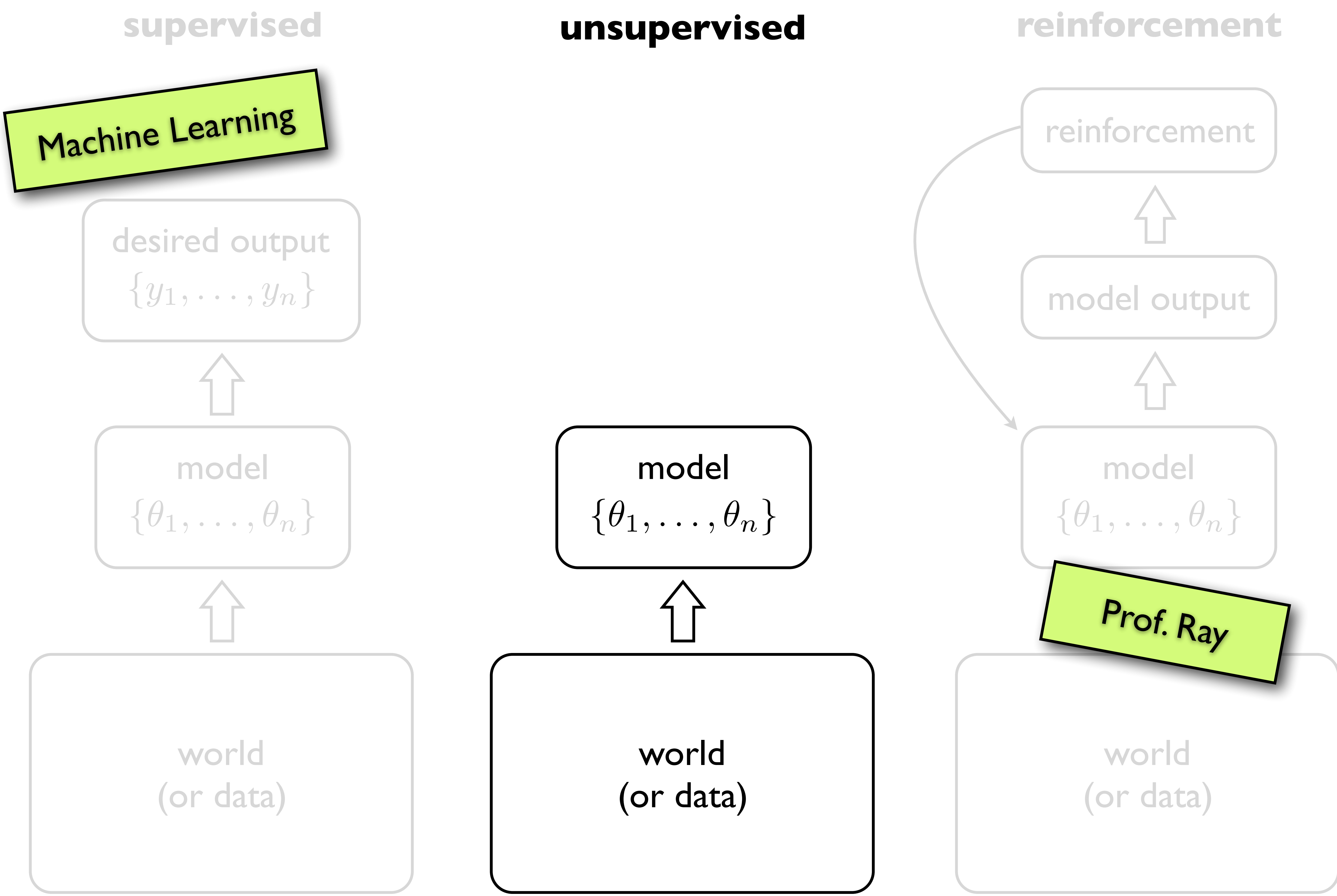
Artificial Intelligence
EECS 491

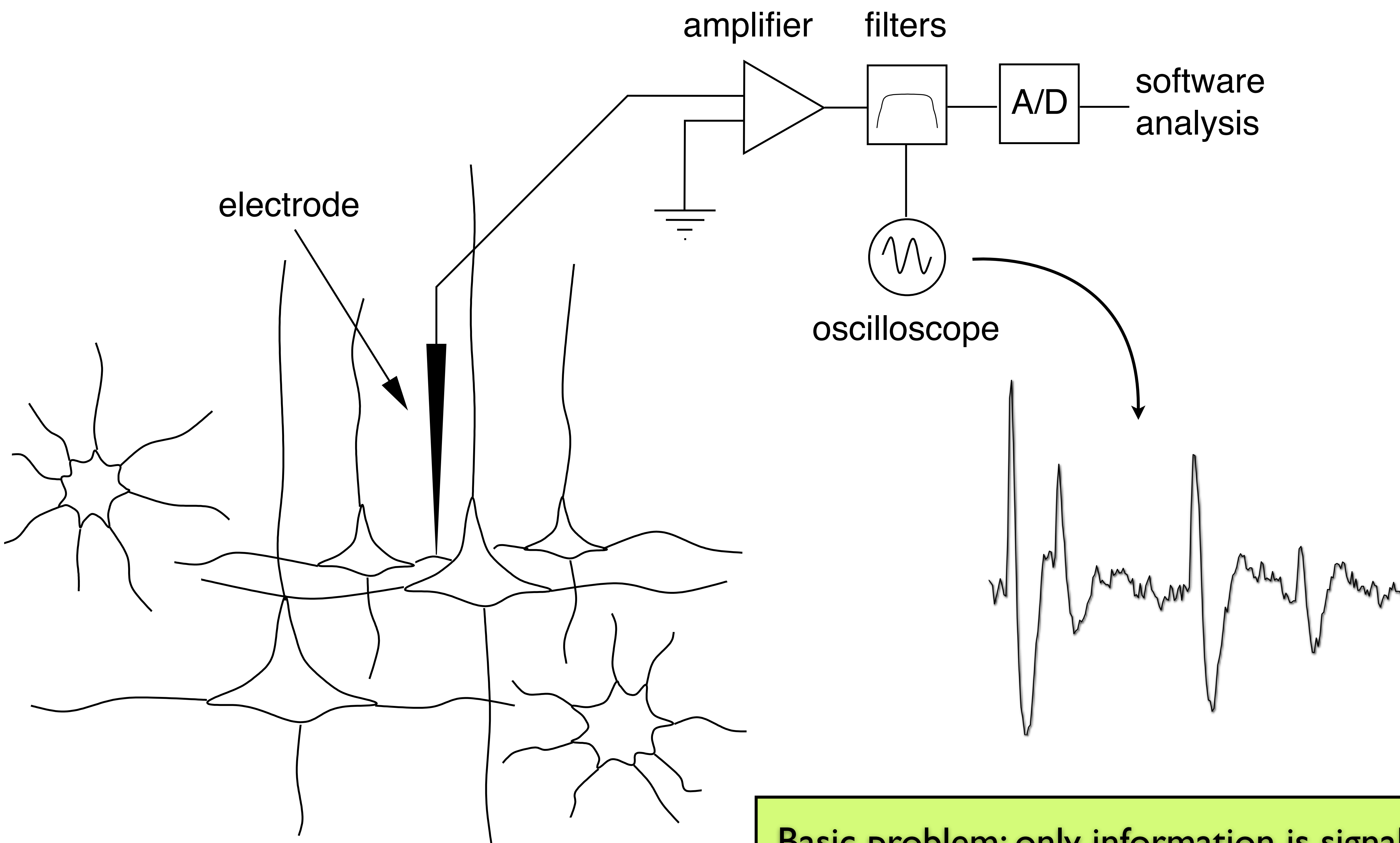
Mar 25, 2019

Multi-variate Gaussians, Dimensionality Reduction and Principal Components Analysis

Types of learning

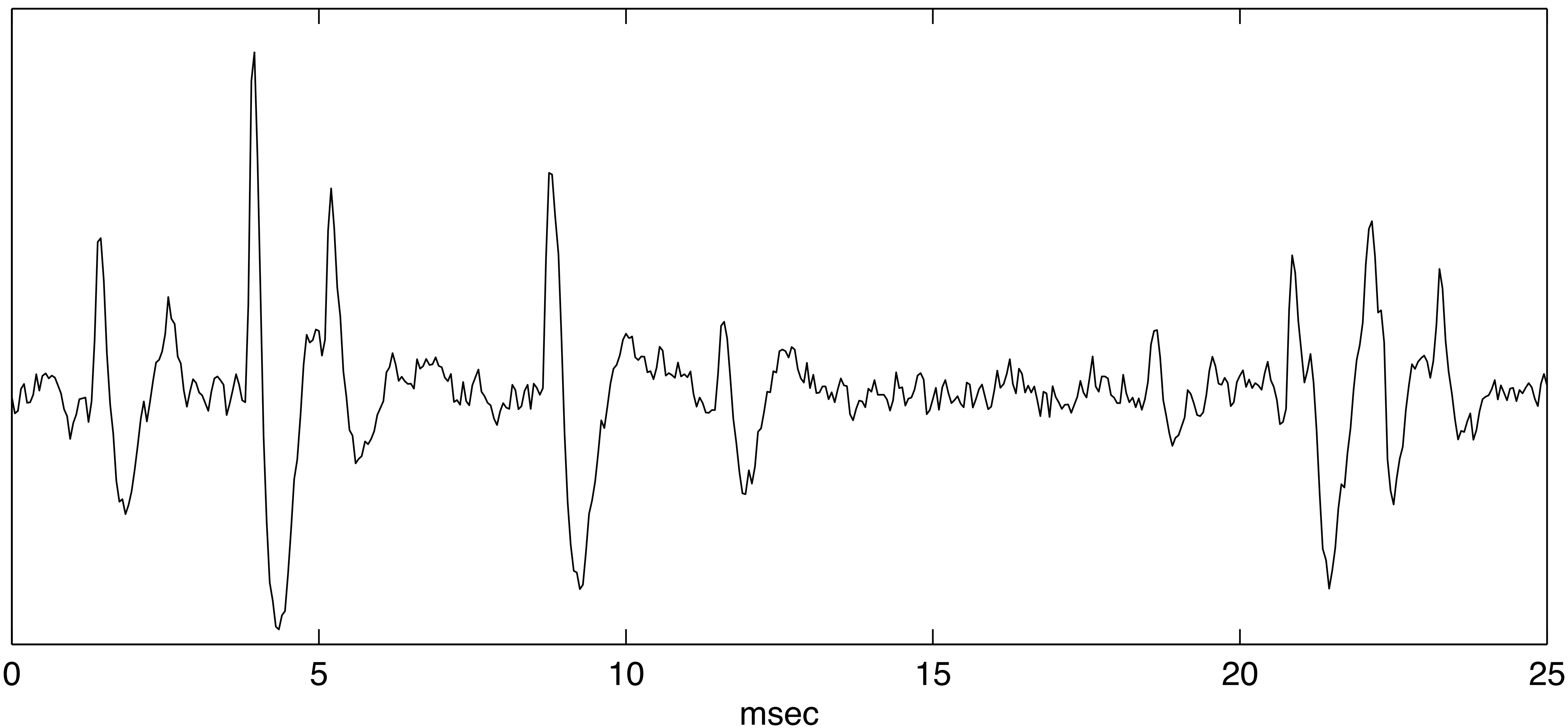


Start with a real example: electrical signals from neurons



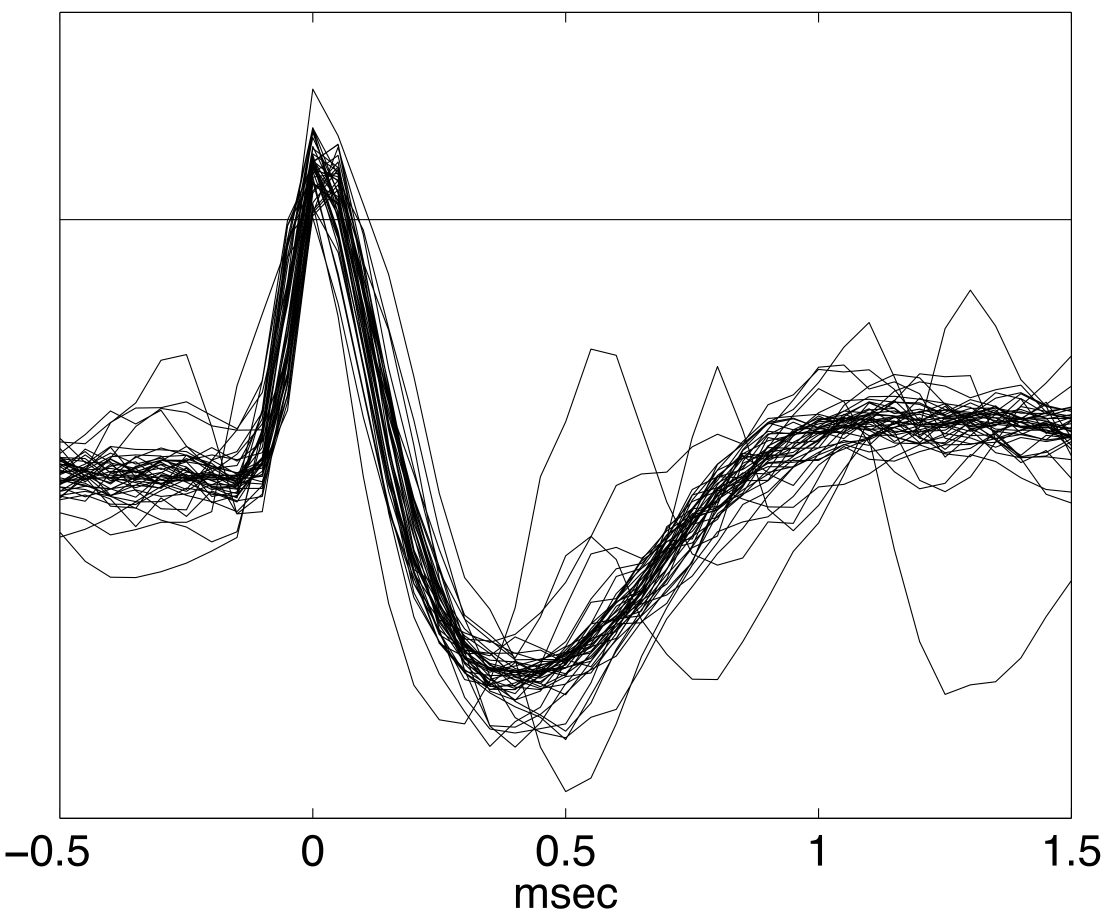
Basic problem: only information is signal.
The true classes are *always* unknown.

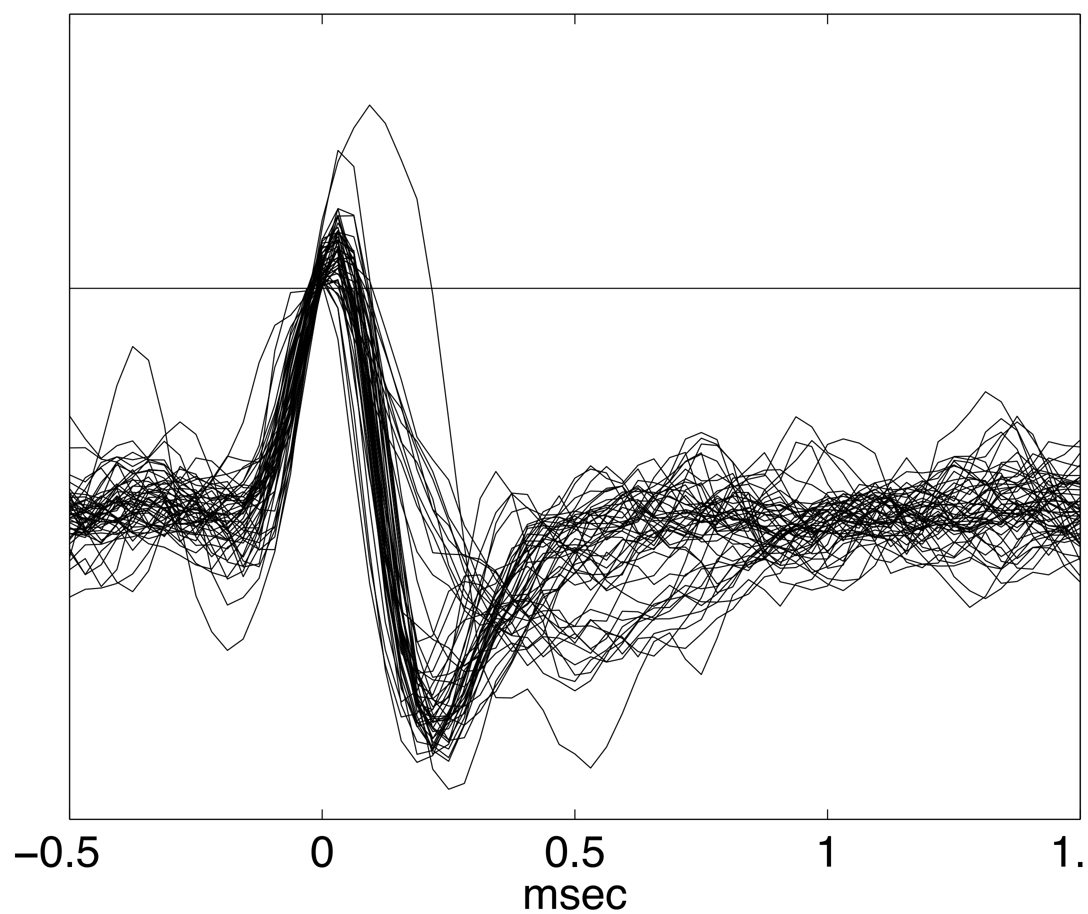
An extracellular waveform with many different neural "spikes"



How do we sort the different spikes?

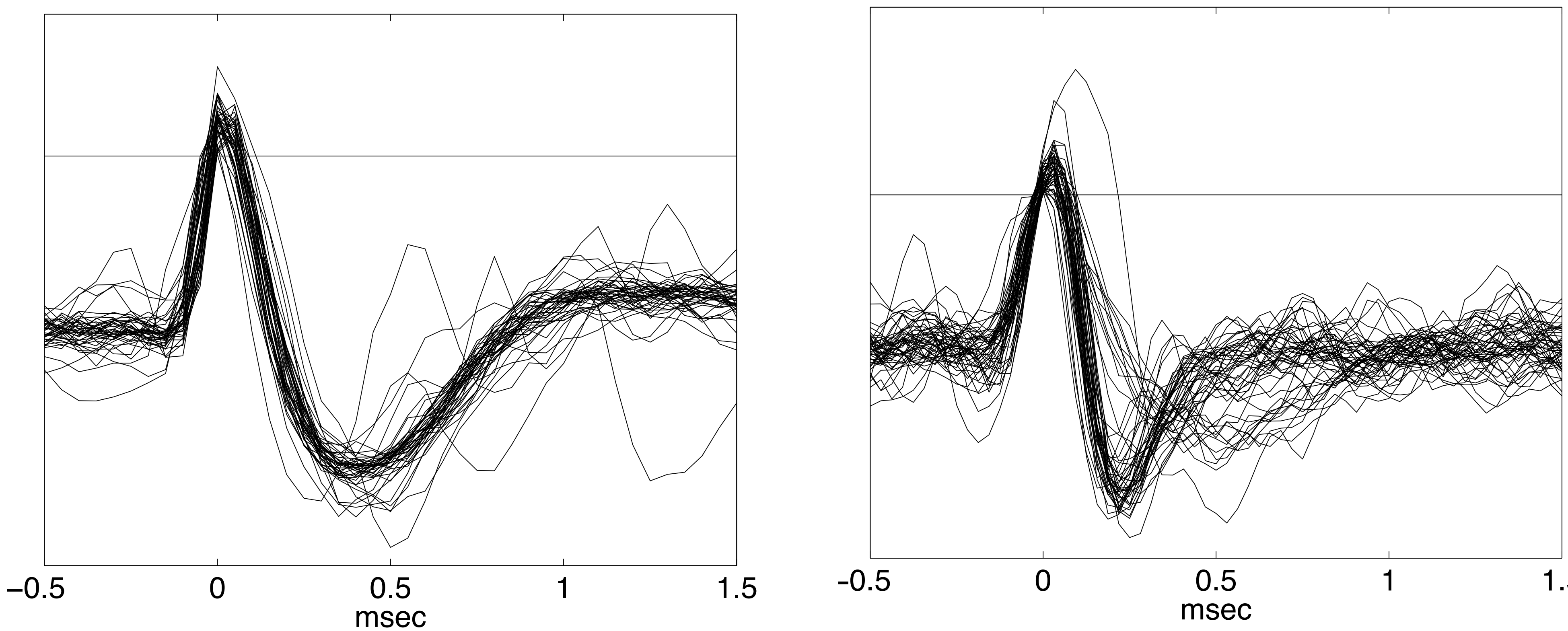
Sorting with level detection

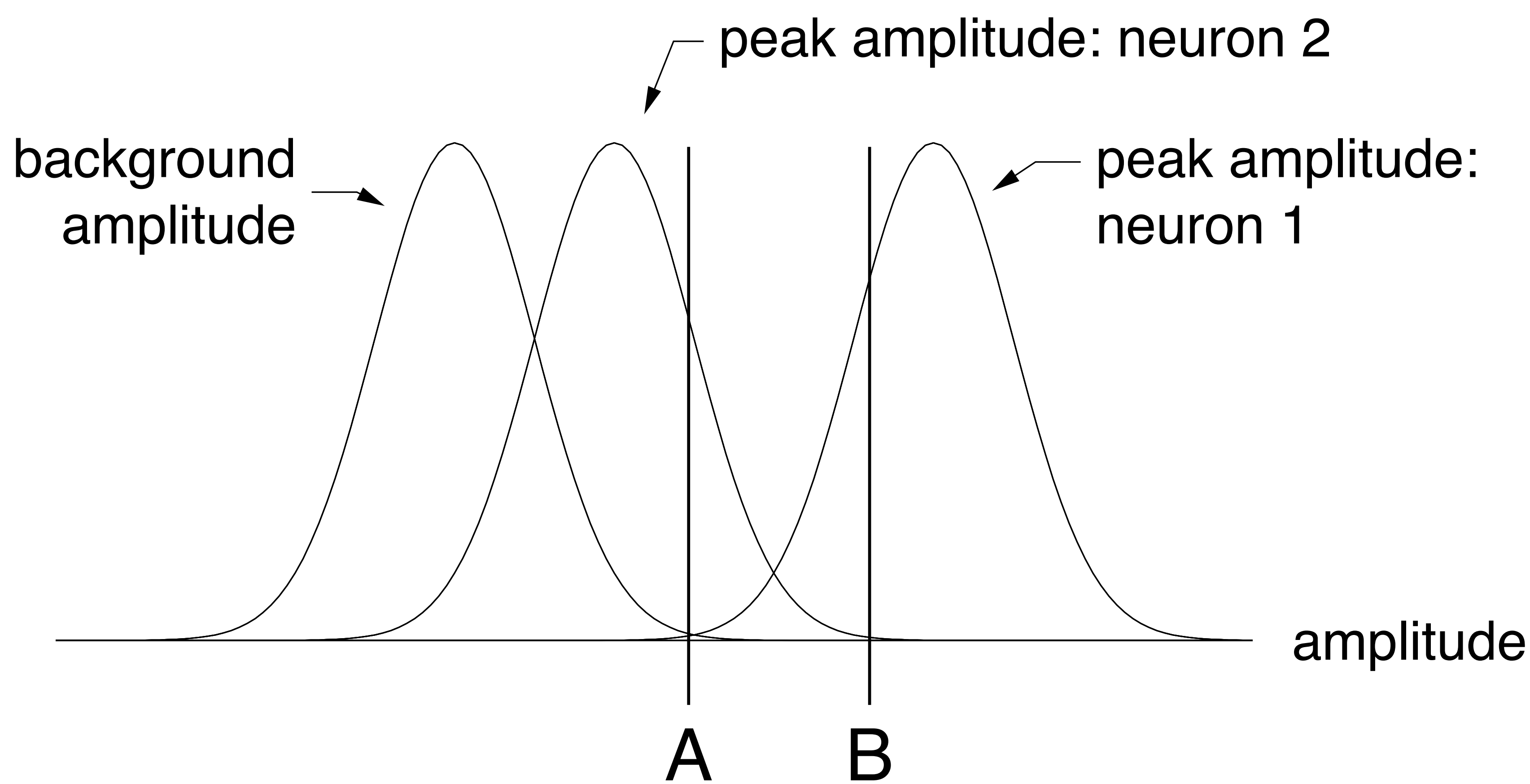




Level detection doesn't always work.

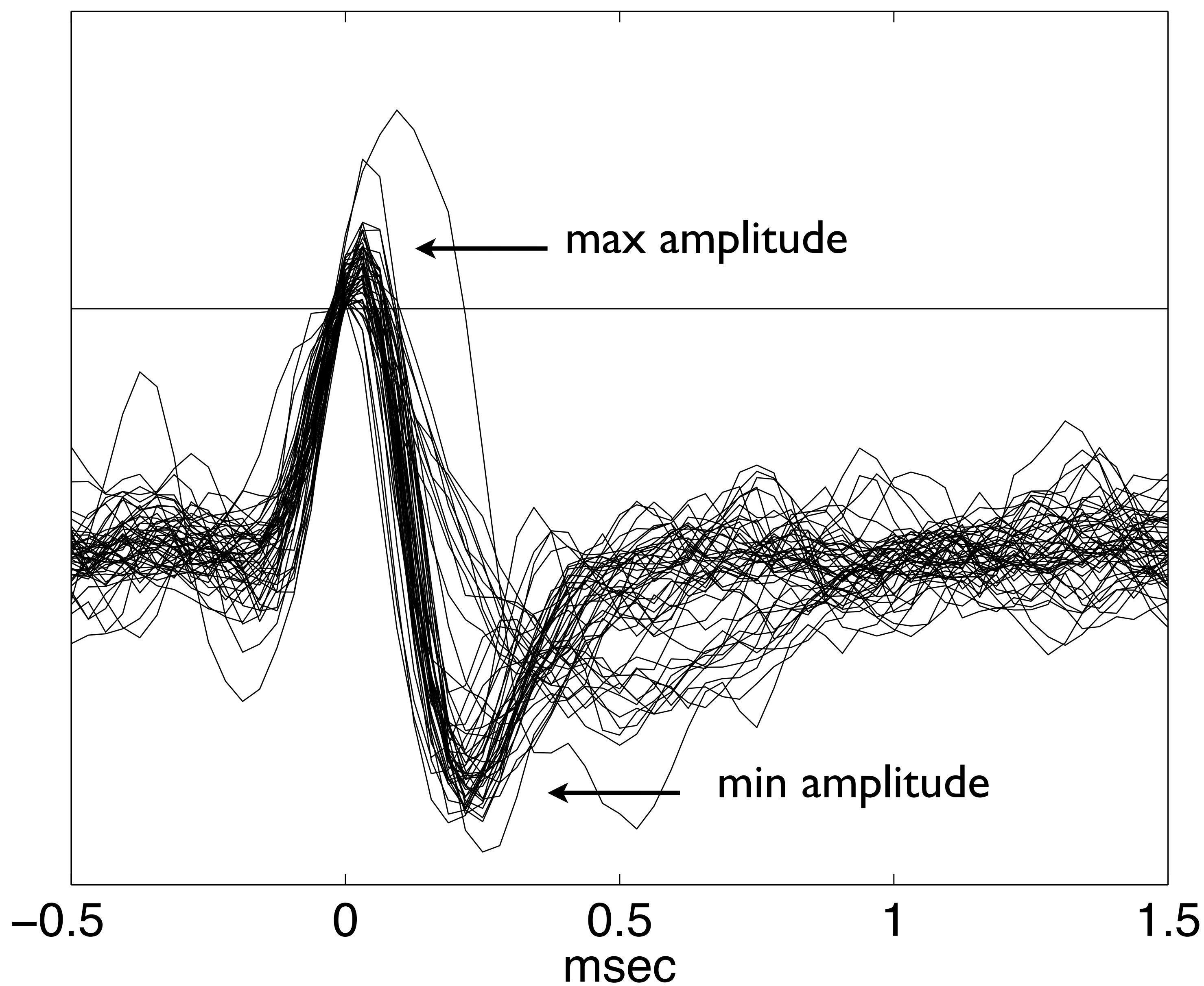
Why level detection doesn't work



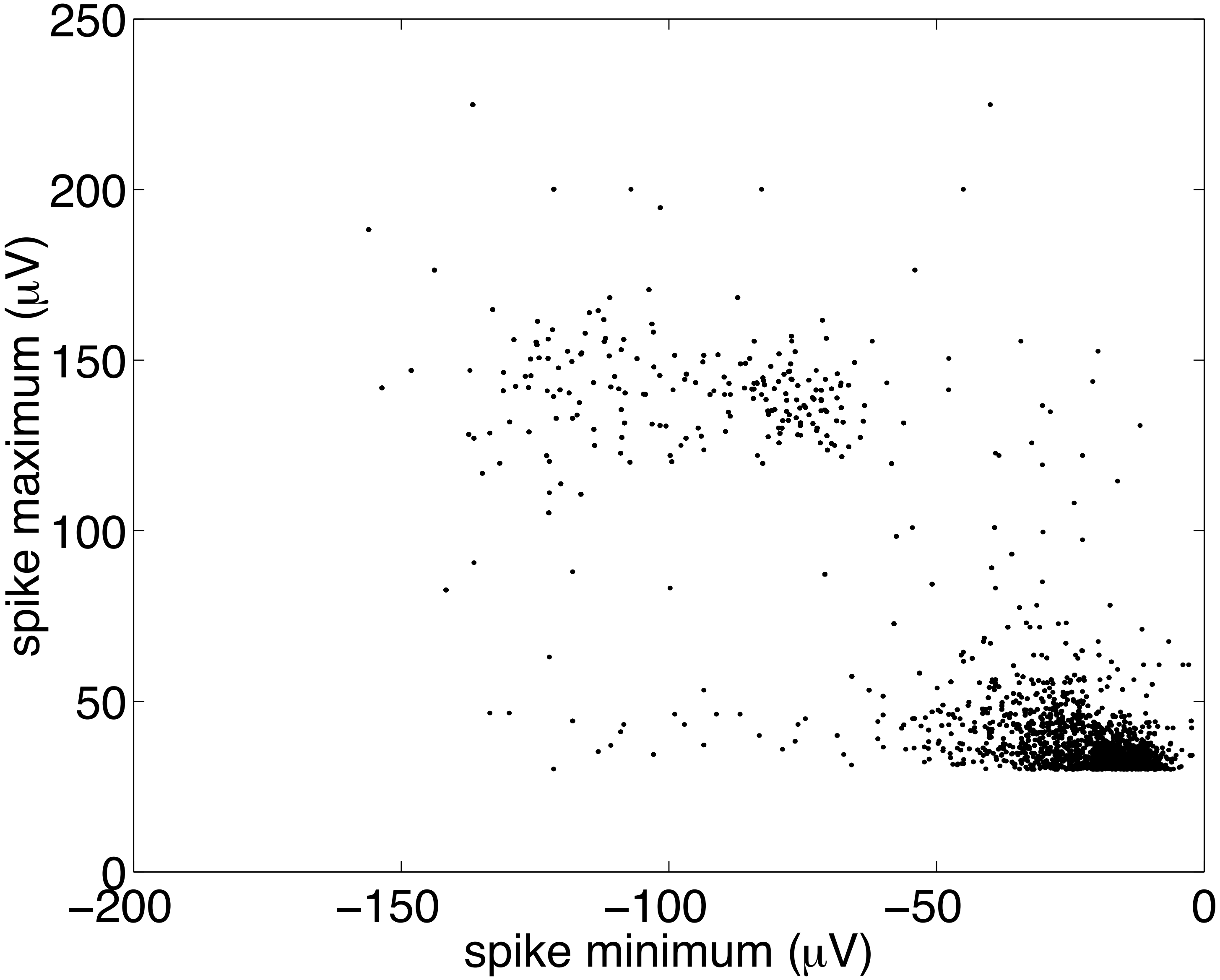


One dimension is not sufficient to separate the spikes.

Idea: try more features

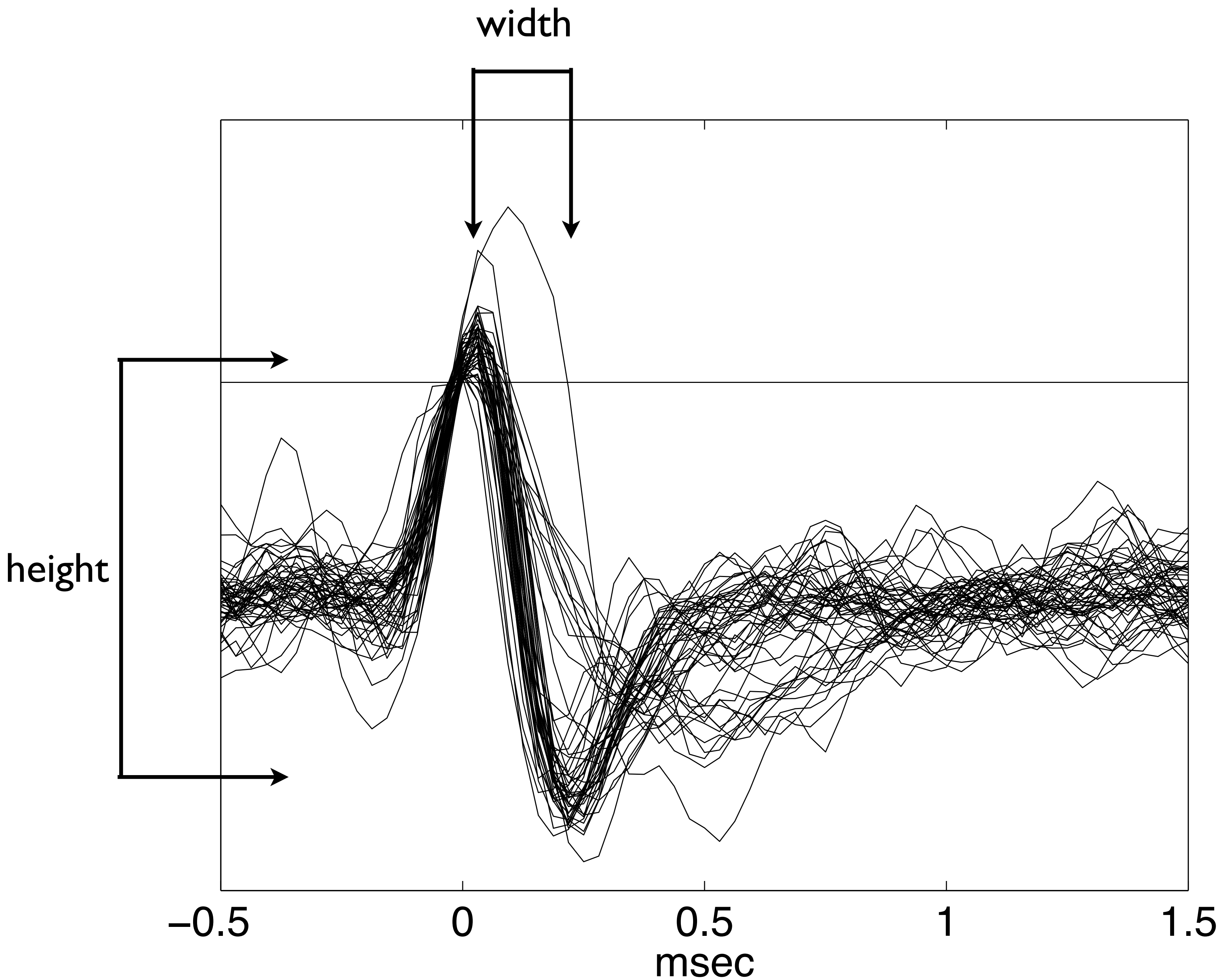


Maximum vs minimum

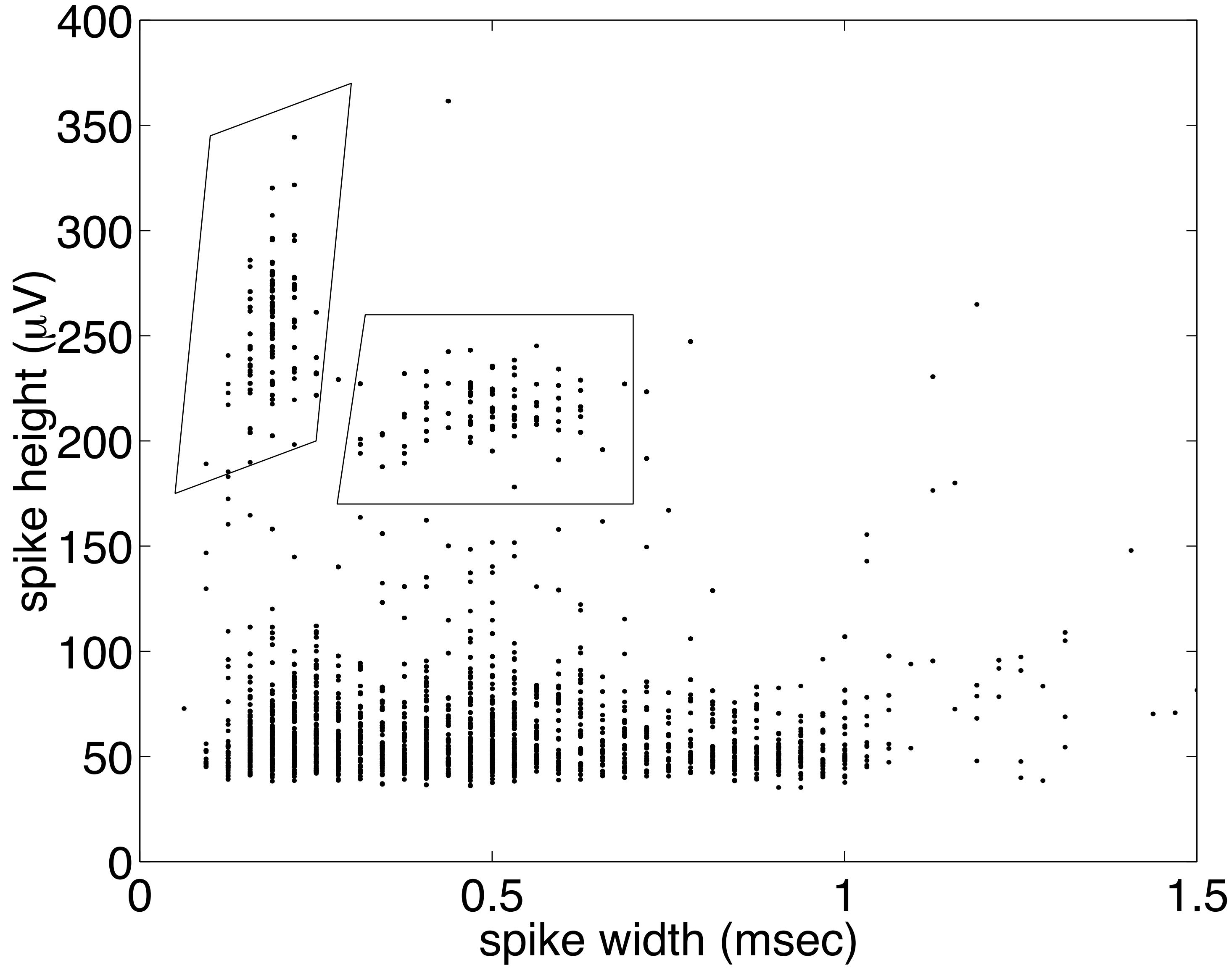


This allows better discrimination than max alone, but is it optimal?

Try different features



Height vs width



This allows allows better discrimination.

How can we choose more objectively?

Foray: dimensionality reduction

(by modeling data with a multi-variate normal distribution)

“Modeling” data with a Gaussian

- a Gaussian (or normal) distribution is “fit” to data with things you’re already familiar with.

$$p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

mean $$\mu = \frac{1}{N}\sum_n x_n$$

variance $$\sigma^2 = \frac{1}{N}\sum_n (x_n - \mu)^2$$

- A multivariate normal is the same but in d-dimensions

$$\mathcal{N}(\boldsymbol{\mu},\boldsymbol{\Sigma}) = (2\pi)^{-d/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

sample mean $$\hat{\boldsymbol{\mu}} = \frac{1}{N}\sum_n \mathbf{x}_n$$

sample covariance $$\hat{\Sigma}_{ij} = \frac{1}{N-1}\sum_n (x_{i,n} - \hat{\boldsymbol{\mu}})(x_{j,n} - \hat{\boldsymbol{\mu}})$$

Multivariate covariance matrices and principal components

Head measurements on two college-age groups (Bryce and Barker):

1) football players (30 subjects)

2) non-football players (30 subjects)

Use six different measurements:

variable	measurement
WDMI	head width at widest dimention
CIRCUM	head circumference
FBEYE	front to back at eye level
EYEHD	eye to top of head
EARHD	ear to top of head
JAW	jaw width

Are these measures independent?

The covariance matrix

$$\mathbf{S} = \begin{bmatrix} .370 & .602 & .149 & .044 & .107 & .209 \\ .602 & 2.629 & .801 & .666 & .103 & .377 \\ .149 & .801 & .458 & .012 & -.013 & .120 \\ .044 & .666 & .011 & 1.474 & .252 & -.054 \\ .107 & .103 & -.013 & .252 & .488 & -.036 \\ .209 & .377 & .120 & -.054 & -.036 & .324 \end{bmatrix}$$

$$S_{ij} = \frac{1}{N-1} \sum_{n=1}^{N} (x_{i,n} - \bar{x_i})(x_{j,n} - \bar{x_j})$$

The corresponding eigenvectors.
In matlab:
```
[V,D]=eig(S); lambda=diag(D);
```

$$\mathbf{V} = \begin{bmatrix} .370 & .602 & .149 & .044 & .107 & .209 \\ .602 & 2.629 & .801 & .666 & .103 & .377 \\ .149 & .801 & .458 & .012 & -.013 & .120 \\ .044 & .666 & .011 & 1.474 & .252 & -.054 \\ .107 & .103 & -.013 & .252 & .488 & -.036 \\ .209 & .377 & .120 & -.054 & -.036 & .324 \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} 3.326 \\ 1.374 \\ 0.475 \\ 0.328 \\ 0.088 \\ 0.157 \end{bmatrix}$$

The eigenvalues

Eigenvalue	Proportion of Variance	Cumulative Proportion
3.323	.579	.579
1.374	.239	.818
.476	.083	.901
.325	.057	.957
.157	.027	.985
.088	.015	1.000

How many PCs should we select?

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How many PCs should we select?

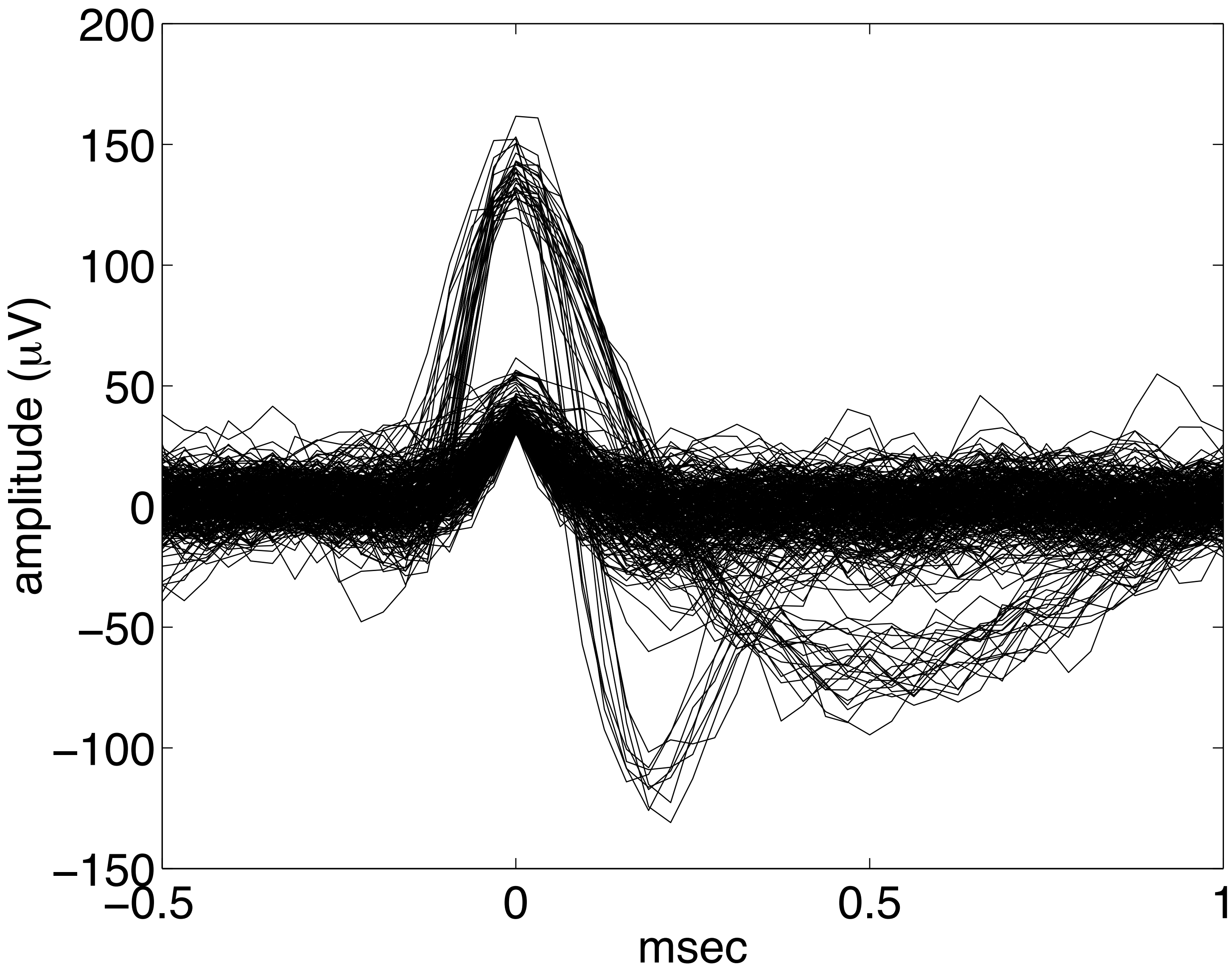
The first two principal components capture 81.8% of the variance.

The corresponding eigenvectors:

variable	$\mathbf{a}_1$	$\mathbf{a}_2$
WDMI	.207	-.142
CIRCUM	.873	-.219
FBEYE	.261	-.231
EYEHD	.326	.891
EARHD	.066	.222
JAW	.128	-.187

Using principal components to characterize the data

- What are the data?

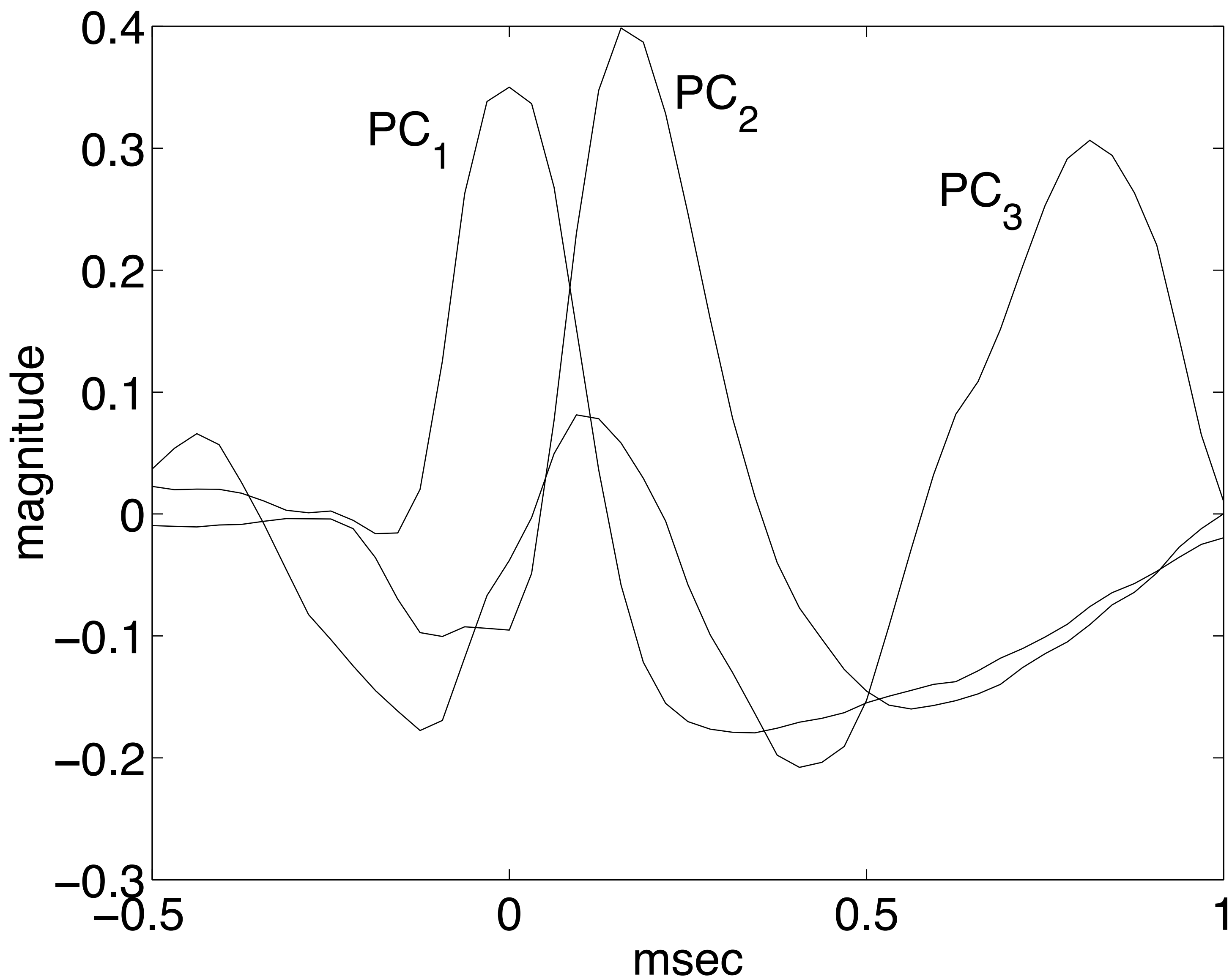


What is the dimensionality of the data?

How many components are there?

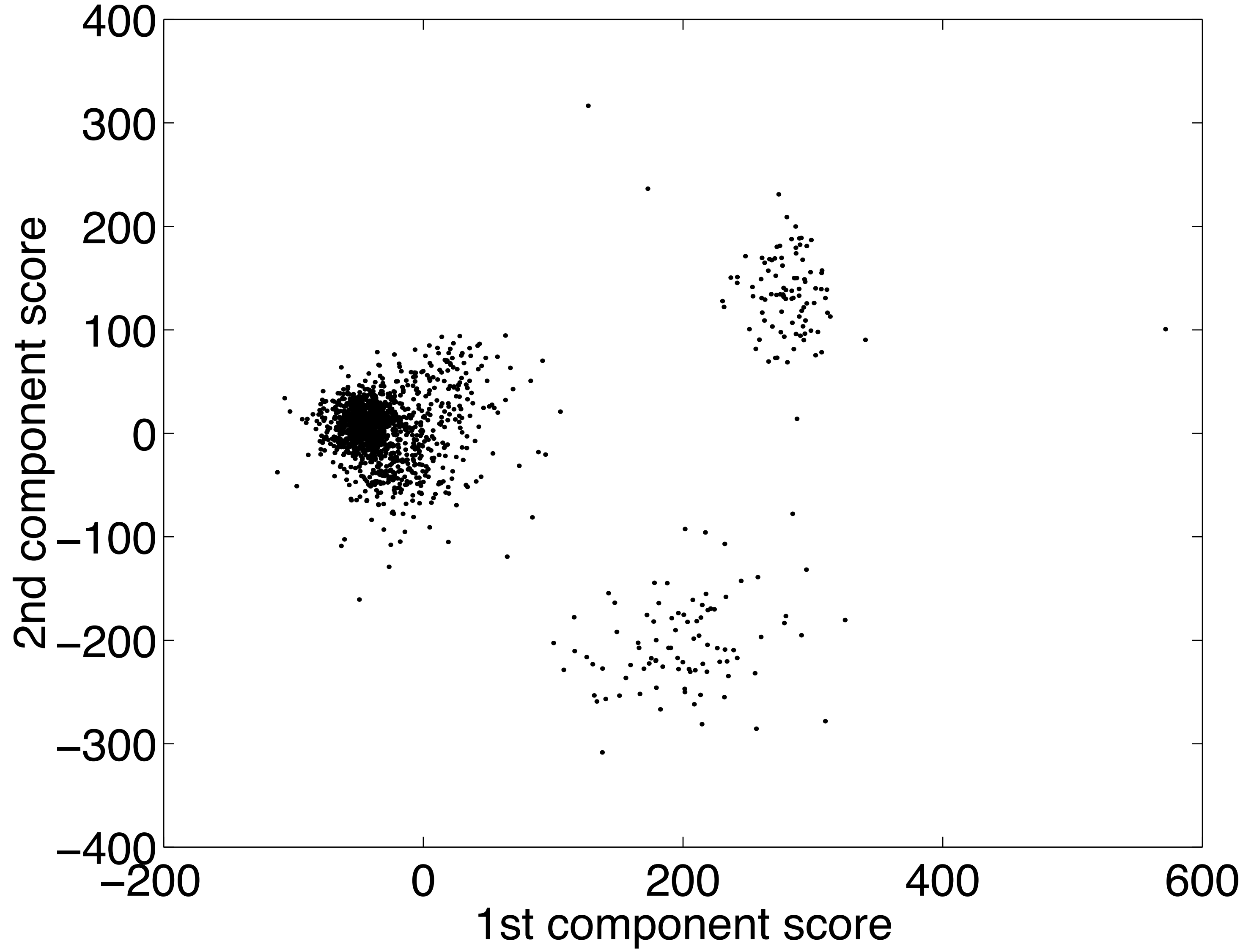
What will the components look like?

The first three principal components of the waveform data



What do you expect when we plot PC1 vs PC2 ?

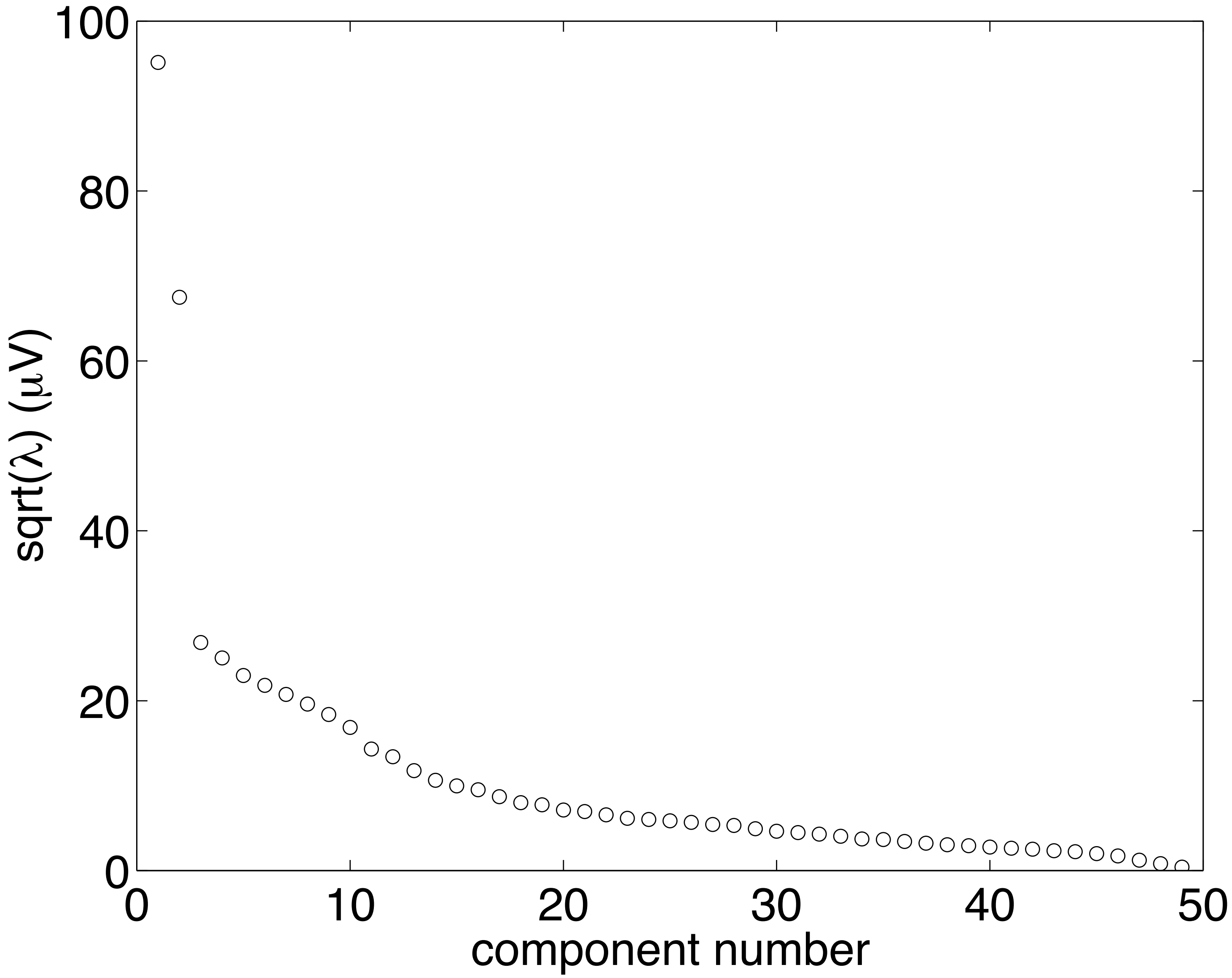
Scatter plot of the first two principal component scores



Now the data are much better separated.

Could we use more PCs? How many?

The eigenspectrum of the waveform data



Recall example from last lecture: waveform data

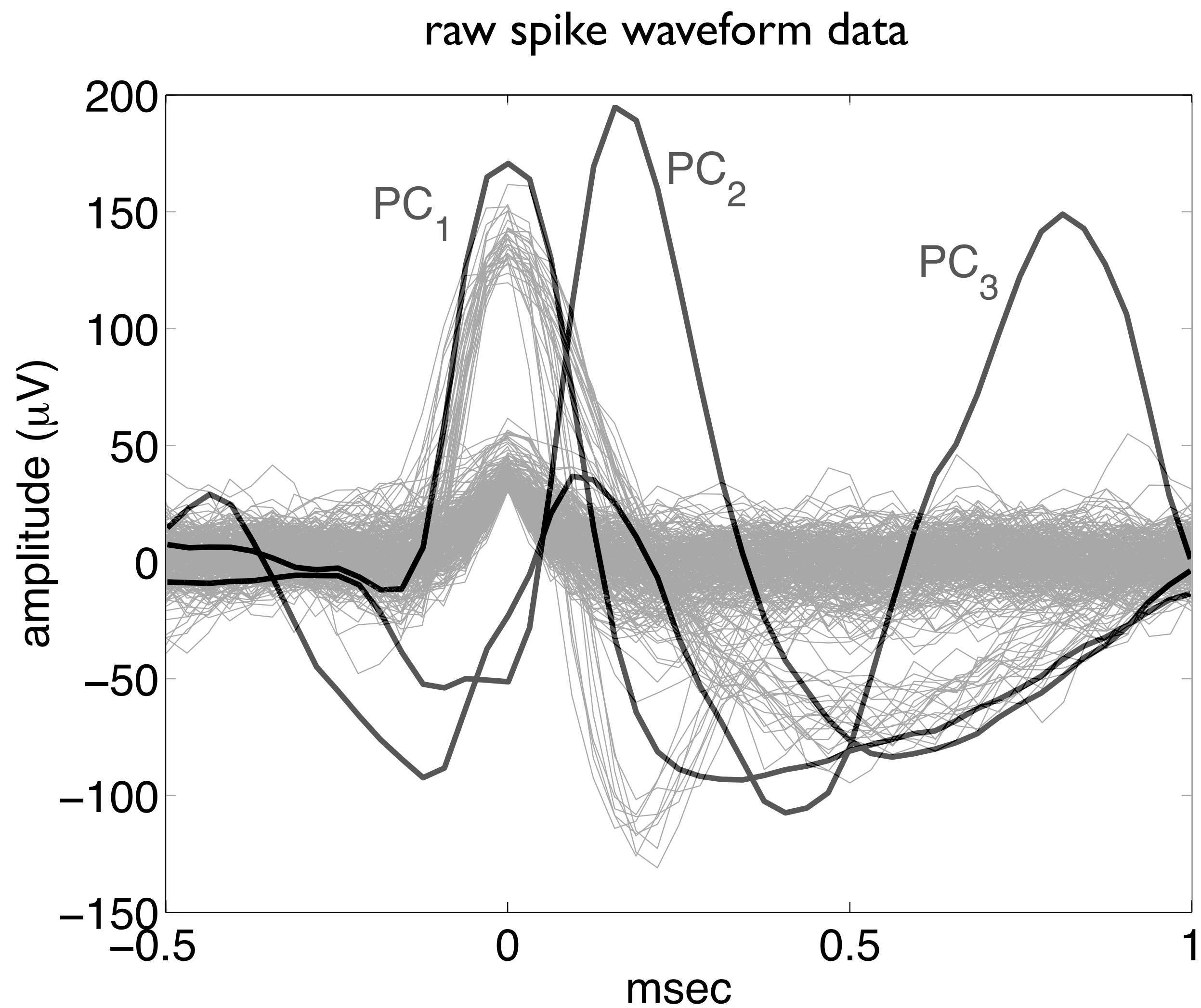
- The waveform x is modeled with a multivariate Gaussian

$$x \sim p(x|\mu, \Sigma)$$

- μ and Σ are mean and covariance of the distribution.
- Principal components can be used to form a low-dimensional approximation

$$x^{(n)} = \sum_{i=1}^{T} c_i^{(n)} \phi_i$$

- The vectors $\{\phi_1, \ldots, \phi_T\}$ are the eigenvectors of Σ.
- keeping on the first two terms in the sum is an adequate approximation of the full T-dimensional density.



Generalized templates: eigenfaces

Idea: Learn a basis for a specific class of images, e.g. faces (Turk and Pentland, 1991).

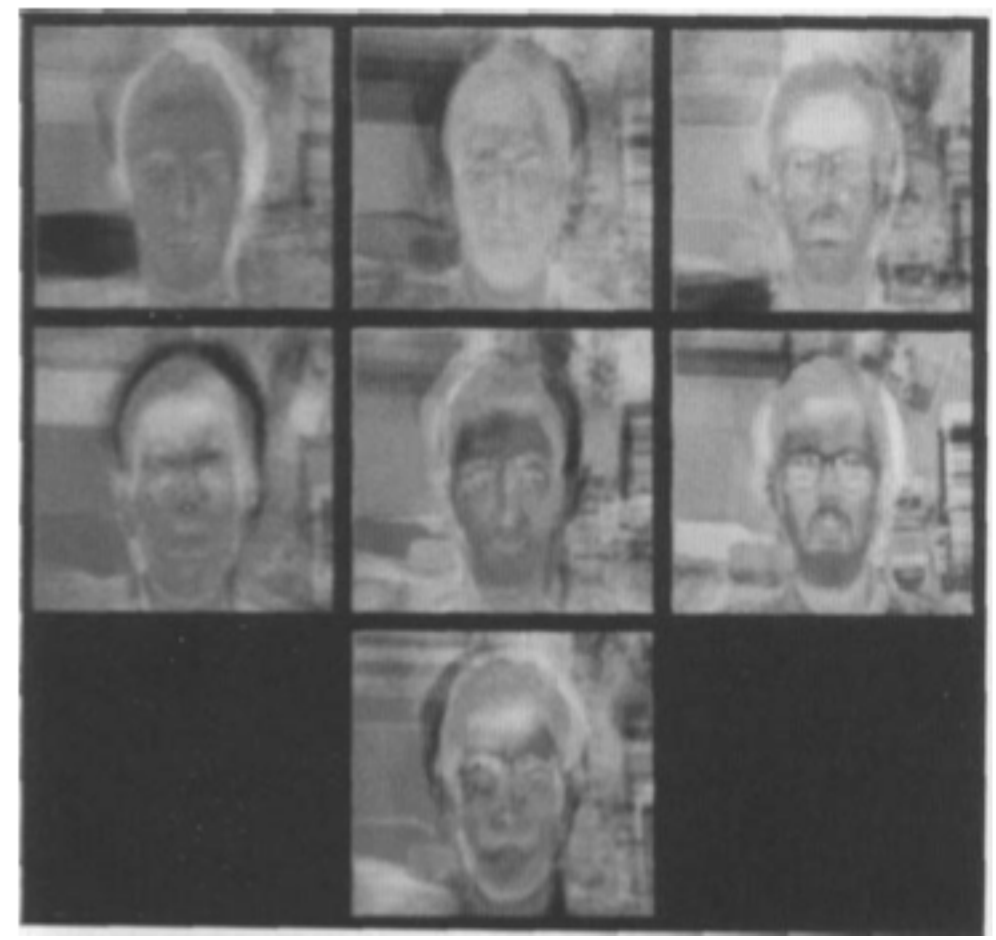
- Normalize object set by position, size, etc. Usually by hand.
- Model the objects using linear superposition:

$$I(x, y) = \sum_i a_i \phi_i(x, y)$$

- Derive optimal basis functions for object set
- Under a Gaussian model, this is PCA, which is feasible for large patterns.

Turk and Pentland (1991)

The average face

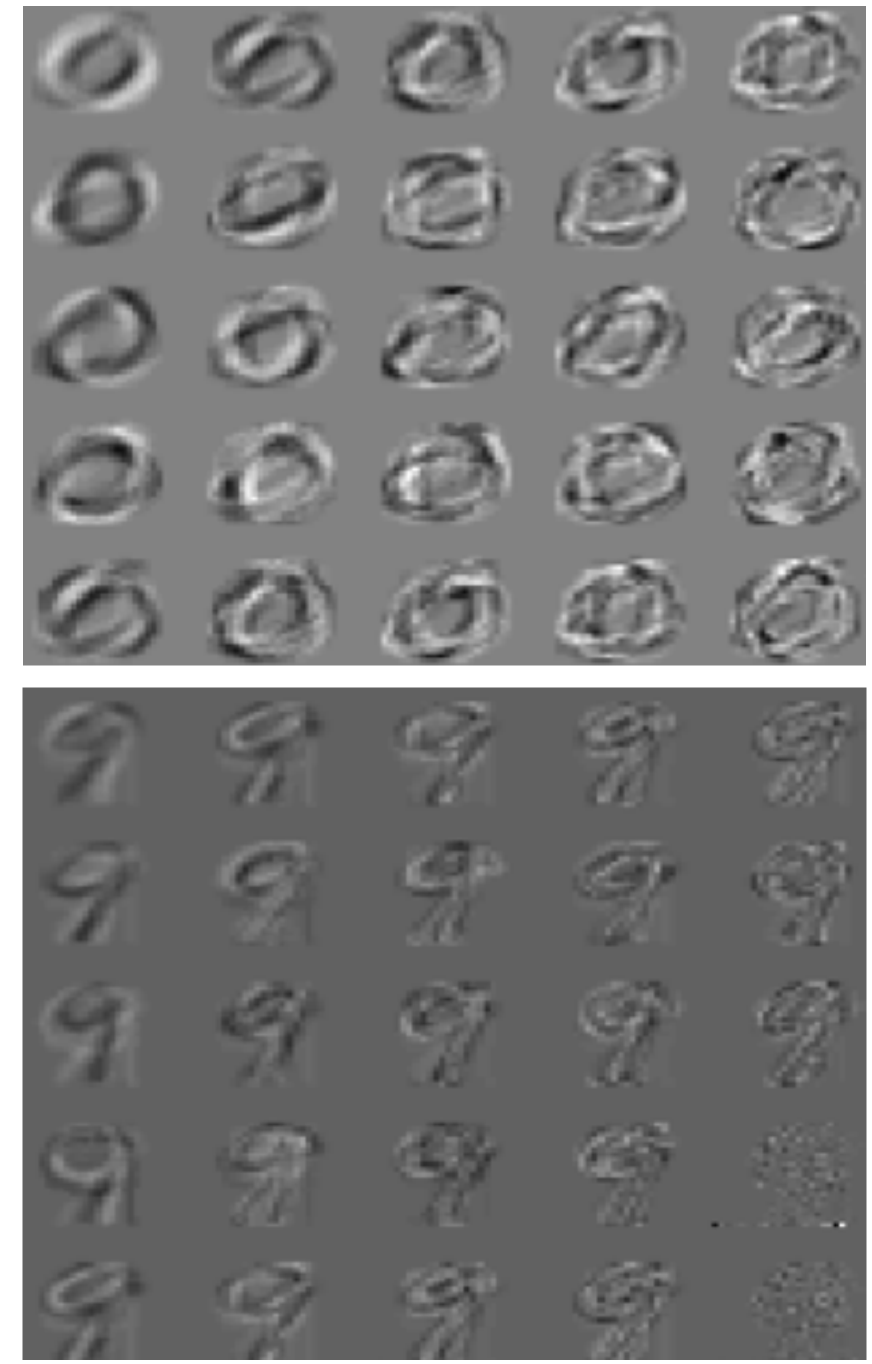


7 eigenfaces

Reconstruction of missing information using eigenfaces

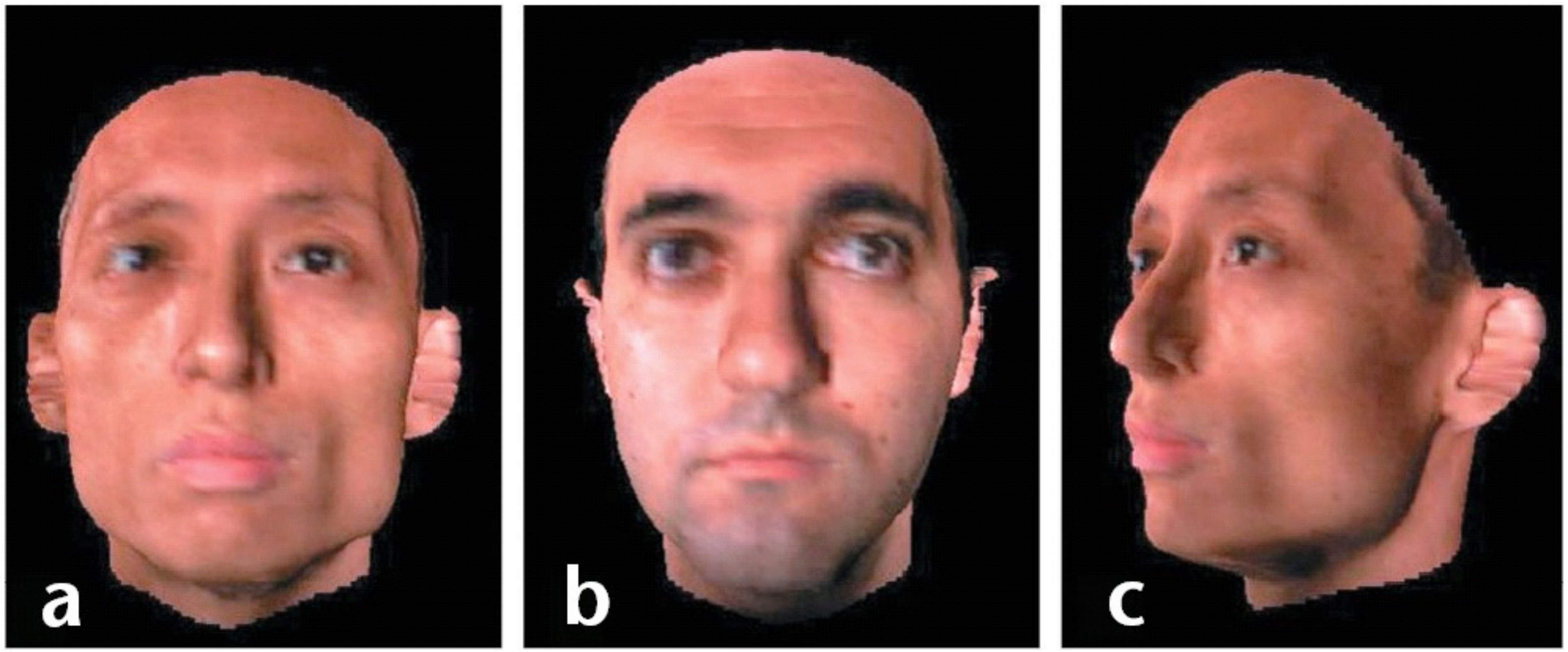
Principal components of digits: Eigendigits

hand-written digits (MNIST)



example eigendigits

Storing templates won’t work



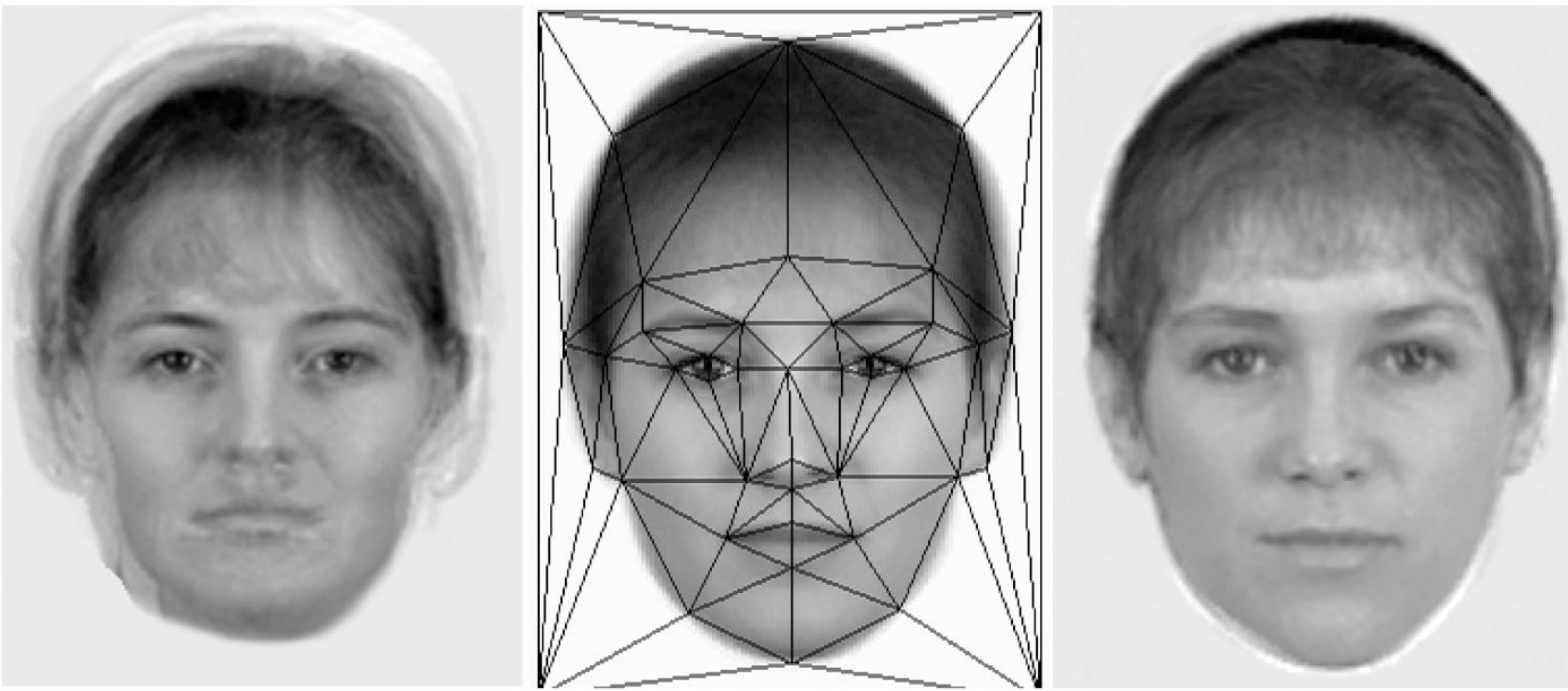
(from Sinha, 2002)

- Which two are more alike?
- What kind of “distance metric” does our visual system use?
- Most computer systems fail to generalize across common transformations:
  - viewpoint
  - lighting
  - expression

Eigenfaces

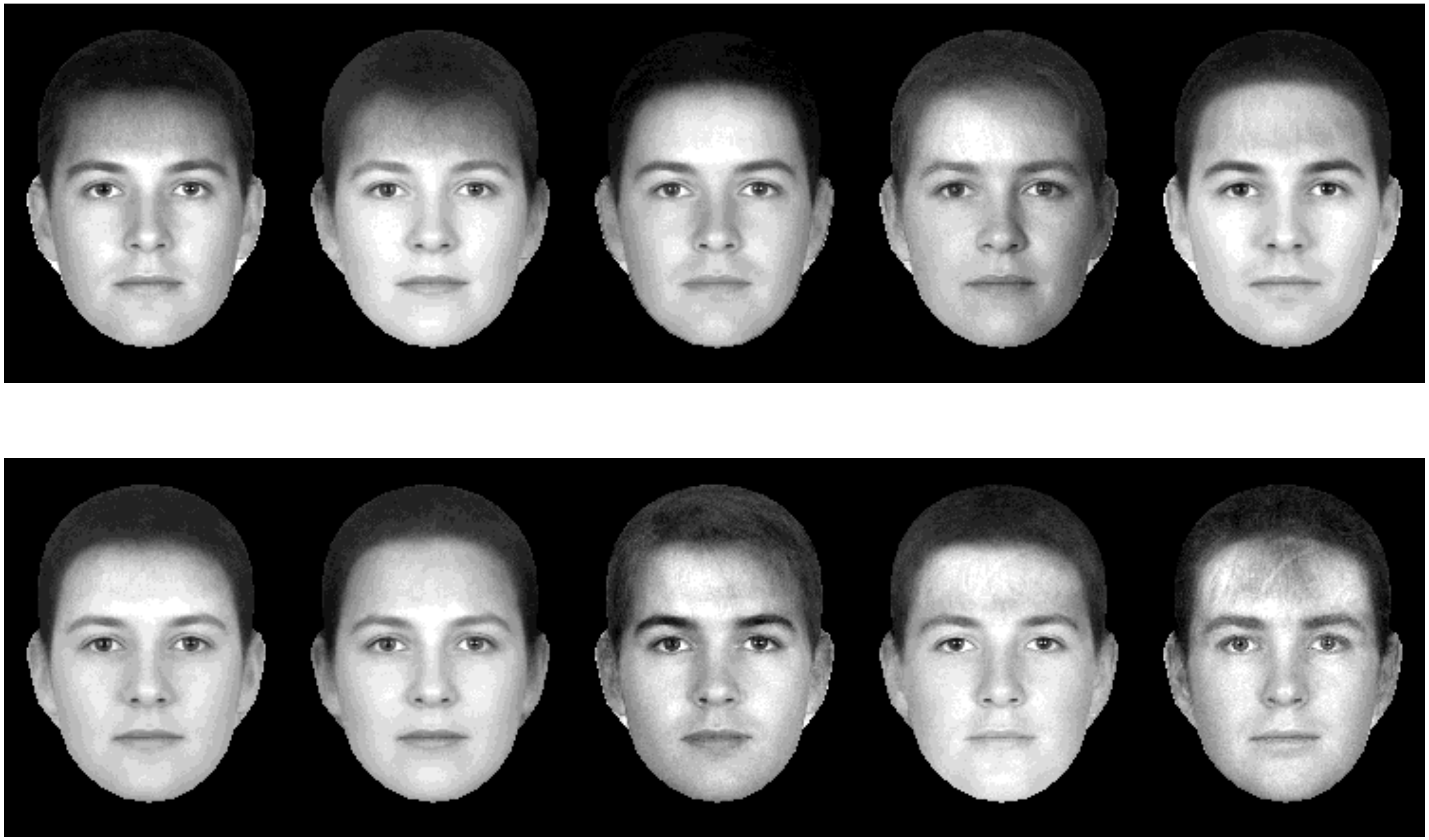
from Hancock, 2000

Eigen face distortions



from Hancock, 2000

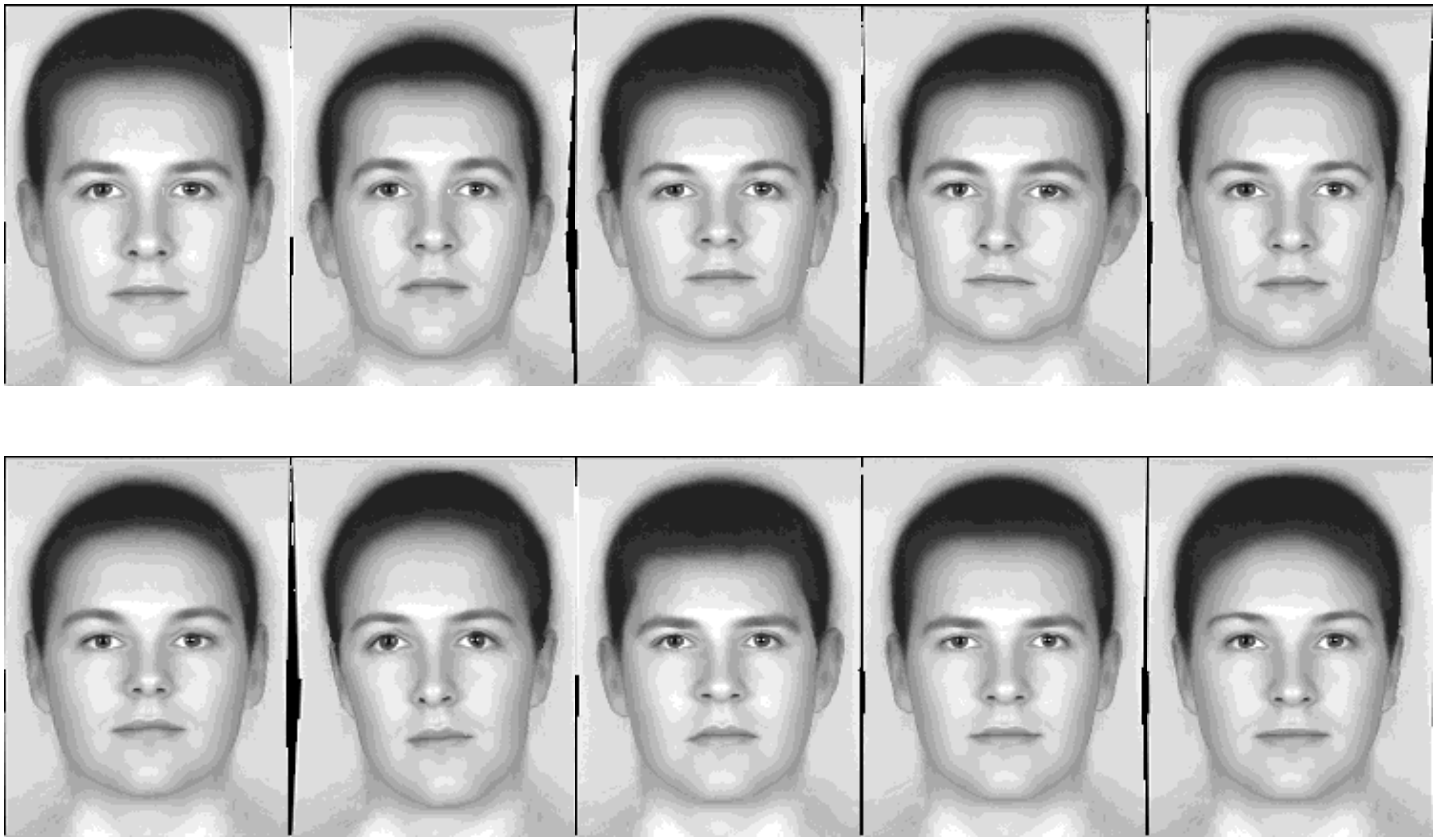
Face image principal components



from Hancock, 2000

The face image components move
back and forth along their axes.

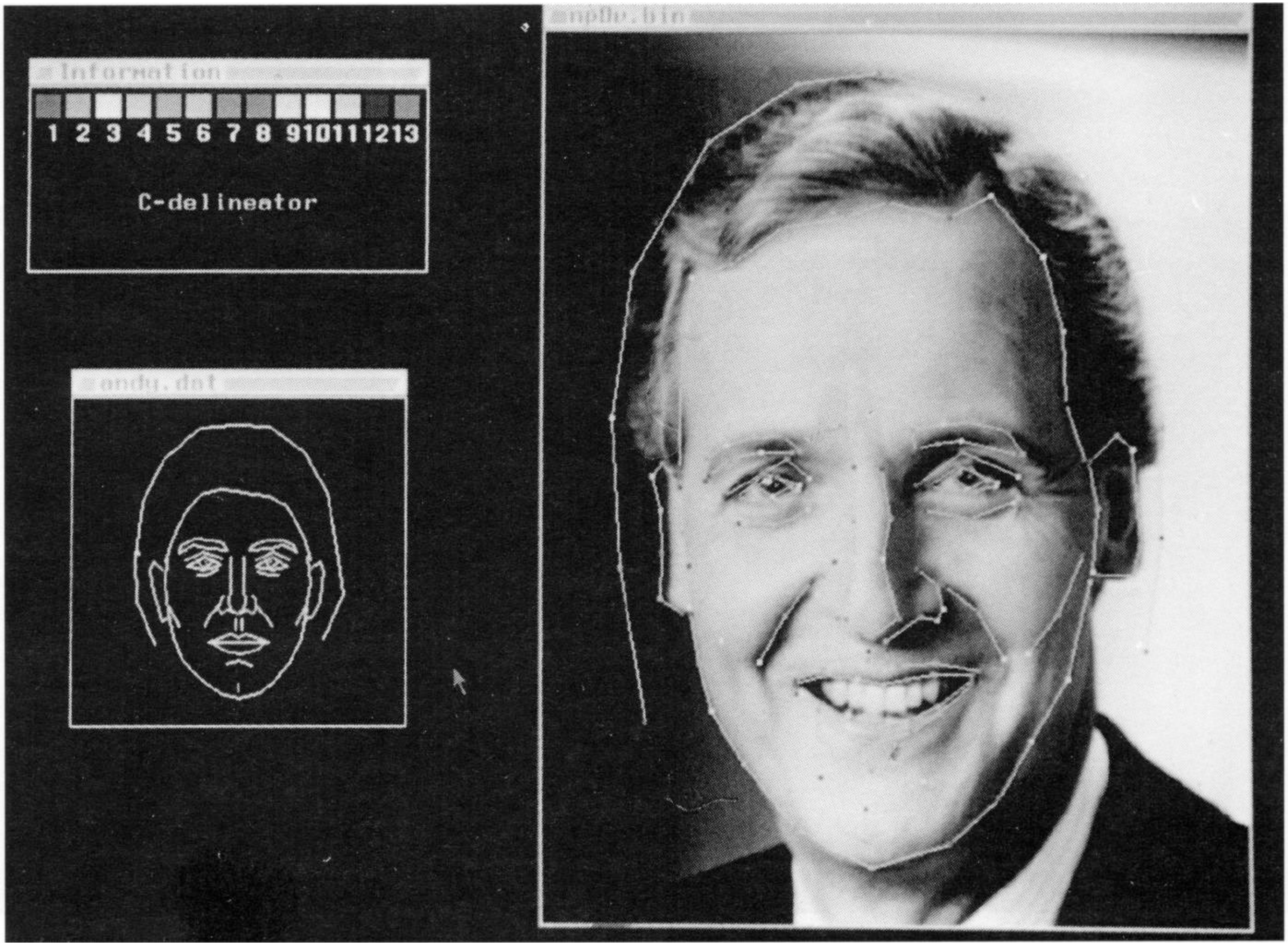
Face shape principal components



from Hancock, 2000

The face shape components move
back and forth along their axes.

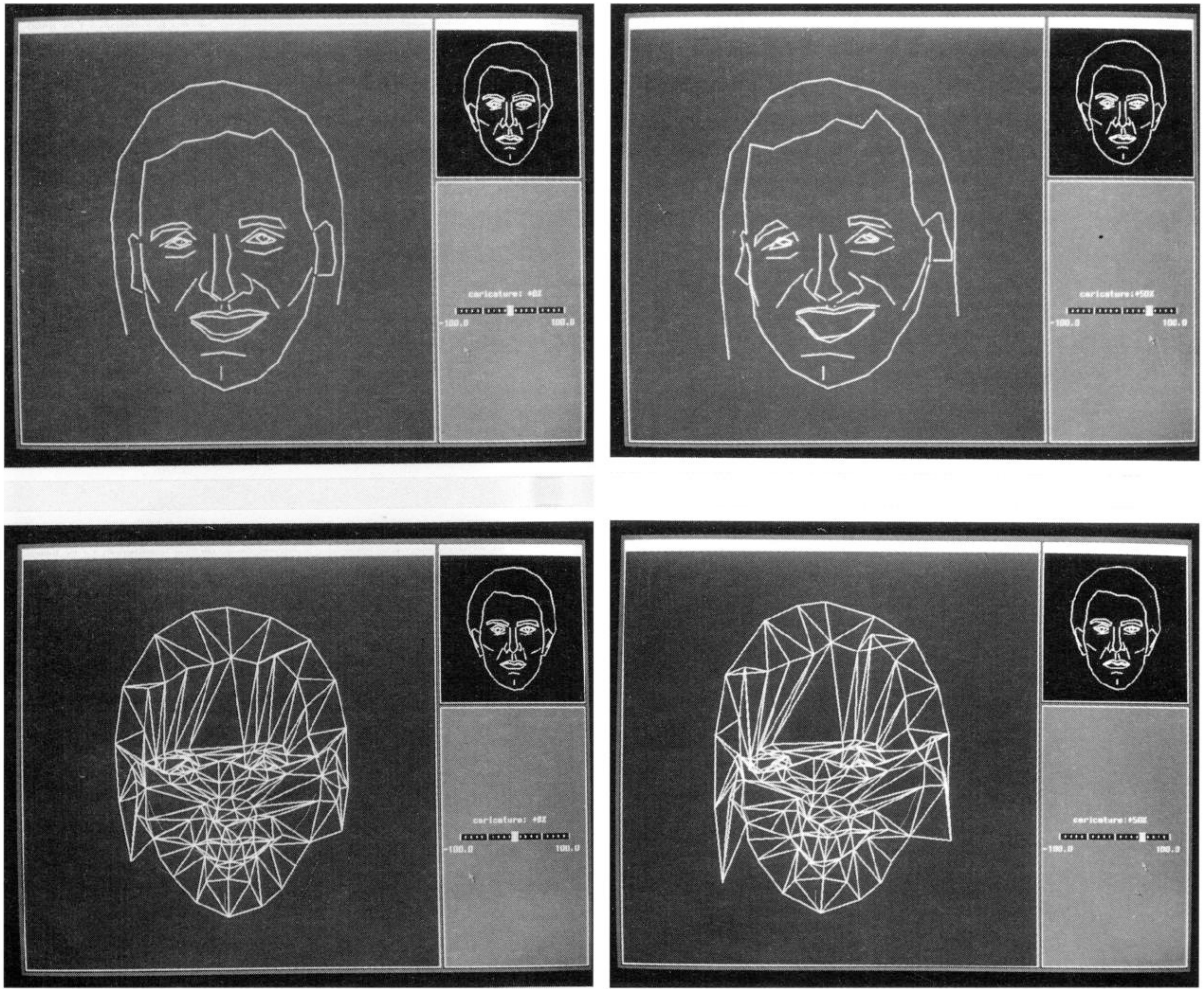
Making a caricature: step 1



from Bruce and Young, 1998

Identify positions of facial features.

Making a caricature: step 2



Distort positions from average.

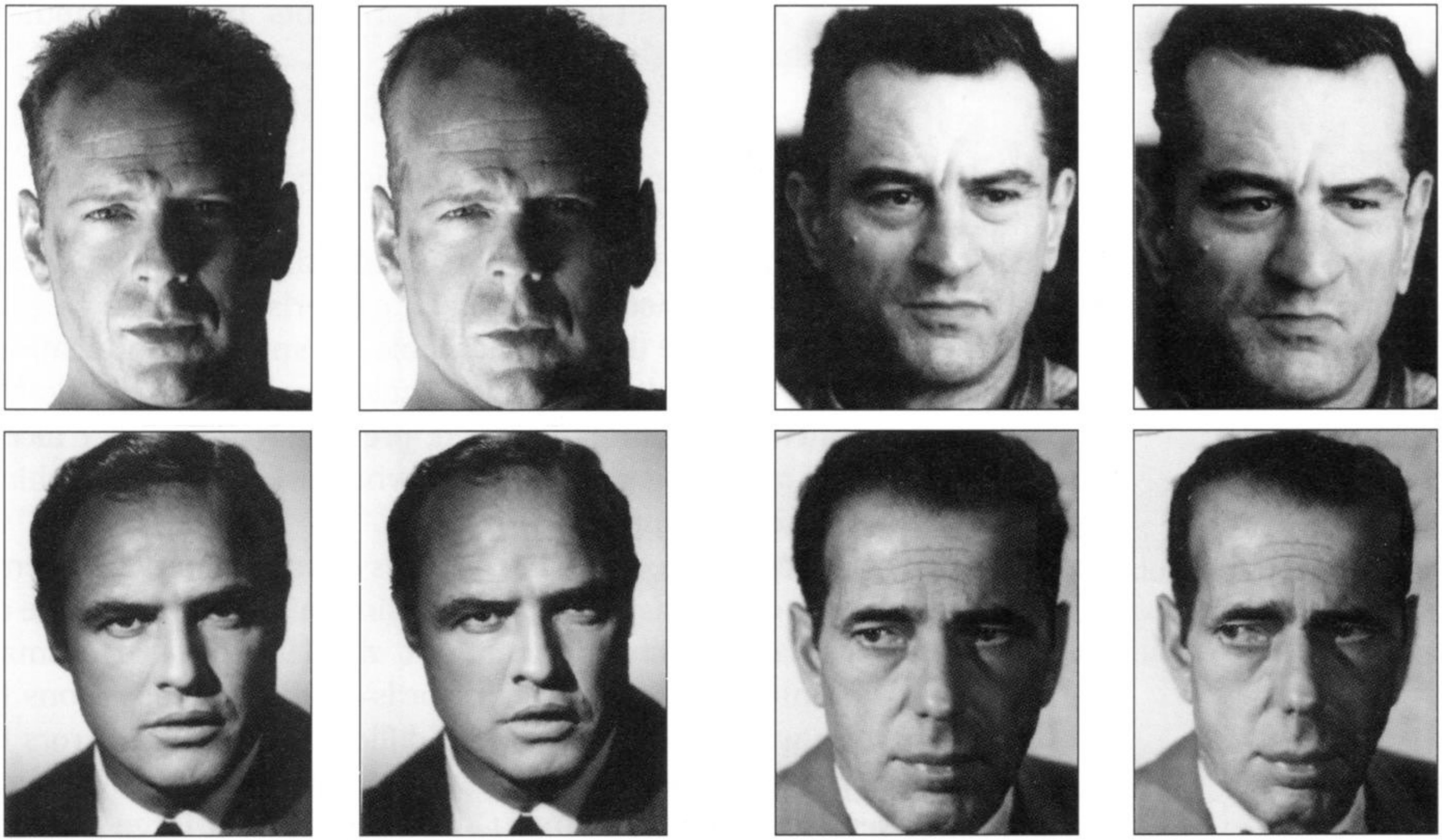
from Bruce and Young, 1998

Making a caricature: step 3

from Bruce and Young, 1998

Morph original face according to caricature distortion.

Computer-generated caricatures of famous actors

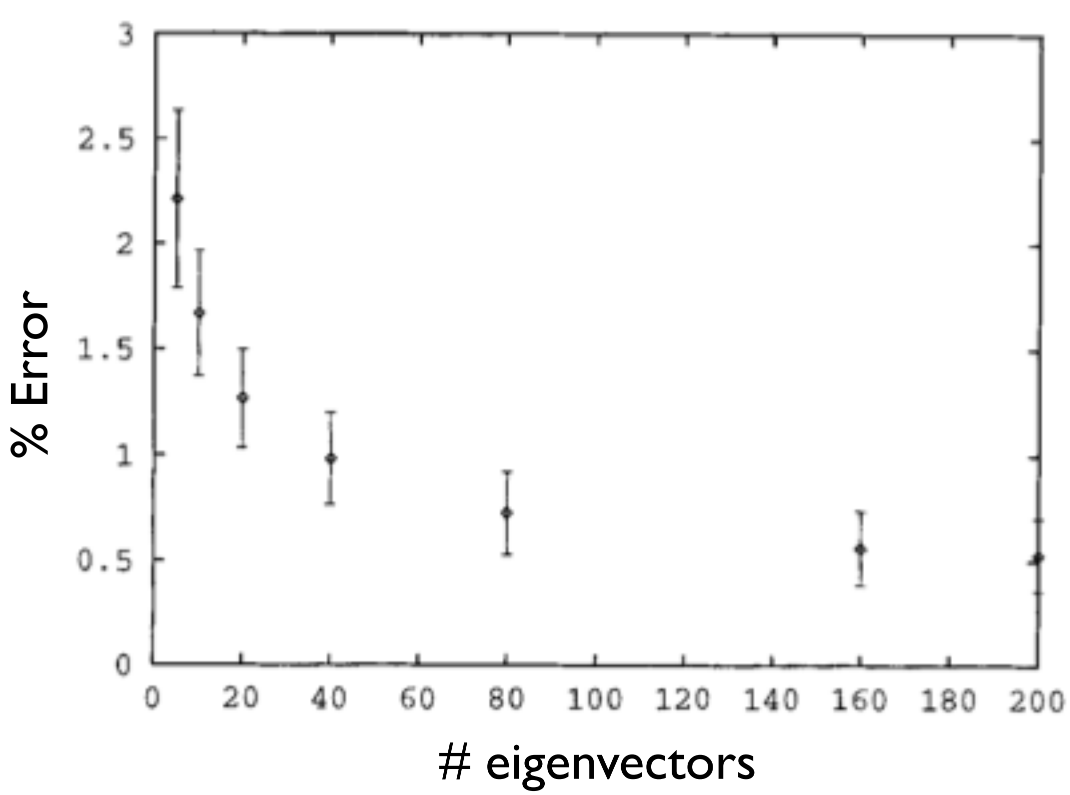


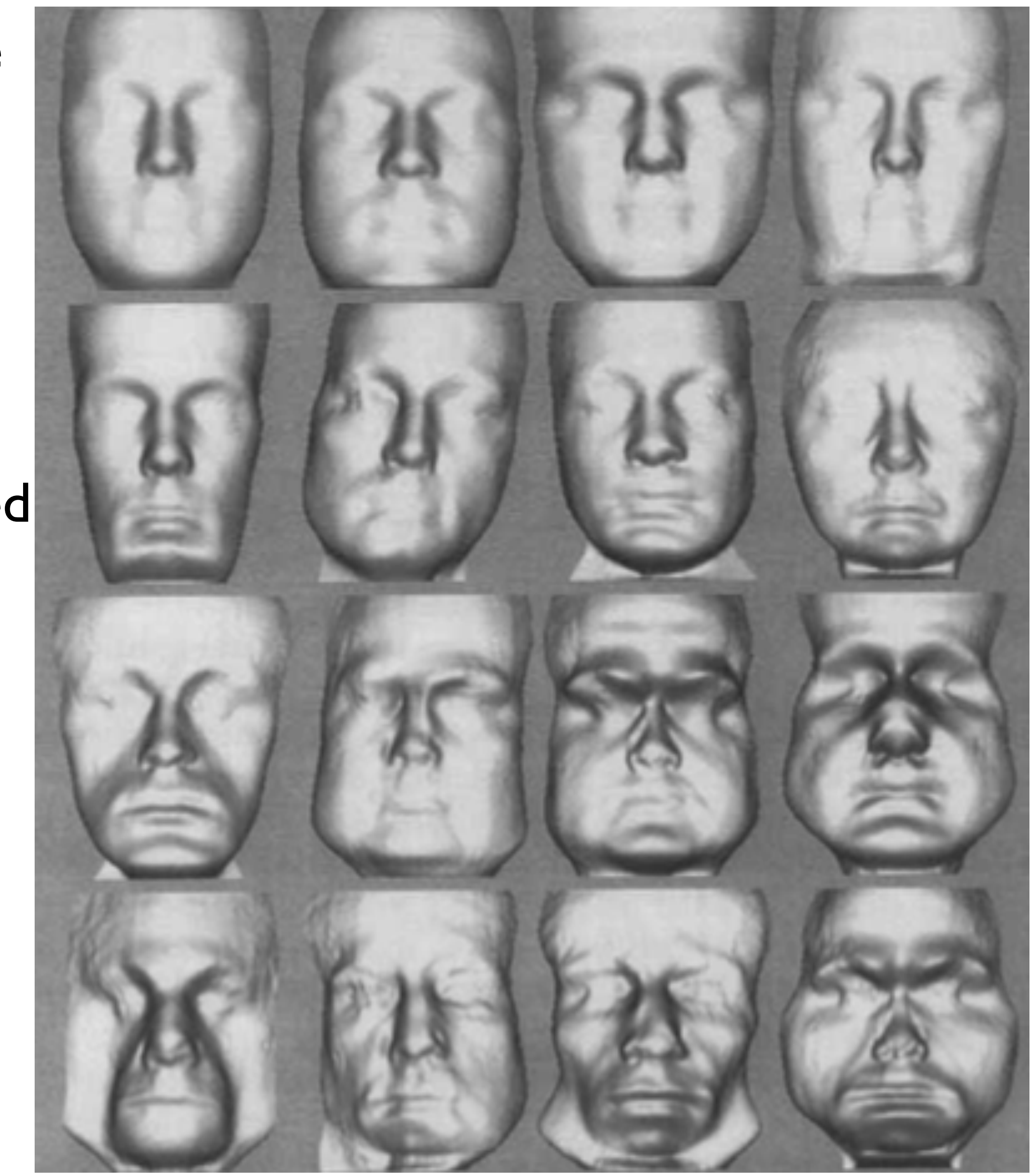
from Bruce and Young, 1998

Caricatures were more recognizable (i.e. faster RT) than originals.

Eigenheads (Atick, Griffin, and Redlich, 1996)

- Statistical Approach to Shape from Shading: Reconstruction of Three-Dimensional Face Surfaces from Single Two-Dimensional Images
- database of laser scans of 347 male heads (256 angular x 200 height)
- image to right shows:
  - mean-head surface (upper left)
  - 15 most significant eigenheads
  - only small percentage are needed

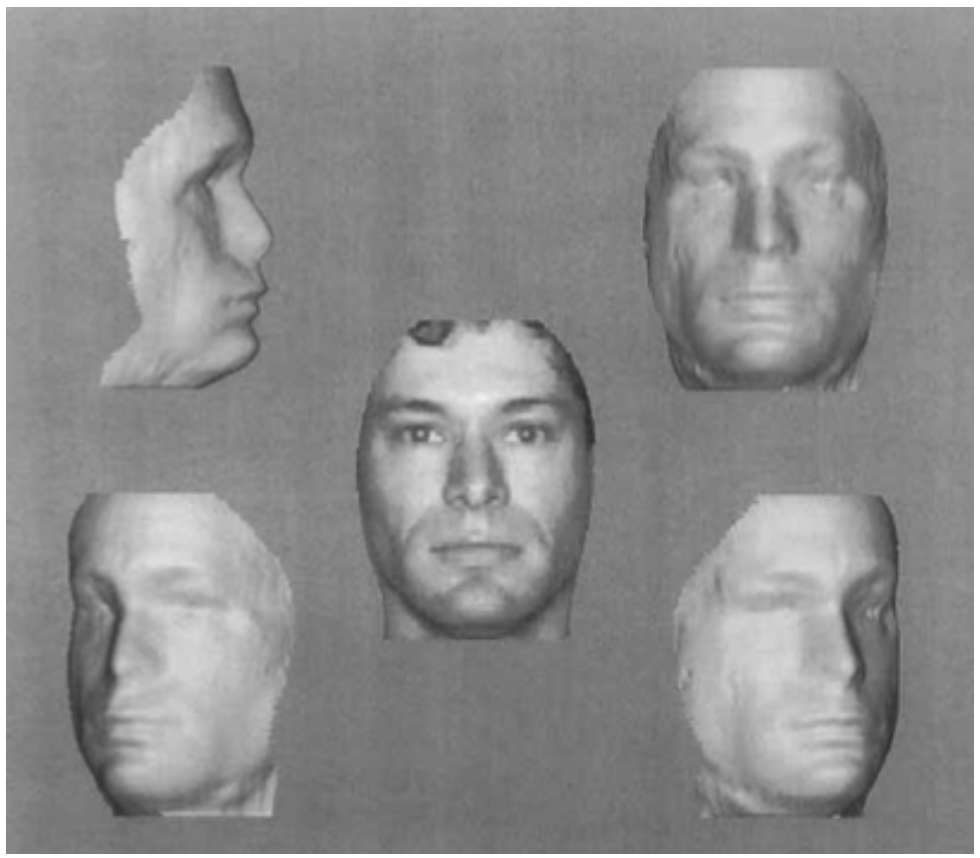




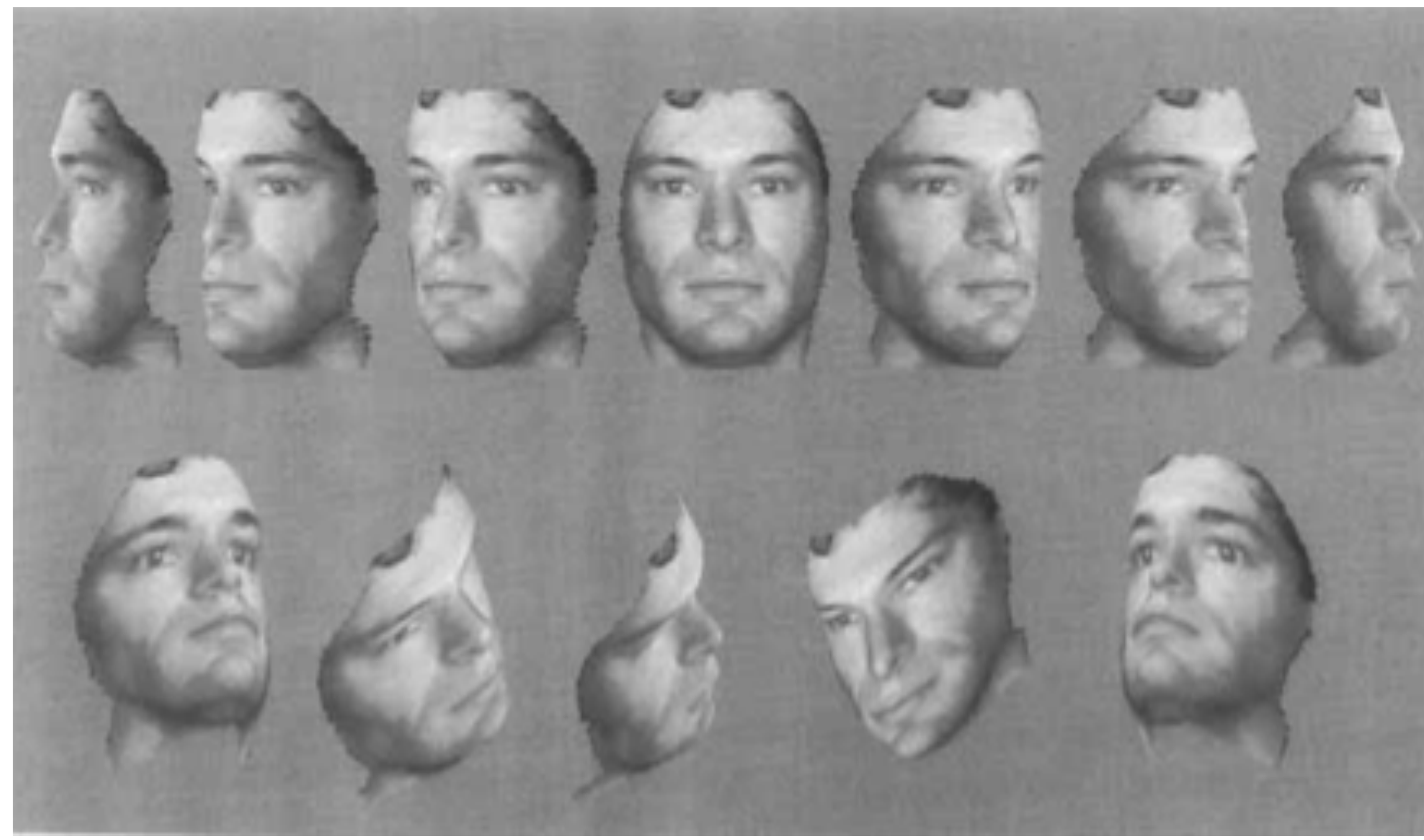
Atick, Griffin, and Redlich (1996)

Reconstructing 3D surface from 2D image

Atick, Griffin, and Redlich (1996)



use eigenheads as prior to reconstruct from 2D image



novel poses generated using inferred head and texture mapping

Blanz and Vetter (2003)

morphable face model based on optic flow

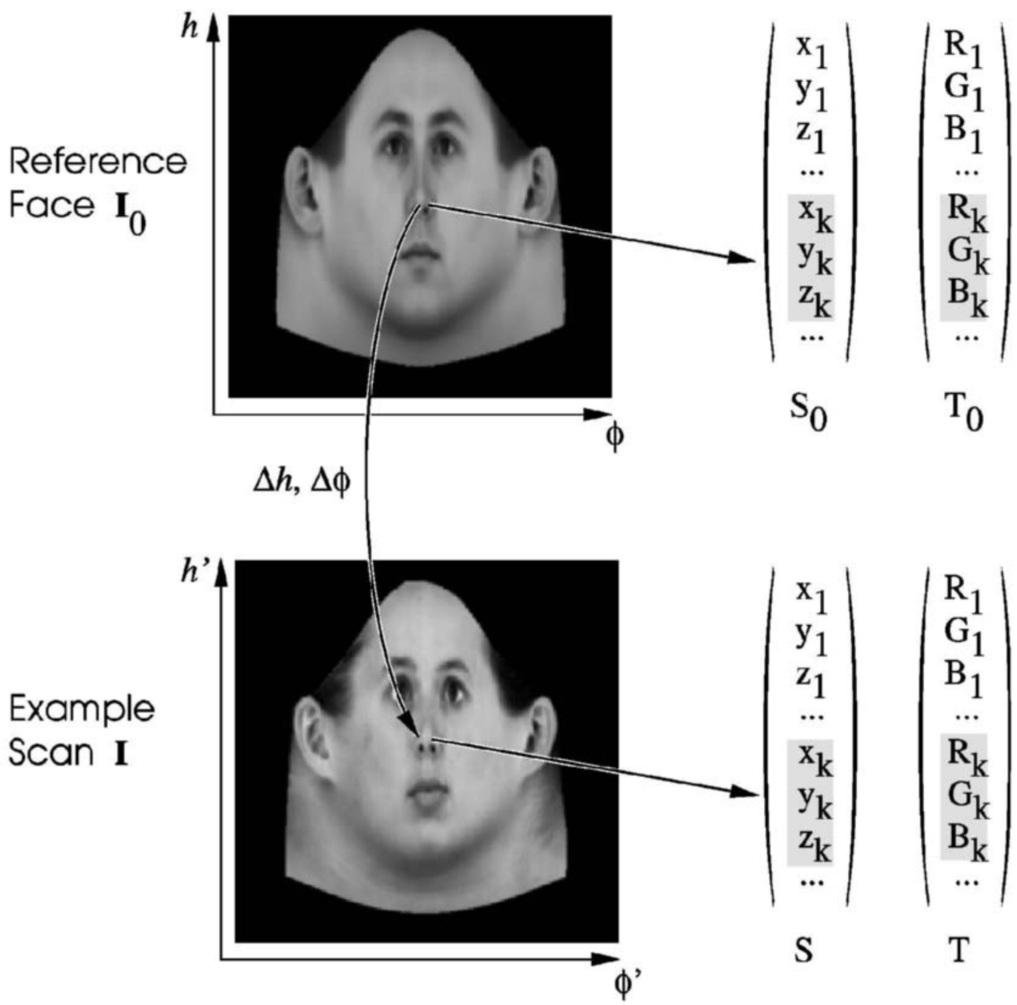


Fig. 3. For 3D laser scans parameterized by cylindrical coordinates (h, ϕ), the flow field that maps each point of the reference face (top) to the corresponding point of the example (bottom) is used to form shape and texture vectors $\mathbf{S}$ and $\mathbf{T}$.

Characterization of face shape and texture using principal components

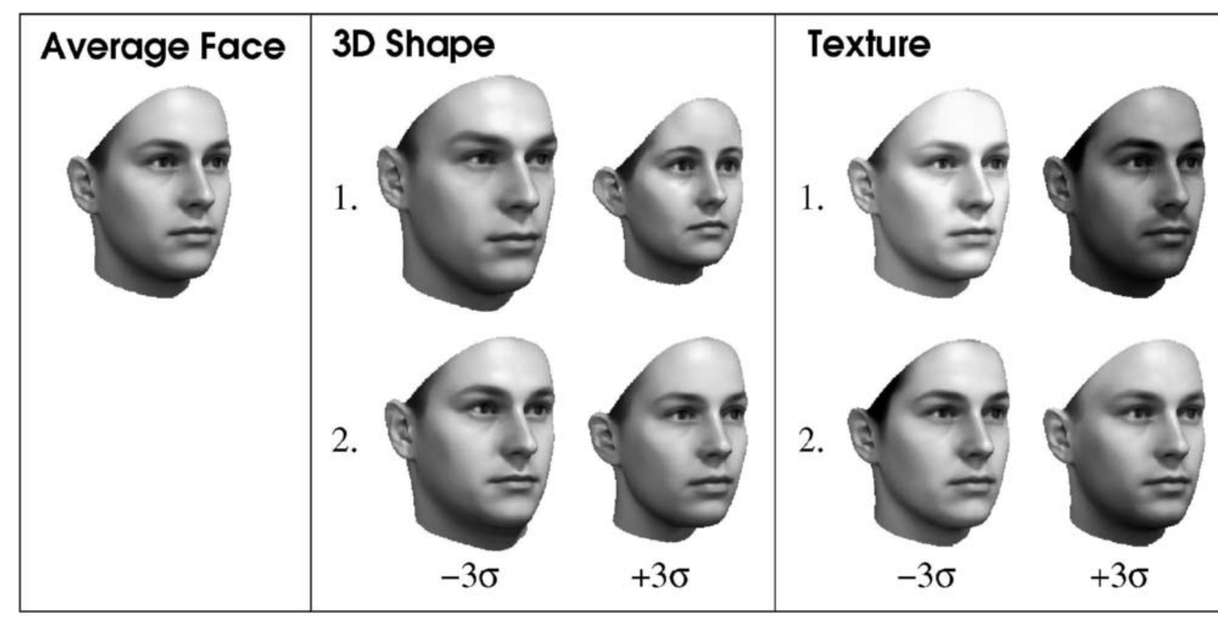
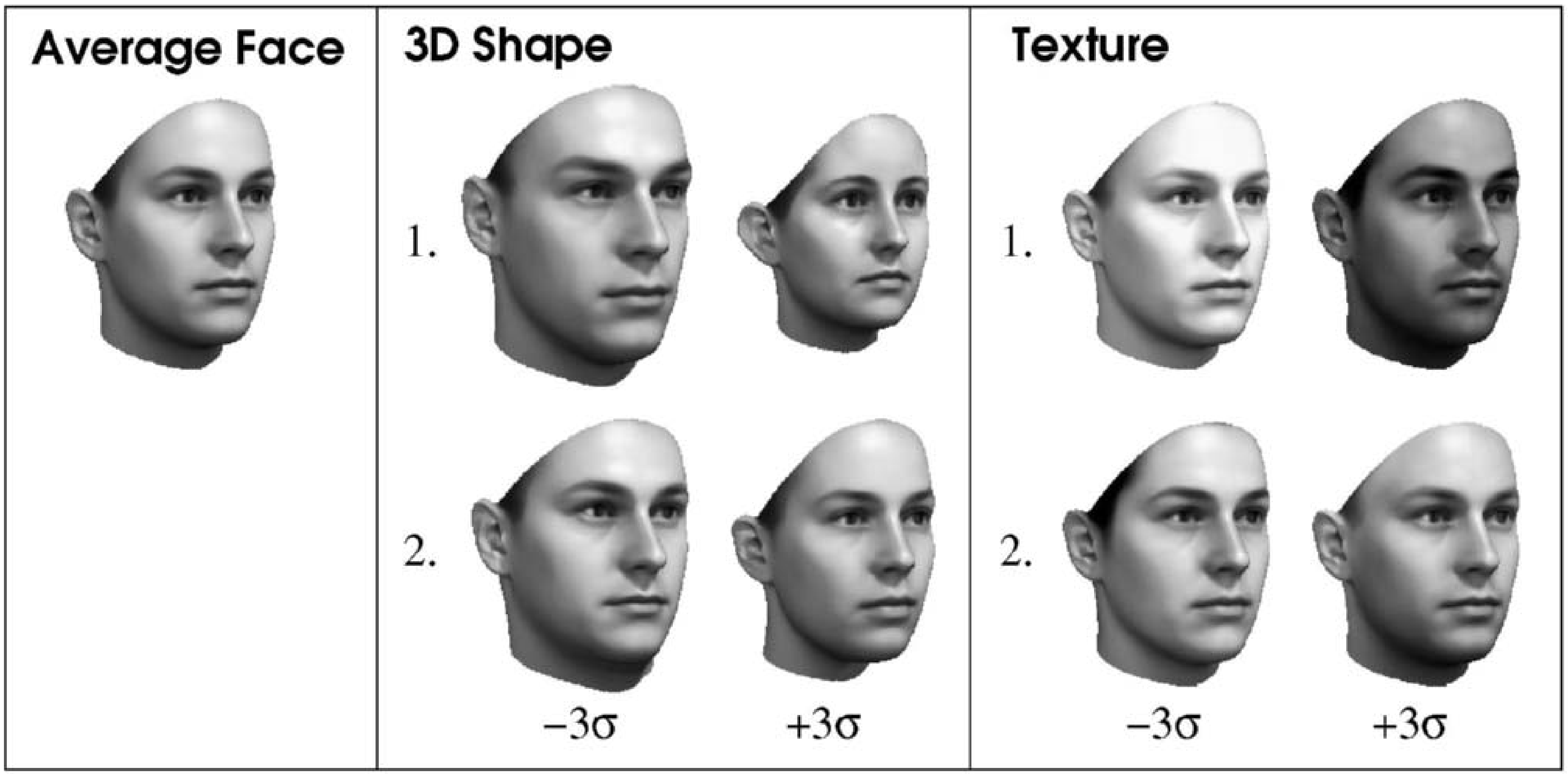


Fig. 4. The average and the first two principal components of a data set of 200 3D face scans, visualized by adding $\pm 3\sigma_{S,i}\mathbf{s}_i$ and $\pm 3\sigma_{T,i}\mathbf{t}_i$ to the average face.

Blanz and Vetter (2003)

Characterization of face shape and texture using principal components



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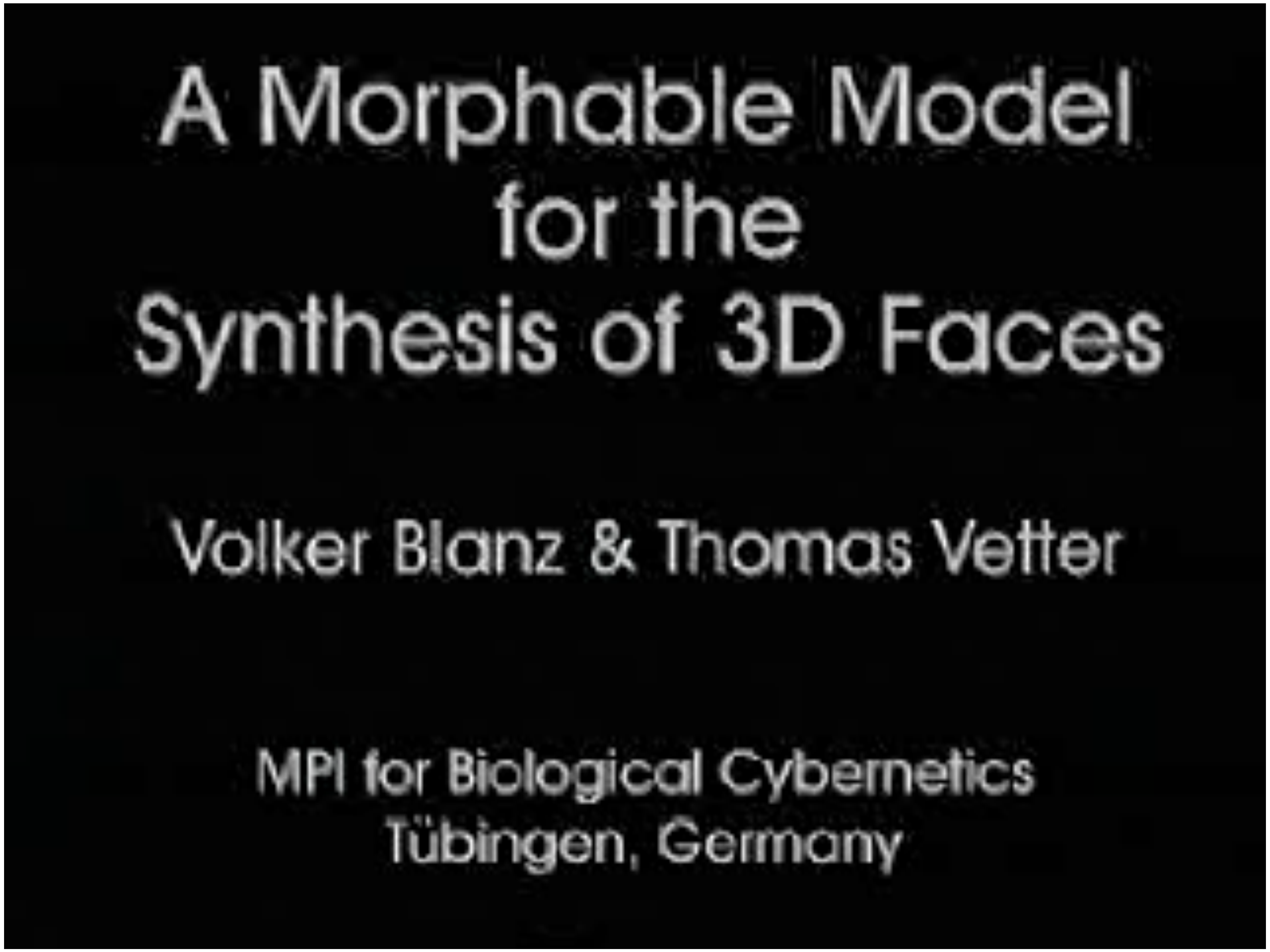
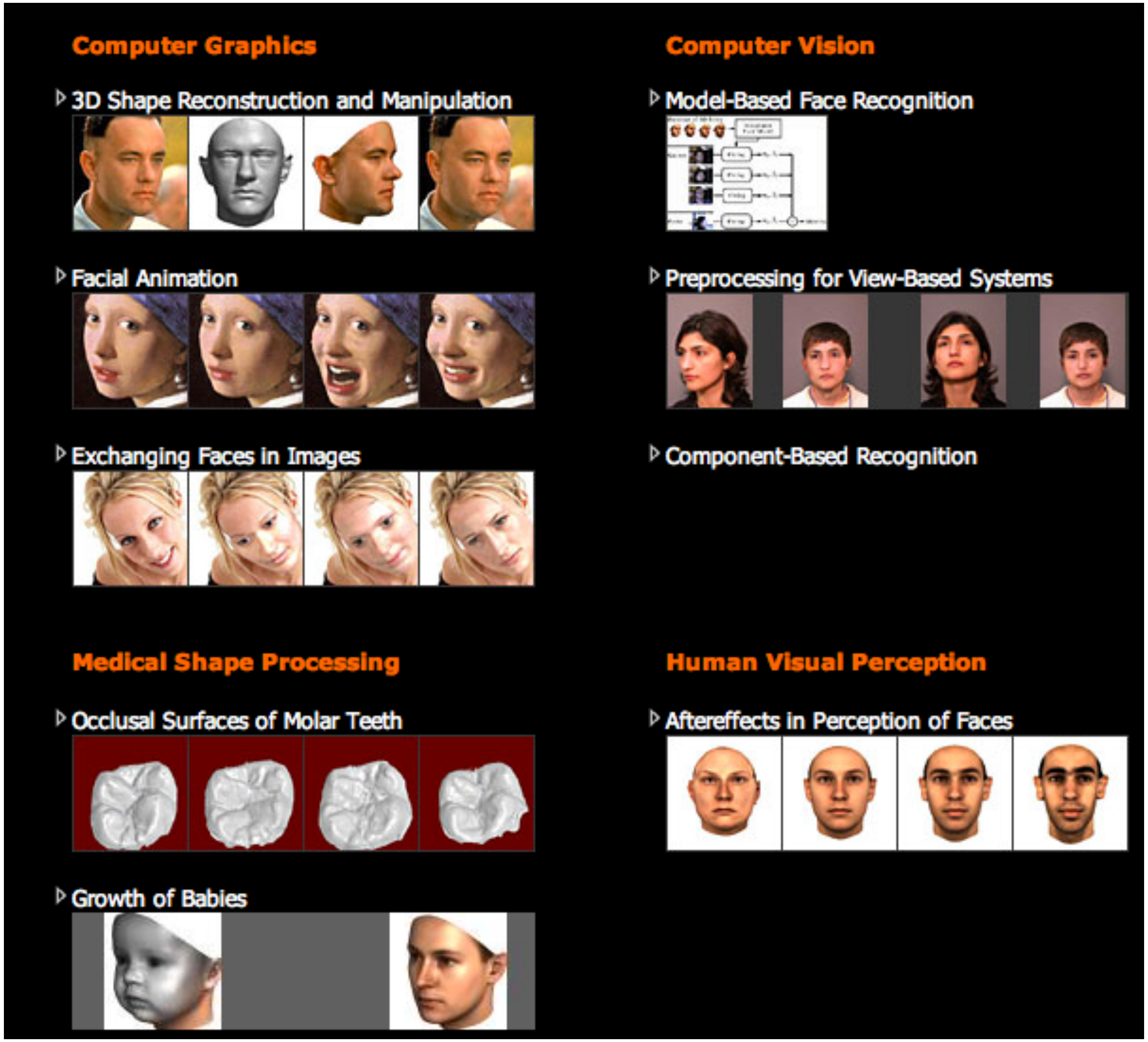
Blanz and Vetter (Siggraph, 1999)



$$E_I = \sum_{x,y} \left\| \mathbf{I}_{input}(x,y) - \mathbf{I}_{model}(x,y) \right\|^2 .$$

Volker Blanz



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