Sato-Wilson formalism for the quantum birational Weyl group actions of type A

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Abstract

We present the the Lax and the Sato-Wilson formalisms for the q-difference version of the quantized birational Weyl group actions of type A_{n-1}, of type A_∞, and of type $A^{(1)}_{n-1}$.

Contents

0 Introduction

In [6], the author canonically the birational Weyl group action arising from a nilpotent Poisson algebra proposed by Noumi and Yamada in [9] and also constructed its q-difference deformation. At that time he was not able to quantize the τ-functions generated by the birational Weyl group action. After that, in [7], he succeeded in quantizing the τ-functions and showed the regularity (or polynomiality) of the quantum τ-functions. However he

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did not mention the Lax and Sato-Wilson formalisms for the quantum birational Weyl group action.

In this paper, we shall present the Lax and the Sato-Wilson formalisms for the q-difference version of the quantum birational Weyl group actions of type A. First we shall construct the Lax and the Sato-Wilson formalisms for the quantum birational Weyl group action of type A_{n-1} (Section 1). Second, taking the inductive limit of them, we shall get the formalisms for the action of type A_∞ (Section 2.1). Third, by the n-periodic reduction, we shall also obtain the formalisms for the actions of type $A^{(1)}_{n-1}$ for $n \geqq 3$ (Section 2.2). Fourth, we shall describe the formalisms of the action of type $A^{(1)}_1$ (Section 2.3).

Notation and Conventions. When $X = \sum_k a_k \otimes b_k$, we set $X^{12} = \sum_k a_k \otimes b_k \otimes 1$, $X^{13} = \sum_k a_k \otimes 1 \otimes b_k$, and $X^{23} = \sum_k 1 \otimes a_k \otimes b_k$. The q-numbers, the q-factorials, and the q-binomial coefficients are defined by

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \qquad [k]_q! = [1]_q[2]_q \cdots [k]_q,$$
$$\begin{bmatrix} a \\ k \end{bmatrix}_q = \frac{[a]_q[a-1]_q[a-2]_q \cdots [a-k+1]_q}{[k]_q!} \quad (k \in \mathbb{Z}_{\geqq 0}).$$

The q-commutator is given by $[A,B]_q = AB - qBA$. The associative algebra is always with unit 1. The matrix units are denoted by E_{ij}. We shall often denote the unit matrix by 1. For any algebra homomorphism f from an algebra R to an algebra R', we shall denote by the same symbol f the induced mapping from the set of matrices over R to the set of matrices over R'. If a and s are mutually commutative elements of an algebra and s is invertible, then we denote $s^{-1}a = as^{-1}$ by the notation of a fraction, a/s.

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1 Quantum birational Weyl group action of type A_{n-1}

1.1 Quantum algebras of type A_{n-1}

In this subsection, we deal with the lower triangular part of the quantum group of type A_{n-1} for a positive integer n.

We define the R-matrix of type A_{n-1} by

$$R = \sum_i qE_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + \sum_{i<j} (q - q^{-1}) E_{ij} \otimes E_{ji}, \tag{1.1}$$

where i and j run through $1, 2, \dots, n$. Then the R-matrix R satisfies the Yang-Baxter equation $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$. Let L be the upper triangular matrix of size n with non-commutative indeterminate entries L_{ij}, namely $L = \sum_{i \leqq j} L_{ij}E_{ij}$.

Let $\mathcal{B}_-$ be the associative algebra over $\mathbb{C}(q)$ generated by L_{ij} $(1 \leqq i \leqq j \leqq n)$ and L_{ii}^{-1} $(1 \leqq i \leqq n)$ with the following fundamental relations:

$$RL^1L^2 = L^2L^1R, \quad L_{ii}L_{ii}^{-1} = L_{ii}^{-1}L_{ii} = 1, \tag{1.2}$$

where $L^1 = L \otimes 1$ and $L^2 = 1 \otimes L$. Explicitly the relation $RL^1L^2 = L^2L^1R$ is equivalent to the following conditions:

$$\begin{aligned}
&i \leqq j < k \leqq l \text{ or } k < i \leqq j < l \implies L_{ij}L_{kl} = L_{kl}L_{ij},\\
&k < i \leqq j \implies L_{ij}L_{kj} = qL_{kj}L_{ij},\\
&i \leqq j < l \implies L_{ij}L_{il} = q^{-1}L_{il}L_{ij},\\
&k < i \leqq l < j \implies L_{ij}L_{kl} - L_{kl}L_{ij} = (q-q^{-1})L_{kj}L_{il}.
\end{aligned}$$

The matrix L is called *the L-operator of type A_{n-1}*.

Remark 1.1. When $L = \sum_{i,j} L_{ij}E_{ij}$ is not upper triangular, the relation $RL^1L^2 = L^2L^1R$ is equivalent to the following relations:

$$\begin{aligned}
&L_{il}L_{kj} = L_{il}L_{kj}, \quad L_{ij}L_{kl} - L_{kl}L_{ij} = (q-q^{-1})L_{il}L_{kj},\\
&L_{ij}L_{il} = qL_{il}L_{ij}, \quad L_{ij}L_{kj} = qL_{kj}L_{ij} \qquad\qquad (k < i, l < j).
\end{aligned}$$

These relations are summarized in the following diagram:

$$\begin{array}{ccc} L_{kl} & \leftarrow & L_{kj} \\ \uparrow & \nwarrow & \uparrow \\ L_{il} & \leftarrow & L_{ij} \end{array}, \qquad (k < i, l < j),$$

where the vertical and horizontal arrows stand for the relations of type $L_{ij}L_{il} = qL_{il}L_{ij}$, the sloping arrow stands for the relation $L_{ij}L_{kl} - L_{kl}L_{ij} = (q-q^{-1})L_{il}L_{kj}$, and the other combination without a connecting arrow commutes. The fundamental relations of $\mathcal{B}_-$ are the specialization of these relations to the case where L is upper triangular. □

Denote by D_L the diagonal part of L, namely $D_L = \sum_i L_{ii}E_{ii}$. We define the unipotent upper triangular matrix $\widetilde{L}$ by $\widetilde{L} = D_L^{-1}L$ and denote its (i,j)-entry by f_{ij}:

$$\widetilde{L} = D_L^{-1}L = 1 + \sum_{i<j} f_{ij}E_{ij}, \quad f_{ij} = L_{ii}^{-1}L_{ij}.$$

We call the matrix $\widetilde{L}$ *the quasi L-operator of type A_{n-1}*.

Let $\mathcal{N}_-$ be the subalgebra of $\mathcal{B}_-$ generated by f_{ij} $(i < j)$. We define $f_i \in \mathcal{N}_-$ by

$$f_i = (q-q^{-1})^{-1} f_{i,i+1}. \tag{1.3}$$

Then we have

$$f_j f_{ij} - q^{-1} f_{ij} f_j = f_{i,j+1} \quad (i < j). \tag{1.4}$$

and hence the algebra $\mathcal{N}_-$ is generated by f_i's. Moreover they satisfy the following q-Serre relations:

$$f_i^2 f_{i\pm1} - (q+q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0, \qquad f_i f_j = f_j f_i \quad (|i-j| \geqq 2). \tag{1.5}$$

Under the condition where L_{ii}'s are invertible, we can derive the fundamental relations of $\mathcal{B}_-$ from (1.3), (1.4), (1.5), and the following relations:

$$L_{ii} f_{ij} = q^{-1} f_{ij} L_{ii}, \quad L_{jj} f_{ij} = q f_{ij} L_{jj}, \quad L_{kk} f_{ij} = f_{ij} L_{kk} \quad (i < j, k \neq i, j).$$

Therefore the algebra $\mathcal{N}_-$ is isomorphic to the upper triangular part $U_q(\mathfrak{n}_-)$ of the q-difference deformation $U_q(\mathfrak{g})$ of the universal enveloping algebra $U_q(\mathfrak{g})$ of the Kac-Moody algebra $\mathfrak{g}$ of type A_{n-1}. (See also Section II of [2].)

In the sequel, we identify $\mathcal{N}_-$ with $U_q(\mathfrak{n}_-)$ and denote $\mathcal{N}_- = U_q(\mathfrak{n}_-)$ simply by U_-. We call the Chevalley generators f_i of U_- *the quantum dependent variables*.

We denote by $Q(R)$ the skew field of fractions of an Ore domain R. The algebra U_- is a Noetherian domain (Sections 7.3 and 7.4 of [5]). Therefore U_- is an Ore domain. We obtain the skew field $K = Q(U_-)$ of fractions of U_-.

1.2 Quantum birational Weyl group action

The Weyl group $W = W(A_{n-1})$ of type A_{n-1} is defined to be the group generated by $s_1, s_2, \dots, s_{n-1}$ with the following fundamental relations:

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \quad (|i-j| \geqq 2), \qquad s_i^2 = 1. \tag{1.6}$$

Then W is isomorphic to the permutation group of $\{1, 2, \dots, n\}$ by sending s_i to the transposition $(i, i+1)$.

Let $Q^\vee$ be the free $\mathbb{Z}$-module generated by $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ and put $P = \mathrm{Hom}(Q^\vee, \mathbb{Z})$. Denote the canonical pairing between $Q^\vee$ and P by $\langle\ ,\ \rangle$. We call $Q^\vee$ the coroot lattice and P the weight lattice. Denote the dual basis of $\{\varepsilon_i^\vee\}_{i=1}^n$ by $\{\varepsilon_i\}_{i=1}^n$. We define the simple coroots $\alpha_i^\vee$ $(i = 1, \dots, n-1)$ by $\alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee$ and the simple roots α_i $(i = 1, \dots, n-1)$ by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. The fundamental weights Λ_i $(i = 0, 1, \dots, n)$ and the Weyl vector ρ are given by $\Lambda_i = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i$ and $\rho = \sum_{i=1}^{n-1} \Lambda_i$. Then we have

$$\langle \alpha_i^\vee, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}, \quad \langle \alpha_i^\vee, \rho \rangle = 1. \tag{1.7}$$

The matrix $[a_{ij}]_{i=1}^{n-1}$ is defined by $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ and called the Cartan matrix of type A_{n-1}.

The Weyl group acts on $Q^\vee$ and P by

$$s_i(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i^\vee \quad (\beta \in Q^\vee), \qquad s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \quad (\lambda \in P).$$

Then the Weyl group actions on $Q^\vee$ and P preserve the canonical pairing between them.

Denote by $K[q^\beta | \beta \in Q^\vee]$ the associative algebra over the skew field $K = Q(U_-)$ generated by symbols q^β for $\beta \in Q^\vee$ with the following fundamental relations:

$$q^\beta f_i = f_i q^\beta, \quad q^\beta q^\gamma = q^{\beta+\gamma}, \quad q^0 = 1 \quad (\beta, \gamma, 0 \in Q^\vee).$$

It coincides with the Laurent polynomial ring over K generated by $\{q^{\pm\varepsilon_i^\vee}\}_{i=1}^n$ and hence is an Ore domain. Denote by K^{pa} the skew filed of fractions of $K[q^\beta | \beta \in Q^\vee] = K[q^{\pm\varepsilon_1^\vee}, \dots, q^{\pm\varepsilon_n^\vee}]$. We call $q^{\varepsilon_i^\vee}$'s *the parameter variables*. The algebra U_-^{pa} is defined to be the subalgebra of K^{pa} generated by U_- and $\{q^{\pm\varepsilon_i^\vee}\}_{i=1}^n$. Here $(\ )^{\mathrm{pa}}$ stands for *an algebra with parameter variables*.

For each $\lambda \in P$, we define the algebra homomorphism $\phi_\lambda : U_-^{\mathrm{pa}} \to U_-$ by $\phi_\lambda(q^\beta) = q^{\langle \beta, \lambda \rangle}$ $(\beta \in Q^\vee)$ and $\phi_\lambda(f_i) = f_i$. Using Theorem 2.1 of [10], we can prove that $S_\lambda = \{\, a \in U_-^{\mathrm{pa}} \mid$

$\phi_\lambda(a) \neq 0\}$ is an Ore set in U_-^{pa}. Therefore we obtain the localization $U_-^{\mathrm{pa}}[S_\lambda^{-1}] \subset K^{\mathrm{pa}}$. Define the algebra $U_{-,(P)}^{\mathrm{pa}}$ to be the intersection of $U_-[S_\lambda^{-1}]$ for all $\lambda \in P$. (For details, see [7].) The algebra homomorphism ϕ_λ is uniquely extended to the algebra homomorphism $\phi_\lambda : U_{-,(P)}^{\mathrm{pa}} \to K$, which substitutes $\langle \beta, \lambda \rangle$ in $\beta \in Q^\vee$.

Let $D(K^{\mathrm{pa}})$ be the associative algebra generated by K^{pa} and $\{\tau^\lambda\}_{\lambda \in P}$ with the following defining relations:

$$\begin{aligned} &\tau^\lambda \tau^\mu = \tau^{\lambda+\mu}, \quad \tau^0 = 1 \quad (\lambda, \mu, 0 \in P), \\ &\tau^\lambda f_i = f_i \tau^\lambda, \quad \tau^\lambda q^\beta = q^{\beta + \langle \beta, \lambda \rangle} \tau^\lambda \quad (\lambda \in P, \beta \in Q^\vee). \end{aligned} \tag{1.8}$$

Then we have $D(K^{\mathrm{pa}}) = \bigoplus_{\lambda \in P} K^{\mathrm{pa}} \tau^\lambda$. The symbol $D(\ ^{\mathrm{pa}})$ stands for *a difference operator algebra with respect to the parameter variables.* We define *the quantum τ-variables* τ_i by $\tau_i = \tau^{\Lambda_i}$ $(i = 0, 1 \ldots, n)$ and call τ^λ's *the quantum Laurent τ-monomials.* (Note that $\tau_0 = 1$.) Let $D(U_-^{\mathrm{pa}})$ (resp. $D(U_{-,(P)}^{\mathrm{pa}})$) be the subalgebra of $D(K^{\mathrm{pa}})$ generated by U_-^{pa} (resp. $U_{-,(P)}^{\mathrm{pa}}$) and $\{\tau^\lambda\}_{\lambda \in P}$.

The following proposition is a special case of the general result of [7] (see also [6]).

Proposition 1.2. *For each $i = 1, 2, \ldots, n-1$, the algebra automorphism $\mathbf{s}_i$ of $D(U_{-,(P)}^{\mathrm{pa}})$ can be given by*

$$\begin{aligned} &\mathbf{s}_i(f_j) = q^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_q [f_i, f_j]_{q^{-1}} f_i^{-1} \quad (j = i \pm 1), \quad \mathbf{s}_i(f_j) = f_j \quad (j \neq i \pm 1), \\ &\mathbf{s}_i(q^\beta) = q^{s_i(\beta)} \quad (\beta \in Q^\vee), \qquad \mathbf{s}_i(\tau^\lambda) = f_i^{\langle \alpha_i^\vee, \lambda \rangle} \tau^{s_i(\lambda)} \quad (\lambda \in P). \end{aligned}$$

Then the mapping $s_i \mapsto \mathbf{s}_i$ defines the Weyl group actions on $D(U_{-,(P)}^{\mathrm{pa}})$, $U_{-,(P)}^{\mathrm{pa}}$, $D(K^{\mathrm{pa}})$, and K^{pa}. □

Definition 1.3. The Weyl group actions on $D(U_{-,(P)}^{\mathrm{pa}})$, $U_{-,(P)}^{\mathrm{pa}}$, $D(K^{\mathrm{pa}})$, and K^{pa} obtained by Proposition 1.2 are called *the quantum birational Weyl group actions of type A_{n-1}.* Denote the action of $w \in W$ on $x \in D(K^{\mathrm{pa}})$ by $w(x)$. □

Remark 1.4. For $w \in W$, we define the algebra automorphism $\widetilde{w}$ of $D(K^{\mathrm{pa}})$ by

$$\widetilde{w}(f_i) = f_i, \quad \widetilde{w}(q^\beta) = q^{w(\beta)}, \quad \widetilde{w}(\tau^\lambda) = \tau^{w(\lambda)} \quad (\beta \in Q^\vee, \lambda \in P).$$

This defines the Weyl group action on $D(K^{\mathrm{pa}})$, which is called *the tilde action.* In [6] and [7], the author constructs the quantum birational Weyl group action by

$$s_i(x) = f_i^{\alpha_i^\vee} \tilde{s}_i(x) f_i^{-\alpha_i^\vee} \quad (x \in D(K^{\mathrm{pa}})),$$

where $f_i^{\alpha_i^\vee}$ denotes a fractional power of f_i. For details, see [7]. □

Remark 1.5. Applying the main result of [7] to the quantum birational Weyl group action of type A_{n-1}, we obtain that $w(\tau_i) \in D(U_-^{\mathrm{pa}})$ for $w \in W$ and $i = 1, 2, \ldots, n-1$. More precisely, there exists a unique non-commutative polynomial $\phi_{i,w}$ in $\{f_i, q^{\pm\alpha_i^\vee}\}_{i=1}^{n-1}$ such that $w(\tau_i) = \phi_{i,w} \tau^{w(\Lambda_i)}$. Such a result is called *the regularity of the quantum τ-functions* in [7]. □

1.3 Lax formalism

For the notational simplicity, we shall denote $q^{-\varepsilon_i^\vee}$ by t_i and $q^{-\alpha_i^\vee}$ by a_i:

$$t_i = q^{-\varepsilon_i^\vee}, \quad a_i = q^{-\alpha_i^\vee} = t_i/t_{i+1}.$$

We should be careful of the minus signs in the exponents. Then we have

$$\begin{aligned}
s_i(f_j) &= a_i f_j + \frac{a_i^{-1}-a_i}{q-q^{-1}}[f_i, f_j]_{q^{-1}} f_i^{-1} = a_i^{-1} f_j + \frac{a_i^{-1}-a_i}{q-q^{-1}}[f_i, f_j]_q f_i^{-1} \\
&= \frac{q a_i - q^{-1}a_i^{-1}}{q-q^{-1}} f_j + \frac{a_i^{-1}-a_i}{q-q^{-1}} f_i f_j f_i^{-1} \qquad (j = i \pm 1), && (1.9) \\
s_i(f_j) &= f_j \qquad (j \ne i \pm 1), && (1.10) \\
s_i(t_i) &= t_{i+1}, \quad s_i(t_{i+1}) = t_i, \quad s_i(t_j) = t_j \quad (j \ne i, i+1), && (1.11) \\
s_i(\tau_i) &= f_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i} \qquad (i = 1, \dots, n-1), \\
s_i(\tau_j) &= \tau_j \qquad (i \ne j).
\end{aligned}$$

These formulas uniquely characterize the action of s_i on $D(K^{\mathrm{pa}})$.

Define the diagonal matrix D_t of the parameter variables by

$$D_t = \sum_{i=1}^n t_i E_{ii} = \mathrm{diag}(t_i)_{i=1}^n.$$

Recall that, in Section 1.1, the quasi L-operator $\widetilde{L}$ is defined by $\widetilde{L} = D_L^{-1} L$, where D_L is the diagonal part of the L-operator L. We introduce *the M-operator* by

$$M = D_t \widetilde{L} D_t = D_t^2 + \sum_{i<j} t_i t_j f_{ij} E_{ij} = [m_{ij}]_{i,j=1}^n.$$

Define *the G-matrices* G_i $(i = 1, 2, \dots, n-1)$ by

$$G_i = 1 + g_i E_{i+1,i}, \quad g_i = \frac{t_i^2 - t_{i+1}^2}{m_{i,i+1}} = \frac{a_i - a_i^{-1}}{f_{i,i+1}} = -\frac{[\alpha_i^\vee]_q}{f_i}.$$

Here we should remember that $a_i = q^{-\alpha_i^\vee}$ (not $a_i = q^{\alpha_i^\vee}$). Then we can obtain the following theorem by straightforward calculations.

Theorem 1.6 (Lax formalism for type A_{n-1})**.** *We have*

$$s_i(M) = G_i M G_i^{-1} \quad \text{for } i = 1, 2, \dots, n-1.$$

These formulas with (1.11) *uniquely characterize the quantum birational Weyl group action of type A_{n-1} on K^{pa}.*

Proof. Using (1.3), (1.4), and (1.5), we can write down the explicit formulas for the actions of s_i on f_{kl}'s as below:

$$\begin{aligned}
s_i(f_{ki}) &= a_i f_{ki} - (a_i - a_i^{-1}) f_{k,i+1} f_{i,i+1}^{-1} \qquad (k < i), \\
s_i(f_{k,i+1}) &= a_i^{-1} f_{k,i+1} \qquad (k < i),
\end{aligned}$$

$$
\begin{aligned}
&s_i(f_{i+1,l}) = a_i^{-1} f_{i+1,l} + (a_i - a_i^{-1}) f_{i,i+1}^{-1} f_{il} \qquad (l > i+1),\\
&s_i(f_{il}) = a_i f_{il} \qquad (l > i+1),\\
&s_i(f_{kl}) = f_{kl} \qquad \text{for other } f_{kl}.
\end{aligned}
$$

These formulas with (1.11) immediately lead to $s_i(M) = G_i M G_i^{-1}$. The last statement of the theorem is clear. □

Remark 1.7 (The $q \to 1$ limit)**.** We shall sketch how to obtain the limit of Theorem 1.6 as $q \to 1$. Do not confuse it with a classical limit. Define x_{ji} by $f_{ij} = (q - q^{-1}) x_{ji}$ $(i < j)$. Then we have $[x_{j+1,j}, x_{ji}]_q = x_{j+1,i}$ $(i < j)$. Therefore, after taking the limit as $q \to 1$, we obtain the relations $[x_{ij}, x_{kl}] = \delta_{jk} x_{il} - \delta_{li} x_{kj}$ $(i < j,\ k < l)$, which are the fundamental relations of the matrix units. Setting $q = e^{\hbar/2}$, we have

$$
M = 1 + \hbar \mathcal{M} + O(\hbar^2), \quad G_i = \mathcal{G}_i + O(\hbar),
$$

where $\mathcal{M}$ and $\mathcal{G}_i$ are defined by

$$
\mathcal{M} = -\sum_i \varepsilon_i^\vee E_{ii} + \sum_{i<j} x_{ji} E_{ij}, \quad \mathcal{G}_i = 1 - \frac{\varepsilon_i^\vee - \varepsilon_{i+1}^\vee}{x_{i+1,i}} E_{i+1,i}.
$$

The limit of the quantum birational Weyl group action of type A_{n-1} as $q \to 1$ can be written in the form $s_i(\mathcal{M}) = \mathcal{G}_i \mathcal{M} \mathcal{G}_i^{-1}$. This is the straightforward canonical quantization of Theorem 7.1 in [8]. Theorem 1.6 is both the quantization and the q-difference analogue of Theorem 7.1 in [8]. □

Remark 1.8 (Quantum unipotent crystal structure on M)**.** In this remark, we shall deform the algebra automorphisms $\mathbf{s}_i$ $(i = 1, 2, \ldots, n-1)$ and construct a quantum unipotent crystal structure on the M-operator.

For any element c of the center of $(K^{\mathrm{pa}})^\times$, we can define the algebra automorphism $\mathbf{e}_i^c$ of K^{pa} as follows. We define the actions of $\mathbf{e}_i^c$ on f_j's by

$$
\begin{aligned}
\mathbf{e}_i^c(f_j) &= c f_j + \frac{c^{-1} - c}{q - q^{-1}} [f_i, f_j]_{q^{-1}} f_i^{-1} = c^{-1} f_j + \frac{c^{-1} - c}{q - q^{-1}} [f_i, f_j]_q f_i^{-1}\\
&= \frac{qc - q^{-1} c^{-1}}{q - q^{-1}} f_j + \frac{c^{-1} - c}{q - q^{-1}} f_j f_j f_j^{-1} \qquad (j = i \pm 1),\\
\mathbf{e}_i^c(f_j) &= f_j \qquad (j \neq i \pm 1).
\end{aligned}
$$

If we formally denote c by $q^{-\gamma}$, then we have $\mathbf{e}_i^c(f_j) = f_i^\gamma f_j f_i^{-\gamma}$. (For the construction of fractional powers of f_i, see [7].) The actions of $\mathbf{e}_i^c$ on t_j's are given by

$$
\mathbf{e}_i^c(t_i) = c^{-1} t_i, \quad \mathbf{e}_i^c(t_{i+1}) = c t_{i+1}, \quad \mathbf{e}_i^c(t_j) = t_j \quad (j \neq i, i+1).
$$

Equivalently, we set $\mathbf{e}_i^c(q^\beta) = c^{\langle \beta, \alpha_i \rangle} q^\beta$ for $\beta \in Q^\vee$. Then the specialization of c at $q^{-\alpha_i^\vee}$ gives $\mathbf{e}_i^c = \mathbf{s}_i$. Putting $\alpha_i(M) = m_{ii}/m_{i+1.i+1}$, $\psi_i(M) = m_{i,i+1}/m_{ii}$, and $y_i(a) = \exp(a E_{i+1,i})$, we obtain

$$
\mathbf{e}_i^c(M) = y_i\left(\frac{c^2 - 1}{\alpha_i(M)\psi_i(M)}\right) \cdot M \cdot y_i\left(\frac{c^{-2} - 1}{\psi_i(M)}\right).
$$

Therefore we can regard the upper triangular matrix M as a quantum unipotent crystal. Compare the above formula with Equation (3.8) of [1]. □

1.4 Sato-Wilson formalism

Since M-operator is the upper triangular matrix with mutually distinct diagonal entries, it can be uniquely diagonalized by the unipotent upper triangular matrix U:

$$M = UD_t^2 U^{-1}, \quad U = 1 + \sum_{i<j} u_{ij} E_{ij},$$

where u_{ij}'s are given by

$$u_{ij} = \sum_{r=1}^{j-i} (-1)^r \sum_{i=i_0<i_1<\cdots<i_r=j} \frac{m_{i_0 i_1} m_{i_1 i_2} \cdots m_{i_{r-1} i_r}}{(t_{i_0}^2 - t_j^2)(t_{i_1}^2 - t_j^2) \cdots (t_{i_{r-1}}^2 - t_j^2)} \quad (i<j). \tag{1.12}$$

In particular, we have $u_{i,i+1} = -m_{i,i+1}/(t_i^2 - t_{i+1}^2) = -g_i^{-1} = f_i/[\alpha_i^\vee]_q$.

The uniqueness of U and Theorem 1.6 show that $s_i(U)$ is written in the following form:

$$s_i(U) = G_i U S_i^g, \quad S_i^g = g_i^{-1} E_{i,i+1} - g_i E_{i+1,i} + \sum_{k \neq i,i+1} E_{kk}. \tag{1.13}$$

In fact, the matrix $G_i U S_i^g$ is unipotent upper triangular and Theorem 1.6 leads to

$$s_i(M) = G_i M G_i^{-1} = G_i U D_t^2 (G_i U)^{-1} = G_i U S_i^g s_i(D_t^2)(G_i U S_i^g)^{-1}.$$

On the other hand, we have $s_i(M) = s_i(U) s_i(D_t^2) s_i(U)^{-1}$. Therefore we obtain $s_i(U) = G_i U S_i^g$. In order to show that $G_i U S_i^g$ is unipotent upper triangular, it is sufficient to calculate its 2-by-2 part for $(i, i+1)$:

$$\begin{bmatrix} 1 & 0 \\ g_i & 1 \end{bmatrix} \begin{bmatrix} 1 & -g_i^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & g_i^{-1} \\ -g_i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -g_i^{-1} \\ g_i & 0 \end{bmatrix} \begin{bmatrix} 0 & g_i^{-1} \\ -g_i & 0 \end{bmatrix} = \begin{bmatrix} 1 & g_i^{-1} \\ 0 & 1 \end{bmatrix}.$$

Define *the z-variables* by $z_i = \tau^{\varepsilon_i}$ $(i = 1, 2, \dots, n)$ and the diagonal matrix D_Z of the z-variables by

$$D_Z = \sum_{i=1}^n z_i E_{ii} = \operatorname{diag}(z_i)_{i=1}^n, \qquad z_i = \tau^{\varepsilon_i} = \frac{\tau_i}{\tau_{i-1}}.$$

We define *the Z-operator* by

$$Z = UD_Z = D_Z + \sum_{i<j} u_{ij} z_j E_{ij} = [z_{ij}]_{i,j=1}^n.$$

Since $z_i t_j = q^{-\delta_{ij}} t_j z_i$, we have $M = q^2 Z D_t^2 Z^{-1}$. The actions of s_i on z_j's are explicitly written in the following form:

$$s_i(z_i) = f_i z_{i+1}, \quad s_i(z_{i+1}) = f_i^{-1} z_i, \quad s_i(z_j) = z_j \quad (j \neq i, i+1). \tag{1.14}$$

Define the matrices S_i $(i = 1, 2, \dots, n-1)$ by

$$S_i = -[\alpha_i^\vee - 1]_q^{-1} E_{i,i+1} + [\alpha_i^\vee + 1]_q E_{i+1,i} + \sum_{k \neq i,i+1} E_{kk}.$$

Theorem 1.9 (Sato-Wilson formalism for type A_{n-1})**.** *We have*

$$s_i(Z) = G_i Z S_i, \quad s_i(D_t) = S_i^{-1} D_t S_i \quad \textit{for } i = 1, 2, \dots, n-1. \tag{1.15}$$

These formulas uniquely characterize the whole of the quantum birational Weyl group action of type A_{n-1} on $D(K^{\mathrm{pa}})$.

Proof. Equation (1.11) is equivalent to $s_i(D_t) = S_i D_t S_i^{-1}$. Because of (1.13), the formula $s_i(Z) = G_i Z S_i$ is equivalent to $s_i(D_Z) = (S_i^g)^{-1} D_Z S_i$ and hence equivalent to (1.14). In order to show the last equivalence, it is sufficient to calculate the 2-by-2 part of $(S_i^g)^{-1} D_Z S_i$ for $(i, i+1)$:

$$\begin{aligned}
&- g_i^{-1} z_{i+1} = [\alpha_i^\vee]_q^{-1} f_i z_{i+1} = f_i z_{i+1} [\alpha_i^\vee + 1]_q^{-1} = s_i(z_i)[\alpha_i^\vee + 1]_q^{-1}, \\
&g_i z_i = -[\alpha_i^\vee]_q f_i^{-1} z_i = -f_i^{-1} z_i [\alpha_i^\vee - 1]_q = -s_i(z_{i+1})[\alpha_i^\vee - 1]_q, \\
&\begin{bmatrix} 0 & -g_i^{-1} \\ g_i & 0 \end{bmatrix} \begin{bmatrix} z_i & 0 \\ 0 & z_{i+1} \end{bmatrix} \begin{bmatrix} 0 & -[\alpha_i^\vee - 1]^{-1} \\ [\alpha_i^\vee + 1] & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -g_i^{-1} z_{i+1} \\ g_i z_i & 0 \end{bmatrix} \begin{bmatrix} 0 & -[\alpha_i^\vee - 1]^{-1} \\ [\alpha_i^\vee + 1] & 0 \end{bmatrix} = \begin{bmatrix} s_i(z_i) & 0 \\ 0 & s_i(z_{i+1}) \end{bmatrix}.
\end{aligned}$$

We have shown Equation (1.15). Equation (1.15) implies $s_i(M) = G_i M G_i^{-1}$. Therefore Theorem 1.6 leads to the last statement of the theorem. □

Remark 1.10. Theorem 1.9 is both the quantization and the q-difference analogue of Theorem 7.3 in [8]. In the classical (and hence commutative) case, the Sato-Wilson formalism of the birational Weyl group action of type A_{n-1} immediately leads to the regularity of the classical τ-functions, because a determinant is a polynomial in its entries. But, in the quantum (and hence non-commutative) case, the Sato-Wilson formalism does not immediately lead to the regularity of the quantum τ-fucntions, because a non-commutative determinant is not always a non-commutative polynomial but a non-commutative rational function in its entries, owing to the theory of quasi-determinants ([3], [4]). In [7], the regularity of the quantum τ-functions are shown by the theory of the translation functors in representation theory. It is an open problem whether or not the quantum τ-functions admit non-commutative polynomial-type determinant representations. □

2 The cases of type A_∞ and of type $A_{n-1}^{(1)}$

2.1 The case of type A_∞

We define the R-matrix R of type A_∞ by the formula (1.1) in which E_{ij}'s denote the matrix units of size $\mathbb{Z}$ and i, j run through all integers. Then R-matrix R satisfies the Yang-Baxter equation. Let L be the upper triangular matrix of size $\mathbb{Z}$ with non-commutative indeterminate entries L_{ij}, namely $L = \sum_{i \leqq j} L_{ij} E_{ij}$. We call L the L-operator of type A_∞.

Let $\mathcal{B}_{-,\infty}$ be the associative algebra over $\mathbb{C}(q)$ generated by L_{ij} ($i, j \in \mathbb{Z}$ with $i \leqq j$) and L_{ii}^{-1} ($i \in \mathbb{Z}$) with fundamental relation (1.2). Denote by D_L the diagonal part of L, namely set $D_L = \sum_{i \in \mathbb{Z}} L_{ii} E_{ii}$. We define the quasi L-operator of type A_∞ by $\widetilde{L} = D_L^{-1} L = 1 + \sum_{i<j} f_{ij} E_{ij}$. Let $\mathcal{N}_{-,\infty}$ be the subalgebra of $\mathcal{B}_{-,\infty}$ generated by f_{ij} ($i < j$). We define $f_i \in \mathcal{N}_{-,\infty}$ by $f_i = (q - q^{-1})^{-1} f_{i,i+1}$.

The algebra $\mathcal{N}_{-,\infty}$ can be regarded as an inductive limit as $n \to \infty$ of the algebras $\mathcal{N}_-$ of type A_{n-1} defined in Section 1. Therefore $\mathcal{N}_{-,\infty}$ is the algebra generated by f_i ($i \in \mathbb{Z}$) with the fundamental relations (1.5) and is an Ore domain. In the sequel, we shall denote $\mathcal{N}_{-,\infty}$ by $U_{-,\infty}$.

The Weyl group $W_\infty = W(A_\infty)$ of type A_∞ is defined to be the group generated by s_i ($i \in \mathbb{Z}$) with the fundamental relations (1.6). The extended Weyl group $\widetilde{W}_\infty$ is given by the semi-direct product $\widetilde{W}_\infty = W_\infty \rtimes \langle \pi \rangle$ with defining relations $\pi s_i = s_{i+1}\pi$ ($i \in \mathbb{Z}$).

Let $Q^\vee_\infty$ be the free $\mathbb{Z}$-module generated by $\delta^\vee$, $\varepsilon^\vee_i$ ($i \in \mathbb{Z}$). Put $\widetilde{P}_\infty = \mathrm{Hom}(Q^\vee_\infty, \mathbb{Z})$ and denote the canonical pairing between $Q^\vee_\infty$ and $\widetilde{P}_\infty$ by $\langle\ ,\ \rangle$. Define $\Lambda_0, \varepsilon_i \in \widetilde{P}_\infty$ ($i \in \mathbb{Z}$) by

$$\langle \varepsilon^\vee_i, \varepsilon_j \rangle = \delta_{ij}, \quad \langle \delta^\vee, \varepsilon_j \rangle = 0, \quad \langle \varepsilon^\vee_i, \Lambda_0 \rangle = \begin{cases} 1 & (i \leqq 0), \\ 0 & (i > 0), \end{cases} \quad \langle \delta^\vee, \Lambda_0 \rangle = 1.$$

Denote by P_∞ the submodule of $\widetilde{P}_\infty$ generated by Λ_0 and ε_i ($i \in \mathbb{Z}$).

The actions of π on $Q^\vee_\infty$ and P_∞ are defined by

$$\pi(\varepsilon^\vee_i) = \varepsilon^\vee_{i+1}, \quad \pi(\delta^\vee) = \delta^\vee, \quad \pi(\varepsilon_i) = \varepsilon_{i+1}, \quad \pi(\Lambda_0) = \Lambda_0 + \varepsilon_1.$$

The actions of π preserve the canonical pairing between $Q^\vee_\infty$ and P_∞. We define the simple coroots $\alpha^\vee_i$, the simple roots α_i, and the fundamental weights Λ_i by

$$\alpha^\vee_i = \varepsilon^\vee_i - \varepsilon^\vee_{i+1}, \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad \Lambda_i = \begin{cases} \Lambda_0 + \varepsilon_1 + \cdots + \varepsilon_i & (i \geqq 0), \\ \Lambda_0 - \varepsilon_0 - \cdots - \varepsilon_{i+1} & (i < 0). \end{cases}$$

Then we have $\pi(\alpha^\vee_i) = \alpha^\vee_{i+1}$, $\pi(\alpha_i) = \alpha_{i+1}$, and $\pi(\Lambda_i) = \Lambda_{i+1}$. The set $\{\Lambda_i\}_{i \in \mathbb{Z}}$ is a free $\mathbb{Z}$-basis of P_∞. We also define the Weyl vector $\rho \in \widetilde{P}_\infty$ by $2\rho = -\sum_{i \in \mathbb{Z}} \varepsilon_i$. Then we obtain the formulas in (1.7). The generalized Cartan matrix $[a_{ij}]_{i \in \mathbb{Z}}$ of type A_∞ is defined by $a_{ij} = \langle \alpha^\vee_i, \alpha_j \rangle$.

The extended Weyl group actions on $Q^\vee_\infty$ and P_∞ by the following formulas with the above of π given above:

$$s_i(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha^\vee_i \quad (\beta \in Q^\vee_\infty), \qquad s_i(\lambda) = \lambda - \langle \alpha^\vee_i, \lambda \rangle \alpha_i \quad (\lambda \in P_\infty).$$

More explicitly, we have

$$\begin{aligned}
&s_i(\varepsilon^\vee_i) = \varepsilon^\vee_{i+1}, \quad s_i(\varepsilon^\vee_{i+1}) = \varepsilon^\vee_i, \quad s_i(\varepsilon^\vee_j) = \varepsilon^\vee_j \quad (j \neq i, i+1), \\
&s_i(\delta^\vee) = \delta^\vee, \\
&s_i(\varepsilon_i) = \varepsilon_{i+1}, \quad s_i(\varepsilon_{i+1}) = \varepsilon_i, \quad s_i(\varepsilon_j) = \varepsilon_j \quad (j \neq i, i+1), \\
&s_i(\Lambda_i) = \Lambda_i - \alpha_i = \Lambda_{i-1} - \Lambda_i + \Lambda_{i+1}, \quad s_i(\Lambda_j) = \Lambda_j \quad (j \neq i).
\end{aligned}$$

We consider the Laurent polynomial ring $U^{\mathrm{pa}}_{-,\infty} = U_{-,\infty}[q^\beta | \beta \in Q^\vee_\infty]$ spanned by $\{q^\beta\}_{\beta \in Q^\vee_\infty}$ over $U_{-,\infty}$. Then $U^{\mathrm{pa}}_{-,\infty}$ is also an Ore domain. We obtain the skew field K^{pa}_∞ of fractions of $U^{\mathrm{pa}}_{-,\infty}$. Let $D(K^{\mathrm{pa}}_\infty)$ be the algebra generated by K^{pa}_∞ and $\{\tau^\lambda\}_{\lambda \in P_\infty}$ with the defining relations (1.8) in which P is replaced by P_∞.

The following proposition immediately follows from Proposition 1.2.

Proposition 2.1. *The algebra automorphism actions of s_i ($i \in \mathbb{Z}$) and π on $D(K_\infty^{\mathrm{pa}})$ can be given by*

$$\begin{aligned}
&s_i(f_j) = q^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_q [f_i, f_j]_{q^{-1}} f_i^{-1} \quad (j = i \pm 1), \quad s_i(f_j) = f_j \quad (j \neq i \pm 1), \\
&s_i(q^\beta) = q^{s_i(\beta)} \quad (\beta \in Q_\infty^\vee), \qquad s_i(\tau^\lambda) = f_i^{\langle \alpha_i^\vee, \lambda \rangle} \tau^{s_i(\lambda)} \quad (\lambda \in P_\infty), \\
&\pi(f_i) = f_{i+1}, \qquad \pi(q^\beta) = q^{\pi(\beta)} \quad (\beta \in Q_\infty^\vee), \qquad \pi(\tau^\lambda) = \tau^{\pi(\lambda)} \quad (\lambda \in P_\infty).
\end{aligned}$$

These actions defines the actions of $\widetilde{W}_\infty$ on $D(K_\infty^{\mathrm{pa}})$. □

For the notational simplicity, we put

$$p = q^{-\delta^\vee}, \quad t_i = q^{-\varepsilon_i^\vee}, \quad a_i = q^{-\alpha_i^\vee} = t_i / t_{i+1}, \quad \tau_i = \tau^{\Lambda_i}, \quad z_i = \tau^{\varepsilon_i} = \tau_i / \tau_{i-1}.$$

Then the algebra $U_{-,\infty}^{\mathrm{pa}}$ is generated by $\{\, f_i, t_i^{\pm 1}, p^{\pm 1} \mid i \in \mathbb{Z} \,\}$, and the algebra $D(K_\infty^{\mathrm{pa}})$ is generated by K_∞^{pa} and $\{\tau_i^{\pm 1}\}_{i \in \mathbb{Z}}$. The diagonal matrices D_t and D_Z are given by

$$D_t = \sum_{i \in \mathbb{Z}} t_i E_{ii}, \quad D_Z = \sum_{i \in \mathbb{Z}} z_i E_{ii}.$$

We define the matrices M, G_i, and Λ by

$$M = D_t \widetilde{L} D_t = [m_{ij}]_{i,j \in \mathbb{Z}}, \quad G_i = 1 + g_i E_{i+1,i}, \quad \Lambda = \sum_{i \in \mathbb{Z}} E_{i,i+1}, \tag{2.1}$$

where $g_i = (t_i^2 - t_{i+1}^2)/m_{i,i+1} = (a_i - a_i^{-1})/f_{i,i+1} = -[\alpha_i^\vee]_q / f_i$. The matrix shall be called *the shift matrix*. There exists a unique upper triangular matrix $U = 1 + \sum_{i<j} u_{ij} E_{ij}$ with $M = U D_t^2 U^{-1}$. Then we have $u_{i,i+1} = -g_i^{-1}$. Put $Z = U D_Z$. We define the matrices S_i^g and S_i by

$$\begin{aligned}
S_i^g &= g_i^{-1} E_{i,i+1} - g_i E_{i+1,i} + \sum_{k \neq i,i+1} E_{kk}, \\
S_i &= -[\alpha_i^\vee - 1]_q^{-1} E_{i,i+1} + [\alpha_i^\vee + 1]_q E_{i+1,i} + \sum_{k \neq i,i+1} E_{kk}.
\end{aligned}$$

The following theorem follows from the results of Sections 1.3 and 1.4.

Theorem 2.2 (Lax and Sato-Wilson formalisms for type A_∞)**.** *We have*

$$\begin{aligned}
&s_i(M) = G_i M G_i^{-1}, \quad s_i(U) = G_i U S_i^g, \\
&s_i(D_Z) = (S_i^g)^{-1} D_Z S_i, \quad s_i(Z) = G_i Z S_i, \quad s_i(D_t) = S_i^{-1} D_t S_i \\
&\pi(X) = \Lambda X \Lambda^{-1} \quad \text{for } X = M, U, D_Z, Z, D_t.
\end{aligned}$$

These relations with $s_i(p) = \pi(p) = p$, $s_i(\tau_0) = f_0 \tau_{-1} \tau_1 / \tau_0$, and $\pi(\tau_0) = \tau_1$ uniquely characterize the extended Weyl group action on $D(K_\infty^{\mathrm{pa}})$. □

2.2 The case of type $A^{(1)}_{n-1}$ for $n \geqq 3$

In this subsection, we assume that $n \geqq 3$. Denote by $\overline{k}$ the image of $k \in \mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$.

The Weyl group $W_n = W(A^{(1)}_{n-1})$ of type $A^{(1)}_{n-1}$ is defined to be the group generated by $\{s_{\bar{i}}\}_{i=0}^{n-1}$ with the fundamental relations $s_{\bar{i}} s_{\overline{i+1}} s_{\bar{i}} = s_{\overline{i+1}} s_{\bar{i}} s_{\overline{i+1}}$, $s_{\bar{i}} s_{\bar{j}} = s_{\bar{j}} s_{\bar{i}}$ ($\bar{j} \neq \overline{i \pm 1}$), and $s_{\bar{i}}^2 = 1$. The extended Weyl group $\widetilde{W}_n$ is given by the semi-direct product $\widetilde{W}_n = W_n \rtimes \langle \pi \rangle$ with defining relations $\pi s_{\bar{i}} = s_{\overline{i+1}} \pi$.

Let $Q^\vee_n$ (resp. P_n) be the quotient lattice of $Q^\vee_\infty$ (resp. P_∞) given by the relations $\varepsilon^\vee_{i+n} = \varepsilon^\vee_i - \delta^\vee$ (resp. $\Lambda_{i+n} = \Lambda_i$) for $i \in \mathbb{Z}$. The images of $\beta \in Q^\vee_\infty$ in $Q^\vee_n$ and $\lambda \in P_\infty$ in P shall be denoted by the same symbols. For example, $\alpha^\vee_i = \varepsilon^\vee_i - \varepsilon^\vee_{i+1}$, $\alpha^\vee_{i+n} = \alpha^\vee_i$ in $Q^\vee_n$ and $\varepsilon_i = \Lambda_{i+1} - \Lambda_i$, $\varepsilon_{i+n} = \varepsilon_i$ in P_n. Denote the induced actions of π on $Q^\vee_n$ and P_n by the same symbols:

$$\pi(\varepsilon^\vee_i) = \varepsilon^\vee_{i+1}, \quad \pi(\delta^\vee) = \delta^\vee \quad \text{in } Q^\vee_n, \qquad \pi(\Lambda_i) = \Lambda_{i+1} \quad \text{in } P_n.$$

The lattice $Q^\vee_n$ (resp. P_n) is the free $\mathbb{Z}$-module spanned by $\delta^\vee, \varepsilon^\vee_1, \dots, \varepsilon^\vee_n$ (resp. $\Lambda_0, \varepsilon_1, \dots, \varepsilon_n$).

We define the paring between $Q^\vee_n$ and P_n by

$$\begin{aligned} &\langle \varepsilon^\vee_i, \varepsilon_j \rangle = \delta_{ij} \quad (1 \leqq i, j \leqq n), \qquad \langle \delta^\vee, \varepsilon_j \rangle = 0, \\ &\langle \varepsilon^\vee_i, \Lambda_0 \rangle = 0 \quad (1 \leqq i \leqq n), \qquad \langle \delta^\vee, \Lambda_0 \rangle = 1. \end{aligned}$$

The induced actions of π on $Q^\vee_n$ and P_n preserve the pairing between them. Then we have

$$\langle \alpha^\vee_i, \alpha_j \rangle = \begin{cases} 2 & (\bar{j} \equiv \bar{i}), \\ -1 & (\bar{j} \equiv \bar{i} \pm \bar{1}), \\ 0 & (\text{otherwise}), \end{cases} \qquad \langle \alpha^\vee_i, \Lambda_j \rangle = \delta_{\bar{i},\bar{j}}.$$

The generalized Cartan matrix $[a_{\bar{i},\bar{j}}]_{i,j=0}^{n-1}$ of type $A^{(1)}_{n-1}$ is defined by $a_{\bar{i},\bar{j}} = \langle \alpha^\vee_i, \alpha_j \rangle$. The actions of the extended Weyl group $\widetilde{W}_n$ on $Q^\vee_n$ and P_n by the following formulas with the induced action of π:

$$s_{\bar{i}}(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha^\vee_i, \quad s_{\bar{i}}(\lambda) = \lambda - \langle \alpha^\vee_i, \lambda \rangle \alpha_i.$$

These actions also preserve the pairing between $Q^\vee_n$ and P_n.

For each $i \in \mathbb{Z}$, let $\overline{s}_i$ be the automorphism of $Q^\vee_\infty \oplus P_\infty$ given by

$$\overline{s}_i(x) = \prod_{k \in \mathbb{Z}} s_{i+nk}(x) \quad (x \in Q^\vee_\infty \oplus P_\infty).$$

This is well-defined because the infinite product in the right-hand side reduces to the finite product for each x. The mapping $s_{\bar{i}} \mapsto \overline{s}_i$ ($i = 0, 1, \dots, n-1$) with the action of π defines the action of $\widetilde{W}_n$ on $Q^\vee_\infty \oplus P_\infty$, which induces the action of $\widetilde{W}_n$ on $Q^\vee_n \oplus P_n$ given above. Furthermore the actions of the infinite product $\overline{s}_i = \prod_{k \in \mathbb{Z}} s_{i+nk}$ for $i \in \mathbb{Z}$ on $D(K^{\mathrm{pa}}_\infty)$ are also well-defined and give the action of $\widetilde{W}_n$ on $D(K^{\mathrm{pa}}_\infty)$ with the action of π. More explicitly, we have

$$\overline{s}_i(f_{j\pm 1}) = a_j f_{j\pm 1} + \frac{a_j^{-1} - a_j}{q - q^{-1}} [f_j, f_{j\pm 1}]_{q^{-1}} f_j^{-1}$$

$$
\begin{aligned}
&= a_j^{-1} f_{j\pm1} + \frac{a_j^{-1} - a_j}{q - q^{-1}} [f_j, f_{j\pm1}]_q f_j^{-1} \\
&= \frac{q a_j - q^{-1} a_j^{-1}}{q - q^{-1}} f_{j\pm1} + \frac{a_i^{-1} - a_i}{q - q^{-1}} f_j f_{j\pm1} f_j^{-1} \qquad (\bar{j} = \bar{i}),
\end{aligned}
$$

$$
\begin{aligned}
&\overline{s}_i(f_k) = f_k \qquad (\bar{k} \neq \overline{i \pm 1}), \\
&\overline{s}_i(t_j) = t_{j+1}, \quad \overline{s}_i(t_{j+1}) = t_j \quad (\bar{j} = \bar{i}), \quad \overline{s}_i(t_k) = t_k \quad (\bar{k} \neq \bar{i}, \overline{i+1}), \quad \overline{s}_i(p) = p, \\
&\overline{s}_i(\tau_j) = f_j \frac{\tau_{j-1}\tau_{j+1}}{\tau_j} \quad (\bar{j} = \bar{i}), \quad \overline{s}_i(\tau_k) = \tau_k \quad (\bar{k} \neq \bar{i}).
\end{aligned}
$$

See also Section 1.3 and recall that $a_i = t_i / t_{i+1}$.

The multiplicative subset S of $U_{-,\infty}^{\mathrm{pa}}$ generated by

$$
f_k, \quad 1 - p^{2k}, \quad t_i^2 - t_j^2 \quad (i, j, k \in \mathbb{Z}, i < j)
$$

is an Ore set in $U_{-,\infty}^{\mathrm{pa}}$. We denote by $\widetilde{U}_{-,\infty}^{\mathrm{pa}}$ the localization of $U_{-,\infty}^{\mathrm{pa}}$ with respect to S:

$$
\begin{aligned}
U_{-,\infty}^{\mathrm{pa}} &= U_{-,\infty}[\, p^{\pm1}, t_k^{\pm1} \mid k \in \mathbb{Z}\,], \\
\widetilde{U}_{-,\infty}^{\mathrm{pa}} &= U_{-,\infty}[\, f_k^{-1}, p^{\pm1}, (1 - p^{2k})^{-1}, t_k^{\pm1}, (t_i^2 - t_j^2)^{-1} \mid i, j, k \in \mathbb{Z}, i < j\,].
\end{aligned}
$$

Define the algebra $D(U_{-,\infty}^{\mathrm{pa}})$ (resp. $D(\widetilde{U}_{-,\infty}^{\mathrm{pa}})$) to be the subalgebra of $D(K_\infty^{\mathrm{pa}})$ generated by $U_{-,\infty}^{\mathrm{pa}}$ (resp. $\widetilde{U}_{-,\infty}^{\mathrm{pa}}$) and $\{\tau_i\}_{i \in \mathbb{Z}}$. For each $i \in \mathbb{Z}$, the infinite product $\overline{s}_i = \prod_{k \in \mathbb{Z}} s_{i+nk}$ maps $D(U_{-,\infty}^{\mathrm{pa}})$ into $D(\widetilde{U}_{-,\infty}^{\mathrm{pa}})$.

~~Let $D(\widetilde{U}_{-,n}^{\mathrm{pa}})$ be the quotient algebra of $D(\widetilde{U}_{-,\infty}^{\mathrm{pa}})$ given by the following relations~~

$$
\cancel{f_{i+n} = f_i, \quad t_{i+n} = p^{-1} t_i, \quad \tau_{i+n} = \tau_i \quad (i \in \mathbb{Z}).} \tag{2.2}
$$

Denote the images, in $D(\widetilde{U}_{-,n}^{\mathrm{pa}})$, of $U_{-,\infty}^{\mathrm{pa}}$, $\widetilde{U}_{-,\infty}^{\mathrm{pa}}$, and $D(U_{-,\infty}^{\mathrm{pa}})$ by $U_{-,n}^{\mathrm{pa}}$, $\widetilde{U}_{-,n}^{\mathrm{pa}}$, and $D(U_{-,n}^{\mathrm{pa}})$, respectively. The algebra $U_{-,n}^{\mathrm{pa}}$ can be identified with the Laurent polynomial ring generated by $p^{\pm1}, t_1^{\pm1}, \dots, t_n^{\pm1}$ over the lower triangular part $U_{-,n}$ of the q-difference deformation of the universal enveloping algebra of the Kac-Moody algebra of type $A_{n-1}^{(1)}$ and hence is an Ore domain ([6]). Denote by K_n^{pa} the skew field of fractions of $U_{-,n}^{\mathrm{pa}}$. Let $D(K_n^{\mathrm{pa}})$ be the algebra generated by K_n^{pa} and $\{\tau^\lambda\}_{\lambda \in P_n}$ with the following defining relations:

$$
\begin{aligned}
&\tau^\lambda \tau^\mu = \tau^{\lambda+\mu}, \quad \tau^0 = 1 \quad (\lambda, \mu, 0 \in P_n) \\
&\tau^\lambda f_i = f_i \tau^\lambda, \quad \tau^\lambda q^\beta = q^{\beta + \langle \beta, \lambda \rangle} \tau^\lambda \quad (\lambda \in P_n, \beta \in Q_n^\vee).
\end{aligned}
$$

We can naturally regard $D(\widetilde{U}_{-,n}^{\mathrm{pa}})$ as a subalgebra of $D(K_n^{\mathrm{pa}})$. We identify $q^{-\varepsilon_i^\vee}, q^{-\delta^\vee}, \tau^{\Lambda_i} \in D(K_n^{\mathrm{pa}})$ with $t_i, p, \tau_i \in D(U_{-,n}^{\mathrm{pa}})$, respectively. The action of π on $D(K_n^{\mathrm{pa}})$ is naturally induced by the action on $D(U_{-,\infty})$.

The injective algebra homomorphism $\overline{s}_i = \prod_{k \in \mathbb{Z}} s_{i+nk} : D(U_{-,\infty}^{\mathrm{pa}}) \to D(\widetilde{U}_{-,\infty}^{\mathrm{pa}})$ induces the algebra automorphism of $D(K_n^{\mathrm{pa}})$, which shall be also denoted by $\overline{s}_i$. The mapping $s_{\bar{i}} \mapsto \overline{s}_i$ with the induced action of π defines the action of $\widetilde{W}_n$ on $D(K_n^{\mathrm{pa}})$.

Denote by $M_{\mathbb{Z}}(R)$ the set of all matrices of size $\mathbb{Z}$ over a ring R. We shall denote the image of a matrix $A \in M_{\mathbb{Z}}(\widetilde{U}_{-,\infty}^{\mathrm{pa}})$ in $M_{\mathbb{Z}}(\widetilde{U}_{-,n}^{\mathrm{pa}})$ by the same symbol.

Recall that the matrices $\widetilde{L}, D_t, M, G_i, U, D_Z, Z, S_i^g, S_i \in M_{\mathbb{Z}}(\widetilde{U}_{-,\infty}^{\mathrm{pa}})$ are given in Section 2.1. We have $\widetilde{L} \in M_{\mathbb{Z}}(U_{-,\infty})$, $D_t, M \in M_{\mathbb{Z}}(U_{-,\infty}^{\mathrm{pa}})$, $G_i, S_i, S_i^g \in M_{\mathbb{Z}}(\widetilde{U}_{-,\infty}^{\mathrm{pa}})$, and

$D_Z \in M_{\mathbb{Z}}(D(U^{\mathrm{pa}}_{-,\infty}))$. From the formula (1.12), we obtain $U \in M_{\mathbb{Z}}(\widetilde{U}^{\mathrm{pa}}_{-,\infty})$ and hence $Z \in M_{\mathbb{Z}}(D(\widetilde{U}^{\mathrm{pa}}_{-,\infty}))$. The relations in (2.2) are equivalent to

$$\Lambda^n \widetilde{L} \Lambda^{-n} = \widetilde{L}, \quad \Lambda^n D_t \Lambda^{-n} = p^{-1} D_t, \quad \Lambda^n D_Z \Lambda^{-n} = D_Z.$$

These relations implies $\Lambda^n M \Lambda^{-n} = p^{-2} M$, $\Lambda^n U \Lambda^{-n} = U$, and $\Lambda^n Z \Lambda^{-n} = Z$.

We define the matrices $\overline{G}_i, \overline{S}^g_i, \overline{S}_i \in M_{\mathbb{Z}}(\widetilde{U}^{\mathrm{pa}}_{-,\infty})$ by

$$\begin{aligned}
\overline{G}_i &= \prod_{j\in i+n\mathbb{Z}} G_j = 1 + \sum_{j\in i+n\mathbb{Z}} g_j E_{j+1,j}, \\
\overline{S}^g_i &= \prod_{j\in i+n\mathbb{Z}} S_j = \sum_{j\in i+n\mathbb{Z}} (g_j^{-1} E_{j,j+1} - g_j E_{j+1,j}) + \sum_{\bar{k}\neq \bar{i}, \overline{i+1}} E_{kk}, \\
\overline{S}_i &= \prod_{j\in i+n\mathbb{Z}} S^g_j = \sum_{j\in i+n\mathbb{Z}} (-[\alpha_j^\vee - 1]_q^{-1} E_{j,j+1} + [\alpha_j^\vee + 1]_q E_{j+1,j}) + \sum_{\bar{k}\neq \bar{i}, \overline{i+1}} E_{kk},
\end{aligned}$$

where $g_j = -[\alpha_j^\vee]_q / f_j$. Since $[\alpha_{j+n}^\vee]_q = [\alpha_j^\vee]_q$ and $g_{j+n} = g_j$ in $\widetilde{U}^{\mathrm{pa}}_{-,n}$, we have $\overline{G}_{i+n} = \overline{G}_i$, $\overline{S}^g_{i+n} = \overline{S}^g_i$, and $\overline{S}_{i+n} = \overline{S}_i$. Theorem 2.2 immediately leads to the following proposition.

Proposition 2.3. *We have, in $M_{\mathbb{Z}}(D(\widetilde{U}^{\mathrm{pa}}_{-,\infty}))$,*

$$\begin{aligned}
&\bar{s}_i(M) = \overline{G}_i M \overline{G}_i^{-1}, \quad \bar{s}_i(U) = \overline{G}_i U \overline{S}^g_i, \\
&\bar{s}_i(D_Z) = (\overline{S}^g_i)^{-1} D_Z \overline{S}_i, \quad \bar{s}_i(Z) = \overline{G}_i Z \overline{S}_i, \quad s_i(D_t) = \overline{S}_i^{-1} D_t \overline{S}_i, \\
&\pi(X) = \Lambda X \Lambda^{-1} \quad \text{for } X = M, U, D_Z, Z, D_t.
\end{aligned}$$

These relations with $s_i(p) = \pi(p) = p$, $s_i(\tau_0) = f_0 \tau_{-1}\tau_1/\tau_0$, and $\pi(\tau_0) = \tau_1$ uniquely characterize the action of the extended Weyl group $\widetilde{W}_n$ on $D(K^{\mathrm{pa}}_n)$. □

The above proposition is the infinite matrix version of the Lax and the Sato-Wilson formalisms for the $A^{(1)}_{n-1}$ case with $n \geqq 3$.

Let R be an associative algebra with unit 1. Assume that c is an invertible elements of the center of R and is not a root of unity. Recall that $\Lambda \in M_{\mathbb{Z}}(R)$ denotes the shift matrix given in (2.1). For each $k \in \mathbb{Z}$, let $M_{\mathbb{Z}}(R)^k_{n,c}$ be the set of all infinite matrices $X = [x_{ij}]_{i,j\in\mathbb{Z}} \in M_{\mathbb{Z}}(R)$ such that $\Lambda^n X \Lambda^{-n} = c^{-k} X$ and there exists an integer N with $x_{ij} = 0$ if $i - j > N$. Then $M_{\mathbb{Z}}(R)_{n,c} = \bigoplus_{k\in\mathbb{Z}} M_{\mathbb{Z}}(R)^k_{n,c}$ can be naturally regarded as an algebra. Define the infinite diagonal matrix $D_{n,c}$ by

$$D_{n,c} = \sum_{k\in\mathbb{Z}} c^{-k} \sum_{i=1}^{n} E_{i+nk,i+nk}.$$

Then we have

$$\Lambda D_{n,c} \Lambda^{-1} = \left(\sum_{i\in\mathbb{Z}} c^{-\delta_{\bar{i},\bar{0}}} E_{ii} \right) D_{n,c}, \quad \Lambda^n D_{n,c} \Lambda^{-n} = c^{-1} D_{n,c}.$$

Therefore we obtain $D_{n,c} \in M_{\mathbb{Z}}(R)^1_{n,c}$ and $M_{\mathbb{Z}}(R)^k_{n,c} = M_{\mathbb{Z}}(R)^0_{n,c} D^k_{n,c}$. Denote by $M_n(R)$ the set of all square matrices of size n over an associative algebra R with 1. We introduce the spectral parameter z. (Do not confuse the spectral parameter z with the z-variables

$z_i = \tau_i/\tau_{i-1}$.) Let $c^{d/dz}$ be the difference operator acting on $R[z^{\pm 1}] = R[z, z^{-1}]$ given by $c^{d/dz}(z) = cz$. We obtain the matrix difference operator algebra $M_n(R[z^{\pm 1}, c^{\pm d/dz}])$. We define *the shift matrix* $\Lambda(z) \in M_n(R[z])$ by

$$\Lambda(z) = \sum_{i=1}^{n-1} E_{i,i+1} + zE_{n1} = \operatorname{diag}(1, \dots, 1, z)\Lambda(1).$$

Then we have, in $M(R[z^{\pm 1}, c^{\pm d/dz}])$,

$$\Lambda(z)c^{d/dz}\Lambda(z)^{-1} = \operatorname{diag}(1, \dots, 1, c^{-1})c^{d/dz}, \quad \Lambda(z)^n c^{d/dz}\Lambda(z)^{-n} = c^{-1}c^{d/dz}.$$

We can define the algebra isomorphism $\iota_{n,c} : M_{\mathbb{Z}}(R)_{n,c} \to M_n(R[z^{\pm 1}, c^{\pm d/dz}])$ by

$$\begin{aligned}
&\iota_{n,c}(D_{n,c}) = c^{d/dz}, \\
&\iota_{n,c}(X) = \sum_{k \in \mathbb{Z}} z^k \sum_{i,j=1}^{n} x_{i,j+nk} E_{ij} \quad (X = [x_{ij}]_{i,j \in \mathbb{Z}} \in M_{\mathbb{Z}}(R)^0_{n,c}).
\end{aligned}$$

In particular, we have $\iota_{n,c}(\Lambda) = \Lambda(z)$.

Let us apply the isomorphism $\iota_{n,c}$ for $R = D(\widetilde{U}^{\mathrm{pa}}_{-,n})$ and $c = p$ to the formulas in Proposition 2.3. Denote the $\iota_{n,p}$-images of $\widetilde{L}$, D_t, M, U, D_Z, Z, $\overline{G}_i$, $\overline{S}^g_i$, and $\overline{S}_i$ by $\widetilde{L}(z)$, $D_{t,n}p^{d/dz}$, $M(z)$, $U(z)$, $D_{Z,n}$, $Z(z)$, $\overline{G}_i(z)$, $\overline{S}_i(z)$, and $\overline{S}_i(z)$, respectively. Then we have

$$\begin{aligned}
&\widetilde{L}(z) = 1 + \sum_{1 \leqq i < j \leqq n} f_{ij}E_{ij} + \sum_{k=1}^{\infty} z^k \sum_{i,j=1}^{n} f_{i,j+nk}E_{ij}, \\
&D_{t,n} = \operatorname{diag}(t_1, \dots, t_n), \\
&M(z) = D_{t,n}p^{d/dz}\widetilde{L}(z)D_{t,n}p^{d/dz} = D_{t,n}\widetilde{L}(pz)D_{t,n}p^{2d/dz}, \\
&U(z) = 1 + \sum_{1 \leqq i < j \leqq n} u_{ij}E_{ij} + \sum_{k=1}^{\infty} z^k \sum_{i,j=1}^{n} u_{i,j+nk}E_{ij}, \\
&M(z) = U(z)D_{t,n}^2 p^{2d/dz}U(z)^{-1} = U(z)D_{t,n}^2 U(p^2 z)^{-1}p^{2d/dz}, \\
&D_{Z,n} = \operatorname{diag}(z_1, \dots, z_n), \quad Z(z) = U(z)D_{Z,n}, \\
&\overline{G}_i(z) = 1 + g_i E_{i+1,i} \quad (i = 1, \dots, n-1), \quad \overline{G}_0(z) = 1 + z^{-1}g_0 E_{1n}, \\
&\overline{S}^g_i(z) = g_i^{-1}E_{i,i+1} - g_i E_{i+1,i} + \sum_{k \neq i,i+1} E_{kk} \quad (i = 1, \dots, n-1), \\
&\overline{S}^g_0(z) = zg_0^{-1}E_{n1} - z^{-1}g_0E_{1n} + \sum_{k=1}^{n-1} E_{kk}, \\
&\overline{S}_i(z) = -[\alpha_i^\vee - 1]_q^{-1}E_{i,i+1} + [\alpha_i^\vee + 1]_q E_{i+1,i} + \sum_{k \neq i,i+1} E_{kk} \quad (i = 1, \dots, n-1), \\
&\overline{S}_0(z) = -z[\alpha_0^\vee - 1]_q^{-1}E_{n1} + z^{-1}[\alpha_0^\vee + 1]_q E_{1n} + \sum_{k=2}^{n-1} E_{kk}.
\end{aligned}$$

Proposition 2.3 immediately leads to the following theorem.

Theorem 2.4 (Lax and Sato-Wilson formalisms for type $A^{(1)}_{n-1}$, $n \geqq 3$)**.** *We have, in* $M_n\left(D(\widetilde{U}^{\mathrm{pa}}_{-,\infty})[z^{\pm 1}, p^{\pm d/dz}]\right)$,

$$
\begin{aligned}
&\overline{s}_i(M(z)) = \overline{G}_i(z)M(z)\overline{G}_i(z)^{-1}, \quad \overline{s}_i(U(z)) = \overline{G}_i(z)U(z)\overline{S}^g_i(z),\\
&\overline{s}_i(D_{Z,n}) = \overline{S}^g_i(z)^{-1}D_{Z,n}\overline{S}_i(z), \quad \overline{s}_i(Z(z)) = \overline{G}_i(z)Z(z)\overline{S}_i(z),\\
&\overline{s}_i(D_{t,n}) = \overline{S}_i(z)^{-1}D_{t,n}\overline{S}_i(pz),\\
&\pi(X) = \Lambda(z)X\Lambda(z)^{-1} \quad \textit{for } X = M(z), U(z), D_{Z,n}, Z(z), D_{t,n}.
\end{aligned}
$$

These relations with $s_i(p) = \pi(p) = p$, $s_i(\tau_0) = f_0\tau_{-1}\tau_1/\tau_0$, *and* $\pi(\tau_0) = \tau_1$ *uniquely characterize the action of the extended Weyl group* $\widetilde{W}_n$ *on* $D(K^{\mathrm{pa}}_n)$. □

2.3 The case of type $A^{(1)}_1$

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