2A

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Exercise 1. gg

Solution. Consider $I_1, \ldots$ such that $B \subset \cup_{k=1}^{\infty} I_k$. and $J_k, \ldots$ s.t. $A \subset \cup_{k=1}^{\infty} J_k$. We can merge both sets of intervals into $L_1, L_2, \ldots$ with the following enumeration

$$L_i = \begin{cases} J_{(i+1)/2}, & i \text{ is odd,} \\ I_{i/2}, & i \text{ is even.} \end{cases}$$

Clearly, $A \cup B \subset \cup_{k=1}^{\infty} L_k = (\cup_{k=1}^{\infty} I_k) \cup (\cup_{k=1}^{\infty} J_k)$.

Then we have

$$|A \cup B| \leq \left(\sum_{k=1}^{\infty} l(L_k) \right) = \left(\sum_{k=1}^{\infty} l(I_k) + l(J_k) \right) = \left(\sum_{k=1}^{\infty} l(I_k) \right) + \left(\sum_{k=1}^{\infty} l(J_k) \right)$$

By taking the infimum on the right side and using the fact that the infimum of the sum of sets is the sum of infimums we get

$$|A \cup B| \leq 0 + |A| = |A|.$$

Another inequality $|A| \leq |A \cup B|$ holds because outer measure preserves order. Thus, $|A| = |A \cup B|$. $\square$

Exercise 2. *Solution.* If $t = 0$, then $tA = \{0\}$, therefore $|tA| = 0$, since it is finite. Also, $|t||A| = 0$ (even if $|A| = \infty$, by convention).

Now assume that $t \neq 0$. Let $I_1, \ldots$ s.t. $A \subset \cup_{k=1}^{\infty} I_k$. Then $tA \subset \cup_{k=1}^{\infty} tI_k$. Easy to see, that $\ell(t \cdot I_k) = |t| \cdot \ell(I_k)$. So $|tA| \leq \sum_{k=1}^{\infty} \ell(tI_k) = |t| \sum_{k=1}^{\infty} \ell(I_k)$. By taking infimum, we obtain $|tA| \leq |t||A|$.

The opposite direction

$$|A| = |\frac{1}{t}(tA)| \leq \frac{1}{|t|}|tA|,$$

where we just used the first inequality with t replaced with $\frac{1}{t}$ and A replaced with tA. Finally, we get $|t||A| \leq |tA|$. $\square$