$$a \to b \to c$$

$$p(a,b,c) = p(c|b)\,p(b|a)\,p(a)$$

Does $p(b|a,c) \overset{?}{\Rightarrow} p(b|c)$?

$$p(b|a,c) = \frac{p(a,b,c)}{p(a,c)} = \frac{p(c|b)\,p(b|a)\,p(a)}{\sum_b p(c|b)\,p(b|a)\,\underbrace{p(a)}_{\text{can pull out of sum}}}$$

$$\Rightarrow \frac{p(c|b)\,p(b|a)\,\cancel{p(a)}}{\cancel{p(a)}\sum_b p(c|b)\,p(b|a)}$$

(Note: we can't just divide by num., b.c. b in num. is fixed by LHS)

$$\neq$$

$$\sum_a p(a) = 1$$

$$p(b|c) = \frac{p(b,c)}{p(c)} = \frac{\sum_a p(a,b,c)}{\sum_{a,b} p(a,b,c)} = \frac{\sum_a p(c|b)\,p(b|c)\,p(a)}{\sum_{a,b} p(c|b)\,p(b|c)\,p(a)}$$

$$= \frac{p(c|b)\,p(b|c)}{\sum_{a,b} p(c|b)\,p(b|c)\,p(a)}$$

Graphs represent independence structure / assumptions

Consider two fully connected DAGs:

("cascade graph")

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left=50] & b \arrow[r] \arrow[rr, bend left] & c \arrow[r] & d \end{tikzcd}$$

$$p(a,b,c,d) = p(d \mid a,b,c)\, p(c \mid a,b)\, p(b \mid a)$$

$$\begin{tikzcd} c \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left=50] & d \arrow[r] \arrow[rr, bend left] & a \arrow[r] & b \end{tikzcd}$$

$$p(a,b,c,d) = p(b \mid c,d,a)\, p(a \mid c,d)\, p(b \mid a)$$

Any joint distr. can be factored like this by repeatedly applying $p(a,b) = p(a \mid b)\, p(b)$.

Contrast this with:

$$a \to b \to c \to d$$

$$p(a,b,c,d) = p(d \mid c)\, p(c \mid b)\, p(b \mid a)$$

The DAG is a statement of the cond. indep. (or assumptions)

- Any joint distributed can be factored in several ways:

$$p(x_1, x_2, x_3) = p(x_{i_1} \mid x_{i_2}, x_{i_3})\, p(x_{i_2} \mid x_{i_3})\, p(x_{i_3})$$

Each represents $p(x_1, x_2, x_3)$ with no independ. assumptions.

(i_1, i_2, i_3) is any of 6 permutations of $(1,2,3)$

If we drop one connection (e.g. between x_1 & x_2), we get:

a) $\begin{tikzcd} x_1 \arrow[dr] & & x_2 \arrow[dl] \\ & x_3 & \end{tikzcd}$ b) $\begin{tikzcd} x_1 & & x_2 \\ & x_3 \arrow[ul] \arrow[ur] & \end{tikzcd}$ c) $\begin{tikzcd} x_1 \arrow[dr] & & x_2 \\ & x_3 \arrow[ur] & \end{tikzcd}$ d) $\begin{tikzcd} x_1 & & x_2 \arrow[dl] \\ & x_3 \arrow[ul] & \end{tikzcd}$

Do any of these represent the same distr.?
That is: Are any of them equivalent?

graph c)

$$p(x_2|x_3)\underbrace{p(x_3|x_1)p(x_1)}_{p(x_3,x_1)} = \frac{p(x_2,x_3)}{p(x_3)} \cdot p(x_3,x_1) = p(x_1|x_3)p(x_2,x_3)$$

$$= \underbrace{p(x_1|x_3)p(x_3|x_2)p(x_2)}_{\text{graph d)}} = \underbrace{p(x_1|x_3)p(x_2|x_3)p(x_3)}_{\text{graph b)}}$$

Graphs b, c, and d represent same CI assumptions:

$$x_1 \perp\!\!\!\perp x_2 \mid x_3$$

Graph a) is different. $p(x_3|x_1,x_2)p(x_1)p(x_2)$ can't be transformed into the others.

Keep in mind: $a \rightarrow b$ says b is dep on a) in general but we could have (define) $p(b|a) = p(a) \Rightarrow a \perp\!\!\!\perp b$

Barber fig 3.5

$$\begin{tikzcd} a \arrow[rr, bend left] \arrow[rrr, bend left] & b \arrow[r] & c & d \end{tikzcd} \qquad p(a,b,c,d) = p(d|a)\,p(c|a,b)\,p(b)\,p(a)$$

$c \perp\!\!\!\perp d \mid a$?

Have to show: $p(c,d|a) = \overset{2}{p(c|a)}\,\overset{1}{p(d|a)}$

$$p(c,d|a) = \frac{1}{p(a)} \sum_b p(d|a)\,p(c|a,b)\,p(b)\,p(a)$$

$$= \frac{\overset{1}{p(d|a)}\,\cancel{p(a)}}{\cancel{p(a)}} \sum_b p(c|a,b)\,p(b)$$

$$p(c|a) = \frac{1}{p(a)} \sum_{b,d} p(d|a)\,p(c|a,b)\,p(b)\,p(a)$$

(rearranging)

$$= \frac{\cancel{p(a)}}{\cancel{p(a)}} \overbrace{\sum_b p(c|a,b)\,p(b)}^{2} \overbrace{\sum_d p(d|a)}^{=1}$$

$$\Rightarrow \quad p(c,d|a) = p(c|a)\,p(d|a)$$