Quantum Painlevé tau-functions

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Summary

- For any symmetrizable GCM, we can introduce the proper non-commutativity of quantum τ-functions by $\tau_i = \exp(\partial/\partial\alpha_i^\vee)$.
- Quantum q-Hirota-Miwa equations for $A_{n-1}^{(1)}$-case.
- Quantized Lax and Sato-Wilson forms of the extended affine Weyl group action for $A_{n-1}^{(1)}$-case.
- Quantized $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$-action for mutually prime m and n.
- An appropriate quantization of $q\mathrm{P_{IV}}$.

General theory of the quantum and q-difference version of τ-functions generated by the Weyl group action for any symmetrizable GCM

We will consider the birational action of the Weyl group (Bäcklund transformations).

Want to construct quantizations of classical τ-functions of Painlevé systems (differrential and q-difference).

Difficulty. How to find the proper non-commutativity of quantized τ-functions?

My Answer.

- parameter variable $\alpha_i^\vee$ $\leftrightarrow$ simple coroot.
- classical τ_i $\leftrightarrow$ exp(fundamental weight)
- In the situation above, the appropriate definition of quantized τ_i is

$$\tau_i = \exp(\partial/\partial\alpha_i^\vee).$$

Quantum Algebra: Definiton

Consider the associative algebra
(precisely the non-commutative field) generated by

- dependent variables: f_i
- parameter variables: $\alpha_i^\vee$
- τ-variables: τ_i

with the relations

- q-Serre relations of f_i.
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$ and f_j.
- τ_i commutes with τ_j and f_j.
- $\tau_i \alpha_j^\vee \tau_i^{-1} = \alpha_j^\vee + \delta_{ij}$. $\quad (\tau_i = \exp(\partial/\partial\alpha_i^\vee))$

Quantum Algebra: q-Serre relations

$[a_{ij}]_{i,j\in I}$: GCM with $d_i a_{ij} = d_j a_{ji}$, $d_i \in \mathbb{Z}_{>0}$.

q: an inderminate.

$$q_i := q^{d_i}, \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}},$$

$$[n]_q! := [1]_q[2]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

q-Serre relations: if $i, j \in I$ and $i \neq j$, then

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0.$$

Quantum Algebra: Relations

- $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j)$,
- $\alpha_i^\vee \alpha_j^\vee = \alpha_j^\vee \alpha_i^\vee$, $\alpha_i^\vee f_j = f_j \alpha_i^\vee$,
- $\tau_i \tau_j = \tau_j \tau_i$, $\tau_i f_j = f_j \tau_i$,
- $\tau_i \alpha_j^\vee \tau^{-1} = \alpha_j^\vee + \delta_{ij}$

τ_i's are the exponentials of the canonical conjugate variables of the parameter variables $\alpha_i^\vee$.

Quantum Algebra: Summary

- f_i $(i \in I) \leftrightarrow$ Chevalley generators of $U_q(\mathfrak{n}_-)$
- $\alpha_i^\vee \leftrightarrow$ simple coroot
- $\tau_i \leftrightarrow$ exp(fundamental weight)

- f_i $(i \in I)$ satisfy the q-Serre relations.
- $\alpha_i^\vee$ and τ_i commute with f_i.
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$.
- τ_i commutes with τ_j.
- $\tau_i \alpha_j^\vee = (\alpha_j^\vee + \delta_{ij})\tau_i \quad (\tau_i = \exp(\partial/\partial\alpha_i^\vee))$.

Weyl group action

Weyl group: $W = \langle s_i \mid i \in I \rangle$.

- $s_i^2 = 1$,
- $a_{ij}a_{ji} = 0 \implies s_is_j = s_js_i$,
- $a_{ij}a_{ji} = 1 \implies s_is_js_i = s_js_is_j$,
- $a_{ij}a_{ji} = 2 \implies (s_is_j)^2 = (s_js_i)^2$,
- $a_{ij}a_{ji} = 3 \implies (s_is_j)^3 = (s_js_i)^3$.

$[A, B]_q := AB - qBA$.

$(\mathrm{ad}_q\, f_i)(x) := [f_i, x]_{q_i^{\langle \alpha_i^\vee, \beta \rangle}}$, where β = the weight of x.

Then $(\mathrm{ad}_q\, f_i)^{k+1}(f_j) = [f_i, (\mathrm{ad}_q\, f_i)^k(f_j)]_{q^{2k+a_{ij}}}$.

Weyl group action (Bäcklund transformations):

- $s_i(f_i) := f_i,$
- $$s_i(f_j) := \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee - k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q f_i)^k (f_j) f_i^{-k} \quad (i \neq j)$$
- $s_i(\alpha_j^\vee) := \alpha_j^\vee - a_{ji}\alpha_i^\vee,$
- $$s_i(\tau_i) := f_i \tau_i \prod_{j \in I} \tau_j^{-a_{ij}} = f_i \tau_i^{-1} \prod_{j \neq i} \tau_j^{-a_{ij}},$$
- $s_i(\tau_j) := \tau_j \quad (i \neq j).$

Remark.

- $\tau_i = \exp(\partial/\partial\alpha_i^\vee) \leftrightarrow$ the fundamental weight Λ_i
- $\tau_i \prod_{j \in I} \tau_j^{-a_{ij}} \leftrightarrow s_i(\Lambda_i) = \Lambda_i - \alpha_i$
  ($\alpha_i = \sum_{j \in I} a_{ji}\Lambda_j$, simple root).

The action of s_i is an algebra automorphism.

Proof. We can define the algebra automorphism $\tilde{s}_i$ by

$$\tilde{s}_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji}\alpha_i^\vee,$$
$$\tilde{s}_i(\tau_i) = \tau_i \prod_{j\in I} \tau_j^{-a_{ij}}, \qquad \tilde{s}_i(\tau_j) = \tau_j \quad (i \neq j),$$
$$\tilde{s}_i(f_j) = f_j.$$

Then we obtain, for $x = f_j, \alpha_j^\vee, \tau_i$,

$$s_i(x) = f_i^{\alpha_i^\vee} \tilde{s}_i(x) f_i^{-\alpha_i^\vee}.$$

This is an algebra automorphism. □

Useful formulas

$$f_i^{\gamma} f_j f_i^{-\gamma} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\gamma-k)} \begin{bmatrix} \gamma \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, f_i)^k (f_j) f_i^{-k} \qquad (i \neq j).$$

$s_i(f_j) = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee}$.

If $a_{ij} = -1$, then

$$\begin{aligned} s_i(f_j) &= q_i^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_{q_i} (f_i f_j - q_i^{-1} f_j f_i) f_i^{-1} \\ &= [1 - \alpha_i^\vee]_{q_i} f_j + [\alpha_i^\vee]_{q_i} f_i f_j f_i^{-1}. \end{aligned}$$

Therefore

$$s_i(f_j) f_i = [1 - \alpha_i^\vee]_{q_i} f_j f_i + [\alpha_i^\vee]_{q_i} f_i f_j.$$

Remark: quantum geometric crystal

Since f_i ($i \in I$) satisfy the Verma relations, for example,

$$f_i^a f_j^{a+b} f_i^b = f_j^b f_i^{a+b} f_j^a \quad \text{if } a_{ij}a_{ji} = 1,$$

we can consider the actions of f_i^γ,

$$e_i(\gamma) : x \mapsto f_i^\gamma x f_i^{-\gamma},$$

as quantum version of a geometric crystal.

For the definition of classical geometric crystal, see Berenstein-Kazhdan `arXiv:math/9912105`, `arXiv:math/0601391`.

Quantum τ-functions: Definition

Fundamental weights: $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$.

Weight lattice: $P := \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$, $P_+ := \sum_{i \in I} \mathbb{Z}_{\geqq 0}\Lambda_i$.

simple roots: $\alpha_j := \sum_{i \in I} a_{ij}\Lambda_i$

Weyl group action on P: $s_i(\Lambda_j) = \Lambda_j - \delta_{ij}\alpha_i$.

τ-monomial: $\tau^\mu := \prod_{i \in I} \tau_i^{\mu_i} \quad (\mu = \sum_{i \in I} \mu_i \Lambda_i \in P_+)$

(lattice) quantum τ-functions:

$$\tau(\lambda) := w(\tau^\mu) \qquad \text{for} \quad \lambda = w(\mu) \in WP_+.$$

Quantum τ-functions: Regularity

Regularity Theorem: All quantum τ-functions $\boldsymbol{\tau(\lambda)}$ ($\lambda \in WP_+$) are (non-commutative) polynomials in the dependent variables f_i. $\square$

Main theorem of `arXiv:1206.3419`.

Proof of the regularity theorem

$\rho := \sum_{i\in I} \Lambda_i$, $w \circ \lambda := w(\lambda+\rho) - \rho$ ($\lambda \in P$, $w \in W$).

Assume $\lambda, \mu \in P_+$ and $w \in W$.

$L(\mu)$: highest weight simple module.
$M(w \circ \lambda)$: Verma module with highest weight $w \circ \lambda$.
$M(w \circ \lambda) \subset M(\lambda)$.

Translation functor: $T_\lambda^\mu(M(\lambda \circ \lambda)) \subset M(w \circ \lambda) \otimes L(\mu)$.

Sketch of the proof: $T_\lambda^\mu(M(w \circ \lambda)) \cong M(w \circ (\lambda + \mu))$ implies the regularity theorem. □

Non-trivial relation between the theory of quantum τ-functions and representation theory!

$A^{(1)}_{n-1}$-case $\quad (n \geqq 3)$

$$i, j \in \mathbb{Z}/n\mathbb{Z}$$

$$a_{ij} = \begin{cases} 2 & (i = j) \\ -1 & (i - j = \pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$d_i = 1, \qquad q_i = q.$$

We will show that the quantum lattice τ-functions satisfy the quantum q-Hirota-Miwa equations.

Quantum algebra

Consider the associative algebra generated by

- dependent variables: f_i
- parameter variables: $\alpha_i^\vee$
- τ-variables: τ_i $\qquad (i \in \mathbb{Z}/n\mathbb{Z})$

with the defining relations

- $f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0.$
- $f_i f_j = f_j f_i \quad (j \neq i \pm 1).$
- $\alpha_i^\vee$ commutes with $\alpha_j^\vee$ and f_j.
- $\tau_i = \exp(\partial/\partial\alpha_i^\vee).$

Weyl group action

Using the useful formulas above, we can show the following formulas:

- $s_i(f_{i\pm1}) = [1-\alpha_i^\vee]_q f_{i\pm1} + [\alpha_i^\vee]_q f_i f_{i\pm1} f_i^{-1}$,
  $s_i(f_j) = f_j \quad (j \neq i \pm 1)$.
- $s_i(\alpha_i^\vee) = -\alpha_i^\vee, \quad s_i(\alpha_{i\pm1}^\vee) = \alpha_{i\pm1}^\vee + \alpha_i^\vee$,
  $s_i(\alpha_j^\vee) = \alpha_j^\vee \quad (j \neq i, i \pm 1)$.
- $s_i(\tau_i) = f_i \dfrac{\tau_{i-1}\tau_{i+1}}{\tau_i}, \quad s_i(\tau_j) = \tau_j \quad (i \neq j)$.

Extended coroot and weight lattices

$Q^\vee := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i^\vee \oplus \mathbb{Z}\delta^\vee, \quad P := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i \oplus \mathbb{Z}\Lambda_0.$

Dual bases: $\varepsilon_i^\vee, \delta^\vee \longleftrightarrow \varepsilon_i, \Lambda_0.$

Assume $\varepsilon_i^\vee = \varepsilon_{i+n}^\vee + \delta^\vee$ and $\varepsilon_{i+n} = \varepsilon_i$.

$\alpha_i^\vee := \varepsilon_i^\vee - \varepsilon_{i+1}^\vee, \quad \alpha_i := \varepsilon_i - \varepsilon_{i+1}.$

$\Lambda_i = \Lambda_0 + \varepsilon_1 + \cdots + \varepsilon_i \quad (i \in \mathbb{Z}_{\geqq 0}).$

Then $P = \bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\varepsilon_{\mathrm{all}}, \quad \varepsilon_{\mathrm{all}} = \varepsilon_1 + \cdots + \varepsilon_n$

$P_+ := \sum_{i=0}^{n-1} \mathbb{Z}_{\geqq 0}\Lambda_i + \mathbb{Z}\varepsilon_{\mathrm{all}}.$

Assume $\Lambda_{i+n} = \Lambda_i + \varepsilon_{\mathrm{all}} \quad (i \in \mathbb{Z}).$

Extended affine Weyl group

$$W := W(A^{(1)}_{n-1}) = \langle s_0, s_1, \dots, s_{n-1} \rangle, \quad s_{i+n} = s_i.$$

$$\pi(s_i) := s_{i+1},$$

$$\widetilde{W} := \widetilde{W}(A^{(1)}_{n-1}) := \langle \pi \rangle \ltimes W = \langle \pi, s_0, \dots, s_{n-1} \rangle.$$

(Do not assume $\pi^n = 1$.)

Assume $\lambda \in P$ and $\beta^\vee \in Q^\vee$.

$$s_i(\lambda) := \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i, \quad \alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}.$$

$$s_i(\beta^\vee) := \beta^\vee - \langle \beta^\vee, \alpha_i \rangle \alpha_i^\vee$$

$$\pi(\Lambda_i) := \Lambda_{i+1}, \quad \pi(\varepsilon_{\text{all}}) := \varepsilon_{\text{all}}.$$

$$\pi(\varepsilon_i^\vee) := \varepsilon_{i+1}^\vee, \quad \pi(\delta^\vee) := \delta^\vee.$$

Translation part of $\widetilde{W}$

$$T_i := s_{i-1} \cdots s_2 s_1 \pi s_{n-1} s_{n-2} \cdots s_i \in \widetilde{W} \quad (i = 1, \dots, n).$$

Assme $\nu = \sum_{i=1}^n \nu_i \varepsilon_i \in \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$.

$T^\nu := \prod_{i=0}^{n-1} T_i^{\nu_i}$.

Then

$$T^\nu(\varepsilon_i^\vee) = \varepsilon_i^\vee - \nu_i \delta^\vee \quad T^\nu(\delta^\vee) = \delta^\vee,$$

$$T^\nu(\alpha_i^\vee) = \alpha_i^\vee - (\nu_i - \nu_{i+1})\delta^\vee,$$

$$T^\nu(\varepsilon_i) = \varepsilon_i, \quad T^\nu(\Lambda_0) = \Lambda_0 + \nu,$$

$$T^\nu(\Lambda_i) = \Lambda_i + \nu.$$

Hirota-Miwa equation (1)

$$
\begin{aligned}
\Lambda_i &= \Lambda_{i-1} + \varepsilon_i, & \Lambda_{i+1} &= \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1}, \\
s_i(\Lambda_i) &= \Lambda_{i-1} + \varepsilon_{i+1}, & s_{i+1}(\Lambda_{i+1}) &= \Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2}, \\
s_{i+1}s_i(\Lambda_i) &= \Lambda_{i-1} + \varepsilon_{i+2}, & s_i s_{i+1}(\Lambda_{i+1}) &= \Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}.
\end{aligned}
$$

$$
\begin{aligned}
\tau_i &= \tau(\Lambda_{i-1} + \varepsilon_i), & \tau_{i+1} &= \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+1}), \\
s_i(\tau_i) &= \tau(\Lambda_{i-1} + \varepsilon_{i+1}), & s_{i+1}(\tau_{i+1}) &= \tau(\Lambda_{i-1} + \varepsilon_i + \varepsilon_{i+2}), \\
s_{i+1}s_i(\tau_i) &= \tau(\Lambda_{i-1} + \varepsilon_{i+2}), & s_i s_{i+1}(\tau_{i+1}) &= \tau(\Lambda_{i-1} + \varepsilon_{i+1} + \varepsilon_{i+2}).
\end{aligned}
$$

Lemma:

$$
\begin{aligned}
&[\alpha_{i+1}^\vee]_q \tau_i \, s_i s_{i+1}(\tau_{i+1}) + [\alpha_i^\vee]_q s_{i+1} s_i(\tau_i) \tau_{i+1} \\
&\qquad = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q s_i(\tau_i) s_{i+1}(\tau_{i+1}).
\end{aligned}
$$

Hirota-Miwa equation (2) Proof of Lemma

Warning: τ_i does not commute with $s_i s_{i+1}(\tau_{i+1})$.

$\tau_i[\alpha_i^\vee]_q = [\alpha_i^\vee + 1]_q \tau_i, \quad \tau_i[1 - \alpha_i^\vee]_q = -[\alpha_i^\vee]_q \tau_i.$

Proof of Lemma:

$$
\begin{aligned}
&\tau_i\, s_i s_{i+1}(\tau_i) \\
&= \tau_i\, s_i \left(f_{i+1} \frac{\tau_i \tau_{i+2}}{\tau_{i+1}} \right) \\
&= \tau_i\, ([1-\alpha_i^\vee]_q f_{i+1} + [\alpha_i^\vee]_q f_i f_{i+1} f_i^{-1}) f_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i} \frac{\tau_{i+2}}{\tau_{i+1}}, \\
&= \tau_i\, ([1-\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee]_q f_i f_{i+1}) \tau_i^{-1}\, \tau_{i-1}\tau_{i+2}, \\
&= (-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1}) \tau_{i-1}\tau_{i+2}.
\end{aligned}
$$

Thus

$$\tau_i\, s_i s_{i+1}(\tau_i) = (-[\alpha_i^\vee]_q f_{i+1} f_i + [\alpha_i^\vee + 1]_q f_i f_{i+1})\tau_{i-1}\tau_{i+2}.$$

Similarly we obtain

$$s_{i+1}s_i(\tau_i)\tau_{i+1} = ([1-\alpha_{i+1}^\vee]_q f_i f_{i+1} + [\alpha_{i+1}^\vee]_q f_{i+1} f_i)\tau_{i-1}\tau_{i+2}$$

$$s_i(\tau_i)s_{i+1}(\tau_{i+1}) = f_i f_{i+1}\tau_{i-1}\tau_{i+2}.$$

q-numbers identity (or addition formula of $\sin$):

$$[\alpha_i^\vee + 1]_q[\alpha_{i+1}^\vee]_q + [\alpha_i^\vee]_q[1-\alpha_{i+1}^\vee]_q = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q.$$

Therefore

$$\begin{aligned}[\alpha_{i+1}^\vee]_q\tau_i\, s_i s_{i+1}(\tau_{i+1}) + [\alpha_i^\vee]_q s_{i+1}s_i(\tau_i)\tau_{i+1} \\ = [\alpha_i^\vee + \alpha_{i+1}^\vee]_q s_i(\tau_i)s_{i+1}(\tau_{i+1}).\end{aligned}$$

In LHS the $f_{i+1}f_i$-terms are canceled. $\square$

Hirota-Miwa equation (3)

Apply T^ν to the formula of Lemma. Then we obtain

Theorem: The quantum τ-functions of type $A^{(1)}_{n-1}$ satisfy the quantum q-Hirota-Miwa equations:

$$
\begin{aligned}
&[\varepsilon^\vee_i(\nu) - \varepsilon^\vee_{i+1}(\nu)]_q \quad \tau_i(\nu+\varepsilon_{i+2})\,\tau_i(\nu+\varepsilon_i+\varepsilon_{i+1}) \\
&+[\varepsilon^\vee_{i+1}(\nu) - \varepsilon^\vee_{i+2}(\nu)]_q\,\tau_i(\nu+\varepsilon_i) \quad \tau_i(\nu+\varepsilon_{i+1}+\varepsilon_{i+2}) \\
&+[\varepsilon^\vee_{i+2}(\nu) - \varepsilon^\vee_{i}(\nu)]_q \quad \tau_i(\nu+\varepsilon_{i+1})\,\tau_i(\nu+\varepsilon_{i+2}+\varepsilon_i) = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_i(\nu) &:= \tau(\Lambda_{i-1}+\nu), \\
\varepsilon^\vee_i(\nu) &:= T^\nu(\varepsilon^\vee_i) = \varepsilon^\vee_i - \nu_i\delta^\vee. \quad \square
\end{aligned}
$$

Lax and Sato-Wilson forms of the affine Weyl group action

The relation between
the $RLL = LLR$ formalism of quantum groups and
the Lax and Sato-Wilson forms of the Painlevé systems
is non-trivial.

Assume that m and n are mutually prime.

Lax form: RLL=LLR

$A^{(1)}_{m-1}$-type R-matrix:
Denote the $m \times m$ matrix units by E_{ij} and

$$R(z) := (q - q^{-1}z) \sum_{i=1}^{m} E_{ii} \otimes E_{ii} + (1 - z) \sum_{i \neq j} E_{ii} \otimes E_{jj}$$
$$+ (q - q^{-1}) \sum_{i<j} (E_{ij} \otimes E_{ji} + zE_{ji} \otimes E_{ij}).$$

Local L-matrices: for $k = 1, \ldots, n$,

$$L_k(z) = \sum_{i=1}^{m} a_{ik} E_{ii} + \sum_{i=1}^{m-1} b_{ik} E_{i,i+1} + z b_{mk} E_{m1}.$$

$$L_k(z)^1 := L_k(z) \otimes 1, \quad L_k(z)^2 := 1 \otimes L_k(z).$$

Fundamental relations:

$$R(z/w) L_k(z)^1 L_k(w)^2 = L_k(w)^2 L_k(z)^1 R(z/w),$$
$$L_k(z)^1 L_l(w)^2 = L_l(w)^2 L_k(z)^1 \quad (k \neq l).$$

Equivalent to the q-commutation relations:

$$a_{ik} b_{ik} = q^{-1} b_{ik} a_{ik}, \quad a_{ik} b_{i+1,k} = q b_{i+1,k} a_{ik},$$
$$a_{ik} a_{jk} = a_{jk} a_{ik}, \quad b_{ik} b_{jk} = b_{jk} b_{ik}, \quad \text{etc.}$$

If $k \neq l$, then a_{ik} and b_{ik} commute with a_{jl} and b_{jl}.

Another form of the bidiagonal matrix $L_k(z)$.

$$a_k := \mathrm{diag}(a_{1k}, \ldots, a_{mk}),$$
$$b_k := \mathrm{diag}(b_{1k}, \ldots, b_{mk}),$$

$$\Lambda(z) := \sum_{i=1}^{m} E_{i,i+1} + zE_{m1} = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ z & & & 0 \end{bmatrix} \quad \text{(shift matrix)}.$$

Then

$$L_k(z) = a_k + b_k\Lambda(z) = \begin{bmatrix} a_{1k} & b_{1k} & & \\ & a_{2k} & \ddots & \\ & & \ddots & b_{m-1,k} \\ zb_{mk} & & & a_{mk} \end{bmatrix}.$$

Lax form: $\widehat{L}(z)$

1. $L(z) := L_1(z) \cdots L_n(z)$, the global L-operator.

$\tilde{a}_i := a_{i1} \cdots a_{in}$.

$\tilde{a} := \mathbf{diag}(\tilde{a}_1, \ldots, \tilde{a}_m)$, the diagonal part of $L(z)$.

2. $\widetilde{L}(z) := L(z)\tilde{a} \leftarrow$ doubling the diagonal part of $L(z)$.

3. $\widehat{L}(z) := \widetilde{C}\widetilde{L}(z)\widetilde{C}^{-1}$.

Here $\widetilde{C} = \mathbf{diag}(\tilde{c}_1, \ldots, \tilde{c}_m)$ is uniquely characterized by

$$\tilde{c}_1 = 1,$$

$$\widehat{L}(z) = \sum_{k=0}^{n-1} \hat{\ell}_k \Lambda(z)^k + \underbrace{\Lambda(z)\Lambda(rz)\cdots\Lambda(r^{n-1}z)}_{\text{highest term}},$$

where $\hat{\ell}_0, \hat{\ell}_1, \ldots, \hat{\ell}_{n-1}$ are diagonal matrices.

$$\widehat{L}(z) = \hat{\ell}_0 + \hat{\ell}_1 \Lambda(z) + \cdots + \Lambda(z)\Lambda(rz)\cdots\Lambda(r^{n-1}z).$$

$t = \mathrm{diag}(t_1, \ldots, t_n) := \tilde{c}\tilde{a}\tilde{c}^{-1}$.

Then $\hat{\ell}_0 = t^2$ and $t_i \widehat{L}(z) = \widehat{L}(z) t_i$.

Define $\hat{b}_i$ and $\hat{f}_i$ by

$$\hat{\ell}_1 = \mathrm{diag}(\hat{b}_i)_{i=1}^n = \mathrm{diag}\left((q^{-1} - q) t_i t_{i+1} \hat{f}_i\right)_{i=1}^n.$$

$r\widehat{L}(z) = \widehat{L}(z) r$.

t_i and r shall be identified with parameter variables.

Assume $t_{i+n} = r^{-1} t_i$ and $\hat{f}_{i+n} = r \hat{f}_i$.

Example ($q\mathrm{P}_{\mathrm{IV}}$ case): $(m, n) = (3, 2)$.

$$\widehat{L}(z) = \begin{bmatrix} t_1^2 & (q^{-1} - q)t_1 t_2 \hat{f}_1 & 1 \\ rz & t_2^2 & (q^{-1} - q)t_2 t_3 \hat{f}_2 \\ rz(q^{-1} - q)t_3 t_4 \hat{f}_3 & z & t_3^2 \end{bmatrix}.$$

Assume $\widetilde{L}(z) = A + B\Lambda(z) + C\Lambda(z)^2$, A, B, C are diagonal, and $C = \mathbf{diag}(c_1, c_2, c_3)$. Then

$$c_i = b_{i1} b_{i+1,2} a_{i+2,1} a_{i+2,2},$$
$$\tilde{c}_1 = 1, \quad \tilde{c}_3 = c_1, \quad \tilde{c}_2 = c_1 c_3, \quad r = c_1 c_3 c_2.$$

$\widetilde{C} = \mathbf{diag}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$, $\widehat{L}(z) = \widetilde{C}\widetilde{L}(z)\widetilde{C}^{-1}$, $r\widehat{L}(z) = \widehat{L}(z)r$.

Lax form: $\widehat{M}(z)$

$T_{z,r} := r^{z\partial/\partial z} : f(z) \mapsto f(rz)$, r-difference operator.

4. $\widehat{M}(z) := \widehat{L}(z) T_{z,r}^n$, matrix coefficient r-difference op.

5. Assume $t_i = q^{-\varepsilon_i^\vee}$ and $r = q^{-\delta^\vee}$.

Then $[\alpha_i^\vee]_q = (t_{i+1}/t_i - t_i/t_{i+1})/(q - q^{-1})$

6. $g_i := (t_i^2 - t_{i+1}^2)/\hat{b}_i = [\alpha_i^\vee]_q/\hat{f}_i$.

$G_i := g_i E_{i+1,\,i} \quad (i = 1, \dots, n-1)$.

$(G_n(z) := r z^{-1} g_n E_{1n}.)$

Lax form: Weyl group action

Consider the algebra generated by the matrix elements of $\widehat{L}(z)$ (precisely of $\hat{\ell}_0, \dots, \hat{\ell}_{n-1}$).

7. Algebra automorphism Weyl group action:

$$\begin{aligned} s_i(\widehat{M}(z)) &:= G_i \widehat{M}(z) G_i^{-1}, \\ \pi(\widehat{M}(z)) &:= (\Lambda(z) T_{z,r}) \widehat{M}(z) (\Lambda(z) T_{z,r})^{-1} \\ &= \Lambda(z) \widehat{L}(rz) \Lambda(r^n z) T_{z,r}^n. \end{aligned}$$

Then

$$\begin{aligned} &s_i(t_i) = t_{i+1}, \quad s_i(t_{i+1}) = t_i, \\ &s_i(\hat{b}_i) = \hat{b}_i, \quad s_i(\hat{b}_{i\pm1}) = \hat{b}_{i\pm1} \pm (t_i^2 - t_{i+1}^2)/\hat{b}_i. \end{aligned}$$

Sato-Wilson form: z-variables

8. Introduce τ_0 and z_i by
   - $\boldsymbol{\tau_0 = \exp(\partial/\partial\delta^\vee)}$: $\quad \tau_0 r = q^{-1} r \tau_0, \quad \tau_0 t_j = t_j \tau_0.$
   - $\boldsymbol{z_i = \exp(\partial/\partial\varepsilon_i^\vee)}$: $\quad z_i r = r z_i, \quad z_i t_j = q^{-\delta_{ij}} t_j z_i.$
   - τ_0 and z_i commute with τ_0, z_j, $\hat{f}_j$.

9. $D_Z := \mathbf{diag}(z_1, \dots, z_n), \quad Z(z) := U(z) D_Z,$ where

$$U(z) = E + \sum_{k=1}^\infty u_k \Lambda(z)^k,$$

$u_1, u_2, \dots$ are diagonal matrices,

$$\widehat{M}(z) = U(z) t^2 T_{z,r}^n U(z)^{-1}.$$

Then

$$\widehat{M}(z) = Z(z) (qt)^2 T_{z,r}^n Z(z)^{-1}.$$

Sato-Wilson form: Weyl group action

10. The Weyl group action can be extended by

$$s_i(U(z)) = G_i U(z) S_i^g, \quad s_i(D_Z) = (S_i^g)^{-1} D_Z S_i,$$

$$s_i(t) = S_i^{-1} t S_i, \quad s_i(Z(z)) = G_i(z) Z(z) S_i,$$

$$\pi(A(z)) = (\Lambda(z) T_{z,r}) A(z) (\Lambda(z) T_{z,r})^{-1},$$

$$(A(z) = U(z), D_Z, t, Z(z))$$

where $g_i = (t_i^2 - t_{i+1}^2)/\hat{b}_i = [\alpha_i^\vee]_q / \hat{f}_i$,

$$S_i^g := g_i^{-1} E_{i,i+1} - g_i E_{i+1,i} + \sum_{j \neq i,i+1} E_{jj},$$

$$S_i := [\alpha_i^\vee + 1]_q E_{i,i+1} - [\alpha_i^\vee - 1]_q^{-1} E_{i+1,i} + \sum_{j \neq i,i+1} E_{jj}.$$

S^g and S_i are permutation matrices $i \leftrightarrow i+1$.

11. Assume $z_{j+m} = z_j$, $\tau_j = \tau_{j-1} z_i$,
and $s_i(\tau_0) = \tau_0$ $(i = 1, 2)$. Then, for $i = 1, 2$,

$$s_i(z_i) = \hat{f}_i z_{i+1}, \quad s_i(z_{i+1}) = \hat{f}_i^{-1} z_i, \quad s_i(\tau_i) = \hat{f}_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i},$$

$$\pi(z_i) = z_{i+1}, \quad \pi(\tau_i) = \tau_{i+1}.$$

Because $g_i = [\alpha_i^\vee]_q / \hat{f}_i$ and $z_i \varepsilon_j^\vee = (\varepsilon_j^\vee + \delta_{ij}) z_i$ implies

$$\begin{bmatrix} 0 & g_i \\ -g_i^{-1} & 0 \end{bmatrix}^{-1} \begin{bmatrix} z_i & 0 \\ 0 & z_{i+1} \end{bmatrix} \begin{bmatrix} 0 & [\alpha_i^\vee + 1]_q \\ -[\alpha_i^\vee - 1]_q^{-1} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} g_i^{-1} z_{i+1} [\alpha_i^\vee + 1]_q & 0 \\ 0 & g_i z_i [\alpha_i^\vee - 1]_q^{-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_i z_{i+1} & 0 \\ 0 & \hat{f}_i^{-1} z_i \end{bmatrix}. \quad \square$$

Quantum $q\mathrm{P_{IV}}$

Both canonically quantized and q-difference.

(m, n)-case: $X_k(z)$

Assume that m and n are mutually prime (gcm = 1).

There exist unique diagonal matrices $\widetilde{C}_1, \dots, \widetilde{C}_1$ such that $\widetilde{C}_1 = \widetilde{C}_{n+1} = \widetilde{C}$ and, for $k = 1, \dots, n-1$,

$$
\begin{aligned}
X_k(r^{k-1}z) &:= \widetilde{C}_k L_k(z) \widetilde{C}_{k+1}^{-1} = x_k + \Lambda(r^{k-1}z), \\
X_n(r^{n-1}z) &:= \widetilde{C}_n L_n(z) \tilde{a} \widetilde{C}_1^{-1} = x_n + \Lambda(r^{n-1}z), \\
x_k &= \mathrm{diag}(x_{1k}, \dots, x_{m,k}).
\end{aligned}
$$

Then

$$
\widehat{L}(z) = X_1(z) X_2(rz) \cdots X_n(r^{n-1}z).
$$

Assume $x_{i+m,k} = r^{-1} x_{ik}$ and $x_{i,k+n} = x_{ik}$.

(m, n)-case: q-commutation relations of x_{ik}

Theorem. $x_{ik}x_{jl} = q^{(m,n)}_{j-i,l-k}x_{jl}x_{ik}$, $q^{(m,n)}_{\mu\nu} \in \{1, q^{\pm 2}\}$. $\square$

Example. If $(m, n) = (2g + 1, 2)$ and $x_i := x_{i1}$, $y_i := x_{i2}$, then

$$x_i y_i = y_i x_i = t_i^2,$$

$$x_i x_{i+\mu} = q^{(-1)^{\mu-1}2} x_{i+\mu} x_i, \qquad x_i y_{i+\mu} = q^{-(-1)^{\mu-1}2} y_{i+\mu} x_i,$$

$$y_i y_{i+\mu} = q^{(-1)^{\mu-1}2} y_{i+\mu} y_i, \qquad y_i x_{i+\mu} = q^{-(-1)^{\mu-1}2} x_{i+\mu} y_i,$$

t_i commutes with t_j, x_j, y_j. $\square$

Example. $(m,n) = (3,5), (5,3)$. $x_{ik}x_{jl} = q^{(m,n)}_{j-i,l-k} x_{jl}x_{ik}$, where $q_{\mu+m,\nu} = q_{\mu\nu}$, $q_{\mu,\nu+n} = q_{\mu\nu}$, and

$$\left[q^{(3,5)}_{\mu\nu}\right] = \begin{bmatrix} 1 & 1 & q^{-2} & q^2 & 1 \\ q^{-2} & q^2 & 1 & q^{-2} & q^2 \\ q^2 & q^{-2} & q^2 & 1 & q^{-2} \end{bmatrix},$$

$$\left[q^{(5,3)}_{\mu\nu}\right] = \begin{bmatrix} 1 & q^{-2} & q^2 \\ 1 & q^2 & q^{-2} \\ q^{-2} & 1 & q^2 \\ q^2 & q^{-2} & 1 \\ 1 & q^2 & q^{-2} \end{bmatrix}.$$

Note that $q^{(5,3)}_{\mu\nu} = q^{(3,5)}_{\nu\mu}$. $\square$

Assume that $0 < \widetilde{m} < n$ and $m\widetilde{m} \equiv 1 \pmod{n}$.
Assume that $0 < \tilde{n} < n$ and $n\tilde{n} \equiv 1 \pmod{m}$.

Theorem. Define $B^{(m,n)}$ and $p_{\mu\nu}^{(m,n)}$ by

$$B^{(m,n)} := \{\, (\mu \bmod m,\ \mu \bmod n) \mid 0 \leqq \mu < \widetilde{m}m \,\}.$$

$$p_{\mu\nu}^{(m,n)} := \begin{cases} q & \text{if } (\mu \bmod m,\ \nu \bmod n) \in B, \\ 1 & \text{if } (\mu \bmod m,\ \nu \bmod n) \notin B. \end{cases}$$

Then

$$q_{\mu\nu}^{(m,n)} = (p_{\mu\nu}/p_{\mu-1,\nu})^2 \in \{1, q^{\pm 2}\}. \qquad \square$$

Cor. (duality) $q_{\mu\nu}^{(m,n)} = q_{\nu\mu}^{(n,m)}$. $\square$

(m, n)-case: Weyl group action

The action of $\widetilde{W}(A^{(1)}_{m-1})$ on t_i, $\hat{f}_i$, etc. can be extended to the one on x_{ik}.

Using the duality above, we can construct the action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ on x_{ik}.

We shall write $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ as

$$\widetilde{W}(A^{(1)}_{m-1}) = \langle \pi, s_0, \ldots, s_{m-1} \rangle,$$
$$\widetilde{W}(A^{(1)}_{n-1}) = \langle \varpi, r_0, \ldots, r_{n-1} \rangle.$$

$q\mathrm{P}_{\mathrm{IV}}$-case: Weyl group action

Example ($q\mathrm{P}_{\mathrm{IV}}$-case) $(m, n) = (3, 2)$. $x_i := x_{i1}$, $y_i := x_{i2}$.

$$s_i(x_i) = (x_i + y_{i+1})x_{i+1}(y_i + x_{i+1})^{-1},$$

$$s_i(x_{i+1}) = (x_i + y_{i+1})^{-1}x_i(y_i + x_{i+1}),$$

$$s_i(y_i) = (y_i + x_{i+1})y_{i+1}(x_i + y_{i+1})^{-1},$$

$$s_i(y_{i+1}) = (y_i + x_{i+1})^{-1}y_i(x_i + y_{i+1}),$$

$$s_i(x_{i+2}) = x_{i+2}, \quad s_i(y_{i+2}) = y_{i+2},$$

$$s_i(t_i) = t_{i+1}, \quad s_i(t_{i+1}) = t_i, \quad s_i(t_{j+2}) = t_{j+2},$$

$$\pi(x_i) = x_{i+1}, \quad \pi(y_i) = y_{i+1}, \quad \pi(t_i) = t_{i+1}.$$

$$Q_i := y_{i+2}y_{i+1} + y_{i+2}x_i + x_{i+1}x_i,$$

$$r_1(x_i) = r^{-1}Q_{i+1}^{-1}y_iQ_i,$$

$$r_1(y_i) = rQ_{i+1}x_iQ_i^{-1},$$

$$r_1(t_i) = t_i, \quad \varpi(x_i) = y_i, \quad \varpi(y_i) = x_i, \quad \varpi(t_i) = t_i.$$

$q\mathrm{P_{IV}}$-case: Lax form

$G'_i := \varpi(G_i)$. Then

$$s_i(X(z)) = G_i X(z) G_i'^{-1},$$
$$s_i(Y(z)) = G'_i Y(z) G_i^{-1},$$
$$\pi(X(z)) = \Lambda(z) X(z) \Lambda(z)^{-1}$$

$$r_1(X(z)Y(rz)) = X(z)Y(rz),$$
$$r_1 : x_{i+2}x_{i+1}x_i \leftrightarrow y_{i+3}y_{i+2}y_{i+1},$$
$$\varpi : X(z) \leftrightarrow Y(z).$$

These relations uniquely characterize the quantized birational action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$.

$q\mathrm{P}_{\mathrm{IV}}$-case: discrete time evolution

$U_1 := r_1\varpi \in$ translation part of $\widetilde{W}(A_1^{(1)})$.

The U_1-action is the discrete time evolution of $q\mathrm{P}_{\mathrm{IV}}$ and the $\widetilde{W}(A_2^{(1)})$-action is its symmetry.

$a_i := t_i/t_{i+1}$ and $F_i := x_{i+1}x_i/(t_{i+1}t_i)$.

Then

$$F_iF_{i+1} = q^2F_{i+1}F_i,$$
$$a_ia_j = a_ja_i,\ a_iF_j = F_ja_i,$$
$$F_{i+3} = F_i,\ a_{i+3} = a_i.$$

Discrete time evolution of quantized $q\mathrm{P_{IV}}$.

$$\begin{aligned} U_1(F_i) &= (1 + q^2 a_{i-1} F_{i-1} + q^2 a_{i-1} a_i F_{i-1} F_i) \\ &\quad \times a_i a_{i+1} F_{i+1} \\ &\quad \times (1 + q^2 a_i F_i + q^2 a_i a_{i+1} F_i F_{i+1})^{-1}, \\ U_1(a_i) &= a_i. \end{aligned}$$

Classical case:

Kajiwara-Noumi-Yamada `arXiv:nlin/0012063`

$$\overline{F_i} = a_i a_{i+1} F_{i+1} \frac{1 + a_{i-1} F_{i-1} + a_{i-1} a_i F_{i-1} F_i}{1 + a_i F_i + a_i a_{i+1} F_i F_{i+1}},$$

$$\overline{a_i} = a_i.$$

Symmetry of quantum $q\mathrm{P}_{\mathrm{IV}}$.

$$
\begin{aligned}
&s_i(F_i) = F_i,\\
&s_i(F_{i-1}) = F_{i-1}\frac{a_i + F_i}{1 + a_i F_i}, \quad s_i(F_{i+1}) = \frac{1 + a_i F_i}{a_i + F_i} F_{i+1},\\
&s_i(a_i) = a_i^{-1}, \quad s_i(a_{i\pm 1}) = a_i a_{i\pm 1}.
\end{aligned}
$$

These formulas coincide with the ones obtained by Koji Hasegawa `arXiv:0703036`, which quantizes Kajiwara-Noumi-Yamada `arXiv:nlin/0012063`.

Summary

- For any symmetrizable GCM, we can introduce the proper non-commutativity of quantum τ-functions by $\tau_i = \exp(\partial/\partial\alpha_i^\vee)$.
- Quantum q-Hirota-Miwa equations for $A_{n-1}^{(1)}$-case.
- Quantized Lax and Sato-Wilson forms of the extended affine Weyl group action for $A_{n-1}^{(1)}$-case.
- Quantized $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$-action for mutually prime m and n.
- An appropriate quantization of $q\mathrm{P_{IV}}$.