

# 1B

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**Exercise 1. Solution.** Clearly,  $L(f, P, [0, 1]) = 0$  for any partition  $P$ , since any interval contains an irrational number, so  $\inf_{[a,b]} f = 0$ , where  $[a, b] \subset [0, 1]$ .

Let  $q$  be a prime number. Then consider a partition  $P_q = [\frac{j-1}{q}, \frac{j}{q}]_{j=0}^q$ . We have  $\sup_{[\frac{j-1}{q}, \frac{j}{q}]} f = \frac{1}{q}$  and  $U(f, P_q, [0, 1]) = q \cdot \frac{1}{q^2} = \frac{1}{q}$ , since we have  $q$  segments in partition of length  $\frac{1}{q}$ . Then by Archimedean property for any  $\varepsilon > 0$  we can find some  $n$  s.t.  $\frac{1}{n} < \varepsilon$ , and we can always find a prime number  $q \geq n$ , since subset of prime numbers is not bounded in the set of natural numbers. Thus for every  $\varepsilon > 0$  we can find a partition  $P_q$  s.t.

$$U(f, P_q, [0, 1]) - L(f, P_q, [0, 1]) < \frac{1}{q} < \varepsilon.$$

Then by problem 1A.3 we have that  $f$  is Riemann integrable and its integral is 0, since  $L(f, [0, 1]) = 0$ . □