

# Quantum Painlevé tau-functions

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# General theory for the quantum and $q$ -difference version of $\tau$ -functions for any symmetrizable GCM

# Quantum Algebra: Definition

Consider the associative algebra generated by

- dependent variables:  $f_i$
- parameter variables:  $\alpha_i^\vee$
- $\tau$ -variables:  $\tau_i$

with the relations

- $q$ -Serre relations of  $f_i$ .
- $\alpha_i^\vee$  commutes with  $\alpha_j^\vee$  and  $f_j$ .
- $\tau_i$  commutes with  $\tau_j$  and  $f_j$ .
- $\tau_i \alpha_j^\vee \tau_i^{-1} = \alpha_j^\vee + \delta_{ij}. \quad (\tau_i = \exp(\partial/\partial \alpha_i^\vee))$

# Quantum Algebra: $q$ -Serre relations

$[a_{ij}]_{i,j \in I}$ : GCM with  $d_i a_{ij} = d_j a_{ji}$ ,  $d_i \in \mathbb{Z}_{>0}$ .

$q$ : an indeterminate.

$$q_i := q^{d_i}, \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}},$$

$$[n]_q! := [1]_q [2]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

$q$ -Serre relations: if  $i, j \in I$  and  $i \neq j$ , then

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q f_i^{1-a_{ij}-k} f_j f_i^k = 0.$$

# Quantum Algebra: Relations

- $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j),$
- $\alpha_i^\vee \alpha_j^\vee = \alpha_j^\vee \alpha_i^\vee, \alpha_i^\vee f_j = f_j \alpha_i^\vee,$
- $\tau_i \tau_j = \tau_j \tau_i, \tau_i f_j = f_j \tau_i,$
- $\tau_i \alpha_j^\vee \tau^{-1} = \alpha_j^\vee + \delta_{ij}$

$\tau_i$ 's are the exponential of the canonical conjugate variables of the parameter variables  $\alpha_i^\vee$ .

# Weyl group action

Weyl group:  $W = \langle s_i \mid i \in I \rangle$ .

- $s_i^2 = 1$ ,
- $a_{ij}a_{ji} = 0 \implies s_i s_j = s_j s_i$ ,
- $a_{ij}a_{ji} = 1 \implies s_i s_j s_i = s_j s_i s_j$ ,
- $a_{ij}a_{ji} = 2 \implies (s_i s_j)^2 = (s_j s_i)^2$ ,
- $a_{ij}a_{ji} = 3 \implies (s_i s_j)^3 = (s_j s_i)^3$ .

$$[A, B]_q := AB - qBA.$$

$$(\mathbf{ad}_q f_i)(x) := [f_i, x]_{q_i^{\langle \alpha_i^\vee, \beta \rangle}}, \text{ where } \beta = \text{the weight of } x.$$

$$\text{Then } (\mathbf{ad}_q f_i)^{k+1}(f_j) = [f_i, (\mathbf{ad}_q f_i)^k(f_j)]_{q^{2k+a_{ij}}}.$$

## Weyl group action (Bäcklund transformations):

- $s_i(f_i) := f_i,$
- $s_i(f_j) := \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee - k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\mathbf{ad}_q f_i)^k (f_j) f_i^{-k} \quad (i \neq j)$
- $s_i(\alpha_j^\vee) := \alpha_j^\vee - a_{ji} \alpha_i^\vee,$
- $s_i(\tau_i) := f_i \tau_i \prod_{j \in I} \tau_j^{-a_{ij}} = f_i \tau_i^{-1} \prod_{j \neq i} \tau_j^{-a_{ij}},$
- $s_i(\tau_j) := \tau_j \quad (i \neq j).$



# The action of $s_i$ is an algebra automorphism.

**Proof.** We can define the algebra automorphism  $\tilde{s}_i$  by

$$\tilde{s}_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji}\alpha_i^\vee,$$

$$\tilde{s}_i(\tau_i) = \tau_i \prod_{j \in I} \tau_j^{-a_{ij}}, \quad \tilde{s}_i(\tau_j) = \tau_j \quad (i \neq j),$$

$$\tilde{s}_i(f_j) = f_j.$$

Then we obtain, for  $x = f_j, \alpha_j^\vee, \tau_i$ ,

$$s_i(x) = f_i^{\alpha_i^\vee} \tilde{s}_i(x) f_i^{-\alpha_i^\vee}.$$

This is an algebra automorphism. □

# Useful formulas

$$f_i^\lambda f_j f_i^{-\lambda} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\lambda-k)} \begin{bmatrix} \lambda \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q f_i)^k (f_j) f_i^{-k} \quad (i \neq j).$$

$$s_i(f_j) = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee}.$$

If  $a_{ij} = -1$ , then

$$\begin{aligned} s_i(f_j) &= q_i^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_{q_i} (f_i f_j - q_i^{-1} f_j f_i) f_i^{-1} \\ &= [1 - \alpha_i^\vee]_{q_i} f_j + [\alpha_i^\vee]_{q_i} f_i f_j f_i^{-1}. \end{aligned}$$

Therefore

$$s_i(f_j) f_i = [1 - \alpha_i^\vee]_{q_i} f_j f_i + [\alpha_i^\vee]_{q_i} f_i f_j.$$

# Quantum $\tau$ -functions: Definition

**Fundamental weights:**  $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$ .

**Weight lattice:**  $P := \bigoplus_{i \in I} \mathbb{Z} \Lambda_i$ ,  $P_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i$ .

**simple roots:**  $\alpha_j := \sum_{i \in I} a_{ij} \Lambda_i$

**Weyl group action on  $P$ :**  $s_i(\Lambda_j) = \Lambda_j - \delta_{ij} \alpha_i$ .

**$\tau$ -monomial:**  $\tau^\mu := \prod_{i \in I} \tau_i^{\mu_i}$  ( $\mu = \sum_{i \in I} \mu_i \Lambda_i \in P_+$ )

**(lattice) quantum  $\tau$ -functions:**

$$\tau(\lambda) := w(\tau^\mu) \quad \text{for } \lambda = w(\mu) \in WP_+.$$

# Quantum $\tau$ -functions: Regularity

**Regularity Theorem:** All quantum  $\tau$ -functions  $\tau(\lambda)$  ( $\lambda \in WP_+$ ) are (non-commutative) polynomials in the dependent variables  $f_i$ . □

Main theorem of arXiv:1206.3419.

# Proof of the regularity theorem

$$\rho := \sum_{i \in I} \Lambda_i, w \circ \lambda := w(\lambda + \rho) - \rho \quad (\lambda \in P, w \in W).$$

Assume  $\lambda, \mu \in P_+$  and  $w \in W$ .

$L(\mu)$ : highest weight simple module.

$M(w \circ \lambda)$ : Verma module with highest weight  $w \circ \lambda$ .

$$M(w \circ \lambda) \subset M(\lambda).$$

**Translation functor:**  $T_\lambda^\mu(M(w \circ \lambda)) \subset M(w \circ \lambda) \otimes L(\mu).$

**Sketch of the proof:**  $T_\lambda^\mu(M(w \circ \lambda)) \cong M(w \circ (\lambda + \mu))$   
implies the regularity theorem. □

Non-trivial relation between the theory of quantum  $\tau$ -functions and representation theory!

# $A_{n-1}^{(1)}$ -case $(n \geq 3)$

$$i, j \in \mathbb{Z}/n\mathbb{Z}$$

$$a_{ij} = \begin{cases} 2 & (i = j) \\ -1 & (i - j = \pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$d_i = 1, \quad q_i = q.$$

# Quantum algebra

Consider the associative algebra generated by

- dependent variables:  $f_i$
- parameter variables:  $\alpha_i^\vee$
- $\tau$ -variables:  $\tau_i$  ( $i \in \mathbb{Z}/n\mathbb{Z}$ )

with the defining relations

- $f_i^2 f_{i\pm 1} - (q + q^{-1}) f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0.$
- $f_i f_j = f_j f_i \quad (j \neq i \pm 1).$
- $\alpha_i^\vee$  commutes with  $\alpha_j^\vee$  and  $f_j.$
- $\tau_i = \exp(\partial / \partial \alpha_i^\vee).$

# Weyl group action

Using the useful formulas above, we obtain the following formulas:

- $s_i(f_{i\pm 1}) = [1 - \alpha_i^\vee]_q f_{i\pm 1} + [\alpha_i^\vee]_q f_i f_{i\pm 1} f_i^{-1},$   
 $s_i(f_j) = f_j \quad (j \neq i \pm 1).$
- $s_i(\alpha_i^\vee) = -\alpha_i^\vee, \quad s_i(\alpha_{i\pm 1}^\vee) = \alpha_{i\pm 1}^\vee + \alpha_i^\vee,$   
 $s_i(\alpha_j^\vee) = \alpha_j^\vee \quad (j \neq i, i \pm 1).$
- $s_i(\tau_i) = f_i \frac{\tau_{i-1} \tau_{i+1}}{\tau_i}, \quad s_i(\tau_j) = \tau_j \quad (i \neq j).$



# Extension of the weight lattice

$$Q^\vee := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i^\vee \oplus \mathbb{Z}\delta^\vee, \quad P := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i \oplus \mathbb{Z}\Lambda_0.$$

Dual bases:  $\varepsilon_i^\vee, \delta^\vee \longleftrightarrow \varepsilon_i, \Lambda_0$ .

Assume  $\varepsilon_i^\vee = \varepsilon_{i+n}^\vee + \delta^\vee$  and  $\varepsilon_{i+n} = \varepsilon_i$ .

$$\alpha_i^\vee := \varepsilon_i^\vee - \varepsilon_{i+1}^\vee, \quad \alpha_i := \varepsilon_i - \varepsilon_{i+1}.$$

$$\Lambda_i = \Lambda_0 + \varepsilon_1 + \cdots + \varepsilon_i \quad (i \in \mathbb{Z}_{\geq 0}).$$

$$\text{Then } P = \bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\varpi_n, \quad \varpi_n = \varepsilon_1 + \cdots + \varepsilon_n$$

$$P_+ := \sum_{i=0}^{n-1} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{Z}\varpi_n.$$

$$\text{Assume } \Lambda_{i+n} = \Lambda_i + \varpi_n \quad (i \in \mathbb{Z}).$$

# Extended affine Weyl group

$$W = W(A_{n-1}^{(1)}) = \langle s_0, s_1, \dots, s_{n-1} \rangle, \quad s_{i+n} = s_i.$$

$$\pi(s_i) := s_{i+1},$$

$$\widetilde{W} = \widetilde{W}(A_{n-1}^{(1)}) = \langle \pi \rangle \ltimes W: \text{the extended aff. Weyl gr.}$$

Assume  $\lambda \in P$  and  $\beta^\vee \in Q^\vee$ .

$$s_i(\lambda) := \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i, \quad \alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}.$$

$$s_i(\beta^\vee) := \beta^\vee - \langle \beta^\vee, \alpha_i \rangle \alpha_i^\vee$$

$$\pi(\Lambda_i) := \Lambda_{i+1}, \quad \pi(\varpi_n) := \varpi_n.$$

$$\pi(\varepsilon_i^\vee) := \varepsilon_{i+1}^\vee, \quad \pi(\delta^\vee) := \delta^\vee.$$

# Translation part of $\widetilde{W}$

$$T_i := s_{i-1} \cdots s_2 s_1 \pi s_{n-1} s_{n-2} \cdots s_i \in \widetilde{W} \quad (i = 1, \dots, n).$$

$$\text{Assume } \nu = \sum_{i=1}^n \nu_i \varepsilon_i \in \bigoplus_{i=1}^n \mathbb{Z} \varepsilon_i.$$

$$T^\nu := \prod_{i=0}^{n-1} T_i^{\nu_i}.$$

Then

$$T^\nu(\varepsilon_i^\vee) = \varepsilon_i^\vee - \nu_i \delta^\vee \quad T^\nu(\delta^\vee) = \delta^\vee,$$

$$T^\nu(\alpha_i^\vee) = \alpha_i^\vee - (\nu_i - \nu_{i+1}) \delta^\vee,$$

$$T^\nu(\varepsilon_i) = \varepsilon_i, \quad T^\nu(\Lambda_0) = \Lambda_0 + \nu,$$

$$T^\nu(\Lambda_i) = \Lambda_i + \nu.$$

# Hirota-Miwa equation (1)

$$\Lambda_i = \Lambda_{i-1} + \varepsilon_i,$$

$$\Lambda_{i+1} = \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1},$$

$$s_i(\Lambda_i) = \Lambda_{i-1} + \varepsilon_{i+1},$$

$$s_{i+1}(\Lambda_{i+1}) = \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2},$$

$$s_{i+1}s_i(\Lambda_i) = \Lambda_{i-1} + \varepsilon_{i+2},$$

$$s_i s_{i+1}(\Lambda_{i+1}) = \Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}.$$

$$\tau_i = \tau(\Lambda_{i-1} + \varepsilon_i),$$

$$\tau_{i+1} = \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1}),$$

$$s_i(\tau_i) = \tau(\Lambda_{i-1} + \varepsilon_{i+1}),$$

$$s_{i+1}(\tau_{i+1}) = \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2}),$$

$$s_{i+1}s_i(\tau_i) = \tau(\Lambda_{i-1} + \varepsilon_{i+2}),$$

$$s_i s_{i+1}(\tau_{i+1}) = \tau(\Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}).$$

**Lemma:**

$$\begin{aligned} & [\alpha_{i+1}^\vee]_q \tau_i s_i s_{i+1}(\tau_{i+1}) + [\alpha_i^\vee]_q s_{i+1} s_i(\tau_i) \tau_{i+1} \\ &= [\alpha_i^\vee + \alpha_{i+1}^\vee]_q s_i(\tau_i) s_{i+1}(\tau_{i+1}). \end{aligned}$$

# Hirota-Miwa equation (2) Proof of Lemma

**Warning:**  $\tau_i$  does not commute with  $s_i s_{i+1}(\tau_{i+1})$ .

$$\tau_i [\alpha_i^\vee]_q = [\alpha_i^\vee + 1]_q \tau_i, \quad \tau_i [1 - \alpha_i^\vee]_q = -[\alpha_i^\vee]_q \tau_i.$$

**Proof of Lemma:**

$$\begin{aligned} & \tau_i s_i s_{i+1}(\tau_i) \\ &= \tau_i s_i \left( f_{i+1} \frac{\tau_i \tau_{i+2}}{\tau_{i+1}} \right) \\ &= \tau_i ([1 - \alpha_i^\vee]_q f_{i+1} + [\alpha_i^\vee]_q f_i f_{i+1} f_i^{-1}) f_i \frac{\tau_{i-1} \tau_{i+1}}{\tau_i} \frac{\tau_{i+2}}{\tau_{i+1}}, \\ &= (-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1}) \tau_{i-1} \tau_{i+2}. \end{aligned}$$

Thus

$$\tau_i s_i s_{i+1}(\tau_i) = (-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1}) \tau_{i-1} \tau_{i+2}.$$

Similarly we obtain

$$s_{i+1} s_i(\tau_i) \tau_{i+1} = ([1 - \alpha_{i+1}^\vee]_q f_i f_{i+1} + [\alpha_{i+1}^\vee]_q f_{i+1} f_i) \tau_{i-1} \tau_{i+2}$$

$$s_i(\tau_i) s_{i+1}(\tau_{i+1}) = f_i f_{i+1} \tau_{i-1} \tau_{i+2}.$$

$q$ -numbers identity (or addition formula of **sin**):

$$[\alpha_i^\vee + 1]_q [\alpha_{i+1}^\vee]_q + [\alpha_i^\vee]_q [1 - \alpha_{i+1}^\vee]_q = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q.$$

The lemma follows from these calculations. □

# Hirota-Miwa equation (3)

Apply  $T^\nu$  to the formula of Lemma. Then we obtain

**Theorem:** The quantum  $\tau$ -functions of type  $A_{n-1}^{(1)}$  satisfy the quantum  $q$ -Hirota-Miwa equations:

$$\begin{aligned} & [\varepsilon_i^\vee(\nu) - \varepsilon_{i+1}^\vee(\nu)]_q \tau_i(\nu + \varepsilon_{i+2}) \tau_i(\nu + \varepsilon_i + \varepsilon_{i+1}) \\ & + [\varepsilon_{i+1}^\vee(\nu) - \varepsilon_{i+2}^\vee(\nu)]_q \tau_i(\nu + \varepsilon_i) \tau_i(\nu + \varepsilon_{i+1} + \varepsilon_{i+2}) \\ & + [\varepsilon_{i+2}^\vee(\nu) - \varepsilon_i^\vee(\nu)]_q \tau_i(\nu + \varepsilon_{i+1}) \tau_i(\nu + \varepsilon_{i+2} + \varepsilon_i) = 0 \end{aligned}$$

where

$$\begin{aligned} \tau_i(\nu) &:= \tau(\Lambda_{i-1} + \nu), \\ \varepsilon_i^\vee(\nu) &:= T^\nu(\varepsilon_i^\vee) = \varepsilon_i^\vee - \nu_i \delta^\vee. \quad \square \end{aligned}$$

# Lax and Sato-Wilson forms of the Weyl group action

The relation between the  $RLL = LLR$  formalism of quantum groups and the Lax and Sato-Wilson forms of the Painlevé systems is non-trivial.

Assume that  $m$  and  $n$  are mutually prime.



# Lax form: $RLL=LLR$

$A_{m-1}^{(1)}$ -type  $R$ -matrix:

$$\begin{aligned} R(z) = & (q - q^{-1}z) \sum_{i=1}^m E_{ii} \otimes E_{ii} + (1 - z) \sum_{i \neq j} E_{ii} \otimes E_{jj} \\ & + (q - q^{-1}) \sum_{i < j} (E_{ij} \otimes E_{ji} + z E_{ji} \otimes E_{ij}). \end{aligned}$$

Local  $L$ -matrices: for  $k = 1, \dots, n$ ,

$$L_k(z) = \sum_{i=1}^m a_{ik} E_{ii} + \sum_{i=1}^{m-1} b_{ik} E_{i,i+1} + z b_{mk} E_{m1}.$$

$$L_k(z)^1 := L_k(z) \otimes 1, \quad L_k(z)^2 := 1 \otimes L_k(z).$$

Fundamental relations:

$$\begin{aligned} R(z/w)L_k(z)^1L_k(w)^2 &= L_k(w)^2L_k(z)^1R(z/w), \\ L_k(z)^1L_l(w)^2 &= L_l(w)^2L_k(z)^1 \quad (k \neq l). \end{aligned}$$

Equivalent to the  $q$ -commutation relations:

$$\begin{aligned} a_{ik}b_{ik} &= q^{-1}b_{ik}a_{ik}, & a_{ik}b_{i+1,k} &= qb_{i+1,k}a_{ik}, \\ a_{ik}a_{jk} &= a_{jk}a_{ik}, & b_{ik}b_{jk} &= b_{jk}b_{ik}, \quad \text{etc.} \end{aligned}$$

If  $k \neq l$ , then  $a_{ik}$  and  $b_{ik}$  commute with  $a_{jl}$  and  $b_{jl}$ .

Another form of the bidiagonal matrix  $L_k(z)$ .

$$a_k := \text{diag}(a_{1k}, \dots, a_{mk}),$$

$$b_k := \text{diag}(b_{1k}, \dots, b_{mk}),$$

$$\Lambda(z) := \sum_{i=1}^m E_{i,i+1} + zE_{m1} \quad (\text{shift matrix}).$$

Then

$$L_k(z) = a_k + b_k \Lambda(z).$$

## Lax form: $\widehat{L}(z)$

1.  $L(z) := L_1(z) \cdots L_n(z)$ , the global  $L$ -operator.

$$\tilde{a}_i := a_{i1} \cdots a_{in}.$$

$\tilde{a} := \text{diag}(\tilde{a}_1, \dots, \tilde{a}_m)$ , the diagonal part of  $L(z)$ .

2.  $\widetilde{L}(z) := L(z)\tilde{a}$ , doubling the diagonal part of  $L(z)$ .

3.  $\widehat{L}(z) := \widetilde{C}\widetilde{L}(z)\widetilde{C}^{-1}$ .

Here  $\widetilde{C} = \text{diag}(\tilde{c}_1, \dots, \tilde{c}_m)$  is uniquely characterized by

$$\tilde{c}_1 = 1,$$

$$\widehat{L}(z) = \hat{\ell}_0 + \hat{\ell}_1 \Lambda(z) + \cdots + \Lambda(z) \Lambda(rz) \cdots \Lambda(r^{n-1}z),$$

where  $\hat{\ell}_0, \hat{\ell}_1, \dots$  are diagonal matrices.

$$\widehat{L}(z) = \hat{\ell}_0 + \hat{\ell}_1 \Lambda(z) + \cdots + \Lambda(z) \Lambda(rz) \cdots \Lambda(r^{n-1}z).$$

$$t = \text{diag}(t_1, \dots, t_n) := \tilde{c} \tilde{a} \tilde{c}^{-1}.$$

Then  $\hat{\ell}_0 = t^2$  and  $t_i \hat{\ell}_j = \hat{\ell}_j t_i$ .

Define  $\hat{b}_i$  and  $\hat{f}_i$  by

$$\hat{\ell}_1 = \text{diag}(\hat{b}_i)_{i=1}^n = \text{diag} \left( (q^{-1} - q) t_i t_{i+1} \hat{f}_i \right)_{i=1}^n.$$

$$r \widehat{L}(z) = \widehat{L}(z) r.$$

Assume  $t_{i+n} = r^{-1} t_i$  and  $\hat{f}_{i+n} = r \hat{f}_i$ .

**Example ( $qP_{\text{IV}}$  case):  $(m, n) = (3, 2)$ .**

$$\widehat{L}(z) = \begin{bmatrix} t_1^2 & (q^{-1} - q)t_1t_2\hat{f}_1 & 1 \\ rz & t_2^2 & (q^{-1} - q)t_2t_3\hat{f}_2 \\ rz(q^{-1} - q)t_3t_4\hat{f}_3 & z & t_3^2 \end{bmatrix}.$$

Assume  $\widetilde{L}(z) = A + B\Lambda(z) + C\Lambda(z)^2$ ,  $A, B, C$  are diagonal, and  $C = \text{diag}(c_1, c_2, c_3)$ . Then

$$c_i = b_{i1}b_{i+1,2}a_{i+2,1}a_{i+2,2},$$

$$\tilde{c}_1 = 1, \quad \tilde{c}_3 = c_1, \quad \tilde{c}_2 = c_1c_3, \quad r = c_1c_3c_2.$$

$$\widetilde{C} = \text{diag}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3), \quad \widehat{L}(z) = \widetilde{C}\widetilde{L}(z)\widetilde{C}^{-1}, \quad r\widehat{L}(z) = \widehat{L}(z)r.$$

# Lax form: $\widehat{M}(z)$

$T_{z,r} := r^{z\partial/\partial z} : f(z) \mapsto f(rz)$ ,  $r$ -difference operator.

4.  $\widehat{M}(z) := \widehat{L}(z)T_{z,r}^n$ , matrix coefficient  $r$ -difference op.

5. Assume  $t_i = q^{-\varepsilon_i^\vee}$  and  $r = q^{-\delta^\vee}$ .

Then  $(q - q^{-1})[\alpha_i^\vee]_q = t_{i+1}/t_i - t_i/t_{i+1}$

6.  $g_i := (t_i^2 - t_{i+1}^2)/\hat{b}_i = [\alpha_i^\vee]_q/\hat{f}_i$ .

$G_i := g_i E_{i+1,i} \quad (i = 1, \dots, n-1).$

$(G_n(z) := rz^{-1}g_n E_{1n}.)$

# Lax form: Weyl group action

Consider the algebra generated by the matrix elements of  $\widehat{L}(z)$  (precisely of  $\widehat{\ell}_0, \dots, \widehat{\ell}_{n-1}$ ).

7. Algebra automorphism Weyl group action:

$$s_i(\widehat{M}(z)) := G_i \widehat{M}(z) G_i^{-1},$$

$$\begin{aligned} \pi(\widehat{M}(z)) &:= (\Lambda(z) T_{z,r}) \widehat{M}(z) (\Lambda(z) T_{z,r})^{-1} \\ &= \Lambda(z) \widehat{L}(rz) \Lambda(r^n z) T_{z,r}^n. \end{aligned}$$

Then

$$s_i(t_i) = t_{i+1}, \quad s_i(t_{i+1}) = t_i,$$

$$s_i(\widehat{b}_i) = \widehat{b}_i, \quad s_i(\widehat{b}_{i\pm 1}) = \widehat{b}_{i\pm 1} \pm (t_i^2 - t_{i+1}^2)/\widehat{b}_i.$$



# Sato-Wilson forms: $z$ -variables

8. Introduce  $\tau_0$  and  $z_i$  by

- $\tau_0 = q^{-r\partial/\partial r}$ :  $\tau_0 r = q^{-1} r \tau_0$ ,  $\tau_0 t_j = t_j \tau_0$ .
- $z_i = q^{-t_i \partial/\partial t_i}$ :  $z_i r = r z_i$ ,  $z_i t_j = q^{-\delta_{ij}} t_j z_i$ .
- $\tau_0$  and  $z_i$  commute with  $\tau_0$ ,  $z_j$ ,  $\hat{f}_j$ .

9.  $D_Z := \text{diag}(z_1, \dots, z_n)$ ,  $Z(z) := U(z)D_Z$ , where

$$U(z) = E + \sum_{k=1}^{\infty} u_k \Lambda(z)^k,$$

$u_1, u_2, \dots$  are diagonal matrices,

$$\widehat{M}(z) = U(z) t^2 T_{z,r}^n U(z)^{-1}.$$

Then

$$\widehat{M}(z) = Z(z) (qt)^2 T_{z,r}^n Z(z)^{-1}.$$

# Sato-Wilson form: Weyl group action

10. The Weyl group action can be extended by

$$s_i(U(z)) = G_i U(z) S_i^g, \quad s_i(D_Z) = (S_i^g)^{-1} D_Z S_i,$$

$$s_i(t) = S_i^{-1} t S_i, \quad \textcolor{red}{s_i(Z(z)) = G_i(z) Z(z) S_i},$$

$$\pi(A(z)) = (\Lambda(z) T_{z,r}) A(z) (\Lambda(z) T_{z,r})^{-1},$$

$$(A(z) = U(z), D_Z, t, Z(z))$$

where  $g_i = (t_i^2 - t_{i+1}^2) / \hat{b}_i = [\alpha_i^\vee]_q / \hat{f}_i,$

$$S_i^g := g_i^{-1} E_{i,i+1} - g_i E_{i+1,i} + \sum_{j \neq i, i+1} E_{jj},$$

$$S_i := [\alpha_i^\vee + 1]_q E_{i,i+1} - [\alpha_i^\vee - 1]_q^{-1} E_{i+1,i} + \sum_{j \neq i, i+1} E_{jj}.$$

$S^g$  and  $S_i$  are permutation matrices  $i \leftrightarrow i + 1$ .

11. Assume  $z_{j+m} = z_j$ ,  $\tau_j = \tau_{j-1}z_i$ ,  
and  $s_i(\tau_0) = \tau_0$  ( $i = 1, 2$ ). Then, for  $i = 1, 2$ ,

$$s_i(z_i) = \hat{f}_i z_{i+1}, \quad s_i(z_{i+1}) = \hat{f}_i^{-1} z_i, \quad s_i(\tau_i) = \hat{f}_i \frac{\tau_{i-1} \tau_{i+1}}{\tau_i},$$

$$\pi(z_i) = z_{i+1}, \quad \pi(\tau_i) = \tau_{i+1}.$$

Because  $g_i = [\alpha_i^\vee]_q / \hat{f}_i$  and  $z_i \epsilon_j^\vee = (\epsilon_j^\vee + \delta_{ij}) z_i$  implies

$$\begin{aligned} & \begin{bmatrix} 0 & g_i \\ -g_i^{-1} & 0 \end{bmatrix}^{-1} \begin{bmatrix} z_i & 0 \\ 0 & z_{i+1} \end{bmatrix} \begin{bmatrix} 0 & [\alpha_i^\vee + 1]_q \\ -[\alpha_i^\vee - 1]_q^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} g_i^{-1} z_{i+1} [\alpha_i^\vee + 1]_q & 0 \\ 0 & g_i z_i [\alpha_i^\vee - 1]_q^{-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_i z_{i+1} & 0 \\ 0 & \hat{f}_i^{-1} z_i \end{bmatrix}. \quad \square \end{aligned}$$

# Quantum $qP_{IV}$

Both canonically quantized and  $q$ -difference.

## $(m, n)$ -case: $X_k(z)$

Assume that  $m$  and  $n$  are mutually prime ( $\text{gcm} = 1$ ).

There exist unique diagonal matrices  $\tilde{C}_1, \dots, \tilde{C}_1$  such that  $\tilde{C}_1 = \tilde{C}_{n+1} = \tilde{C}$  and, for  $k = 1, \dots, n-1$ ,

$$X_k(r^{k-1}z) := \tilde{C}_k L_k(z) \tilde{C}_{k+1}^{-1} = x_k + \Lambda(r^{k-1}z),$$

$$X_n(r^{n-1}z) := \tilde{C}_n L_n(z) \tilde{C}_1^{-1} = x_n + \Lambda(r^{n-1}z),$$

$$x_k = \text{diag}(x_{1k}, \dots, x_{mk}).$$

Then

$$\widehat{L}(z) = X_1(z) X_2(rz) \cdots X_n(r^{n-1}z).$$

Assume  $x_{i+m,k} = r^{-1}x_{ik}$  and  $x_{i,k+n} = x_{ik}$ .

## $(m, n)$ -case: $q$ -commutation relations of $x_{ik}$

**Theorem.**  $x_{ik}x_{jl} = q_{j-i, l-k}^{(m,n)} x_{jl}x_{ik}$ ,  $q_{\mu\nu}^{(m,n)} \in \{1, q^{\pm 2}\}$ .  $\square$

**Example.** If  $(m, n) = (2g + 1, 2)$  and  $x_i := x_{i1}$ ,  $y_i := x_{i2}$ , then

$$x_i y_i = y_i x_i = t_i^2,$$

$$x_i x_{i+\mu} = q^{(-1)^{\mu-1}2} x_{i+\mu} x_i, \quad x_i y_{i+\mu} = q^{-(-1)^{\mu-1}2} y_{i+\mu} x_i,$$

$$y_i y_{i+\mu} = q^{(-1)^{\mu-1}2} y_{i+\mu} y_i, \quad y_i x_{i+\mu} = q^{-(-1)^{\mu-1}2} x_{i+\mu} y_i,$$

$t_i$  commutes with  $t_j, x_j, y_j$ .  $\square$

**Example.**  $(m, n) = (5, 3), (3, 5)$ .  $x_{ik}x_{jl} = q_{j-i, l-k}^{(m,n)} x_{jl}x_{ik}$ ,  
 where  $q_{\mu+m, \nu} = q_{\mu\nu}$ ,  $q_{\mu, \nu+n} = q_{\mu\nu}$ , and

$$\begin{bmatrix} q_{\mu\nu}^{(5,3)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & q^{-2} & q^2 & \mathbf{1} \\ q^{-2} & \mathbf{q^2} & 1 & q^{-2} & q^2 \\ q^2 & q^{-2} & q^2 & 1 & q^{-2} \end{bmatrix},$$

$$\begin{bmatrix} q_{\mu\nu}^{(3,5)} \end{bmatrix} = \begin{bmatrix} 1 & q^{-2} & q^2 \\ 1 & \mathbf{q^2} & q^{-2} \\ q^{-2} & 1 & q^2 \\ q^2 & q^{-2} & 1 \\ \mathbf{1} & q^2 & q^{-2} \end{bmatrix}.$$

Note that  $q_{\mu\nu}^{(5,3)} = q_{\nu\mu}^{(3,5)}$ .  $\square$

Assume that  $0 < \tilde{m} < n$  and  $m\tilde{m} \equiv 1 \pmod{n}$ .

Assume that  $0 < \tilde{n} < n$  and  $n\tilde{n} \equiv 1 \pmod{m}$ .

**Theorem.** Define  $B^{(m,n)}$  and  $p_{\mu\nu}^{(m,n)}$  by

$$B^{(m,n)} := \{ (\mu \bmod m, \mu \bmod n) \mid 0 \leq \mu < \tilde{m}m \}.$$

$$p_{\mu\nu}^{(m,n)} := \begin{cases} q & \text{if } (\mu \bmod m, \nu \bmod n) \in B, \\ 1 & \text{if } (\mu \bmod m, \nu \bmod n) \notin B. \end{cases}$$

Then

$$q_{\mu\nu}^{(m,n)} = (p_{\mu\nu} / p_{\mu-1,\nu})^2 \in \{1, q^{\pm 2}\}.$$

□

**Cor. (duality)**  $q_{\mu\nu}^{(m,n)} = q_{\nu\mu}^{(n,m)}$ .

□



## $(m, n)$ -case: Weyl group action

The action of  $\widetilde{W}(A_{m-1}^{(1)})$  on  $t_i, \hat{f}_i, \tau_i$ , etc. can be extended to the one on  $x_{ik}$ .

Using the duality above, we can construct the action of  $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$  on  $x_{ik}$ .

We shall write  $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$  as

$$\widetilde{W}(A_{m-1}^{(1)}) = \langle \pi, s_0, \dots, s_{m-1} \rangle,$$

$$\widetilde{W}(A_{n-1}^{(1)}) = \langle \varpi, r_0, \dots, r_{n-1} \rangle.$$

# $qP_{IV}$ -case: Weyl group action

**Example ( $qP_{IV}$ -case)  $(m, n) = (3, 2)$ .**     $x_i := x_{i1}, \quad y_i := x_{i2}.$

$$s_i(x_i) = (x_i + y_{i+1})x_{i+1}(y_i + x_{i+1})^{-1},$$

$$s_i(x_{i+1}) = (x_i + y_{i+1})^{-1}x_i(y_i + x_{i+1}),$$

$$s_i(y_i) = (y_i + x_{i+1})y_{i+1}(x_i + y_{i+1})^{-1},$$

$$s_i(y_{i+1}) = (y_i + x_{i+1})^{-1}y_i(x_i + y_{i+1}),$$

$$s_i(x_{i+2}) = x_{i+2}, \quad s_i(y_{i+2}) = y_{i+2},$$

$$s_i(t_i) = t_{i+1}, \quad s_i(t_{i+1}) = t_i, \quad s_i(t_{j+2}) = t_{j+2},$$

$$\pi(x_i) = x_{i+1}, \quad \pi(y_i) = y_{i+1}, \quad \pi(t_i) = t_{i+1}.$$

$$Q_i := y_{i+2}y_{i+1} + y_{i+2}x_i + x_{i+1}x_i,$$

$$r_1(x_i) = r^{-1}Q_{i+1}^{-1}y_iQ_i,$$

$$r_1(y_i) = rQ_{i+1}x_iQ_i^{-1},$$

$$r_1(t_i) = t_i, \quad \varpi(x_i) = y_i, \quad \varpi(y_i) = x_i, \quad \varpi(t_i) = t_i.$$

## $qP_{IV}$ -case: Lax form

$$G'_i := \varpi(G_i).$$

Then

$$s_i(X(z)) = G_i X(z) G_i'^{-1},$$

$$s_i(Y(z)) = G_i' Y(z) G_i^{-1},$$

$$\pi(X(z)) = \Lambda(z) X(z) \Lambda(z)^{-1}$$

$$r_1(X(z)Y(rz)) = X(z)Y(rz),$$

$$r_1 : x_{i+2}x_{i+1}x_i \leftrightarrow y_{i+3}y_{i+2}y_{i+1},$$

$$\varpi : X(z) \leftrightarrow Y(z).$$

These relations uniquely characterizes the actions.

# $qP_{IV}$ -case: discrete time evolution

$$U_1 := r_1 \varpi \in \widetilde{W}(A_1^{(1)}).$$

The  $U_1$ -action is the discrete time evolution of  $qP_{IV}$  and the  $\widetilde{W}(A_2^{(1)})$ -action is its symmetry.

$$a_i := t_i/t_{i+1} \text{ and } F_i := x_{i+1}x_i/(t_{i+1}t_i).$$

Then

$$F_i F_{i+1} = q^2 F_{i+1} F_i,$$

$$a_i a_j = a_j a_i, a_i F_j = F_j a_i,$$

$$F_{i+3} = F_i, a_{i+3} = a_i.$$

## Time evolution of quantized $qP_{IV}$ .

$$\begin{aligned}U_1(F_i) &= (1 + q^2 a_{i-1} F_{i-1} + q^2 a_{i-1} a_i F_{i-1} F_i) \\&\quad \times a_i a_{i+1} F_{i+1} \\&\quad \times (1 + q^2 a_i F_i + q^2 a_i a_{i+1} F_i F_{i+1})^{-1}, \\U_1(a_i) &= a_i.\end{aligned}$$

### Classical case:

Kajiwara-Noumi-Yamada arXiv:nlin/0012063

$$\begin{aligned}\overline{F_i} &= a_i a_{i+1} F_{i+1} \frac{1 + a_{i-1} F_{i-1} + a_{i-1} a_i F_{i-1} F_i}{1 + a_i F_i + a_i a_{i+1} F_i F_{i+1}}, \\ \overline{a_i} &= a_i.\end{aligned}$$

# $qP_{IV}$ -case: symmetry

## Symmetry of quantum $qP_{IV}$ .

$$s_i(F_i) = F_i,$$

$$s_i(F_{i-1}) = F_{i-1} \frac{a_i + F_i}{1 + a_i F_i}, \quad s_i(F_{i+1}) = \frac{1 + a_i F_i}{a_i + F_i} F_{i+1},$$

$$s_i(a_i) = a_i^{-1}, \quad s_i(a_{i\pm 1}) = a_i a_{i\pm 1}.$$

These formulas coincide with the ones obtained by Koji Hasegawa [arXiv:0703036](#), which quantizes Kajiwar-Noumi-Yamada [arXiv:nlin/0012063](#).