{i\in I} a_{ij}\Lambda_i$

**Weyl group action on $P$:** $s_i(\Lambda_j) = \Lambda_j - \delta_{ij}\alpha_i$.

**$\tau$-monomial:** $\tau^\mu := \prod_{i\in I} \tau_i^{\mu_i} \quad (\mu = \sum_{i\in I} \mu_i\Lambda_i \in P_+)$

**(lattice) quantum $\tau$-functions:**

$$\tau(\lambda) := w(\tau^\mu) \qquad \text{for} \quad \lambda = w(\mu) \in WP_+.$$

# Quantum $\tau$-functions: Regularity

**Regularity Theorem:** All quantum $\tau$-functions $\boldsymbol{\tau}(\boldsymbol{\lambda})$ ($\lambda \in WP_+$) are (non-commutative) polynomials in the dependent variables $f_i$. □

Main theorem of `arXiv:1206.3419`.

# Proof of the regularity theorem

$\rho := \sum_{i\in I} \Lambda_i$, $w \circ \lambda := w(\lambda + \rho) - \rho$ $(\lambda \in P, w \in W)$.

Assume $\lambda, \mu \in P_+$ and $w \in W$.

$L(\mu)$: highest weight simple module.
$M(w \circ \lambda)$: Verma module with highest weight $w \circ \lambda$.
$M(w \circ \lambda) \subset M(\lambda)$.

**Translation functor:** $T_\lambda^\mu(M(\lambda \circ \lambda)) \subset M(w \circ \lambda) \otimes L(\mu)$.

**Sketch of the proof:** $T_\lambda^\mu(M(w \circ \lambda)) \cong M(w \circ (\lambda + \mu))$ implies the regularity theorem. □

Non-trivial relation between the theory of quantum $\tau$-functions and representation theory!

# $A^{(1)}_{n-1}$-case $\quad (n \geqq 3)$

$$i, j \in \mathbb{Z}/n\mathbb{Z}$$

$$a_{ij} = \begin{cases} 2 & (i = j) \\ -1 & (i - j = \pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$d_i = 1, \qquad q_i = q.$$

# Quantum algebra

Consider the associative algebra generated by

- dependent variables: $f_i$
- parameter variables: $\alpha_i^\vee$
- $\tau$-variables: $\tau_i$ $\qquad$ $(i \in \mathbb{Z}/n\mathbb{Z})$

with the defining relations

- $f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0.$
- $f_i f_j = f_j f_i \quad (j \neq i \pm 1).$
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$ and $f_j$.
- $\tau_i = \exp(\partial/\partial\alpha_i^\vee).$

# Weyl group action

Using the useful formulas above, we can show the following formulas:

- $s_i(f_{i\pm1}) = [1-\alpha_i^\vee]_q f_{i\pm1} + [\alpha_i^\vee]_q f_i f_{i\pm1} f_i^{-1}$,
  $s_i(f_j) = f_j \quad (j \neq i \pm 1)$.
- $s_i(\alpha_i^\vee) = -\alpha_i^\vee, \quad s_i(\alpha_{i\pm1}^\vee) = \alpha_{i\pm1}^\vee + \alpha_i^\vee$,
  $s_i(\alpha_j^\vee) = \alpha_j^\vee \quad (j \neq i, i \pm 1)$.
- $s_i(\tau_i) = f_i \dfrac{\tau_{i-1}\tau_{i+1}}{\tau_i}, \quad s_i(\tau_j) = \tau_j \quad (i \neq j)$.

# Extended coroot and weight lattices

$Q^\vee := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i^\vee \oplus \mathbb{Z}\delta^\vee, \quad P := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i \oplus \mathbb{Z}\Lambda_0.$

Dual bases: $\varepsilon_i^\vee, \delta^\vee \longleftrightarrow \varepsilon_i, \Lambda_0$.

Assume $\varepsilon_i^\vee = \varepsilon_{i+n}^\vee + \delta^\vee$ and $\varepsilon_{i+n} = \varepsilon_i$.

$\alpha_i^\vee := \varepsilon_i^\vee - \varepsilon_{i+1}^\vee, \quad \alpha_i := \varepsilon_i - \varepsilon_{i+1}.$

$\Lambda_i = \Lambda_0 + \varepsilon_1 + \cdots + \varepsilon_i \quad (i \in \mathbb{Z}_{\geqq 0}).$

Then $P = \bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\varpi_n, \quad \varpi_n = \varepsilon_1 + \cdots + \varepsilon_n$

$P_+ := \sum_{i=0}^{n-1} \mathbb{Z}_{\geqq 0}\Lambda_i + \mathbb{Z}\varpi_n.$

Assume $\Lambda_{i+n} = \Lambda_i + \varpi_n \quad (i \in \mathbb{Z}).$

# Extended affine Weyl group

$$W = W(A^{(1)}_{n-1}) = \langle s_0, s_1, \dots, s_{n-1} \rangle, \quad s_{i+n} = s_i.$$

$$\pi(s_i) := s_{i+1},$$

$\widetilde{W} = \widetilde{W}(A^{(1)}_{n-1}) = \langle \pi \rangle \ltimes W$: the extended aff. Weyl gr.

Assume $\lambda \in P$ and $\beta^\vee \in Q^\vee$.

$$s_i(\lambda) := \lambda - \langle \alpha^\vee_i, \lambda \rangle \alpha_i, \quad \alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}.$$

$$s_i(\beta^\vee) := \beta^\vee - \langle \beta^\vee, \alpha_i \rangle \alpha^\vee_i$$

$$\pi(\Lambda_i) := \Lambda_{i+1}, \quad \pi(\varpi_n) := \varpi_n.$$

$$\pi(\varepsilon^\vee_i) := \varepsilon^\vee_{i+1}, \quad \pi(\delta^\vee) := \delta^\vee.$$

# Translation part of $\widetilde{W}$

$$T_i := s_{i-1} \cdots s_2 s_1 \pi \, s_{n-1} s_{n-2} \cdots s_i \in \widetilde{W} \quad (i = 1, \ldots, n).$$

Assme $\nu = \sum_{i=1}^{n} \nu_i \varepsilon_i \in \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$.

$T^\nu := \prod_{i=0}^{n-1} T_i^{\nu_i}$.

Then

$$T^\nu(\varepsilon_i^\vee) = \varepsilon_i^\vee - \nu_i \delta^\vee \quad T^\nu(\delta^\vee) = \delta^\vee,$$

$$T^\nu(\alpha_i^\vee) = \alpha_i^\vee - (\nu_i - \nu_{i+1})\delta^\vee,$$

$$T^\nu(\varepsilon_i) = \varepsilon_i, \quad T^\nu(\Lambda_0) = \Lambda_0 + \nu,$$

$$T^\nu(\Lambda_i) = \Lambda_i + \nu.$$

# Hirota-Miwa equation (1)

$$
\begin{aligned}
\Lambda_i &= \Lambda_{i-1} + \varepsilon_i, & \Lambda_{i+1} &= \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1}, \\
s_i(\Lambda_i) &= \Lambda_{i-1} + \varepsilon_{i+1}, & s_{i+1}(\Lambda_{i+1}) &= \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2}, \\
s_{i+1}s_i(\Lambda_i) &= \Lambda_{i-1} + \varepsilon_{i+2}, & s_i s_{i+1}(\Lambda_{i+1}) &= \Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}.
\end{aligned}
$$

$$
\begin{aligned}
\tau_i &= \tau(\Lambda_{i-1} + \varepsilon_i), & \tau_{i+1} &= \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1}), \\
s_i(\tau_i) &= \tau(\Lambda_{i-1} + \varepsilon_{i+1}), & s_{i+1}(\tau_{i+1}) &= \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2}), \\
s_{i+1}s_i(\tau_i) &= \tau(\Lambda_{i-1} + \varepsilon_{i+2}), & s_i s_{i+1}(\tau_{i+1}) &= \tau(\Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}).
\end{aligned}
$$

**Lemma:**

$$
\begin{aligned}
&[\alpha_{i+1}^\vee]_q \tau_i \, s_i s_{i+1}(\tau_{i+1}) + [\alpha_i^\vee]_q s_{i+1} s_i(\tau_i) \tau_{i+1} \\
&\qquad = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q s_i(\tau_i) s_{i+1}(\tau_{i+1}).
\end{aligned}
$$

# Hirota-Miwa equation (2) Proof of Lemma

**Warning:** $\tau_i$ does not commute with $s_i s_{i+1}(\tau_{i+1})$.

$$\tau_i[\alpha_i^\vee]_q = [\alpha_i^\vee + 1]_q \tau_i, \quad \tau_i[1 - \alpha_i^\vee]_q = -[\alpha_i^\vee]_q \tau_i.$$

**Proof of Lemma:**

$$\begin{aligned}
&\tau_i\, s_i s_{i+1}(\tau_i) \\
&= \tau_i\, s_i \left( f_{i+1} \frac{\tau_i \tau_{i+2}}{\tau_{i+1}} \right) \\
&= \tau_i \left( [1 - \alpha_i^\vee]_q f_{i+1} + [\alpha_i^\vee]_q f_i f_{i+1} f_i^{-1} \right) f_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i} \frac{\tau_{i+2}}{\tau_{i+1}}, \\
&= \left( -[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1} \right) \tau_{i-1} \tau_{i+2}.
\end{aligned}$$

Thus

$$\tau_i\, s_i s_{i+1}(\tau_i) = (-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1})\tau_{i-1}\tau_{i+2}.$$

Similarly we obtain

$$s_{i+1} s_i(\tau_i)\tau_{i+1} = ([1 - \alpha_{i+1}^\vee]_q f_i f_{i+1} + [\alpha_{i+1}^\vee]_q f_{i+1} f_i)\tau_{i-1}\tau_{i+2}$$

$$s_i(\tau_i) s_{i+1}(\tau_{i+1}) = f_i f_{i+1}\tau_{i-1}\tau_{i+2}.$$

$q$-numbers identity (or addition formula of **sin**):

$$[\alpha_i^\vee + 1]_q[\alpha_{i+1}^\vee]_q + [\alpha_i^\vee]_q[1 - \alpha_{i+1}^\vee]_q = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q.$$

Therefore

$$\begin{aligned}[\alpha_{i+1}^\vee]_q \tau_i\, s_i s_{i+1}(\tau_{i+1}) + [\alpha_i^\vee]_q s_{i+1} s_i(\tau_i)\tau_{i+1} \\ = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q s_i(\tau_i) s_{i+1}(\tau_{i+1}).\end{aligned}$$

□

# Hirota-Miwa equation (3)

Apply $T^\nu$ to the formula of Lemma. Then we obtain

**Theorem:** The quantum $\tau$-functions of type $A^{(1)}_{n-1}$ satisfy the quantum $q$-Hirota-Miwa equations:

$$
\begin{aligned}
&[\varepsilon^\vee_i(\nu) - \varepsilon^\vee_{i+1}(\nu)]_q \quad \tau_i(\nu + \varepsilon_{i+2})\,\tau_i(\nu + \varepsilon_i + \varepsilon_{i+1}) \\
+ &[\varepsilon^\vee_{i+1}(\nu) - \varepsilon^\vee_{i+2}(\nu)]_q\, \tau_i(\nu + \varepsilon_i) \quad \tau_i(\nu + \varepsilon_{i+1} + \varepsilon_{i+2}) \\
+ &[\varepsilon^\vee_{i+2}(\nu) - \varepsilon^\vee_i(\nu)]_q \quad \tau_i(\nu + \varepsilon_{i+1})\,\tau_i(\nu + \varepsilon_{i+2} + \varepsilon_i) = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_i(\nu) &:= \tau(\Lambda_{i-1} + \nu), \\
\varepsilon^\vee_i(\nu) &:= T^\nu(\varepsilon^\vee_i) = \varepsilon^\vee_i - \nu_i \delta^\vee. \quad \square
\end{aligned}
$$

# Lax and Sato-Wilson forms of the affine Weyl group action

The relation between
the $RLL = LLR$ formalism of quantum groups and
the Lax and Sato-Wilson forms of the Painlevé systems
is non-trivial.

Assume that $m$ and $n$ are mutually prime.

# Lax form: RLL=LLR

$A^{(1)}_{m-1}$-type $R$-matrix:

$$R(z) = (q - q^{-1}z) \sum_{i=1}^{m} E_{ii} \otimes E_{ii} + (1 - z) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} (E_{ij} \otimes E_{ji} + z E_{ji} \otimes E_{ij}).$$

Local $L$-matrices: for $k = 1, \ldots, n$,

$$L_k(z) = \sum_{i=1}^{m} a_{ik} E_{ii} + \sum_{i=1}^{m-1} b_{ik} E_{i,i+1} + z b_{mk} E_{m1}.$$

$$L_k(z)^1 := L_k(z) \otimes 1, \quad L_k(z)^2 := 1 \otimes L_k(z).$$

Fundamental relations:

$$R(z/w)L_k(z)^1 L_k(w)^2 = L_k(w)^2 L_k(z)^1 R(z/w),$$
$$L_k(z)^1 L_l(w)^2 = L_l(w)^2 L_k(z)^1 \quad (k \neq l).$$

Equivalent to the $q$-commutation relations:

$$a_{ik}b_{ik} = q^{-1}b_{ik}a_{ik}, \quad a_{ik}b_{i+1,k} = qb_{i+1,k}a_{ik},$$
$$a_{ik}a_{jk} = a_{jk}a_{ik}, \quad b_{ik}b_{jk} = b_{jk}b_{ik}, \quad \text{etc.}$$

If $k \neq l$, then $a_{ik}$ and $b_{ik}$ commute with $a_{jl}$ and $b_{jl}$.

Another form of the bidiagonal matrix $L_k(z)$.

$$a_k := \mathrm{diag}(a_{1k}, \dots, a_{mk}),$$
$$b_k := \mathrm{diag}(b_{1k}, \dots, b_{mk}),$$
$$\Lambda(z) := \sum_{i=1}^{m} E_{i,i+1} + zE_{m1} \quad \text{(shift matrix)}.$$

Then

$$L_k(z) = a_k + b_k\Lambda(z).$$

# Lax form: $\widehat{L}(z)$

1. $L(z) := L_1(z) \cdots L_n(z)$, the global $L$-operator.

$\tilde{a}_i := a_{i1} \cdots a_{in}$.

$\tilde{a} := \mathrm{diag}(\tilde{a}_1, \ldots, \tilde{a}_m)$, the diagonal part of $L(z)$.

2. $\widetilde{L}(z) := L(z)\tilde{a}$, doubling the diagonal part of $L(z)$.

3. $\widehat{L}(z) := \widetilde{C}\widetilde{L}(z)\widetilde{C}^{-1}$.

Here $\widetilde{C} = \mathrm{diag}(\tilde{c}_1, \ldots, \tilde{c}_m)$ is uniquely characterized by

$$\tilde{c}_1 = 1,$$
$$\widehat{L}(z) = \hat{\ell}_0 + \hat{\ell}_1 \Lambda(z) + \cdots + \Lambda(z)\Lambda(rz) \cdots \Lambda(r^{n-1}z),$$

where $\hat{\ell}_0, \hat{\ell}_1, \ldots$ are diagonal matrices.

$$\widehat{L}(z) = \hat{\ell}_0 + \hat{\ell}_1 \Lambda(z) + \cdots + \Lambda(z)\Lambda(rz)\cdots\Lambda(r^{n-1}z).$$

$t = \mathrm{diag}(t_1, \ldots, t_n) := \tilde{c}\tilde{a}\tilde{c}^{-1}$.

Then $\hat{\ell}_0 = t^2$ and $t_i \hat{\ell}_j = \hat{\ell}_j t_i$.

Define $\hat{b}_i$ and $\hat{f}_i$ by

$$\hat{\ell}_1 = \mathrm{diag}(\hat{b}_i)_{i=1}^n = \mathrm{diag}\left((q^{-1} - q) t_i t_{i+1} \hat{f}_i\right)_{i=1}^n.$$

$r\widehat{L}(z) = \widehat{L}(z) r$.

Assume $t_{i+n} = r^{-1} t_i$ and $\hat{f}_{i+n} = r \hat{f}_i$.

## Example ($q\mathrm{P}_{\mathrm{IV}}$ case): $(m,n)=(3,2)$.

$$\widehat{L}(z)=\begin{bmatrix} t_1^2 & (q^{-1}-q)t_1t_2\hat{f}_1 & 1 \\ rz & t_2^2 & (q^{-1}-q)t_2t_3\hat{f}_2 \\ rz(q^{-1}-q)t_3t_4\hat{f}_3 & z & t_3^2 \end{bmatrix}.$$

Assume $\widetilde{L}(z)=A+B\Lambda(z)+C\Lambda(z)^2$, $A,B,C$ are diagonal, and $C=\mathbf{diag}(c_1,c_2,c_3)$. Then

$$c_i=b_{i1}b_{i+1,2}a_{i+2,1}a_{i+2,2},$$
$$\tilde{c}_1=1,\quad \tilde{c}_3=c_1,\quad \tilde{c}_2=c_1c_3,\quad r=c_1c_3c_2.$$

$\widetilde{C}=\mathbf{diag}(\tilde{c}_1,\tilde{c}_2,\tilde{c}_3)$, $\widehat{L}(z)=\widetilde{C}\widetilde{L}(z)\widetilde{C}^{-1}$, $r\widehat{L}(z)=\widehat{L}(z)r$.

# Lax form: $\widehat{M}(z)$

$T_{z,r} := r^{z\partial/\partial z} : f(z) \mapsto f(rz)$, $r$-difference operator.

4. $\widehat{M}(z) := \widehat{L}(z) T^n_{z,r}$, matrix coefficient $r$-difference op.

5. Assume $t_i = q^{-\varepsilon_i^\vee}$ and $r = q^{-\delta^\vee}$.

Then $(q - q^{-1})[\alpha_i^\vee]_q = t_{i+1}/t_i - t_i/t_{i+1}$

6. $g_i := (t_i^2 - t_{i+1}^2)/\hat{b}_i = [\alpha_i^\vee]_q/\hat{f}_i$.

$G_i := g_i E_{i+1,\, i} \quad (i = 1, \ldots, n-1)$.

$(G_n(z) := r z^{-1} g_n E_{1n}.)$

# Lax form: Weyl group action

Consider the algebra generated by the matrix elements of $\widehat{L}(z)$ (precisely of $\hat{\ell}_0, \dots, \hat{\ell}_{n-1}$).

7. Algebra automorphism Weyl group action:

$$s_i(\widehat{M}(z)) := G_i \widehat{M}(z) G_i^{-1},$$

$$\begin{aligned} \pi(\widehat{M}(z)) &:= (\Lambda(z) T_{z,r}) \widehat{M}(z) (\Lambda(z) T_{z,r})^{-1} \\ &= \Lambda(z) \widehat{L}(rz) \Lambda(r^n z) T_{z,r}^n. \end{aligned}$$

Then

$$s_i(t_i) = t_{i+1}, \quad s_i(t_{i+1}) = t_i,$$

$$s_i(\hat{b}_i) = \hat{b}_i, \quad s_i(\hat{b}_{i\pm1}) = \hat{b}_{i\pm1} \pm (t_i^2 - t_{i+1}^2)/\hat{b}_i.$$

# Sato-Wilson forms: $z$-variables

8. Introduce $\tau_0$ and $z_i$ by
   - $\tau_0 = q^{-r\partial/\partial r}$: $\quad \tau_0 r = q^{-1} r \tau_0, \quad \tau_0 t_j = t_j \tau_0.$
   - $z_i = q^{-t_i \partial/\partial t_i}$: $\quad z_i r = r z_i, \quad z_i t_j = q^{-\delta_{ij}} t_j z_i.$
   - $\tau_0$ and $z_i$ commute with $\tau_0$, $z_j$, $\hat{f}_j$.

9. $D_Z := \mathbf{diag}(z_1, \dots, z_n), \quad Z(z) := U(z) D_Z, \quad$ where

$$U(z) = E + \sum_{k=1}^{\infty} u_k \Lambda(z)^k,$$

$u_1, u_2, \dots$ are diagonal matrices,

$$\widehat{M}(z) = U(z) t^2 T_{z,r}^n U(z)^{-1}.$$

Then

$$\widehat{M}(z) = Z(z) (qt)^2 T_{z,r}^n Z(z)^{-1}.$$

# Sato-Wilson form: Weyl group action

10. The Weyl group action can be extended by

$$s_i(U(z)) = G_i U(z) S_i^g, \quad s_i(D_Z) = (S_i^g)^{-1} D_Z S_i,$$

$$s_i(t) = S_i^{-1} t S_i, \quad s_i(Z(z)) = G_i(z) Z(z) S_i,$$

$$\pi(A(z)) = (\Lambda(z) T_{z,r}) A(z) (\Lambda(z) T_{z,r})^{-1},$$
$$(A(z) = U(z), D_Z, t, Z(z))$$

where $\quad g_i = (t_i^2 - t_{i+1}^2)/\hat{b}_i = [\alpha_i^\vee]_q / \hat{f}_i,$

$$S_i^g := g_i^{-1} E_{i,i+1} - g_i E_{i+1,i} + \sum_{j \neq i,i+1} E_{jj},$$

$$S_i := [\alpha_i^\vee + 1]_q E_{i,i+1} - [\alpha_i^\vee - 1]_q^{-1} E_{i+1,i} + \sum_{j \neq i,i+1} E_{jj}.$$

$S^g$ and $S_i$ are permutation matrices $i \leftrightarrow i+1$.

11. Assume $z_{j+m} = z_j$, $\tau_j = \tau_{j-1} z_i$, and $s_i(\tau_0) = \tau_0$ $(i = 1, 2)$. Then, for $i = 1, 2$,

$$s_i(z_i) = \hat{f}_i z_{i+1}, \quad s_i(z_{i+1}) = \hat{f}_i^{-1} z_i, \quad s_i(\tau_i) = \hat{f}_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i},$$

$$\pi(z_i) = z_{i+1}, \quad \pi(\tau_i) = \tau_{i+1}.$$

Because $g_i = [\alpha_i^\vee]_q / \hat{f}_i$ and $z_i \varepsilon_j^\vee = (\varepsilon_j^\vee + \delta_{ij}) z_i$ implies

$$\begin{bmatrix} 0 & g_i \\ -g_i^{-1} & 0 \end{bmatrix}^{-1} \begin{bmatrix} z_i & 0 \\ 0 & z_{i+1} \end{bmatrix} \begin{bmatrix} 0 & [\alpha_i^\vee + 1]_q \\ -[\alpha_i^\vee - 1]_q^{-1} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} g_i^{-1} z_{i+1} [\alpha_i^\vee + 1]_q & 0 \\ 0 & g_i z_i [\alpha_i^\vee - 1]_q^{-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_i z_{i+1} & 0 \\ 0 & \hat{f}_i^{-1} z_i \end{bmatrix}. \quad \square$$

# Quantum $q\mathrm{P}_{\mathrm{IV}}$

Both canonically quantized and $q$-difference.

# $(m, n)$-case: $X_k(z)$

Assume that $m$ and $n$ are mutually prime (gcm = 1).

There exist unique diagonal matrices $\widetilde{C}_1, \dots, \widetilde{C}_1$ such that $\widetilde{C}_1 = \widetilde{C}_{n+1} = \widetilde{C}$ and, for $k = 1, \dots, n-1$,

$$X_k(r^{k-1}z) := \widetilde{C}_k L_k(z) \widetilde{C}_{k+1}^{-1} = x_k + \Lambda(r^{k-1}z),$$

$$X_n(r^{n-1}z) := \widetilde{C}_n L_n(z) \tilde{a} \widetilde{C}_1^{-1} = x_n + \Lambda(r^{n-1}z),$$

$$x_k = \operatorname{diag}(x_{1k}, \dots, x_{m,k}).$$

Then

$$\widehat{L}(z) = X_1(z) X_2(rz) \cdots X_n(r^{n-1}z).$$

Assume $x_{i+m,k} = r^{-1} x_{ik}$ and $x_{i,k+n} = x_{ik}$.

# $(m,n)$-case: $q$-commutation relations of $x_{ik}$

**Theorem.** $x_{ik}x_{jl} = q^{(m,n)}_{j-i,l-k} x_{jl}x_{ik}$, $\quad q^{(m,n)}_{\mu\nu} \in \{1, q^{\pm 2}\}$. $\square$

**Example.** If $(m,n) = (2g+1, 2)$ and $x_i := x_{i1}$, $y_i := x_{i2}$, then

$$x_i y_i = y_i x_i = t_i^2,$$

$$x_i x_{i+\mu} = q^{(-1)^{\mu-1}2} x_{i+\mu} x_i, \qquad x_i y_{i+\mu} = q^{-(-1)^{\mu-1}2} y_{i+\mu} x_i,$$

$$y_i y_{i+\mu} = q^{(-1)^{\mu-1}2} y_{i+\mu} y_i, \qquad y_i x_{i+\mu} = q^{-(-1)^{\mu-1}2} x_{i+\mu} y_i,$$

$t_i$ commutes with $t_j$, $x_j$, $y_j$. $\square$

**Example.** $(m,n) = (5,3), (3,5)$. $x_{ik}x_{jl} = q^{(m,n)}_{j-i,l-k} x_{jl}x_{ik}$, where $q_{\mu+m,\nu} = q_{\mu\nu}$, $q_{\mu,\nu+n} = q_{\mu\nu}$, and

$$\left[q^{(5,3)}_{\mu\nu}\right] = \begin{bmatrix} 1 & 1 & q^{-2} & q^2 & 1 \\ q^{-2} & q^2 & 1 & q^{-2} & q^2 \\ q^2 & q^{-2} & q^2 & 1 & q^{-2} \end{bmatrix},$$

$$\left[q^{(3,5)}_{\mu\nu}\right] = \begin{bmatrix} 1 & q^{-2} & q^2 \\ 1 & q^2 & q^{-2} \\ q^{-2} & 1 & q^2 \\ q^2 & q^{-2} & 1 \\ 1 & q^2 & q^{-2} \end{bmatrix}.$$

Note that $q^{(5,3)}_{\mu\nu} = q^{(3,5)}_{\nu\mu}$. $\square$

Assume that $0 < \widetilde{m} < n$ and $m\widetilde{m} \equiv 1 \pmod{n}$.
Assume that $0 < \tilde{n} < n$ and $n\tilde{n} \equiv 1 \pmod{m}$.

**Theorem.** Define $B^{(m,n)}$ and $p_{\mu\nu}^{(m,n)}$ by

$$B^{(m,n)} := \{\, (\mu \bmod m,\ \mu \bmod n) \mid 0 \leqq \mu < \widetilde{m}m \,\}.$$

$$p_{\mu\nu}^{(m,n)} := \begin{cases} q & \text{if } (\mu \bmod m,\ \nu \bmod n) \in B, \\ 1 & \text{if } (\mu \bmod m,\ \nu \bmod n) \notin B. \end{cases}$$

Then

$$q_{\mu\nu}^{(m,n)} = (p_{\mu\nu}/p_{\mu-1,\nu})^2 \in \{1, q^{\pm 2}\}. \qquad \square$$

**Cor. (duality)** $q_{\mu\nu}^{(m,n)} = q_{\nu\mu}^{(n,m)}$. $\square$

# $(m, n)$-case: Weyl group action

The action of $\widetilde{W}(A^{(1)}_{m-1}$ on $t_i$, $\hat{f}_i$, $\tau_i$, etc. can be extended to the one on $x_{ik}$.

Using the duality above, we can construct the action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ on $x_{ik}$.

We shall write $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ as

$$\widetilde{W}(A^{(1)}_{m-1}) = \langle \pi, s_0, \dots, s_{m-1} \rangle,$$
$$\widetilde{W}(A^{(1)}_{n-1}) = \langle \varpi, r_0, \dots, r_{n-1} \rangle.$$

# $q\mathrm{P}_{\mathrm{IV}}$-case: Weyl group action

**Example ($q\mathrm{P}_{\mathrm{IV}}$-case)** $(m,n) = (3,2)$. $\quad x_i := x_{i1}, \quad y_i := x_{i2}.$

$$s_i(x_i) = (x_i + y_{i+1})x_{i+1}(y_i + x_{i+1})^{-1},$$

$$s_i(x_{i+1}) = (x_i + y_{i+1})^{-1}x_i(y_i + x_{i+1}),$$

$$s_i(y_i) = (y_i + x_{i+1})y_{i+1}(x_i + y_{i+1})^{-1},$$

$$s_i(y_{i+1}) = (y_i + x_{i+1})^{-1}y_i(x_i + y_{i+1}),$$

$$s_i(x_{i+2}) = x_{i+2}, \quad s_i(y_{i+2}) = y_{i+2},$$

$$s_i(t_i) = t_{i+1}, \quad s_i(t_{i+1}) = t_i, \quad s_i(t_{j+2}) = t_{j+2},$$

$$\pi(x_i) = x_{i+1}, \quad \pi(y_i) = y_{i+1}, \quad \pi(t_i) = t_{i+1}.$$

$$Q_i := y_{i+2}y_{i+1} + y_{i+2}x_i + x_{i+1}x_i,$$

$$r_1(x_i) = r^{-1}Q_{i+1}^{-1}y_iQ_i,$$

$$r_1(y_i) = rQ_{i+1}x_iQ_i^{-1},$$

$$r_1(t_i) = t_i, \quad \varpi(x_i) = y_i, \quad \varpi(y_i) = x_i, \quad \varpi(t_i) = t_i.$$

# $q\mathrm{P}_{\mathrm{IV}}$-case: Lax form

$G'_i := \varpi(G_i)$.

Then

$$
\begin{aligned}
s_i(X(z)) &= G_i X(z) G_i'^{-1},\\
s_i(Y(z)) &= G'_i Y(z) G_i^{-1},\\
\pi(X(z)) &= \Lambda(z) X(z) \Lambda(z)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
r_1(X(z)Y(rz)) &= X(z)Y(rz),\\
r_1 &: x_{i+2}x_{i+1}x_i \leftrightarrow y_{i+3}y_{i+2}y_{i+1},\\
\varpi &: X(z) \leftrightarrow Y(z).
\end{aligned}
$$

These relations uniquely characterizes the actions.

# $q\mathrm{P_{IV}}$-case: discrete time evolution

$U_1 := r_1\varpi \in \widetilde{W}(A_1^{(1)})$.

The $U_1$-action is the discrete time evolution of $q\mathrm{P_{IV}}$ and the $\widetilde{W}(A_2^{(1)})$-action is its symmetry.

$a_i := t_i/t_{i+1}$ and $F_i := x_{i+1}x_i/(t_{i+1}t_i)$.

Then
$F_iF_{i+1} = q^2F_{i+1}F_i$,
$a_ia_j = a_ja_i$, $a_iF_j = F_ja_i$,
$F_{i+3} = F_i$, $a_{i+3} = a_i$.

# Time evolution of quantized $q\mathrm{P}_{\mathrm{IV}}$.

$$
\begin{aligned}
U_1(F_i) = (1 &+ q^2 a_{i-1} F_{i-1} + q^2 a_{i-1} a_i F_{i-1} F_i) \\
&\times a_i a_{i+1} F_{i+1} \\
&\times (1 + q^2 a_i F_i + q^2 a_i a_{i+1} F_i F_{i+1})^{-1}, \\
U_1(a_i) &= a_i.
\end{aligned}
$$

## Classical case:

Kajiwara-Noumi-Yamada `arXiv:nlin/0012063`

$$
\begin{aligned}
\overline{F_i} &= a_i a_{i+1} F_{i+1} \frac{1 + a_{i-1} F_{i-1} + a_{i-1} a_i F_{i-1} F_i}{1 + a_i F_i + a_i a_{i+1} F_i F_{i+1}}, \\
\overline{a_i} &= a_i.
\end{aligned}
$$

# $q\mathrm{P}_{\mathrm{IV}}$-case: symmetry

**Symmetry of quantum $q\mathrm{P}_{\mathrm{IV}}$.**

$$
\begin{aligned}
&s_i(F_i) = F_i,\\
&s_i(F_{i-1}) = F_{i-1}\frac{a_i + F_i}{1 + a_i F_i}, \quad s_i(F_{i+1}) = \frac{1 + a_i F_i}{a_i + F_i} F_{i+1},\\
&s_i(a_i) = a_i^{-1}, \quad s_i(a_{i\pm 1}) = a_i a_{i\pm 1}.
\end{aligned}
$$

These formulas coincide with the ones obtained by Koji Hasegawa `arXiv:0703036`, which quantizes Kajiwara-Noumi-Yamada `arXiv:nlin/0012063`.

# Summary

- For any symmetrizable GCM, we can introduce the proper non-commutativity of quantum $\tau$-functions. $\tau_i = \exp(\partial/\partial\alpha_i^\vee)$.
- Quantum $q$-Hirota-Miwa equations for $A^{(1)}_{n-1}$-case.
- Quantized Lax and Sato-Wilson forms of the extended affine Weyl group action for $A^{(1)}_{n-1}$-case.
- Quantized $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A(1)_{n-1})$-action for mutually prime $m$ and $n$.
- An appropriate quantization of $q\mathrm{P}_{\mathrm{IV}}$.