

2C

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Exercise 10. Give an example of a measure space (X, S, μ) and a decreasing sequence $E_1 \supseteq E_2 \supseteq \dots$ of sets in S such that

$$\mu \left(\bigcap_{k=1}^{\infty} E_k \right) \neq \lim_{k \rightarrow \infty} \mu(E_k).$$

Solution. Consider \mathbb{R} with a counting measure μ which is defined on each $E \subset \mathbb{R}$ as

$$\mu(E) = \begin{cases} n, & \text{if } E \text{ is finite and has } n \text{ elements,} \\ \infty, & \text{otherwise.} \end{cases}$$

Choose $E_k = (k, \infty)$.

Clearly $\bigcap_{k=1}^{\infty} E_k = \emptyset$ (Suppose it is not true. Then there exists a real number $x \in \bigcap_{k=1}^{\infty} E_k$, that is, $x \in (k, \infty)$ for every $k \in \mathbb{N}$. By Archimedean property, there exists a natural n s.t. $n > x$, so $x \notin (n, \infty)$, which leads to contradiction). Thus, $\mu \left(\bigcap_{k=1}^{\infty} E_k \right) = 0$. But $\mu(E_k) = \infty$ for every k , therefore $\lim_{k \rightarrow \infty} \mu(E_k) = \infty$ which gives the desired inequality. \square