Quantum geometric crystals

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Abstract

We investigate quantization of geometric crystals.

Contents

0 Introduction

In this paper, rings and fields are possibly non-commutative. We denote the set of non-negative (resp. non-positive) integers by $\mathbb{Z}_{\geqq 0}$ (resp. $\mathbb{Z}_{\leqq 0}$). Let $\mathbb{F}$ be the commutative rational function field $\mathbb{Q}(q)$ over $\mathbb{Q}$. We shall deal with $\mathbb{F} = \mathbb{Q}(q)$ as a base field. A (possibly non-commutative) field $\mathcal{K}$ is called a field over $\mathbb{F}$ if $\mathcal{K}$ is an algebra over $\mathbb{F}$, that is, the center of $\mathcal{K}$ includes $\mathbb{F}$ as a subring.

1 Notation and definitions

1.1 Cartan datum and root datum

Let $[a_{ij}]_{i,j\in I}$ be a symmetrizable generalized Cartan matrix (GCM) with an index set I ([6]). That is, we assume that (1) $a_{ii} = 2$, (2) a_{ij} is a non-positive integer if $i \neq j$, (3) $a_{ij} = 0$ if and only if $a_{ji} = 0$, (4) there exists positive integers d_i $(i \in I)$ such that $d_i a_{ij} = d_j a_{ji}$ for $i, j \in I$. Then $C = ([a_{ij}]_{i,j\in I}, \{d_i\}_{i\in I})$ is called a *Cartan datum*. For any subset J of I, we define the Cartan datum C_J by $C_J = ([a_{ij}]_{i,j\in J}, \{d_i\}_{i\in J})$ and call it the restriction of C to J.

A symmetrizable GCM $[a_{ij}]_{i,j\in I}$ with a finite index set I is of finite (resp. affine) type if and only if the symmetric matrix $[d_i a_{ij}]_{i,j\in I}$ is positive definite (resp. semi-possitive definite of rank $|I| - 1$, where $|I|$ denotes the cardinality of I).

Let $Q^\vee$, P be finitely generated free $\mathbb{Z}$-modules and $\langle\,,\,\rangle : Q^\vee \times P \to \mathbb{Z}$ a perfect bilinear pairing. Assume that subsets $\{h_i\}_{i\in I}$ of $Q^\vee$ and $\{\alpha_i\}_{i\in I}$ of P satisfy $\langle h_i, \alpha_j\rangle = a_{ij}$ for $i, j \in I$. Then h_i is called a simple coroot, α_i a simple root, $Q^\vee$ a coroot lattice, and P a weight lattice. We call $R = (\langle\,,\,\rangle : Q^\vee \times P \to \mathbb{Z}, \{h_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ a *root datum* of type $C = ([a_{ij}]_{i,j\in I}, \{d_i\}_{i\in I})$ (see [7] Section 2.2).

The root lattice Q is defined to be the free $\mathbb{Z}$-module generated by all simple roots: $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$. Put $Q^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geqq 0}\alpha_i$. If $\{\alpha_i\}_{i\in I}$ is linearly independent over $\mathbb{Z}$ in P, then Q is naturally identified with the $\mathbb{Z}$-submodule of P generated by $\{\alpha_i\}_{i\in I}$. Let $(\,|\,) : Q \times Q \to \mathbb{Z}$ be the symmetric bilinear form given by $(\alpha_i|\alpha_j) = d_i a_{ij}$.

For any subset J of I, we put

$$Q_J^\vee = \sum_{i\in J} \mathbb{Z}h_i, \quad P_J = \mathrm{Hom}(Q_J^\vee, \mathbb{Z}), \quad \langle\,,\,\rangle : Q^\vee \times P_J \to \mathbb{Z} \text{ (natural pairing)},$$
$$h_{J,i} = h_i, \quad \alpha_{J,i} = (\text{the image of } \alpha_i \text{ in } P) \quad \text{for } i \in J, \quad A_J = [a_{ij}]_{i,j\in J}.$$

Then $R_J = (\langle\,,\,\rangle : Q_J^\vee \times P_J \to \mathbb{Z}, \{h_{J,i}\}_{i\in J}, \{\alpha_{J,i}^\vee\}_{i\in J})$ is a root datum of type $C_J = ([a_{ij}]_{i,j\in J}, \{d_i\}_{i\in J})$ and called the restriction of R to J. Put $Q_J = \bigoplus_{i\in J} \mathbb{Z}\alpha_i$ and $Q_J^+ \bigoplus_{i\in J} \mathbb{Z}_{\geqq 0}\alpha_i$.

1.2 Group algebra of the weight lattice

Denote by $\mathbb{F}[q^P]$ the group algebra of the weight lattice P given by

$$\mathbb{F}[q^P] = \bigoplus_{\lambda\in P} \mathbb{F}q^\lambda, \quad q^\lambda q^\mu = q^{\lambda+\mu} \quad (\lambda, \mu \in P).$$

Put $q_i = q^{d_i}$ and define $\alpha_i^\vee \in P_\mathbb{Q} = P \otimes \mathbb{Q}$ by $\alpha_i^\vee = d_i^{-1}\alpha_i$ which is also called a simple coroot. We define q^P to be the subset of $\mathbb{F}[q^P]$ consisting of all q^λ for $\lambda \in P$. For any subset J of I, we define $\mathbb{F}[q^{P_J}]$ and q^{P_J} by the same way.

For $v = q^n$ with $n \in \mathbb{Z}$ and a non-negative integer k, we put

$$[x]_v = \frac{v^x - v^{-x}}{v - v^{-1}}, \quad [k]_v! = [1]_v[2]_v \cdots [v]_v,$$
$$\begin{bmatrix} x \\ k \end{bmatrix}_v = \frac{[x]_v[x-1]_v \cdots [x-k+1]_v}{[k]_v!}.$$

Since $q_i^{\pm\alpha_i^\vee} = q^{\pm\alpha_i} \in \mathbb{F}[q^P]$, we have $\begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} \in \mathbb{F}[q^P]$ for $k \in \mathbb{Z}_{\geqq 0}$.

1.3 Weyl group

Let W be the group defined by generators $\{s_i\}_{i\in I}$ and the definig relations (1) $s_i^2 = 1$, (2) if $a_{ij}a_{ji} = 0, 1, 2, 3$, then $s_is_js_i \cdots = s_js_is_j \cdots$ where the both sides have 2,3,4,6 factors respectively. We call W the Weyl group of type $[a_{ij}]_{i,j\in I}$. Then $(W, \{s_i\}_{i\in I})$ is a Coxeter group.

The *length* $\ell(w)$ of $w \in W$ is defined to be the minumum of the non-negative ingtegers m such that there exists a word $(j_1, \dots, j_m) \in I^m$ with $w = s_{j_1}s_{j_2}\cdots s_{j_m}$. A word $\mathbf{i} = (i_1, i_2, \dots, i_N) \in I^N$ of length N is called a *reduced word* for w if $N = \ell(w)$ and $w = s_{i_1}s_{i_2}\cdots s_{i_N}$. Then $w = s_{i_1}s_{i_2}\cdots s_{i_N}$ is called a *reduced expression* of w. Denote by $R(w)$ the set of all reduced words for $w \in W$.

The Weyl group W acts on the weight lattice P by $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle\alpha_i$ for $i \in I$ and $\lambda \in P$. This is naturally extended to the action of W on the group algebra $\mathbb{F}[q^P]$ by $w(q^\lambda) = q^{w(\lambda)}$ for $w \in W$ and $\lambda \in P$.

Put $a_i = q^{\alpha_i} = q_i^{\alpha_i^\vee}$ for $i \in I$. Then we have $s_i(a_j) = a_i^{-a_{ij}}a_j$.

For any subset J of I, the Weyl group W_J of type $[a_{ij}]_{i,j\in J}$ is identified with the subgroup of W generated by $\{s_i\}_{i\in J}$.

1.4 Ore domain and field of fractions

A (possibly non-commutative) ring A is called an integral domain or a domain if $ab \neq 0$ for any non-zero $a, b \in A$. A ring A is called an Ore domain if A is an integral domain and $Aa \cap Ab \neq 0$, $aA \cap bA \neq 0$ for every non-zero $a, b \in A$.

A ring A is an Ore domain if and only if there exists a field K such that K includes A as a subring and $K = \{\, as^{-1} \mid a, s \in A,\ s \neq 0\,\} = \{\, s^{-1}a \mid a, s \in A,\ s \neq 0\,\}$. Such a field K has the following universal mapping property: if L is a ring, $f : A \to L$ is a ring homomorphism, and $f(s)$ is invertible in L for any non-zero $s \in A$, then there exists a unique ring homomorphism $\phi : K \to L$ such that $\phi(a) = f(a)$ for all $a \in A$. Therefore K is uniquely determined by A up to canonical isomorphisms. We shall denote K by $Q(A)$ and call it the *quotient field* or the *field of fractions* of A.

Let A be an Ore domain, $a, b, c, d \in A$, and $b, d \neq 0$. Then there exists non-zero $b', d' \in A$ with $bd' = db'$. We call $bd' = db'$ a common (left) denominator of ab^{-1} and cd^{-1}. Moreover $ab^{-1} = cd^{-1}$ in $Q(A)$ if and only if there exist non-zero $b', d' \in A$ such that $bd' = db'$ and $ad' = cb'$.

Let A be a (possibly non-commutative) ring, k a (possibly non-commutative) subfield of A, and $\{F_nA\}_{n=0}^\infty$ a family of left (resp. right) k-subspaces of A. We call $\{F_nA\}_{n=0}^\infty$ a left (resp. right) k-filtration of A if $1 \in F_0A$, $F_nA \subset F_{n+1}A$, $F_mAF_nA \subset F_{m+n}A$ for $m, n = 0, 1, 2, \dots$, and $\bigcup_{n=0}^\infty F_nA = A$. A left or right k-filtration $\{F_nA\}_{n=0}^\infty$ of A is said to be *slowly increasing* if F_nA is finite dimensional over k for any n and the convergence radius of the power series $\sum_{n=0}^\infty(\dim_k F_nA)z^n$ is greater than or equal to 1.

Lemma 1.1. *Let A be an integral domain and k a subfield of A. If there exist both a slowly increasing left k-filtration and a slowly increasing right k-filtration of A, then A is an Ore domain.*

Proof. Let $\{F_nA\}_{n=1}^{\infty}$ be a left k-filtration of A. Assume that there exist non-zero $a, b \in A$ such that $Aa \cap Ab = 0$. Since $\bigcup_{n=0}^{\infty} F_nA = A$, there exists N with $a, b \in F_NA$. Since $1 \in F_0A$, $F_nAF_NA \subset F_{n+N}A$, and $Aa \cap Ab = 0$, we have

$$\dim_k F_0A \geqq 1, \quad \dim_k F_{n+N}A \geqq \dim_k((F_nA)a + (F_nA)b) = 2\dim_k F_nA.$$

Therefore $\dim_k F_{mN}A \geqq 2^m$ for $m \in \mathbb{Z}_{\geq 0}$. Since the convergence radius of $\sum_{m=0}^{\infty} 2^m z^{mN}$ is less than 1, that of $\sum_{n=0}^{\infty}(\dim_k F_nA)z^n$ is also less than 1. This means that $\{F_nA\}_{n=1}^{\infty}$ is not slowly increasing. Therefore if there exists a slowly increasing left k-filtration of A, then $Aa \cap Ab \neq 0$ for any non-zero $a, b \in A$. Similarly if there exists a slowly increasing right k-filtration of A, then $aA \cap aA \neq 0$ for any non-zero $a, b \in A$. □

As applications of this lemma, we obtain the following examples of Ore domains.

The polynomial ring $K[t_1, \dots, t_N]$ over a (possibly non-commutative) field K is an Ore domain. Denote the field of fractions $Q(K[t_1, \dots, t_N])$ by $K(t_1, \dots, t_N)$. Elements of $K(t_1, \dots, t_N)$ are called rational functions of $t_1, \dots, t_N$ over K.

Let $[c_{\mu\nu}]_{\mu,\nu=1}^{N}$ be a skew-symmetric integer matrix and $\mathcal{A}$ the associative algebra over $\mathbb{F}$ given by generators $\{x_\nu\}_{\nu=1}^{N}$ and the defining relations $x_\nu x_\mu = q^{c_{\mu\nu}} x_\mu x_\nu$ $(1 \leqq \mu, \nu \leqq N)$. Then $\mathcal{A}$ is an Ore domain and the field of fractions $Q(\mathcal{A})$ is called a *rational function field of quantum torus* or simply a *quantum torus*.

A quantum enveloping algebra U_q of finite or affine type is an Ore domain and any quotient integral domain of any subalgebra of U_q is also an Ore domain ([5]).

1.5 Substitution

Let K be a (possibly non-commutative) field, $c_1, \dots, c_N$ central elements of K, and $t_1, \dots, t_N$ indeterminates. Put $t = (t_1, \dots, t_N)$ and $c = (c_1, \dots, c_N)$ and denote $K(t_1, \dots, t_N)$ by $K(t)$ and $K[t_1, \dots, t_n]$ by $K[t]$.

A rational fucntion $f(t) \in K(t)$ is said to be regular at $t = c$ if there exist polynomials $g(t), h(t) \in K[t]$ such that $h(c) \neq 0$ and $f(t) = g(t)h(t)^{-1}$. Then $g(c)h(c)^{-1}$ does not depend on the choice of $g(t)$ and $h(t)$. Therefore $f(c) \in K$ is well-defined. Denote by $K[t]_c$ the subset of $K(t)$ consisting of all rational functions regular at $t = c$. Then $K[t]_c$ is a subring of $K(t)$. Thus we obtain the substitution ring homomorphism $K[t]_c \to K$, $f(t) \mapsto f(c)$. For any $f(t) \in K[t]_c$, $f(t)$ is invertible in $K[t]_c$ if and only if $f(c) \neq 0$. We call $K[t]_c$ the local ring at $t = c$.

Let $t'_1, \dots, t'_N$ be indeterminates and put $t' = (t'_1, \dots, t'_N)$. Then any rational function in $K(t, t')$ regular at $(t, t') = (c, c)$ is regular at $t' = t$. That is, $K[t, t']_{(c,c)} \subset K(t)[t']_t$. Therefore we have the ring homomorphism $K[t, t']_{(c,c)} \to K(t)$, $f(t, t') \mapsto f(t, t)$.

Let $\phi^t : K \to K(t)$ be a ring homomorphism.

If $\phi^t(K) \subset K[t]_c$, then we call ϕ^t regulat at $t = c$ and can define the ring homomorphism $\phi^c : K \to K$ to be the composition of $\phi^t : K \to K[t]_c$ and the substitution ring homomorphism at $t = c$.

For subsets $C_1, \dots, C_N$ of the center of K, we define the substitution ring homomorphism at $C = C_1 \times \cdots \times C_N$ to be the mapping $\bigcap_{c \in C} K[t]_c \to K^C$, $f(t) \mapsto (f(c))_{c \in C}$. If $C_1, \dots, C_N$ are all infinite, then the substitution ring homomorphism at C is injective.

Let $C_1, \ldots, C_N$ be infinite subsets of the center of K and put $C = C_1 \times \cdots \times C_N$. Assume that ϕ^t is regulat at $t = c$ for any $c \in C$. Extend ϕ^t to the ring homomorphism $K[t] \to K(t)$ by $\phi^t(t) = t$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} K[t] & \xrightarrow{\phi^t} & \bigcap_{c\in C} K[t]_c & \xrightarrow{\text{inclusion}} & K(t), \\ \downarrow & & \downarrow & & \\ K^C & \xrightarrow{\prod_{c\in C}\phi^c} & K^C & & \end{array}$$

where the vertical arrows are the evalutation ring homomorphisms at C and hence injective. Since $\prod_{c\in C^n} \phi^c$ is injective, $\phi^t : K[t] \to K(t)$ is also injective. Therefore $\phi^t : K[t] \to K(t)$ is uniquely extended to the ring homomorphism $K(t) \to K(t)$, which shall be also denoted by ϕ^t.

For each $\nu = 1, \ldots, N$, let $\phi_\nu^{t_\nu} : K \to K(t_\nu)$ be a ring homomorphism and extend it to the ring homomorphism $K(t_{\nu+1}, \ldots, t_N) \to K(t_\nu, t_{\nu+1}, \ldots, t_N)$ by $\phi_\nu^{t_\nu}(t_\mu) = t_\mu$ for $\mu > \nu$. Denote by $\phi^{t_1,\ldots,t_N}$ the composition ring homomorphism $\phi_1^{t_1} \cdots \phi_N^{t_N} : K \to K(t) = K(t_1, \ldots, t_N)$. Let $(c_1, \ldots, c_N) \in C_1 \times \cdots \times C_N$. Assume that $\phi_\nu^{t_\nu}$ is regular at $t = c'_\nu$ for any $c'_\nu \in C_\nu$ and $\phi_\nu^{c_\nu}(C_\mu) \subset C_\mu$ for $\mu > \nu$. Put $c'_1 = c_1$, $c'_2 = \phi_1^{c_1}(c_2)$, $\ldots$, $c'_N = \phi_1^{c_1}(\phi_2^{c_2}(\cdots(\phi_{N-1}^{c_{N-1}}(c_N))\cdots))$. Then $\phi^{t_1,\ldots,t_N}(K)$ is regulat at $(t_1, \ldots, t_N) = (c'_1, \ldots, c'_N)$ and $\phi_1^{c_1} \cdots \phi_N^{c_N} = \phi^{c'_1,\ldots,c'_N}$.

Let $[k_{\mu\nu}]_{\mu,\nu=1}^N$ be an integer matrix and $n_1, \ldots, n_N$ integers. Put $t'_\nu = t_1^{k_{1\nu}} \cdots t_N^{k_{N\nu}}$ and $t' = (t'_1, \ldots, t'_N)$. Assume that $\phi_\nu^{t_\nu}$ is regulat at $t_\nu = 1$ for $\nu = 1, \ldots, N$. Then $\phi^{t_1,\ldots,t_N}(K) \subset K[t_1, \ldots, t_N]_{(1,\ldots,1)}$. Therefore the ring homomorphism $\phi^{t'_1,\ldots,t'_N} : K \to K(t_1, \ldots, t_N)$ is well-defined and shall be denoted by $\phi_1^{t'_1} \cdots \phi_N^{t'_N}$.

2 Quantum geometric semicrystals and crystals

2.1 Definition of quantum geometric semicrystals

Let $C = ([a_{ij}]_{i,j\in I}, \{d_i\}_{i\in I})$ be a Cartan datum.

Let $\mathcal{K}$ be a (possibly non-commutative) field and $\mathbf{e}_i^t : \mathcal{K} \to \mathcal{K}(t)$ a ring homomorphism for each $i \in I$. Assume the following conditions:

(1) For any $i \in I$, $\mathbf{e}_i^t$ is regular at $t = 1$, $\mathbf{e}_i^1 = \mathrm{id}_{\mathcal{K}}$, and $\mathbf{e}_i^{t_1}\mathbf{e}_i^{t_2} = \mathbf{e}_i^{t_1t_2}$.

(2) For any $i, j \in I$ with $i \neq j$,

$$\begin{aligned}
(a_{ij}, a_{ji}) = (0, 0) &\implies \mathbf{e}_i^{t_1}\mathbf{e}_j^{t_2} = \mathbf{e}_j^{t_2}\mathbf{e}_i^{t_1}, \\
(a_{ij}, a_{ji}) = (-1, -1) &\implies \mathbf{e}_i^{t_1}\mathbf{e}_j^{t_1t_2}\mathbf{e}_i^{t_2} = \mathbf{e}_j^{t_2}\mathbf{e}_i^{t_1t_2}\mathbf{e}_j^{t_1}, \\
(a_{ij}, a_{ji}) = (-1, -2) &\implies \mathbf{e}_i^{t_1}\mathbf{e}_j^{t_1t_2}\mathbf{e}_i^{t_1t_2^2}\mathbf{e}_j^{t_2} = \mathbf{e}_j^{t_2}\mathbf{e}_i^{t_1t_2^2}\mathbf{e}_j^{t_1t_2}\mathbf{e}_i^{t_1}, \\
(a_{ij}, a_{ji}) = (-1, -3) &\implies \mathbf{e}_i^{t_1}\mathbf{e}_j^{t_1t_2}\mathbf{e}_i^{t_1^2t_2^3}\mathbf{e}_j^{t_1t_2^2}\mathbf{e}_i^{t_1t_2^3}\mathbf{e}_j^{t_2} = \mathbf{e}_j^{t_2}\mathbf{e}_i^{t_1t_2^3}\mathbf{e}_j^{t_1t_2^2}\mathbf{e}_i^{t_1^2t_2^3}\mathbf{e}_j^{t_1t_2}\mathbf{e}_i^{t_1}.
\end{aligned}$$

These relations are called the *Verma relations*.

Then $(\mathcal{K}, \{e_i^t\}_{i\in I})$ is called a *quantum geometric semicrystal* of type C. For any subset J of I, replacing I with J, we define a *quantum geometric J-semicrystal* by the same way.

Let $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in I})$ be a quantum geometric semicrystal.

Let $[k_{\mu\nu}]_{\mu,\nu=1}^N$ be an integer matrix and $i_1, \ldots, i_N \in I$. Put $t'_\nu = t_1^{k_{1\nu}} \cdots t_N^{k_{N\nu}}$. Then the conditoin (1) implies that $\mathbf{e}_{i_1}^{t'_1} \cdots \mathbf{e}_{i_N}^{t'_N} : \mathcal{K} \to \mathcal{K}(t_1, \ldots, t_N)$ is well-defined and can be extended to the ring automorphism of $\mathcal{K}(t_1, \ldots, t_N)$. In particular, $\mathbf{e}_i^t \mathbf{e}_i^{t^{-1}} : \mathcal{K} \to \mathcal{K}(t)$ can be extended to the identity map of $\mathcal{K}(t)$ for $i \in I$.

Let $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in J})$ and $(\mathcal{K}', \{\mathbf{e}_i'^t\}_{i\in J})$ be quantum geometric semicrystals. An algebra homomorphism $f : \mathcal{K} \to \mathcal{K}'$ is a morphism of quantum geometric semicrystals if $\mathbf{e}_i'^t \circ f = f \circ \mathbf{e}_i^t$ for any $i \in I$, where we denote by the same symbol f the extension of f to the algebra homomorphism $\mathcal{K}(t) \to \mathcal{K}'(t)$ characterized by $f(t) = t$. Denote by $\mathcal{SC}(C)$ the category of quantum geometric semicrystals and morphisms of quantum geometric semicrystals associated to the Cartan datum C.

Recall that $C_J = ([a_{ij}]_{i,j\in J}, \{d_i\}_{i\in J})$ is called the restriction of the Cartan datum C to J. We call $\mathcal{SC}(C_J)$ the category of quantum geometric J-semicrystals. We have the forgetful functor from $\mathcal{SC}(C)$ to $\mathcal{SC}(C_J)$ which maps a quantum geometric semicrystal $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in I})$ to the restriction $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in J})$ to J. We also have the trivial extension functor from $\mathcal{SC}(C_J)$ to $\mathcal{SC}(C)$ which maps a quantum geometric J-semicrystal $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in J})$ to the quantum geometric semicrystal $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in I})$ given by $e_i^t(a) = a$ for $i \in I - J$ and $a \in \mathcal{K}$.

2.2 Definition of quantum geometric crystals

Let $C = ([a_{ij}]_{i,j\in I}, \{d_i\}_{i\in I})$ be a Cartan datum, $(\mathcal{K}, \{e_i^t\}_{i\in I})$ a quantum geometric semicrystal in $\mathcal{SC}(C)$, and $R = (\langle\ ,\ \rangle : Q^\vee \times P \to \mathbb{Z},\ \{h_i\}_{i\in I},\ \{\alpha_i^\vee\}_{i\in I})$ a root datum of type C.

Assume that a ring homomorphism $\gamma : \mathbb{F}[q^P] \to \mathcal{K}$ satisfies the following conditions:

(3) $\gamma(\mathbb{F}[q^P])$ is included in the center of $\mathcal{K}$ and $\mathbf{e}_i^t$ is regular at $t = q^\lambda$ for $i \in I$, $\lambda \in P$.

(4) $\mathbf{e}_i^t(\gamma(q^\lambda)) = t^{-\langle h_i,\lambda\rangle}\gamma(q^\lambda)$ for $i \in I$ and $\lambda \in P$.

Then $(\mathcal{K}, \{e_i^t\}_{i\in I}, \gamma)$ is called a *quantum geometric crystal*. For any subset J of I, replacing the root datum R with its restriction $R_J = (\langle\ ,\ \rangle : Q_J^\vee \times P_J \to \mathbb{Z},\ \{h_{J,i}\}_{i\in J},\ \{\alpha_{J,i}^\vee\}_{i\in J})$, we define a *quantum geometric J-crystal* by the same way.

Let $(\mathcal{K}, \{e_i^t\}_{i\in I}, \gamma)$ be a quantum geometric crystal. The conditions (1) and (3) imply that the ring homomorphism $\mathbf{e}_i^{\gamma(q_i^\beta)} = \mathbf{e}_i^{\gamma(q^{d_i\beta})}$ is well-defined and is a ring automorphism of $\mathcal{K}$ for any $\beta \in d_i^{-1}P$. For example, since $\alpha_i^\vee = d_i^{-1}\alpha_i \in d_i^{-1}P$, the ring automorphism $\mathbf{e}_i^{\gamma(q_i^{\alpha_i^\vee})} = \mathbf{e}_i^{\gamma(q^{\alpha_i})}$ of $\mathcal{K}$ is well-defined and

$$\mathbf{e}_i^{\gamma(q_i^{\alpha_i^\vee})}(\gamma(q^\lambda)) = \mathbf{e}_i^{\gamma(q^{\alpha_i})}(\gamma(q^\lambda)) = \gamma(q^{\lambda-\langle h_i,\lambda\rangle\alpha_i}) = \gamma(q^{s_i(\lambda)}) \quad \text{for } \lambda \in P.$$

Let $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in J}, \gamma)$ and $(\mathcal{K}', \{\mathbf{e}_i'^t\}_{i\in J}, \gamma')$ be quantum geometric crystals. An algebra homomorphism $f : \mathcal{K} \to \mathcal{K}'$ is a morphism of quantum geometric crystals if $\mathbf{e}_i'^t \circ f = f \circ \mathbf{e}_i^t$ for any $i \in J$ and $\gamma' = f \circ \gamma$. Denote by $\mathcal{C}(R)$ the category of quantum geometric crystals and morphisms of quantum geometric crystals associated to the root datum R.

Let $(\mathcal{K}, \{e_i^t\}_{i\in I})$ be a quantum geometric semicrystal and assume that $\mathcal{K}$ is a (possibly non-commutative) field over $\mathbb{F}$. Put $\mathcal{K}[q^P] = \mathcal{K} \otimes_{\mathbb{F}} \mathbb{F}[q^P]$ and $\mathcal{K}(q^P) = Q(\mathcal{K}[q^P])$. Extend $\mathbf{e}_i^t$ to the ring homomorphism $\mathcal{K}(q^P) \to \mathcal{K}(q^P)(t)$ by

$$\mathbf{e}_i^t(q^\lambda) = t^{-\langle h_i,\lambda\rangle} q^\lambda \quad \text{for } \lambda \in P.$$

Denote by γ the canonical inclusion $\mathbb{F}[q^P] \to \mathcal{K}(q^P)$. Then $(\mathcal{K}(q^P), \{e_i^t\}_{i\in I}, \gamma)$ is a quantum geometric crystal, which shall be called the associated quantum geometric crystal of $(\mathcal{K}, \{e_i^t\}_{i\in I})$.

Remark 2.1. A quantum geometric crystal $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in I}, \gamma)$ with commutative $\mathcal{K}$ essentially coincides with a geometric crystal in the sense of Berenstein-Kazhdan [2], [3]. □

2.3 Weyl group action on a quantum geometric crystal

The following is an immediate consequence of the definition of quantum geometric crystals.

Lemma 2.2 (Weyl group action). *Let $(\mathcal{K}, \{e_i^t\}_{i\in I}, \gamma)$ be a quantum geometric crystal and put $a_i = q^{\alpha_i}$ for $i \in I$. For each $i \in I$, define the action of s_i on $\mathcal{K}$ by*

$$s_i(x) = \mathbf{e}_i^{\gamma(a_i)}(x) \quad \textit{for } x \in \mathcal{K}.$$

Then the action of $\{s_i\}_{i\in I}$ satisfies the defining relations of the Weyl group and generates the action of the Weyl group on $\mathcal{K}$. □

3 Quantum groups and quantum geometric semicrystals

Let $C = ([a_{ij}]_{i,j\in I}, \{d_i\}_{i\in I})$ be a Cartan datum and $R = (\langle\ ,\ \rangle : Q^\vee \times P \to \mathbb{Z}, \{h_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ a root datum of type C. Recall that the root lattice Q is defined by $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$.

3.1 Quantum enveloping algebra U_q

The quantum enveloping algebra U_q is defined to be the associative algebra given by generators $\{\, E_i^+, E_i^-, K_h \mid i \in I,\, h \in Q^\vee \,\}$ and the following defining relations:

$$\begin{aligned}
&K_0 = 1, \quad K_h K_{h'} = K_{h+h'} \quad \text{for } h, h' \in Q^\vee,\\
&K_h E_j^\pm = q^{\pm\langle h,\alpha_j\rangle} E_j^\pm K_h \quad \text{for } h \in Q^\vee,\ j \in I,\\
&E_i^+E_j^- - E_j^-E_i^+ = \delta_{ij}\frac{K_{d_ih_i} - K_{-d_ih_i}}{q_i - q_i^{-1}} \quad \text{for } i, j \in I,\\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (E_i^\pm)^{1-a_{ij}-k} E_j^\pm (E_i^\pm)^k = 0 \quad \text{for } i, j \in I \text{ with } i \neq j.
\end{aligned}$$

The last relations are called the *q-Serre relations*. In particular, we have $K_{d_ih_i}E_j^\pm = q^{\pm d_ia_{ij}} E_j^\pm K_{d_ih_i}$.

The Cartan part U_q^0 of U_q is defined to be the subalgebra generated by $\{K_h\}_{h\in Q^\vee}$. The upper and lower parts $U_q^\pm$ of U_q are defined to be the subalgebras generated by $\{E_i^\pm\}_{i\in I}$. The upper Borel part $U_q^{\geqq 0}$ (resp. the lower Borel part $U_q^{\leqq 0}$) of U_q is defined to be the subalgebras generated U_q^0 and U_q^+ (resp. U_q^0 and U_q^-).

The Q-gradation of U_q is defined by $\deg K_h = 0$, $\deg E_i = \alpha_i$, and $\deg F_i = -\alpha_i$. For $\beta \in Q$, denote by U_q^β the degree-β part of U_q. Put $U_q^{\pm,\beta} = U_q^\beta \cap U_q^\pm$ for $\beta \in Q$. Then we have $U_q^\pm = \bigoplus_{\beta\in Q_+} U_q^{\pm,\pm\beta}$. For each $\beta \in Q_+$, we put $U_q^{+,\geqq\beta} = \bigoplus_{\gamma\in\beta+Q_+} U_q^{\pm,\gamma}$ and $U_q^{-,\leqq-\beta} = \bigoplus_{\gamma\in\beta+Q_+} U_q^{\pm,-\gamma}$.

For the sake of simplicity, we shall denote $K_{d_ih_i}$ by K_i, E_i^+ by E_i, and E_i^- by F_i.

Define algebra homomorphisms $\Delta : U_q \to U_q \otimes U_q$, $\varepsilon : U_q \to \mathbb{F}$, and an algebra anti-homomorphism $S : U_q \to U_q$ by

$$\begin{aligned}
&\Delta(K_h) = K_h \otimes K_h,\\
&\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,\\
&\varepsilon(K_h) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,\\
&S(K_h) = K_h^{-1}, \quad S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i.
\end{aligned}$$

Then U_q is a Hopf algebra with the product Δ, the counit ε, and the antipode S. Then the Borel parts $U_q^{\geqq 0}$, $U_q^{\leqq 0}$ and the Cartan part U_q^0 are Hopf subalgebras of U_q.

3.2 Quantum function algebra V_q

Let V_q be the associative algebra given by generators $\{ X_i^{\pm}, Z_\lambda \mid i \in I,\, \lambda \in P \}$ and the following defining relations:

$$
\begin{aligned}
& Z_0 = 1, \quad Z_\lambda Z_\mu = Z_{\lambda+\mu} \quad \text{for } \lambda, \mu \in P, \\
& Z_\lambda X_j^{\pm} = q^{\langle d_j h_j, \lambda \rangle} X_j^{\pm} Z_\lambda \quad \text{for } \lambda \in P,\, i \in I, \\
& X_i^+ X_j^- - X_j^- X_i^+ = 0 \quad \text{for } i, j \in I, \\
& \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^{1-a_{ij}-k} X_j (X_i^{\pm})^k = 0 \quad \text{for } i, j \in I \text{ with } i \neq j.
\end{aligned}
$$

Then, since $d_j a_{ji} = d_i a_{ij}$, we have $Z_{\alpha_i} X_j^{\pm} = q^{d_i a_{ij}} X_j^{\pm} Z_{\alpha_i}$.

The Cartan part V_q^0 of V_q is defined to be the subalgebra generated by $\{Z_\lambda\}_{\lambda \in P}$. The upper and lower parts $V_q^{\pm}$ are defined to be the subalgebras generated by $\{ X_i^{\pm} \mid i \in I \}$. The upper and lower Borel parts $V_q^{\geqq 0}$,$V_q^{\leqq 0}$ are defined to be the subalgebras generated by $\{ Z_\lambda, X_i^{\pm} \mid \lambda \in P,\, i \in I \}$.

The Q-gradation of V_q is defined by $\deg Z_\lambda = 0$, $\deg X_i = \alpha_i$, and $\deg Y_i = -\alpha_i$. For $\beta \in Q$, denote by $V_{q,\beta}$ the degree-β part of V_q. Put $V_{q,\beta}^{\pm} = V_{q,\beta} \cap V_q^{\pm}$ for $\beta \in Q$.

Denote Z_{α_i} by Z_i, X_i^+ by X_i, and X_i^- by Y_i.

The Hopf algebra stucture of V_q is given by

$$
\begin{aligned}
& \Delta(Z_\lambda) = Z_\lambda \otimes Z_\lambda, \\
& \Delta(X_i) = X_i \otimes 1 + Z_i \otimes X_i, \quad \Delta(Y_i) = Y_i \otimes Z_i + 1 \otimes Y_i, \\
& \varepsilon(Z_\lambda) = 1, \quad \varepsilon(X_i) = \varepsilon(X_i) = 0, \\
& S(Z_\lambda) = K_\lambda^{-1}, \quad S(X_i) = -K_i^{-1} X_i, \quad S(Y_i) = -Y_i Z_i^{-1}.
\end{aligned}
$$

3.3 Drinfeld pairing

There exists a unique biliner form $\tau : V_q^{\geqq 0} \times U_q^{\leqq 0} \to \mathbb{F}$ with the following properties:

$$
\begin{aligned}
& \tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) && \text{for } x \in V_q^{\geqq 0},\, y_1, y_2 \in U_q^{\leqq 0}, \\
& \tau(x_1 x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) && \text{for } x_1, x_2 \in V_q^{\geqq 0},\, y \in U_q^{\leqq 0}, \\
& \tau(Z_\lambda, K_h) = q^{-\langle h, \lambda \rangle} && \text{for } \lambda \in P,\, h \in Q^\vee, \\
& \tau(Z_\lambda, F_i) = \tau(X_i, K_h) = 0 && \text{for } \lambda \in P,\, i \in I,\, h \in Q^\vee, \\
& \tau(X_i, F_j) = \delta_{ij} && \text{for } i, j \in I.
\end{aligned}
$$

Then for each $\beta \in Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geqq 0} \alpha_i$ the restriction of τ on $V_{q,\beta}^+ \times U_{q,-\beta}^-$ is non-degenerate. We call τ the Drinfeld pairing.

We can identify the Hopf subalgebra of $V_q^{\geqq 0}$ generated by $\{Z_i, X_i\}_{i \in I}$ and the Hopf subalgebra of $U_q^{\geqq 0}$ generated by $\{K_i, E_i\}_{i \in I}$ by $Z_i = K_i$ and $X_i = -(q_i - q_i^{-1}) E_i$ for $i \in I$. Then the restriction to the subalgebras of the bilnear pairing τ coincides with the Drinfeld pairing between the upper and lower Borel parts of the quantum universal enveloping algebra.

3.4 Quotient Ore domain $\mathcal{A}_q$ of V_q^+ and $\mathcal{K}_q = Q(\mathcal{A}_q)$

Let $\mathcal{A}_q$ be a quotient algebra of V_q^+ and denote by ξ_i the image of X_i in $\mathcal{A}_q$ for $i \in I$. Let J be the subset of I consisting of all $i \in I$ with $\xi_i \neq 0$. Then $\{\xi_i\}_{i\in J}$ generates $\mathcal{A}_q$ over $\mathbb{F}$ and satisfies the q-Serre relations. Assume that $\mathcal{A}_q$ is an Ore domain and put $\mathcal{K}_q = Q(\mathcal{A}_q)$.

For $i, j \in J$ with $i \neq j$ and $k \in \mathbb{Z}_{\geqq 0}$, we define $(\mathrm{ad}_q\, \xi_i)^k(\xi_j)$ by

$$(\mathrm{ad}_q\, \xi_i)^0(\xi_j) = \xi_j, \quad (\mathrm{ad}_q\, \xi_i)^{k+1}(\xi_j) = [\xi_i, (\mathrm{ad}_q\, \xi_i)^k(\xi_j)]_{q_i^{2k+a_{ij}}}.$$

where $[a, b]_v = ab - vba$. Then we have

$$(\mathrm{ad}_q\, \xi_i)^k(\xi_j) = \sum_{\nu=0}^{k} (-1)^\nu q_i^{\nu(k-1+a_{ij})} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q_i} \xi_i^{k-\nu}\xi_j\xi_i^\nu.$$

The q-Serre relations for $\{\xi_i\}_{i\in I}$ are equivalent to $(\mathrm{ad}_q\, \xi_i)^k(\xi_j) = 0$ for $i \neq j$ and $k > -a_{ij}$.

3.5 Quantum geometric semicrystal structure on $\mathcal{K}_q$

Let us construct a quantum geometric J-semicrystal structure on $\mathcal{K}_q$.

For $n \in \mathbb{Z}$, by induction on $|n|$, we obtain

$$\xi_i^n \xi_j \xi_i^{-n} = \begin{cases} \xi_i & \text{if } i = j, \\ \displaystyle\sum_{k=0}^{-a_{ij}} q_i^{(n-k)(k+a_{ij})} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, \xi_i)^k(\xi_j)\xi_i^{-k} & \text{if } i \neq j. \end{cases}$$

Therefore $\xi_i^n \xi_j \xi_i^{-n}$ is an n-independent rational function of q_i^n. Replacing q_i^n with an indeterminate t, we define $\mathbf{e}_i^t(\xi_j) \in \mathcal{K}_q(t)$ by

$$\mathbf{e}_i^t(\xi_j) = \begin{cases} \xi_i & \text{if } i = j, \\ \displaystyle\sum_{k=0}^{-a_{ij}} t^{k+a_{ij}} q_i^{-k(k+a_{ij})} \begin{bmatrix} t; 0 \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, \xi_i)^k(\xi_j)\xi_i^{-k} & \text{if } i \neq j, \end{cases}$$

where

$$\begin{bmatrix} t; x \\ k \end{bmatrix}_v = \frac{[t; x]_v [t; x-1]_v \cdots [t; x-k+1]}{[k]_v!}, \quad [t; x]_v = \frac{tv^x - t^{-1}v^{-x}}{v - v^{-1}}.$$

Lemma 3.1. *For each $i \in J$, the mapping $\mathbf{e}_i^t : \{\xi_j\}_{j\in J} \to \mathcal{K}_q(t)$ can be extended to the algebra homomorphism $\mathcal{K}_q \to \mathcal{K}_q(t)$ also denoted by $\mathbf{e}_i^t$. Then $(\mathcal{K}_q, \{\mathbf{e}_i\}_{i\in J})$ is a quantum geometric J-semicrystal.* □

3.6 Weyl group action on $\mathcal{K}_q(q^P)$

Let $R = (\langle\ ,\ \rangle : Q^\vee \times P \to \mathbb{Z}, \{h_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be a root datum of type $C = ([a_{ij}]_{i,j\in I}, \{d_i\}_{i\in I})$ and $R_J = (\langle\ ,\ \rangle : Q_J^\vee \times P_J \to \mathbb{Z}, \{h_{J,i}\}_{i\in J}, \{\alpha_{J,i}^\vee\}_{i\in J})$ the restriction of R to J. Let $(\mathcal{K}_q, \{\mathbf{e}_i\}_{i\in J})$ be the quantum geometric semicrystal given in Section 3.5.

Lemma 3.2. *Let $(\mathcal{K}_q(q^{P_J}), \{\mathbf{e}_i^t\}_{i\in J}, \gamma)$ be the associated quantum geometric J-crystal of $(\mathcal{K}_q, \{\mathbf{e}_i\}_{i\in J})$ and put $a_i = q^{\alpha_i}$ for $i \in J$. Then the Weyl group W_J of type $[a_{ij}]_{i,j\in J}$ acts on $\mathcal{K}_q(q^{P_J})$ by $s_i(x) = \mathbf{e}_i^{a_i}(x)$ for $i \in J$ and $x \in \mathcal{K}_q(q^{P_J})$.* □

Remark 3.3. The Weyl group action of the corollary was first constructed by the author in [5]. It is both a q-difference analogue and a quantization of the birational Weyl group action constructed by Noumi-Yamada [8]. They also proposed that the birational action of the lattice part of the affine Weyl group can be regarded as a difference analogue of a Painlevé system. Therefore the action of the lattice part of the affine Weyl group on the quantum geometric crystal $\mathcal{K}_q(q^{P_J})$ can be regarded as a q-difference analogue of a quantum Painlevé system. □

4 Standard quantum geometric semicrystals

4.1 Quantum algebra $\mathcal{A}_{\mathbf{i}}$ and $\mathcal{K}_{\mathbf{i}} = Q(\mathcal{A}_{\mathbf{i}})$

Let $([a_{ij}]_{i,j\in I},\ \{d_i\}_{i\in I})$ be a Cartan datum and W the Weyl group of type $[a_{ij}]_{i,j\in I}$ and put $b_{ij} = (\alpha_i|\alpha_j) = d_i a_{ij}$ for $i, j \in I$.

For any $\mathbf{i} = (i_1, \dots, i_N) \in I^N$, define the algebra $\mathcal{A}_{\mathbf{i}}$ to be the associative algebra over $\mathbb{F}$ given by generators $\{x_{\mathbf{i},\nu}\}_{i=1}^N$ and the defining relations $x_{\mathbf{i},\nu}x_{\mathbf{i},\mu} = q^{b_{i_\mu i_\nu}} x_{\mathbf{i},\mu}x_{\mathbf{i},\nu}$ for $\mu < \nu$. Then $\mathcal{A}_{\mathbf{i}}$ is an Ore domain. Put $\mathcal{K}_{\mathbf{i}} = Q(\mathcal{A}_{\mathbf{i}})$.

Let $\mathcal{K}_{\mathbf{i}}^+$ be the semi-subfield of $\mathcal{K}_{\mathbf{i}}$ generated by $q, x_{\mathbf{i},1}, \dots, x_{\mathbf{i},N}$. That is, $\mathcal{K}_{\mathbf{i}}^+$ is the minimum subset of $\mathcal{K}_{\mathbf{i}}$ which contains $0, 1, q, x_{\mathbf{i},1}, \dots, x_{\mathbf{i},N}$ and is closed under the addition, the multiplication, and the division by non-zero elements. Elements of $\mathcal{K}_{\mathbf{i}}^+$ are said to be *subtraction-free* or *positive*.

For $\mathbf{i}, \mathbf{j} \in I^N$, an algebra isomorphism $f : \mathcal{K}_{\mathbf{j}} \to \mathcal{K}_{\mathbf{i}}$ is said to be *subtraction-free* or *positive* if $f(\mathcal{K}_{\mathbf{j}}^+) \subset \mathcal{K}_{\mathbf{i}}^+$. The isomorphism f is positive if and only if $f(x_{\mathbf{j},\nu}) \in \mathcal{K}_{\mathbf{i}}^+$ for every $\nu = 1, \dots, N$.

Similarly let $\mathcal{K}_{\mathbf{i}}(t)^+$ be the semi-subfield of $\mathcal{K}_{\mathbf{i}}(t)$ generated by $q, t, x_{\mathbf{i},1}, \dots, x_{\mathbf{i},N}$. An algebra homomorphism $f : \mathcal{K}_{\mathbf{i}} \to \mathcal{K}_{\mathbf{i}}(t)$ is called *subtraction-free* or *positive* if $f(\mathcal{K}_{\mathbf{i}}^+) \subset \mathcal{K}_{\mathbf{i}}(t)^+$.

4.2 Transition isomorphism and its positivity

Recall that U_q^- denotes the lower part of the quantum universal enveloping algebra generated by $\{F_i\}_{i\in I}$. The v-exponential function $\exp_v(x)$ is defined by

$$\exp_v(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k)_v!}, \quad (k)_v = \frac{1-v^k}{1-v}, \quad (k)_v! = (1)_v(2)_v \cdots (k)_v.$$

We define the q-exponential function $e_q(x)$ by

$$e_q(x) = \exp_{q^2}(x) = \sum_{k=0}^{\infty} q^{-k(k-1)/2} \frac{x^k}{[k]_q!}.$$

Then, for any $a \in \mathcal{K}_{\mathbf{i}}$, the q-exponential $e_{q^k}(aF_i)$ is well-defined as an element of the completion $\mathcal{K}_{\mathbf{i}}\widehat{\otimes}U_q^- = \operatorname{proj\,lim}_{\beta\in Q_+} \mathcal{K}_{\mathbf{i}} \otimes \left(U_q^-/U_q^{-,\leq -\beta}\right)$

Let $\mathbf{i} = (i_1, \dots, i_N)$ and $\mathbf{j} = (j_1, \dots, j_N)$ be reduced words for a same element $w \in W$: $\mathbf{i}, \mathbf{j} \in R(w)$. Denote $x_{\mathbf{i},\nu} \in \mathcal{K}_{\mathbf{i}}$ by x_ν and $x_{\mathbf{j},\nu} \in \mathcal{K}_{\mathbf{j}}$ by y_ν.

In [1], Berenstein shows that there exists a unique algebra isomorphism $f : \mathcal{K}_{\mathbf{j}} \to \mathcal{K}_{\mathbf{i}}$ such that

$$e_{q_{i_1}}(x_1F_{i_1}) \cdots e_{q_{i_N}}(x_NF_{i_N}) = e_{q_{j_1}}(f(y_1)F_{j_1}) \cdots e_{q_{j_N}}(f(y_N)F_{j_N}). \tag{4.1}$$

We call f the *transition isomorphism*. The explicit formulae of f for the $A_1 \times A_1$, A_2, and B_2 cases can be found in [1], Proposition 2.8. These formulae show that the transition isomorphisms f are positive for the $A_1 \times A_1$, A_2, and B_2 cases.

Lemma 4.1. *For any reduced words* $\mathbf{i}, \mathbf{j}$ *of length* N *for a same element* $w \in W$*, the transition isomorphism* $f : \mathcal{K}_{\mathbf{j}} \to \mathcal{K}_{\mathbf{i}}$ *characterized by* (4.1) *is always positive and satisfies* $f(\sum_{i_\nu = i} y_\nu) = \sum_{i_\nu = i} x_\nu$ *for* $i \in I$. $\square$

The proof of Lemma 4.1 reduces to the cases of finite type with rank 2. Since the $A_1 \times A_1$, A_2, and B_2 cases are shown by Berenstein [1], it is sufficient to show the G_2 case only.

Before proceeding to prove it, for the convenience of readers we write down the explicit formulae of the transition isomorphisms for the $A_1 \times A_1$, A_2, and B_2 cases.

Case $A_1 \times A_1$. Let $[a_{ij}]_{i,j=1}^2$ be the Cartan matrix of type A_2: $a_{11} = a_{22} = 2$, $a_{12} = a_{21} = 0$. Put $d_1 = d_2 = 1$. Let $f : \mathcal{K}_{(1,2)} \to \mathcal{K}_{(2,1)}$ be the transition isomorphism uniquely characterized by $e_q(x_1 F_1) e_q(x_2 F_2) = e_q(f(y_1) F_2) e_q(f(y_2) F_1)$. Since F_1 and F_2 commute, we have

$$f(y_1) = x_2, \quad f(y_2) = x_1.$$

These formulae mean that $f(\sum_{i_\nu = i} y_\nu) = \sum_{i_\nu = i} x_\nu$ for $i = 1, 2$. Clearly both f and f^{-1} are positive.

Case A_2. Let $[a_{ij}]_{i,j=1}^2$ be the Cartan matrix of type A_2: $a_{11} = a_{22} = 2$, $a_{12} = a_{21} = -1$. Put $d_1 = d_2 = 1$. Let $f : \mathcal{K}_{(2,1,2)} \to \mathcal{K}_{(1,2,1)}$ be the transition isomorphism uniquely characterized by

$$e_q(x_1 F_1) e_q(x_2 F_2) e_q(x_3 F_1) = e_q(f(y_1) F_2) e_q(f(y_2) F_1) e_q(f(y_3) F_2). \tag{4.2}$$

Comparison of the coefficients of F_1, F_2, and $F_2 F_1$ in the both-sides leads to

$$f(y_2) = x_1 + x_3, \quad f(y_1 y_2) = x_2 x_3, \quad f(y_1 + y_3) = x_2.$$

The first and the third equations mean that $f(\sum_{i_\nu = i} y_\nu) = \sum_{i_\nu = i} x_\nu$ for $i = 1, 2$. Solving the equations, we obtain

$$f(y_1) = x_2 x_3 (x_1 + x_3)^{-1}, \quad f(y_2) = x_1 + x_3, \quad f(y_3) = x_2 x_1 (x_1 + x_3)^{-1}.$$

The anti-algebra isomorphism $\rho : \mathcal{K}_{(1,2,1)} \otimes U_q^- \to \mathcal{K}_{(2,1,2)} \otimes U_q^-$ is given by $\rho(x_\nu) = y_{4-\nu}$, $\rho(F_1) = F_2$, and $\rho(F_2) = F_1$. Applying ρ to (4.2), we have $f^{-1} = \rho \circ f \circ \rho^{-1}$. Therefore both f and f^{-1} are positive.

Case B_2. Let $[a_{ij}]_{i,j=1}^2$ be the Cartan matrix of type B_2: $a_{11} = a_{22} = 2$, $a_{12} = -1$, $a_{21} = -2$. Put $d_1 = 2$ and $d_2 = 1$. Let $f : \mathcal{K}_{(2,1,2,1)} \to \mathcal{K}_{(1,2,1,2)}$ be the transition isomorphism uniquely characterized by

$$\begin{aligned} & e_{q^2}(x_1 F_1) e_q(x_2 F_2) e_{q^2}(x_3 F_1) e_q(x_4 F_2) \\ & \quad = e_q(f(y_1) F_2) e_{q^2}(f(y_2) F_1) e_q(f(y_3) F_2) e_{q^2}(f(y_4) F_1). \end{aligned} \tag{4.3}$$

Comparing the coefficients of F_1 and F_2 in the both-sides, we obtain $f(y_1 + y_3) = x_2 + x_4$ and $f(y_2 + y_4) = x_1 + x_3$, which mean that $f(\sum_{i_\nu = i} y_\nu) = \sum_{i_\nu = i} x_\nu$ for $i = 1, 2$. Comparing the coefficients of $F_2 F_1 F_2$, $F_1 F_2$, $F_1 F_2^2 F_2$, and $F_1 F_2^2$, we also obtain

$$\begin{aligned} & p_1 := f(y_1 y_2 y_3) = x_2 x_3 x_4, \quad p_2 := f(y_2 y_3) = x_1 x_2 + x_1 x_4 + x_3 x_4, \\ & p_3 := f(y_2 y_3^2 y_4) = x_1 x_2^2 x_3, \quad p_4 := f(y_2 y_3^2) = x_1 (x_2 + x_4)^2 + x_3 x_4^2. \end{aligned}$$

These equations lead to

$$f(y_1) = p_1p_2^{-1}, \quad f(y_2) = p_2p_4^{-1}p_2, \quad f(y_3) = p_2^{-1}p_4, \quad f(y_4) = p_4^{-1}p_3.$$

The anti-algebra isomorphism $\rho : \mathcal{K}_{(1,2,1,2)} \otimes U_q^- \to \mathcal{K}_{(2,1,2,1)} \otimes U_q^-$ is given by $\rho(x_\nu) = y_{5-\nu}$ and $\rho(F_i) = F_i$. Applying ρ to (4.3), we have $f^{-1} = \rho \circ f \circ \rho^{-1}$. Therefore both f and f^{-1} are positive.

Let us prove Lemma 4.1 for the G_2 case.

Case G_2. Let $[a_{ij}]_{i,j=1}^2$ be the Cartan matrix of type G_2: $a_{11} = a_{22} = 2$, $a_{12} = -1$, $a_{21} = -3$. Put $d_1 = 3$ and $d_2 = 1$. Let $f : \mathcal{K}_{(2,1,2,1,2,1)} \to \mathcal{K}_{(1,2,1,2,1,2)}$ be the tansition isomorphism uniquely characterized by

$$\begin{aligned}&e_{q^3}(x_1F_1)e_q(x_2F_2)e_{q^3}(x_3F_1)e_q(x_4F_2)e_{q^3}(x_5F_1)e_q(x_6F_2)\\&\quad= e_q(f(y_1)F_2)e_{q^3}(f(y_2)F_1)e_q(f(y_3)F_2)e_{q^3}(f(y_4)F_1)e_q(f(y_5)F_2)e_{q^3}(f(y_6)F_1). \qquad (4.4)\end{aligned}$$

Comparing the coefficients of F_1 and F_2, we obtain $f(y_1 + y_3 + y_5) = x_2 + x_4 + x_6$ and $f(y_2 + y_4 + y_6) = x_1 + x_3 + x_5$, which mean that $f(\sum_{i_\nu=i} y_\nu) = \sum_{i_\nu=i} x_\nu$ for $i = 1, 2$.

We can realize the Chevalley generators of the finite-dimensional simple Lie algebra of type G_2 by

$$\begin{aligned}
e_1 &= E_{23} + E_{56}, & e_2 &= E_{12} + E_{34} + 2E_{45} + E_{67},\\
f_1 &= E_{32} + E_{65}, & f_2 &= E_{21} + 2E_{43} + E_{54} + E_{76},\\
h_1 &= E_{22} - E_{33} + E_{55} - E_{66}, & h_2 &= E_{11} - E_{22} + 2E_{33} - 2E_{55} + E_{66} - E_{77},
\end{aligned}$$

where E_{ij} denotes the (i, j)-matrix unit of size 7. In this realization, we have $f_1^2 = f_2^3 = f_1f_2f_1 = f_2^2f_1f_2^2 = 0$ and the same relations for e_1, e_2. These formulae imply both the Serre and the q-Serre relations for f_1, f_2 and e_1, e_2. Namely we have

$$\begin{aligned}
&f_1^2f_1 - [2]_{q^3}f_1f_2f_1 + f_2f_1^2 = 0,\\
&f_2^4f_1 - [3]_qf_2^3f_1f_2 + \begin{bmatrix}4\\2\end{bmatrix}_q f^2f_1f^2 - [3]_qf_2f_1f_2^3 + f_1f_2^4 = 0
\end{aligned}$$

and the same relations for e_1, e_2. Therefore we obtain the 7-dimensional representation of the lower part U_q^- of the quantum universal enveloping algebra of type G_2.

Then the transition isomorphism f satisfies

$$\begin{aligned}&e_{q^3}(x_1f_1)e_q(x_2f_2)e_{q^3}(x_3f_1)e_q(x_4f_2)e_{q^3}(x_5f_1)e_q(x_6f_2)\\&\quad= e_q(f(y_1)f_2)e_{q^3}(f(y_2)f_1)e_q(f(y_3)f_2)e_{q^3}(f(y_4)f_1)e_q(f(y_5)f_2)e_{q^3}(f(y_6)f_1). \qquad (4.5)\end{aligned}$$

Denote by m_{ij} the (i, j)-entry of the both sides. First write down m_{ij} by y_ν:

$$\begin{aligned}
c/2 \cdot m_{71} &= f(y_1y_2y_3^2y_4y_5),\\
c/2 \cdot m_{61} &= f(y_2y_3^2y_4y_5),\\
c/2 \cdot m_{72} &= f(y_1y_2y_3^2y_4 + y_1y_2y_3^2y_6 + y_1y_2y_5^2y_6 + y_1y_4y_5^2y_6 + y_3y_4y_5^2y_6 + cy_1y_2y_3y_5y_6),\\
c/2 \cdot m_{62} &= f(y_2y_3^2y_4 + y_2y_3^2y_6 + y_2y_5^2y_6 + y_4y_5^2y_6 + cy_2y_3y_5y_6),\\
c/2 \cdot m_{73} &= f(y_1y_2y_3^2 + y_1y_2y_5^2 + y_1y_4y_5^2 + y_3y_4y_5^2 + cy_1y_2y_3y_5),\\
c/2 \cdot m_{63} &= f(y_2y_3^2 + y_2y_5^2 + y_4y_5^2 + cy_2y_3y_5),\\
1/2 \cdot m_{41} &= f(y_1y_2y_3 + y_1y_2y_5 + y_1y_4y_5 + y_3y_4y_5),\\
m_{31} &= f(y_2y_3 + y_2y_5 + y_4y_5),\\
m_{21} &= f(y_1 + y_3 + y_5),
\end{aligned}$$

where $c = 1 + q^2$. Therefore we have

$$\begin{aligned}
p_1 &:= f(y_1y_2y_3^2y_4y_5) = c/2 \cdot m_{71},\\
p_2 &:= f(y_2y_3^2y_4y_5) = c/2 \cdot m_{61},\\
p_3 &:= f(y_2y_3^3y_4^2y_5^3y_6) = q^4c/2 \cdot (p_2m_{72} - p_1m_{62}),\\
p_4 &:= f(y_2y_3^3y_4^2y_5^3) = q^4c/2 \cdot (p_2m_{73} - p_1m_{63}),\\
p_5 &:= f(y_2y_3^3y_4^2y_5^2) = q^4(p_2m_{41}/2 - p_1m_{31}),\\
p_6 &:= f(y_2^2y_3^6y_4^3y_5^3) = q^{18}(p_5m_{21}p_2 - c/2 \cdot p_5m_{71} - p_4p_2).
\end{aligned}$$

Solving these equations, we obtain

$$\begin{aligned}
&f(y_1) = p_1p_2^{-1}, && f(y_2) = q^{24}p_2^3p_6^{-1}, && f(y_3) = q^{-18}p_6p_5^{-1}p_2^{-1},\\
&f(y_4) = q^{15}p_5^3p_6^{-1}p_4^{-1}, && f(y_5) = q^{-3}p_4p_5^{-1}, && f(y_6) = p_3p_4^{-1}.
\end{aligned}$$

Second write down m_{ij} by x_ν:

$$\begin{aligned}
c/2 \cdot m_{71} &= x_2x_3x_4^2x_5x_6 = p_1,\\
c/2 \cdot m_{61} &= x_3x_4^2x_5x_6 + x_1(x_2 + x_4)^2x_5x_6 + x_1x_2^2x_3(x_4 + x_6) = p_2,\\
c/2 \cdot m_{72} &= x_2x_3x_4^2x_5,\\
c/2 \cdot m_{62} &= x_1x_2^2x_3 + x_3x_4^2x_5 + x_1(x_2 + x_4)^2x_5,\\
c/2 \cdot m_{73} &= x_2x_5x_6^2 + x_4x_5x_6^2 + x_2x_3(x_4 + x_6)^2,\\
c/2 \cdot m_{63} &= x_5x_6^2 + x_3(x_4 + x_6)^2 + x_1(x_2 + x_4 + x_6)^2,\\
1/2 \cdot m_{41} &= x_2x_3x_4 + x_2x_3x_6 + x_2x_5x_6 + x_4x_5x_6,\\
m_{31} &= x_1x_2 + x_1x_4 + x_3x_4 + x_1x_6 + x_3x_6 + x_5x_6,\\
m_{21} &= x_2 + x_4 + x_6,
\end{aligned}$$

By straightforward (but very tedious) calculation, we can prove that $p_1, \ldots, p_6$ are polynomials of $q, x_1, \ldots, x_6$ with non-negative integer coefficients. Explicitly we have

$$\begin{aligned}
p_1 &= x_2x_3x_4^2x_5x_6,\\
p_2 &= x_3x_4^2x_5x_6 + x_1(x_2 + x_4)^2x_5x_6 + x_1x_2^2x_3(x_4 + x_6),\\
p_3 &= x_1x_2^3x_3^2x_4^3x_5,\\
p_4 &= x_3x_4^3x_5^2x_6^3 + x_1(x_2 + x_4)^3x_5^2x_6^3 + x_1x_2^3x_3^2(x_4 + x_6)^3\\
&\quad + q^4x_1x_2^2x_3x_5x_6(q^4(3)_{q^2}x_2x_4x_6 + q^2(3)_{q^2}x_4^2x_6 + (3)_{q^2}x_4x_6^2 + (2)_{q^6}x_2x_6^2),\\
p_5 &= x_3x_4^3x_5^2x_6^2 + x_1(x_2 + x_4)^3x_5^2x_6^2 + x_1x_2^3x_3^2(x_4 + x_6)^2\\
&\quad + q^4x_1x_2^2x_3x_5x_6(q^4(2)_{q^2}x_2x_4 + q^2(2)_{q^2}x_4^2 + (2)_{q^6}x_2x_6 + (3)_{q^2}x_4x_6),\\
p_6 &= q^{15}\big(x_3x_4^3x_5x_6 + (2)_{q^6}x_1(x_2 + x_4)^3x_5x_6\\
&\qquad + qx_1x_2^2x_3(q^2(3)_{q^2}x_2x_4 + (3)_{q^2}x_4^2 + q^2(3)_{q^2}x_4x_6 + q^2(2)_{q^6}x_2x_6)\big)x_3x_4^3x_5^2x_6^2\\
&\quad + q^{15}x_1^2((x_2 + x_4)^2x_5x_6 + x_2^2x_3(x_4 + x_6))^3.
\end{aligned}$$

The anti-algebra isomorphism $\rho : \mathcal{K}_{(1,2,1,2,1,2)} \otimes U_q^- \to \mathcal{K}_{(2,1,2,1,2,1)} \otimes U_q^-$ is given by $\rho(x_\nu) = y_{7-\nu}$ and $\rho(F_i) = F_i$. Applying ρ to (4.4), we have $f^{-1} = \rho \circ f \circ \rho^{-1}$. Therefore both f and f^{-1} are positive.

This completes the proof of Lemma 4.1.

Remark 4.2. The above formulae for the G_2 case are quatum analogue of Theorem 3.1 (c) of Berenstein-Kazhdan [4]. □

4.3 Quantum geometric semicrystal structure on $\mathcal{K}_{\mathbf{i}}$

For $\mathbf{i} = (i_1, \dots, i_N) \in I^N$, let $\mathcal{A}_{\mathbf{i}}$ be the algebra given in Section 4.1 and $\mathcal{K}_{\mathbf{i}} = Q(\mathcal{A}_{\mathbf{i}})$ its field of fractions. Let J be the subset of I consisting of $i \in I$ with $i_\nu = 0$ for some $\nu = 1, \dots, N$. We call J the *support of* $\mathbf{i}$. If $\mathbf{i} = (i_1, \dots, i_N)$ is a reduced word for $w \in W$, then the support of $\mathbf{i}$ depends only on w and J is called the *support of* w.

Let us construct a quantum geometric semicrystal structure on $\mathcal{K}_{\mathbf{i}}$.

For $i \in I$, define $\xi_i \in \mathcal{A}_{\mathbf{i}}$ by

$$\xi_i = \sum_{1 \leqq \nu \leqq N,\, i_\nu = i} x_\nu.$$

Then $\xi_i \neq 0$ is equivalent to $i \in J$ and $\{\xi_i\}_{i \in J}$ satisfies the q-Serre relations.

For $i \in J$ and $\nu = 1, \dots, N$, we define $X, Y \in \mathcal{A}_{\mathbf{i}}$ by

$$X = \sum_{i_\nu = i,\, \mu < \nu} x_\mu, \quad Y = \sum_{i_\nu = i,\, \mu > \nu} x_\mu.$$

Then we have $\xi_i = X + \delta_{i i_\nu} x_\nu + Y$. For $n \in \mathbb{Z}$, by induction on $|n|$, we can show that

$$\xi_i^n x_\nu \xi_i^{-n} = \begin{cases} q_i^{-2n} x_\nu \dfrac{1 + q_i^2 (x_\nu + q_i^2 Y) X^{-1}}{1 + q_i^{2(1-n)} (x_\nu + q_i^2 Y) X^{-1}} \dfrac{1 + q_i^2 Y (X + x_\nu)^{-1}}{1 + q_i^{2(1-n)} Y (X + x_\nu)^{-1}} & \text{if } i_\nu = i, \\ q_i^{-a_{i i_\nu} n} x_\nu \displaystyle\prod_{k=0}^{-a_{i i_\nu} - 1} \dfrac{1 + q_i^{-2(n+k-1)} Y X^{-1}}{1 + q_i^{-2(k-1)} Y X^{-1}} & \text{if } i_\nu \neq i. \end{cases}$$

Therefore $\xi_i^n x_\nu \xi_i^{-n}$ is an n-independent rational function of q_i^n. Replacing q_i^n by an indeteminate t, we define $\mathbf{e}_i^t(x_\nu) \in \mathcal{K}_{\mathbf{i}}(t)$ by

$$\mathbf{e}_i^t(x_\nu) = \begin{cases} t^{-2} x_\nu \dfrac{1 + q_i^2 (x_\nu + q_i^2 Y) X^{-1}}{1 + q_i^2 t^{-2} (x_\nu + q_i^2 Y) X^{-1}} \dfrac{1 + q_i^2 Y (X + x_\nu)^{-1}}{1 + q_i^2 t^{-2} Y (X + x_\nu)^{-1}} & \text{if } i_\nu = i, \\ t^{-a_{i i_\nu}} x_\nu \displaystyle\prod_{k=0}^{-a_{i i_\nu} - 1} \dfrac{1 + q_i^{-2(k-1)} t^{-2} Y X^{-1}}{1 + q_i^{-2(k-1)} Y X^{-1}} & \text{if } i_\nu \neq i. \end{cases}$$

Lemma 4.3. *For each $i \in J$, the mapping $\mathbf{e}_i^t : \{\xi_i\}_{i \in J} \to \mathcal{K}_q(t)$ can be extended to the positive algebra homomorphism $\mathcal{K}_{\mathbf{i}} \to \mathcal{K}_{\mathbf{i}}(t)$ also denoted by $\mathbf{e}_i^t$. Then $(\mathcal{K}_{\mathbf{i}}, \{\mathbf{e}_i\}_{i \in J})$ is a quantum geometric J-semicrystal.* □

We call $(\mathcal{K}_{\mathbf{i}}, \{\mathbf{e}_i\}_{i \in J})$ the *standard quantum geometric semicrystal* for $\mathbf{i} \in I^N$.

Lemma 4.4. *Let $\mathbf{i} = (i_1, \dots, i_N)$, $\mathbf{i}' = (i'_1, \dots, i'_N)$ be reduced words for a same element $w \in W$, J the support of w, and $f : \mathcal{K}_{\mathbf{i}'} \to \mathcal{K}_{\mathbf{i}}$ the transition isomorphism. Then f is an isomorphism of quantum geometric J-semicrystals. That is, f commutes with $\mathbf{e}_i^t$ for $i \in J$.* □

Proof. Since the transition isomorphism f preserves ξ_i, the algebra isomorphism $x \mapsto \xi_i^n x \xi_i^{-n}$ commutes with f. This leads to the commutativity of f and $\mathbf{e}_i^t$. □

Let $\mathbf{i} = (i_1, \dots, i_N)$, $\mathbf{i}' = (i'_1, \dots, i'_N)$ be reduced words for a same element $w \in W$ and J the support of w. Then the quantum geometric J-semicrystals $\mathcal{K}_{\mathbf{i}}$ and $\mathcal{K}_{\mathbf{i}'}$ can be identified via the transition isomorphism. Then $\mathcal{K}_{\mathbf{i}} = \mathcal{K}_{\mathbf{i}'}$ is denoted by $\mathcal{K}_w$ and called the *standard quantum geometric semicrystal* for $w \in W$.

5 Quantum toric semicrystals

5.1 Definition of quantum toric semicrystals

5.2 Quantum torus and positive structure

Let $\mathcal{K}$ be a (possibly non-commutative) field over $\mathbb{F}$. We call $\mathcal{K}$ a *rational function field of quantum torus* or simply a *quantum torus* if there exists a finite family $\{x_\nu\}_{\nu=1}^N$ of elements in $\mathcal{K}$ such that $\mathcal{K}$ is generated by $\{x_\nu\}_{\nu=1}^N$ as a field over $\mathbb{F}$ and all the relations of $\{x_\nu\}_{\nu=1}^N$ are generated by $x_\nu x_\mu = q^{c_{\mu\nu}} x_\mu x_\nu$ $(1 \leqq \mu, \nu \leqq N)$ for some skew-symmetric integer matrix $[c_{\mu\nu}]_{\mu,\nu=1}^N$. We call $\mathbf{x} = \{x_\nu\}_{\nu=1}^N$ a *chart* of the quantum torus $\mathcal{K}$. Quantum tori with charts are quantum analogue of split algebraic tori.

Let $\mathcal{K}$ be a quantum torus with a chart $\{x_\nu\}_{\nu=1}^N$ and $\mathcal{A}$ the subalgebra generated by $\{x_\nu\}_{\nu=1}^N$. Then $\mathcal{A}$ is an Ore domain and hence $\mathcal{K}$ is identified with the field of fractions $Q(\mathcal{A})$.

For example, for each $\mathbf{i} \in I^N$, the standard quantum geometric semicrystal $\mathcal{K}_{\mathbf{i}}$ is a quantum torus with a chart $\{x_{\mathbf{i},\nu}\}_{\nu=1}^N$.

If $\mathcal{K}$ is a quantum torus with a chart $\{x_\nu\}_{\nu=1}^N$, then the rational function field $\mathcal{K}(t_1, \ldots, t_n)$ over $\mathcal{K}$ is naturally a quantum torus with a chart $\{t_k\}_{k=1}^n \cup \{x_\nu\}_{\nu=1}^N$.

A subset S of $\mathcal{K}$ is called a *semi-subfield* of $\mathcal{K}$ if S contains $0, 1$ and is closed under the addition, the multiplication, and the division by non-zero elements in S. For any subset X of $\mathcal{K}$, the semi-subfield of $\mathcal{K}$ generated by X is defined to be the minimum semi-subfield of $\mathcal{K}$ which includes X. An element of $\mathcal{K}$ is positive if and only if it can be written in a subtraction-free expression of elements in X.

Let $\mathcal{K}$ be a quantum torus with a chart $\mathbf{x} = \{x_\nu\}_{\nu=1}^N$. The *positive structure* $\mathcal{K}^{\mathbf{x},+}$ on $\mathcal{K}$ is defined to be the semi-subfield of $\mathcal{K}$ generated by q and $\{x_\nu\}_{\nu=1}^N$. More generally, the positive structure $\mathcal{K}(t_1, \ldots, t_n)^{\mathbf{x},+}$ on the rational function field $\mathcal{K}(t_1, \ldots, t_n)$ is defined to be the positive structure given by the chart $\{t_k\}_{k=1}^n \cup \{x_\nu\}_{\nu=1}^N$.

Let $\mathcal{K}$, $\mathcal{K}'$ be quantum tori with charts $\mathbf{x} = \{x_\nu\}_{\nu=1}^N$, $\mathbf{x}' = \{x'_{\nu'}\}_{\nu'=1}^{N'}$ respectively. An algebra homomorphism $f : \mathcal{K}' \to \mathcal{K}$ is said to be *positive* if $f(\mathcal{K}'^{\mathbf{x}',+}) \subset \mathcal{K}^{\mathbf{x},+}$. An algebra homomorphism $f : \mathcal{K}' \to \mathcal{K}$ is positive if and only if $f(x'_{\nu'})$ can be written in a subtraction-free expression of $\mathbf{x} = \{x_\nu\}_{\nu=1}^N$ for each $\nu' = 1, \ldots, N'$.

Denote by $\mathcal{T}^+$ the category of quantum tori with charts and positive algebra homomorphisms between them. Note that a positive algebra isomorphism $f : \mathcal{K}' \to \mathcal{K}$ is not necessarily an isomorphism in $\mathcal{T}^+$. For example, if $f : \mathbb{F}(t_1, t_2) \to \mathbb{F}(t_1, t_2)$ is the positive algebra isomorphism given by $f(t_1) = t_1 + t_2$ and $f(t_2) = t_2$, then its inverse is not positive.

5.3 Product and dual

6 Appendix

References

[1] Berenstein, Arkady. Group-like elements in quantum groups and Feigin's conjecture. http://arxiv.org/abs/q-alg/9605016

[2] Berenstein, Arkady and Kazhdan, David. Geometric and unipotent crystals. GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part I, 188–236. `http://arxiv.org/abs/math/9912105`

[3] Berenstein, Arkady and Kazhdan, David. Geometric and unipotent crystals: II: From unipotent bicrystals to crystal bases. `http://arxiv.org/abs/math/0601391`

[4] Berenstein, Arkady and Zelevinsky, Andrei. Total positivity in Schubert varieties.

[5] Kuroki, Gen. Quantum groups and quantization of Weyl group symmetries of Painlevé systems. Preprint 2008, to appear in Advanced Studies in Pure Mathematics, Proceedings of "Exploration of New Structures and Natural Constructions in Mathematical Physics", Nagoya University, March 5–8, 2007. `http://arxiv.org/abs/0808.2604`

[6] Kac, Victor G., Infinite-dimensional Lie algebras, Third edition, Cambridge University Press, Cambridge, 1990, xxii+400 pp.

[7] Lusztig, George, Introduction to quantum groups, Progress in Mathematics, 110, Birkhäuser Boston, Inc., Boston, MA, 1993, xii+341 pp.

[8] Noumi, Masatoshi and Yamada, Yasuhiko. Birational Weyl group action arising from a nilpotent Poisson algebra. Physics and combinatorics 1999 (Nagoya), 287–319, World Sci. Publ., River Edge, NJ, 2001. `http://arxiv.org/abs/math/0012028`