Application for Duality Theorem: Binary Type Screening (Recitation 3)

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Consider the following problem:

- A seller tries to sell a product (that has already been produced, without cost) to a buyer
- Buyer may value the product high($v_H = 20$) or low($v_L = 5$), each of which happens with a half probability
- If transaction happens and price t is paid, buyer gets payoff $v - t$, seller gets payoff t; otherwise, both gets 0
- Everyone cares about expected payoff when evaluating uncertainty
- Buyer knows the value and seller does not
- Seller tries to use a mechanism to sell the product

Definition 1 (Mechanism)**.** A mechanism is a triplet $\mathcal{M} = (M, q, t)$, such that

- M is a set of messages

- $q : M \mapsto [0, 1]$, $q(m)$ specifies the probability of getting the product if message m is sent

- $t : M \mapsto \mathbb{R}$, $t(m)$ specifies how much to pay if message m is sent (No matter get the product or not, the payment has to be paid. If you prefer the setting where paying the money only when getting the product, t can be thought as the expected payment.)

- There must be a message rejecting transaction m_0, where $q(m_0) = t(m_0) = 0$

Example 1. $M = \{1, 2, ..., 10\}, q(m) = 0.1m, t(m) = m$, choosing option m, buyer gets

$$0.1mv - m = (0.1v - 1)m$$

If $v > 10$, buyer should choose $m = 10$; if $v < 10$, buyer should choose $m = 1$.

Among options, there will be one that the high value buyer will choose, and there will be one that the low value buyer will choose. Let us just focus on the options that buyer will choose. Actually, it is without loss to focus on mechanism that only contains three options (Revelation Principal):

- one for high value buyer q_H, t_H

- one for low value buyer q_L, t_L

- one for rejecting $0, 0$

We can now write a (complicated) LP primal problem

$$
\begin{aligned}
\max_{q_H, q_L, t_H, t_L} \; & \frac{1}{2} t_H + \frac{1}{2} t_L \\
\text{s.t. } & v_H q_H - t_H \geq v_H q_L - t_L \\
& v_H q_H - t_H \geq 0 \\
& v_L q_L - t_L \geq v_L q_H - t_H \\
& v_L q_L - t_L \geq 0 \\
& q_H \leq 1 \\
& q_L \leq 1 \\
& q_H, q_L \geq 0
\end{aligned}
$$

Consider canonical form

$$
\max_{(q_H,q_L,t_H,t_L)} \left(0,0,\frac{1}{2},\frac{1}{2}\right)\begin{pmatrix} q_H \\ q_L \\ t_H \\ t_L \end{pmatrix}
$$

$$
\text{s.t.} \quad \begin{pmatrix} -v_H & v_H & 1 & -1 \\ -v_H & 0 & 1 & 0 \\ v_L & -v_L & -1 & 1 \\ 0 & -v_L & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_H \\ q_L \\ t_H \\ t_L \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
$$

$$
q_H, q_L \geq 0
$$

Consider its dual

$$\min_{(y_1,y_2,y_3,y_4,y_5,y_6)} (y_1,y_2,y_3,y_4,y_5,y_6)\begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}$$

$$\text{s.t.}\quad (y_1,y_2,y_3,y_4,y_5,y_6)\begin{pmatrix}-v_H & v_H\\ -v_H & 0\\ v_L & -v_L\\ 0 & -v_L\\ 1 & 0\\ 0 & 1\end{pmatrix} \geq (0,0)$$

$$(y_1,y_2,y_3,y_4,y_5,y_6)\begin{pmatrix}1 & -1\\ 1 & 0\\ -1 & 1\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix} = \left(\frac{1}{2},\frac{1}{2}\right)$$

$$y_1,y_2,y_3,y_4,y_5,y_6 \geq 0$$

Interestingly, this is solvable. The two equalities show that

$$y_1 + y_2 = y_3 + \frac{1}{2}$$

$$y_3 + y_4 = y_1 + \frac{1}{2}$$

We can plug them back into the two inequalities to eliminate y_2, y_4

$$y_5 \geq (v_H - v_L)y_3 + \frac{1}{2}v_H$$

$$y_6 \geq -(v_H - v_L)y_1 + \frac{1}{2}v_L$$

We still need to consider $y_2, y_4 \geq 0$, which gives

$$|y_1 - y_3| \leq \frac{1}{2}$$

Notice we want y_5, y_6 as small as possible. That means $y_3 < y_1$. Otherwise, if $y_3 \geq y_1$, there could be two cases. Case 1: $y_3 = y_1 = 0$, then increasing y_1 will decrease the lower bound for y_6 and give a better solution to dual. Case 2: $y_3 > 0$, decreasing y_3 will decrease the lower bound for y_6 and give a better solution. Thus, we know for sure $y_1, y_4, y_5 > 0$. The rest y_2, y_3, y_6 we are not very sure, but this is enough. By complementary slackness, we know in primal problem

$$v_H q_H - t_H = v_H q_L - t_L$$

$$v_L q_L - t_L = 0$$

$$q_H = 1$$

The primal problem is then reduced to a problem with only q_L changing. It is very easy to solve. When $v_L \geq \frac{1}{2}v_H$, $t_L = t_H = 1, q_H = q_L = 1$ is the solution; when $v_L \leq \frac{1}{2}v_H$, $t_L = q_L = 0, t_H = v_H, q_H = 1$ is the solution. Notice this suggests, the optimal mechanism for seller is always posting a price, either v_H or v_L, depending on which price is going to give a higher profit.

The result is proved for any distribution of v, i.e. posting a take it or leave it price (price may vary with distribution of v) is always the optimal mechanism in this setting. In this setting, it reduces the complicated mechanism design problem to a monopoly pricing problem. One can find a proof about it in Manelli and Vincent (2007).