Application for Duality Theorem: Decentralized Matching (Recitation 4)

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Consider the following problem:

- There are n workers in X and n firms in Y
- Each firm needs exactly one worker, and if firm y matches with worker x, they generates value $f(x, y)$, productivity of x and y

Primal problem: A social planner designs a matching algorithm $\mu(x, y) \in \{0, 1\}$, whether x and y match, to maximize the total productivity, subject to each worker works for one firm and each firm hires one worker.

$$
\begin{aligned}
&\max_{\mu(x,y)} f(x,y)\mu(x,y) \\
\text{s.t. } &\sum_{y \in Y} \mu(x,y) = 1, \forall x \in X \\
&\sum_{x \in X} \mu(x,y) = 1, \forall y \in Y \\
&\mu(x,y) \in \{0,1\}
\end{aligned}
$$

Let's relax it a bit and consider the following problem. If the solution of this problem exhibits $\mu(x,y) \in \{0,1\}$, it is the solution of orginal problem.

$$
\begin{aligned}
&\max_{\mu(x,y)} f(x,y)\mu(x,y) \\
\text{s.t. } &\sum_{y \in Y} \mu(x,y) = 1, \forall x \in X \\
&\sum_{x \in X} \mu(x,y) = 1, \forall y \in Y \\
&\mu(x,y) \in [0,1]
\end{aligned}
$$

I claim it is equivalent to the following problem. With the following constraints, $\mu(x,y)$ already cannot exceeds 1.

$$
\begin{aligned}
&\max_{\mu(x,y)} f(x,y)\mu(x,y) \\
&\text{s.t. } \sum_{y \in Y} \mu(x,y) = 1, \forall x \in X \\
&\quad\ \ \sum_{x \in X} \mu(x,y) = 1, \forall y \in Y \\
&\quad\ \ \mu(x,y) \geq 0
\end{aligned}
$$

Consider the dual of this problem

$$
\begin{aligned}
&\min_{w(x),v(y)} \sum_{x \in X} w(x) + \sum_{y \in Y} v(y) \\
&\text{s.t. } w(x) + v(y) \geq f(x,y), \forall x \in X, y \in Y
\end{aligned}
$$

- Dual problem says I give a wage offer $w(x)$ to workers and a profit offer $v(y)$ to firms, such that they will all accept, in the sense that worker (firm) cannot find a firm (worker) that he cooperate with to make higher total profit to share
- Duality theorem says the least money you are going to spend is equal to the maximum matching profit
- Another way of thinking it is, you keep finding matching and making offer that you are not going to lose money. Once you find a matching and a corresponding offer that everyone agrees, it must mean you have achieved the optimal

matching. This process can be decentralized