# Application for Supporting Hyperplane Theorem: Never Best Response and Strictly Dominated Strategies in Normal Form Game (Recitation 1 & 2)

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Additional knowledge:

**Theorem 1** (Supporting Hyperplane Theorem). *Let $C$ be a convex set in $\mathbb{R}^n$ and $x_0$ be a point on the boundary of $C$, there exists $a \in \mathbb{R}^n$, s.t. $ax \leq ax_0, \forall x \in C$*

Here begins the application.

**Definition 1** (Normal Form Game). A Normal Form Game is a triplet $(N, A, u)$

- $N = \{1, 2, ..., n\}$ is a finite set of player
- $A = A_1 \times A_2 \times ... \times A_n$ and $A_i$ is set of actions of player $i$
- $u = (u_1, u_2, ..., u_n)$ and $u_i : A \mapsto \mathbb{R}$ is pay-off function of player $i$

For any (measurable) space $X$, let $\Delta(X)$ be the set of probability measures defined on $X$.

**Definition 2** (Strategy)**.** A (mixed) strategy $\sigma_i$ of player $i$ is a measure defined on $A_i$; i.e. $\sigma_i \in \Delta(A_i)$

Let $A_{-i} = A_1 \times A_2 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_n$

**Definition 3** (Belief)**.** A belief $\sigma_{-i}$ of player $i$ is a measure defined on $A_{-i}$; i.e. $\sigma_{-i} \in \Delta(A_{-i})$. A belief is independent if $\sigma_{-i} \in \Delta(A_1) \times \Delta(A_2) ... \times \Delta(A_{i-1}) \times \Delta(A_{i+1}) \times ... \times \Delta(A_n)$.

Note: This is abuse of notation. Conventionally, $\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$ denotes the strategy profile of all players but $i$. Yet here, it is just the belief of player $i$. They usually coincides in Nash Equilibrium.

Think of von Neumann–Morgenstern preference over $\Delta(A)$; that is, player $i$ cares about $u_i(\sigma) = \mathbb{E}_\sigma[u_i] = \sum_{a \in A} \sigma(a) u_i(a)$ of strategy profile $\sigma$ or equivalently, strategy profile $\sigma$ is better than $\sigma'$ to player $i$ when $u_i(\sigma) \geq u_i(\sigma')$.

**Example 1** (Prisoner Dilemma)**.** $N = 2, A_i = \{C_i, D_i\}$ and $u$ described by

| | $C_2$ | $D_2$ |
|---|---|---|
| $C_1$ | 1,1 | -1,2 |
| $D_1$ | 2,-1 | 0,0 |

Example Strategies of player 1:

- Always cooperates: $\sigma_1(C_1) = 1$, we tend to write $C_1$
- Always defects: $\sigma_1(D_1) = 1$, we tend to write $D_1$
- Half-half: $\sigma_1(C_1) = \sigma(D_1) = \frac{1}{2}$, we tend to write $\frac{1}{2}C_1 + \frac{1}{2}D_1$

Example Beliefs of player 1:

- Always cooperates: $\sigma_{-1}(C_2) = 1$

- Always defects: $\sigma_{-1}(D_2) = 1$

- Half-half: $\sigma_{-1}(C_2) = \sigma_{-1}(D_2) = \frac{1}{2}$

Some strategy profiles $(C_1, C_2), (\frac{1}{2}C_1 + \frac{1}{2}D_1, D_2), \ldots.$

Player 1 prefers strategy profile $(\frac{1}{2}C_1 + \frac{1}{2}D_1, D_2)$ than $(C_1, C_2)$ as

$$u_1(\frac{1}{2}C_1 + \frac{1}{2}D_1, C_2) = \frac{1}{2}u_1(C_1, C_2) + \frac{1}{2}u_1(D_1, C_2) = \frac{3}{2}$$

$$u_1(C_1, C_2) = 1$$

If someone else have a belief about what strategy profile will be played in this game, belief $\frac{1}{2}(C_1, C_2) + \frac{1}{2}(D_1, D_2)$ is not independent, while belief $\frac{1}{4}(C_1, C_2) + \frac{1}{4}(C_1, D_2) + \frac{1}{4}(D_1, C_2) + \frac{1}{4}(D_1, D_2) = (\frac{1}{2}C_1 + \frac{1}{2}D_1) \times (\frac{1}{2}C_2 + \frac{1}{2}D_2)$ is independent.

**Definition 4** (Best Response)**.** A strategy $\sigma_i$ of player $i$ is a best response to belief $\sigma_{-i}$ of player $i$ if for arbitrary strategy $\sigma'_i$ of player $i$, $u(\sigma_i, \sigma_{-i}) \geq u(\sigma'_i, \sigma_{-i})$

**Definition 5.** A strategy $\sigma_i$ of player $i$ is strictly dominated if there exists a strategy $\sigma'_i$, such that $u(\sigma'_i, a_{-i}) > u(\sigma_i, a_{-i}), \forall a_{-i} \in A_{-i}$

**Theorem 2.** *In a finite (action) game, a strategy is never a best response to any (potentially correlated) belief if and only if it is strictly dominated.*

*Proof.* "If" side is trivial. If a strategy $\sigma_i$ is strictly dominated by $\sigma'_i$, $\sigma_i$ is never best response to any belief as $\sigma'_i$ is always going to be a strictly better response.

"Only if" side is equivalent to the statement "any strategy that is not strictly dominated is a best response to some (potentially correlated) belief". Suppose a strategy $\sigma_i^0$ is not strictly dominated, we are going to construct such a belief to which $\sigma_i^0$ is best response.

Consider the following set

$$C = \{x \in \mathbb{R}^{|A_{-i}|} : \exists \sigma_i \in \Delta(A_i), x \leq u_i(\sigma_i) = (u(\sigma_i, a_{-i}))_{a_{-i} \in A_{-i}}\}$$

$C$ is convex. If $x_1, x_2 \in C$, then there exists $\sigma_i^{(1)}, \sigma_i^{(2)} \in \Delta(A_i)$, such that $x_1 \leq u(\sigma_i^{(1)}), x_2 \leq u(\sigma_i^{(2)})$. Then,

$$\alpha x_1 + (1-\alpha)x_2 \leq \alpha u(\sigma_i^{(1)}) + (1-\alpha)u(\sigma_i^{(2)}) = u(\alpha\sigma_i^{(1)} + (1-\alpha)\sigma_i^{(2)}), \forall \alpha \in [0,1]$$

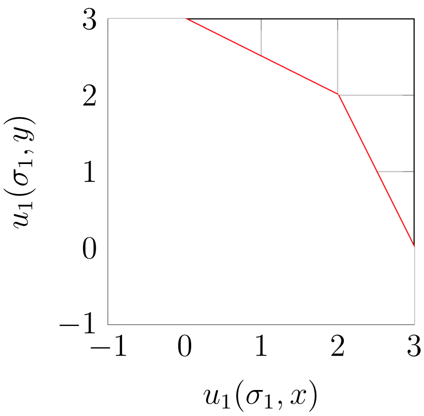
and $\alpha\sigma_i^{(1)} + (1-\alpha)\sigma_i^{(2)} \in \Delta(A_i)$. That is to say, $\alpha x_1 + (1-\alpha)x_2 \in C$.

$u(\sigma_i^0)$ is on the boundary of $C$. Otherwise, $u(\sigma_i^0)$ is in the interior of $C$. But if it is in the interior of $C$, then there exists $\varepsilon, x \in \mathbb{R}^{|A_{-i}|}$, where $x >> 0$ and $u_i(\sigma_i^0) = x + \varepsilon \leq u_i(\sigma_i^x) + \varepsilon$ for some $\sigma_i^x \in \Delta(A_1)$ determined by $x$. That is to say, $\sigma_i^0$ is strictly dominated by $\sigma_i^x$. This conflicts with $\sigma_0^i$ is not strictly dominated.

Combining the previous two observations, by Supporting Hyperplane Theorem, we can find $a \in \mathbb{R}^{|A_{-i}|}$, such that $au_i(\sigma_0^i) \geq ax, \forall x \in C$. As in every direction, $-\infty$ is in $C$, we have $a > 0$. The belief can then be constructed by $\frac{a}{||a||}$. $\sigma_i^0$ is a best response to $\frac{a}{||a||}$. □

**Example 2** (Numerical Example for the Proof)**.** Consider a case where $n = 2, A_1 = \{a, b, c, d\}, A_2 = \{x, y\}$

| $u_1(a_1, a_2)$ | $x$ | $y$ |
|---|---|---|
| $a$ | 3 | 0 |
| $b$ | 0 | 3 |
| $c$ | 2 | 2 |
| $d$ | 1 | 1 |




White area (extending outside the graph) depicts $C$. Payoff vectors of strategies that are not strictly dominated lie on the red curve. Belief is a direction on the graph.

**Example 3** (Strategies Never Best Response to Independent Belief may not be Strictly Dominated)**.** $N = 2, A_1 = \{U, D\}, A_2 = \{L, R\}, A_3 = \{a, b, c, d\}$ and $u_3$ is described by

| $a$ | $L$ | $R$ |
|---|---|---|
| $U$ | 9 | 0 |
| $D$ | 0 | 0 |

| $b$ | $L$ | $R$ |
|---|---|---|
| $U$ | 0 | 9 |
| $D$ | 9 | 0 |

| $c$ | $L$ | $R$ |
|---|---|---|
| $U$ | 0 | 0 |
| $D$ | 0 | 9 |

| $d$ | $L$ | $R$ |
|---|---|---|
| $U$ | 6 | 0 |
| $D$ | 0 | 6 |

$d$ is not best response to any **independent** belief. Suppose Player 1 uses $pU + (1-p)D$ and player 2 uses $qL + (1-q)D$. Need to show $d$ cannot be better than $a, b, c$ at the same time. Expected payoff of using $d$ is $6(pq + (1-p)(1-q))$; expected payoff of using $a$ is $9pq$; expected payoff of using $b$ is $9p(1-q)+9q(1-p)$; expected payoff of using $c$ is $9(1-p)(1-q)$. $d$ better than $a$ requires

$$6(pq+(1-p)(1-q)) \geq 9pq \Rightarrow \frac{pq}{(1-p)(1-q)} \leq 2$$

$d$ better than $c$ requires

$$6(pq+(1-p)(1-q)) \geq 9(1-p)(1-q) \Rightarrow \frac{pq}{(1-p)(1-q)} \geq \frac{1}{2}$$

$d$ better than $b$ requires

$$6(pq+(1-p)(1-q)) \geq 9p(1-q)+9q(1-p) \Rightarrow 3\frac{p}{1-p}+3\frac{q}{1-q}-2\frac{pq}{(1-p)(1-q)} \leq 2$$

The last one implies

$$6\sqrt{\frac{pq}{(1-p)(1-q)}}-2\frac{pq}{(1-p)(1-q)} \leq 2$$

It can be easily verified the three inequalies cannot hold at the same time.

And $d$ is not strictly dominated. To strictly dominate $d$, a strategy needs to put more than $\frac{2}{3}$ probability on both $a$ and $c$, but that is more than 1.