EECS 491
Probabilistic Graphical Models

Reasoning with Continuous Variables

Recall Inference with Binary Variables

- Probability: *precise representation of uncertainty*
- Probability theory: *optimal updating of knowledge based on new information*

- Bayesian Inference with Boolean variables

likelihood *prior*

posterior
$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\bar{D})P(\bar{D})}$$

normalizing constant

- Inferences combines sources of knowledge

$$P(D|T) = \frac{0.9 \times 0.001}{0.9 \times 0.001 + 0.1 \times 0.999} = 0.0089$$

- Inference is sequential

$$P(D|T_1, T_2) = \frac{P(T_2|D)P(T_1|D)P(D)}{P(T_2)P(T_1)}$$

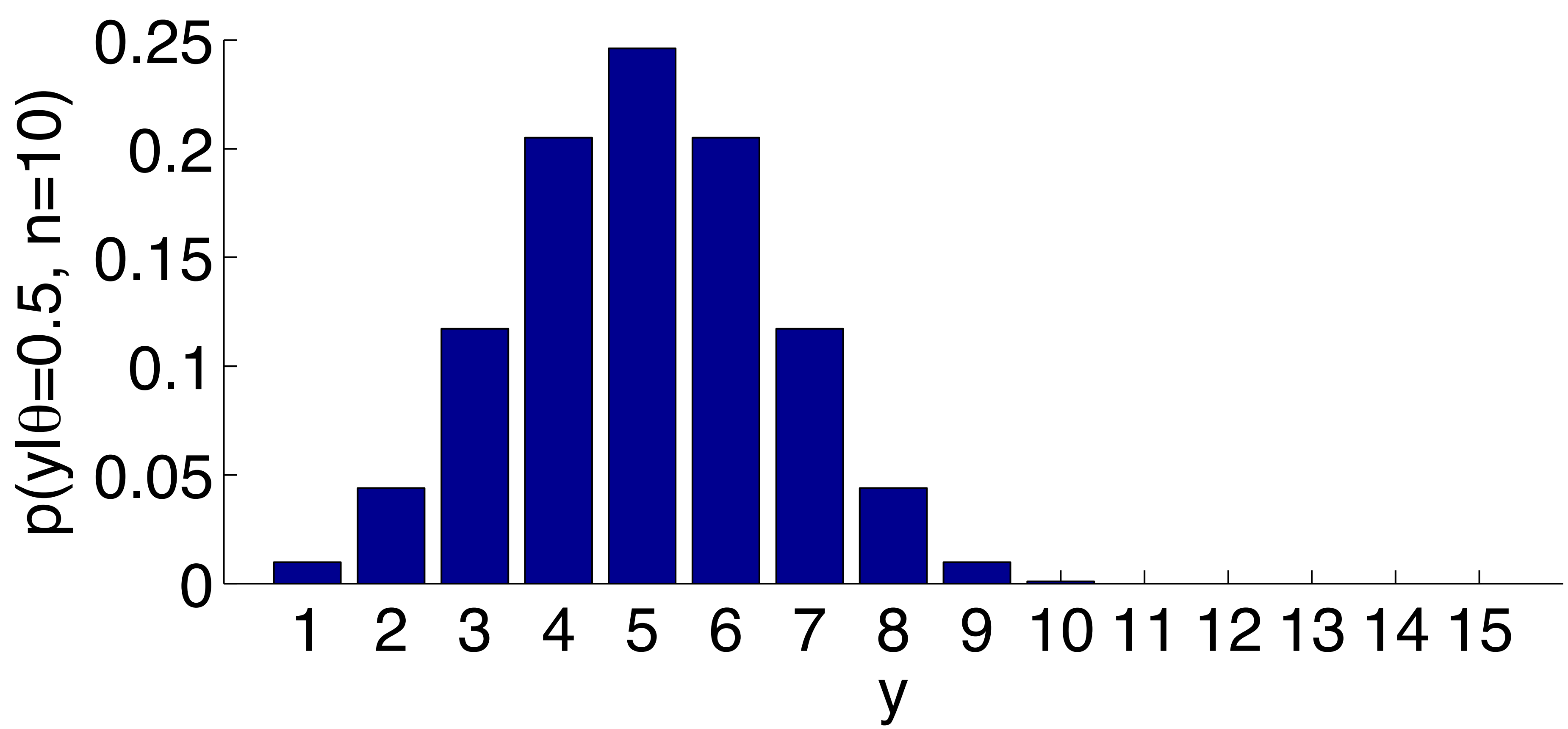
Bayesian inference for distributions

- The simplest case is true or false propositions
- The basic computations are the same for distributions

An example with distributions: coin flipping

- In Bernoulli trials, each sample is either 1 (e.g. heads) with probability θ, or 0 (tails) with probability 1 − θ.
- The binomial distribution specifies the probability of the total # of heads, y, out of n trials:

$$p(y|\theta, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$



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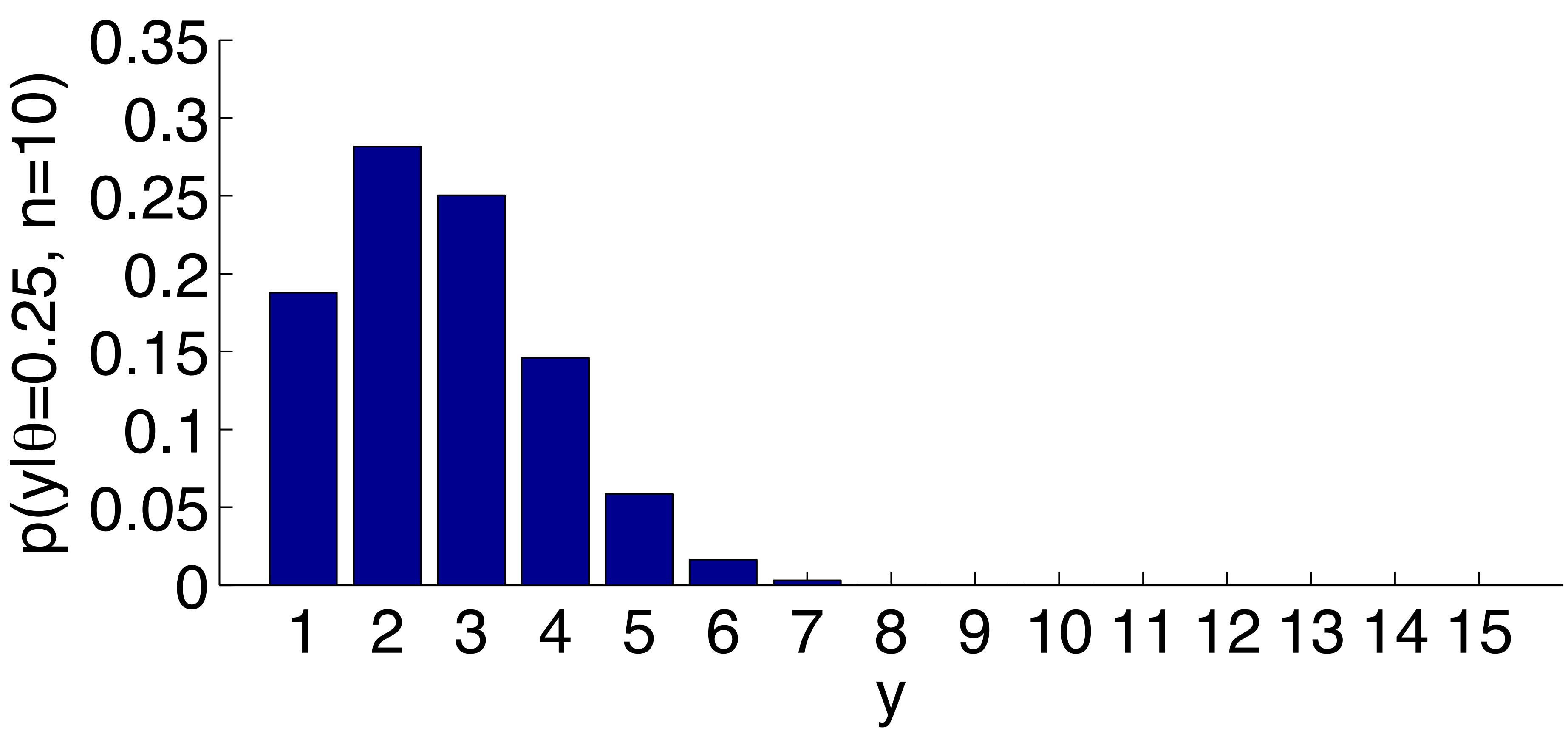

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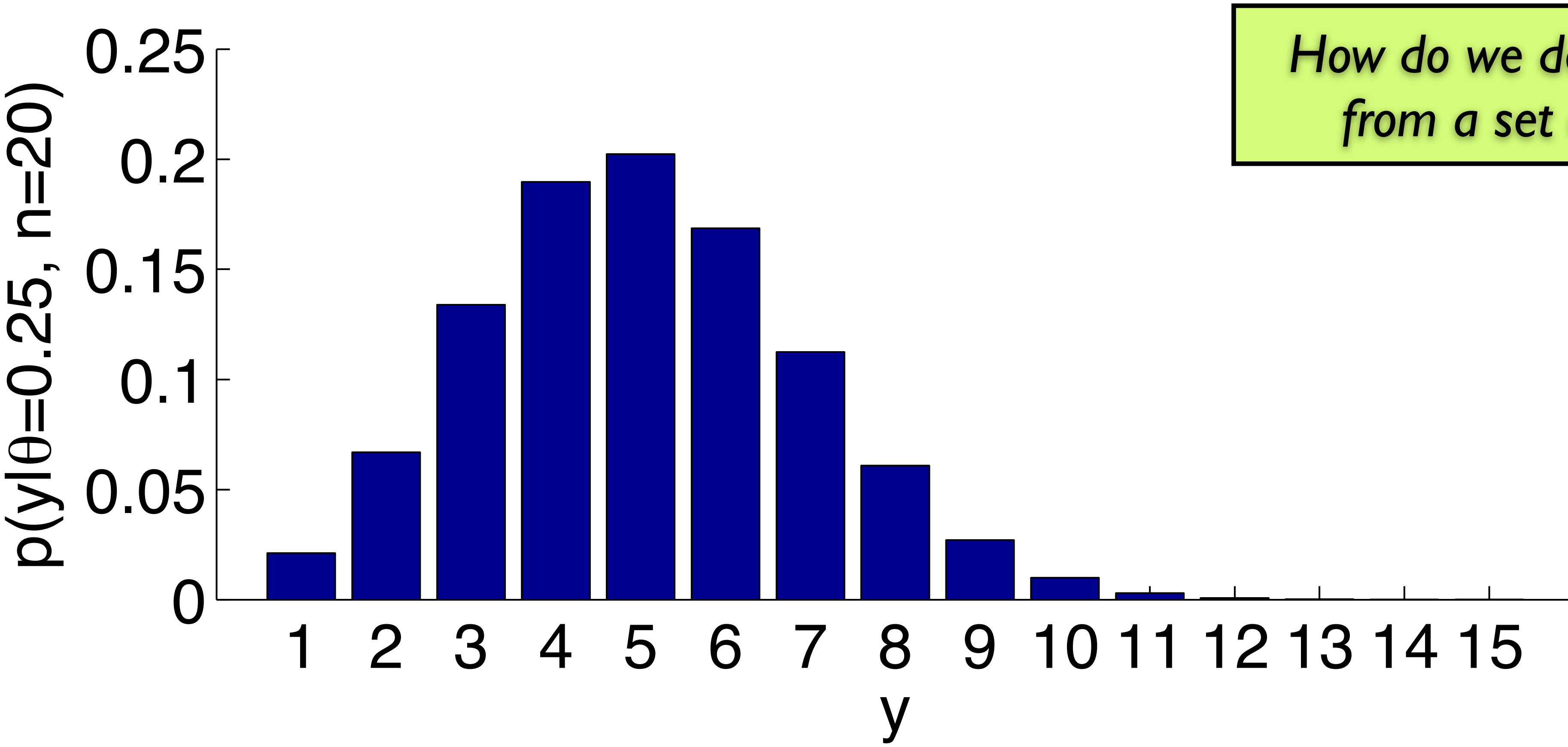


How do we determine θ from a set of trials?

Applying Bayes' rule

- Given n trials with k heads, what do we know about θ?
- We can apply Bayes' rule to see how our knowledge changes as we acquire new observations:

$$\underset{\text{posterior}}{p(\theta|y,n)} = \frac{\overset{\text{likelihood}}{p(y|\theta,n)}\,\overset{\text{prior}}{p(\theta|n)}}{\underset{\text{normalizing constant}}{p(y|n)}}$$

$$p(y|n) = \int p(y|\theta,n)p(\theta|n)d\theta$$

- We know the likelihood, what about the prior?
- Uniform on [0, 1] is a reasonable assumption, i.e. "we don't know anything".
- What is the form of the posterior?
- In this case, the posterior is just proportional to the likelihood:

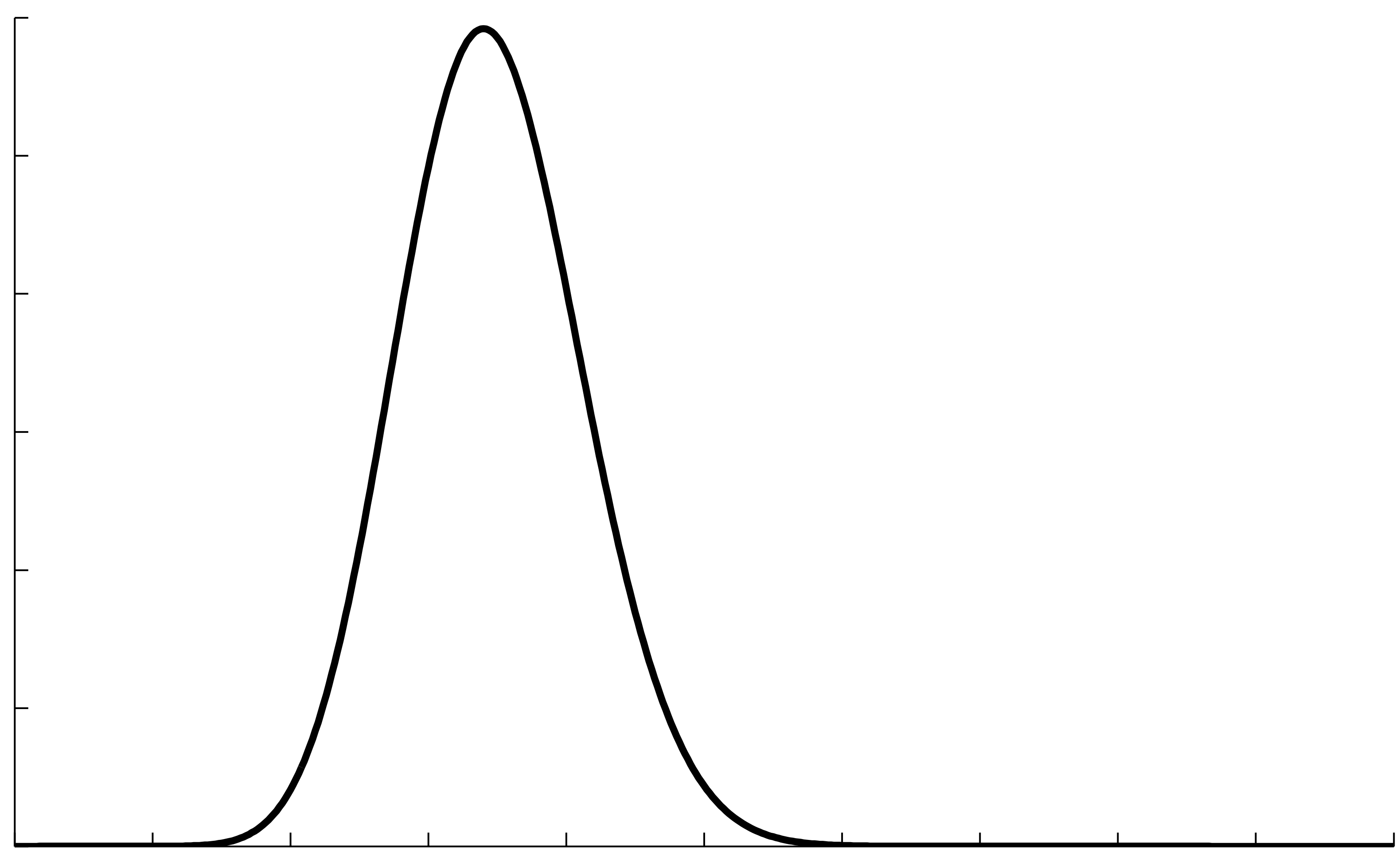
$$p(\theta|y,n) \propto \binom{n}{y}\theta^y(1-\theta)^{n-y}$$

Updating our knowledge with new information

- Now we can evaluate the poster just by plugging in different values of y and n.

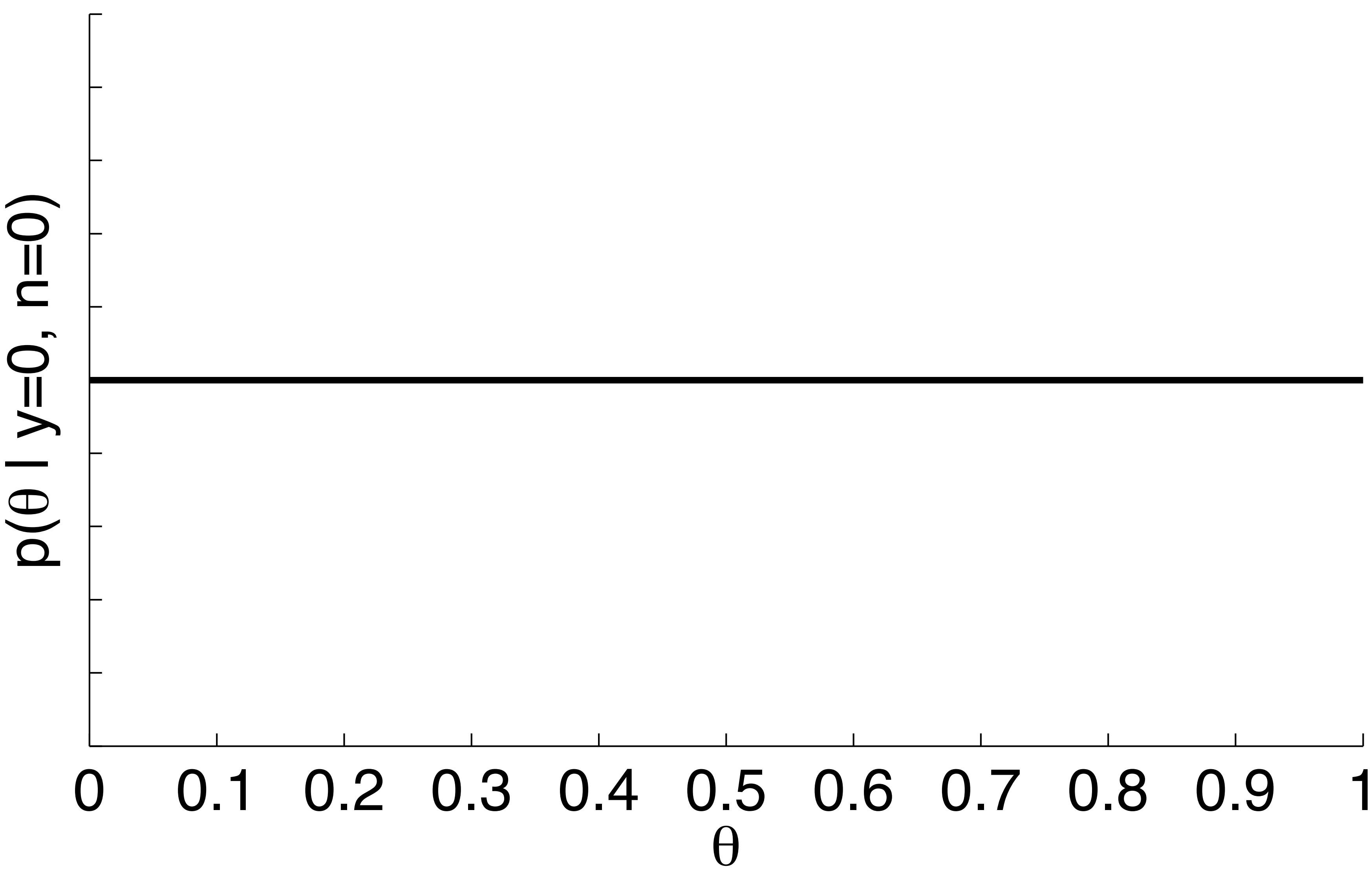
$$p(\theta|y,n) \propto \binom{n}{y}\theta^y(1-\theta)^{n-y}$$

- Check: What goes on the axes?



Evaluating the posterior

- What do we know initially, before observing any trials?

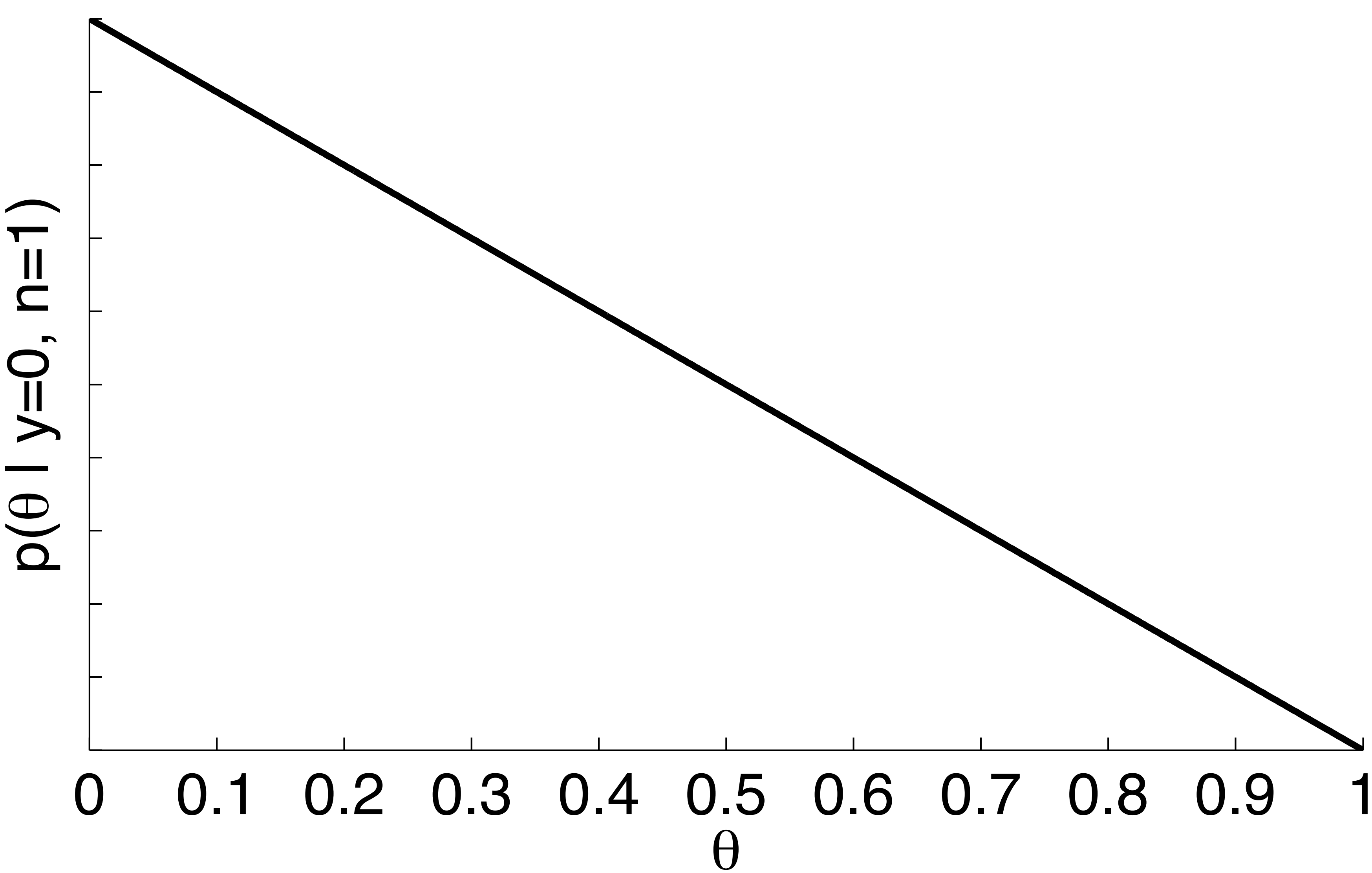


Coin tossing

- What is our belief about θ after observing one "tail" ? *How would you bet?*

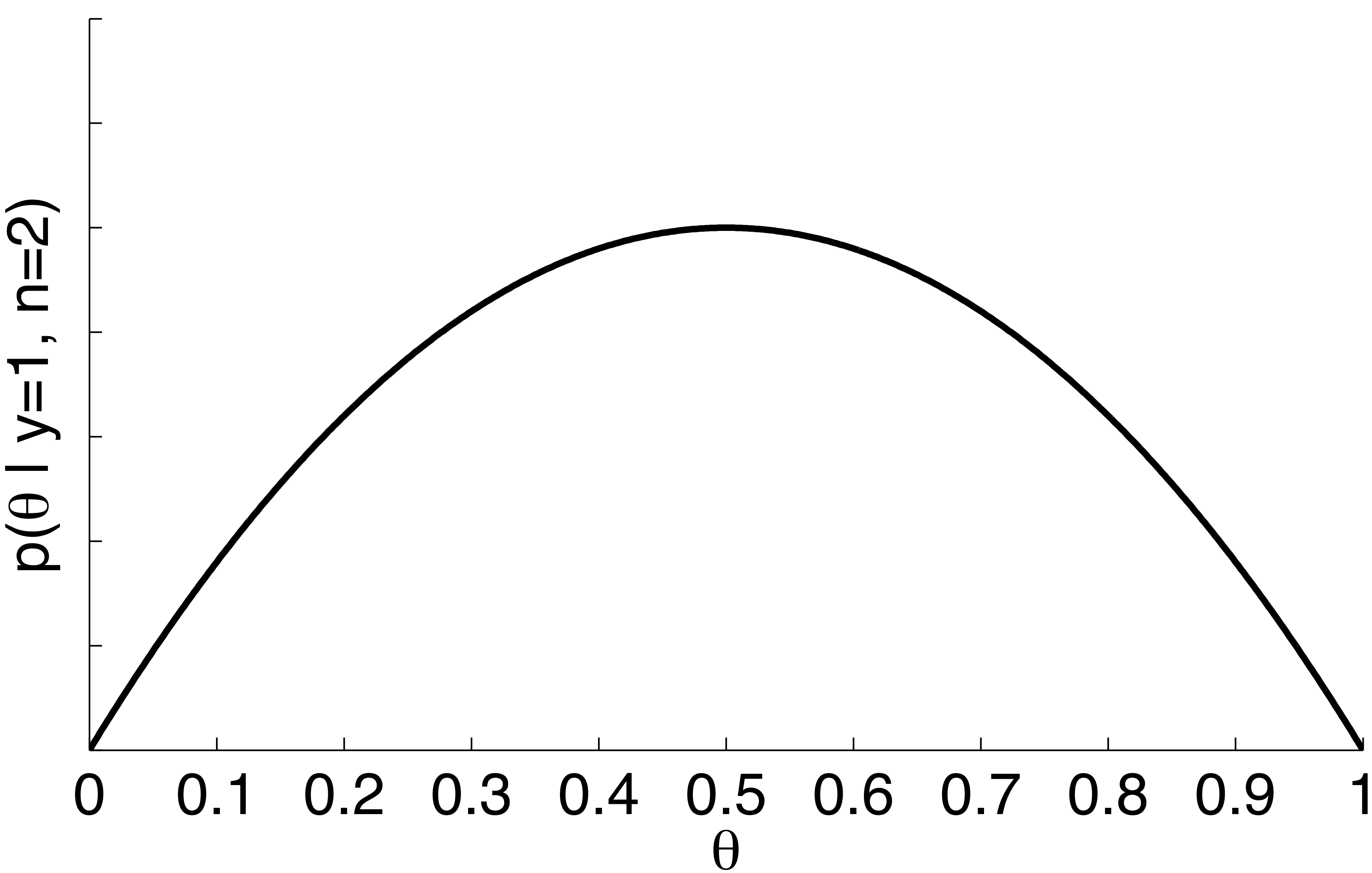
Is the p(θ >0.5) less or greater than 0.5?

What about p(θ >0.3)?



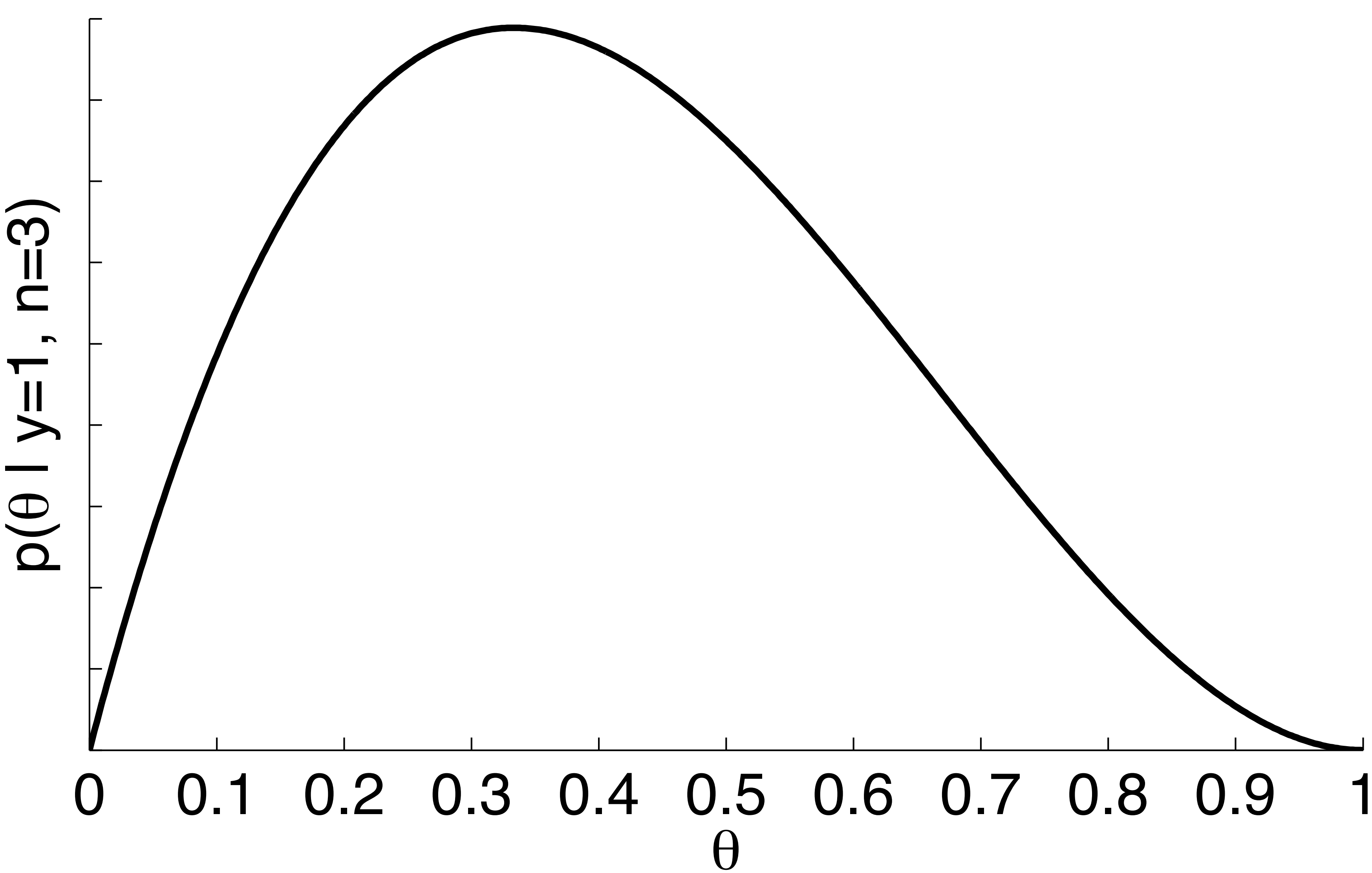
Coin tossing

- Now after two trials we observe 1 head and 1 tail.



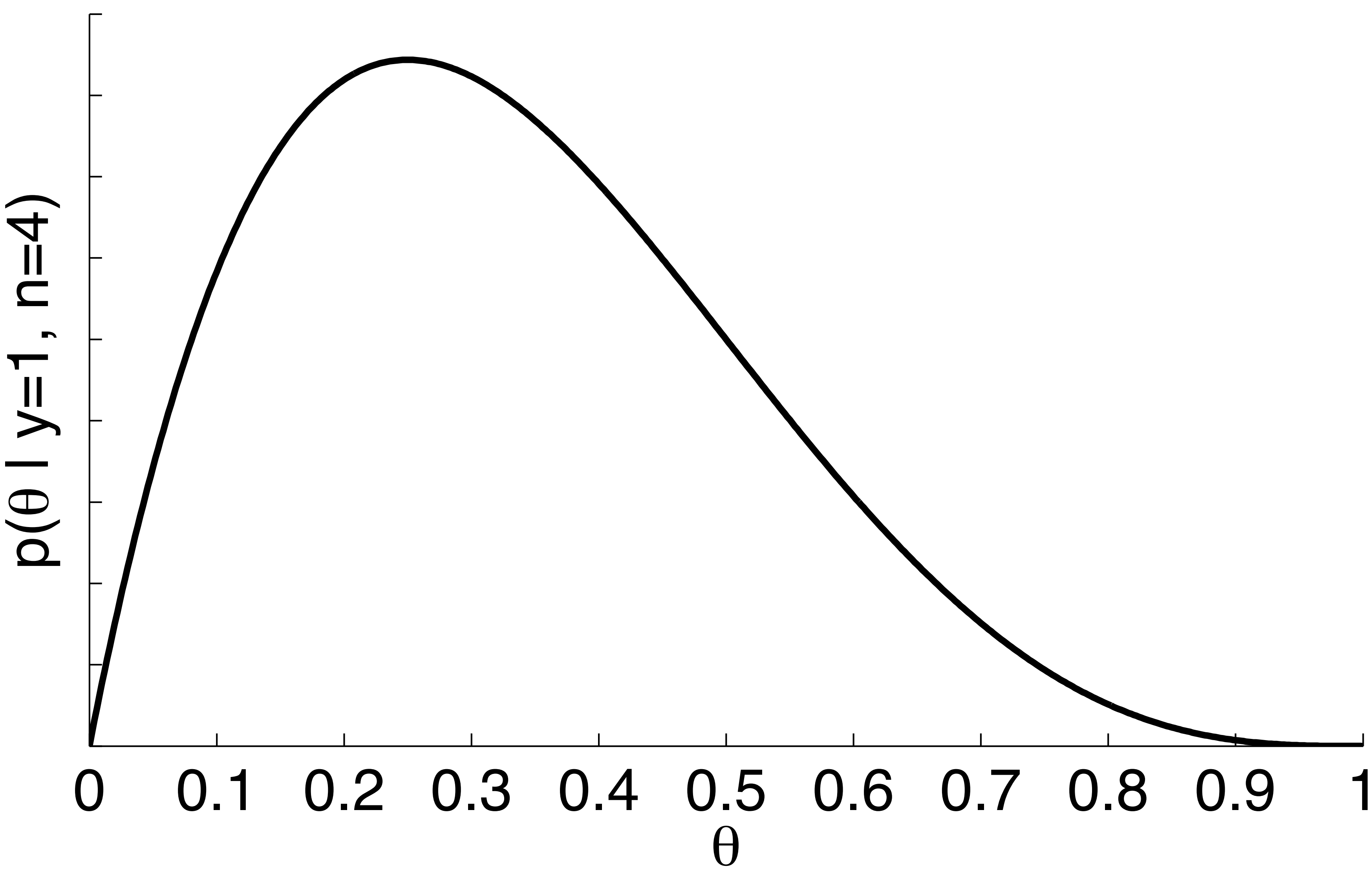
Coin tossing

- 3 trials: 1 head and 2 tails.



Coin tossing

- 4 trials: 1 head and 3 tails.



Coin tossing

- 5 trials: 1 head and 4 tails.

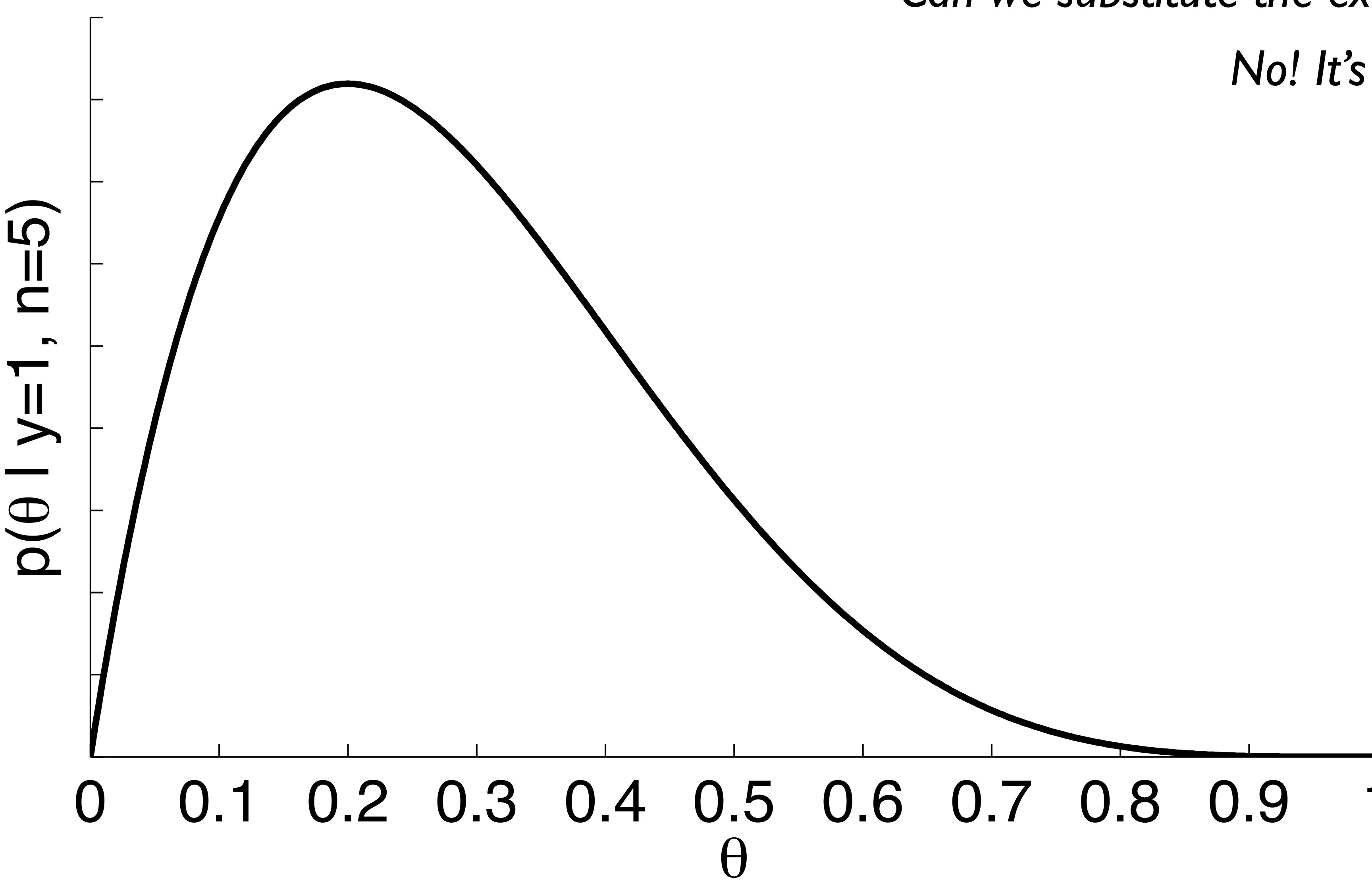
Do we have good evidence that this coin is biased?

How would you quantify this statement?

$$p(\theta > 0.5) = \int_{0.5}^{1.0} p(\theta|y,n)d\theta$$

Can we substitute the expression above?

No! It's not normalized.



Evaluating the normalizing constant

- To get proper probability density functions, we need to evaluate p(y|n):

$$p(\theta|y,n) = \frac{p(y|\theta,n)p(\theta|n)}{p(y|n)}$$

- Bayes in his original paper in 1763 showed that:

$$\begin{aligned} p(y|n) &= \int_0^1 p(y|\theta,n)p(\theta|n)d\theta \\ &= \frac{1}{n+1} \end{aligned}$$

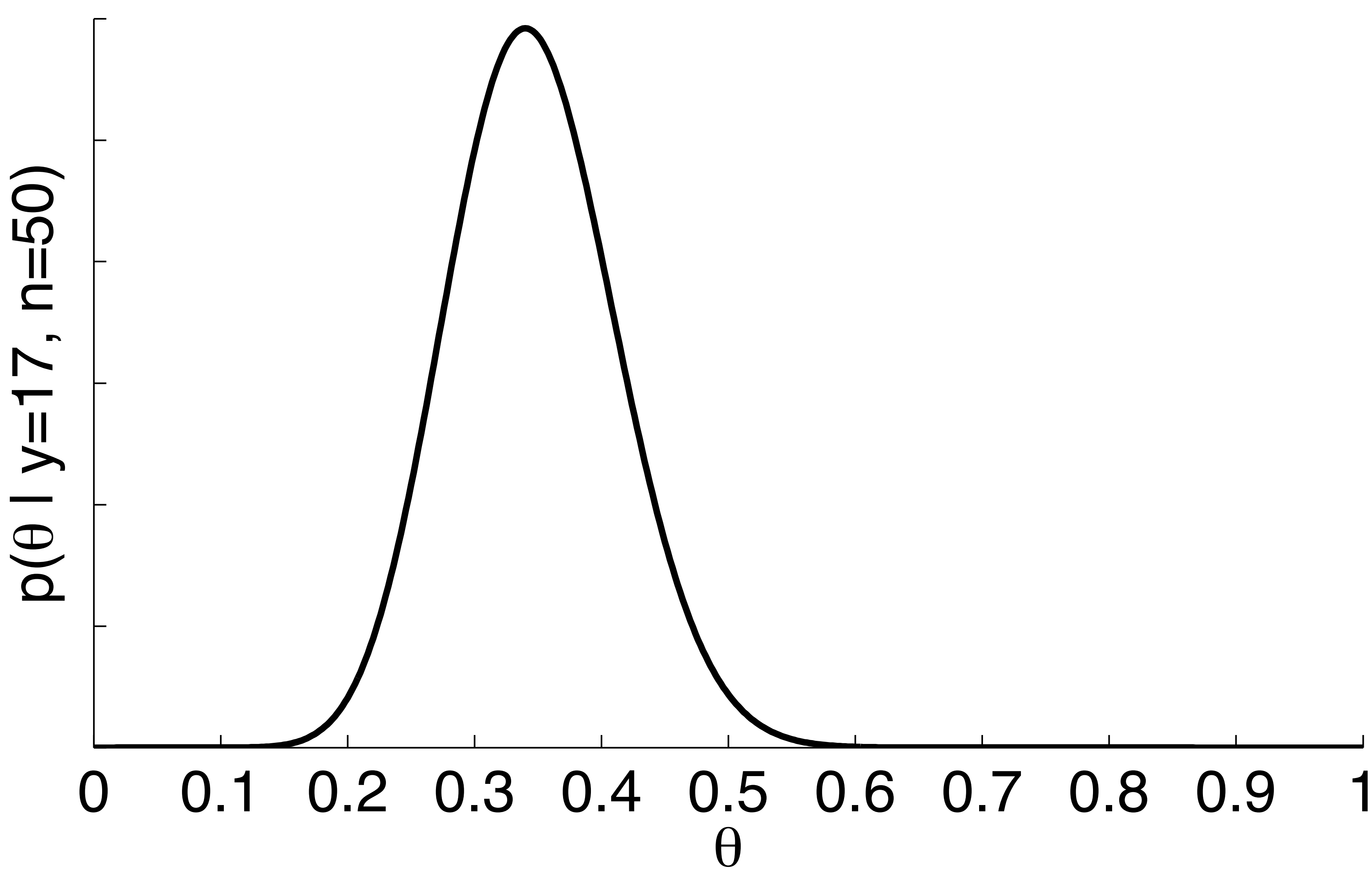
$$\Rightarrow p(\theta|y,n) = \binom{n}{y}\theta^y(1-\theta)^{n-y}(n+1)$$

More coin tossing

- After 50 trials: 17 heads and 33 tails.

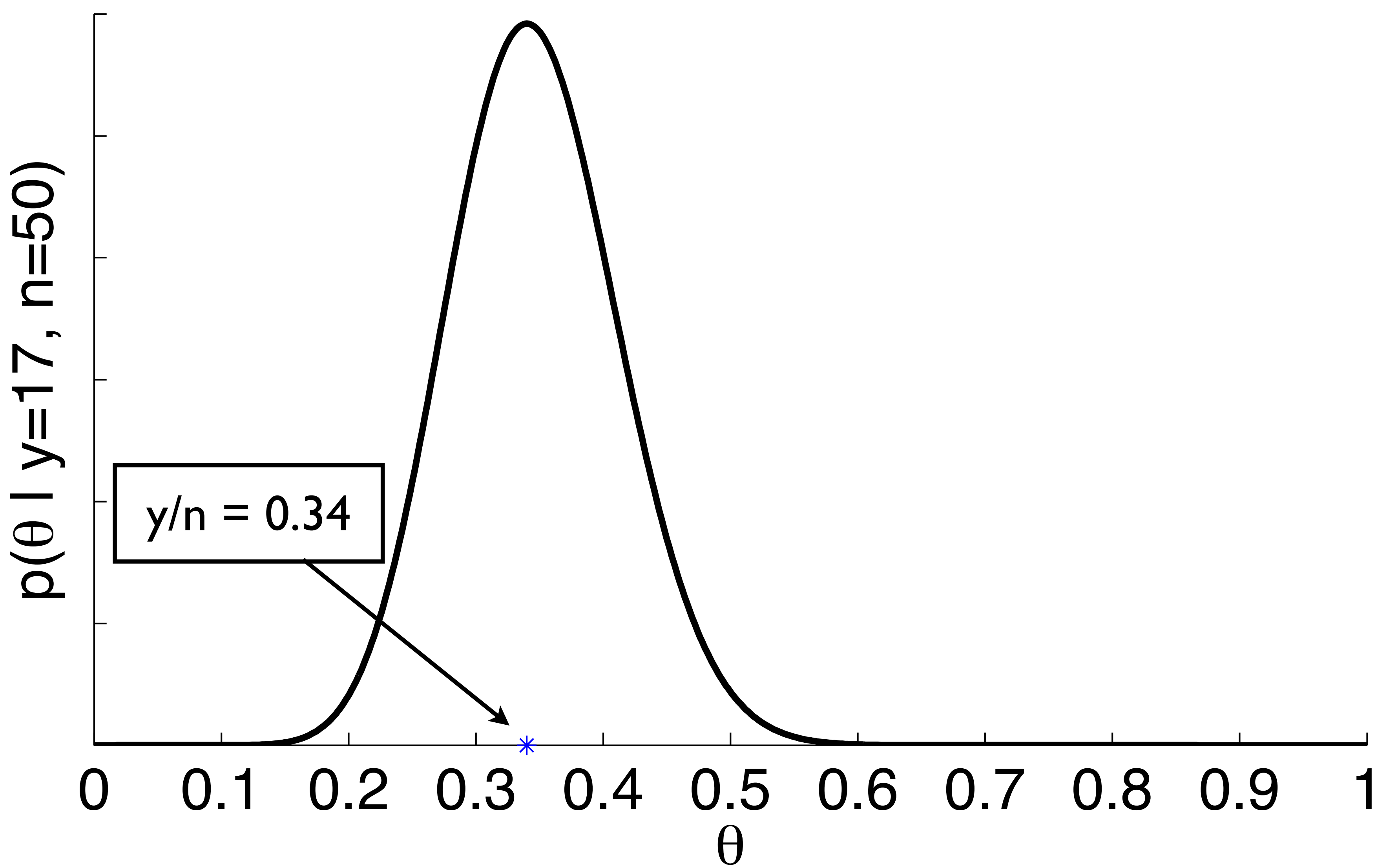
What's a good estimate of θ*?*

- There are many possibilities.



A ratio estimate

- Intuitive estimate: just take ratio θ = 17/50 = 0.34



Estimates for parameter values

- maximum likelihood estimate (MLE)
  - derive by taking derivative of likelihood, setting result to zero, and solving

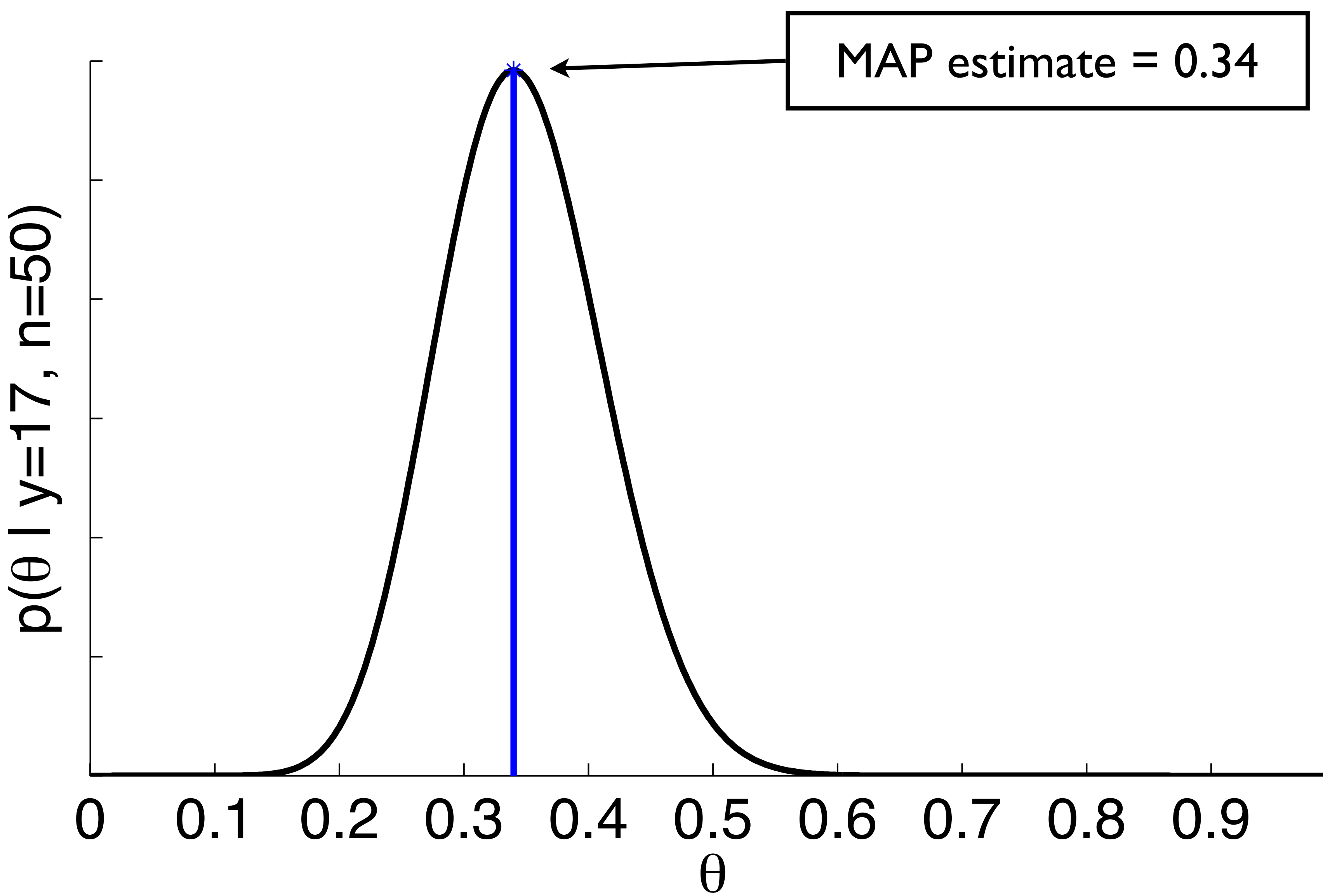
$$\frac{\partial \mathcal{L}}{\theta} = \binom{n}{y} \theta^{y} (1-\theta)^{n-y} = 0$$

  - ignores prior (or assumes uniform prior)
  - $\theta^{\mathrm{ML}} = \frac{y}{n}$ (derived on board)

- Maximum a posteriori (MAP)
  - derive by taking derivative of posterior, setting result to zero, and solving

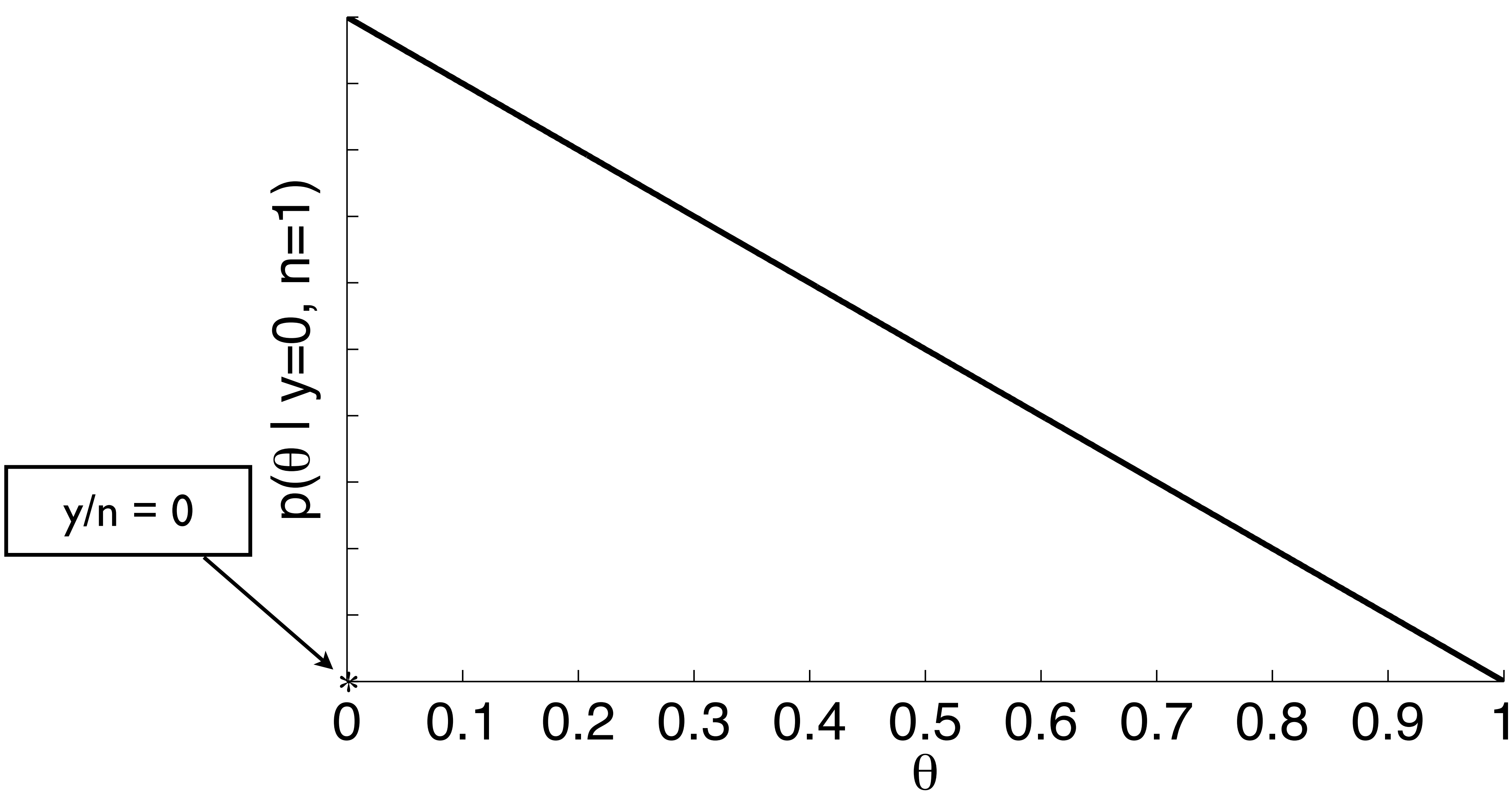
The maximum a posteriori (MAP) estimate

- This just picks the location of maximum value of the posterior
- In this case, maximum is also at $\theta = 0.34$.



A different case

- What about after just one trial: 0 heads and 1 tail?
- MAP and ratio estimate would say 0. *Does this make sense?*
- What would a better estimate be?



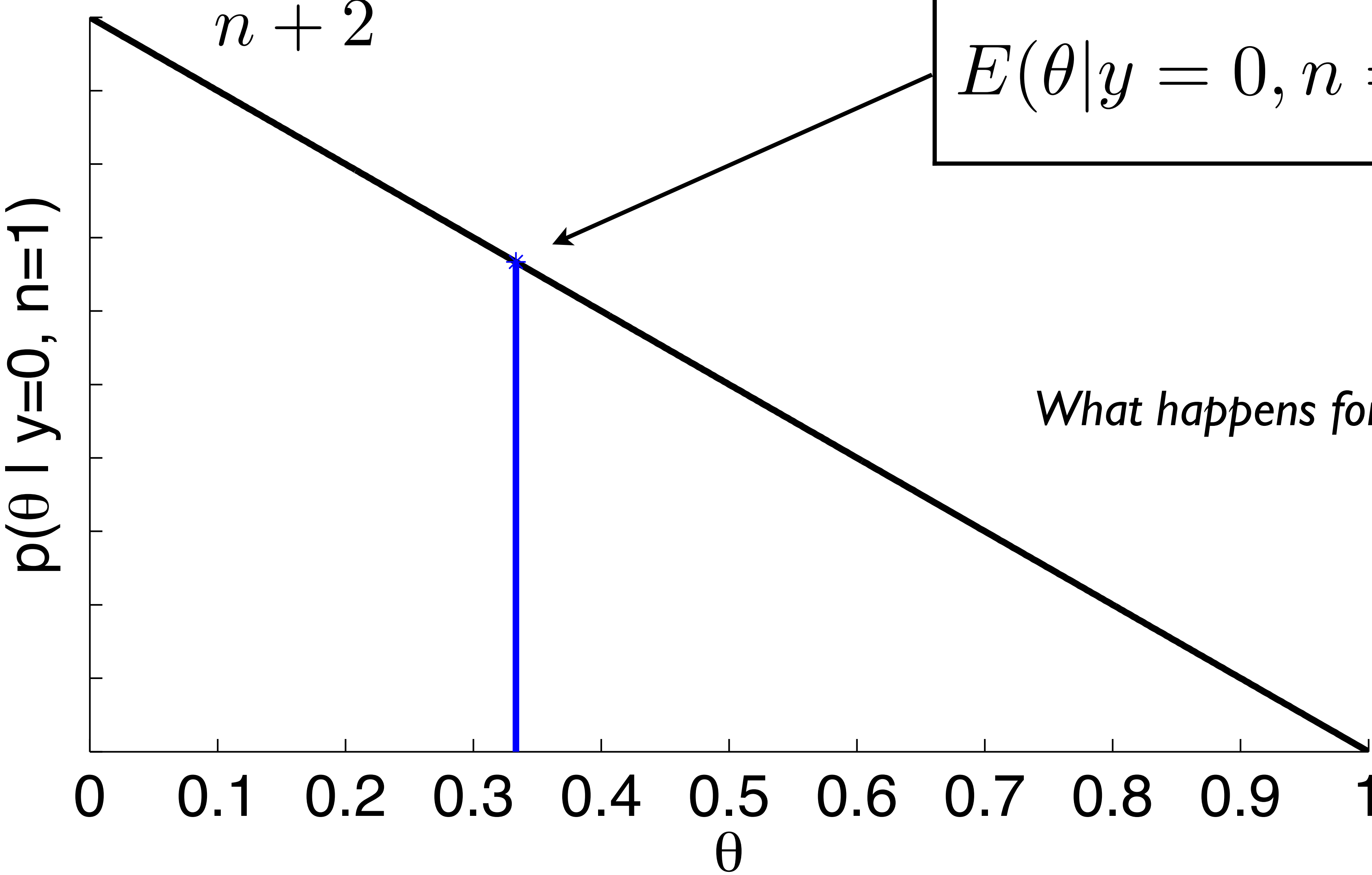
The expected value estimate

- We defined the expected value of a pdf in the previous lecture:

$$E(\theta|y,n) = \int_0^1 \theta p(\theta|y,n) d\theta$$

$$= \frac{y+1}{n+2}$$

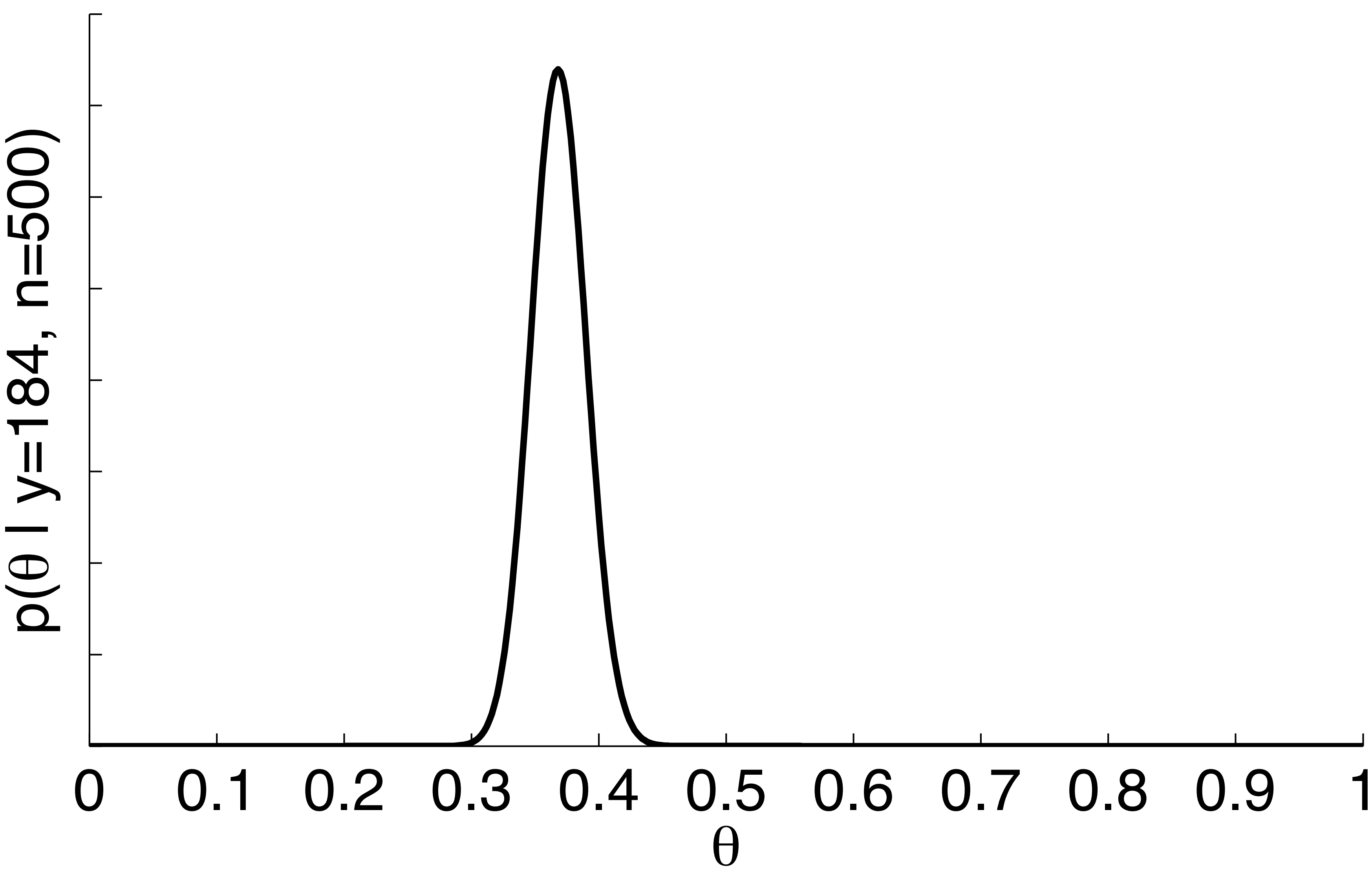
$$\boxed{E(\theta|y=0,n=1) = \frac{1}{3}}$$



What happens for zero trials?

Much more coin tossing

- After 500 trials: 184 heads and 316 tails. *What's your guess of* θ*?*

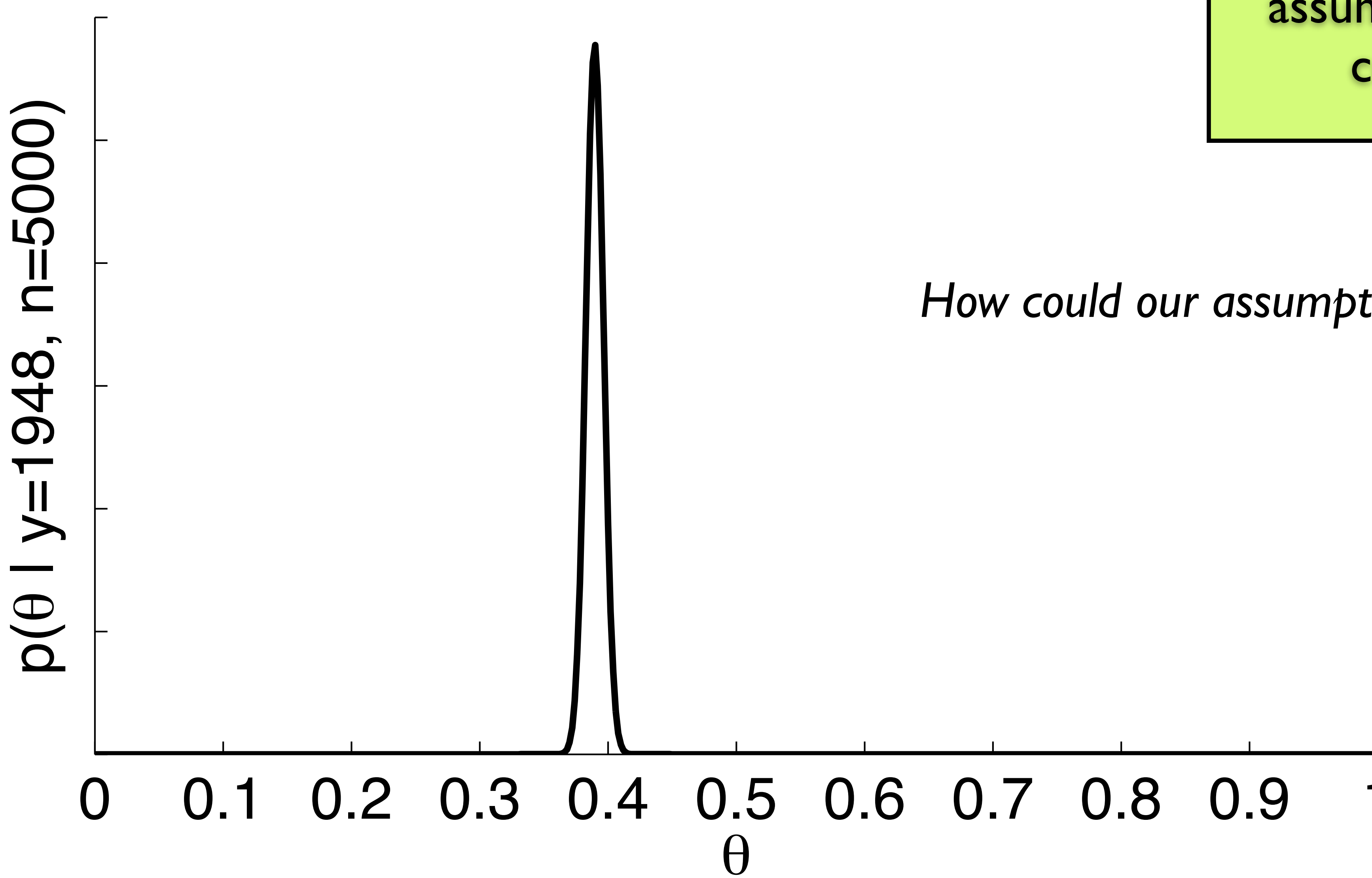


Much more coin tossing

- After 5000 trials: 1948 heads and 3052 tails.
- Posterior contains true estimate.

True value is 0.4.

Is this always the case?



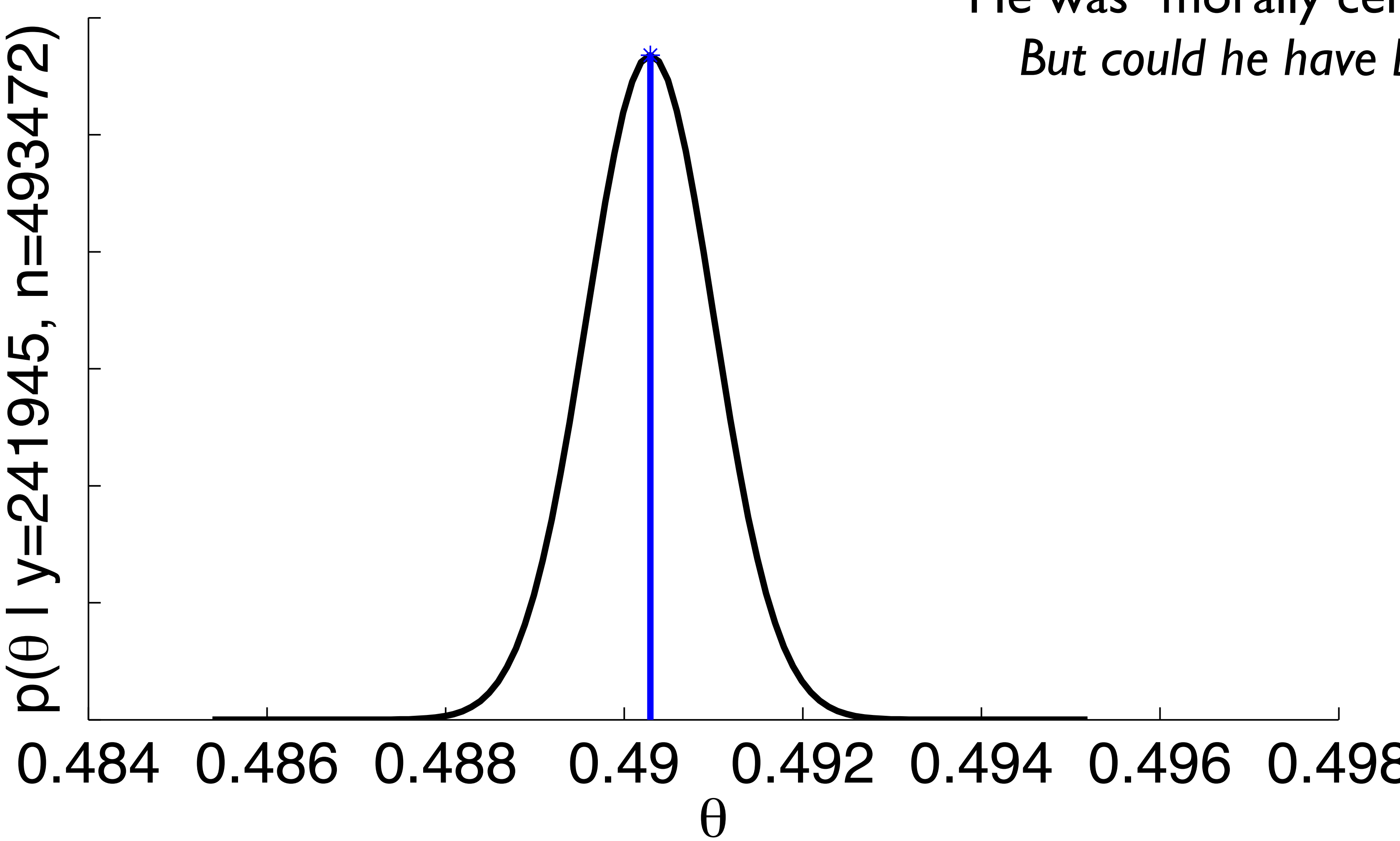
NO! Only if the assumptions are correct.

How could our assumptions be wrong?

Laplace's example: proportion female births

- A total of 241,945 girls and 251,527 boys were born in Paris from 1745-1770.
- Laplace was able to evaluate the following

$$p(\theta > 0.5) = \int_{0.5}^{1.0} p(\theta|y,n)d\theta \approx 1.15 \times 10^{-42}$$



He was "morally certain" θ < 0.5.
But could he have been wrong?

Laplace and the mass of Saturn

- Laplace used "Bayesian" inference to estimate the mass of Saturn and other planets. For Saturn he said:

  It is a bet of 11000 to 1 that the error in this result is not within 1/100th of its value

Mass of Saturn as a fraction of the mass of the Sun	
Laplace (1815)	NASA (2004)
3512	3499.1

(3512 - 3499.1) / 3499.1 = 0.0037

Laplace is still wining.

Applying Bayes' rule with an informative prior

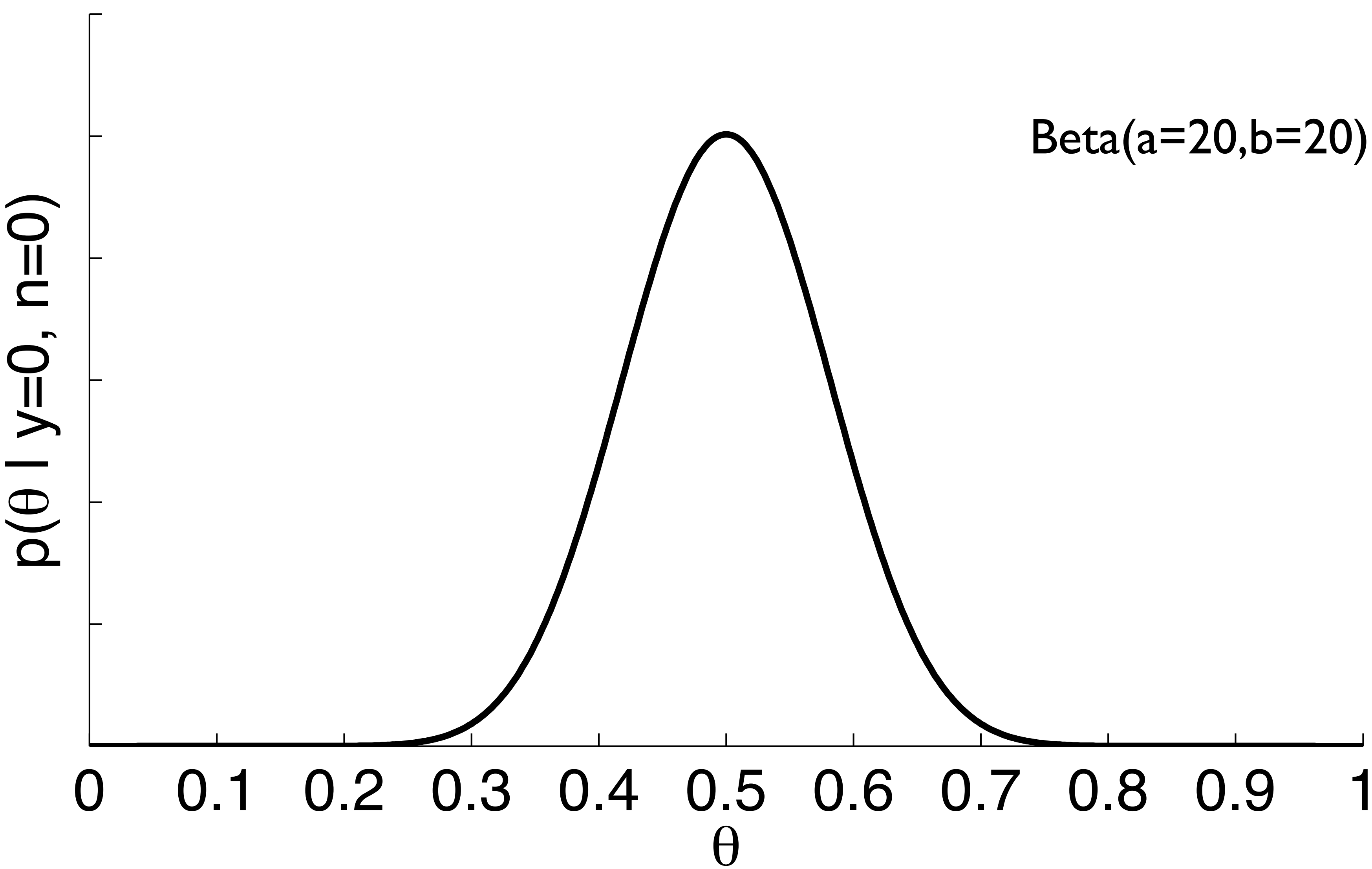
- What if we already know something about θ?
- We can still apply Bayes' rule to see how our knowledge changes as we acquire new observations:

$$p(\theta|y,n) = \frac{p(y|\theta,n)p(\theta|n)}{p(y|n)}$$

- But now the prior becomes important.
- Assume we know biased coins are never below 0.3 or above 0.7.
- To describe this we can use a beta distribution for the prior.

A beta prior

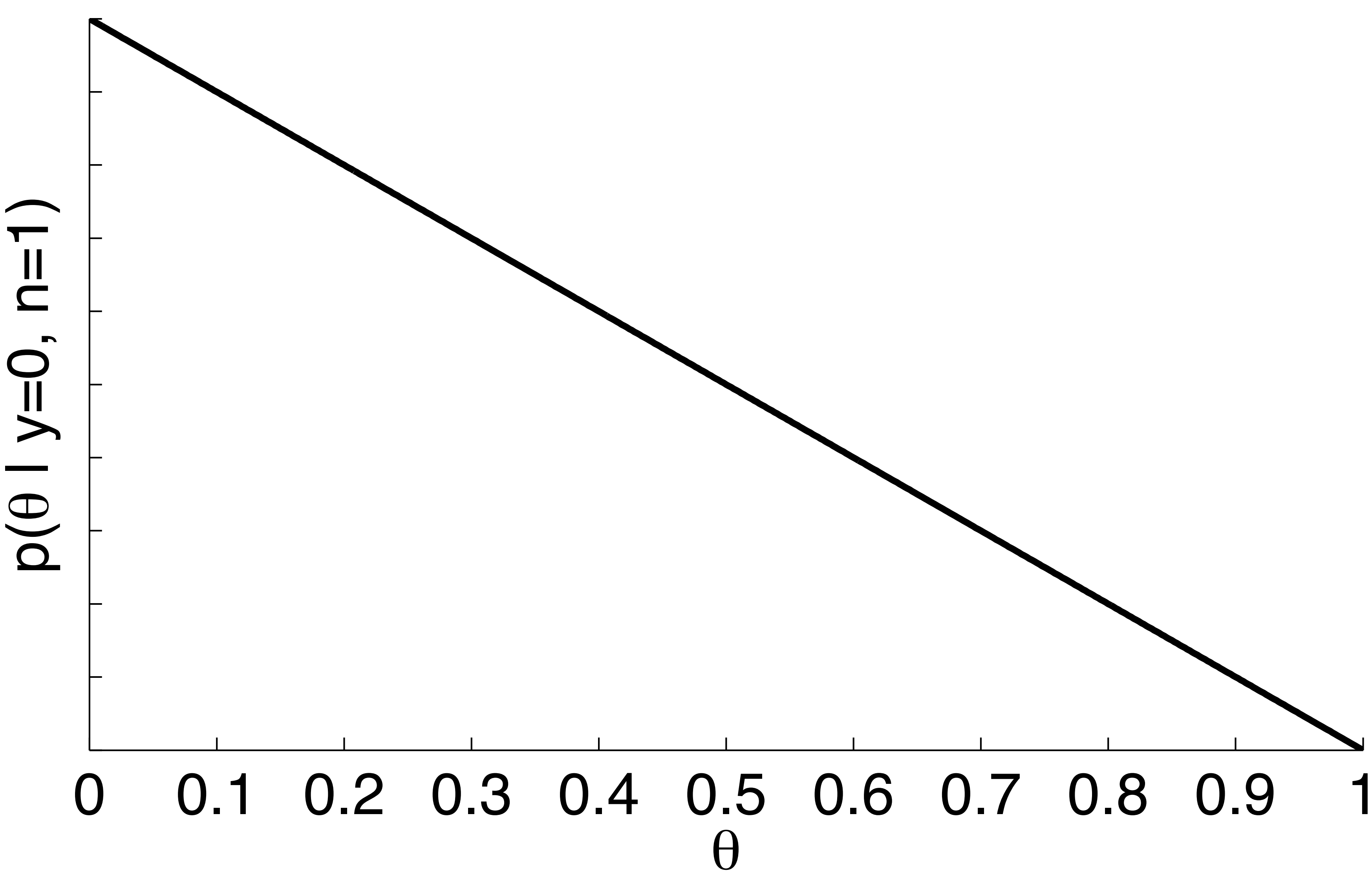
- In this case, before observing any trials our prior is not uniform:



Coin tossing revisited

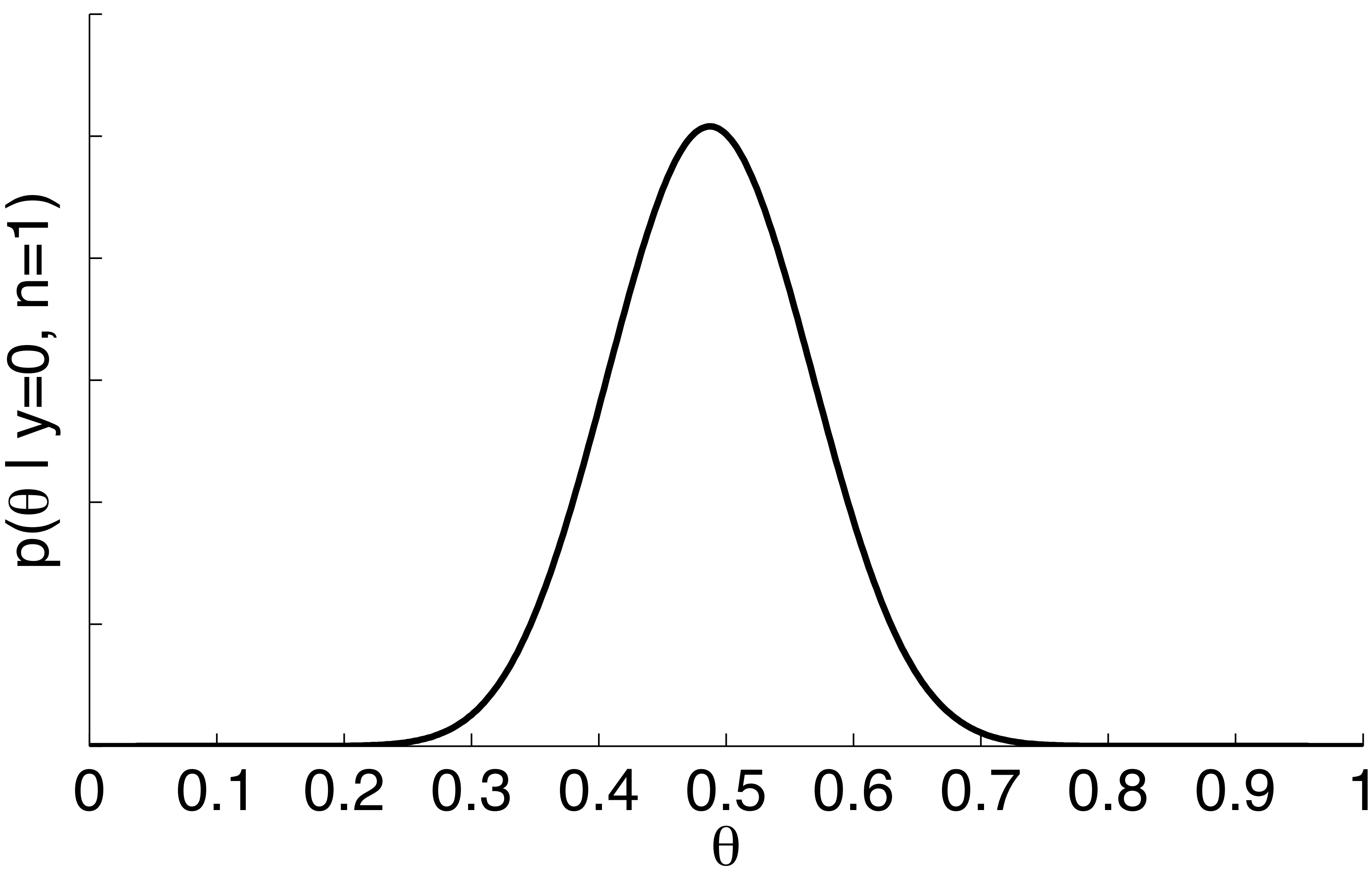
- What is our belief about θ after observing one "tail" ?
- With a uniform prior it was:

What will it look like with our prior?



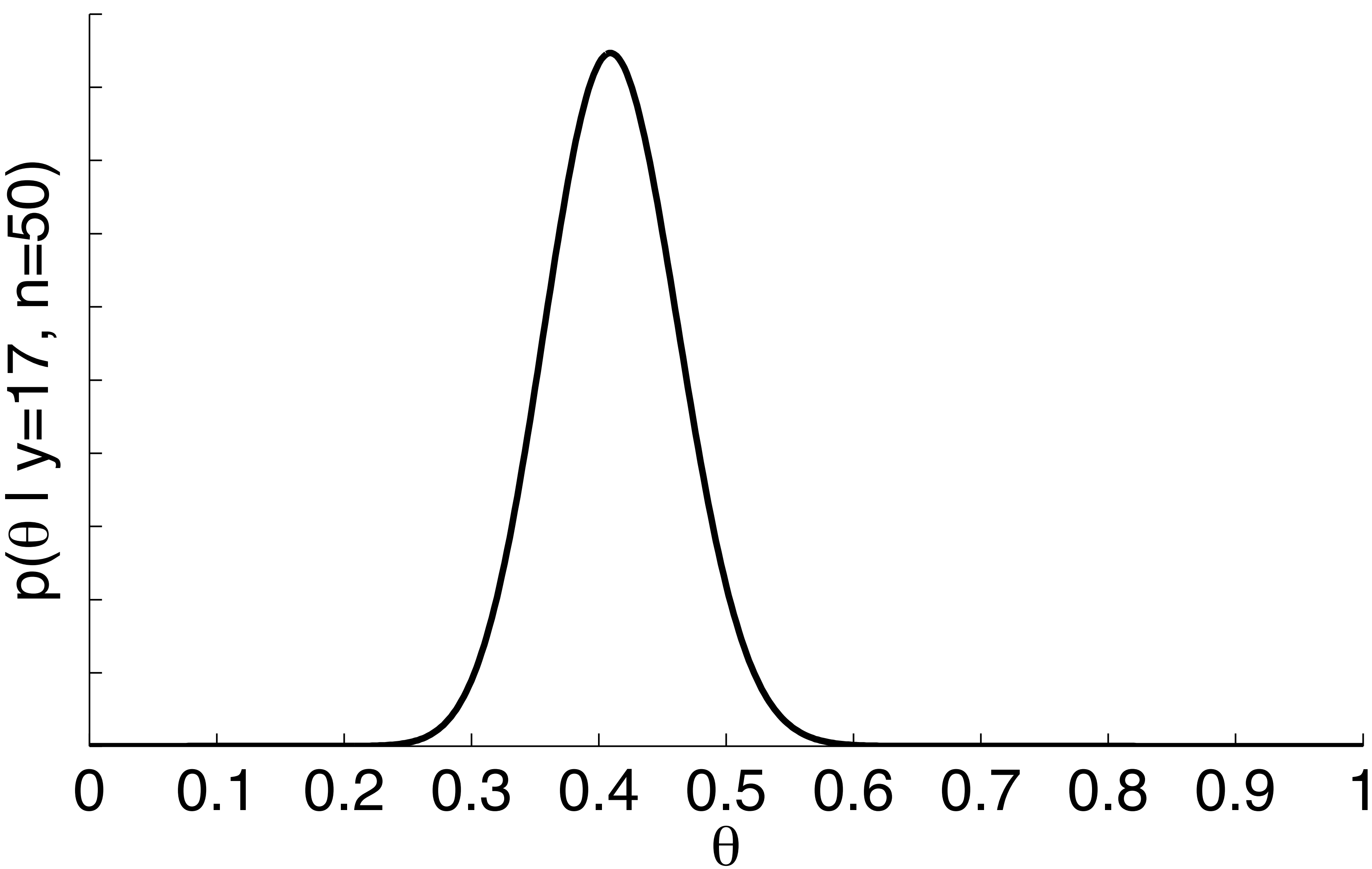
Coin tossing with prior knowledge

- Our belief about θ after observing one “tail” hardly changes.



Coin tossing

- After 50 trials, it's much like before.



Coin tossing

- After 5,000 trials, it's virtually identical to the uniform prior.

What did we gain?