Exercises A

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Exercise 1. *Solution.* **(a)** Let $a, b, c \in \mathbb{F}$ where b, c are additive inverses of a. Then

$$b = b + 0 = b + (a + c) = (a + b) + c = 0 + c = c.$$

That's why the cancellation property holds:

$$x + y = x + z \implies -x + x + y = -x + x + z \implies y = z.$$

(b) We have

$$\begin{gathered} a + (-a) = 0 \\ -(-a) + (-a) = 0. \end{gathered}$$

Earlier we showed that the additive inverse is unique, therefore $a = -(-a)$.

(c) We have

$$(-1)a + a = (-1)a + 1a = (-1 + 1)a = 0a = 0.$$

By **(a)** we get $(-1)a = -a$. $\square$

Exercise 2. *Solution.* **(a)** Let $a, b, c \in \mathbb{F}$, where b, c are multiplicative inverses of a. Then we have

$$b = 1b = (ac)b = (ab)c = 1c = c.$$

(b) We have

$$\begin{gathered} a \cdot a^{-1} = 1 \\ (a^{-1})^{-1} \cdot a^{-1} = 1. \end{gathered}$$

By uniqueness of an inverse we obtain $a = (a^{-1})^{-1}$. $\square$

Exercise 3. *Solution.* First, note that $(-1)(-1) = -(-1) = 1$. Then we have $(-a)(-b) = (-1)(-1)ab = ab$. $\square$

Exercise 13. *Solution.* Suppose $i \in \mathbb{F}$ s.t. $i^2 = -1$. Certainly, $i \neq 0$, since $0^2 = 0 \neq -1$. Therefore, by properties of an ordered field, either $i \in P$ or $i \notin P$. If $i \in P$, then $i^2 \in P$, and we reach contradiction since $-1 \notin P$. Now suppose $i \notin P$ and consider $i^2 \cdot i^2 = (-1)(-1) = 1 \in P$, and we again reach contradiction. Thus, such element doesn't exist. $\square$