2C

Gennady Laptev

Exercise 1. Explain why there does not exist a measure space (X, S, μ) with the property $\{\mu(E) \mid E \in S\} = [0, 1)$.

Solution. Suppose it is true. Let's denote $M = \{\mu(E) \mid E \in S\}$.

Since S is a σ-algebra on X, then $X \in S$. Suppose $\mu(X) = x$ and, by hypothesis, $x < 1$. Since measures preserve order, then for any $A \in S$ holds $\mu(A) \leq \mu(X)$. Assume $x > 0$, otherwise $M = \{0\}$. Then $x = \sup M < 1$. Therefore, for any $\varepsilon > 0$ there exists $y \in M$ s.t. $y + \varepsilon > x$. Taking $\varepsilon = \frac{1-x}{2}$, we get

$$y + \varepsilon = y + \frac{1-x}{2} \leq x + \frac{1-x}{2} < x + 1 - x = 1.$$

So there exist $z = y + \varepsilon$ s.t. $z \notin M$ and $z \in [0, 1)$. Thus, $M \neq [0, 1)$. $\square$

Exercise 4. Give an example of a measure space (X, S, μ) such that

$$\{\mu(E) \mid E \in S\} = \{\infty\} \cup \bigcup_{k=1}^{\infty} [3k, 3k+1]$$

Solution. $\square$

Exercise 6. Find all $c \in [3, \infty)$ s.t. there exists a measure space (X, S, μ) with

$$\{\mu(E) \mid E \in S\} = [0, 1] \cup [3, c]$$

Solution. Since measures preserve order, then we have $\mu(X) = c$. Suppose $c < 4$. Then by hypothesis there exists $E \subset X$ s.t $\mu(E) = 1$, therefore $\mu(X - E) = \mu(X) - \mu(E) = c - 1 < 3$, contradiction. So, $c \geq 4$.

Assume $c = 4$. Then there exists $E \subset X$ s.t $\mu(E) = 3 + a, a \in [\frac{1}{2}, 1)$, and there exists $F \subset X$ s.t. $\mu(F) = b > a, b \in (\frac{1}{2}, 1]$. If $F \subset E$, then $\mu(E - F) = 3 + a - b < 3$, contradiction, therefore, $F \cap E = \emptyset$. Thus, $\mu(E \cup F) = 3 + a + b > 4$, so $c > 4$.

Now assume $c = k \geq 5 \in \mathbb{N}$. Then there exist $E, F \subset X$ s.t. $\mu(E) = k - \varepsilon, \varepsilon \in (0, 1)$, $\mu(F) = k - 2$. Assume $F \subset E$, then we get $\mu(E - F) = k - \varepsilon - k + 2 = 2 - \varepsilon$, contradiction. So, $E \cap F = \emptyset$, therefore $\mu(E \cup F) = k - \varepsilon + k - 2 = 2k - 2 - \varepsilon > k + 1$, since $k > 2$. We get that c is greater than any natural number, and by Archimedean property, greater than any real number. Thus, $c = \infty$. $\square$

Exercise 10. Give an example of a measure space (X, S, μ) and a decreasing sequence $E_1 \supseteq E_2 \supseteq \ldots$ of sets in S such that

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k \to \infty} \mu(E_k).$$

Solution. Consider $\mathbb{R}$ with a counting measure μ which is defined on each $E \subset \mathbb{R}$ as

$$\mu(E) = \begin{cases} n, & \text{if } E \text{ is finite and has } n \text{ elements,} \\ \infty, & \text{otherwise.} \end{cases}$$

Choose $E_k = (k, \infty)$.

Clearly $\bigcap_{k=1}^{\infty} E_k = \emptyset$ (Suppose it is not true. Then there exists a real number $x \in \bigcap_{k=1}^{\infty} E_k$, that is, $x \in (k, \infty)$ for every $k \in \mathbb{N}$. By Archimedean property, there exists a natural n s.t. $n > x$, so $x \notin (n, \infty)$, which leads to contradiction). Thus, $\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = 0$. But $\mu(E_k) = \infty$ for every k, therefore $\lim_{k \to \infty} \mu(E_k) = \infty$ which gives the desired inequality. $\square$