Chapter 7: Operators on Inner Product Spaces

Linear Algebra Done Right (4th Edition), by Sheldon Axler

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Contents

7A: Self-Adjoint and Normal Operators

Definition 1 (adjoint, T^*)**.** *Suppose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^*\colon W \to V$ such that*

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and every $w \in W$.

Corollary 2. *If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$. In other words, the adjoint of a linear map is a linear map.*

Corollary 3 (properties of the adjoint)**.** *Suppose $T \in \mathcal{L}(V, W)$. Then*

(a) $(S + T)^* = S^* + T^*$ *for all* $S \in \mathcal{L}(V, W)$;

(b) $(\lambda T)^* = \overline{\lambda} T^*$ *for all* $\lambda \in \mathbb{F}$;

(c) $(T^*)^* = T$;

(d) $(ST)^* = T^* S^*$ *for all* $S \in \mathcal{L}(W, U)$ *(here U is a finite-dimensional inner product space over $\mathbb{F}$).*

(e) $I^* = I$;

(f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Theorem 4 (null space and range of T^*)**.** *Suppose $T \in \mathcal{L}(V, W)$. Then*

(a) $\text{null } T^* = (\text{range } T)^\perp$;

(b) $\text{range } T^* = (\text{null } T)^\perp$;

(c) $\text{null } T = (\text{range } T^*)^\perp$;

(d) $\text{range } T = (\text{null } T^*)^\perp$.

Definition 5 (conjugate transpose, A^*)**.** *The **conjugate transpose** of an m-by-n matrix A is the n-by-m matrix A^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry. In other words, if $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then*

$$(A^*)_{j,k} = \overline{A_{k,j}}$$

Remark 6. *We denote $A^\top$ (transpose) when we know the matrix is real. Note that wrt. nonorthonormal bases, the matrix of T^* does not necessarily equal the conjugate transpose of the matrix of T.*

Theorem 7 (matrix of T^* equals conjugate transpose of matrix of T)**.** *Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W. Then $\mathcal{M}(T^*, (f_1, \dots, f_m), (e_1, \dots, e_n))$ is the conjugate transpose of $\mathcal{M}(T, (e_1, \dots, e_n), (f_1, \dots, f_m))$. In other words,*

$$\mathcal{M}(T^*) = (\mathcal{M}(T))^*$$

Remark 8. *Orthogonal complements and adjoints are concepts that are easier to work with than annihilators and dual maps in the context of inner product spaces.*

Definition 9 (self-adjoint)**.** *An operator $T \in \mathcal{L}(V)$ is called **self-adjoint** if $T = T^*$. In other words, an operator $T \in \mathcal{L}(V)$ is self-adjoint if and only if*

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

for all $v, w \in V$.

Corollary 10. *Every eigenvalue of a self-adjoint operator is real.*

Corollary 11. *Suppose V is a **complex** inner product space and $T \in \mathcal{L}(V)$. Then*

$$\langle Tv, v\rangle = 0 \text{ for every } v \in V \Longleftrightarrow T = 0.$$

Corollary 12. *Suppose V is a **complex** inner product space and $T \in \mathcal{L}(V)$. Then*

$$T \text{ is self-adjoint} \Longleftrightarrow \langle Tv, v\rangle \in \mathbb{R} \text{ for every } v \in V$$

Remark 13. *The above two corollaries do not hold for real inner product spaces.*

Theorem 14. *Suppose T is a self-adjoint operator on V. Then*

$$\langle Tv, v\rangle = 0 \text{ for every } v \in V \Longleftrightarrow T = 0$$

Definition 15 (normal)**.** *An operator on an inner product space is called **normal** if it commutes with its adjoint. In other words, $T \in \mathcal{L}(V)$ is normal if $TT^* = T^*T$.*

Remark 16. *Every self-adjoint opeartor is normal, but not vice versa.*

Theorem 17. *Suppose $T \in \mathcal{L}(V)$. Then*

$$T \text{ is normal } \Longleftrightarrow \|Tv\| = \|T^*v\| \text{ for every } v \in V$$

Corollary 18 (range, null space, and eigenvectors of a normal operator)**.** *Suppose $T \in \mathcal{L}(V)$ is normal. Then*

(a) null T = null T^*;

(b) range T = range T^*;

(c) V = null $T \oplus$ range T;

(d) $T - \lambda I$ is normal for every $\lambda \in \mathbb{F}$;

(e) if $v \in V$ and $\lambda \in \mathbb{F}$, then $Tv = \lambda v$ if and only if $T^*v = \overline{\lambda} v$.

Theorem 19. *Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*

Theorem 20. *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then T is normal if and only if there exist commuting self-adjoint operators A and B such that $T = A + iB$.*

Problem 1

Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1})$$

Find a formula for $T^*(z_1, \ldots, z_n)$.

Proof. We have that

$$\begin{aligned} \langle T(x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle &= \langle (0, x_1, \ldots, x_{n-1}), (y_1, \ldots, y_n) \rangle \\ &= x_1 y_2 + \cdots + x_{n-1} y_n \\ &= \langle (x_1, \ldots, x_n), (y_2, \ldots, y_n, 0) \rangle \\ &= \langle (x_1, \ldots, x_n), T^*(y_1, \ldots, y_n) \rangle \end{aligned}$$

where $T^*(z_1, \ldots, z_n) = (z_2, \ldots, z_n, 0)$. □

Problem 2

Suppose $T \in \mathcal{L}(V, W)$. Prove that

$$T = 0 \Longleftrightarrow T^* = 0 \Longleftrightarrow T^*T = 0 \Longleftrightarrow TT^* = 0$$

Proof.

$$\begin{aligned} T = 0 &\Longleftrightarrow \langle Tv, v \rangle = 0 \text{ for all } v \\ &\Longleftrightarrow \langle v, T^*v \rangle = 0 \text{ for all } v \\ &\Longleftrightarrow T^* = 0 \\ &\Longleftrightarrow T^*T = T^*(0) = 0 \\ &\Longleftrightarrow TT^* = 0 \end{aligned}$$

□

Problem 4

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that

$$U \text{ is invariant under } T \Longleftrightarrow U^\perp \text{ is invariant under } T^*$$

Proof. Let $u \in U$ and $w \in U^\perp$, then we know that

$$\langle Tu, w \rangle = \langle u, T^*w \rangle$$

Thus

$$\langle Tu, w \rangle = 0 \Longleftrightarrow \langle u, T^*w \rangle = 0$$

which implies the desired result. □

Problem 5

Suppose $T \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W. Prove that

$$\|Te_1\|^2 + \cdots + \|Te_n\|^2 = \|T^* f_1\|^2 + \cdots + \|T^* f_m\|^2$$

Proof. Note that first we have

$$\|Te_k\|^2 = \langle Te_k, Te_k \rangle = \left\langle Te_k, \sum_{j=1}^{m} \langle Te_k, f_j \rangle f_j \right\rangle = \sum_{j=1}^{m} |\langle Te_k, f_j \rangle|^2$$

At the same time, we have

$$\langle Te_k, f_j \rangle = \langle e_k, T^* f_j \rangle$$

Combining the two equations yields that

$$\sum_{i=1}^{n} \|Te_i\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} |\langle Te_i, f_j \rangle|^2 = \sum_{j=1}^{m} \sum_{i=1}^{n} |\langle e_i, T^* f_j \rangle|^2 = \sum_{j=1}^{m} \left\|T^* f_j\right\|^2$$

□

Problem 9

Prove that the product of two self-adjoint operators on V is self-adjoint if and only if the two operatos commute.

Proof. Let T and S be two self-adjoint operators on V. First if $ST = TS$, then $(ST)^* = (TS)^*$. It suffices to show the forward direction. Let's suppose WLOG that ST is self-adjoint. Then we know that $(ST)^* = T^* S^* = S^* T^* = (TS)^*$. This implies that for arbitrary $v, w \in V$, we have

$$\begin{aligned} \langle v, T^* S^* w \rangle = \langle v, S^* T^* w \rangle &= \langle TSv, w \rangle \\ &= \langle STv, w \rangle \end{aligned}$$

This implies that $TS = ST$. □

Problem 10

Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if

$$\langle Tv, v \rangle = \langle T^* v, v \rangle$$

for all $v \in V$.

Proof.

$$T \text{ self-adjoint} \quad \iff \quad T = T^* \quad \iff \quad \langle Tv, v\rangle = \langle T^*v, v\rangle$$

□

Problem 12

An operator $B \in \mathcal{L}(V)$ is called **skew** if

$$B^* = -B$$

Suppose that $T \in \mathcal{L}(V)$. Prove that T is normal if and only if there exist commuting operators A and B such that A is self-adjoint, B is a skew operator, and $T = A + B$.

Proof. $\Rightarrow$ Take $A = \frac{T+T^*}{2}$ and $B = \frac{T-T^*}{2}$, so it holds that $T = A + B$. We first verify that A is self-adjoint.

$$4AA^* = (T + T^*)(T + T^*)^* = (T + T^*)(T^* + T) = TT^* + TT + T^*T^*T^*T$$
$$4A^*A = (T + T^*)^*(T + T^*) = (T^* + T)(T + T^*) = T^*T + T^*T^* + TT + TT^*$$

Therefore $AA^* = A^*A$ and A is self-adjoint. For B, we have that

$$B^* = \frac{T^* - T}{2} = -B$$

$\Leftarrow$ We have that $T^* = (A + B)^* = A^* - B = A - B$

$$TT^* = (A + B)(A - B) = A^2 - AB + BA - B^2 = A^2 - B^2$$
$$T^*T = (A - B)(A + B) = A^2 + AB - BA - B^2 = A^2 - B^2$$

since A and B commute. □

Problem 15

Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

(a) T is self-adjoint $\iff$ T^{-1} is self-adjoint;

(b) T is normal $\iff$ T^{-1} is normal.

Proof. (a) T is self-adjoint $\iff T = T^* \iff T^{-1} = (T^*)^{-1} = (T^{-1})^* \iff T^{-1}$ is self-adjoint.

(b) T is normal $\iff TT^* = T^*T \iff (T^*)^{-1}T^{-1} = T^{-1}(T^*)^{-1} \iff T^{-1}$ is normal. □

Problem 19

Suppose $T \in \mathcal{L}(V)$ and $\|T^*v\| \leq \|Tv\|$ for every $v \in V$. Prove that T is normal.

Proof. Note that we have

$$\begin{aligned}\|T^*v\|^2 \leq \|Tv\|^2 &\iff \langle T^*v, T^*v\rangle \leq \langle Tv, Tv\rangle \\ &\iff \langle TT^*v, v\rangle \leq \langle T^*Tv, v\rangle \\ &\iff \langle (T^*T - TT^*)v, v\rangle \leq 0 \\ &\iff T^*T - TT^* \leq 0\end{aligned}$$

We want to borrow the fact from later chapters to make the proof substantially easier:

$$\operatorname{tr}(T^*T - TT^*) = 0$$

which means that the sum of the eigenvalues are 0. However, by the fact derived above, let (λ, v) be arbitrary eigenpair of $T^*T - TT^*$, we have that

$$\lambda\langle v, v\rangle = \langle (T^*T - TT^*)v, v\rangle \leq 0$$

meaning that $\lambda \leq 0$ and therefore all $\lambda = 0$. Notice that $T^*T - TT^*$ is self-adjoint as

$$(T^*T - TT^*)^* = (T^*T)^* - (TT^*)^* = T^*T - TT^*$$

Hence, we can claim that $T^*T - TT^* = 0$, and by the result 7.20 T is normal. □

Problem 20

Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the following are equivalent.

(a) P is self-adjoint.

(b) P is normal.

(c) There is a subspace U of V such that $P = P_U$.

Proof. $(a) \Rightarrow (b)$ Trivial.

$(b) \Rightarrow (c)$ Take arbitrary $v \in V$, then we have that

$$P(v - Pv) = Pv - P^2v = 0$$

which means that $v - Pv \in \operatorname{null} P$. Since P is normal, we know that $V = \operatorname{null} P \oplus \operatorname{range} P$ and thus every v can be written as

$$v = Pv + (v - Pv)$$

which means that $P = P_{\operatorname{range} P}$.

$(c) \Rightarrow (a)$ Take any $v_1, v_2 \in V$, then we know that $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ for $u_1, u_2 \in U$ and $w_1, w_2 \in U^\perp$. It suffices to show that $\langle Pv_1, v_2\rangle = \langle v_1, Pv_2\rangle$.

$$\begin{aligned}\langle Pv_1, v_2\rangle &= \langle u_1, u_2 + w_2\rangle \\ &= \langle u_1, u_2\rangle + \langle w_1, u_2\rangle \\ &= \langle v_1, Pv_2\rangle\end{aligned}$$

□

Problem 23

Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$\|v\| = \|w\| = 2, \quad Tv = 3v, \quad Tw = 4w$$

Show that $\|T(v+w)\| = 10$.

Proof. We note that v and w are distinct eigenvectors with different eigenvalues. As T is normal, v and w are thus orthogonal. We have that

$$\|T(v+w)\|^2 = \|Tv\|^2 + \|Tw\|^2 = 9\|v\|^2 + 16\|w\|^2 = 100$$

which shows the desired result. □

Problem 25

Suppose $T \in \mathcal{L}(V)$. Prove that T is diagnoalizable if and only if T^* is diagnoalizable.

Proof. We know that (λ, v) is an eigenpair of T if and only if $(\overline{\lambda}, v)$ is an eigenpair of T^*. Since T and T^* shares the same eigenvectors, V has eigenbasis consisting of T's eigenvectors iff V has eigenbasis consisting of T^*'s eigenvectors, completing the proof. □

Problem 27

Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\text{null } T^k = \text{null } T \text{ and range } T^k = \text{range } T$$

for every positive integer k.

Proof. It is obvious that null $T \subseteq$ null T^k and range $T^k \subseteq$ range T. We aim at proving the other direction.

If T is self-adjoint, then the inclusion for other direction of null space is simpler to prove. Suppose $v \in$ null T^k, then

$$\langle T^k v, T^{k-2} v\rangle = \langle T^{k-1} v, T^{k-1} v\rangle = 0$$

which shows that $T^{k-1}v = 0$. Doing this recursively gives that $Tv = 0$.

Now let's consider the normal operator T, then

$$(T^*T)^k v = (T^*)^k T^k v = 0$$

So $v \in$ null $(T^*T)^k$. Notice that (T^*T) is a self-adjoint operator and applying the fact proved above yields that $v \in$ null T^*T. Hence we have that

$$\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = 0$$

which shows that $v \in$ null T.

For proving the range-related inclusion, this can relate to the null space, where we have that

$$\text{range } (T^k) = (\text{null } (T^k)^*)^\perp = (\text{null } (T^*)^k)^\perp = (\text{null } T^*)^\perp = \text{range } T$$

$\square$

Problem 29

Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_k\| = \|T^*e_k\|$ for each $k = 1, \ldots, n$, then T is normal.

Proof. We will give a counterexample to the statement. Let $v \in V$. Then we have

$$\|Tv\|^2 = \langle Tv, Tv\rangle = \left\langle T\left(\sum_{j=1}^{n}\langle v, e_j\rangle e_j\right), T\left(\sum_{k=1}^{n}\langle v, e_k\rangle e_k\right)\right\rangle$$

$$= \left\langle \sum_{j=1}^{n}\langle v, e_j\rangle Te_j, \sum_{k=1}^{n}\langle v, e_k\rangle Te_k\right\rangle = \sum_{j=1}^{n}\sum_{k=1}^{n}\langle v, e_j\rangle\overline{\langle v, e_k\rangle}\langle Te_j, Te_k\rangle.$$

Similarly, $\|T^*v\|^2 = \sum_{j=1}^{n}\sum_{k=1}^{n}\langle v, e_j\rangle\overline{\langle v, e_k\rangle}\langle T^*e_j, T^*e_k\rangle$. So, in order to give a counterexample we need to find some T such that it satisfies the hypothesis and $\sum_{j\neq k}\langle Te_j, Te_k\rangle \neq \sum_{j\neq k}\langle T^*e_j, T^*e_k\rangle$.

Now let $V = \mathbb{R}^2$, e_1, e_2 be a standart basis of $\mathbb{R}^2$ and $a, b, c, d \in \mathbb{R}$. Consider

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ T^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

and suppose that T satisfies the conditions above. Namely

$$\begin{aligned}\|Te_1\|^2 &= \|T^*e_1\|^2 \\ \|Te_2\|^2 &= \|T^*e_2\|^2 \\ \langle Te_1, Te_2\rangle = \frac{1}{2}\langle Te_1, Te_2\rangle + \frac{1}{2}\langle Te_2, Te_1\rangle &\neq \\ \frac{1}{2}\langle T^*e_1, T^*e_2\rangle + \frac{1}{2}\langle T^*e_2, T^*e_1\rangle &= \langle T^*e_1, T^*e_2\rangle.\end{aligned}$$

The identities in the third line hold because $\mathbb{F} = \mathbb{R}$. Plugging T we get

$$a^2 + c^2 = a^2 + b^2 \quad \text{and} \quad b^2 + d^2 = c^2 + d^2 \implies b^2 = c^2$$
$$ab + cd \neq ac + bd.$$

To ensure $T \neq T^*$ we take $b = -c$. Then $ab - bd = b(a-d) \neq -ab + bd = -b(a-d)$. So picking $a \neq d$ and $b \neq 0$ will suffice. For example, take $a = 0,\ b = 1,\ c = -1,\ d = 1$. Then we get the following matrices

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},\ T^* = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

You can easily verify that $\|Te_k\| = \|T^*e_k\|$ for $k = 1, 2$, but $TT^* \neq T^*T$. Thus, T satisfies the hypothesis but is not normal. □

Problem 30

Suppose that $T \in \mathcal{L}(\mathbb{F}^3)$ is normal and $T(1,1,1) = (2,2,2)$. Suppose $(z_1, z_2, z_3) \in \operatorname{null} T$. Prove that $z_1 + z_2 + z_3 = 0$.

Proof. We can see that $(1,1,1)$ is an eigenvector of T with 2 to be the eigenvalue. Then z_1, z_2, z_3 is orthogonal to $(1,1,1)$, so we have that

$$z_1 + z_2 + z_3 = 0$$

by taking the inner product. □

Problem 32

Suppose $T\colon V \to W$ is a linear map. Show that under the standard identification of V with V' and the corresponding identification of W with W', the adjoint map $T^*\colon W \to V$ corresponds to the dual map $T'\colon W' \to V'$. More precisely, show that

$$T'(\varphi_w) = \varphi_{T^*w}$$

for all $w \in W$, where φ_w and φ_{T^*w} are defined previously in the book.

Proof. We first know that $T'(\varphi_w) = \varphi_w \circ T$, so

$$T'(\varphi_w)(u) = \langle Tu, w \rangle = \langle u, T^* w \rangle = \varphi_{T^* w}(u)$$

$\square$

7B: Spectral Theorem

Lemma 21 (invertible quadratic expressions)**.** *Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 < 4c$. Then*

$$T^2 + bT + cI$$

is an invertible operator.

Lemma 22 (minimal polynomial of self-adjoint operator)**.** *Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Then the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$.*

Theorem 23 (REAL SPECTRAL THEOREM)**.** *Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) T is self-adjoint.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

Theorem 24 (COMPLEX SPECTRAL THEOREM)**.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) T is normal.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

Problem 1

Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

Proof. $\Rightarrow$ Assume the normal operator T is self-adjoint, then we know that $T = T^*$ and their eigenvalue and eigenvectors are the same. However, we know that (λ, v) is an eigenpair of T if and only if $(\overline{\lambda}, v)$ is an eigenpair of T^*. This means that $\lambda = \overline{\lambda}$ and thus all eigenvalues are real.

$\Leftarrow$ Conversely, since T is normal, T has a diagnoal entry wrt some orthonormal basis of V where the entries are the eigenvalues. Since the eigenvalues are real, the conjugate transpose of the matrix equals itself, therefore completing the proof. $\square$

Problem 3

Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal. Prove that the set of eigenvalues of T is contained in $\{0, 1\}$ if and only if there is a subspace U of V such that $T = P_U$.

Proof. Since T is normal, there exists a diagonal matrix representation of T wrt. some orthonormal basis $e_1, \ldots, e_n$. Then we have that

$$\begin{aligned}\text{eigenvalues of } T \text{ contained in } \{0,1\} &\iff Te_k = 0 \text{ or } Te_k = 1\\ &\iff \exists U = \operatorname{span}\{e_i\}_{i \in J} \text{ for some index set}\\ &\quad J \text{ s.t. } Tv \in U \text{ for all } v \in V\end{aligned}$$

$\square$

Problem 4

Prove that a normal operator on a complex inner product space is skew (meaning it equals the negative of its adjoint) if and only if all its eigenvalues are purely imaginary (meaning that they have real part equal to 0).

Proof. Suppose T is normal, then we know that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

$$\begin{aligned}T \text{ is skew } &\iff -\langle Tv, v\rangle = \langle T^* v, v\rangle = -\lambda\langle v, v\rangle\\ &\iff -\lambda = \overline{\lambda}\\ &\iff \lambda \text{ is purely imaginary}\end{aligned}$$

$\square$

Problem 6

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Proof. Let (λ, v) be an eigenpair of T. Then we know that

$$T^9 v = \lambda^9 v = T^8 v = \lambda^8 v$$

Therefore $\lambda^9 = \lambda^8$ and thus $\lambda = \{0, 1\}$, then by P1 we know T is self-adjoint and by P3 we know that T is a projection operator and thus $T^2 = T$. $\square$

Problem 8

Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if every eigenvector of T is also an eigenvector of T^*.

Proof. $\Rightarrow$ This follows by the theorem on the book.

$\Leftarrow$ Since we are concerned with the field $\mathbb{C}$, there exists a matrix of T that is upper-triangular wrt. some orthonormal basis $e_1, \ldots, e_n$ such that $Te_j = \sum_{i=1}^{j} a_{ij} e_i$. In particular, e_1 is an eigenvector of T and it is also an eigenvector of T^*. So the first column of T^* only has the first term nonzero. We can repeat the statement for other eigenvectors of T to that of T^*. Note that since the field is taken over $\mathbb{C}$, T is guaranteed to have n eigenvalues and therefore the corresponding eigenvectors. This means that T and T^* are simultaneously diagonalizable and thus they commute, completing the proof. $\square$

Problem 10

Suppose V is a complex inner product space. Prove that every normal operator on V has a square root. (Note that an operator $S \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $S^2 = T$).

Proof. Suppose T is a normal operator on V. By complex spectral theorem, $T = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ wrt. to some orthonormal bases $e_1, \ldots, e_n$. We can naturally define an operator S such that

$$Se_k = \sqrt{\lambda_k} e_k$$

Then we have that

$$S^2(v) = S\left(\sum_{i=1}^{n} a_i S(e_i)\right) = S\left(\sum_{i=1}^{n} a_i \sqrt{\lambda_i} e_i\right) = \sum_{i=1}^{n} a_i \lambda_i e_i = \sum_{i=1}^{n} a_i T(e_i) = T(v)$$

$\square$

Problem 11

Prove that every self-adjoint operator on V has a cube root.

Proof. We can do the similar proof as above. The only difference is that the self-adjoint operators have real eigenvales and therefore their cube root is also real and uniquely determined. $\square$

Problem 12

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is normal. Prove that if S is an operator on V that commutes with T, then S commutes with T^*.

Proof. Let v be an eigenvector of T with eigenvalue of λ. We have that

$$T(Sv) = STv = S(\lambda v) = \lambda(Sv)$$

So Sv is an eigenvector of T with eigenvalue λ. This means that it is also an eigenvector of T^* with eigenvalue $\overline{\lambda}$. Hence, we have

$$T^*(Sv) = \overline{\lambda}Sv = S(\overline{\lambda}v) = ST^*v$$

completing the proof. $\square$

Problem 14

Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.

Proof. $\Rightarrow$ Given an self-adjoint operator T, we know that there exists an orthonormal basis of V consisting of eigenvectors of T and thus we know that $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Let's say we have arbitrary eigenpairs (λ_1, v_1) and (λ_2, v_2) with $\lambda_1 \neq \lambda_2$. Then

$$0 = \langle Tv_1, v_2 \rangle - \langle v_1, Tv_2 \rangle = (\lambda_1 - \lambda_2)\langle v_1, v_2 \rangle$$

Therefore, we have $\langle v_1, v_2 \rangle = 0$.

$\Leftarrow$ Conversely, Such eigenvectors form a basis of V and thus T is self-adjoint. $\square$

Problem 16

Suppose $\mathbb{F} = \mathbb{C}$ and $\mathcal{E} \subseteq \mathcal{L}(V)$. Prove that there is an orthonormal basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if S and T are commuting normal operators for all $S, T \in \mathcal{E}$.

Proof. $\Rightarrow$ Since $S, T \in \mathcal{E}$ are diagonal, they commute and are normal.

$\Leftarrow$ This side follows as what we've done in ch5E P2. $\square$

Problem 19

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T.

(a) Prove that $U^{\perp}$ is invariant under T.

(b) Prove that $T|_U \in \mathcal{L}(V)$ is self-adjoint.

(c) Prove that $T|_{U^{\perp}} \in \mathcal{L}(U^{\perp})$ is self-adjoint.

Proof. (a) Let $u \in U$ and $w \in U^{\perp}$, then

$$\langle u, Tw \rangle = \langle Tu, w \rangle = 0$$

since $Tu \in U$. Thus $Tw \in U^{\perp}$.

(b) Take $u_1, u_2 \in U$, then we know that $u_1, u_2 \in V$. Since T is self-adjoint, then

$$\langle Tu_1, u_2 \rangle = \langle u_1, Tu_2 \rangle$$

which shows that $T|_U$ is self-adjoint.

(c) This follows similarly. $\square$

Problem 21

Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that

$$T^2 - 5T + 6I = 0$$

Proof. Since T is self-adjoint, it is diagonalizable, therefore its minimal polynomial p has the form of $(z - \lambda_1) \dots (z - \lambda_n)$, where $\lambda_1, ..., \lambda_n$ are distinct (by 5.62). Since roots of a minimal polynomial are exactly eigenvalues of T (by 5.27), then we have $p(z) = (z-2)(z-3)$. Thus

$$0 = p(T) = (T - 2I)(T - 3I) = T^2 - 5T + 6I.$$

$\square$

Problem 23

Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbb{F}$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon$$

Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

Proof. Since T is self-adjoint, there exists an orthonormal eigenbasis $e_1, \ldots, e_n$ with the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ and every λ_i is real. Then we know that any $v \in V$ can be expressed as $v = \sum_{i=1}^n c_i e_i$, where $c_i = \langle v, e_i \rangle$. We have

$$1 = \|v\|^2 = \sum_{i=1}^n |c_i|^2 \text{ (by 6.24)},$$
$$Tv = \sum_{i=1}^n c_i \lambda_i e_i.$$

Further, we have

$$\begin{aligned}\|Tv - \lambda v\|^2 &= \left\langle \sum_{i=1}^n c_i(\lambda_i - \lambda)e_i, \sum_{j=1}^n c_j(\lambda_j - \lambda)e_j \right\rangle \\ &= \sum_{i=1}^n |c_i|^2(\lambda_i - \lambda)^2 = \sum_{i=1}^n |c_i|^2(\lambda - \lambda_i)^2 < \epsilon^2.\end{aligned}$$

Now suppose that $|\lambda - \lambda_i| \geq \epsilon$ for all $i = 1, \ldots n$. Then

$$\|Tv - \lambda v\|^2 = \sum_{i=1}^n |c_i|^2(\lambda - \lambda_i)^2 \geq \sum_{i=1}^n |c_i|^2 \epsilon^2 = \epsilon^2 \sum_{i=1}^n |c_i|^2 = \epsilon^2,$$

which leads to contradiction. Thus, there exists some eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$. $\square$

7C: Positive Operators

Definition 25 (positive operator)**.** *An operator $T \in \mathcal{L}(V)$ is called **positive** if T is self-adjoint and*

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$.

Remark 26. *Positive operators are also known as positive definite operators.*

Definition 27 (square root)**.** *An operator R is called a **squared root** of an operator T if $R^2 = T$.*

Theorem 28 (characterizations of positive operators)**.** *Let $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) T is a positive operator.

(b) T is self-adjoint and all eigenvalues of T are nonnegative.

(c) With respect to some orthonormal basis of V, the matrix of T is a diagonal matrix with only nonnegative numbers on the diagonal.

(d) T has a positive square root.

(e) T has a self-adjoint square root.

(f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

Theorem 29. *Every positive operator on V has a unique positive square root.*

Remark 30. *For T a positive operator, $\sqrt{T}$ denotes the unique positive square root of T.*

Corollary 31. *Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.*

Problem 1

Suppose $T \in \mathcal{L}(V)$. Prove that if both T and $-T$ are positive operators, then $T = 0$.

Proof. This means that

$$\langle Tv, v\rangle \geq 0 \quad \langle Tv, v\rangle \leq 0$$

for all $v \in V$, so $T = 0$. □

Problem 3

Suppose n is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is the operator whose matrix (wrt. the standard basis) consists of all 1's. Show that T is a positive operator.

Proof. Suppose $v = (v_1, \ldots, v_n) \in V$, then we have that

$$\langle Tv, v\rangle = \left\langle \left(\sum_{i=1}^n v_i, \ldots, \sum_{i=1}^n v_i\right), (v_1, \ldots, v_n)\right\rangle = \left(\sum_{i=1}^n v_i\right)^2 \geq 0$$

□

Problem 6

Prove that the sum of two positive operators on V is a positive operator.

Proof. Let S, T be two positive operator and take $v \in V$. Then

$$\langle (S+T)v, v\rangle = \langle Sv, v\rangle + \langle Tv, v\rangle \geq 0$$

□

Problem 7

Suppose $S \in \mathcal{L}(V)$ is an invertible positive operator and $T \in \mathcal{L}(V)$ is a positive operator. Prove that $S + T$ is invertible.

Proof. Take any nonzero $v \in V$, then

$$\langle (S+T)v, v\rangle = \langle Sv, v\rangle + \langle Tv, v\rangle > 0$$

So null $(S+T) = \{0\}$ and thus $S+T$ is injective and therefore invertible. □

Problem 9

Suppose $T \in \mathcal{L}(V)$ is a positive operator and $S \in \mathcal{L}(W, V)$. Prove that S^*TS is a positive operator on W.

Proof. We have that

$$\langle S^*TSw, w\rangle = \langle T(Sw), (Sw)\rangle \geq 0$$

as T is positive. $\square$

Problem 10

Suppose T is a positive operator on V. Suppose $v, w \in V$ are such that

$$Tv = w \quad Tw = v$$

Prove that $v = w$.

Proof. We have that

$$T^2 v = T(Tv) = Tw = v$$

Thus $(T^2 - I)v = 0$. This means that either $v = 0$ or $T = \pm I$. In the first case, $w = Tv = 0 = v$ and $Tw = 0 = v$. In the second case, if $T = I$, then $v = w$. Note that T cannot be $-I$ as T is a positive operator (all eigenvalues nonnegative). $\square$

Problem 12

Suppose $T \in \mathcal{L}(V)$ is a positive operator. Prove that T^k is a positive operator for every positive integer k.

Proof. If k is even, then

$$\langle T^k v, v\rangle = \langle T^{\frac{k}{2}} v, T^{\frac{k}{2}} v\rangle \geq 0$$

If k is odd, then

$$\langle T^k v, v\rangle = \langle T(T^{\frac{k-1}{2}} v), (T^{\frac{k-1}{2}} v)\rangle \geq 0$$

as T is positive. $\square$

Problem 13

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $\alpha \in \mathbb{R}$.

(a) Prove that $T - \alpha I$ is a positive operator if and only if α is less than or equal to every eigenvalue of T.

(b) Prove that $\alpha I - T$ is a positive operator if and only if α is greater than or equal to every eigenvalue of T.

Proof. Let $v_1, \ldots, v_n$ be the eigenbasis of V with corresponding sorted eigenvalues $\lambda_1, \ldots, \lambda_n$ (λ_1 being smallest). Note that λ is a eigenvalue of T iff $T - \alpha I$ is an eigenvalue of $T - \alpha I$ iff $\alpha - \lambda$ is an eigenvalue of $\alpha I - T$. You may verify this.

(a) $T - \alpha I$ positive $\iff \lambda_1 - \alpha \geq 0 \iff \alpha \leq \lambda_i$ for all i. (b) $\alpha I - T$ positive $\iff \alpha - \lambda_n \geq 0 \iff \alpha \geq \lambda_i$ for all i. □

Problem 14

Suppose T is a positive operator on V and $v_1, \ldots, v_m \in V$. Prove that

$$\sum_{j=1}^{m} \sum_{k=1}^{m} \langle T v_k, v_j \rangle \geq 0$$

Proof.

$$\sum_{j=1}^{m} \sum_{k=1}^{m} \langle T v_k, v_j \rangle = \left\langle T \left(\sum_{i=1}^{m} v_i \right), \left(\sum_{i=1}^{m} v_i \right) \right\rangle \geq 0$$

□

Problem 15

Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that there exist positive operators $A, B \in \mathcal{L}(V)$ such that

$$T = A - B \text{ and } \sqrt{T^* T} = A + B \text{ and } AB = BA = 0$$

Proof. Define $|T| = \sqrt{T^2}$ and that

$$A = \frac{|T| + T}{2} \quad B = \frac{|T| - T}{2}$$

So $A - B = T$ and $A + B = |T| = \sqrt{T^* T}$. Finally, we have

$$AB = \frac{|T| + T}{2} \frac{|T| - T}{2} = \frac{|T|^2 - |T|T + T|T| - T^2}{4} = 0$$

As $|T|^2 = T^2$ and $|T|T = T|T|$. Similarly, $BA = 0$. □

Problem 16

Suppose T is a positive operator on V. Prove that

$$\text{null } \sqrt{T} = \text{null } T \text{ and range } \sqrt{T} = \text{range } T$$

Proof. We've shown that the unique square root $\sqrt{T}$ is obtained from simply taking the square root of the eigenvalues of T. This means that $\sqrt{T}$ and T share the same eigenbasis, which determines the respective null and range space and completes the proof. $\square$

Problem 18

Suppose S and T are positive operators on V. Prove that ST is a positive operator if and only if S and T commute.

Proof. $\Rightarrow$ Given ST is positive, then

$$ST = (ST)^* = T^*S^* = TS$$

$\Leftarrow$ Given $ST = TS$, then $(ST)^* = T^*S^* = TS = ST$. In addition,

$$\langle STv, v\rangle = \langle S\sqrt{T}v, \sqrt{T}v\rangle \geq 0$$

$\square$

Problem 22

Suppose $T \in \mathcal{L}(V)$ is a positive operator and $u \in V$ is such that $\|u\| = 1$ and $\|Tu\| \geq \|Tv\|$ for all $v \in V$ with $\|v\| = 1$. Show that u is an eigenvector of T corresponding to the largest eigenvalue of T.

Proof. Let $e_1, \ldots, e_n$ be the orthogonal eigenbasis of V and the corresponding eigenvalues to be $\lambda_1, \ldots, \lambda_n$ with sorted from smallest to largest. Then we have that

$$\|Tu\| = \left\| \sum_{i=1}^{n} a_i \lambda_i e_i \right\| = \sqrt{\sum_{i=1}^{n} |a_i|^2 \lambda_i^2}$$

We can take $v = e_n$, then we have that

$$\|Tu\|^2 = \sum_{i=1}^{n} |a_i|^2 \lambda_i^2 \geq \sum_{i=1}^{n} |a_i|^2 \lambda_n^2 = \lambda_n^2 = \|Tv\|^2$$

which shows the desired conclusion. $\square$

7D: Isometries, Unitary Operators, and Matrix Factorization

Definition 32 (isometry)**.** *A linear map $S \in \mathcal{L}(V, W)$ is called an **isometry** if*

$$\|Sv\| = \|v\|$$

for every $v \in V$. In other words, a linear map is an isometry if it preserves norms.

Remark 33. *Every isometry is injective.*

Theorem 34 (characterizations of isometries)**.** *Suppose $S \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Then the following are equivalent.*

(a) S is an isometry.

(b) $S^*S = I$.

(c) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$.

(d) $Se_1, \ldots, Se_n$ is an orthonoral list in W.

(e) The columns of $\mathcal{M}(S, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$ form an orthonormal list in $\mathbb{F}^m$ with respect to the Euclidean inner product.

Definition 35 (Unitary Operators)**.** *An operator $S \in \mathcal{L}(V)$ is called **unitary** if S is an invertible isometry.*

Remark 36. *Note that every isometry is injective and therefore invertible on a finite-dimensional vector space. The author makes the distinction simply to avoid confusion in more abstract stages. One might think of unitary operators being equivalent to isometries in finite-dimensional space.*

Theorem 37 (characterizations of unitary operators)**.** *Suppose $S \in \mathcal{L}(V)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V. Then the following are equivalent.*

(a) S is a unitary operator.

(b) $S^*S = SS^* = I$.

(c) S is invertible and $S^{-1} = S^*$.

(d) $Se_1, \ldots, Se_n$ is an orthonoral list in W.

(e) The rows of $\mathcal{M}(S, (e_1, \ldots, e_n))$ form an orthonormal basis of $\mathbb{F}^m$ with respect to the Euclidean inner product.

(f) S^* is a unitary operator.

Corollary 38. *Suppoe λ is an eigenvalue of a unitary operator. Then $|\lambda| = 1$.*

Corollary 39. *Suppose $\mathbb{F} = \mathbb{C}$ and $S \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) S is a unitary operator.

(b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

Definition 40 (unitary matrix)**.** *An n-by-n matrix is called **unitary** if its columns form an orthonormal list in $\mathbb{F}^n$.*

Theorem 41 (characterizations of unitary matrices)**.** *Suppose Q is an n-by-n matrix. Then the following are equivalent.*

(a) Q is a unitary matrix.

(b) The rows of Q form an orthonormal list in $\mathbb{F}^n$.

(c) $\|Qv\| = \|v\|$ for every $v \in \mathbb{F}^n$.

(d) $Q^*Q = QQ^* = I_n$.

Theorem 42 (QR factorization)**.** *Suppose A is a square matrix with linearly independent columns. Then there exists unique matrices Q and R such that Q is unitary, R is upper triangular with only positive numbers on its diagonal, and*

$$A = QR.$$

Lemma 43 (positive invertible operator)**.** *A self-adjoint operator $T \in \mathcal{L}(V)$ is a positive invertible operator if and only if $\langle Tv, v\rangle > 0$ for every nonzero $v \in V$.*

Definition 44 (positive definite)**.** *A matrix $B \in \mathbb{F}^{n,n}$ is called **positive definite** if $B^* = B$ and*

$$\langle Bx, x\rangle > 0$$

for every nonzero $x \in \mathbb{F}^n$.

Theorem 45 (Cholesky factorization)**.** *Suppose B is a positive definite matrix. Then there exists a unique upper-triangular matrix R with only positive numbers on its diagonal such that*

$$B = R^*R$$

Problem 1

Suppose $\dim V \geq 2$ and $S \in \mathcal{L}(V, W)$. Prove that S is an isometry if and only if Se_1, Se_2 is an orthonormal list in W for every orthonormal list e_1, e_2 of length two in V.

Proof. ($\Longrightarrow$) Let e_1, e_2 be an orthonormal list in V. Since S is an isometry, then $S^*S = I$. Hence

$$\langle Se_j, Se_k \rangle = \langle e_j, S^*Se_k \rangle = \langle e_j, e_k \rangle = \delta_{jk},$$

where $j = 1, 2, k = 1, 2$. Thus, Se_1, Se_2 is an orthonormal list in W.

($\Longleftarrow$) Let $e_1, \ldots, e_n$ be an orthonormal basis of V. Then any list (e_j, e_k), where $j \neq k$ and $j = 1, \ldots, n,\ k = 1, \ldots, n$ is orthonormal. By hypothesis, (Se_j, Se_k) is an orthonormal in W. Hence, $\langle Se_j, Se_k \rangle = \delta_{jk}$ for $j = 1, \ldots, n,\ k = 1, \ldots, n$. Therefore, $Se_1, \ldots, Se_n$ is an orthonormal list in W. Thus, S is an isometry. $\square$

Problem 2

Suppose $T \in \mathcal{L}(V, W)$. Prove that T is a scalar multiple of an isometry if and only if T preserves orthogonality.

Proof. $\Rightarrow$ if $T = aS$ for S to be an isometry, then suppose $\langle u, v \rangle = 0$,

$$\langle Tu, Tv \rangle = |a|^2 \langle Su, Sv \rangle = |a|^2 \langle u, v \rangle = 0$$

$\Leftarrow$ Suppose for all $\langle u, v \rangle = 0$, $\langle Tu, Tv \rangle = 0$. Take an orthonormal basis $\{e_1, \ldots, e_n\}$ of V. Then we have that

$$\langle e_i + e_j, e_i - e_j \rangle = \|e_i\|^2 - \langle e_i, e_j \rangle + \langle e_j, e_i \rangle - \left\|e_j\right\|^2 = 1 - 1 = 0$$

This means that

$$\langle T(e_i + e_j), T(e_i - e_j) \rangle = \|Te_i\|^2 - \left\|Te_j\right\|^2 = 0$$

which means that $\|Te_k\| = a$ for some scalar a. Take any $v \in V$, then

$$\|Tv\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2 \|Te_i\|^2 + \sum_{i \neq j} |\langle v, e_i \rangle \langle v, e_j \rangle| \langle Te_i, Te_j \rangle = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2 |a|^2 = \|av\|^2$$

This means that

$$\|Tv\| = a\|v\|$$

for some a and thus completes the proof ($\frac{1}{a}T$ is an isometry). $\square$

Problem 3

(a) Show that the product of two unitary operators on V is a unitary operator.

(b) Show that the inverse of a unitary operator on V is a unitary operator.

Proof. Let T, S be two unitary operators.
(a) we have

$$(TS)^*(TS) = S^*T^*TS = S^*IS = S^*S = I$$

(b) since $S^{-1} = S^*$ and S^* is a unitary operator, S^{-1} is also unitary. $\square$

Problem 5

Suppose $S \in \mathcal{L}(V)$. Prove that the following are equivalent.

(a) S is a self-adjoint unitary operator.

(b) $S = 2P - I$ for some orthogonal projection P on V.

(c) There exists a subspace U of V such that $Su = u$ for every $u \in U$ and $Sw = -w$ for every $w \in U^\perp$.

Proof. (a) $\Rightarrow$ (b), (c) Since S is unitary, its eigenvalues are either -1 or 1. Since S is self-adjoint, we can decompose V into its eigenspace $E(-1)$ and $E(1)$ such that $V = E(-1) \oplus E(1)$. This means that for all $w \in E(-1), Sw = -w$ and $u \in E(1), Su = u$ where $E(1) = U$. So part (c) is completed. For each $v \in V$, $v = v_1 + v_{-1}$ where $v_1 \in E(1), v_{-1} \in E(-1)$. Then define $P(v) = v_1$ and we have that $Sv = Sv_1 + Sv_{-1} = v_1 - v_{-1} = P(v) - (I - P)(v) = (2P - I)v$. Part(b) is also completed.

(b), (c) $\Rightarrow$ (a) Conversely, first assume (b) holds, then this means that

$$S^* = (2P - I)^* = 2P - I = S$$

so S is self-adjoint. To see that it is unitary,

$$SS^* = S^2 = (2P - I)(2P - I) = 4P^2 - 2P - 2P + I = I$$

Next assume (c) holds, then Let P define to be the projection operator into U and then $(I-P)$ is the projection into $U^\perp$, then we have that $S = P-(I-P) = 2P-I$, so applying the previous proof finishes the problem. $\square$

Problem 6

Suppose T_1, T_2 are both normal operators on $\mathbb{F}^3$ with 2,5,7 as eigenvalues. Prove that there exists a unitary operator $S \in \mathcal{L}(\mathbb{F}^3)$ such that $T_1 = S^*T_2S$.

Proof. By complex spectral theorem, T_1 has $\{a_1, a_2, a_3\}$ as a set of orthonormal eigenvectors that form a basis with eigenvalues 2,5,7. Similarly, T_2 has $\{b_1, b_2, b_3\}$. What we want to do here is simply define a change-of-basis operator that is also unitary. Define $S \in \mathcal{L}(\mathbb{F}^3)$ s.t.

$$Sa_i = b_i$$

Clearly, S is unitary. Now, it suffices to prove that the desired equation holds. We also have that $S^{-1} = S^*$ so $S^*b_i = a_i$. Take any $v = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$, then

$$\begin{aligned}
S^*T_2S(v) &= S^*T_2S\left(\sum_{i=1}^{3} \alpha_i a_i\right) \\
&= S^*T_2\left(\sum_{i=1}^{3} \alpha_i S(a_i)\right) \\
&= S^*\left(\sum_{i=1}^{3} \alpha_i T_2(b_i)\right) \\
&= S^*(\alpha_1 2b_1 + \alpha_2 5b_2 + \alpha_3 7b_3) \\
&= 2\alpha_1 a_1 + 5\alpha_2 a_2 + 7\alpha_3 a_3
\end{aligned}$$

Note that we also have that

$$\begin{aligned}
T_1(v) &= T_1\left(\sum_{i=1}^{3} \alpha_i a_i\right) \\
&= 2\alpha_1 a_1 + 5\alpha_2 a_2 + 7\alpha_3 a_3
\end{aligned}$$

□

Problem 9

Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Suppose every eigenvalue of T has absolute value 1 and $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that T is a unitary operator.

Proof. Since T has no eigenvalue 0, it is invertible. We also have that

$$\begin{aligned}\|Tv\| \leq \|v\| &\iff \langle Tv, Tv\rangle \leq \langle v, v\rangle \\ &\iff \langle T^*Tv, v\rangle \leq \langle v, v\rangle \\ &\iff \langle (T^*T - I)v, v\rangle \leq 0\end{aligned}$$

which gives that $T^*T \leq I$. Since the eigenvalue of T^*T are all one and it is self-adjoint, there exists eigenbasis all with eigenvalue one. This implies that $T^*T = I$ and thus T is unitary. $\square$

Problem 10

Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator such that $\|Tv\| \leq \|v\|$ for all $v \in V$.

(a) Show that $I - T^2$ is a positive operator.

(b) Show that $T + i\sqrt{I - T^2}$ is a unitary operator.

Proof. (a) take any $v \in V$, then

$$\begin{aligned}\langle (I - T^2)v, v\rangle &= \langle v, v\rangle - \langle Tv, T^*v\rangle \\ &= \langle v, v\rangle - \langle Tv, Tv\rangle \\ &= \langle (I - T)v, (I - T)v\rangle \geq 0\end{aligned}$$

(b) Let $A = T, B = \sqrt{I - T^2}$. Then we have that

$$(A + Bi)(A + Bi)^* = A^2 - iAB + iBA + B^2$$

We know that A is self-adjoint so A^2 is as well. By (a), B^2 is also self-adjoint. Note that $AB = BA$ algebraically. Thus, we have that

$$(A + Bi)(A + Bi)^* = A^2 + B^2 = I$$

which shows that $A + Bi$ is unitary. $\square$

Problem 11

Suppose $S \in \mathcal{L}(V)$. Prove that S is a unitary operator if and only if

$$\{Sv \colon v \in V \text{ and } \|v\| \leq 1\} = \{v \in V \colon \|v\| \leq 1\}$$

Proof. $\Rightarrow$ take any $v \in r.h.s.$, then since S is invertible and thus surjective, there exists $u \in V$ such that $Sv = u$. As $\|v\| \leq 1$ and $\|Su\| = \|u\| = \|v\| \leq 1$, so $v \in l.h.s.$. Conversely, take any $Sv \in l.h.s.$, then $\|Sv\| = \|v\| \leq 1$, so $Sv \in r.h.s.$. Hence we complete this direction.

$\Leftarrow$ Take any $v \in V$ and consider $u \coloneqq \frac{v}{\|v\|}$. Then we know that there exists w with $\|w\| \leq 1$ such that $Sw = u$. $\square$

Problem 12

Prove or give a counterexample: If $S \in \mathcal{L}(V)$ is invertible and $\left\|S^{-1}v\right\| = \|Sv\|$ for every $v \in V$, then S is unitary.

Proof. Note that $\|v\| = \left\|S^{-1}(Sv)\right\| = \left\|S^2 v\right\|$. Hence, S^2 is an isometry and S is its square root. As a counterexample we can use a non-unitary square root of identity operator (which is trivially an isometry). Consider

$$S = \begin{pmatrix} i & 2 \\ 1 & -i \end{pmatrix}.$$

You can easily verify that $S^2 = I \iff S^{-1} = S$, and that it is not unitary. (This counterexample is taken from this paper). □

Problem 13

Explain why the columns of a square matrix of complex numbers form an orthonormal list in $\mathbb{C}^n$ if and only if the rows of the matrix form an orthonormal list in $\mathbb{C}^n$.

Proof. A square matirx Q is unitary iff Q^* is unitary, completing the proof. □

Problem 18

A square matrix A is called *symmetric* if it equals its transpose. Prove that if A is a symmetric matrix with real entries, then there exists a unitary matrix Q with real entries such that Q^*AQ is a diagonal matrix.

Proof. So we know that A is self-adjoint, so it is orthogonally diagonalizable. There exists eigenbasis $v_1, \ldots, v_n$ that are orthogonal. We can thus define Q to be consisting of such eigenvectors as

$$Av_i = \lambda_i v_i$$

Writing this in matrix form gives that

$$AQ = QD$$

for unitary matrix Q and diagonal matrix D with entries to be the corresponding eigenvalues. Hence, we get that

$$Q^*AQ$$

is diagonal. □

Problem 19

Suppose n is a positive integer. For this exercise, we adopt the notation that a typical element z of $\mathbb{C}^n$ is denoted by $z = (z_0, z_1, \ldots, z_{n-1})$. Define linear functionals $\omega_0, \omega_1, \ldots, \omega_{n-1}$ on $\mathbb{C}^n$ by

$$\omega_j(z) = \omega_j(z_0, z_1, \ldots, z_{n-1}) = \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} z_m e^{(-2\pi i) j \frac{m}{n}}.$$

The *discrete Fourier transform* is the operator $\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\mathcal{F} z = (\omega_0(z), \omega_1(z), \ldots, \omega_{n-1}(z)).$$

(a) Show that $\mathcal{F}$ is a unitary operator on $\mathbb{C}^n$.

(b) Show that if $(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$ and z_n is defined to equal z_0, then

$$\mathcal{F}^{-1}(z_0, z_1, \ldots, z_{n-1}) = \mathcal{F}(z_n, z_{n-1}, \ldots, z_1).$$

(c) Show that $\mathcal{F}^4 = I$.

Proof. (a) Let $f_0, \ldots, f_{n-1}$ be a standart basis on $\mathbb{C}^n$. Then

$$\omega_j(f_k) = \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} \delta_{m,k} e^{(-2\pi i) j \frac{m}{n}} = \frac{1}{\sqrt{n}} e^{(-2\pi i) j \frac{k}{n}},$$
$$\mathcal{F}(f_k) = (\omega_0(f_k), \omega_1(f_k), \ldots, \omega_{n-1}(f_k)) =$$
$$\frac{1}{\sqrt{n}} (e^{(-2\pi i \frac{k}{n})0}, e^{(-2\pi i \frac{k}{n})1}, \ldots, e^{(-2\pi i \frac{k}{n})(n-1)}).$$

Therefore

$$\langle \mathcal{F}(f_k), \mathcal{F}(f_l) \rangle =$$
$$\frac{1}{n} \sum_{j=0}^{n-1} e^{(-2\pi i \frac{k}{n})j} \overline{e^{(-2\pi i \frac{l}{n})j}} = \frac{1}{n} \sum_{j=0}^{n-1} e^{(-2\pi i \frac{k}{n})j} e^{(2\pi i \frac{l}{n})j} = \frac{1}{n} \sum_{j=1}^{n-1} e^{(\frac{2\pi i}{n})j(l-k)}.$$

If $k = l$, then

$$\langle \mathcal{F}(f_k), \mathcal{F}(f_l) \rangle = \frac{1}{n} \sum_{j=0}^{n-1} e^0 = \frac{1}{n} \sum_{j=0}^{n-1} 1 = 1.$$

Now suppose $k \neq l$. Note that $e^{\frac{2\pi i}{n}}$ is the nth root of unity which we denote as α. Then $\alpha^{(l-k)} \neq 1$ is also the nth root of unity, therefore

$\alpha^{(l-k)} = e^{2\pi i \frac{p}{n}}$ for some integer p. We have

$$\langle \mathcal{F}(f_k), \mathcal{F}(f_l) \rangle = \frac{1}{n} \sum_{j=0}^{n-1} \alpha^{(l-k)j} = \frac{1 - \alpha^{(l-k)n}}{n(1 - \alpha^{(l-k)})} = \frac{1 - 1}{n(1 - e^{2\pi i \frac{p}{n}})} = 0,$$

where the second identity comes from the formula for the partial sum of of the geometric series and the third identity comes from

$$\alpha^n = e^{2\pi i \frac{p}{n} n} = e^{2\pi i p} = e^0 = \ 1.$$

The equation above implies that $\mathcal{F}f_1, \ldots, \mathcal{F}f_n$ is an orthonormal list in $\mathbb{C}^n$. Since it has the length $n = \mathbb{C}^n$, it is an orthonormal basis of $\mathbb{C}^n$. Thus by 7.53(d) $\mathcal{F}$ is unitary.

(b) Let $\alpha = e^{-\frac{2\pi i}{n}}$. We can write the matrix of $\mathcal{F}$ w.r.t. the standart basis as

$$\frac{1}{\sqrt{n}} \begin{pmatrix} \alpha^{0\cdot 0} & \alpha^{0\cdot 1} & \ldots & \alpha^{0\cdot n} \\ \alpha^{1\cdot 0} & \alpha^{1\cdot 1} & \ldots & \alpha^{1\cdot (n-1)} \\ \vdots & \ddots & & \\ \alpha^{(n-1)\cdot 0} & \alpha^{(n-1)\cdot 1} & \ldots & \alpha^{(n-1)\cdot(n-1)} \end{pmatrix} =$$

$$\frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \alpha & \ldots & \alpha^{(n-1)} \\ \vdots & \ddots & & \\ 1 & \alpha^{(n-1)} & \ldots & \alpha^{(n-1)(n-1)} \end{pmatrix}.$$

Since $\mathcal{F}$ is unitary, then $\mathcal{F}^{-1} = \mathcal{F}^*$ and the matrix of $\mathcal{F}^{-1}$ is just the conjugate transpose of the matrix of $\mathcal{F}$. Therefore

$$\mathcal{F}^{-1}(z_0, z_1, \ldots, z_{n-1}) = (\overline{\omega}_0(z), \overline{\omega}_1(z), \ldots, \overline{\omega}_{n-1}(z)),$$

where

$$\overline{\omega}_j(z) = \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} z_m \overline{\alpha^{jm}}.$$

Consider the jth coordinate of $\mathcal{F}(z_n, z_{n-1}, \ldots, z_1)$, multiplied by $\sqrt{n}$

$$\sqrt{n}\mathcal{F}(z_n, z_{n-1}, \ldots, z_1)_j = \sum_{m=0}^{n-1} z_{n-m} \alpha^{jm} \overset{(1)}{=} \sum_{k=1}^{n} z_k \alpha^{j(n-k)} \overset{(2)}{=}$$

$$\sum_{k=1}^{n} z_k \alpha^{-jk} = \sum_{k=1}^{n} z_k \overline{\alpha^{jk}} = \sum_{k=1}^{n-1} z_k \overline{\alpha^{jk}} + z_n \overline{\alpha^{jn}} \overset{(3)}{=} \sum_{k=1}^{n-1} z_k \overline{\alpha^{jk}} + z_0 \overline{\alpha^0} = \sum_{k=0}^{n-1} z_k \overline{\alpha^{jk}}$$

$$= \sqrt{n}\mathcal{F}^{-1}(z_0, z_1, \ldots, z_{n-1})_j,$$

in (1) we made the substitution $k = m - n$ and, correspondingly, $m = n - k$; in (2) and (3) we used $\alpha^n = 1$; in (3) we used $z_n = z_0$. Thus $\mathcal{F}^{-1}(z_0, z_1, \ldots, z_{n-1}) = \mathcal{F}(z_n, z_{n-1}, \ldots, z_1)$.

(c) Let $A = \sqrt{n}\mathcal{M}(\mathcal{F})$, $B = A^2$, $C = B^2$. We will be enumerating matrix rows and columns with $0, \dots, n-1$. We have

$$B_{ij} = \sum_{k=0}^{n-1} A_{ik}A_{kj} = \sum_{k=0}^{n-1} \alpha^{ik}\alpha^{kj} = \sum_{k=0}^{n-1} (\alpha^{i+j})^k.$$

We've seen earlier taht the partial sum of this geometric series equals 0 if $\alpha^{i+j} \neq 1$ and n otherwise. Since α is the nth root of unity, then $\alpha^{i+j} = 1$ iff $i + j \equiv 0 \bmod n$. Hence

$$B_{ij} = \begin{cases} n, & \text{if } i + j \equiv 0 \bmod n, \\ 0, & \text{otherwise.} \end{cases}$$

Further, we have

$$C_{pq} = \sum_{m=0}^{n-1} B_{pm}B_{mq}.$$

Clearly,

$$B_{pm}B_{mq} \neq 0 \iff p + m \equiv 0 \bmod n \text{ and } q + m \equiv 0 \bmod n \iff p = q = n - m.$$

The last identity holds because of our summation limits. Also, for the same reason if $p = q$, then only one element in the sum above is nonzero. Therefore,

$$C_{pq} = \begin{cases} n^2, & p = q \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Finally, we have

$$n^2 I = C = B^2 = A^4 = n^2\mathcal{M}(\mathcal{F})^4.$$

Thus, $\mathcal{M}(\mathcal{F})^4 = I$.

□

7E: Singular Value Decomposition

Lemma 46 (properties of T^*T)**.** *Suppose $T \in \mathcal{L}(V, W)$. Then*

(a) T^*T is a positive operator on V;

(b) null $T^*T =$ null T;

(c) range $T^*T =$ range T^*;

(d) $\dim$ range $T = \dim$ range $T^* = \dim$ range T^*T.

Definition 47 (singular values)**.** *Suppose $T \in \mathcal{L}(V, W)$. The **singular values** of T are the nonnegative square root of the eigenvalues of T^*T, listed in decreasing order, each included as many times as the dimension of the corresponding eigenspace of T^*T.*

Theorem 48 (role of positive singular values)**.** *Suppose that $T \in \mathcal{L}(V, W)$. Then*

(a) T is injective $\iff$ 0 is not a singular value of T;

(b) the number of positive singular values of T equals $\dim$ range T;

(c) T is surjective $\iff$ number of positive singular values of T equals $\dim W$.

Corollary 49. *Suppose $S \in \mathcal{L}(V, W)$. Then*

$$S \text{ is an isometry } \iff \text{ all singular values of } S \text{ equal } 1.$$

Theorem 50 (SINGULAR VALUE DECOMPOSITION)**.** *Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Then there exist orthonormal lists $e_1, \ldots, e_m$ in V and $f_1, \ldots, f_m$ in W such that*

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_m\langle v, e_m\rangle f_m$$

for every $v \in V$.

Remark 51. *In the proof, we first let $s_1, \ldots, s_n$ to denote the singular values of T (thus $n = \dim V$). Then the spectral theorem implies that there exists an orthonormal list $e_1, \ldots, e_n$ of V with*

$$T^*Te_k = s_k^2 e_k$$

For each $k = 1, \ldots, m$, define

$$f_k = \frac{Te_k}{s_k}$$

Remark 52. *Suppose $T \in \mathcal{L}(V, W)$, the positive singular values of T are $s_1, \ldots, s_m$ and $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ are as in the singular decomposition above. Then the two orthonormal lists can both be extended to basis of the respective vector space. Where we can now define*

$$Te_k = \begin{cases} s_k f_k & \text{if } 1 \le k \le m, \\ 0 & \text{if } m < k \le \dim V \end{cases}$$

Definition 53 (diagonal matrix)**.** *An M-by-N matrix A is called a **diagonal matrix** if all entries of the matrix are 0 except possibly $A_{k,k}$ for $k = 1, \ldots \min\{M, N\}$.*

Theorem 54 (singular value decomposition of adjoint and pseudoinverse)**.** *Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Suppose $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ are orthonormal list in V and W such that*

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_m \rangle f_m$$

for every $v \in V$. Then

$$T^* w = s_1 \langle w, f_1 \rangle e_1 + s_m \langle w, f_m \rangle e_m$$

and

$$T^\dagger w = \frac{\langle w, f_1 \rangle}{s_1} e_1 + \cdots + \frac{\langle w, f_m \rangle}{s_m} e_m$$

for every $w \in W$.

Theorem 55 (matrix version of SVD)**.** *Suppose A is a p-by-n matrix of rank $m \geq 1$. Then there exist a p-by-m matrix B with orthonormal columns, an m-by-m diagonal matrix D with positive numbers on the diagonal, and an n-by-m matrix C with orthonormal columns such that*

$$A = BDC^*$$

Remark 56. *A is a $p \times n$ matrix while BDC^* has a total of $m(p + m + n)$ entries, which could be considerably smaller than A.*

Problem 1

Suppose $T \in \mathcal{L}(V, W)$. Show that $T = 0$ if and only if all singular values of T are 0.

Proof. $\Rightarrow$ if $T = 0$, then $T^*T = 0$ and the singular values have to be all 0.

$\Leftarrow$ if all singular values are 0, then $\dim \operatorname{range} T = 0$ and thus $T = 0$. $\square$

Problem 2

Suppose $T \in \mathcal{L}(V, W)$ and $s > 0$. Prove that s is a singular value of T if and only if there exists nonzero vectors $v \in V$ and $w \in W$ such that

$$Tv = sw \text{ and } T^*w = sv$$

Proof. $\Rightarrow$ This means that there exists nonzero eigenvector $v \in V$ such that

$$T^*Tv = s^2v$$

let $w = \frac{Tv}{s}$, then

$$Tv = s\left(\frac{Tv}{s}\right) = sw$$

and

$$T^*w = \frac{T^*Tv}{s} = sv$$

$\Leftarrow$ we have that

$$T^*Tv = T^*(sw) = s^2v$$

Therefore, completing the proof. $\square$

Problem 4

Suppose that $T \in \mathcal{L}(V, W)$, s_1 is the largest singular value of T, and s_n is the smallest value of T. Prove that

$$\{\|Tv\| : v \in V \text{ and } \|v\| = 1\} = [s_n, s_1]$$

Proof. Our proof goes two-fold. First we prove that for any $v \in V$ s.t. $\|v\| = 1$, we have

$$s_1 \leq \|Tv\| \leq s_n$$

Second, we will show that the $A \coloneqq \{\|Tv\| : v \in V \text{ and } \|v\| = 1\}$ is a closed interval.

To prove the first part, let v be a unit norm vector in V, we know that $v = \sum_{i=1}^n a_i e_i$ and $\|v\|^2 = \sum_{i=1}^n |a_i|^2$ for some orthonormal basis $e_1, \ldots, e_n$. Then we have

$$\begin{aligned}
\|Tv\|^2 &= \langle Tv, Tv \rangle \\
&= \langle v, T^*Tv \rangle \\
&= \left\langle \sum_{i=1}^n a_i e_i, \sum_{i=1}^n s_i^2 a_i e_i \right\rangle \\
&= \sum_{i=1}^n a_i \sum_{j=1}^n \overline{a_j} \overline{s_j^2} \langle e_i, e_j \rangle \\
&= \sum_{i=1}^n |a_i|^2 s_i^2
\end{aligned}$$

Note that the last identity is $\leq \|v\|^2 s_1^2$ and $\geq \|v\|^2 s_n^2$, which gets our desired equality. For the second part, we know that the norm function $\|\cdot\| : v \to Tv$ is continuous on the unit sphere $\{v \in V : \|v\| = 1\}$, which is compact. The continuous image of the compact set is therefore also compact. Hence, we prove that

$$A = [s_n, s_1]$$

$\square$

Problem 9

Suppose $T \in \mathcal{L}(V, W)$. Show that T and T^* have the same positive singular values.

Proof. We know that T^*T is self-adjoint, and that its eigenvalue equals the eigenvalue of its adjoint. Therefore the squared root of the eigenvalue of $T^*T = TT^*$, and thus T and T^* have the same positive singular values.

Problem 10 follows similarly. $\square$

Problem 11

Suppose that $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is an orthonormal basis of V. Let $s_1, \ldots, s_n$ denote the singular values of T.

(a) Prove that $\|Tv_1\|^2 + \cdots + \|Tv_n\|^2 = s_1^2 + \cdots + s_n^2$.

(b) Prove that if $W = V$ and T is a positive operator, then

$$\langle Tv_1, v_1 \rangle + \cdots + \langle Tv_n, v_n \rangle = s_1 + \cdots + s_n$$

Proof. We first know from the previous problems that for any two orthonormal basis, $e_1, \ldots, e_n$ in V and $f_1, \ldots, f_m$ in W, then

$$\sum_{i=1}^{n} \|Te_i\|^2 = \sum_{j=1}^{m} \left\|T^* f_j\right\|^2$$

(a) In the proof of SVD, we have shown that by letting $f_k = \frac{Te_k}{s_k}$, then we have $f_1, \ldots, f_n$ also to be an orthonormal list in W. Then applying the fact above yields that

$$\sum_{i=1}^{n} \|Tv_i\|^2 = \sum_{i=1}^{n} \|T^* f_i\|^2 = \sum_{i=1}^{n} \left\| \frac{T^*Te_i}{s_i} \right\|^2 = \sum_{i=1}^{n} \|s_i e_i\|^2 = \sum_{i=1}^{n} s_i^2$$

where $e_1, \ldots, e_n$ are the orthonormal eigenbasis that diagonalizes T^*T.

(b) Note that since T is positive, the singular value is eigenvalue, as $\lambda(T^*T) = \lambda(T^2)$ and the eigenvalue is positive. Therefore, for each i, $Tv_i = s_i v_i$ and we have that

$$\sum_{i=1}^{n} \langle Tv_i, v_i \rangle = \sum_{i=1}^{n} s_i$$

□

Problem 13

Suppose $T_1, T_2 \in \mathcal{L}(V)$. Prove that T_1 and T_2 have the same singular values if and only if there exist unitary operators $S_1, S_2 \in \mathcal{L}(V)$ such that $T_1 = S_1 T_2 S_2$.

Proof. $\Rightarrow$ Let A be the matrix of T_2 and B be the matrix of T_1 then we know that $A = U_1 D V_1^*$ and $B = U_2 D V_2^*$ for unitary matrix U_1, V_1, U_2, V_2 and diagonal matrix D. Then we have

$$B = \underbrace{(U_1 U_2^*)}_{S_1} \underbrace{(U_2 D V_2^*)}_{A} \underbrace{(V_2 V_1^*)}_{S_2}$$

where we have S_1, S_2 to be unitary.

$\Leftarrow$ Let A be the matrix of T_2 and let $A = UD^*$ be the SVD of A where U and V^* is unitary, then we have that (let B be the matrix of T_1)

$$B = (S_1 U) D (V^* S_2)$$

where $S_1 U$ and $V^* S_2$ are both unitary, therefore B (T_2) also have the singular values as in the diagonal of D. Therefore, T_1 and T_2 shares same eigenvalues. □

Problem 15

Suppose $T \in \mathcal{L}(V)$ and $s_1 \geq \cdots \geq s_n$ are the singular values of T. Prove that if λ is an eigenvalue of T, then $s_1 \geq |\lambda| \geq s_n$.

Proof. In the proof of Problem 4, we have shown that for every $v \in V$, we have

$$s_n\|v\| \leq \|Tv\| \leq s_1\|v\|$$

Since this holds for all vectors, substitute any eigenvector v we can get that

$$s_n\|v\| \leq |\lambda|\|v\| \leq s_1\|v\|$$

which is the desired result. □

7F: Consequences of Singular Value Decomposition

Theorem 57 (upper bound for $\|Tv\|$)**.** *Suppose $T \in \mathcal{L}(V, W)$. Let s_1 be the largest singular value of T. Then*

$$\|Tv\| \leq s_1 \|v\|$$

for all $v \in V$.

Definition 58 (norm of a linear map, $\|\cdot\|$)**.** *Suppose $T \in \mathcal{L}(V, W)$. Then the **norm** of T, denoted by $\|T\|$, is defined by*

$$\|T\| = \max\{\|Tv\| : v \in V \text{ and } \|v\| \leq 1\}$$

Remark 59. *For a linear map T, $\|T\| = \sigma_{\max}$ (using the more common notation). Also note that $\|T\| \neq \sqrt{\langle T, T \rangle}$. Now we have two different uses of the word **norm** and the notation $\|\cdot\|$.*

Corollary 60 (basic properties of norms of linear maps)**.** *Suppose $T \in \mathcal{L}(V, W)$. Then*

(a) $\|T\| \geq 0$*;*

(b) $\|T\| = 0 \iff T = 0$*;*

(c) $\|\lambda T\| = |\lambda| \|T\|$ *for all* $\lambda \in \mathbb{F}$*;*

(d) $\|S + T\| \leq \|S\| + \|T\|$ *for all* $S \in \mathcal{L}(V, W)$*.*

Remark 61. *For $S, T \in \mathcal{L}(V, W)$, the quantity $\|S - T\|$ is often called the distance between S and T. Informally, think of the condition that $\|S - T\|$ is a small number as meaning S and T are close together.*

Theorem 62 (alternative formulas for $\|T\|$)**.** *Suppose $T \in \mathcal{L}(V, W)$. Then*

(a) $\|T\|$ *= the largest eigenvalue of* T*;*

(b) $\|T\| = \max\{\|Tv\| : v \in V \text{ and } \|v\| = 1\}$*;*

(c) $\|T\| =$ *the smallest number* c *such that* $\|Tv\| \leq c\|v\|$ *for all* $v \in V$*.*

Remark 63. *An important inequality during the proof:*

$$\|Tv\| \leq \|T\| \|v\|$$

for all $v \in V$ and $v \neq 0$.

Corollary 64 (norm of the adjoint)**.** *Suppose $T \in \mathcal{L}(V, W)$. Then $\|T\| = \|T^*\|$.*

Theorem 65 (best approximation by linear map whose range has dimension $\leq k$). *Suppose $T \in \mathcal{L}(V, W)$ and $s_1 \geq \cdots \geq s_m$ are the singular values of T. Suppose $1 \leq k < m$. Then*

$$\min\{\|T - S\| : S \in \mathcal{L}(V, W) \text{ and } \dim \operatorname{range} S \leq k\} = s_{k+1}$$

Furthermore, if

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_m\langle v, e_m\rangle f_m$$

is a singular value decomposition of T and $T_k \in \mathcal{L}(V, W)$ is defined by

$$T_k v = s_1\langle v, e_1\rangle f_1 + \cdots + s_k\langle v, e_k\rangle f_k$$

for each $v \in V$, then $\dim \operatorname{range} T_k = k$ and $\|T - T_k\| = s_{k+1}$.

Theorem 66 (polar decomposition). *Suppose $T \in \mathcal{L}(V)$. Then there exists a unitary operator $S \in \mathcal{L}(V)$ such that*

$$T = S\sqrt{T^*T}$$

Remark 67. *This holds for both $\mathbb{C}$ and $\mathbb{R}$.*

Definition 68 (ball, B). *The **ball** in V of radius 1 centered at 0, is defined by*

$$B = \{v \in V : \|v\| \leq 1\}$$

Definition 69 (ellipsoid, $E(s_1 f_1, \ldots, s_n f_n)$, principle axes). *Suppose that $f_1, \ldots, f_n$ is an orthonormal basis of V and $s_1, \ldots, s_n$ are positive numbers. The **ellipsoid** $E(s_1 f_1, \ldots, s_n f_n)$ with **principle axes** $s_1 f_1, \ldots, s_n f_n$ is defined by*

$$E(s_1 f_1, \ldots, s_n f_n) = \{v \in V : \frac{|\langle v, f_1\rangle|^2}{s_1} + \cdots + \frac{|\langle v, f_n\rangle|^2}{s_n} < 1\}$$

Remark 70. *If $\dim V = 2$, the word "disk" is sometimes used to denote ball and the word "ellipse" is sometimes used to denote ellipsoid.*

Definition 71 ($T(\Omega)$). *For a function T defined on V and $\Omega \subseteq V$, define $T(\Omega)$ by*

$$T(\Omega) = \{Tv : v \in \Omega\}$$

Proposition 72 (invertible map takes ball to ellipsoid). *Suppose $T \in \mathcal{L}(V)$ is invertible. Then T maps the ball B in V onto an ellipsoid in V.*

Proposition 73 (invertible map takes ellipsoid to ellipsoid). *Suppose $T \in \mathcal{L}(V)$ is invertible and E is an ellipsoid in V. Then $T(E)$ is an ellipsoid in V.*

Definition 74 ($P(v_1, \ldots, v_n)$, parallelepiped). *Suppose $v_1, \ldots, v_n$ is a basis of V. Let*

$$P(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots + a_n v_n : a_1, \ldots, a_n \in (0, 1)\}$$

*A **parallelepiped** is a set of the form $u + P(v_1, \ldots, v_n)$ for some $u \in V$. The vectors $v_1, \ldots, v_n$ are called the **edges** of the parallelepiped.*

Proposition 75 (invertible operator takes parallelepiped to parallelepiped)**.** *Suppose $u \in V$ and $v_1, \ldots, v_n$ is a basis of V. Suppose $T \in \mathcal{L}(V)$ is invertible. Then*

$$T(u + P(v_1, \ldots, v_n)) = Tu + P(Tv_1 \ldots, Tv_n)$$

Definition 76 (box)**.** *A **box** is of the form*

$$u + P(r_1 e_1, \ldots, r_n e_n)$$

where $u \in V$, $r_1, \ldots, r_n$ are positive numbers and $e_1, \ldots, e_n$ is an orthonormal basis of V.

Rest of Notes are omitted.

Problem 1

Prove that if $S, T \in \mathcal{L}(V, W)$, then $|\|S\| - \|T\|| \leq \|S - T\|$.

Proof. We have

$$\begin{aligned}\|S\| &= \|S - T + T\| \leq \|S - T\| + \|T\| \\ \|T\| &= \|T - S + S\| \leq \|T - S\| + \|S\|\end{aligned}$$

This means that

$$\|S - T\| \geq |\|S\| - \|T\||$$

□

Problem 2

Suppose that $T \in \mathcal{L}(V)$ is self-adjoint and or that $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal, then

$$\|T\| = \{\max |\lambda| : \lambda \text{ is an eigenvalue of } T\}$$

Proof. Given by the conditions we know that the eigenvalues of T are its singular values, therefore $\|T\| = s_1 = \max(\lambda)$ □

Problem 3

Suppose that $T \in \mathcal{L}(V)$ and $v \in V$. Prove that

$$\|Tv\| = \|T\|\|v\| \iff T^*Tv = \|T\|^2 v$$

Proof. $\Rightarrow$ We have that $\|Tv\| = \left\|Tv/\|v\|\right\|\|v\| = \|T\|\|v\|$. Therefore $\left\|Tv/\|v\|\right\| = \|T\|$ which gives that $v/\|v\|$ being the vector corresponds to the largest singular value of T. This gives that

$$T^*Tv/\|v\| = \left\|T^2\right\| v/\|v\|$$

Multiplying $\|v\|$ on both sides solves the problem.

$\Leftarrow$ Given $T^*Tv = \|T\|^2 v$, then we know that v is the eigenvector of T^*T corresponding to its largest eigenvalue s_1^2, (i.e. it is the vector that corresponds the largest singular value of T). Therefore, we have $\|Tv\| = \left\|Tv/\|v\|\right\|\|v\| = \|T\|\|v\|$. □

Problem 4

Suppose $T \in \mathcal{L}(V, W), v \in V$, and $\|Tv\| = \|T\|\|v\|$. Prove that if $u \in V$ and $\langle u, v\rangle = 0$, then $\langle Tu, Tv\rangle = 0$.

Proof. By P3 we know that $T^*Tv = \|T\|^2 v$, then

$$\langle Tu, Tv \rangle = \langle T^*Tu, v \rangle = \|T\|^2 \langle u, v \rangle = 0$$

$\square$

Problem 5

Suppose U is a finite-dimensional inner product space, $T \in \mathcal{L}(V, U)$, and $S \in \mathcal{L}(U, W)$. Prove that

$$\|ST\| \leq \|S\| \|T\|$$

Proof. Take $v \in V$ with $\|v\| \leq 1$ and $\|ST\| = \|STv\|$, then

$$\|ST\| = \left\|(ST)v\right\| = \left\|S(Tv)\right\| \leq \|S\| \|Tv\| \leq \|S\| \|T\| \|v\| \leq \|S\| \|T\|$$

$\square$

Problem 7

Show that defining $d(S, T) = \|S - T\|$ for $S, T \in \mathcal{L}(V, W)$ makes d a metric on $\mathcal{L}(V, W)$.

Proof. We will examine the definitions one by one:

- $d(S, S) = \|S - S\| = 0$.
- If $S \neq T$, then $d(S, T) = \|S - T\| = \sigma_{\max}(S - T)$. Since $S - T \neq 0$, its largest singular value is nonzero and therefore $d(S, T) > 0$.
- $d(S, T) = \|S - T\| = \|T - S\| = d(T, S)$.
- $d(S, G) = \|S - G\| = \left\|(S - T) + (T - G)\right\| \leq \|S - T\| + \|T - G\| = d(S, T) + d(T, G)$.

$\square$

Problem 8

(a) Prove that if $T \in \mathcal{L}(V)$ and $\|I - T\| < 1$, then T is invertible.

(b) Suppose that $S \in \mathcal{L}(V)$ is invertible. Prove that if $T \in \mathcal{L}(V)$ and $\|S - T\| < 1/\left\|S^{-1}\right\|$, then T is invertible.

Proof. (a) Let $S = I - T$ and $S_n = \sum_{i=1}^{n} S^i$. Then

$$(I - S)S_n = I - S^{n+1} \to I$$

as $n \to \infty$ because $\|S\| < 1$. Therefore, the operator

$$S_\infty = \sum_{i=1}^{\infty} S_i$$

is also bounded. This gives that

$$(I - S)S_\infty = S_\infty(I - S) = I$$

and thus S is invertible, which leads to that T is also invertible.

(b) Equivalently,

$$\left\|I - TS^{-1}\right\| \leq \|S - T\|\left\|S^{-1}\right\| < 1$$

So TS^{-1} is invertible. Since we already know that S is invertible, T is therefore also invertbile. □

Problem 9

Suppose $T \in \mathcal{L}(V)$. Prove that for every $\epsilon > 0$ there exists an invertible operator $S \in \mathcal{L}(V)$ such that $0 < \|T - S\| < \epsilon$.

Proof. Define $S = T + \delta I$ for some $\epsilon > \delta > 0$. Then we have

$$\|T - S\| = \|\delta I\| = \delta$$

which satisifes the desired condition. Note that if T is invertible, we can simply choose a sufficiently small $\delta < 1/\left\|T^{-1}\right\|$; if not, then any δ in $(0, 1)$ can make S invertible. □

Problem 12

Suppose $T \in \mathcal{L}(V)$ is a positive operator. Show that $\left\|\sqrt{T}\right\| = \sqrt{\|T\|}$.

Proof. Let $\|T\| = \sigma_1$, then $\left\|\sqrt{T}\right\| = \sqrt{\sigma_1} = \sqrt{\|T\|}$. □

Problem 17

Prove that if $u \in V$ and φ_u is the linear functional on V defined by the equation $\varphi_u(v) = \langle v, u\rangle$, then $\|\varphi_u\| = \|u\|$.

Proof. We have that

$$\|\varphi_u\| = \sup_{\|v\|=1} |\langle v, u\rangle| = \left|\left\langle \frac{u}{\|u\|}, u \right\rangle\right| = \|u\|$$

□

Problem 18

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $T \in \mathcal{L}(V, W)$.

(a) Prove that $\max\{\|Te_1\|, \ldots, \|Te_n\|\} \le \|T\| \le (\|Te_1\|^2 + \cdots + \|Te_n\|^2)^{1/2}$.

(b) Prove that $\|T\| = (\|Te_1\|^2 + \cdots + \|Te_n\|^2)^{1/2}$ if and only if $\dim \operatorname{range} T \le 1$.

Proof. (a) We first know that $\|Te_i\| \le \|T\|$ by definition of $\|\cdot\|$. In the proof of equivalent characterizations, we showed that there exists some orthonormal basis (corresponding to SVD) such that $\|T(f_n)\| = s_1$ for some f_n. This finishes the l.h.s. To see the r.h.s., let $v \in V$ with $\|v\| = 1$ to be such that $\|Tv\| = \|T\|$. Then we have

$$\|T\|^2 = \|Tv\|^2 = \left\|\sum_{i=1}^n a_i T e_i\right\|^2 \le \left(\sum_{i=1}^n |a_i|^2\right)\left(\sum_{i=1}^n \|Te_i\|^2\right) = \sum_{i=1}^n \|Te_i\|^2$$

Taking the square root solves the problem.

(b) $\Rightarrow$ By the Cauchy-Schwarz inequality, we know that this is an equality if and only if Te_i must be proportional to each other. That is, there exists a unit vector w such that $Te_i = a_i w$. So $\dim \operatorname{range} T \le 1$.

$\Leftarrow$ If $\dim \operatorname{range} T \le 1$, then there exists a unit vector w and $\beta_i \ge 0$ s.t.

$$Te_i = \beta_i w$$

We now have that for some v with unit norm,

$$\|Tv\| = \left\|\sum_{i=1}^n a_i T e_i\right\| = \left\|\sum_{i=1}^n a_i \beta_i w\right\| \le \left|\sum_{i=1}^n a_i \beta_i\right|$$

We know that

$$\left|\sum_{i=1}^n a_i \beta_i\right| \le \left(\sum_{i=1}^n a_i^2\right)^{1/2} \left(\sum_{i=1}^n \beta_i^2\right)^{1/2} = \left(\sum_{i=1}^n \beta_i^2\right)^{1/2}$$

with equality obtained when $\beta_i = ca_i$ for all i. That is, the sup of $\|Tv\|$ occurs under this condition and that we have

$$\|T\| = \left(\sum_{i=1}^{n} \beta_i^2\right)^{1/2} = \left(\sum_{i=1}^{n} \|Te_i\|^2\right)^{1/2}$$

□

Problem 24

Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

$$\left\|T^{-1}\right\| = \|T\|^{-1} \iff \frac{T}{\|T\|} \text{ is a unitary operator}$$

Proof. $\Rightarrow$ Let $U = \frac{T}{\|T\|}$, then $\|U\| = 1$ and $\left\|U^{-1}\right\| = 1$. Then

$$\|Ux\| \leq \|U\|\|x\| = \|x\|$$

and

$$\|x\| = \left\|U^{-1}Ux\right\| \leq \left\|U^{-1}\right\|\|Ux\| = \|Ux\|$$

Therefore, we have $\|Ux\| = \|x\|$ and therefore it is an invertible isometry and thus unitary.

$\Leftarrow$ Conversely, we have $\|U\| = 1 = \left\|U^{-1}\right\|$ by it being unitary. Therefore, we have $\left\|T^{-1}\right\| = \|T\|^{-1}$ □