Stoch1

gennadycantor

September 2019

1

In class, we ran an experiment. 30 people flipped a coin. We began at some initial point and set two destinations. One was to go broke and the other was to win the game. Everytime someone flipped a coin, we would move one unit to the right toward "win the game". Else, we would move one unit to the left toward "go broke". My initial prediction of the outcome of the game was that it would terminate. However, result showed otherwise. Perhaps 30 was not sufficient. I claim that for n-many people, for n large enough, for any initial point between and not including "go broke" and "win", the game terminates.

More formally, let the following be given.

x := initial point, x $\in \mathbb{Z}$

0 := "go broke"; I think not necessarily 0, but any integer would work.

W := "win", W $\in \mathbb{Z}$

where $0 < x < W$.

In words, what we need to show is that as n $->$ ∞ (n is number of flips or trials), we must see W consecutive heads. (worst case is that before we encounter this consecutive W heads, we are at 1, but W-1 $<$ W so we will still reach W. By the way, the game also ends if we go broke, the argument that I will give can be applied in the exact same manner to argue for the other case under which the game terminates at a loss i.e. we went broke).

Let $(X_{Wj-W+1}, ..., X_{Wj})$ be a random vector and each element in the random vector is a random variable that records the outcome of a coin flip. Thus our random variable X_i is 0 if we flip tails. Else, it is 1. I think this is also called an indicator random variable. Mathematically, we need to show:

$P((X_1, ..., X_W) \neq (1, ..., 1), ..., (X_{Wj-W+1}, ..., X_{Wj}) \neq (1, ..., 1), ..., (X_{Wn-W+1}, ..., X_{Wn}) \neq (1, ..., 1))$ $->0$ as n $->\infty$

The reason why I partitioned the events into n-many sequences of M random variables (or just random vector) will become apparent at the end of the proof.

We assume $X_i's$ are iid random variables. (Proof doesn't work otherwise and I think this is realistic)

Observe:

Let $u_1, ..., u_{Wn} \in \{0, 1\}$. Define $U_1 = (u_1, ..., u_W)$,..., $U_j = (U_{Wj-W+1}, ..., U_{Wn})$ for $1 \leq j \leq n$.

$P((X_1, ..., X_W) \neq (1, ..., 1), ..., (X_{Wj-W+1}, ..., X_{Wj}) \neq (1, ..., 1), ..., (X_{Wn-W+1}, ..., X_{Wn}) \neq (1, ..., 1))$

$=$

$\sum_{U_1,...,Un \neq (1,..,1)} P((X_1, ..., X_W) \neq U_1, ..., (X_{Wj-W+1}, ..., X_{Wj}) \neq U_j, ..., (X_{Wn-W+1}, ..., X_{Wn}) \neq U_n)$

$=$

$\sum_{U_1,...,Un \neq (1,..,1)} \prod_{i=1}^{Wn} P(X_i = u_i) = \sum_{U_1,...,Un \neq (1,..,1)} \prod_{j=1}^{n} P((X_{Wj-W+1}, ..., X_{Wj}) = U_j)$

$=$

$\prod_{j=1}^{n} \sum_{U_j \neq (1,...,1)} P((X_{Wj-W+1}, ..., W_{Wj}) = U_j)$

$=$

$\prod_{j=1}^{n}(1 - P((X_{Wj-W+1}, ..., W_{Wj}) = (1, ..., 1)))$

$=$

$(1 - P^W)^n$. And $P^W > 0$ implies $(1 - P^W) < 1$ implies $(1 - P^W)^n- > 0$ as $n- > \infty$.

This means that P((1,...,1) never appears) = 0, which implies eventually, we see W consecutive heads.

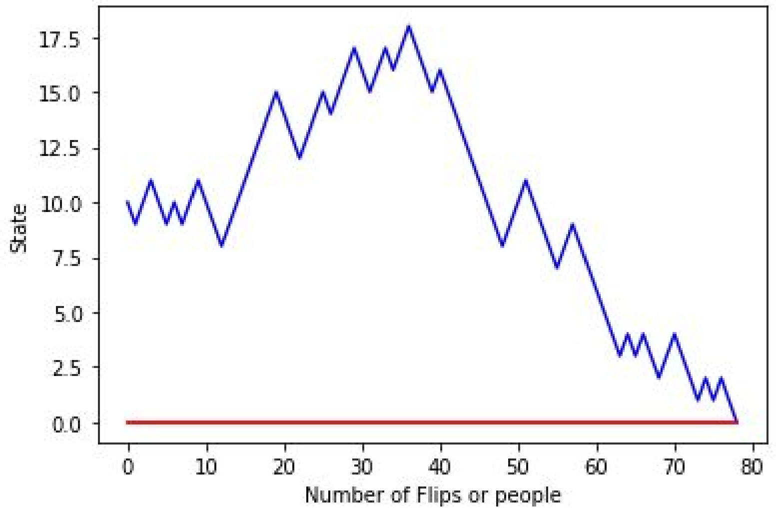
Now that we have shown that this process ends, we can try and simulate this with a computer program. The Program can be written in any language. I have written mine in python, but here is the pseudo-code in case python is not a preferred choice of one's programming language. This code is best to transfer to MATLAB or Julia. I do not recommend C/C++ unless one likes to do extra work.

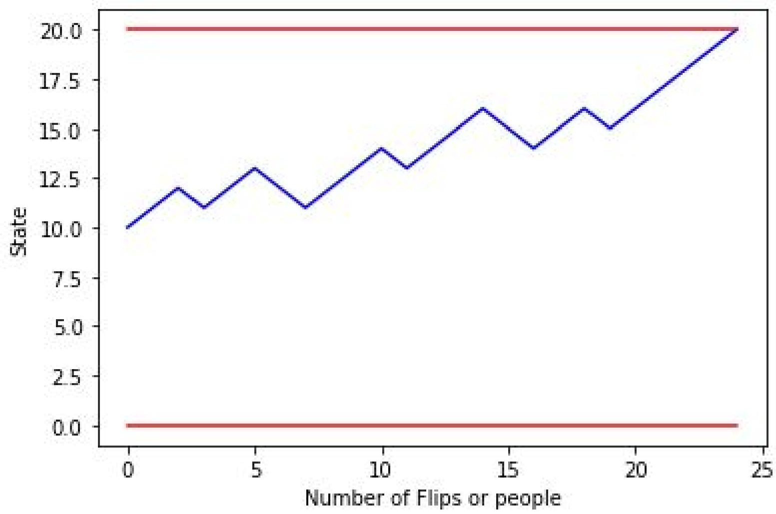
This experiment actually has a name. It is an instance of something that is called a simple random walk. Here is the pseudo-code. This code is only for simulating the experiment. Full python code (including plotting) is on my github, https://github.com/gennadycantor.

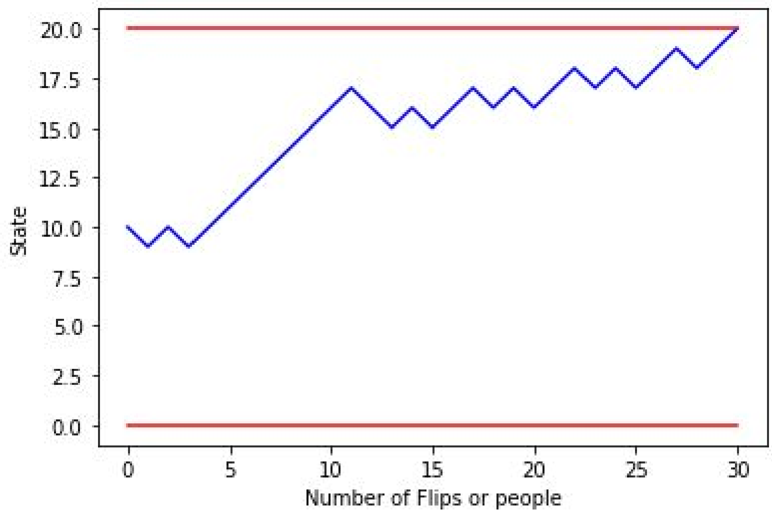
Also, followed by the pseudo-code are the simulation results. Of course, you can simulate this as many times as you want, but I ran 10 simulations to plot the result, and 500 simulations to compute the mean number of people/flips for the game to terminate. Horizontal axis is number of flips or people and vertical is where we are at after each person's flip.

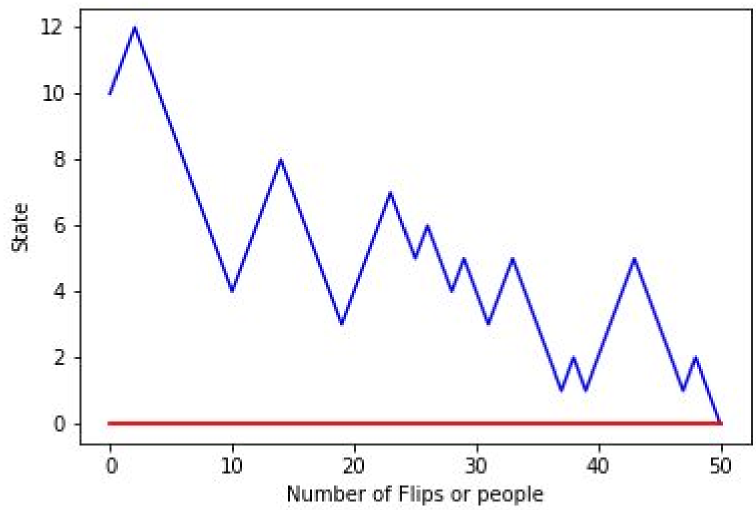
```
input: initial Point, Winning Point
output: number of flips for convergence
Function randomWalk(initialPoint, WinAt):
   % initialize number of flips to 0
   numFlips = 0
   % while loop for iteration until convergence
   while initialPoint ~= WinAt AND initialPoint ~= 0:
      % sample from Uniform distribution, U[0,1]
      flip $<-$ x~U[0,1]
      if flip < 0.5:
         initialPoint = initialPoint - 1
      else:
         initialPoint = initialPoint + 1
      numFlips = numFlips + 1
   return numFlips
```

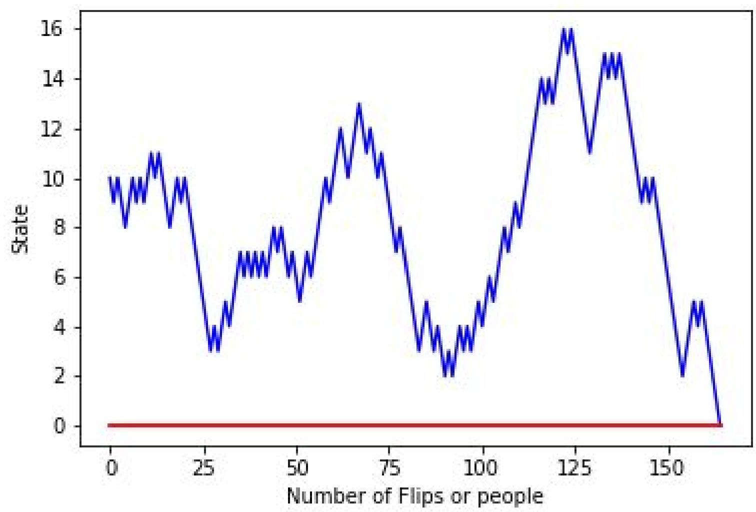
And here are the simulations.

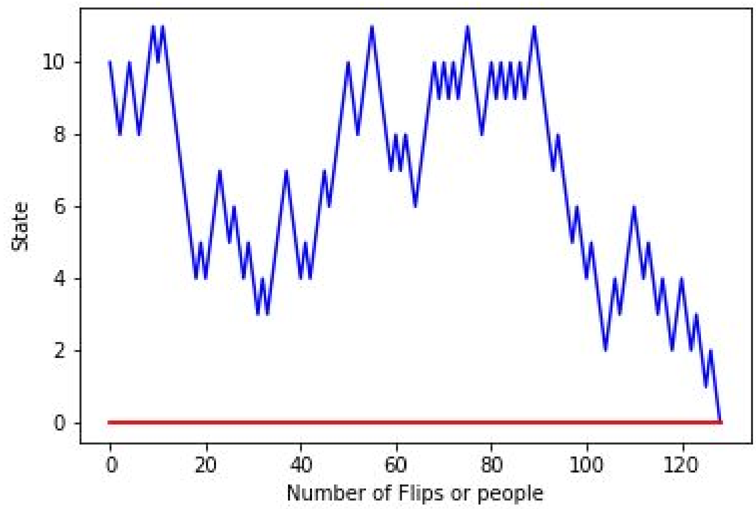




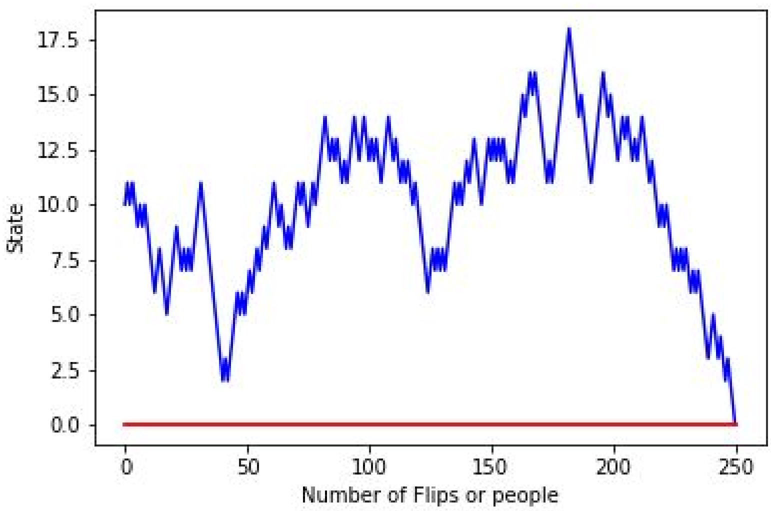

State
Number of Flips or people


State
Number of Flips or people

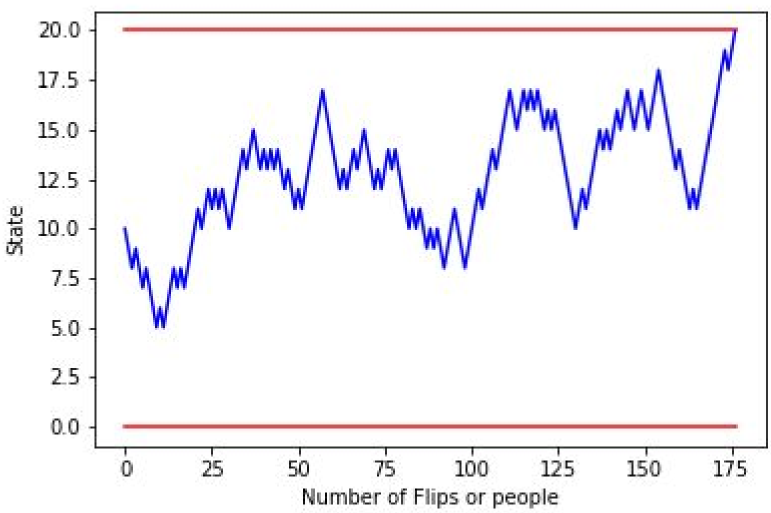

State
Number of Flips or people



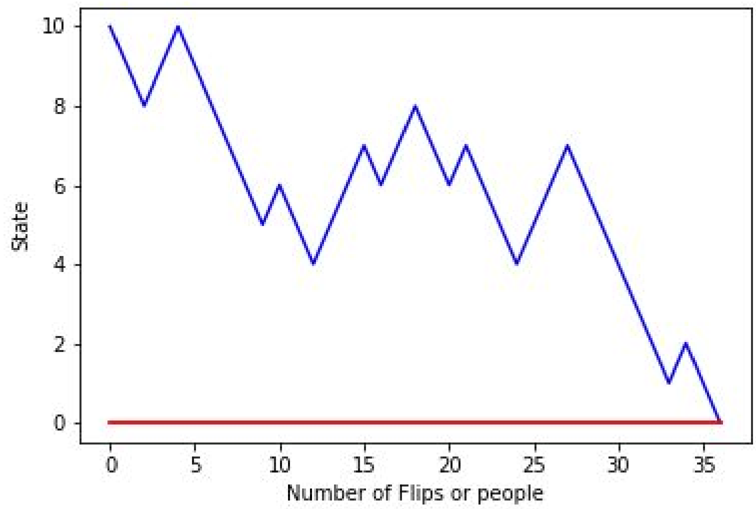
10
8
6
4
2
0
State
0
20
40
60
80
100
120
Number of Flips or people

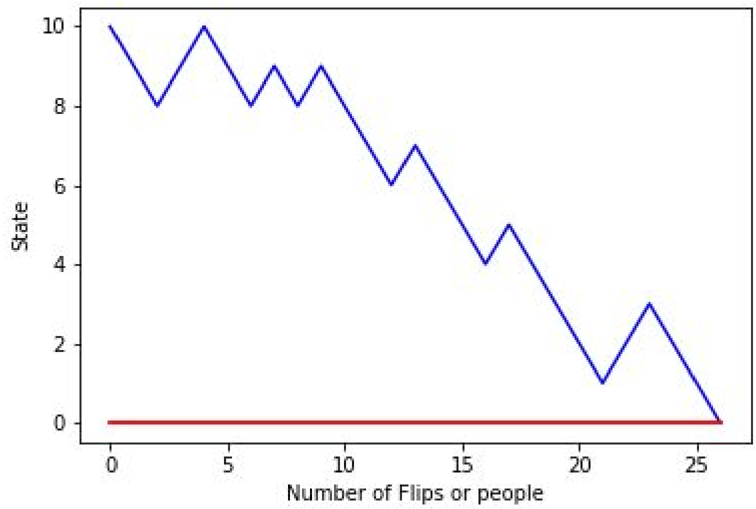


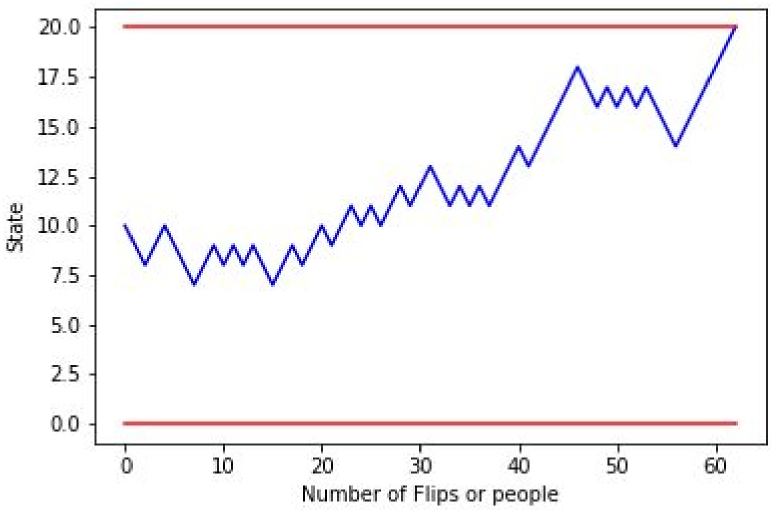
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
State
0
50
100
150
200
250
Number of Flips or people

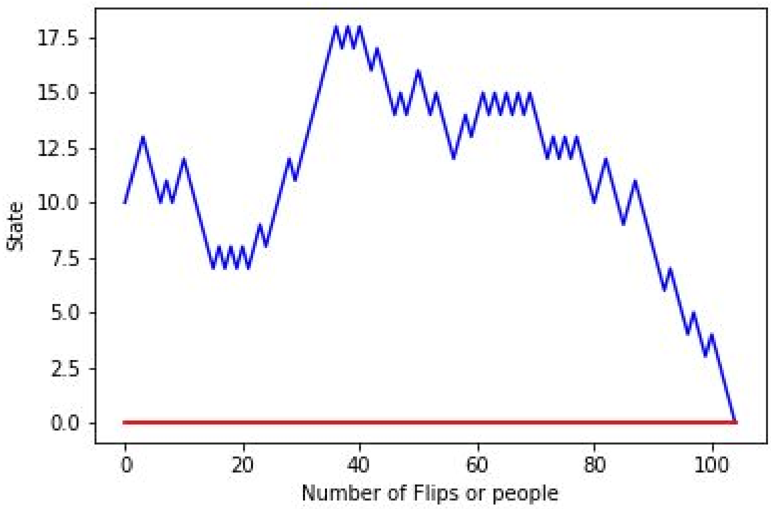


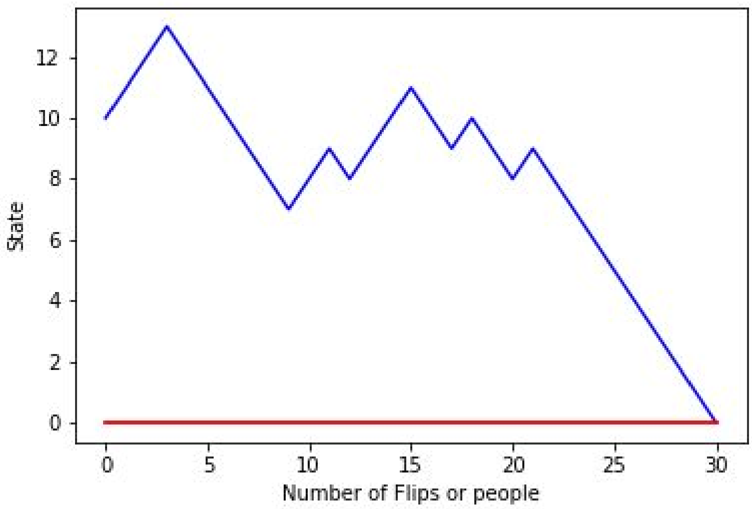
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
State
0
25
50
75
100
125
150
175
Number of Flips or people


State
Number of Flips or people


State
Number of Flips or people

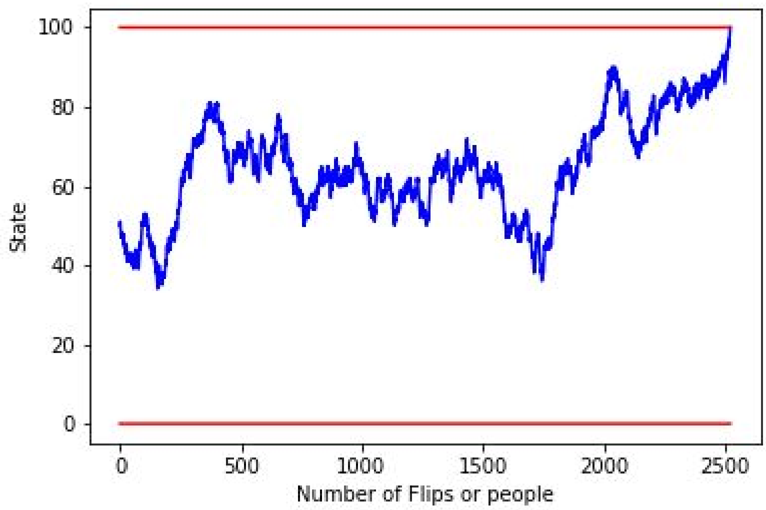

State
Number of Flips or people

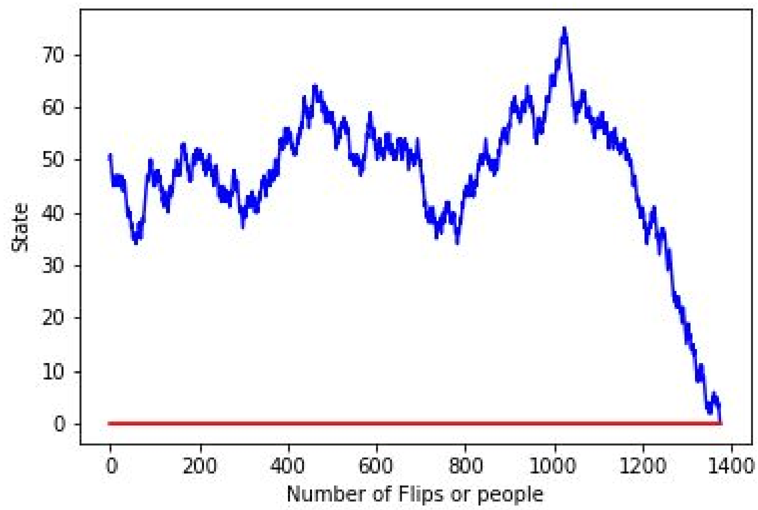


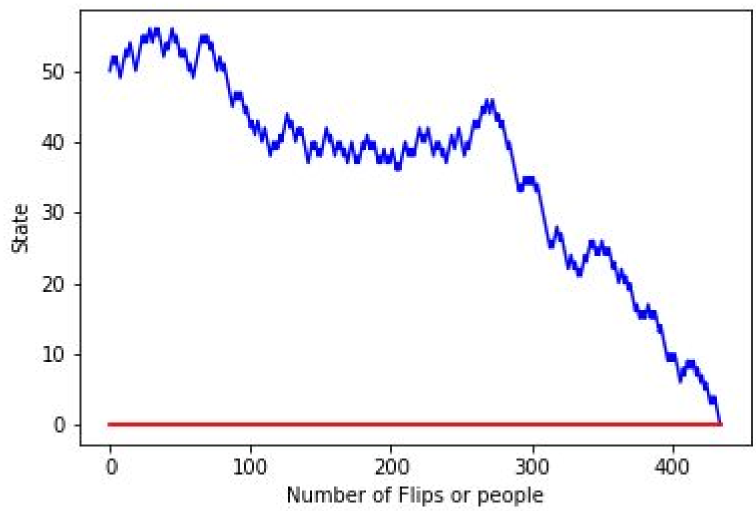


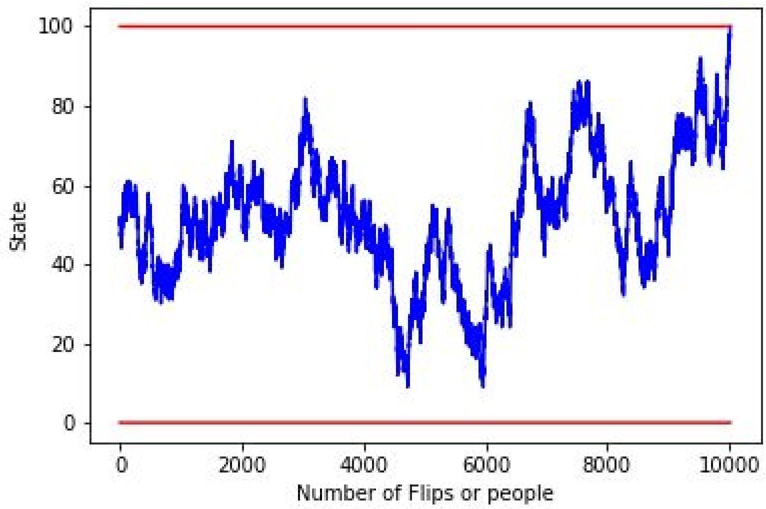
X axis is the number of flips and Y axis is the state at which we are in after each person's flip. Also, mean number of flips for convergence came out to be 100 when I ran 500 simulations. Apparently, n need not be large. I ran this 500 simulations about 2,3 times, and it does not get stuck in the infamous infinite loop. So it is safe to conclude our experiment confirms our theory.

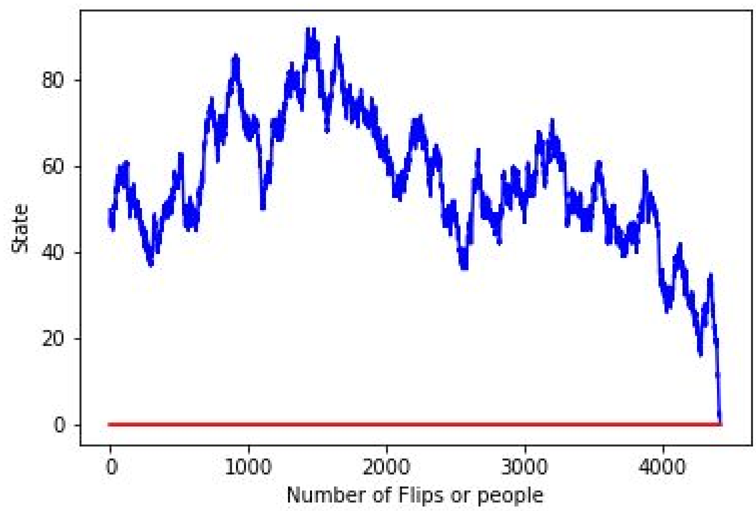
Here are the simulations if I set initial position to 50, and winning position to 100.

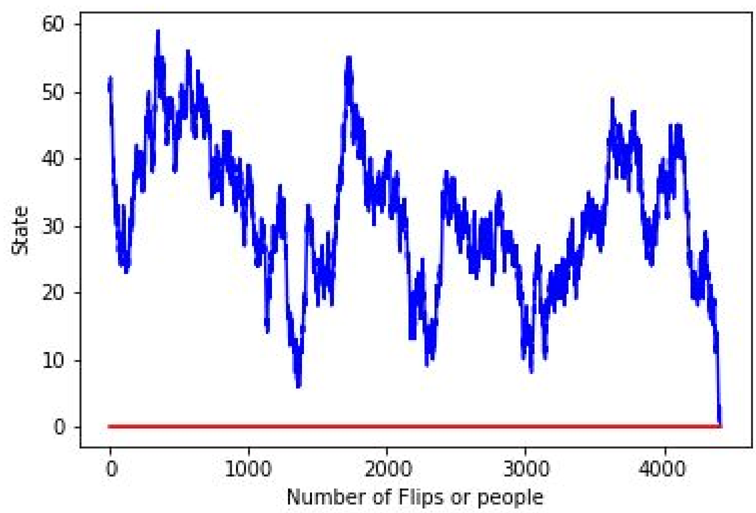


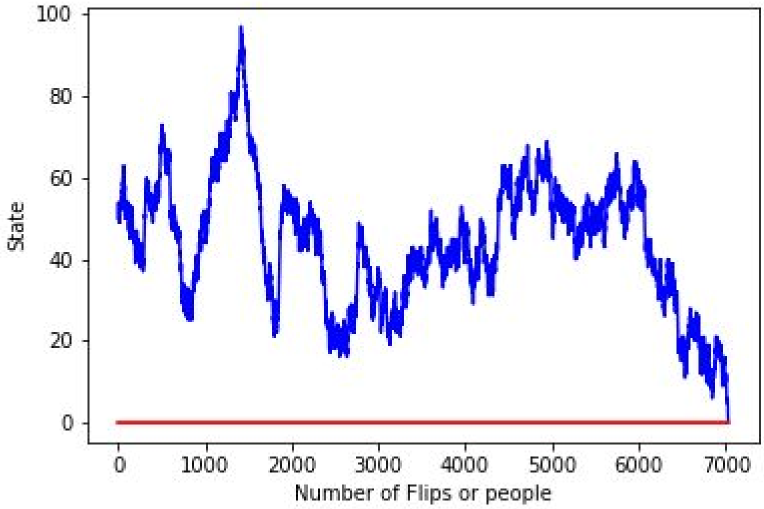


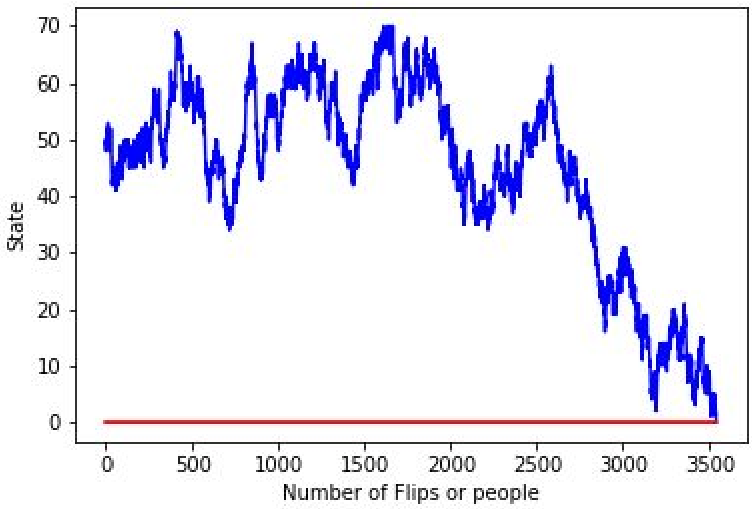


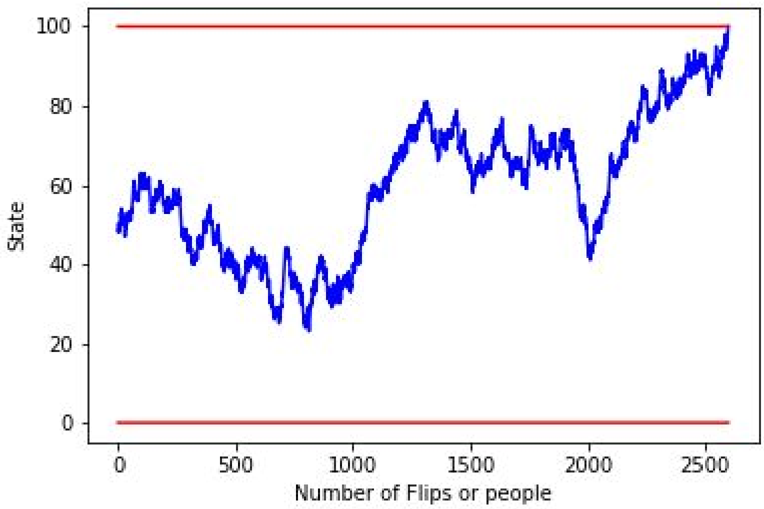

State
Number of Flips or people
0
20
40
60
80
100
0
2000
4000
6000
8000
10000

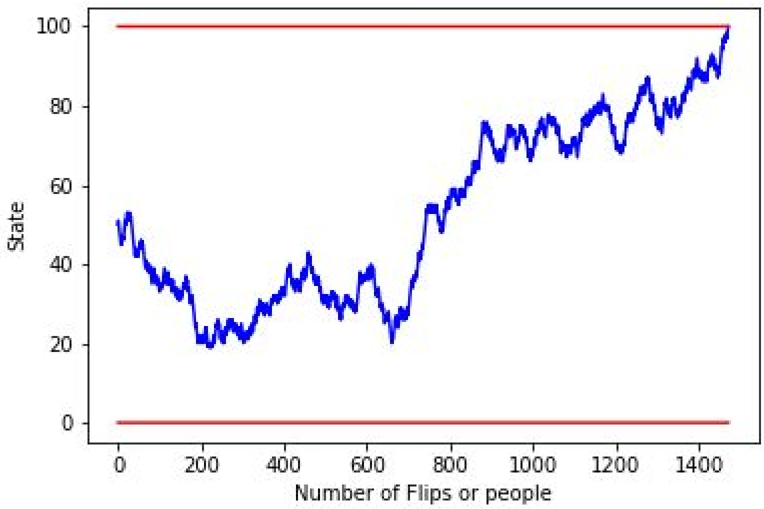

State
Number of Flips or people
0
20
40
60
80
0
1000
2000
3000
4000


State
Number of Flips or people
0
10
20
30
40
50
60
0
1000
2000
3000
4000


100
80
60
40
20
0
State
0
1000
2000
3000
4000
5000
6000
7000
Number of Flips or people


70
60
50
40
30
20
10
0
State
0
500
1000
1500
2000
2500
3000
3500
Number of Flips or people





The average number of flips for convergence was: 3781 flips.

Stoch1

gennadycantor

September 2019

1

Now that we have confirmed the game ends both in theory and experiment, we can answer a more interesting question. One of them might be what our chances are of winning the game. In other words, we reach the winning point before we go broke. This is actually much simpler to show than whether the game ends or not.

First let's consider all possible positions from x, our initial state. If we start at some position x, we can go to x + 1 if we flip heads, or go to x - 1 if we flips tails. So really, the game restarts at x - 1 or x + 1.

More formally, we have

$P_x(win) = 0.5 * P_x(win|firstflip = H) + 0.5 * P_x(win|firstflip = T) = 0.5 * P_{x+1}(win) + 0.5 * P_{x-1}(win)$.

Now let $P_j(win) = p_j$.

Then we have: $p_x = 0.5 * p_{x+1} + 0.5 * p_{x-1}$ for $0 \leq x \leq W$.

<=>

$2p_x = p_{x+1} + p_{x-1}$

<=>

(0) $p_{x+1} - p_x = p_x - p_{x-1}$ where clearly, $p_0 = 0$ and $p_W = 1$

(1) $1 = P_W - P_0 = \sum_{x=1}^{W}(p_x - p_{x-1}) = M(p_k - p_{k-1})$ for k = 1,...,W

So we have: 1/W = $p_k - p_{k-1}$

(2) which means for x = 1,...,W-1 we have $p_x = p_x - p_0 = \sum_{k=1}^{x}(p_k - p_{k-1}) = x/W$

Where in (1) and (2) we took advantage of the fact that our sequence p_x is an arithmetic sequence as can be seen in (0).

So we can conclude that the probability of winning before going broke is: x/W where x is the initial point at which we start and W is the point at which we have won the game.

References: My Math531 notes from the past semesters.

I believe this textbook was freely online when I took the course. If anyone is interested in the textbook (I thought it was a good textbook for an introduction to probability), please visit Professor Valko's website.