# 2C

Gennady Laptev

**Exercise 1.** Explain why there does not exist a measure space $(X, S, \mu)$ with the property $\{\mu(E) \mid E \in S\} = [0, 1)$.

*Solution.* Suppose it is true. Let's denote $M = \{\mu(E) \mid E \in S\}$.

Since $S$ is a $\sigma$-algebra on $X$, then $X \in S$. Suppose $\mu(X) = x$ and, by hypothesis, $x < 1$. Since measures preserve order, then for any $A \in S$ holds $\mu(A) \leq \mu(X)$. Assume $x > 0$, otherwise $M = \{0\}$. Then $x = \sup M < 1$. Therefore, for any $\varepsilon > 0$ there exists $y \in M$ s.t. $y + \varepsilon > x$. Taking $\varepsilon = \frac{1-x}{2}$, we get

$$y + \varepsilon = y + \frac{1-x}{2} \leq x + \frac{1-x}{2} < x + 1 - x = 1.$$

So there exist $z = y + \varepsilon$ s.t. $z \notin M$ and $z \in [0, 1)$. Thus, $M \neq [0, 1)$. $\square$

**Exercise 10.** Give an example of a measure space $(X, S, \mu)$ and a decreasing sequence $E_1 \supseteq E_2 \supseteq \ldots$ of sets in S such that

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k\to\infty} \mu(E_k).$$

*Solution.* Consider $\mathbb{R}$ with a counting measure $\mu$ which is defined on each $E \subset \mathbb{R}$ as

$$\mu(E) = \begin{cases} n, & \text{if } E \text{ is finite and has } n \text{ elements,} \\ \infty, & \text{otherwise.} \end{cases}$$

Choose $E_k = (k, \infty)$.

Clearly $\bigcap_{k=1}^{\infty} E_k = \emptyset$ (Suppose it is not true. Then there exists a real number $x \in \bigcap_{k=1}^{\infty} E_k$, that is, $x \in (k, \infty)$ for every $k \in \mathbb{N}$. By Archimedean property, there exists a natural $n$ s.t. $n > x$, so $x \notin (n, \infty)$, which leads to contradiction). Thus, $\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = 0$. But $\mu(E_k) = \infty$ for every $k$, therefore $\lim_{k\to\infty} \mu(E_k) = \infty$ which gives the desired inequality. $\square$