1B

Gennady Laptev

Exercise 1. *Solution.* Clearly, $L(f, P, [0,1]) = 0$ for any partition P, since any interval contains an irrational number, so $\inf_{[a,b]} f = 0$, where $[a,b] \subset [0,1]$.

Let q be a prime number. Then consider a partition $P_q = [\frac{j-1}{q}, \frac{j}{q}]_{j=0}^{q}$. We have $\sup_{[\frac{j-1}{q}, \frac{j}{q}]} f = \frac{1}{q}$ and $U(f, P_q, [0,1]) = q \cdot \frac{1}{q^2} = \frac{1}{q}$, since we have q segments in partition of length $\frac{1}{q}$. Then by Archimedean property for any $\varepsilon > 0$ we can find some n s.t. $\frac{1}{n} < \varepsilon$, and we can always find a prime number $q \geq n$, since subset of prime numbers is not bounded in the set of natural numbers. Thus for every $\varepsilon > 0$ we can find a partition P_q s.t.

$$U(f, P_q, [0,1]) - L(f, P_q, [0,1]) < \frac{1}{q} < \varepsilon.$$

Then by problem 1A.3 we have that f is Riemann integrable and its integral is 0, since $L(f, [0,1]) = 0$. $\square$

Exercise 2. *Solution.* ($\Longrightarrow$) We have

$$\begin{gathered} L(-f, [a,b]) = \sup L(-f, P, [a,b]) = \sup -L(f, P, [a,b]) = \\ \inf L(f, P, [a,b]) = \inf(U(f, P, [a,b]) + \varepsilon) = U(f, [a,b]) \end{gathered}$$

$\square$