The Efficiency Trade-Off in Recommendation Systems

Genyu Li*

September 6, 2025

Abstract

This paper studies how recommendation systems affect market outcomes when sellers set a uniform price for multiple perfectly substitutable products. A buyer receives a costless signal about values of all products that reveals both which product is most likely to be the best fit and the expected value of that product, thereby shaping the buyer's willingness-to-pay. Unlike standard models, here not only by the spread but also the expectation of the distribution of willingness-to-pay can be influenced by recommendation systems, making conventional characterizations inapplicable. We introduce a tractable class of signal structures, rank-free signals with noisy recommendation (RFSNR), and show that it is without loss of generality to restrict attention to this class. Using RFSNR, we characterize feasible distributions of willingness-to-pay as mean-preserving contractions of pseudo-priors, which allows us to compute the full set of implementable market outcomes. Our analysis reveals that recommendation systems maximizing buyer surplus are often inefficient. Manipulating buyer surplus proceeds in two phases: initially, surplus is shifted from sellers to buyers without efficiency loss by spreading the willingness-to-pay distribution; subsequently, further gains for buyers require sacrificing total surplus by reducing recommendation accuracy to lower prices. These results highlight a fundamental trade-off between consumer surplus and market efficiency in the design of recommendation systems.

*University of North Carolina at Chapel Hill. genyuli@ad.unc.edu

1 Introduction

Developments in technology makes it possible for consumers to acquire recommendations before making purchasing choices, which in many scenarios not only include whether to trade or not but also which product to purchase among all the alternative substitutes. The recommendation over alternative products may also influence how much a consumer would like to pay for a trade. For example, a consumer who wants to purchase a pair of glasses may check Youtube videos for what frame fits him most. Seeing such an option in store, he may be willing to pay a lot for the "most suitable" glasses. More accurate recommendation available will surely help every single consumer make better decision on purchasing. By helping consumers find suitable products, accurate recommendation will also increase the general efficiency of trading. However, consumers as a group may be harmed by accurate recommendation even if it is free, as it will increase the overall willingness-to-pay and thus invite the seller to raise the price. The question now becomes: how the existence of recommendation influence the outcome of market including price, buyer surplus, seller surplus and total surplus? This paper contributes to this problem by characterizing possible distributions of willingness-to-pay in markets where consumers only consider buying the best (most suitable) product post receiving a signal (recommendation). Such market is often the case when the buyer pays a uniform price before making choice among alternatives like customize clothes or over-the-top media service. Building on the characterization, we further characterize all possible outcomes of such market that can be implemented by some signal structure.

We consider a model where seller sets a uniform price to sell one of multiple substitutable products to a buyer. The buyer receives a costless signal about true joint values of those products. Such signal usually contains two pieces of information that we should pay attention to. One is which product has the highest expected value (recommended product) and the other is the expected value of the recommended product (willingness-to-pay of the buyer). Accuracy of recommendation, measured by how frequent the recommended product is indeed the product with highest true value, has an influence on the expectation of willingness-to-pay. Thus, changing signal structure usually involves both changing how spread the distribution of willingness-to-pay is and the expectation of willingness-to-pay. Conventional characterization that says distribution of willingness-to-pay must have the same expectation is not applicable here. We suggest a class of signal structure, rank-free signal with a noisy recommendation (RFSNR), separates the two forces and prove it is without loss to only consider this class of signal structure. We then restore the conventional characterization by introducing a class of distribution, pseudo-priors, and uses RFSNR to show any distribution of willingness-to-pay must be mean preserving contraction of a pseudo-prior. This characterization makes it computable to calculate the possible outcomes of market.

We find the signal structure maximizing buyer surplus is not efficient in many cases. For most prior value distributions, using signal structure to manipulate buyer surplus from zero to maximum possible amount will go through two phases. In the first phase, the total surplus will not change and the surplus is moved from the seller to the buyer. Signals always recommend the product with highest true value in this phase, so the total surplus stays at highest. The surplus transfer is done through making the distribution of willingness-to-pay more spread and inviting the seller to set a low price to accommodate every consumer. In the second phase, increasing buyer surplus will sacrifice total surplus; that is, to increase a certain amount of buyer surplus, seller surplus will decrease more that that amount. To increase buyer surplus, signal will recommend less accurately to decrease overall willingness-to-pay to reduce price.

Related Literature. We identify Roesler and Szentes (2017) to be the most related literature. They solve out the buyer optimal information structure when there is only one product with vertical difference. They find the buyer optimal information structure will induce an equilibrium where every buyer buys the product. It can be inefficient for the buyer, as some of the buyers pay a price higher than their value. However, the trading is efficient as only buyer values the product. The main difference of our paper to Roesler and Szentes (2017) is we introduce multiple substitute products. With multiple products, buyer optimal information structure exhibits a new source of inefficiency. The buyer optimal information structure will induce an equilibrium where still every buyer buys, but they may not choose the best match. The trading thus is generally inefficient.

Armstrong and Zhou (2022) stands for another class of related literature investigating information structure for multiple products. Their paper investigates the buyer optimal information structure in a market with multiple competitive firms, each of which supplies one product. This is similar to our paper in the sense that in their setting there is also multiple products that are substitutes to each other. Our paper differs from their paper because in our paper the products are supplied by a simple monopolist. We have to admit that the monopolist question is much simpler than their oligopolist problem. But on the other hand, our paper is more general as we allow more than two products and an outside option for the buyer, while what they can do with multiple products is very limited. Ichihashi (2020) also falls in this classification. It discusses the effect of information in a similar setting but more about the commitment power of information structure rather than information structure itself.

Our paper also contributes to the techniques of multi-dimensional information design. Kleiner, Moldovanu, Strack, and Whitmeyer (2024) is one most relevant paper caring about the general multi-dimensional information design problem. Our paper does not follow into their scope. They mainly care about the situation where there is a sender (information designer or regulator) and a receiver (buyer) who will simply reacts to the signal and receives a payoff. In our paper, we do not stress the role played by the information designer, though we can assume one. More importantly, there is a seller who directly reacts to the information structure rather than the realized signal. This makes their tool invalid in our question.

Some other papers also have similarities with our paper, but more on other topics. One of the most related paper is Matějka and McKay (2015). It has a almost identical setting as our model but with a cost on information acquisition. This difference makes their paper go a completely different way compared to ours. They mainly strengthen how people acquire information when facing a cost, while our paper strengthen without cost, which surplus can be implemented by a signal structure. Our paper is also similar to Strack and Yang (2024) in the sense that we are also trying to find Blackwell boundary in a certain setting.

Potentially, the technique and intuition provided by our paper may extend to some other settings. For example, Ravid, Roesler, and Szentes (2022) studies a buyer information acquisition problem in a setting with a single seller and a single product. Our techniques can be applied on this setting to analyze a similar problem but with multiple substitutable products. Intuition provided by our paper may be helpful in robust multi-dimensional screening problem like Deb and Roesler (2024). Consider a setting where a seller wants to seller a set of substitutable products, it is suggested by our paper that the seller may only need to consider mechanisms with two prices, one of which is for fully random product and the other is for the most suitable product.

2 Model

A seller has a set $J = \{1, 2, ..., n\}$ of perfectly substitute products to sell to a buyer. The buyer values each product $j \in J$ potentially different, v_j. Values of all products $v = (v_1, v_2, ..., v_n)$ are jointly distributed according to a symmetric[1] cumulated distribution function (CDF) F supported on $V = [0, 1]^n$. We call F prior. The buyer observes a signal s about v. The signal s is generated according to an signal structure (S, q), where S is a set of signals and $q : V \mapsto \Delta(S)$ specifies the distribution according to which signals are generated. The joint distribution of v and s is common knowledge. The seller then gives a take-it-or-leave-it offer to the buyer, p. If the offer is accepted, the buyer will choose one product $j \in J$ to consume and pay the price p. When offer p accepted and product j chosen, the buyer gets payoff $v_j - p$ and the seller gets payoff p.[2] If the offer is rejected, both gets payoff 0. Both the seller and the buyer are von Neumann-Morgenstern expected payoff maximizers. In what follows, we fix the CDF F and analyze those signal structures that maximizes the buyer's expected payoff.

A sufficient statistics, posterior maximum expected value, is essential to the question. Let us denote posterior expected value of product j to be $m_j = \mathbb{E}[v_j|s]$. The buyer makes the decision about whether to accept the offer or not according to the comparison between $\max m_j$ and the price p. For simplicity, let us assume the buyer mechanically accepts the offer if and only if $\max m_j \geq p$.[3] The seller then faces the demand function $D(p) = \mathbb{P}[\max_j m_j \geq p]$, the probability of accepting an offer with price p, and decides the offered price p accordingly. The demand and the price both only depends on the distribution of $\max_j m_j$, denoted $G(x) = \mathbb{P}[\max_j m_j \leq x]$, so buyer's expected payoff $\int_p^1 (x-p) dG(x)$ also only depends on G. That is to say, for any signal structure, we only need to pay attention to the distribution G of $\max_j m_j$ it corresponds to. We call G posterior maximum distribution. Let $\mathcal{G}_F$ be the set of posterior maximum distributions that have a corresponding signal structure given prior distribution is F, our question can then be stated as follows:

$$\max_{G \in \mathcal{G}_F} \int_{p^*}^1 (x - p^*) dG(x)$$
$$\text{subject to } p^* \in \arg\max_p p D_G(p)$$

[1]A CDF F is symmetric if and only if for any permutation σ, $F \circ \sigma = F$, where we use permutation σ on v meaning changing the order of arguments of v, i.e. $\sigma(v) = (v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, ..., v_{\sigma^{-1}(n)})$.

[2]This implies the seller has 0 marginal cost for all products.

[3]Similar to Roesler and Szentes (2017), assuming a nonstrategic buyer at the last stage simplifies the analysis without influencing the result.

3 Implementable Distribution

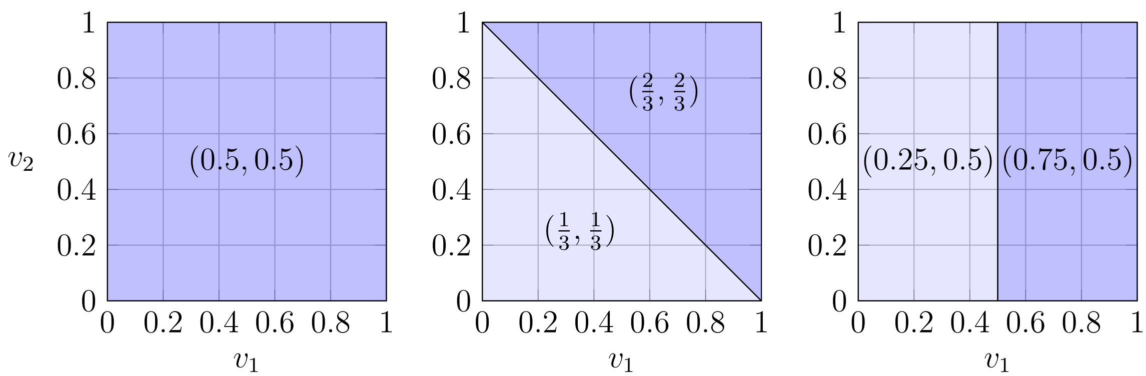
This section aims to characterize $\mathcal{G}_F$. Formally,

Definition 1. A posterior maximum distribution G is implemented by (S, q) if

$$G(x) = \mathbb{P}_{F,(S,q)}\{\max_j m_j \leq x\}$$

If posterior maximum distribution G is implemented by some (S, q), it is posterior maximum implementable, implementable for short. Let $\mathcal{G}_F$ be the set of implementable posterior maximum distributions.

To determine what posterior maximum distribution is implementable, the method provided by Blackwell (1953) is not applicable. Indeed, corresponding to some signal structure, the distribution of $m = (m_1, m_2, ..., m_n)$ must be a mean preserving contraction of prior F, yet it does not simply extend to the distribution of $\max_j m_j$ because the maximum we take after expectation. We do not even know what distribution should be considered as the prior distribution of $\max_j m_j$.



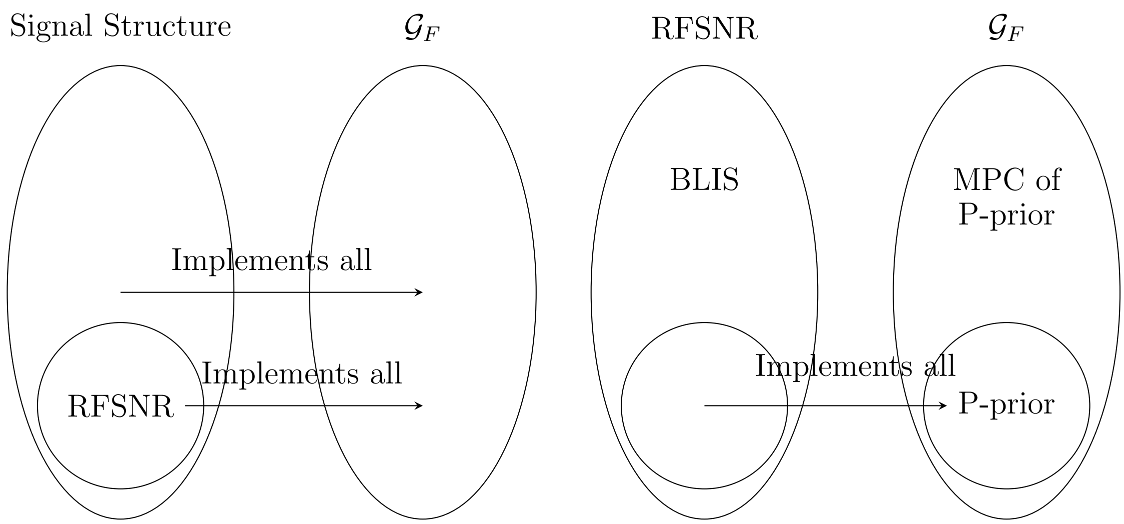
(a) Non-informative signal (b) Rank-free Signal (c) Arbitrary Signal

Figure 1: Prior F determined by $n = 2, v_1, v_2 \sim_{\text{i.i.d.}} U[0, 1]$

The difficulty of characterizing all posterior maximum distribution is that when increasing (decreasing) the informativeness of signal structure, two things can happen simultaneously. One is similar to Blackwell (1953), being more (less) informative about the value of $\max_j m_j$, while the others, emerging from the multi-dimensional nature of the problem, is increasing (decreasing) the expectation of $\max_j m_j$. Consider the example described by Figure 1. Figure 1a shows a signal structure that always sends a uniform signal, which is not informative at all. Receiving the signal, the buyer forms a posterior estimation of the value of the product $(0.5, 0.5)$, and the posterior maximum distribution is a degenerate distribution on 0.5. The expectation of such distribution is trivially 0.5.

Figure 1b shows a more informative signal structure. A signal $(\frac{2}{3}, \frac{2}{3})$ is sent if $v_1 + v_2 \geq 1$ and $(\frac{1}{3}, \frac{1}{3})$ is sent otherwise. Though more informative, this signal structure only involves the first driven force mentioned above, making the value of $\max_j m_j$ more informed without changing the expectation of $\max_j m_j$. Receiving signal $(\frac{2}{3}, \frac{2}{3})$, the buyer will form an estimation that both products have expected value $\frac{2}{3}$, and receiving signal $(\frac{1}{3}, \frac{1}{3})$, both products have expected value $\frac{1}{3}$. It will not help the buyer choose a product, since both products always have the same expected value; yet, it will help the buyer decide whether to accept a price or not. We call such signal rank-free signal, signal structure providing no information about the rank of the products.

Figure 1c describes a more generic signal structure that sends two signals, $(0.75, 0.5)$ is sent when $v_1 \geq 0.5$ and $(0.25, 0.5)$ is sent otherwise. Receiving signal $(0.75, 0.5)$ ($(0.25, 0.5)$) gives posterior value estimation $(0.75, 0.5)$ ($(0.25, 0.5)$). The distribution of $\max_j m_j$ is now a two-point distribution over 0.75 and 0.5, each with $\frac{1}{2}$ probability. This signal structure is not only more informative about the value of $\max_j m_j$ than the non-informative one, but also increases the expectation of $\max_j m_j$ to 0.625. The reason is receiving signal $(0.75, 0.5)$ ($(0.25, 0.5)$), the buyer not only knows the value of the best (in expectation) product is 0.75 (0.5), but also knows he should choose product 1 (2) as the expected value of product 1 (2) is higher. Thus, buyers facing the this signal structure make overall better decision than those facing the non-informative signal structure, resulting in a higher expectation of $\max_j m_j$ for this signal structure. This effect of changing the expectation of $\max_j m_j$ almost always happens and makes determining which posterior maximum distribution is implementable difficult.



(a) $G \in \mathcal{G}_F$ implemented by RFSNR

(b) $\mathcal{G}_F$ characterized by MPC of P-prior

Figure 2: Characterizing $\mathcal{G}_F$ Road Map

We provide a signal structure called rank-free signal with a noisy recommendation (RFSNR) as a tool to separate the two forces. The two pieces of information, rank-free signal and noisy recommendation, respectively influences the informativeness about the value of $\max_j m_j$ and the quality of the recommendation $\mathbb{E}[\max_j m_j]$. We prove that any implementable posterior maximum distribution can be implemented by RFSNR. (As shown in Figure 2a) Thus, we can consider only those posterior maximum distributions implemented by RFSNR. By considering RFSNR, we find some signal structure is never most Blackwell informative among those implementing the posterior maximum distributions with the same expectation. We call these signal structures Blackwell less informative signal structure (BLISS)[4]. We then ignore all posterior maximum distributions implemented by BLISS and collect all posterior maximum distributions remaining

[4]BLISS is not a full characterization of signal structures that are less informative than some other

to build a set of distributions, called pseudo-priors (P-prior)[5], which can generate all implementable posterior maximum distributions via mean preserving contraction operation[6]. (As shown in Figure 2b) Therefore, the set of all implementable posterior maximum distributions is characterized by mean preserving contraction of P-prior. Then, to solve our problem, we can exhaust every implementable distributions by exhausting every mean preserving contraction of every P-priors.

3.1 Rank-free Signal

In this subsection, we introduce the formal definition of rank-free signal. Let $\mathcal{S}_n$ be the symmetric group of degree n, i.e. the set of all permutations of degree n.

Definition 2. A signal structure (S, q) is a rank-free signal structure if and only if q is symmetric, i.e. $q \circ \sigma = q, \forall \sigma \in \mathcal{S}_n$. We use (C, ψ) as a generic notation for a rank-free signal.

The most important property of a rank-free signal is receiving any such signal will not reveal any information about the rank of products. The symmetric property guarantees that for any rank-free signal $c \in C$, $\mathbb{E}[v|c] = \mathbb{E}[\sigma(v)|c], \forall \sigma \in \mathcal{S}_n$[7] and thus $m_j(c) = \mathbb{E}[v_j|c] = \mathbb{E}[v_k|c] = m_k(c), \forall j, k \in J$. That is to say, post receiving c, the expected value of all products still seem to be the same for the buyer. It will preserve the expected value of the to-be-chosen product $\mathbb{E}[\max_j m_j(c)] = \mathbb{E}[m_1(c)] = \mathbb{E}[v_1]$. One more thing to notice is that though rank-free signal gives no information about the rank of products, it still contains information about the value of $\max_j m_j$. Using different rank-free signal allows us to change the informativeness of signal structure without altering $\mathbb{E}[\max_j m_j]$.

Obviously, only rank-free signal structures are not enough to implement all implementable posterior maximum distribution. As shown in Figure 1c, $\mathbb{E}[\max_j m_j]$ can change if a different signal structure is used, while rank-free signal will never change it. To implement all implementable posterior maximum distributions, rank-free signal needs a counter-part that will change the expectation of $\mathbb{E}[\max_j m_j]$. A very natural idea is we can add a recommendation in addition to a rank-free signal. It seems that a recommendation about which product is better perfectly compensates rank-free signals' lack of information about rank. This gives us rank-free signal with a noisy recommendation, RFSNR.

Definition 3 (rank-free signal with a noisy recommendation, RFSNR)**.** A rank-free signal with a noisy recommendation is a signal structure $(S_{(C,\psi,r)}, q_{(C,\psi,r)})$ determined by elements (C, ψ, r), where

signal structure.

[5]A P-prior may be strict mean preserving contraction of other P-prior.

[6]We show later that mean preserving contraction of implementable distribution is still implementable.

[7]It can be heuristically verified by

$$\begin{aligned}\mathbb{E}[v|c] &= \frac{\int_V v f(v)\psi[v](c)dv}{\int_V f(v)\psi[v](c)dv} = \frac{\int_V v f(\sigma(v))\psi[\sigma(v)](c)dv}{\int_V f(\sigma(v))\psi[\sigma(v)](c)dv} \\ &= \frac{\int_V \sigma^{-1}(v) f(v)\psi[v](c)d\sigma^{-1}(v)}{\int_V f(v)\psi[v](c)d\sigma^{-1}(v)} = \mathbb{E}[\sigma^{-1}(v)|c], \forall \sigma \in \mathcal{S}_n\end{aligned}$$

1. (C, ψ) is a rank-free signal structure
2. $r : C \mapsto [0, 1]$ is a function of recommendation accuracy

Let $M(v) = \arg\max_j\{v_j\}$, $(S_{(C,\psi,r)}, q_{(C,\psi,r)})$ is determined by

1. $S_{(C,\psi,r)} = C \times \{1, 2, ..., n\}$ with generic signals (c, j)
2. $q_{(C,\psi,r)}[v](c, j) = \psi[v](c)(\frac{r(c)}{|M(v)|}\mathbb{1}\{j \in M(v)\} + \frac{1-r(c)}{n})$

Sometimes, we use (C, ψ, r) instead of $(S_{(C,\psi,r)}, q_{(C,\psi,r)})$ to denote RFSNR signal structure.

Remark 1. Consider a specific v, the signal structure first determines which rank-free signal c should be sent according to distribution $\psi[v]$. Then, according to c, it determines one product to recommend. For those products that are one of the best given v, i.e. $j \in M(v)$, they are recommended with probability $\frac{r(c)}{|M(v)|} + \frac{1-r(c)}{n}$; for those product that are not one of the best, they are recommended with probability $\frac{1-r(c)}{n}$. That is to say, all products share $1 - r(c)$ probability to be recommended and in extra those best products share $r(c)$ more probability to be recommended. Whenever $r(c)$ is large, the recommendation accuracy is higher since those best products are recommended with higher probability.

Remark 2. Let $\mathcal{S}_n$ denote the symmetric group of degree n, a set collecting all permutations of degree n. Let $v\mathcal{S}_n$ denote a equivalent calss of v with respect to permutations, i.e. $v\mathcal{S}_n = \{v'|\exists\sigma \in \mathcal{S}_n, v' = \sigma(v)\}$. Let $V/\mathcal{S}_n$ denote the quotinent of V by $\mathcal{S}_n$, the set collecting all equivalent classes, i.e. $V/\mathcal{S}_n = \{v\mathcal{S}_n|v \in V\}$. There is a one-to-one mapping between symmetric $\psi : V \mapsto \Delta(C)$ and $\hat{\psi} : V/\mathcal{S}_n \mapsto \Delta(C)$. That is because $\psi[v] = \psi[v'], \forall v' \in v\mathcal{S}_n$, so we can let $\hat{\psi}[v\mathcal{S}_n] = \psi[v], \forall v\mathcal{S}_n \in V/\mathcal{S}_n$. Thus, it is equivalent to design $\hat{\psi}$ and to design symmetric ψ. Designing $\hat{\psi}$ is more intuitive. One just needs to separate the measure on $V/\mathcal{S}_n$ to different rank-free signals similar to separating measure on V to different signals in conventional literature. The most important intuition carrying to here is finer separation would give a more mean preserving spread distribution. It is going to eventually help build up our results.

RFSNR always sends a pair of signals. c is a rank-free signal, giving no information about the rank, but contains information about generally how good all products are, the most important of which is the expectation of mean value $\mathbb{E}[\frac{1}{n}\sum_j v_j|c]$ and the expectation of maximum value $\mathbb{E}[\max_j v_j|c]$. This information is utilized combined with a recommended product j, the second part of the signal. The accuracy of the recommendation is determined by a number $r(c)$. If $r(c) = 1$, the recommended product j must be the best product. The buyer can then combine his information obtained from c to form a posterior estimation of the value of the recommended product, $\mathbb{E}[\max_j v_j|c]$. If $r(c) = 0$, a uniformly random product is recommended and the value of the recommended product is just the mean conditional on c, $\mathbb{E}[\frac{1}{n}\sum_j v_j|c]$. If $r(c) \in (0, 1)$, the recommended product is one of the best with probability $r(c)$ and uniformly random with probability $1 - r(c)$. A linear combination of $\mathbb{E}[\max_j v_j|c]$ and $\mathbb{E}[\frac{1}{n}\sum_j v_j|c]$ will be the estimated value of the recommended product. The estimated value of the recommended product will be posterior maximum expected value, as compared to other products, the recommended product has a weakly higher chance to be the best product and thus has a higher posterior expected value.

3.2 Main Result

It turns out adding the noisy recommendation is enough to allow the rank-free signal to implement all the implementable posterior maximum distributions. The following theorem provides the foundation.

Theorem 1. *A posterior maximum distribution is implementable if and only if it can be implemented by RFSNR.*

The implication of the theorem is very straightforward. It says as long as a posterior maximum distribution can be implemented by any signal structure, it can be implemented by RFSNR and vice versa. This allows us to consider RFSNR only when designing the signal structure.

The greatest merits of considering RFSNR is it allows us to design $\hat{\psi}$ and r separately. It separates the two important ideas, designing more informative signal structure about the value of the recommended product without making more accurate recommendation and making more accurate recommendation. If one change $\hat{\psi}$ and cut the space $V/\mathcal{S}_n$ finer and finer but keeps r[8], the resulted signal structure will be Blackwell more informative about $\max_j m_j$ but not changing $\mathbb{E}[\max_j m_j]$ as it does not make better recommendation. This idea will help us eventually characterize a set of Blackwell most informative distributions of $\max_j m_j$ for every possible level of $\mathbb{E}[\max_j m_j]$, and allow us to apply the method provided by Roesler and Szentes (2017) and Blackwell (1953) to solve the model.

Sketch of the proof with an example. "If" side of Theorem 1 is straight forward. Since RFSNR is a signal structure, if a posterior maximum distribution is implemented by RFSNR, it is implementable. We prove "only if" side by construction; that is, for any posterior maximum distribution implemented by some signal structures, we construct an RFSNR to implement it. To illustrate the main idea of the construction, we go through the construction process for a simple signal structure with a simple prior distribution. One can easily apply the same process for any signal structure over any symmetric prior distribution.

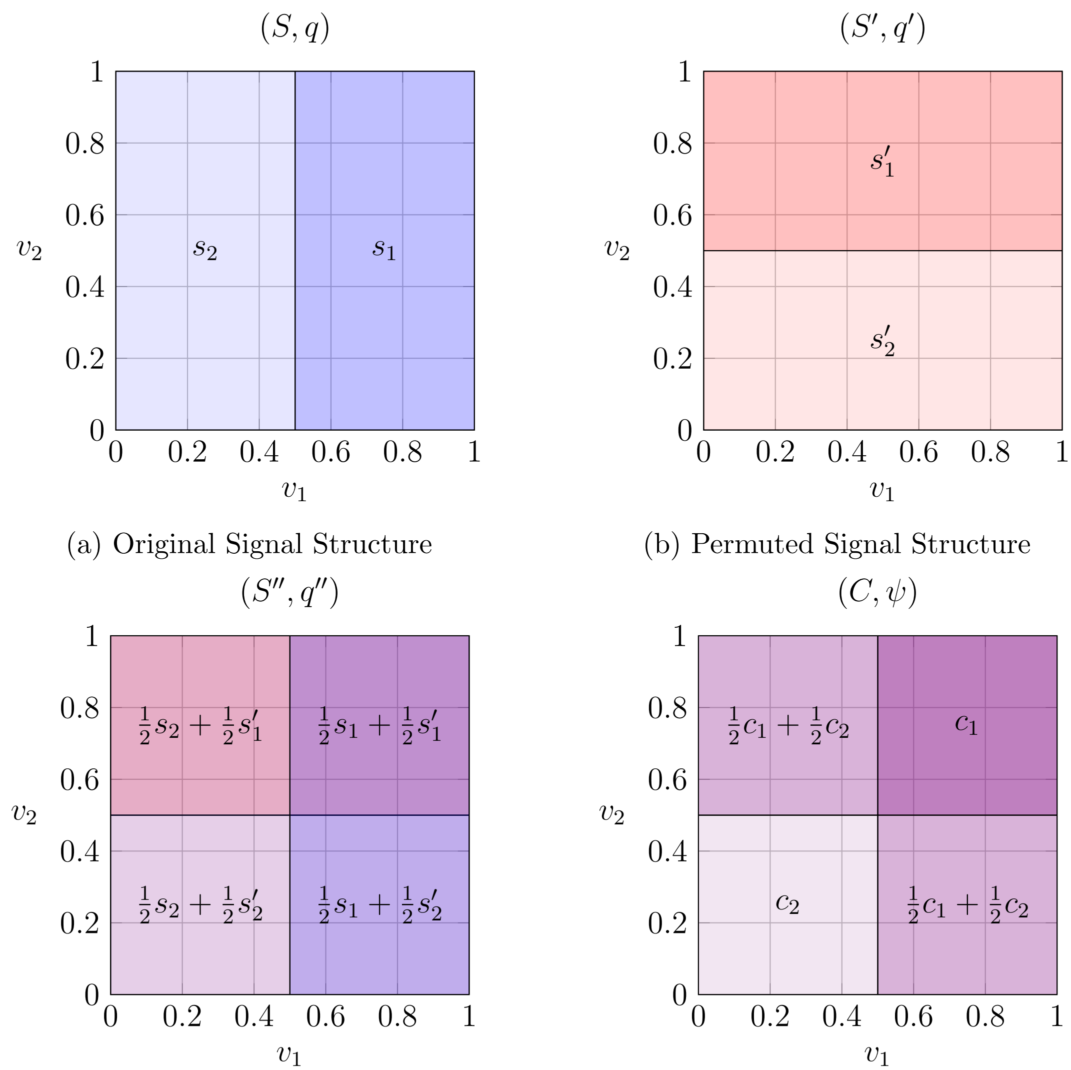
Let us consider a prior distribution F defined by $n = 2$ and $v_1, v_2 \sim_{\text{i.i.d.}} U[0, 1]$ and a signal structure consisted of two signals s_1, s_2 where s_1 is sent whenever $v_1 \geq 0.5$ and s_2 is sent otherwise as shown in Figure 3a. We call it the original signal structure (S, q) in this part. The posterior value is $(0.75, 0.5)$ when s_1 is received happening with a half probability and with the other half probability s_2 is received and the posterior value is $(0.25, 0.5)$. Thus, the posterior maximum value is with a half probability to be 0.75 and the other half probability to be 0.5. The posterior maximum distribution implemented by (S, q) is therefore

$$G(x) = \begin{cases} 0 & x < 0.5 \\ 0.5 & 0.5 \leq x < 0.5 \\ 1 & x \geq 0.75 \end{cases}$$

Another fact worth noting is that conditional on receiving s_1, the expected maximum value is $\mathbb{E}[\max\{v_1, v_2\}|s_1] = \frac{19}{24}$ and the expected mean value is $\mathbb{E}[\frac{v_1+v_2}{2}|s_1] = \frac{5}{8}$; and conditional on receiving s_1, the expected maximum value is $\mathbb{E}[\max\{v_1, v_2\}|s_1] = \frac{13}{24}$ and the expected mean value is $\mathbb{E}[\frac{v_1+v_2}{2}|s_1] = \frac{3}{8}$. One may observe that the posterior

[8] If one separate a rank-free signal c to c_1, c_2, keeping r means $r(c_1) = r(c_2) = r(c)$.

maximum value conditional on receiving s_1 is between the expected maximum value and expected mean value conditional on s_1. A similar relation also holds for those numbers conditional on s_2. This is generally true for any signal of any signal structure guaranteed by Jensen's inequality. We are now going to use the following steps to implement G with RFSNR.



(a) Original Signal Structure

(b) Permuted Signal Structure

(c) Symmetric Signal Structure

(d) Rank-free Signal Structure

Figure 3: Signal Operation

Step 1: Construct rank-free signal preserving the expected mean value and expected maximum value. To achieve this goal, we first permute the original signal structure. When $n = 2$, there is only one possible permutation. The permuted signal structure (S', q') is shown in Figure 3b. The signal in (S', q') corresponding to s_1 (s_2) is called s'_1 (s'_2). s'_1 (s'_2) has the same posterior maximum value, expected maximum value and expected mean value as s_1 (s_2) as (S', q') can be thought as the same signal structure as (S, q) with a reorder of products. We can then construct a symmetric signal structure (S'', q'') by applying (S, q) and (S', q') each with a half probability as shown in Figure 3c. Receiving any signal in (S'', q'') will give the buyer exactly the same information as if they receive the corresponding signal in (S, q) or (S', q'). The symmetry is in the sense that reordering the products will not change the joint posterior distribution of m_1, m_2. Yet, it is not rank-free. That is because reordering products will change the name of signals and rank-free signal requires reordering products preserves the name of signals. It is thus natural to combine the signals s_1 (s_2) in original structure with the corresponding ones s'_1 (s'_2) in the permuted signal structure and call the combination of s_1, s'_1 (s_2, s'_2) to be c_1 (c_2) as shown in Figure 3d. This completes the construction of rank-free signal. It indeed preserves the expected mean and maximum value in the sense that for any $j \in \{1, 2\}$

$$\mathbb{E}[\frac{v_1 + v_2}{2}|c_j] = \mathbb{E}[\frac{v_1 + v_2}{2}|s_j]$$

$$\mathbb{E}[max\{v_1, v_2\}|c_j] = \mathbb{E}[\max\{v_1, v_2\}|s_j]$$

Step 2: Determine the recommendation accuracy for each rank-free signal. Now, what's left is to determine $r(c_1)$ and $r(c_2)$. Let us first pay attention on determining $r(c_1)$. As mentioned above, the conditional expected maximum and mean value is preserved between c_1 and s_1, and the posterior maximum value conditional on receiving s_1 must be between the conditional expected maximum and mean value. It is then guaranteed that the following equation has a solution $r(c_1) \in [0, 1]$

$$r(c_1)\mathbb{E}[\max\{v_1, v_2\}|c_1] + (1 - r(c_1))\mathbb{E}[\frac{v_1 + v_2}{2}|c_1] = \max\{\mathbb{E}[v_1|s_1], \mathbb{E}[v_2|s_1]\}$$

We get $r(c_1) = \frac{3}{4}$. When the buyer receives signal c_1 and a recommendation j, he will think that product j is with probability $r(c_1)$ to be the best product with posterior value $\mathbb{E}[\max\{v_1, v_2\}|c_1]$ and probability $1-r(c_1)$ to be a uniformly random product with posterior value $\mathbb{E}[\frac{v_1+v_2}{2}|c_1]$. The resulted posterior value of the recommended product is thus equal to $\max\{\mathbb{E}[v_1|s_1], \mathbb{E}[v_2|s_1]\}$, the posterior maximum value conditional on receiving s_1 in the original signal structure. When being recommended product j, product j indeed has a higher posterior expected value than other product j' as j is with higher probability to be one of the best products than j'. Similarly, we can solve $r(c_2)$. This completes the construction of RFSNR.

This RFSNR indeed implements G. As we argued above, when using RFSNR, whenever c_1 (c_2) is sent with a recommendation j, the posterior value of the recommended product j is the posterior maximum value and is by construction equal to the posterior maximum value when receiving s_1 (s_2) in original signal structure (S, q). Also, by construction, the probability of rank-free signal c_1 (c_2) is sent in RFSNR is equal to the probability s_1 (s_2) is sent in the original signal structure (S, q). Combining them, it is not hard to see, the constructed RFSNR implements the same posterior maximum distribution as the original signal structure.

The procedure can also be applied to any symmetric prior and signal structure. For multiple product, instead of finding one permuted signal structure, we will need to find all possible permutation and construct permuted signal structure corresponding to each of them respectively. After that, we can construct symmetric signal structure by using each of them with equal probability. And the rank-free signal can be constructed by combining all signals in permuted signal structures corresponding to the the same signal in the original signal structure. Jensen's inequality will still guarantee that we can find an appropriate recommendation accuracy for every rank-free signal.

The guarantee of RFSNR implementation implies the following corollary.

Corollary 1. *Mean preserving contraction of an implementable posterior maximum distribution is implementable.*

If a posterior maximum distribution G is implementable, then it can be implemented by RFSNR. In RFSNR, conditional on product j is recommended, the value of product j follows exactly the same distribution G. By garbling conditional on product j is recommended, distribution of value of product j can be chosen as any mean preserving contraction of G. Mean preserving contraction of G can then be implemented by applying this garbling conditional on every product is recommended.

3.3 Implementable Posterior Maximum Distribution (Subject to change)

As mentioned above, we use RFSNR to identify and eliminate posterior maximum distributions that cannot be most mean-preserving spread among those distributions with the same expectation. We call distributions surviving the elimination pseudo-priors and denote the set of pseudo-priors to be $\mathcal{G}^{PP}$. Finally, we characterize the set of implementable distributions $\mathcal{G}$ by

$$\mathcal{G} = \{G | \exists G^{PP} \in \mathcal{G}^{PP}, G \text{ is mean preserving contraction of } G^{PP}\}$$

Let us first describe the criterion for pseudo priors and then explain what eliminations they have gone through.

Definition 4 (Pseudo Prior)**.** A posterior maximum distribution is a pseudo prior if it can be implemented by RFSNR (C^*, ψ^*, r^*) satisfying

1. $C^* = (V/\mathcal{S}_n) \times \{0, 1\}$
2. $\psi^*[v] \in \Delta(\{v\mathcal{S}_n\} \times \{0, 1\})$, denote $\hat{\varphi}(v\mathcal{S}_n) = \psi^*[v](v\mathcal{S}_n, 1)$
3. $r^*(v\mathcal{S}_n, 0) = 0, r^*(v\mathcal{S}_n, 1) = 1$
4. $\forall v^{(1)}, v^{(2)}$ that satisfies $\max_j\{v_j^{(1)}\} - \frac{1}{n}\sum_j v_j^{(1)} = \max_j\{v_j^{(2)}\} - \frac{1}{n}\sum_j v_j^{(2)}$ and $\max_j\{v_j^{(1)}\} > \max_j\{v_j^{(2)}\}$, we have $\varphi(v^{(2)}) > 0$ only if $\varphi(v^{(1)}) = 1$

We call (C^*, ψ^*, r^*) pseudo prior signal.

The first three lines gives two requirements for pseudo prior signals. First, each rank-free signal c must come with a recommendation that either always recommend the best product $r(c) = 1$ or always recommend a uniformly random product $r(c) = 0$. Second, each rank-free signal must identify what $v\mathcal{S}_n$ the realized v is in.

If RFSNR (C, ψ, r) implementing G violates the first requirement, i.e. for some $\tilde{c} \in C$, $r(\tilde{c}) \in (0, 1)$, we can construct RFSNR $(C^\dagger, \psi^\dagger, r^\dagger)$ implementing a distribution $G^\dagger$ that is mean preserving spread of G. The idea is separating $\tilde{c}$ into two rank-free signals $\tilde{c}^0$ and $\tilde{c}^1$ where $r(\tilde{c}^0) = 0, r(\tilde{c}^1) = 1$. Whenever $\tilde{c}$ is sent in (C, ψ, r), we send $\tilde{c}^1$ with probability $r(\tilde{c})$ and send $\tilde{c}^0$ with probability $1 - r(\tilde{c})$ in $(C^\dagger, \psi^\dagger, r^\dagger)$ instead. The constructed signal structure is more informative. And at the same time, the expected recommendation accuracy is preserved, i.e. no matter the realization of v, the expected probability of the best product and uniformly random product being recommended are the same between (C, ψ, r) and $(C^{\dagger}, \psi^{\dagger}, r^{\dagger})$. It then guarantees the expectation of $G^{\dagger}$ is the same as G and therefore is mean preserving spread of G.

If RFSNR $(C^{\dagger}, \psi^{\dagger}, r^{\dagger})$ implementing $G^{\dagger}$ does not violate the first requirement but there exists a signal $\hat{c} \in C^{\dagger}$, which is sent with positive probability when v is realized in two different $v\mathcal{S}_n$, we are able to construct $(C^{\ddagger}, \psi^{\ddagger}, r^{\ddagger})$ implementing $G^{\ddagger}$, mean preserving spread of $G^{\dagger}$. If $v\mathcal{S}_n \neq v'\mathcal{S}_n$ both involve sending signal $\hat{c}$, we can separate $\hat{c}$ to two signals $\hat{c}^{v}$ and $\hat{c}^{v'}$. For cases $\hat{c}$ is sent in $(C^{\dagger}, \psi^{\dagger}, r^{\dagger})$, in $(C^{\ddagger}, \psi^{\ddagger}, r^{\ddagger})$, we send $\hat{c}^{v}$ instead when v is realized in $v\mathcal{S}_n$ and $\hat{c}^{v'}$ when v is realized in $v'\mathcal{S}_n$ and keep $r(\hat{c}) = r(\hat{c}^{v}) = r(\hat{c}^{v'})$. $(C^{\dagger}, \psi^{\dagger}, r^{\dagger})$ is then a garbling of $(C^{\ddagger}, \psi^{\ddagger}, r^{\ddagger})$. The distribution of $\mathbb{E}[\frac{1}{n}\sum_j v_j | c^{\dagger}]$ ($\mathbb{E}[\max_j v_j | c^{\dagger}]$) implemented by $(C^{\dagger}, \psi^{\dagger}, r^{\dagger})$ is then mean preserving contraction of the distribution of $\mathbb{E}[\frac{1}{n}\sum_j v_j | c^{\ddagger}]$ ($\mathbb{E}[\max_j v_j | c^{\ddagger}]$) implemented by $(C^{\ddagger}, \psi^{\ddagger}, r^{\ddagger})$. As both $(C^{\dagger}, \psi^{\dagger}, r^{\dagger})$ and $(C^{\ddagger}, \psi^{\ddagger}, r^{\ddagger})$ still satisfies the first requirement, $G^{\dagger}$ ($G^{\ddagger}$) is a linear combination of the distributions of $\mathbb{E}[\frac{1}{n}\sum_j v_j | c^{\dagger}]$ and $\mathbb{E}[\max_j v_j | c^{\dagger}]$ ($\mathbb{E}[\frac{1}{n}\sum_j v_j | c^{\ddagger}]$ and $\mathbb{E}[\max_j v_j | c^{\ddagger}]$). Thus, $G^{\dagger}$ is mean preserving contraction of $G^{\ddagger}$.

The last line in the definition gives additional restriction to reduce the size of $\mathcal{G}^{PP}$. It says on any hyperplane $H(W) = \{v \in V \,|\, \max_j\{v_j\} - \frac{1}{n}\sum_j v_j = W\}$, $v\mathcal{S}_n$ with larger mean and maximum must always recommend the best product if $v'\mathcal{S}_n$ with smaller mean and maximum recommends best with product with positive probability. More intuitively, there is a cut-off point $Q(W)$ for each hyperplane $H(W)$, pseudo prior signal sent in every $v\mathcal{S}_n \subset H(W)$ must always recommend the best product if $\max_j v_j$ is greater than $Q(W)$; it must always recommend a uniformly random product if $\max_j v_j$ is smaller than $Q(W)$; it may randomize between recommending the best product and uniformly random product if $\max_j v_j$ is equal to $Q(W)$, but the result of randomization must be informed to the buyer as required above.

If this requirement is violated, one may move value of $\hat{\varphi}(v\mathcal{S}_n)$ from $v\mathcal{S}_n$ with smaller mean and maximum to $v'\mathcal{S}_n$ with larger mean and maximum to construct new RFSNR. And it can be verified, as the move of recommendation accuracy happens on the same hyper plane $H(W)$, it does not influence the expectation of G being implemented, while the change will make the distribution G more spread.

Thus, we have the following Theorem.

Theorem 2. *A posterior maximum distribution is implementable if and only if it is a mean preserving contraction of pseudo prior.*

The "only if" side is argued above by showing that every implementable distribution is a mean preserving contraction of pseudo prior. The "if" side is implied by Corollary 1. Since pseudo priors are implementable the mean preserving contraction of them are also implementable.

3.4 A Parametric Example (Subject to change)

As an example, we show how the set of pseudo priors look like when $n = 2$ with a prior satisfying $v_j \in \{0, 0.5, 1\}$. Possible priors can be described by the following table.

As we require the prior to be symmetric, the parameters have to satisfy the following three equalities $p_{ml} = p_{lm}, p_{hl} = p_{lh}, p_{hm} = p_{mh}$. Each pseudo prior corresponds to a $\hat{\varphi}$.

1	p_{lh}	p_{mh}	p_{hh}
0.5	p_{lm}	p_{mm}	p_{hm}
0	p_{ll}	p_{ml}	p_{hl}
v_2, v_1	0	0.5	1

To see how the set of pseudo priors looks like, we need exhaust all the possible $\hat{\varphi}$. We need to determine value of

$$\hat{\varphi}(\{(0,0)\}), \hat{\varphi}(\{(0.5,0.5)\}), \hat{\varphi}(\{(1,1)\})$$

$$\hat{\varphi}(\{(0,0.5),(0.5,0)\}), \hat{\varphi}(\{(0,1),(1,0)\}), \hat{\varphi}(\{(0.5,1),(1,0.5)\})$$

Notice that no matter how we change the three values in the first line, there is no influence on the implemented pseudo prior, so there are only three free parameters that we can change. Let

$$\hat{\varphi}(\{(0,0.5),(0.5,0)\}) = \varphi_1, \hat{\varphi}(\{(0,1),(1,0)\}) = \varphi_2, \hat{\varphi}(\{(0.5,1),(1,0.5)\}) = \varphi_3$$

There is one extra requirement for them determined by requirement 4 in the definition of pseudo prior. That is,

$$\varphi_1 + \varphi_3 \geq 1 \Rightarrow \varphi_3 = 1$$

$$\varphi_1 + \varphi_3 \leq 1 \Rightarrow \varphi_1 = 0$$

The posterior maximum distribution implemented by $\varphi_1, \varphi_2, \varphi_3$ is

$$G^{PP}(x; \varphi_1, \varphi_2, \varphi_3)$$
$$= \begin{cases} 0 & x < 0 \\ p_{ll} & x \in [0, 0.25) \\ p_{ll} + (1-\varphi_1)(p_{lm} + p_{ml}) & x \in [0.25, 0.5) \\ p_{ll} + p_{mm} + (p_{lm} + p_{ml}) + (1-\varphi_2)(p_{lh} + p_{hl}) & x \in [0.5, 0.75) \\ 1 - p_{hh} - \varphi_2(p_{lh} + p_{hl}) - \varphi_3(p_{mh} + p_{hm}) & x \in [0.75, 1) \\ 1 & x \geq 1 \end{cases}$$

The set of all pseudo priors is then

$$\{G^{PP}(x; \varphi_1, \varphi_2, \varphi_3) | (\varphi_1 = 0 \text{ or } \varphi_3 = 1) \text{ and } \varphi_1, \varphi_2, \varphi_3 \in [0,1]\}$$

One thing worth noting is that this set does not only contain those "most mean preserving spread" posterior maximum distribution; that is, some distributions in this set is strictly mean preserving contraction of the others. Yet, it is not hard to get rid of all of them. The set only containing "most mean preserving spread" posterior maximum distribution is the following.

$$\{G^{PP}(x; \varphi_1, \varphi_2, \varphi_3) | (\varphi_1 = \varphi_2 = 0 \text{ or } (\varphi_3 = 1, \varphi_1 = 0) \text{ or } \varphi_2 = \varphi_3 = 1) \\ \text{and } \varphi_1, \varphi_2, \varphi_3 \in [0,1]\}$$

A good property of this set is for each expectation of G^{PP}, there is only one corresponding $\varphi_1, \varphi_2, \varphi_3$ in the set. In other words, it is the smallest set containing all "most mean preserving spread" posterior maximum distributions. We can rewrite it using the expectation of G^{PP} as parameter. Let

$$\begin{aligned}
\mu_4 &= (p_{lh} + p_{mh} + p_{hh} + p_{hm} + p_{hl}) + 0.5(p_{lm} + p_{mm} + p_{ml}) \\
\mu_3 &= (p_{lh} + p_{mh} + p_{hh} + p_{hm} + p_{hl}) + 0.5p_{mm} + 0.25(p_{lm} + p_{ml}) \\
\mu_2 &= (p_{mh} + p_{hh} + p_{hm}) + 0.5(p_{lh} + p_{mm} + p_{hl}) + 0.25(p_{lm} + p_{ml}) \\
\mu_1 &= p_{hh} + 0.75(p_{mh} + p_{hm}) + 0.5(p_{lh} + p_{mm} + p_{hl}) + 0.25(p_{lm} + p_{ml})
\end{aligned}$$

When $\mu \in [\mu_1, \mu_2]$, we have

$$G^{PP}(x;\mu) = \begin{cases} 0 & x < 0 \\ p_{ll} & x \in [0, 0.25) \\ p_{ll} + p_{lm} + p_{ml} & x \in [0.25, 0.5) \\ p_{ll} + p_{mm} + p_{lm} + p_{ml} + p_{lh} + p_{hl} & x \in [0.5, 0.75) \\ 1 - p_{hh} - 4(\mu - \mu_1) & x \in [0.75, 1) \\ 1 & x \geq 1 \end{cases}$$

When $\mu \in [\mu_2, \mu_3]$, we have

$$G^{PP}(x;\mu) = \begin{cases} 0 & x < 0 \\ p_{ll} & x \in [0, 0.25) \\ p_{ll} + p_{lm} + p_{ml} & x \in [0.25, 0.5) \\ p_{ll} + p_{mm} + p_{lm} + p_{ml} + p_{lh} + p_{hl} - 2(\mu - \mu_2) & x \in [0.5, 0.75) \\ 1 - p_{hh} - p_{mh} - p_{hm} - 2(\mu - \mu_2) & x \in [0.75, 1) \\ 1 & x \geq 1 \end{cases}$$

When $\mu \in [\mu_3, \mu_4]$, we have

$$G^{PP}(x;\mu) = \begin{cases} 0 & x < 0 \\ p_{ll} & x \in [0, 0.25) \\ p_{ll} + p_{lm} + p_{ml} - 4(\mu - \mu_3) & x \in [0.25, 0.5) \\ p_{ll} + p_{mm} + p_{lm} + p_{ml} & x \in [0.5, 0.75) \\ 1 - p_{hh} - p_{mh} - p_{hm} - p_{hl} - p_{hl} & x \in [0.75, 1) \\ 1 & x \geq 1 \end{cases}$$

This finalizes the characterization of all the "most mean preserving spread" posterior maximum distributions without redundancy

$$\mathcal{G}^{PP} = \{G^{PP}(x;\mu) | \mu \in [\mu_1, \mu_4]\}$$

and also all the implementable posterior maximum distributions

$$\mathcal{G} = \{G | \exists G^{PP} \in \mathcal{G}^{PP}, G \text{ is mean preserving contraction of } G^{PP}\}$$

4 Surplus Characterization (To be written)

Appendix: Proofs in Section 3

References

Armstrong, M., and J. Zhou (2022): "Consumer information and the limits to competition," *American Economic Review*, 112(2), 534–577.

Blackwell, D. (1953): "Equivalent comparisons of experiments," *The annals of mathematical statistics*, pp. 265–272.

Deb, R., and A.-K. Roesler (2024): "Multi-Dimensional Screening: Buyer-Optimal Learning and Informational Robustness," *The Review of Economic Studies*, 91(5), 2744–2770.

Ichihashi, S. (2020): "Online privacy and information disclosure by consumers," *American Economic Review*, 110(2), 569–595.

Kleiner, A., B. Moldovanu, P. Strack, and M. Whitmeyer (2024): "The Extreme Points of Fusions," *arXiv preprint arXiv:2409.10779*.

Matějka, F., and A. McKay (2015): "Rational inattention to discrete choices: A new foundation for the multinomial logit model," *American Economic Review*, 105(1), 272–298.

Ravid, D., A.-K. Roesler, and B. Szentes (2022): "Learning before trading: on the inefficiency of ignoring free information," *Journal of Political Economy*, 130(2), 346–387.

Roesler, A.-K., and B. Szentes (2017): "Buyer-optimal learning and monopoly pricing," *American Economic Review*, 107(7), 2072–2080.

Strack, P., and K. H. Yang (2024): "Privacy-Preserving Signals," *Econometrica*, 92(6), 1907–1938.