Fock space representations of twisted affine Lie algebras

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May 20, 2008

0. Introduction

Fock space representations of Wakimoto type for non-twisted affine Lie algebras are constructed in [W], [FeFr1], [FeFr2], and [Kur]. (There are in fact many other references by physicists.) In this note, Fock space representations for *twisted* affine Lie algebras are researched.

Section 1 shall be devoted to the definition of a twisted affine Lie algebra $\widehat{\mathfrak{g}}$ associated to a finite dimensional simple Lie algebra $\mathfrak{g}$ and its diagram automorphism σ. In Section 2, we shall explain a finite dimensional counterpart of Fock space representations, which is a realization of $\mathfrak{g}$ by first order differential operators on an open cell in the flag variety of $\mathfrak{g}$. Fock space representations are constructed by replacing the differential operators by Bosonic fields which is defined in Section 3. In Section 4, we shall construct Fock space representations of the twisted affine Lie algebras. The main result is Theorem 4.7. We shall also deduce several corollaries from this theorem. In particular, we shall explain how to prove the Kac-Kazhdan conjecture [KK] for the Kac-Moody Lie algebras of affine type (in twiseted cases as well as in non-twisted ones).

1. Twisted affine Lie algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of type X_r over $\mathbb{C}$ and $h^\vee$ its dual Coxeter number. Denote its triangular decomposition by $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}_\pm$ are maximal nilpotent subalgebras of $\mathfrak{g}$. Let σ be a diagram automorphism of $\mathfrak{g}$. The order N of σ is 1, 2, or 3. ($N = 2$ for $X_r = A_r, D_r, E_6$ and $N = 3$ for $X_r = D_4$.) Denote the eigenspace decomposition of $\mathfrak{g}$ with respect to σ by $\mathfrak{g} = \bigoplus_{i=0}^{N-1} \mathfrak{g}_i$, where we put

$$\mathfrak{g}_i := \{\, X \in \mathfrak{g} \mid \sigma(X) = \exp(\tfrac{2\pi\sqrt{-1}}{N} i) X \,\} \qquad \text{for} \quad i \in \mathbb{Z}.$$

For any vector subspace V of $\mathfrak{g}$, denote $V \cap \mathfrak{g}_i$ by V_i. For example, $\mathfrak{n}_{\pm,i} = \mathfrak{n}_\pm \cap \mathfrak{g}_i$ and $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$. Then $\mathfrak{g}_0$ becomes a simple Lie algebra and $\mathfrak{h}_0$ is its Cartan subalgebra.

Let us define the affine Lie algebra $\widehat{\mathfrak{g}}$ associated to the pair $(\mathfrak{g}, \sigma)$. When $N = 1$, $\widehat{\mathfrak{g}}$ is called a non-twisted affine Lie algebra. In the other cases ($N = 2, 3$), $\widehat{\mathfrak{g}}$ is called a twisted one. In the following, we shall mainly consider the twisted cases. Put $R_i := t^{i/N}\mathbb{C}[t, t^{-1}]$. Define the subalgebra $\mathrm{L}\mathfrak{g}$ of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t^{1/N}, t^{-1/N}]$ by

$$\mathrm{L}\mathfrak{g} := \bigoplus_{i=0}^{N-1} \mathfrak{g}_i \otimes R_i,$$

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which is called a twisted loop Lie algebra. For any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ stable under the action of σ, put $\mathrm{L}\mathfrak{a} := \bigoplus_{i=0}^{N-1} \mathfrak{a}_i \otimes R_i$. Let d be a derivation $t\frac{d}{dt}$ acting on $\mathrm{L}\mathfrak{g}$. Then $\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}$ possesses a natural semidirect Lie algebra structure. As a vector space, we define $\widehat{\mathfrak{g}}$ by

$$\widehat{\mathfrak{g}} := \mathrm{L}\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}\mathrm{d}.$$

Its Lie algebra structure is defined by

$$\begin{aligned}
&[X \otimes t^m, Y \otimes t^n] = [X,Y] \otimes t^{m+n} + (X|Y)m\delta_{m+n,0}K, \\
&[\mathrm{d}, X \otimes t^m] = mX \otimes t^m, \\
&K \in \text{center of } \widehat{\mathfrak{g}},
\end{aligned}$$

where $(.|.)$ is a non-degenerate symmetric bilinear form on $\mathfrak{g}$ defined by

$$\mathrm{trace}_{\mathfrak{g}}(\mathrm{ad}\, X\, \mathrm{ad}\, Y) = 2h^\vee (X|Y) \qquad \text{for} \quad X, Y \in \mathfrak{g}.$$

Then $\widehat{\mathfrak{g}}$ is a central extension of $\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}$ and is isomorphic to the Kac-Moody Lie algebra of type $X_r^{(N)}$ ([Kac] Chapter 8).

2. Finite dimensional counter part

Put $\mathfrak{b}_- := \mathfrak{n}_- \oplus \mathfrak{h}$ (the lower Borel subalgebra of $\mathfrak{g}$). Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, B_- the Lie subgroup corresponding to $\mathfrak{b}_-$, and U_+ the one corresponding to $\mathfrak{n}_+$. The flag variety is defined by $F := B_- \backslash G$ and its origin is defined by $o := B_- \backslash B_-$. Then the U_+-orbit $o\,U_+$ is a Zariski open cell in F and is isomorphic to U_+ as a right homogeneous G-space. Moreover the exponential map from $\mathfrak{n}_+$ to U_+ is an isomorphism of algebraic varieties. Thus $o\,U_+$ is naturally isomorphic to $\mathfrak{n}_+$ as an algebraic variety. Let λ be a Lie algebra character of $\mathfrak{b}_-$ (*i.e.* a Lie algebra homomorphism from $\mathfrak{b}_-$ into $\mathbb{C}$). In general, such a λ is a trivial extension of an element of $\mathfrak{h}^*$, where $\mathfrak{h}^*$ denotes the dual vector space of $\mathfrak{h}$. Define the actions L and R of $\mathfrak{g}$ on the structure ring $\mathbb{C}[B_-U_+]$ of B_-U_+ by

$$\begin{aligned}
(L(X)f)(x) &:= \left.\frac{d}{ds}\right|_{s=0} f(\exp(-sX)x) \\
(R(X)f)(x) &:= \left.\frac{d}{ds}\right|_{s=0} f(x\exp(sX)) \qquad \text{for } x \in B_-U_+ \text{ and } X \in \mathfrak{g}.
\end{aligned}$$

Put $M_\lambda^* := \{\, f \in \mathbb{C}[B_-U_+] \mid L(Y)f = -\lambda(Y)f \text{ for } Y \in \mathfrak{b}_- \,\}$. Then, since $R(X)$ preserves M_λ^* for $X \in \mathfrak{g}$, we obtain a natural left $\mathfrak{g}$-module structure on M_λ^*. The action of $R(X)$ on M_λ^* shall be denoted by $R_\lambda(X)$. Let v_λ be a unique function in M_λ^* which takes the constant value 1 on U_+. Then v_λ is a highest weight vector of M_λ^*.

Remark 2.1. *In fact, M_λ^* is isomorphic to the contragredient dual representation of the lowest weight left Verma module of $\mathfrak{g}$. Namely, for any $\mathfrak{g}$-module V which possesses a highest weight vector v with weight λ, there is a unique $\mathfrak{g}$-homomorphism from M_λ^* into V which sends v_λ to v.*

M_λ^* is a free $\mathbb{C}[o\,U_+]$-module of rank one: $M_\lambda^* = \mathbb{C}[o\,U_+]v_\lambda$. The structure ring $\mathbb{C}[o\,U_+]$ is isomorphic to a polynomial ring generated by $\dim \mathfrak{n}_+$ variables. Hence $o\,U_+ \simeq \mathbb{C}^{\dim \mathfrak{n}_+}$. For $X \in \mathfrak{g}$ the operator $R_\lambda(X)$ can be represented by a first order differential operator with coefficients in the polynomial ring. This is a finite dimensional counterpart of Fock space representations of Wakimoto type. Fock space representations shall be constructed by replacing differential operators by operators of Bosonic fields.

For any $\alpha \in \mathfrak{h}_0^*$, we put

$$\begin{aligned} \mathfrak{n}_{+,i,\alpha} &:= \{\, X \in \mathfrak{n}_{+,i,\alpha} \mid [H, X] = \alpha(H)X \text{ for } H \in \mathfrak{h}_0 \,\}, \\ \Delta_{+,i} &:= \{\alpha \in \mathfrak{h}_0^* \mid \mathfrak{n}_{+,i,\alpha} \neq 0\}. \end{aligned}$$

Then, for $\alpha \in \Delta_{+,i}$ we can write $\mathfrak{n}_{+,i,\alpha} = \mathbb{C}e_{+,i,\alpha}$, because $\dim \mathfrak{n}_{+,i,\alpha} = 1$ (see [Kac]). Since $\mathfrak{n}_+ = \bigoplus_{i=0}^{N-1} \bigoplus_{\alpha\in\Delta_{+,i}} \mathfrak{n}_{+,i,\alpha}$, we can define a coordinate system $x = (x_{i,\alpha})$ on $\mathfrak{n}_+$ by

$$X = \sum_{i=0}^{N-1} \sum_{\alpha\in\Delta_{+,i}} x_{i,\alpha}(X) e_{i,\alpha} \quad \text{for } X \in \mathfrak{n}_+.$$

Recall that we have the natural isomorphism $oU \simeq \mathfrak{n}_+$. We obtain a coordinate system (also denoted by x) on oU_+. Under the coordinate system, for $X \in \mathfrak{g}$ the operator $R_\lambda(X)$ can be written in the following form:

$$R_\lambda(X) = \sum_{i,\alpha} R_{i,\alpha}(X; x) \frac{\partial}{\partial x_{i,\alpha}} + \sum_{i,a} \rho_{i,a}(X; x) \lambda(H_{i,a}), \tag{2.1}$$

where $\{H_{i,a} \mid a = 1, \dots, \dim \mathfrak{h}_i\}$ is a basis of $\mathfrak{h}_i$ and the summations run over $i = 0, \dots, N-1$, $\alpha \in \Delta_{+,i}$, $a = 1, \dots, \dim \mathfrak{h}_i$.

3. Bosonic fields and Fock spaces

Let κ be a complex number. Define the associative algebra $\mathcal{A}$ with 1 by the following conditions:

- $\mathcal{A}$ is generated by the following set:

$$\begin{aligned} A := \{\, & x_{i,\alpha}[-m], \delta_{i,\alpha}[m], p_{i,a}[m] \mid \\ & i = 0, \cdots, N-1,\ \alpha \in \Delta_{+,i},\ a = 1, \cdots, \dim \mathfrak{h}_i,\ m \in \mathbb{Z} + \tfrac{i}{N} \,\}; \end{aligned}$$

- Then $\mathcal{A}$ is a quotient of the tensor algebra generated by A. The relations of $\mathcal{A}$ are generated by the following ones:

$$\begin{aligned} &[\delta_{i,\alpha}[m], x_{j,\beta}[n]] = \delta_{i,j}\delta_{\alpha,\beta}\delta_{m+n,0}, \\ &[p_{i,a}[m], p_{j,b}[n]] = \kappa(H_{i,a}|H_{j,b}) m \delta_{m+n,0}, \\ &(\text{other commutators}) = 0. \end{aligned}$$

Denote by $\mathcal{O}$ the subalgebra of $\mathcal{A}$ generated by the set of all $x_{i,\alpha}[m]$'s.

Remark 3.1. *If $\kappa \neq 0$, then the center of $\mathcal{A}$ is equal to the subalgebra generated by $\{p_{0,a}[0] \mid a = 1, \cdots, \dim \mathfrak{h}_0\}$. But $\kappa = 0$ implies that the center of $\mathcal{A}$ is generated by the set of all $p_{i,a}[m]$'s.*

In order to define a triangular decomposition of $\mathcal{A}$, we put

$$\begin{aligned} A_+ &:= \{x_{i,\alpha}[m], \delta_{i,\alpha}[n], p_{i,a}[m] \in \mathcal{A} \mid m > 0, n \geq 0\}, \\ A_- &:= \{x_{i,\alpha}[m], \delta_{i,\alpha}[n], p_{i,a}[n] \in \mathcal{A} \mid m \leq 0, n < 0\}, \\ A_0 &:= \{p_{0,a}[0] \mid a = 1, \cdots, \dim \mathfrak{h}_0\}. \end{aligned}$$

Let $\mathcal{A}_\pm$ be the subalgebra of $\mathcal{A}$ generated by $A_\pm$ and $\mathcal{A}_0$ the one generated by A_0. Each of them is isomorphic to polynomial rings: $\mathcal{A}_\pm \simeq \mathbb{C}[A_\pm]$, $\mathcal{A}_0 \simeq \mathbb{C}[A_0]$. It follows that $\mathcal{A}_- \otimes \mathcal{A}_0 \otimes \mathcal{A}_+$ can be identified with the polynomial ring $\mathbb{C}[A]$ generated by A as an algebra. Moreover we have the following isomorphism of vector spaces:

$$\mathbb{C}[A] = \mathcal{A}_- \otimes \mathcal{A}_0 \otimes \mathcal{A}_+ \to \mathcal{A}, \quad a_- \otimes a_0 \otimes a_+ \mapsto a_- a_0 a_+.$$

This isomorphism defines a triangular decomposition of $\mathcal{A}$ and is called normal product. The normal product of $a \in \mathbb{C}[A]$ shall be denoted by $\vdots a \vdots$.

For $\lambda \in \mathfrak{h}_0^*$, let I_λ be the left ideal generated by A_+ and $\{\, p_{0,a} - \lambda(H_{0,a})1 \mid a = 1, \cdots, \dim \mathfrak{h}_0 \,\}$. The Fock space $\mathcal{F}_\lambda$ with highest weight λ is defined by

$$\mathcal{F}_\lambda := \mathcal{A}/I_\lambda.$$

Denoting the vector $1 \bmod I_\lambda$ in $\mathcal{F}_\lambda$ by $|\lambda\rangle$, we obtain the following:

$$\mathcal{F}_\lambda = \mathcal{A}|\lambda\rangle, \quad A_+|\lambda\rangle = 0, \quad p_{0,a}[0]|\lambda\rangle = \lambda(H_{0,a})|\lambda\rangle.$$

These relations characterize the left $\mathcal{A}$-module structure of $\mathcal{F}_\lambda$ if $\kappa \neq 0$.

In fact the algebra $\mathcal{A}$ is too small to realize in it the twisted affine Lie algebra $\widehat{\mathfrak{g}}$. It is necessary to extend $\mathcal{A}$. Let $\widehat{\mathcal{A}}$ be an extension of $\mathcal{A}$ and $\widehat{\mathcal{O}}$ an associated extension of $\mathcal{O}$. But, if $\widehat{\mathcal{A}}$ is too large, then we can not calculate a certain 2-cocycle ω (see bellow) of the twisted loop algebra $\mathrm{L}\mathfrak{g}$ with coefficients in $\widehat{\mathcal{O}}$. An appropriate extension $\widehat{\mathcal{A}}$ shall be defined as follows.

Let Θ be the $\mathbb{C}$-derivation Θ acting on $\mathcal{A}$ which is characterized by

$$\Theta(a[m]) := m\, a[m] \qquad \text{for} \quad a[m] = x_{i,\alpha}[m], \delta_{i,\alpha}[m], p_{i,a}[m].$$

Putting $\mathcal{A}[m] := \{a \in \mathcal{A} \mid \Theta(a) = ma\}$, we have a gradation of $\mathcal{A}$:

$$\mathcal{A} = \bigoplus_{m \in \frac{1}{N}\mathbb{Z}} \mathcal{A}[m], \qquad \mathcal{A}[m]\mathcal{A}[n] \subseteq \mathcal{A}[m+n].$$

Generally, for any vector subspace V of $\mathcal{A}$, we shall denote $V \cap \mathcal{A}[m]$ by $V[m]$. (For example, $\mathcal{O}[m] = \mathcal{O} \cap \mathcal{A}[m]$.) Since Θ preserves and acts on $\mathcal{O}$, it follows that $\mathcal{O} = \bigoplus_{m \in \frac{1}{N}\mathbb{Z}} \mathcal{O}[m]$. Define the decreasing filtration $\mathcal{A}^\bullet[m]$ of $\mathcal{A}[m]$ by

$$\mathcal{A}^n[m] := \bigoplus_{l \geq n} \mathcal{A}_-[m-l]\mathcal{A}_0\mathcal{A}_+[l] \qquad \text{for} \quad n \in \tfrac{1}{N}\mathbb{Z}.$$

The completion of $\mathcal{A}[m]$ with respect to this filtration shall be denoted by $\widehat{\mathcal{A}}[m]$:

$$\widehat{\mathcal{A}}[m] := \operatorname*{proj\,lim}_{n \to \infty} \mathcal{A}[m]/\mathcal{A}^n[m].$$

Put $\widehat{\mathcal{A}} := \bigoplus_m \widehat{\mathcal{A}}[m]$. Then $\mathcal{A}$ is dense in $\widehat{\mathcal{A}}$ under the linear topology induced by the filtration. Since the multiplication map from $\mathcal{A}[m] \otimes \mathcal{A}[n]$ into $\mathcal{A}[m+n]$ is continuous, the algebra structure of $\mathcal{A}$ is naturally extended to that of $\widehat{\mathcal{A}}$. Denote by $\widehat{\mathcal{O}}$ the closure of $\mathcal{O}$ in $\widehat{\mathcal{A}}$. The action of $\mathcal{A}$ on $\mathcal{F}_\lambda$ can be continuously extended to that of $\widehat{\mathcal{A}}$.

The derivation Θ on $\mathcal{A}$ is naturally extended to that on $\widehat{\mathcal{A}}$. We shall consider the algebra generated by $\widehat{\mathcal{A}}$ and Θ. More precisely, let $\widehat{\mathcal{A}}[\Theta]$ be the associative algebra with 1 defined by the following conditions:

- $\widehat{\mathcal{A}}[\Theta]$ is isomorphic to $\widehat{\mathcal{A}} \otimes \mathbb{C}[\Theta]$ as a vector space;

- $\Theta a = a\Theta + ma \quad$ for $a \in \widehat{\mathcal{A}}[m]$, $\quad$ *i.e.* $[\Theta, a] = \Theta(a)$.

Put $\xi := (\lambda, c)$,where $\lambda \in \mathfrak{h}_0^*$ and $c \in \mathbb{C}$. Then there exists a unique extension of the action of $\widehat{\mathcal{A}}$ on $\mathcal{F}_\lambda$ to that of $\widehat{\mathcal{A}}[\Theta]$ with the property $\Theta|\lambda\rangle = c|\lambda\rangle$. When we regard $\mathcal{F}_\lambda$ as a left $\widehat{\mathcal{A}}[\Theta]$-module, we denote it by $\mathcal{F}_\xi = \mathcal{F}_{\lambda,c}$. We shall construct a Lie algebra homomorphism from the twisted affine Lie algebra $\widehat{\mathfrak{g}}$ into $\widehat{\mathcal{A}}[\Theta]$ so that we shall obtain a representation of $\widehat{\mathfrak{g}}$ in $\mathcal{F}_\xi$.

As formal Laurent series in z, we define Bosonic fields $x_{i,\alpha}(z)$, $\delta_{i,\alpha}(z)$, $p_{i,a}(z)$ by

$$\begin{aligned} x_{i,\alpha}(z) &:= \sum_{m\in\mathbb{Z}-\frac{i}{N}} z^{-m} x_{i,\alpha}[m], \\ \delta_{i,\alpha}(z) &:= \sum_{m\in\mathbb{Z}+\frac{i}{N}} z^{-m-1} \delta_{i,\alpha}[m], \\ p_{i,a}(z) &:= \sum_{m\in\mathbb{Z}+\frac{i}{N}} z^{-m-1} p_{i,a}[m]. \end{aligned}$$

Let each of $a_1(z), \dots, a_n(z)$ be one of the Bosonic fields $x_{i,\alpha}(z)$, $\delta_{i,\alpha}(z)$, $p_{i,a}(z)$, $\partial x_{i,\alpha}(z)$, $\partial\delta_{i,\alpha}(z)$, $\partial p_{i,a}(z)$, $\cdots$, where ∂ denotes $\frac{\partial}{\partial z}$. Put $a(z_1, \dots, z_n) := {}^{\bullet}_{\bullet} a_1(z_1)\cdots a_n(z_n) {}^{\bullet}_{\bullet}$. Then $a(z, \dots, z)$ is well-defined as a formal Laurent series with coefficients in $\widehat{\mathcal{A}}$.

4. Fock space representations

Let $U = \bigoplus_m U[m]$ and $V = \bigoplus_m V[m]$ be graded vector spaces, where m runs over $\frac{1}{N}\mathbb{Z}$. We define the restricted hom set $\widetilde{\mathrm{Hom}}_{\mathbb{C}}(U, V)$ by

$$\begin{aligned} \widetilde{\mathrm{Hom}}_{\mathbb{C}}(U, V)[m] &:= \{f \in \mathrm{Hom}_{\mathbb{C}}(U, V) \mid f(U[n]) \subseteq V[m+n] \text{ for } n \in \tfrac{1}{N}\mathbb{Z}\}, \\ \widetilde{\mathrm{Hom}}_{\mathbb{C}}(U, V) &:= \bigoplus_m \widetilde{\mathrm{Hom}}_{\mathbb{C}}(U, V)[m]. \end{aligned}$$

Let $\mathfrak{a} = \bigoplus_m \mathfrak{a}[m]$ be a graded Lie algebra and V a graded left $\mathfrak{a}$-module. The exterior product $\bigwedge^p \mathfrak{a}$ possesses the induced natural gradation. We can define a cochain complex $(\widetilde{C}^{\bullet}, d)$ as follows:

$$\begin{aligned} \widetilde{C}^p &:= \widetilde{\mathrm{Hom}}_{\mathbb{C}}(\textstyle\bigwedge^p \mathfrak{a}, V), \\ (df)(l_1, \cdots, l_{p+1}) &:= \sum_{1\le i\le p+1} (-1)^{i-1} l_i(f(l_1, \cdots, \widehat{l_i}, \cdots, l_{p+1})) \\ &\quad + \sum_{1\le i<j\le p+1} (-1)^{i+j} f([l_i, l_j], l_1, \cdots, \widehat{l_i}, \cdots, \widehat{l_j}, \cdots, l_{p+1}), \end{aligned}$$

where $f \in \widetilde{C}^p$, $l_i \in \mathfrak{a}$, and the hats $\widehat{\ }$ means eliminations. We shall denote the p-th coboundary, cocycle, and cohomology group of this complex by $\widetilde{B}^p(\mathfrak{a}, V)$, $\widetilde{Z}^p(\mathfrak{a}, V)$, and $\widetilde{H}^p(\mathfrak{a}, V)$ respectively. We call $\widetilde{H}^p(\mathfrak{a}, V)$ the restricted Lie algebra cohomology of $\mathfrak{a}$ with coefficients in V.

Recall that we have the realization (2.1) of $\mathfrak{g}$ by first order differential operators with polynomial coefficients. For $X \in \mathfrak{g}_i$, we define $\widetilde{X}(z)$ by

$$\widetilde{X}(z) := \sum_{i,\alpha} {}^{\bullet}_{\bullet} R_{i,\alpha}(X; x(z))\delta_{i,\alpha}(z) {}^{\bullet}_{\bullet} + \sum_{i,a} {}^{\bullet}_{\bullet} \rho_{i,a}(X; x(z)) p_{i,a}(z) {}^{\bullet}_{\bullet}.$$

We can expand $\widetilde{X}(z)$ in the following form:

$$\widetilde{X}(z) = \sum_{m\in\mathbb{Z}+\frac{i}{N}} z^{-m-1} \widetilde{\pi}(X \otimes t^m) \qquad \text{for} \quad X \in \mathfrak{g}_i,$$

where we obtain $\widetilde{\pi}(X \otimes t^m) \in \widehat{\mathcal{A}}$. We also set $\widetilde{\pi}(\mathrm{d}) := \Theta$. Thus we obtain a linear map $\widetilde{\pi}$ from $\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}$ into $\widehat{\mathcal{A}}[\Theta]$. For $a, b \in \mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}$, define $\omega(a,b) \in \widehat{\mathcal{A}}$ by

$$\omega(a,b) := [\widetilde{\pi}(a), \widetilde{\pi}(b)] - \widetilde{\pi}([a,b]),$$

where the bracket $[a,b]$ in the left-hand side is the commutator in the Lie algebra $\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}$ not in $\widehat{\mathfrak{g}}$. The Wick theorem proves $\omega(a,b) \in \widehat{\mathcal{O}}$. Therefore we can define the left $(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d})$-module structure on $\widehat{\mathcal{O}}$ by

$$(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}) \times \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}, \quad (a,b) \mapsto [\widetilde{\pi}(a), b].$$

Since $\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}$ and $\widehat{\mathcal{O}}$ have natural $\frac{1}{N}\mathbb{Z}$-gradations, we can consider the restricted Lie algebra cohomology $\widetilde{H}^{\bullet}(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$. Moreover ω is a 2-cocycle in $\widetilde{Z}(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$ and defines a cohomology class $[\omega] \in \widetilde{H}^2(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$. The standard 2-cocycle c_2 of $\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}$, which gives a realization of $\widehat{\mathfrak{g}}$, is defined by

$$\begin{aligned} &c_2(X \otimes t^m, Y \otimes t^n) := (\kappa - h^\vee)(X|Y)m\delta_{m+n,0}, \\ &c_2(\mathrm{d}, X \otimes t^m) := 0. \end{aligned}$$

Then c_2 defines a cohomology class $[c_2]$ in $\widetilde{H}^2(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$.

Lemma 4.1. *The values of ω and c_2 coincide on $\bigwedge^2(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d})$.*

The proof is followed from the Wick theorem. □

Lemma 4.2. *Consider the canonical inclusions $\mathrm{L}\mathfrak{h} \to \mathrm{L}\mathfrak{b}_+ \to \mathrm{L}\mathfrak{g}$ and the canonical projection $\widehat{\mathcal{O}} \to \mathbb{C}$. Then they induce canonical isomorphisms bellow:*

$$\widetilde{H}^p(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}}) \simeq \widetilde{H}^p(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}}) \simeq \widetilde{H}^p(\mathrm{L}\mathfrak{h} \oplus \mathbb{C}\mathrm{d}, \mathbb{C}).$$

The standard spectral sequence technique shows this lemma. (The detail of non-twisted case can be found in [Kur].) □

For $\xi \in (\mathfrak{h}_0 \oplus \mathbb{C}\mathrm{d})^*$ we can define the algebra automorphism τ_ξ of $\widehat{\mathcal{A}}[\Theta]$ by

$$\tau_\xi(p_{0,a}[0]) := p_{0,a}[0] + \xi(H_{0,a}), \qquad \tau_\xi(\Theta) := \Theta + \xi(\mathrm{d}).$$

Then, putting $f_\xi := \tau_\xi \circ \widetilde{\pi} - \widetilde{\pi}$, we have $f_\xi \in \widetilde{Z}^1(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$ and

$$\begin{aligned} f_\xi(l) &= \xi(l) \qquad &&\text{for} \quad l \in \mathfrak{h}_0 \oplus \mathbb{C}\mathrm{d} = \mathfrak{h}_0 \otimes 1 \oplus \mathbb{C}\mathrm{d}, \\ f_\xi(l) &= 0 &&\text{for} \quad l \in (\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d})', \end{aligned}$$

where $(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d})'$ is the derived Lie subalgebra of $\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}$. In general, the derived Lie subalgebra of $\mathfrak{a}$ is defined by $\mathfrak{a}' := [\mathfrak{a}, \mathfrak{a}]$ Note that $(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d})/(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d})'$ is canonically isomorphic to $\mathfrak{h}_0 \oplus \mathbb{C}\mathrm{d}$. Denote by g_ξ the restriction of f_ξ on $\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}$. Then f_ξ and g_ξ give the cohomology classes $[f_\xi] \in \widetilde{H}^1(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$ and $[g_\xi] \in \widetilde{H}^1(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$.

Lemma 4.3. *The map $\xi \mapsto [f_\xi]$ is an isomorphism from $(\mathfrak{h}_0 \oplus \mathbb{C}\mathrm{d})^*$ onto $\widetilde{H}^1(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$.*

Lemma 4.4. *The map $\xi \mapsto [g_\xi]$ is an isomorphism from $(\mathfrak{h}_0 \oplus \mathbb{C}\mathrm{d})^*$ onto $\widetilde{H}^1(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}})$.*

The above two lemmas are deduced from Lemma 4.2. □

Lemma 4.5. $\widetilde{H}^0(\mathrm{L}\mathfrak{n}_+, \widehat{\mathcal{O}}) \simeq \widehat{\mathcal{O}}^{\mathrm{L}\mathfrak{n}_+} = \mathbb{C}$, *where* $\widehat{\mathcal{O}}^{\mathrm{L}\mathfrak{n}_+} := \{\, a \in \widehat{\mathcal{O}} \mid [\widetilde{\pi}(\mathrm{L}\mathfrak{n}_+), a] = 0 \,\}$.

The similar method of the poof of Lemma 4.2 shows that the inclusion $0 \to \mathrm{L}\mathfrak{n}_+$ and the projection $\widehat{\mathcal{O}} \to \mathbb{C}$ induce an isomorphism from $\widetilde{H}^0(\mathrm{L}\mathfrak{n}_+, \widehat{\mathcal{O}}\,)$ onto $\widetilde{H}^0(0, \mathbb{C}) = \mathbb{C}$. $\square$

Lemma 4.6. *There exists a unique* $\Gamma \in \widetilde{\mathrm{Hom}}_{\mathbb{C}}(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}}\,)$ *which satisfies the following conditions:*

$$c_2 = \omega + d\Gamma, \tag{4.1}$$

$$\Gamma = 0 \qquad \text{on} \quad \mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}. \tag{4.2}$$

Proof. Existence. It follows from Lemmas 4.1 and 4.2 that ω and c_2 give a same cohomology class in $\widetilde{H}^2(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}}\,)$. Namely there exists $\widetilde{\Gamma} \in \widetilde{\mathrm{Hom}}_{\mathbb{C}}(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}}\,)$ such that $c_2 = \omega + d\widetilde{\Gamma}$. But Lemma 4.1 and the definition of c_2 imply that $d\widetilde{\Gamma} = 0$ on $\bigwedge^2(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d})$. Therefore the restriction of $\widetilde{\Gamma}$ on $\bigwedge^2(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d})$ becomes a 1-cocycle in $\widetilde{Z}^1(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}}\,)$. By Lemma 4.4 we can choose some $\xi \in (\mathfrak{h}_0 \oplus \mathbb{C}\mathrm{d})^*$ and $a \in \widehat{\mathcal{O}}$ so that $\widetilde{\Gamma} = g_\xi + da$ on $\bigwedge^2(\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d})$. Put $\Gamma := \widetilde{\Gamma} - f_\xi - da$. Then Γ satisfies (4.1) and (4.2).
Uniqueness. Assume that $\Gamma' \in \widetilde{\mathrm{Hom}}_{\mathbb{C}}(\mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \widehat{\mathcal{O}}\,)$ is also satisfies (4.1) and (4.2) and put $u := \Gamma' - \Gamma$. Then u satisfies (i) $du = 0$ and (ii) $u = 0$ on $\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}$. By (i) and Lemma 4.3 we can take some $\xi \in (\mathfrak{h}_0 \oplus \mathbb{C}\mathrm{d})^*$ and $a \in \widehat{\mathcal{O}}$ so that $u = f_\xi + da$. The definition of f_ξ implies that $f_\xi = 0$ on $\mathrm{L}\mathfrak{n}_+$. Therefore it follows from (ii) that $da = 0$ on $\mathrm{L}\mathfrak{n}_+$. Since Lemma 4.5 implies $a \in \mathbb{C}$ and hence $da = 0$. This and (ii) imply that $f_\xi = 0$ on $\mathrm{L}\mathfrak{b}_+ \oplus \mathbb{C}\mathrm{d}$. Then $\xi = 0$ and hence $f_\xi = 0$. We have just proved $u = 0$. $\square$

Using the linear map Γ in the theorem, we define the linear map $\pi\colon \widehat{\mathfrak{g}} \to \widehat{\mathcal{A}}\,[\Theta]$ by

$$\begin{aligned} &\pi(l) := \widetilde{\pi}(l) + \Gamma(l) \qquad \text{for} \quad l \in \mathrm{L}\mathfrak{g} \oplus \mathbb{C}\mathrm{d}, \\ &\pi(K) := \kappa - h^\vee. \end{aligned}$$

From (4.1) we can immediately obtain the following main result of this article.

Theorem 4.7. *The linear map* π *is a Lie algebra homomorphism from* $\widehat{\mathfrak{g}}$ *into* $\widehat{\mathcal{A}}\,[\Theta]$*. Hence the Fock space* $\mathcal{F}_\xi$ *can be regarded as a left* $\widehat{\mathfrak{g}}$*-module with level* $\kappa - h^\vee$*.*

We call the $\widehat{\mathfrak{g}}$-modules $\mathcal{F}_\xi$ Fock space representations of the twisted affine Lie algebra.

Recall that $\widehat{\mathfrak{g}}$ can be regarded as a Kac-Moody Lie algebra of affine type. Its Cartan subalgebra is equal to $\mathrm{L}\mathfrak{h} \oplus \mathbb{C}\mathrm{d} \oplus \mathbb{C}K$. A pair of $k \in \mathbb{C}$ and $\xi \in (\mathrm{L}\mathfrak{h} \oplus \mathbb{C}\mathrm{d})^*$ can be identified with a weight of $\widehat{\mathfrak{g}}$. Namely the corresponding weight in $(\mathrm{L}\mathfrak{h} \oplus \mathbb{C}\mathrm{d} \oplus \mathbb{C}K)^*$ is defined by

$$a \mapsto \xi(a) \quad (\text{for } a \in \mathrm{L}\mathfrak{h} \oplus \mathbb{C}\mathrm{d}), \qquad K \mapsto k$$

Then (4.2) deduces the following result.

Corollary 4.8. *For* $\kappa \in \mathbb{C}$ *and* $\xi \in \mathrm{L}\mathfrak{h} \oplus \mathbb{C}\mathrm{d}$*, the Fock space representation* $\mathcal{F}_\xi$ *of* $\widehat{\mathfrak{g}}$ *possesses a highest weight* $(\kappa - h^\vee, \xi)$*.*

It is easy to calculate the formal character of the Fock spaces. Thus we obtain the following.

Corollary 4.9. *The formal character of* $\mathcal{F}_\xi$ *is equal to that of the left Verma module* $\mathcal{M}_\xi$ *of* $\widehat{\mathfrak{g}}$ *with the same highest weight. Hence, if the pair of* κ *and* ξ *is enough generic, then* $\mathcal{F}_\xi$ *is isomorphic to* $\mathcal{M}_\xi$*.*

Let us consider the case of $\kappa = 0$, in which the level of a Fock space representation is critical, *i.e.* level $= -h^\vee$. If $\kappa = 0$, then the center of $\widehat{\mathcal{A}}$ is very large (Remark 3.1). Hence we can find many singular vectors with the following forms:

$$p_{i_1,a_1}[m_1] \cdots p_{i_M,a_M}[m_M]\,|\xi\rangle, \qquad m_1 \leq \cdots \leq m_M < 0,$$

where $|\xi\rangle$ is a highest vector of $\mathcal{F}_\xi$. Denote by $\overline{\mathcal{F}}_\xi$ the quotient space of $\mathcal{F}_\xi$ divided by the $\widehat{\mathfrak{g}}$-submodule generated by the singular vectors mentioned above. Note that there is another construction of $\overline{\mathcal{F}}_\xi$. Namely, if we start without $p_{i,a}[m]$'s, then we shall obtain $\overline{\mathcal{F}}_\xi$ using the construction similar to that of $\mathcal{F}_\xi$. Anyway a simple $\widehat{\mathfrak{g}}$ module $\mathcal{L}_\xi$ with highest weight $(-h^\vee, \xi)$ is isomorphic to some subquotient of $\overline{\mathcal{F}}_\xi$. If ξ is enough generic, then $\overline{\mathcal{F}}_\xi$ is isomorphic to $\mathcal{L}_\xi$. Making the above argument precise (cf. [H]), we can obtain the following result.

Corollary 4.10. *The Kac-Kazhdan conjecture* [KK] *is true for the Kac-Moody Lie algebras of affine type (in twisted cases as well as in non-twisted ones).*

References

[DGK] Doedhar, V. V., Gabber, O., Kac, V.: Structure of some categories of representations of infinite-dimensional Lie algebras. Adv. Math. **45**, 92–116 (1982)

[FeFr1] Feigin, B., Frenkel, E.: Representation of affine Kac-Moody algebras, bosonization and resolutions. In: Brink, L., Friedan, D., Polyakov, A.M. (eds.) Physics and Mathematics of Strings. Memorial volume for Vadim Knizhnik, pp. 271–316. Singapore, New Jersey, London, Hong Kong: World Scientific 1990

[FeFr2] Feigin, B., Frenkel, E.: Affine Kac-Moody algebras and semi-infinite flag manifolds. Commun. Math. Phys. **128**, 161–189 (1990)

[GW] Goodman, R., Wallach, N. R.': Higher-order Sugawara operators for affine Lie algebras. Trans. A. M. S. **315**, No. 1, 1–55 (1989)

[H] Hayashi, T.: Sugawara operators and Kac-Kazhdan conjecture. Invent. math. **94**, 13–52 (1988)

[Kac] Kac, V. G.: Infinite dimensional Lie algebras (Second Edition). Cambridge, London, New York, New Rochelle, Melbourne Sydney: Cambridge University Press (1985)

[KK] Kac, V. G., Kazhdan, D. A.: Structure of representations with highest weight of infinite-dimensional Lie algebras. Adv. in Math. **34**, 97–108 (1979)

[Ku] Ku, J. M.: Structure of the Verma module $M(-\rho)$ over Euclidean Lie algebras. J. Alg. **124**, 367–387 (1989)

[Kur] Kuroki, G.: Fock space representations of affine Lie algebras and integral representations in the Wess-Zumino-Witten models. Commun. Math. Phys. **141**, 511–542 (1991)

[W] Wakimoto, N.: Fock representations of the affine Lie algebra $A_1^{(1)}$. Commun. Math. Phys. **104**, 605–609 (1986)