

# Solving Continuous Problems with Guarantees

Till Miltzow



# Algorithms

# Algorithms

Shortest Path

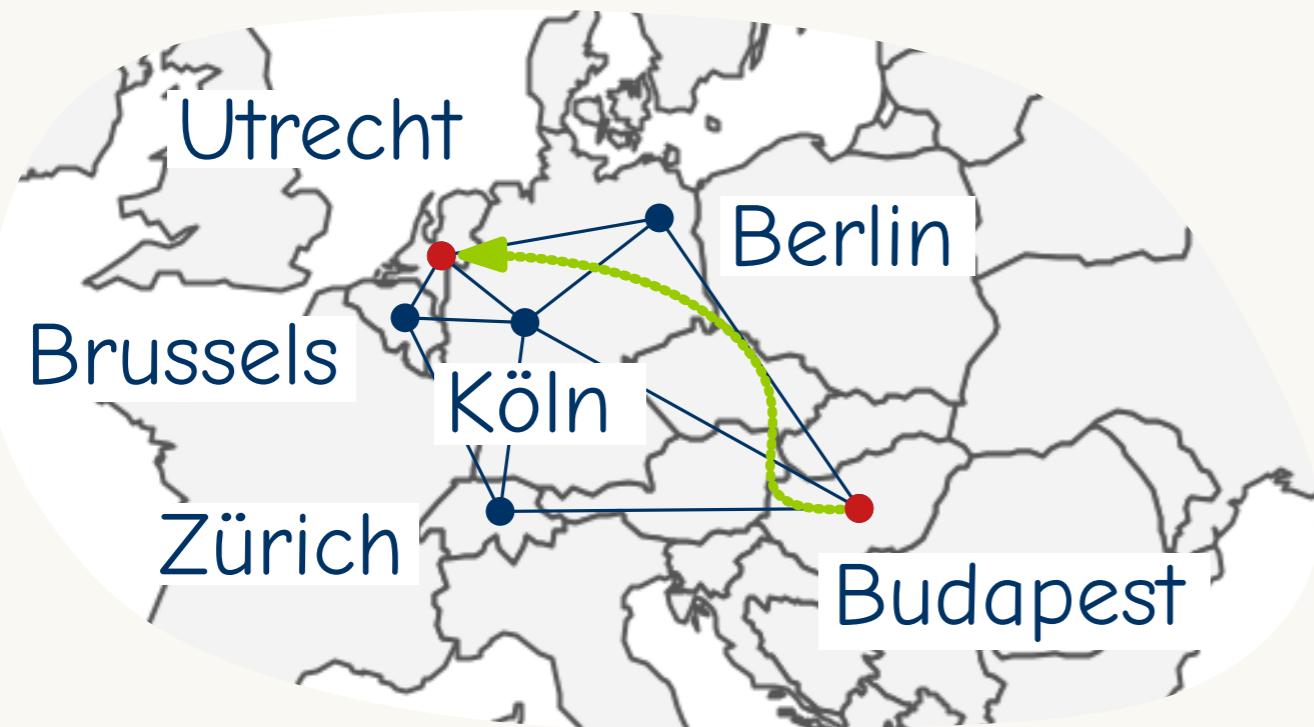
Problem

# Algorithms

Shortest Path

Problem

Instance



# Algorithms

Shortest Path

Problem

Instance



Solution

# Algorithms

Shortest Path



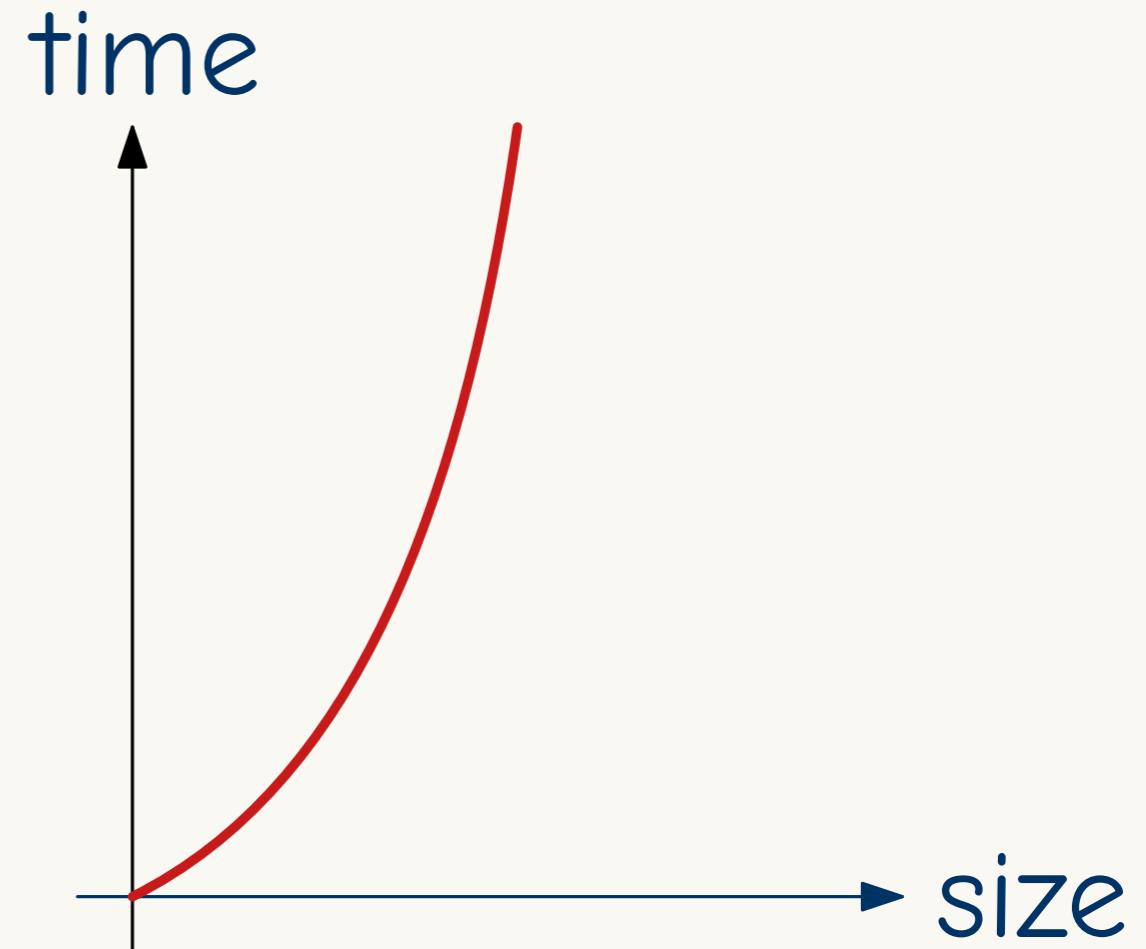
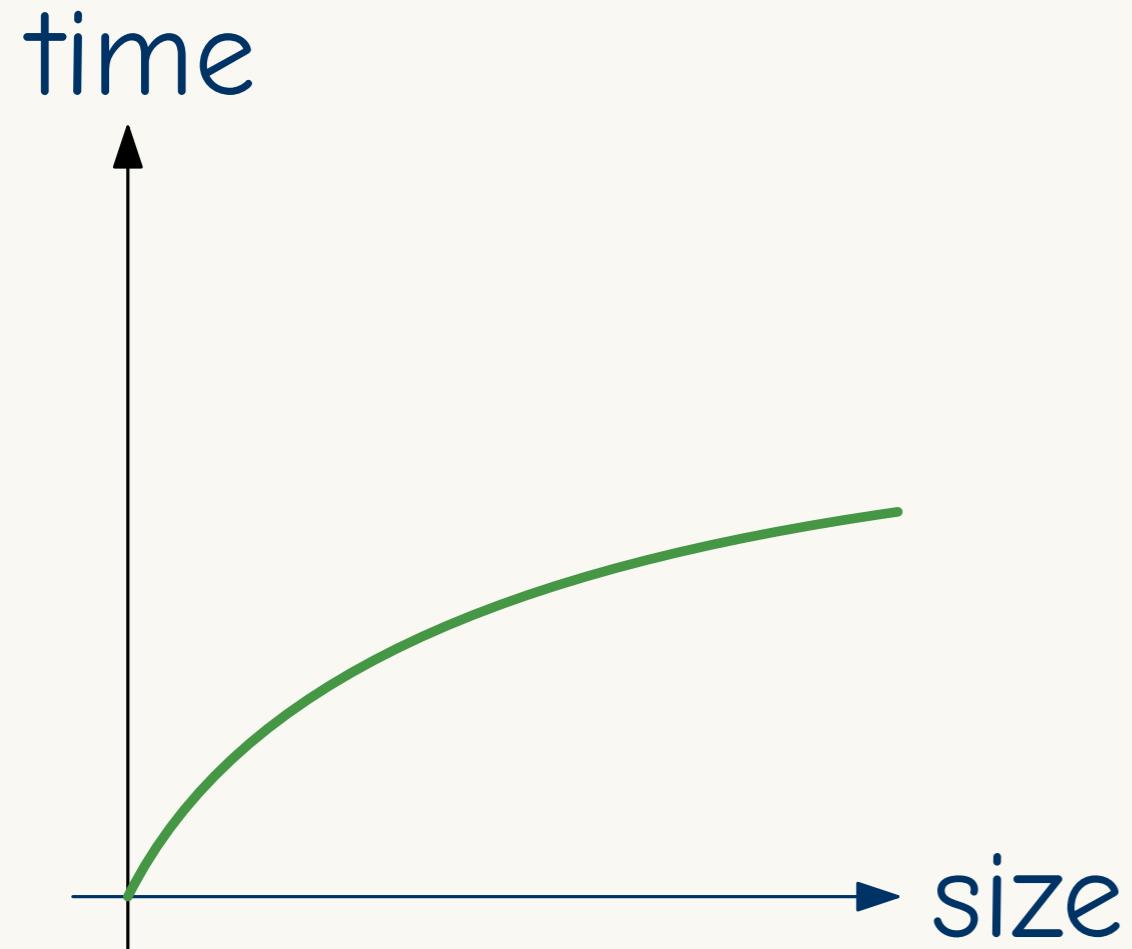
Problem

Instance

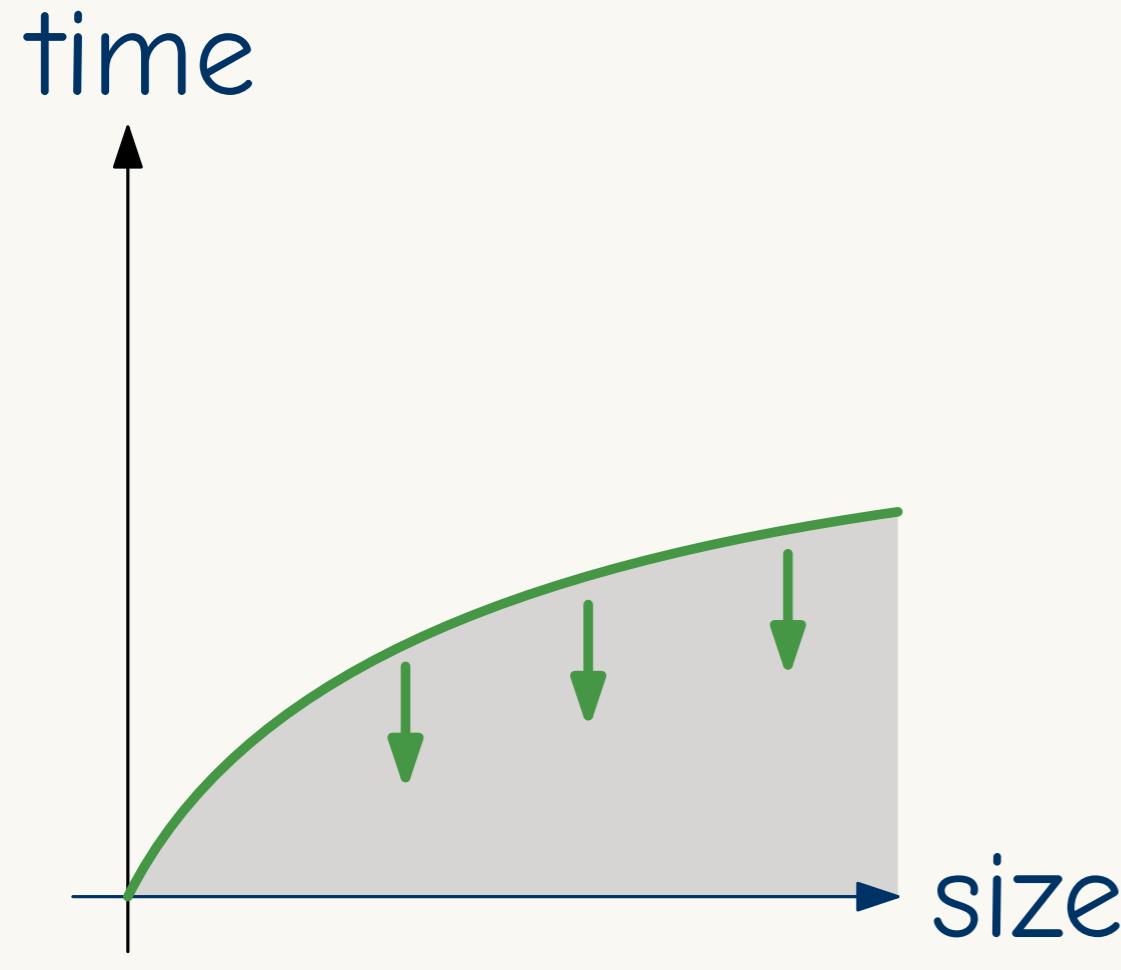
Algorithm

Solution

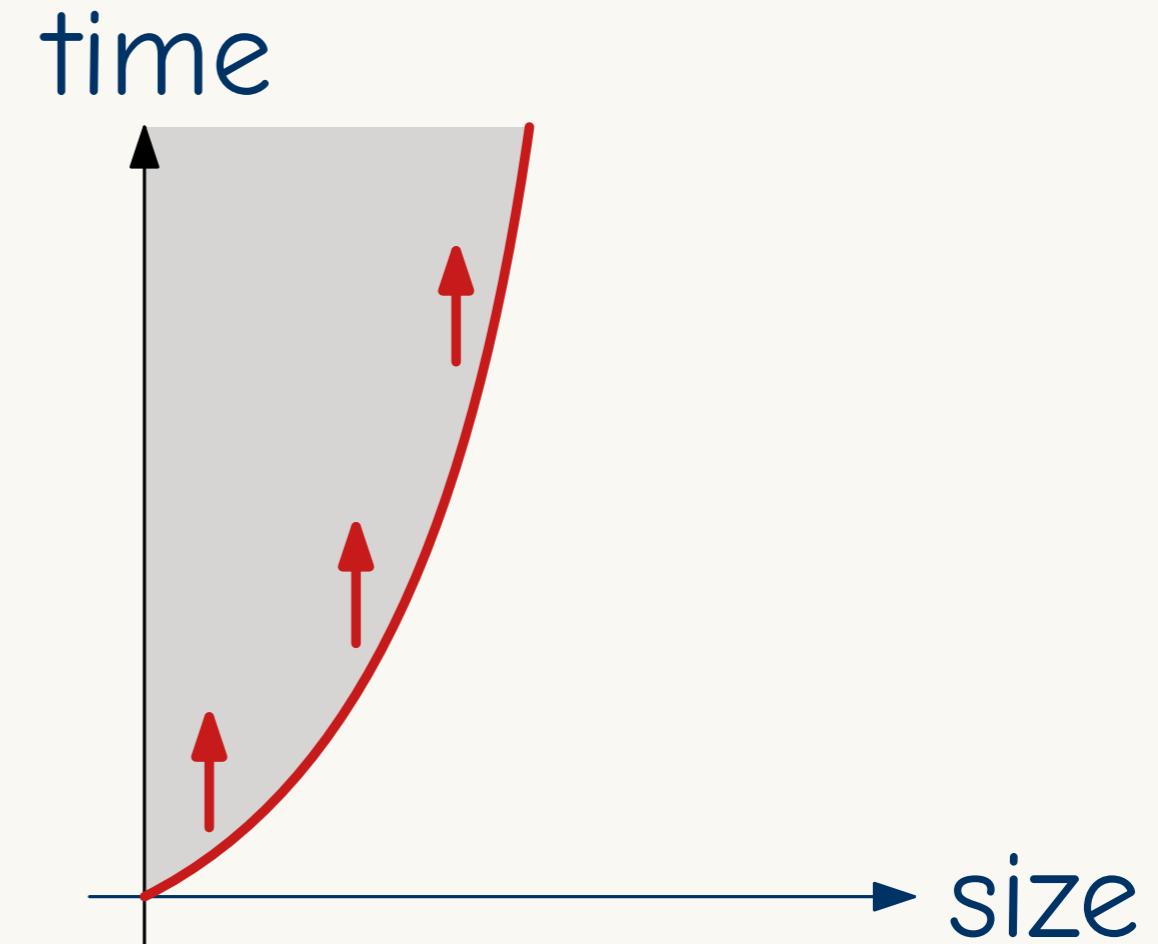
# Efficiency



# Efficiency



Upper Bounds



Lower Bounds

# Efficient Computability

# Efficient Computability

P

“easy”

# Efficient Computability

P

“easy”

NP

combinatorial  
explosion

# Efficient Computability

P

“easy”

NP

combinatorial  
explosion

awareness  
techniques

# Efficient Computability

P

“easy”

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explosion

awareness  
techniques

ER

continuous  
non-linear

# Efficient Computability

P

“easy”

NP

combinatorial  
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techniques

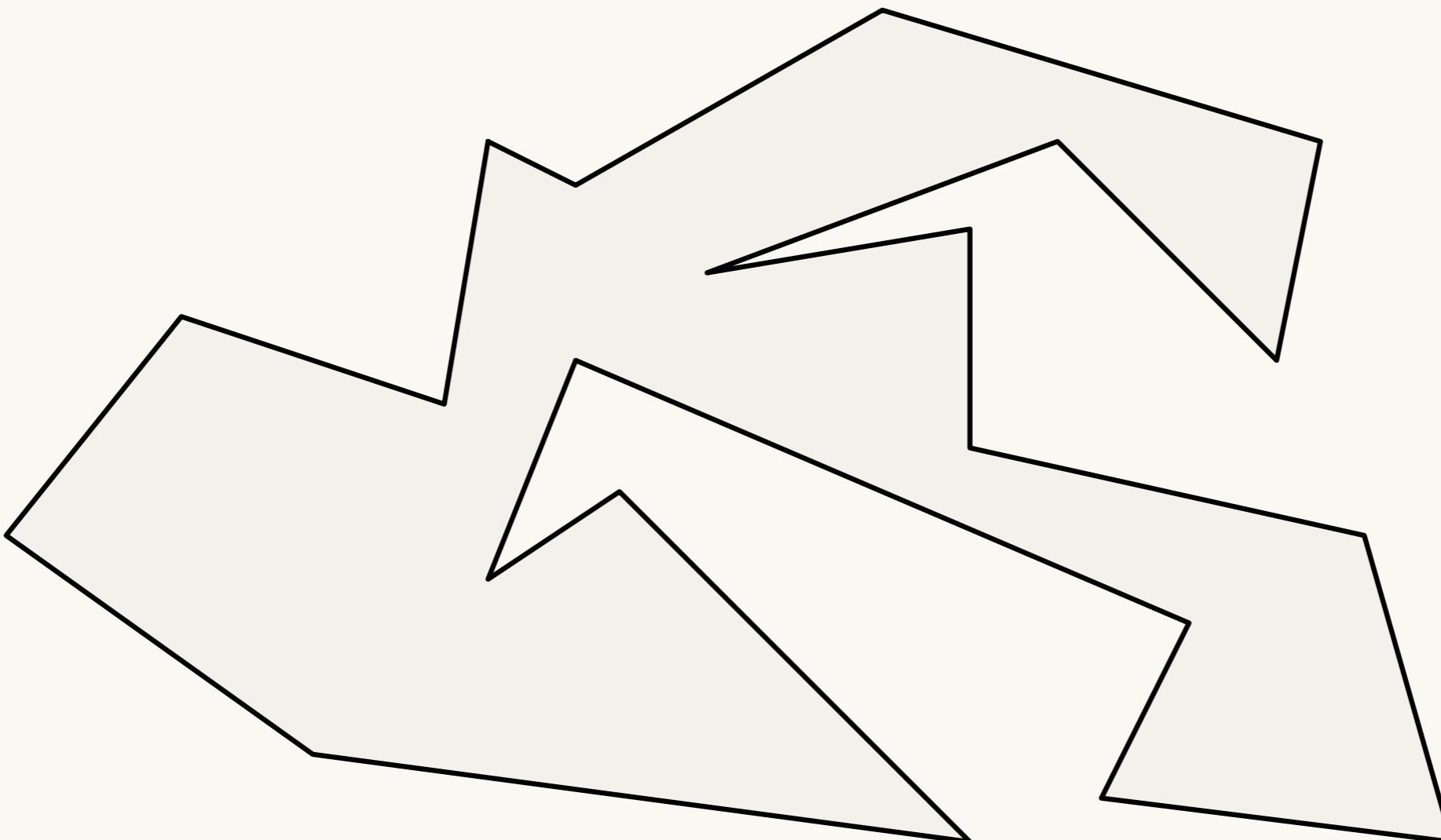
NP

continuous  
non-linear

awareness  
techniques

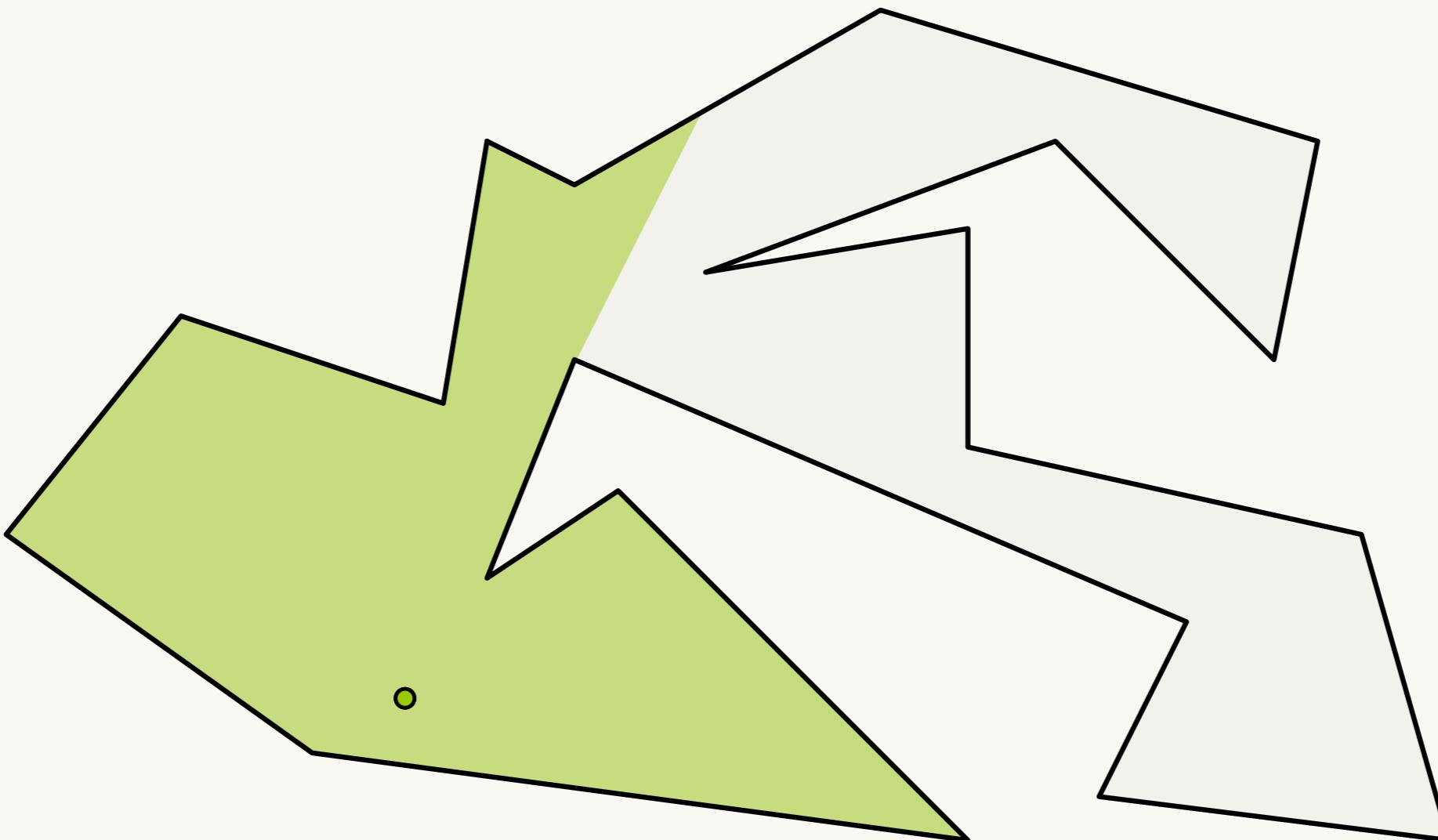
# Continuous Problems

## Art Gallery Problem



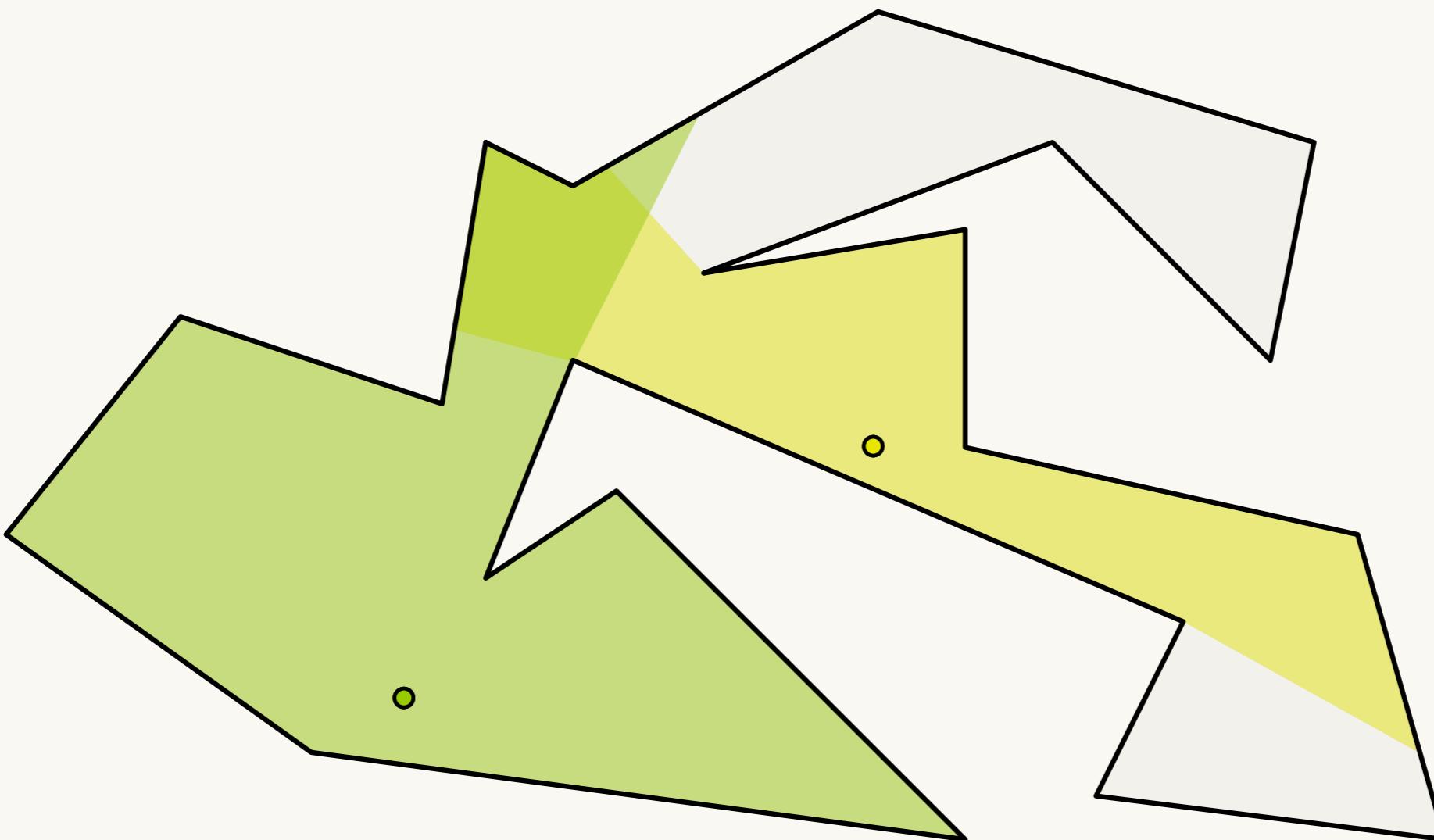
# Continuous Problems

## Art Gallery Problem



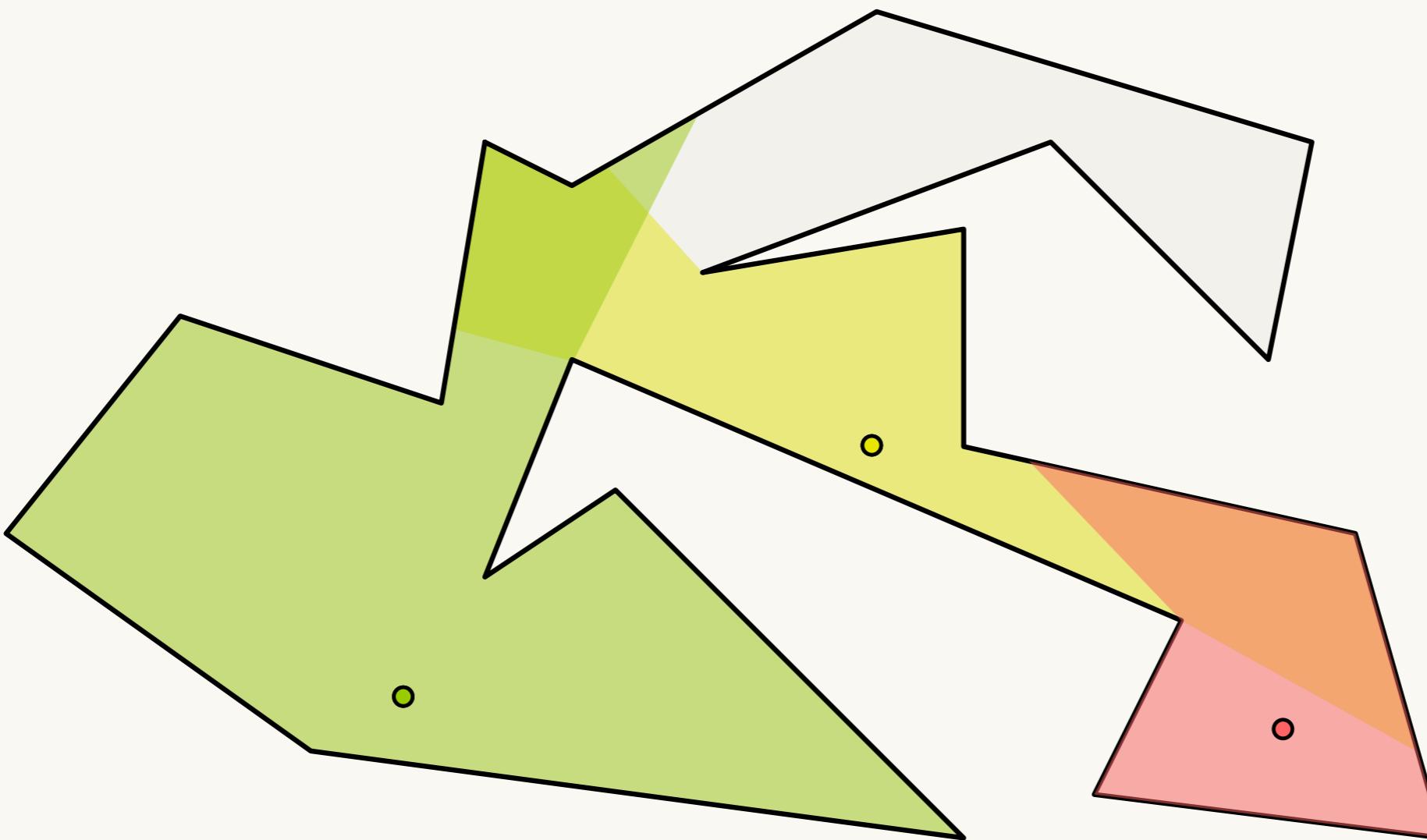
# Continuous Problems

## Art Gallery Problem



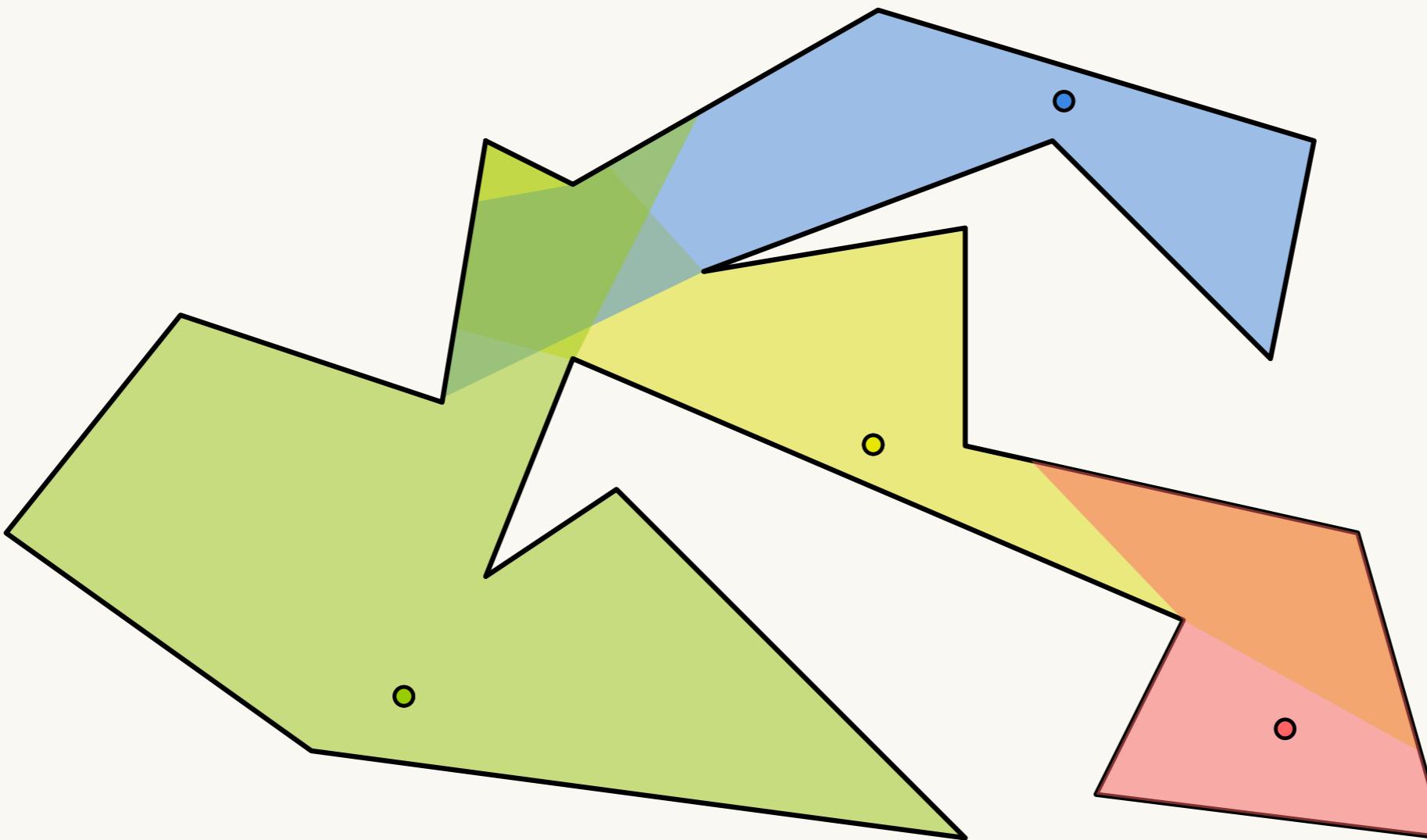
# Continuous Problems

## Art Gallery Problem



# Continuous Problems

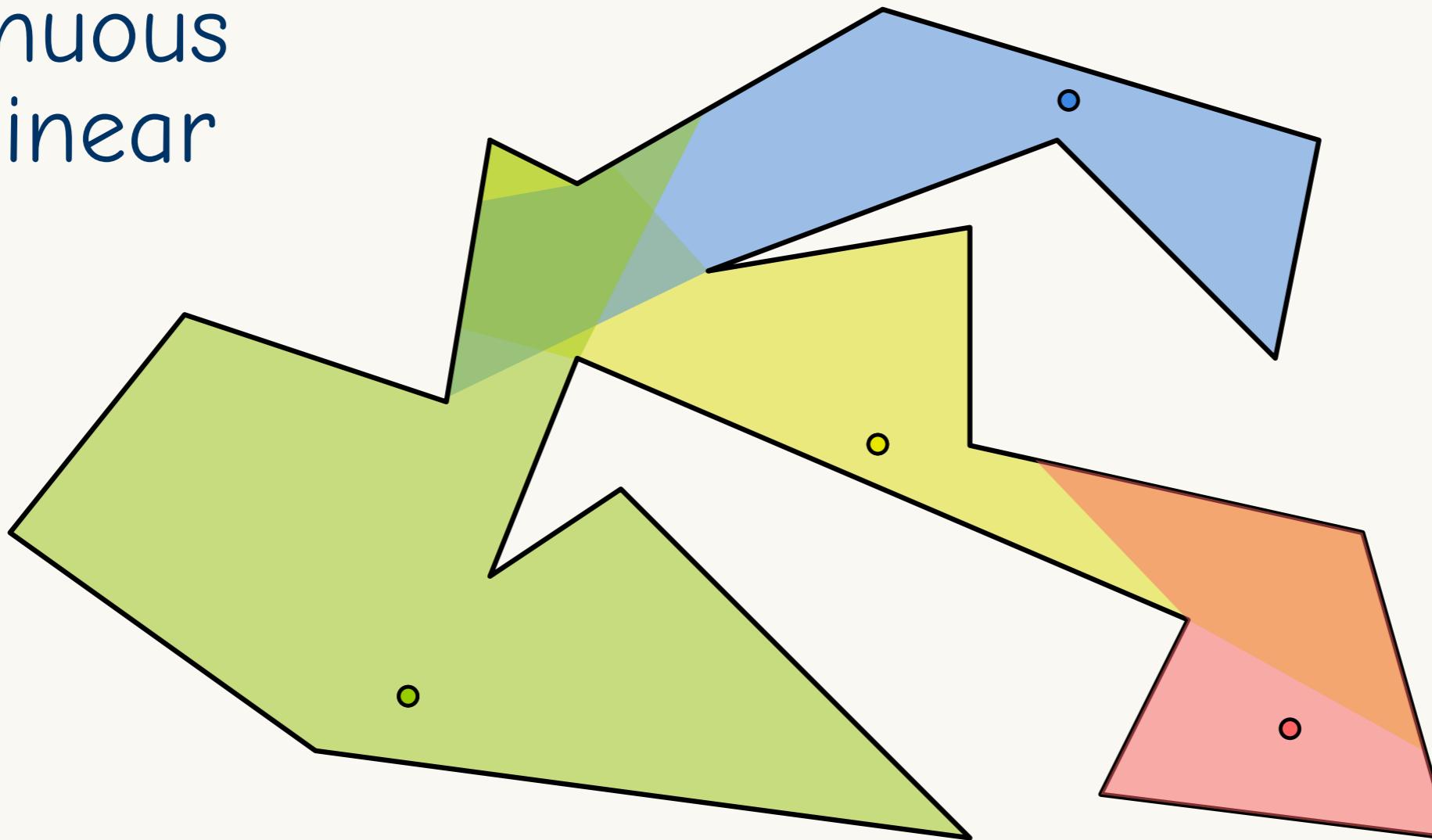
## Art Gallery Problem



# Continuous Problems

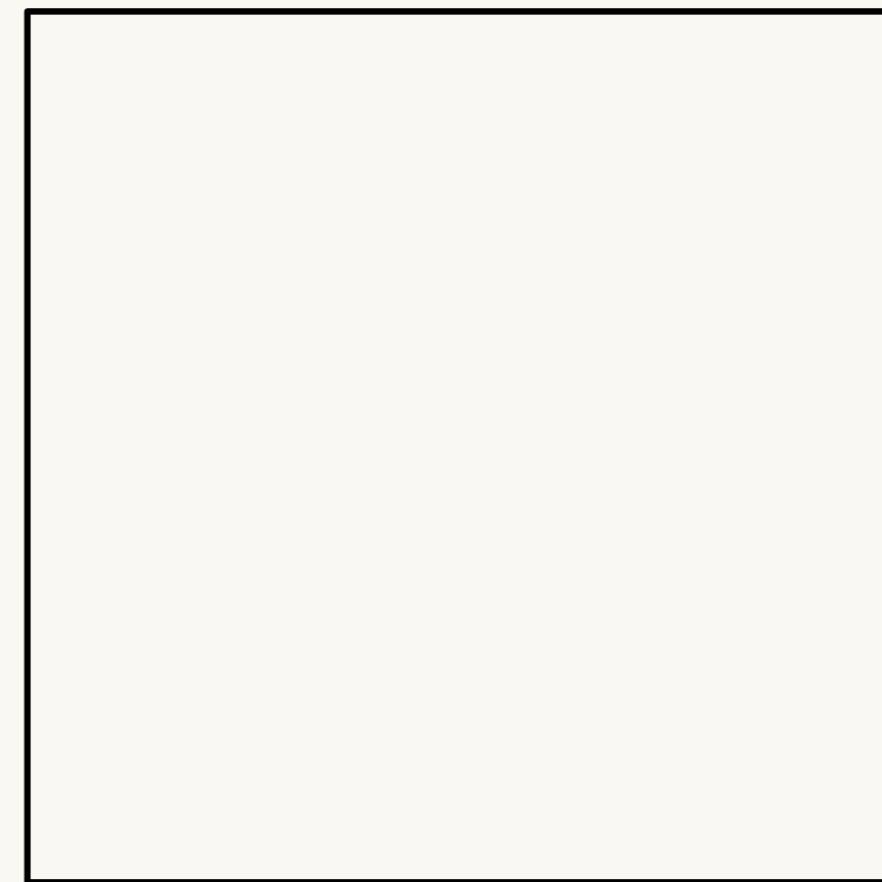
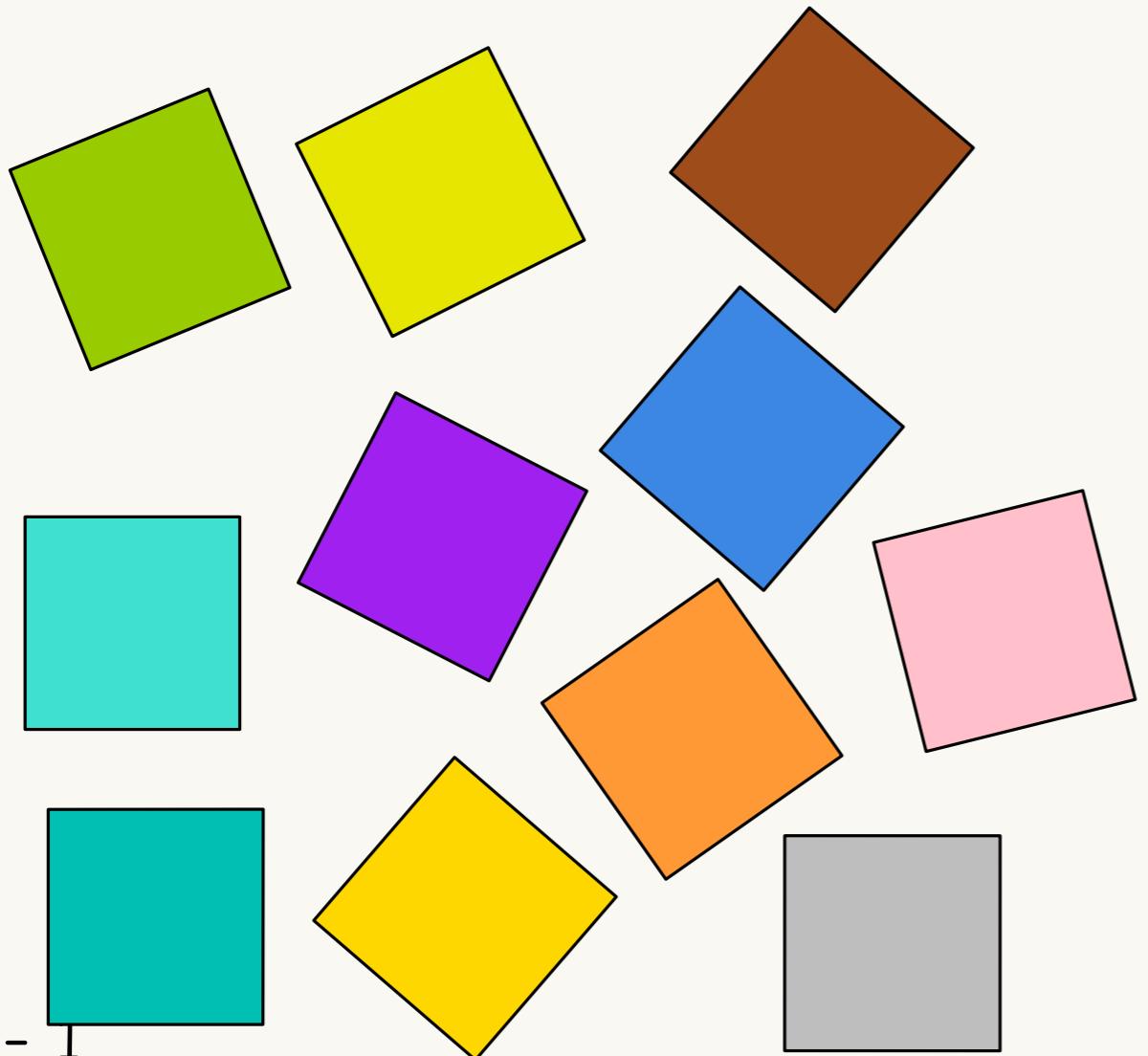
## Art Gallery Problem

continuous  
non-linear



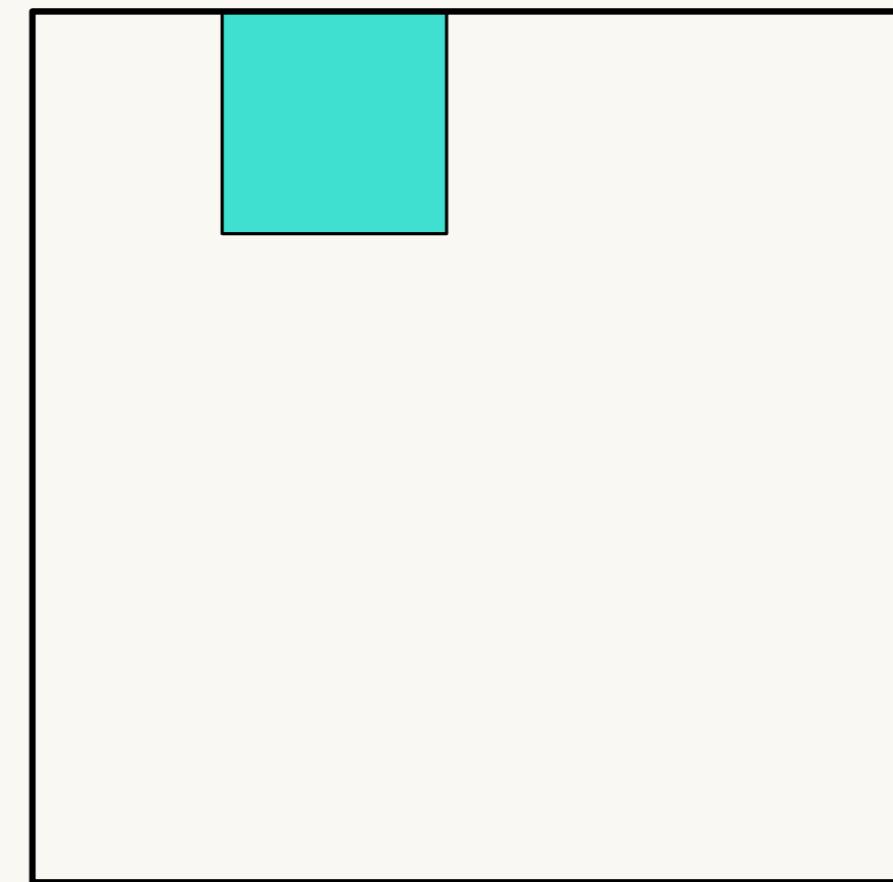
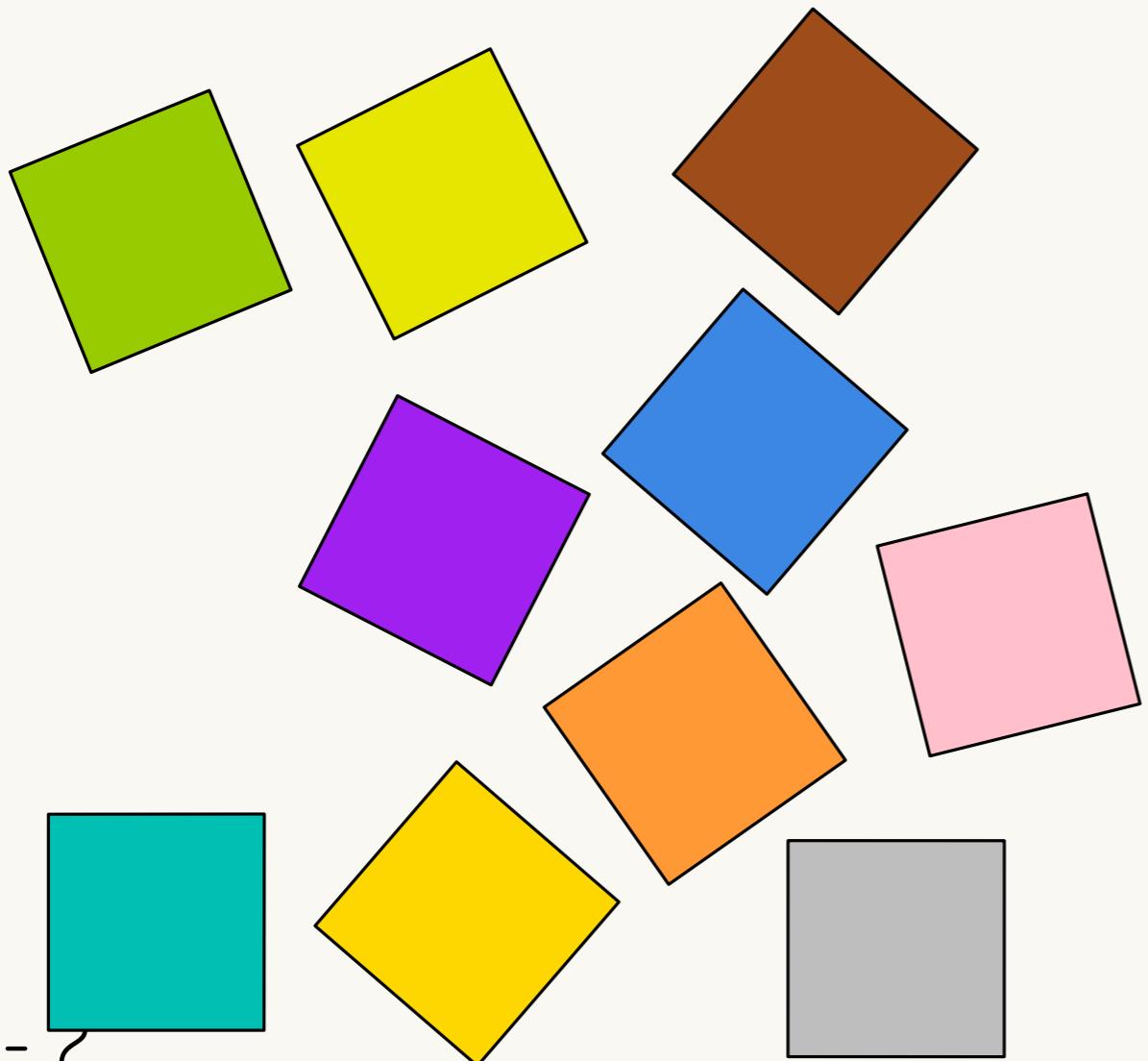
# Continuous Problems

## Packing



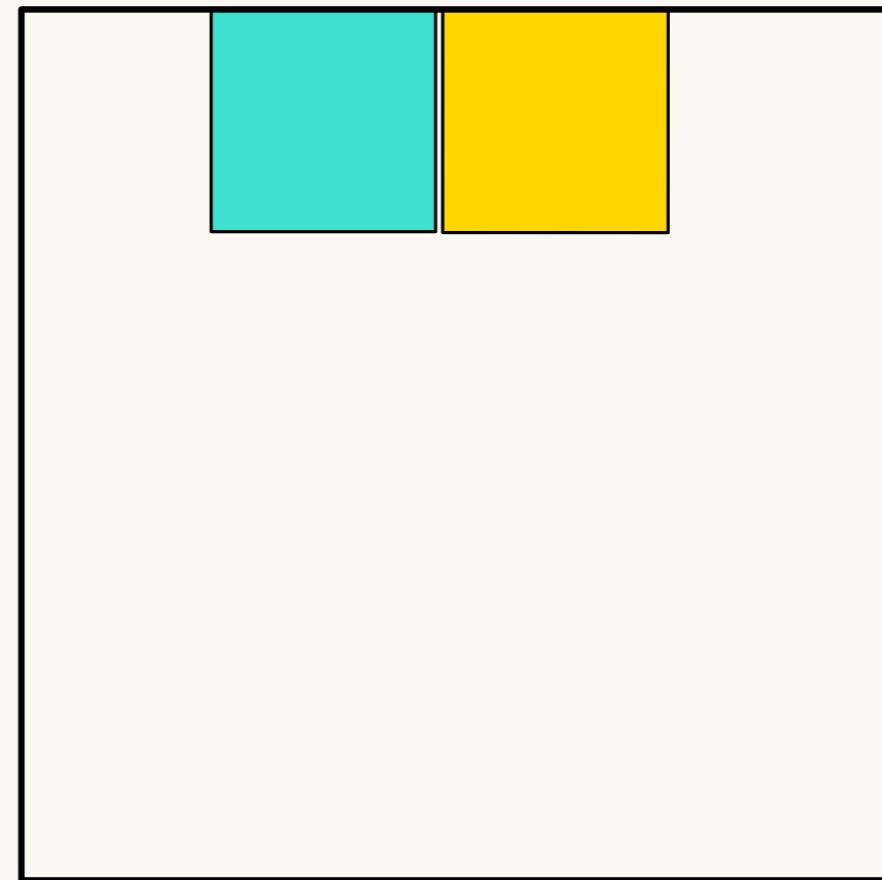
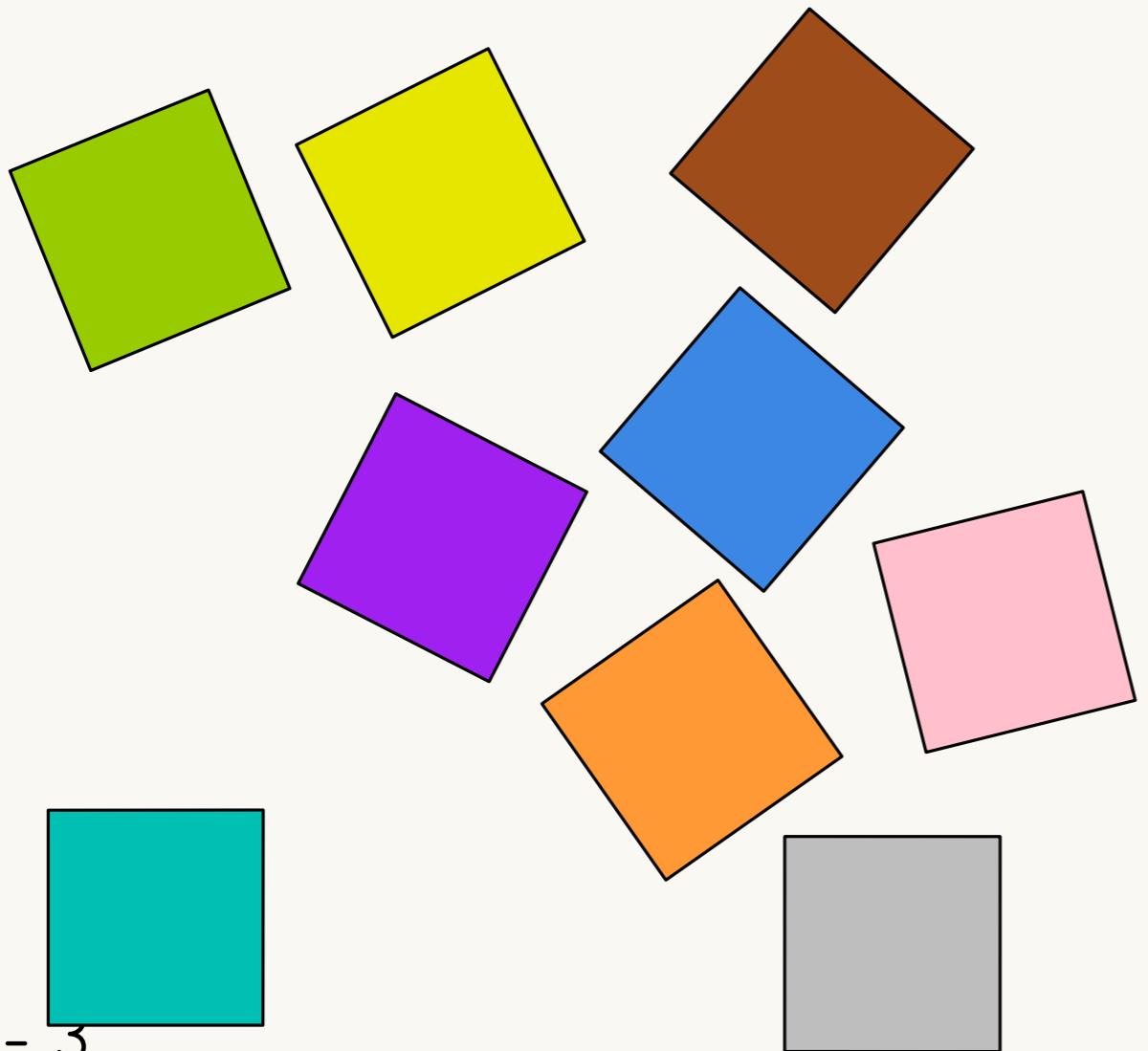
# Continuous Problems

## Packing



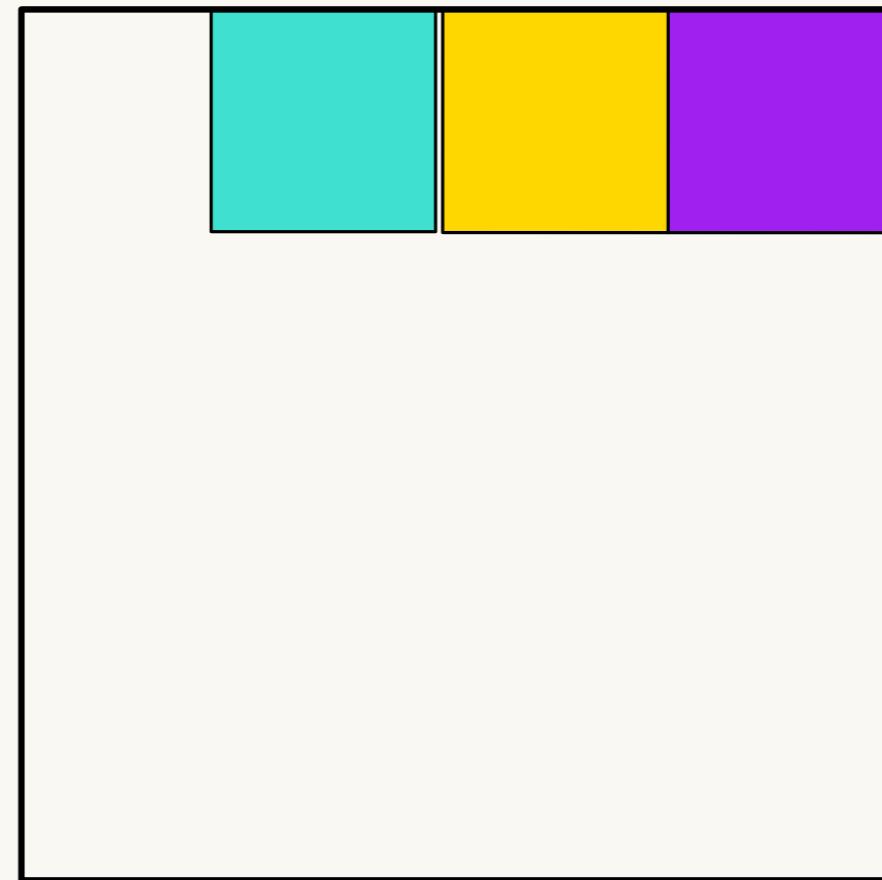
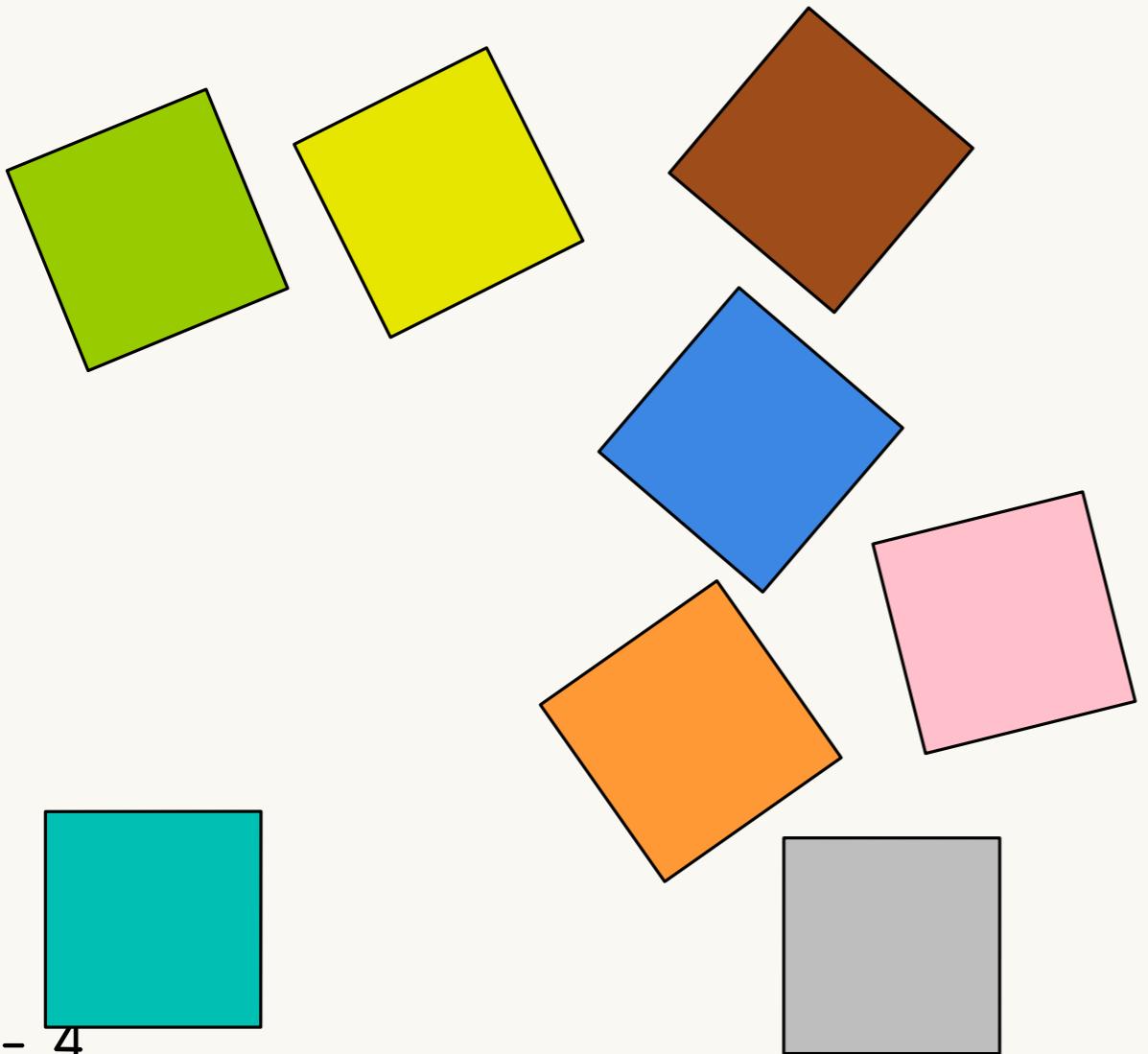
# Continuous Problems

## Packing



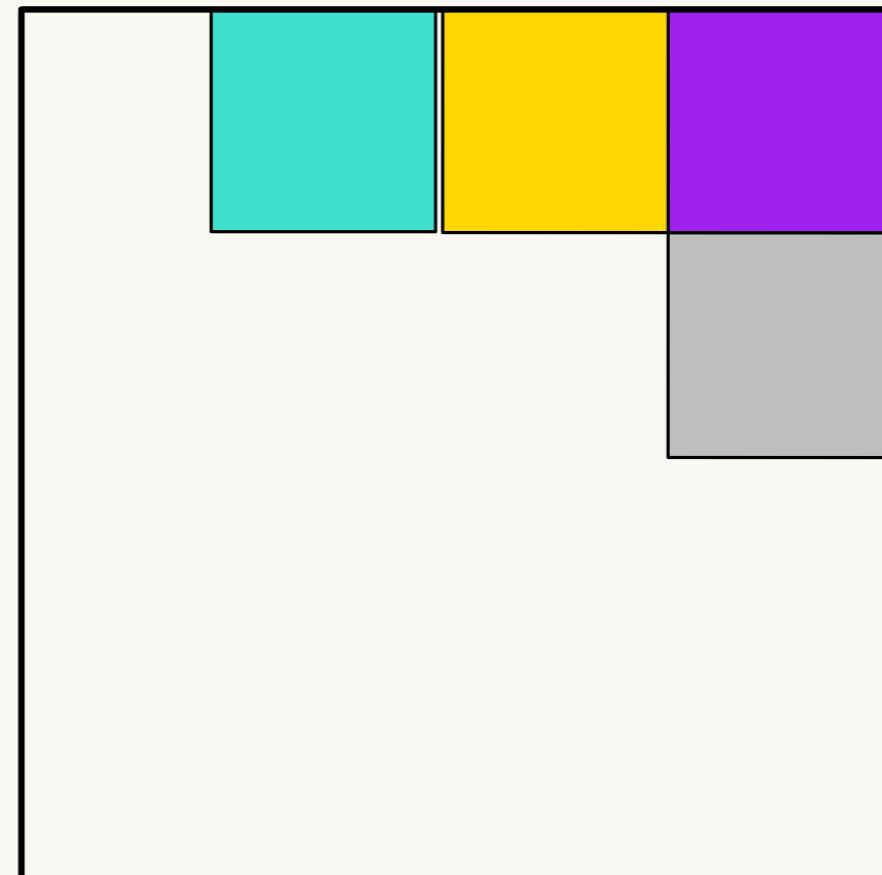
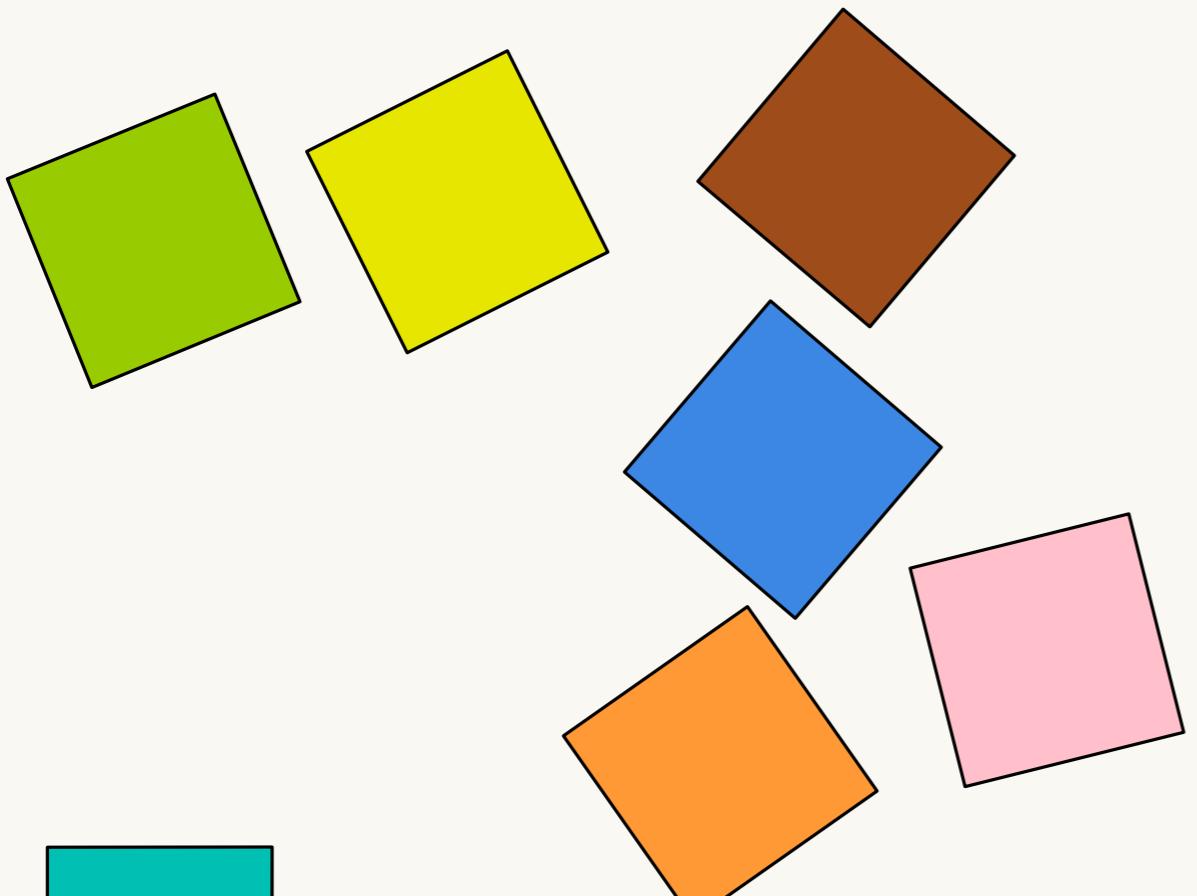
# Continuous Problems

## Packing



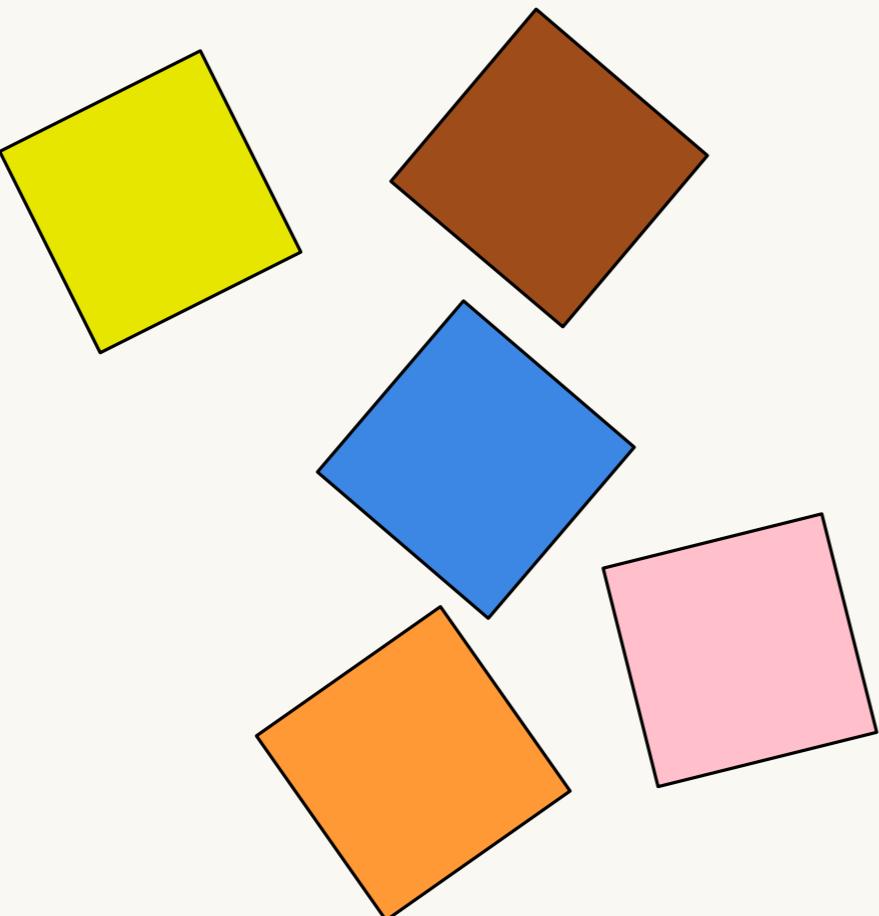
# Continuous Problems

## Packing

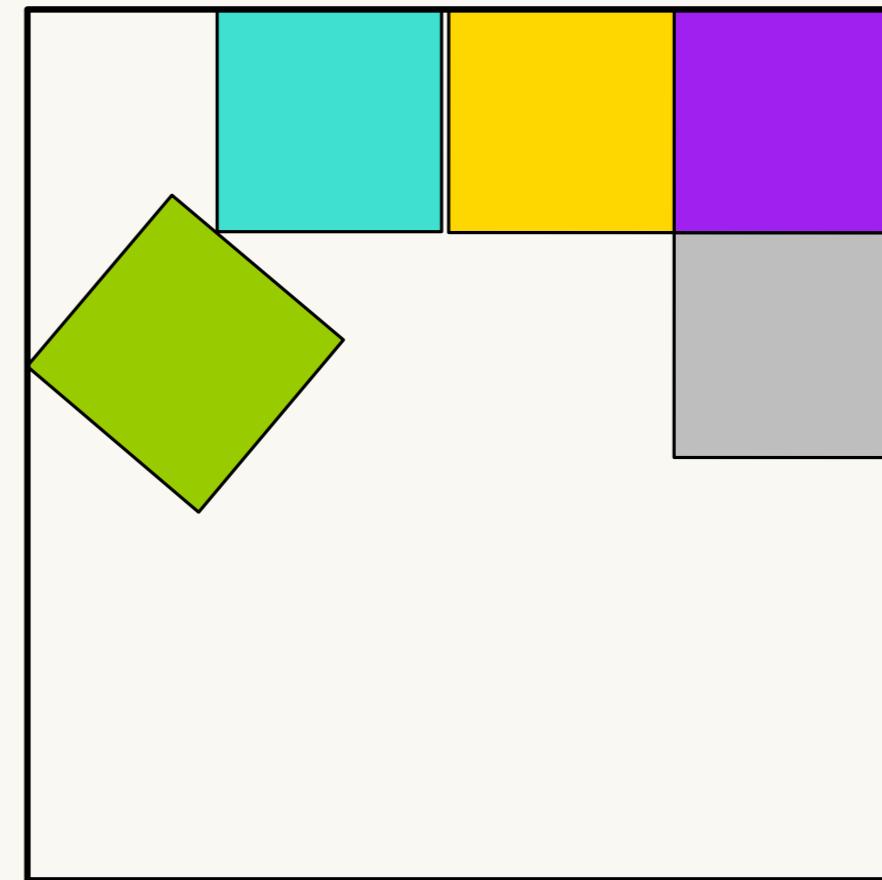


# Continuous Problems

## Packing

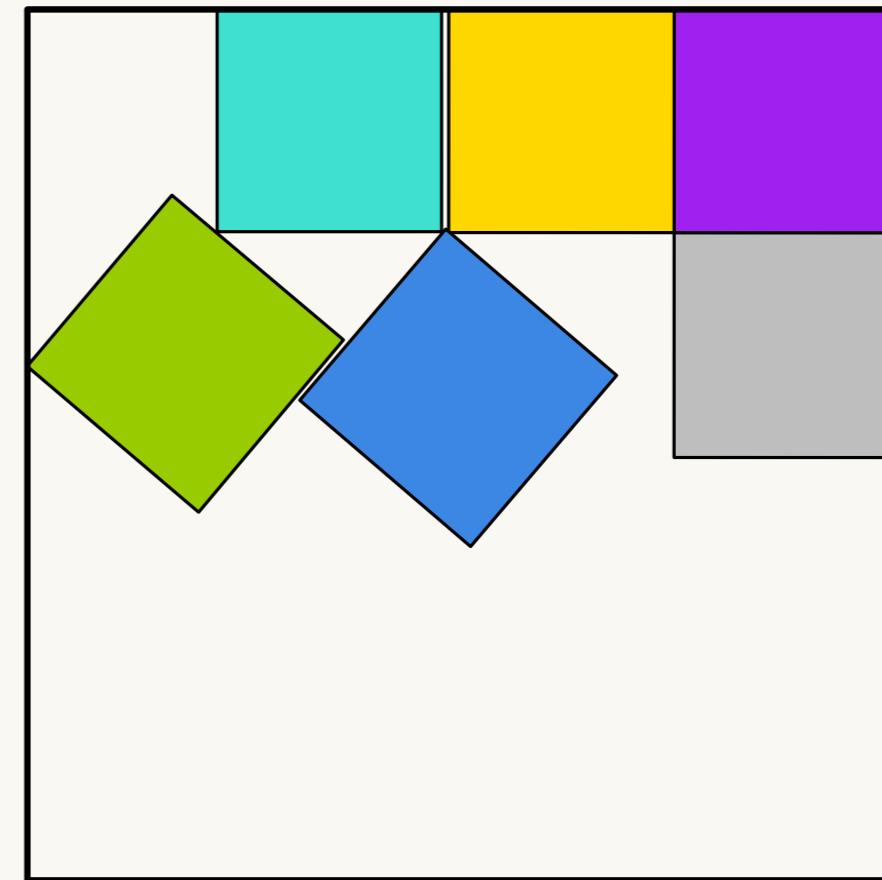
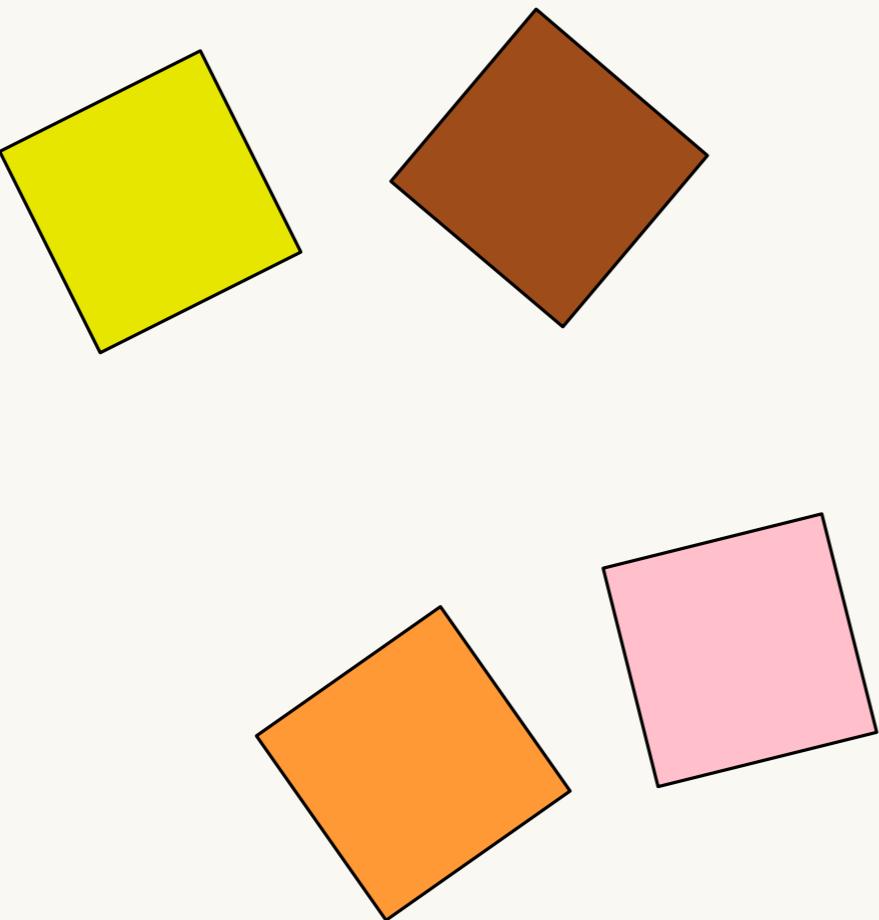


7 - 6



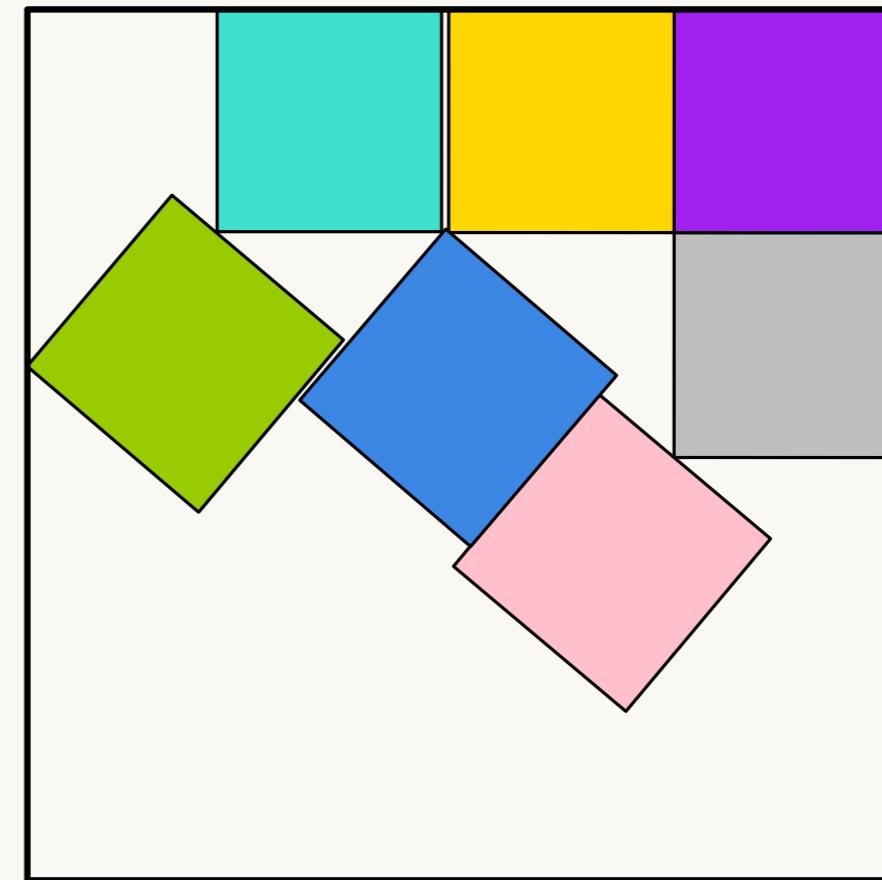
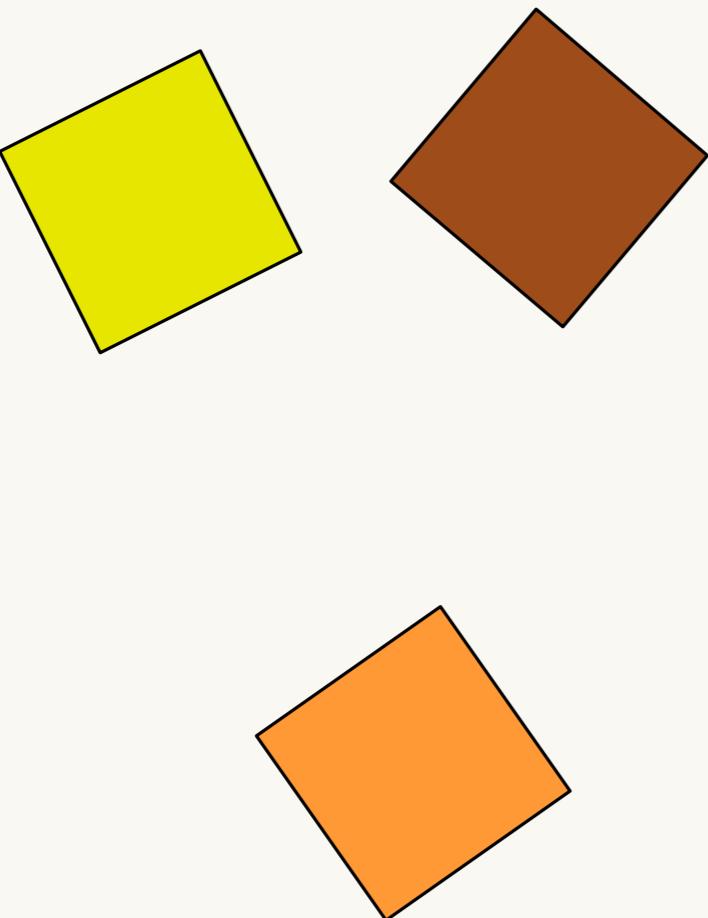
# Continuous Problems

## Packing



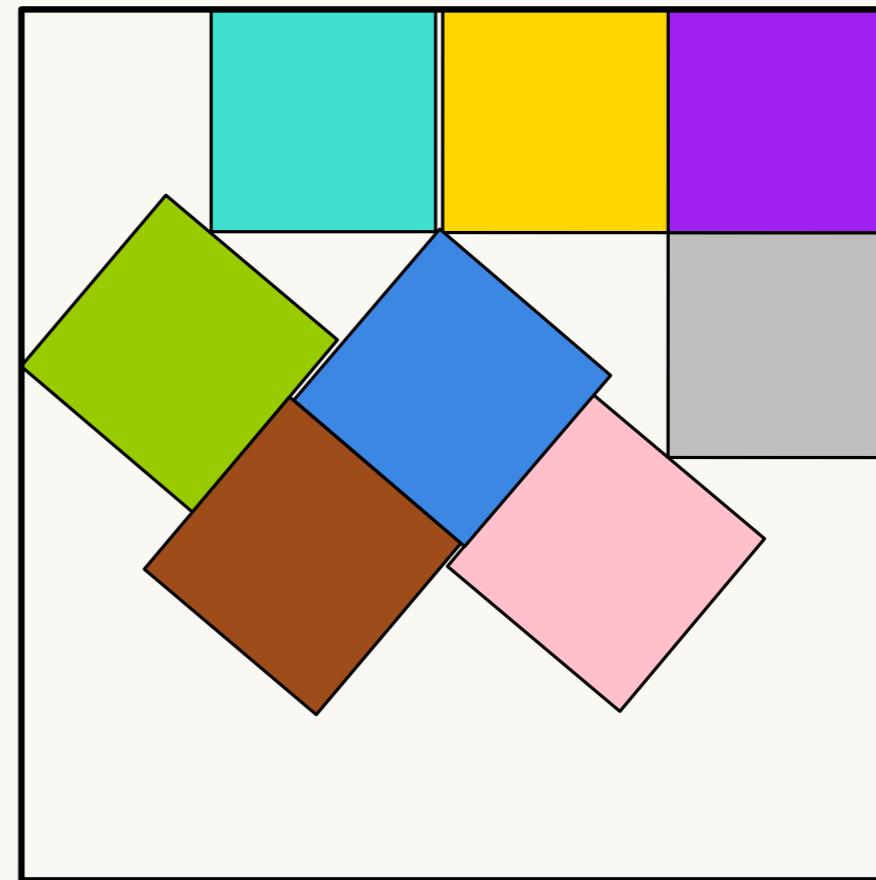
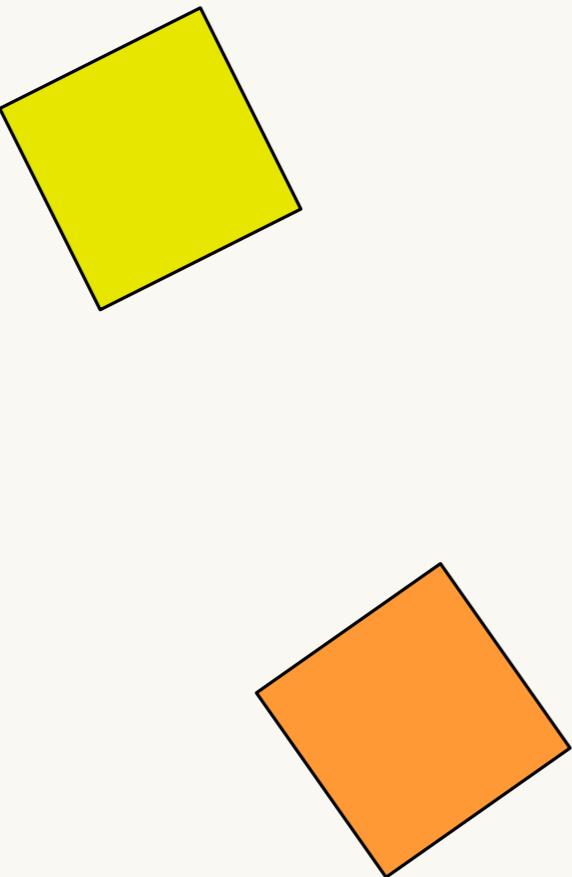
# Continuous Problems

## Packing



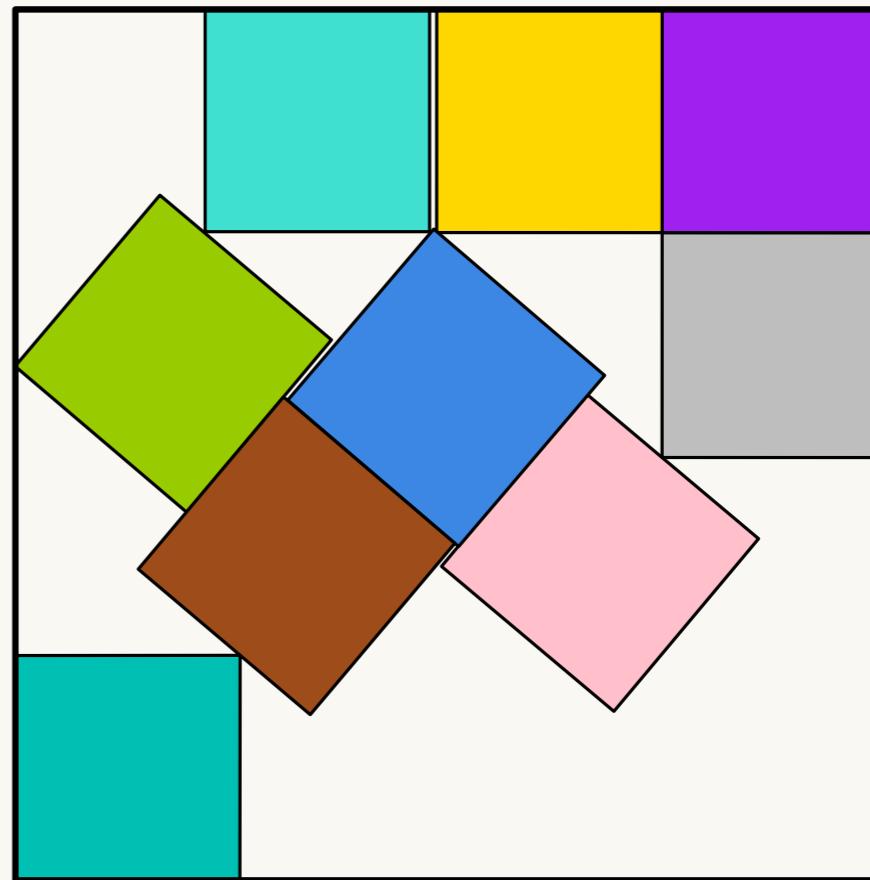
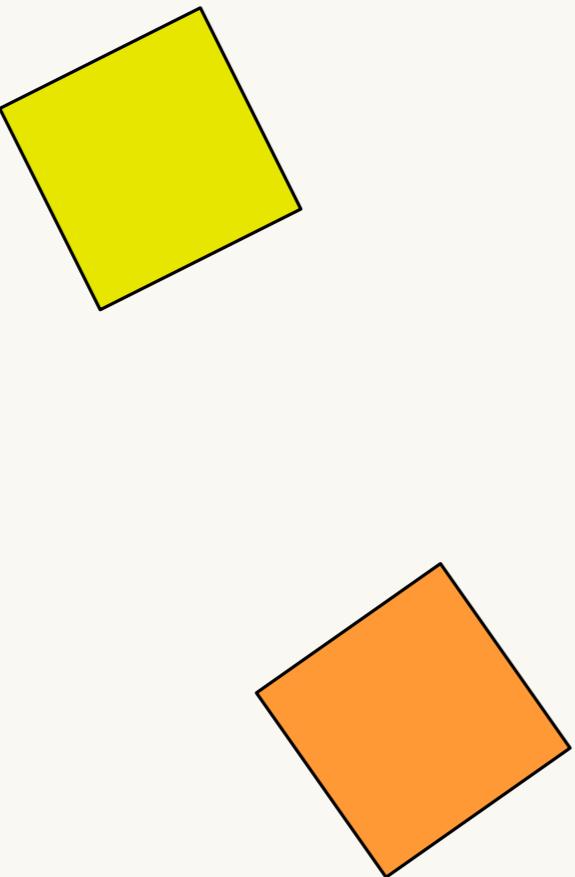
# Continuous Problems

## Packing



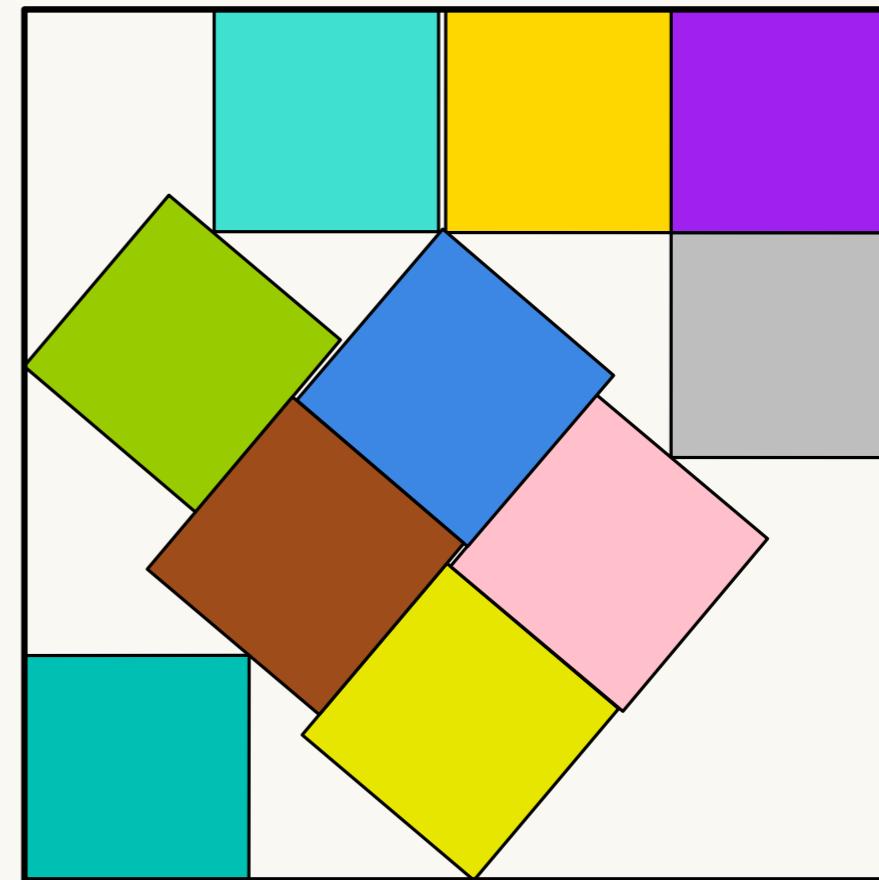
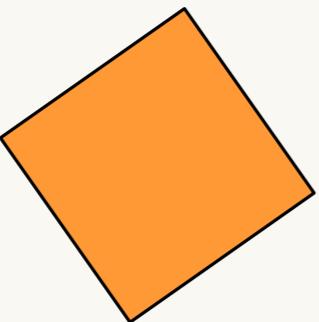
# Continuous Problems

## Packing



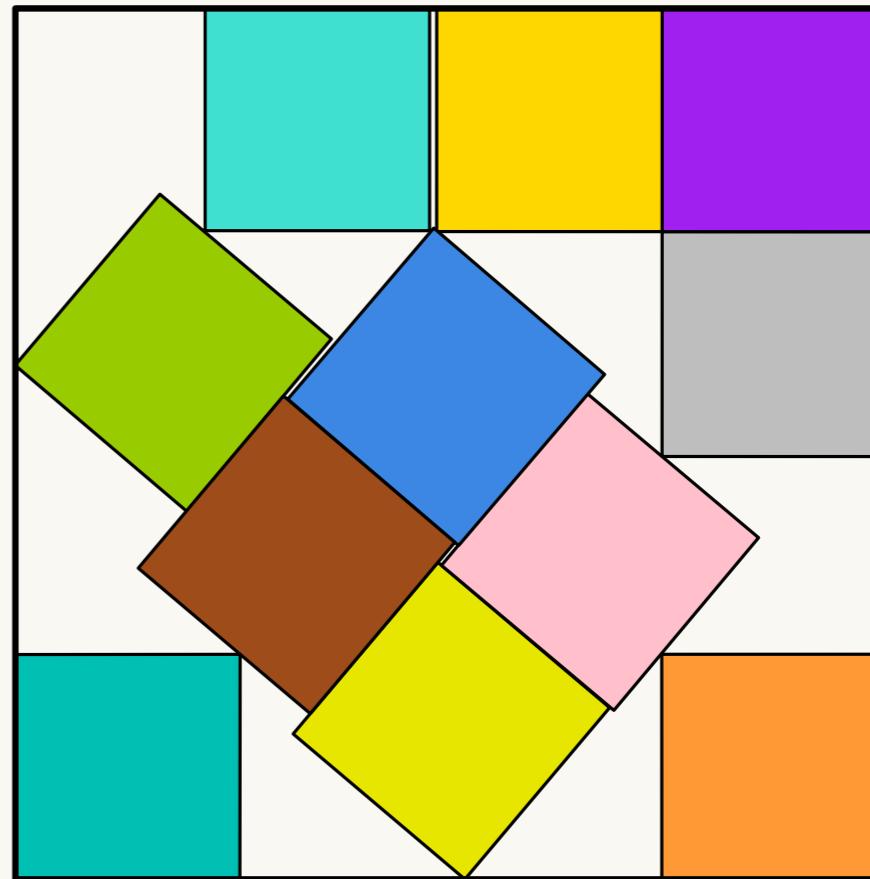
# Continuous Problems

## Packing



# Continuous Problems

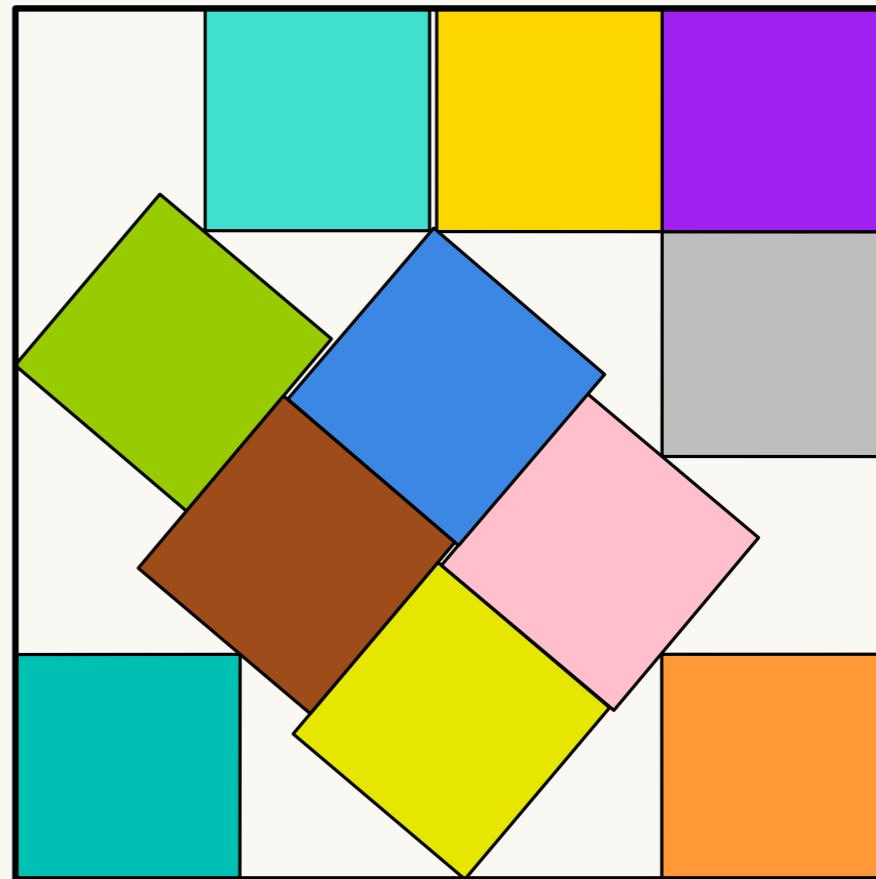
## Packing



# Continuous Problems

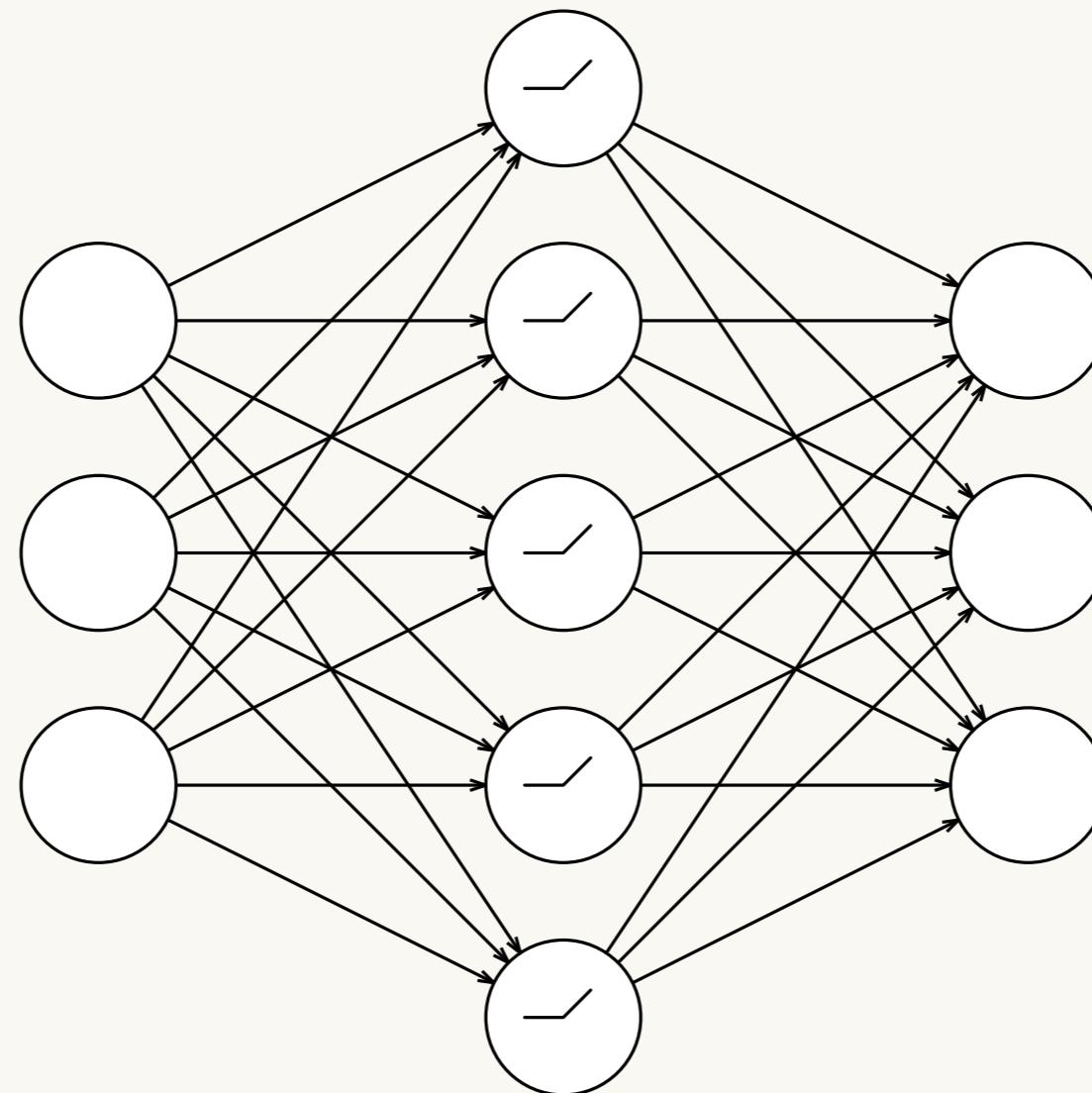
## Packing

continuous  
non-linear



# Continuous Problems

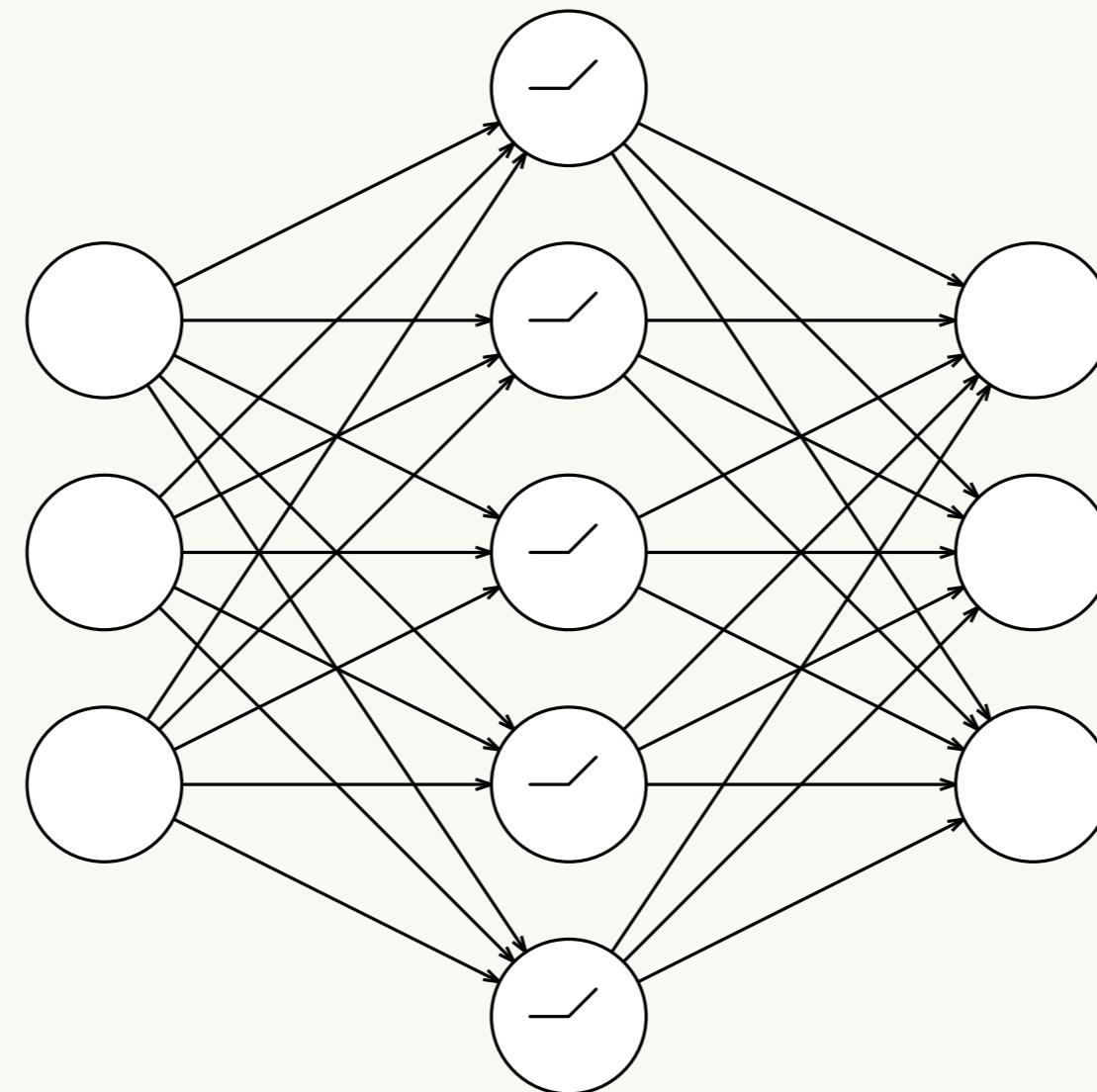
## Neural Networks



# Continuous Problems

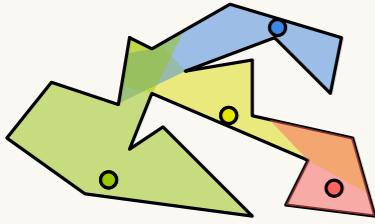
## Neural Networks

continuous  
non-linear



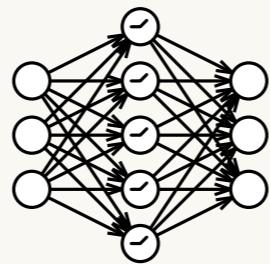
# Continuous Problems

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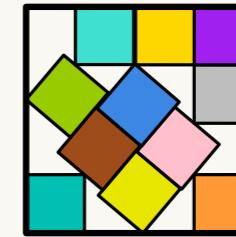
ER

JACM



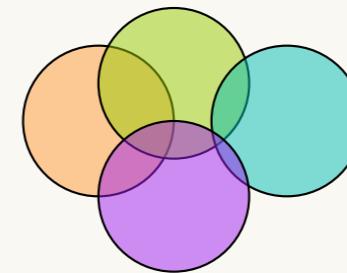
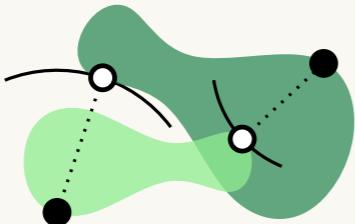
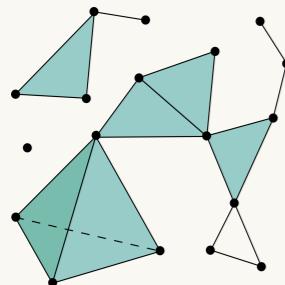
ER

NeurIPS

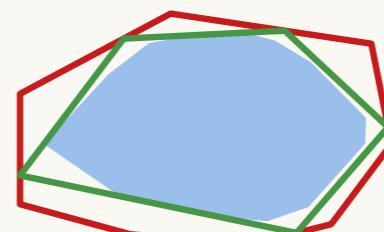
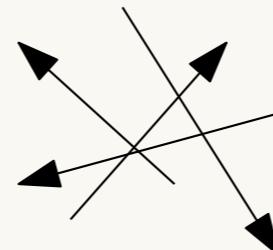
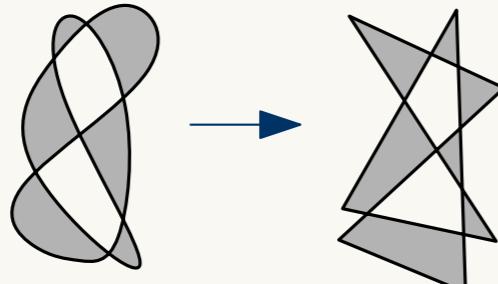
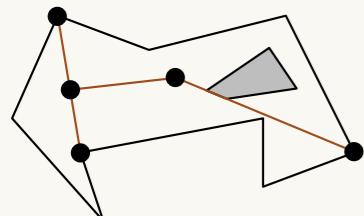


ER

FOCS

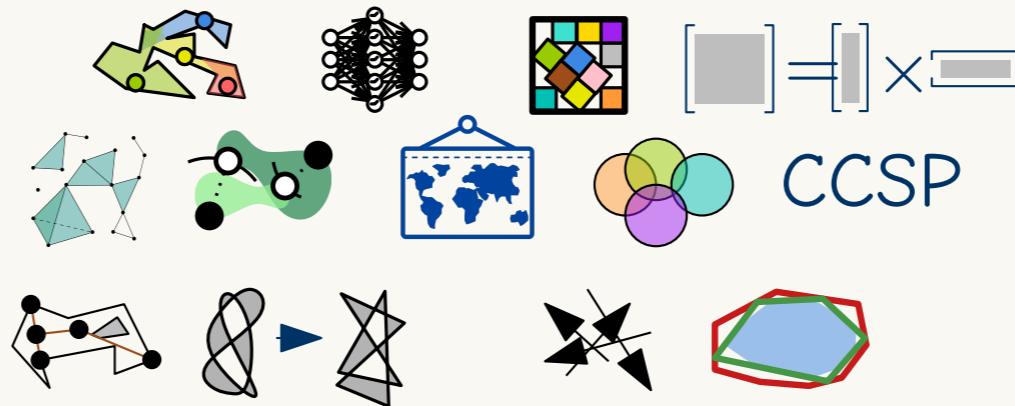


$$\begin{bmatrix} \geq 0 \end{bmatrix} = \begin{bmatrix} \cdot \end{bmatrix} \times \begin{bmatrix} \cdot \end{bmatrix}$$

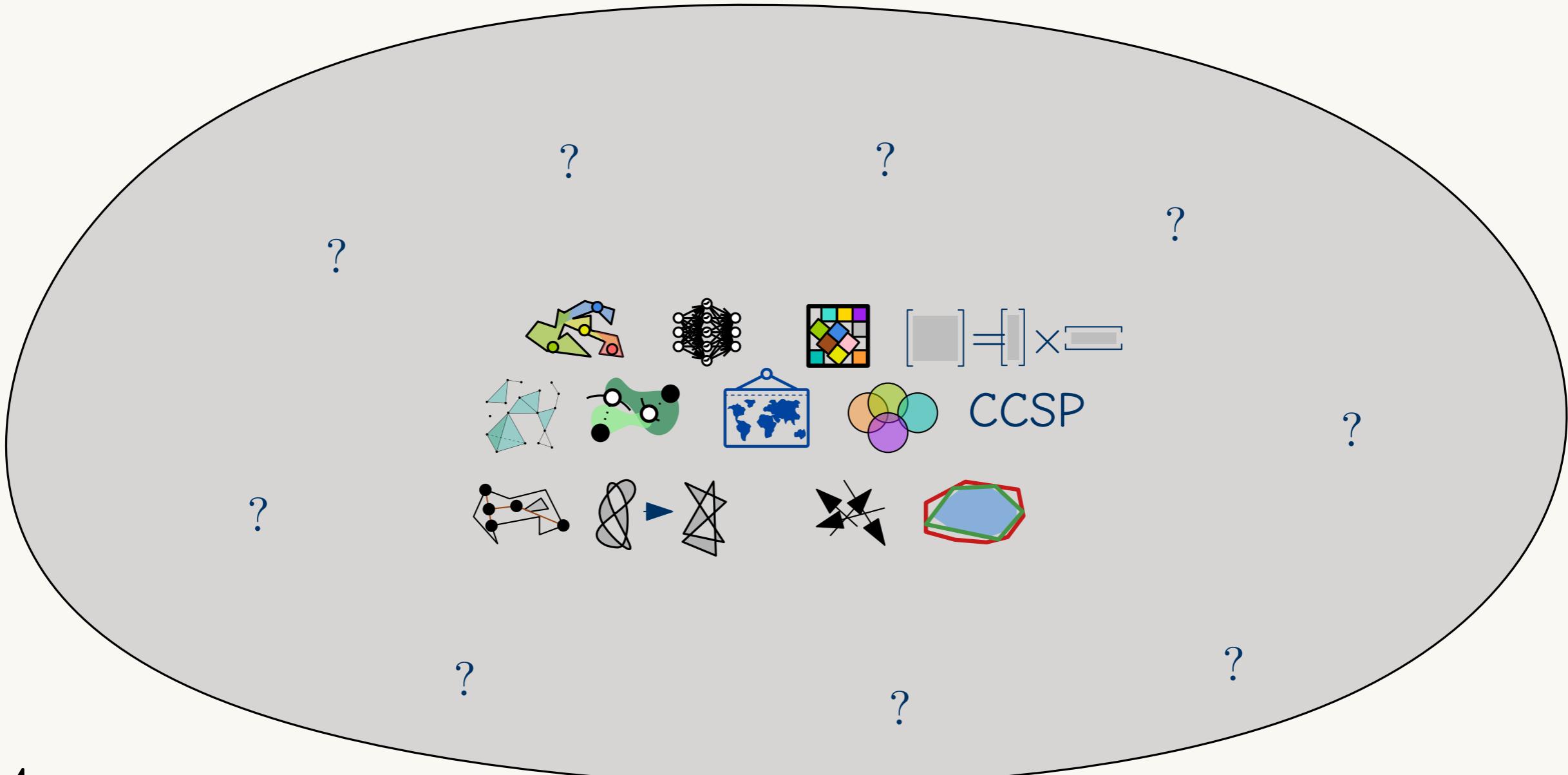


CCSP

# Continuous Problems



# Continuous Problems



Task 1:  
Unveil  $\exists R$

# Audience Participation

Potential  $\exists R$ -problem in your area?

NP

VS

ER

NP

VS

ER



Travelling Salesperson

NP

vs

ER



Travelling Salesperson

NP

VS

ER



Travelling Salesperson

NP

VS

ER



Travelling Salesperson

NP

VS

ER



Travelling Salesperson  
finite possibilities

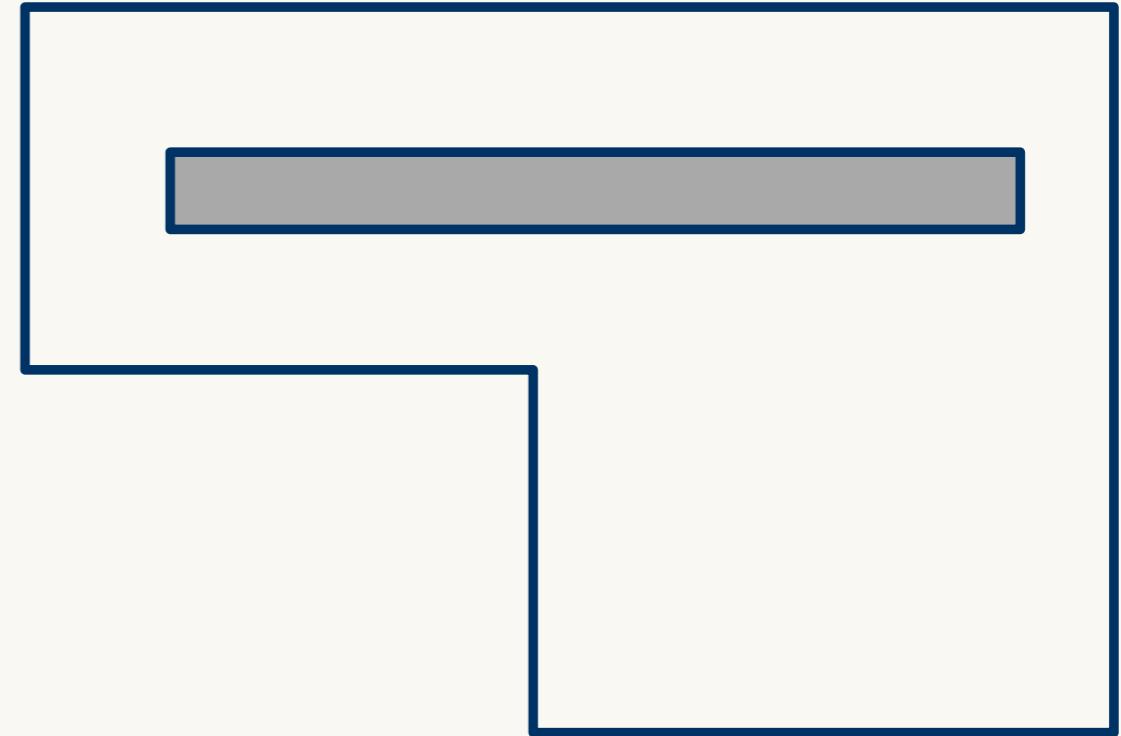
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

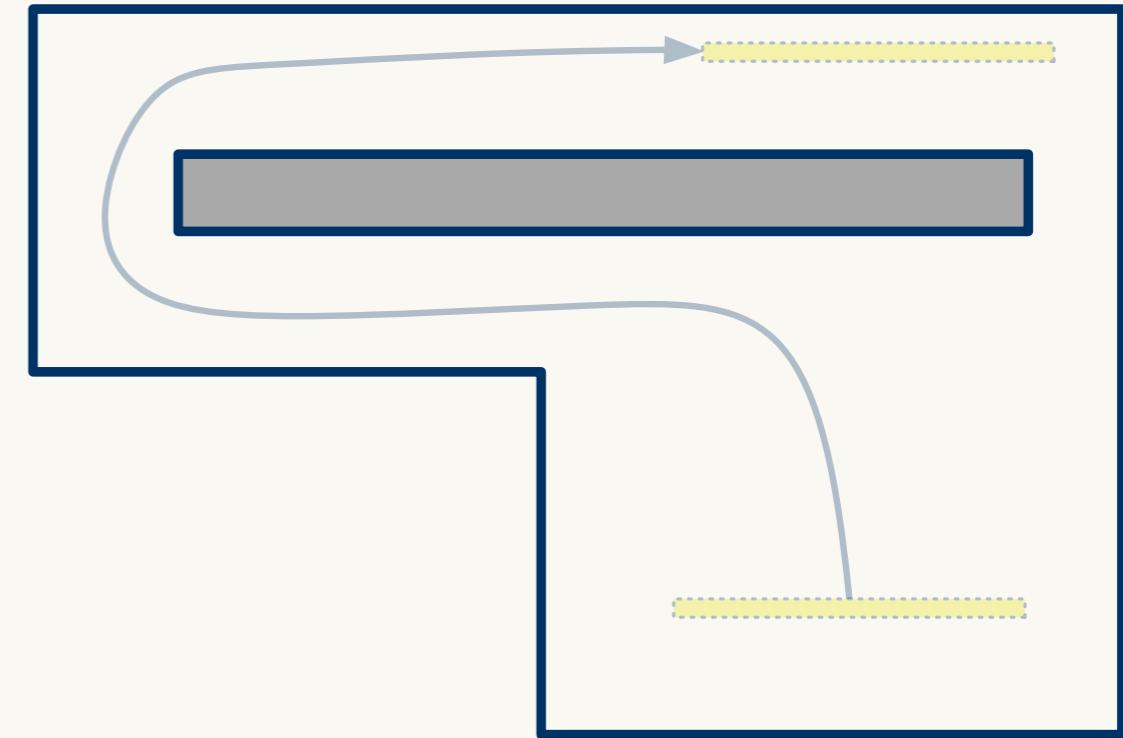
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

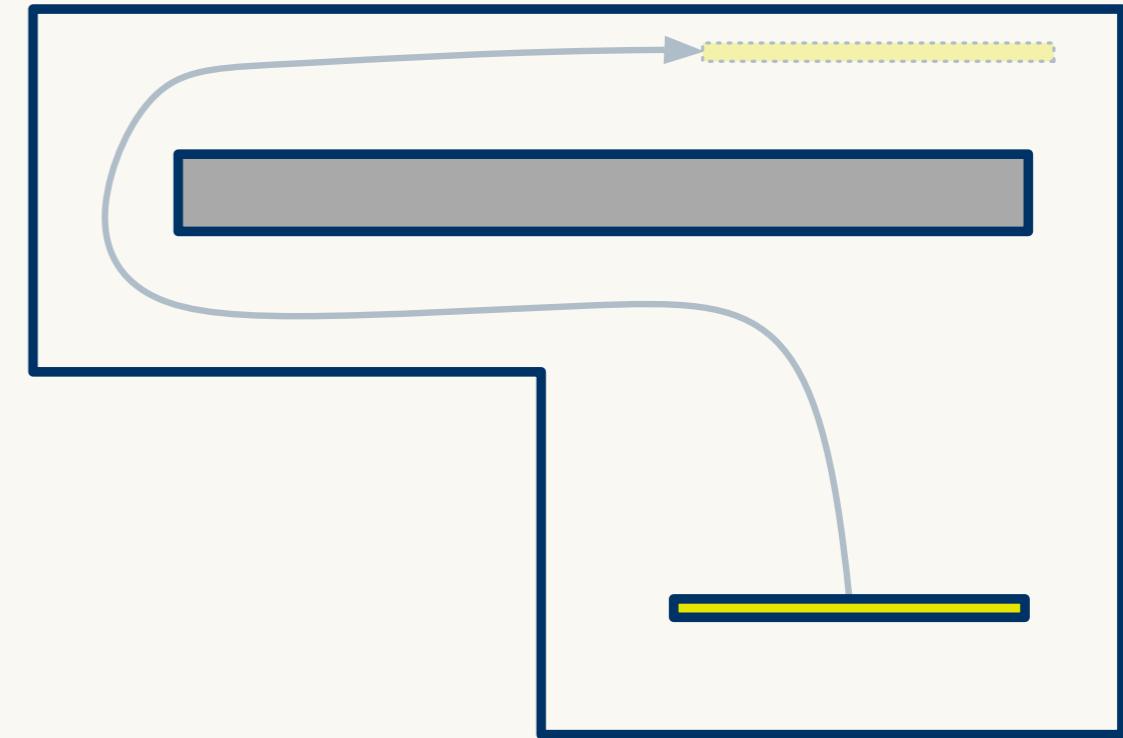
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

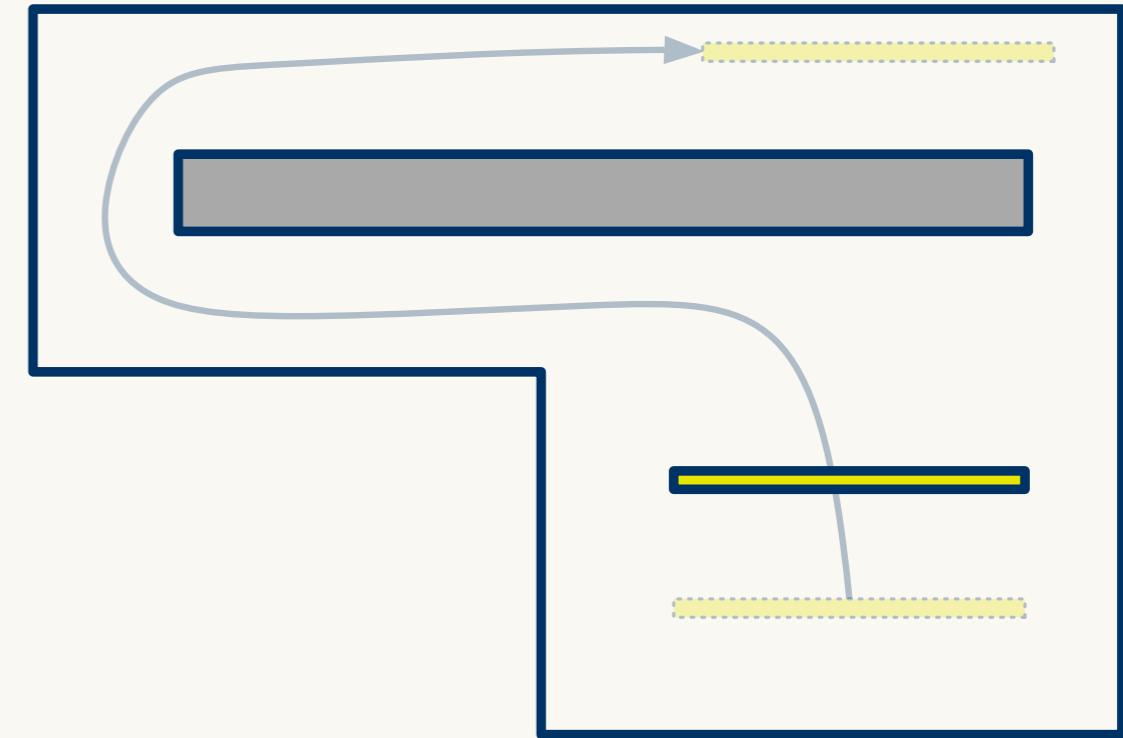
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

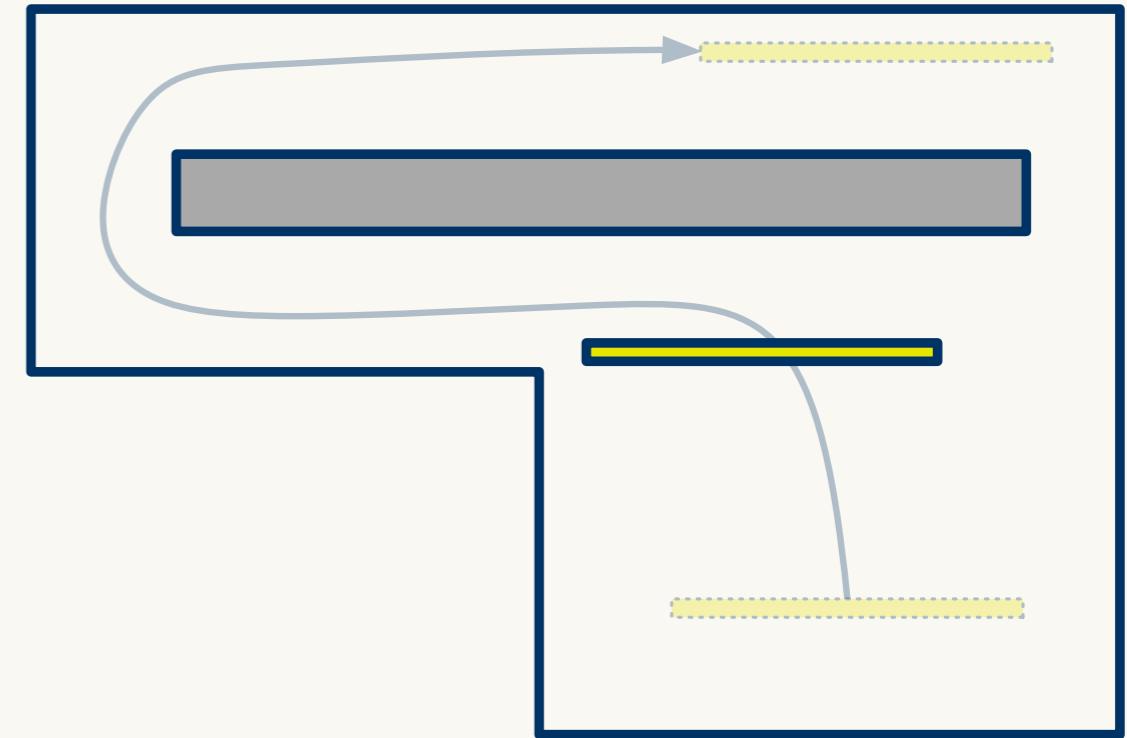
NP

vs

ER



Travelling Salesperson  
finite possibilities



Motion Planning

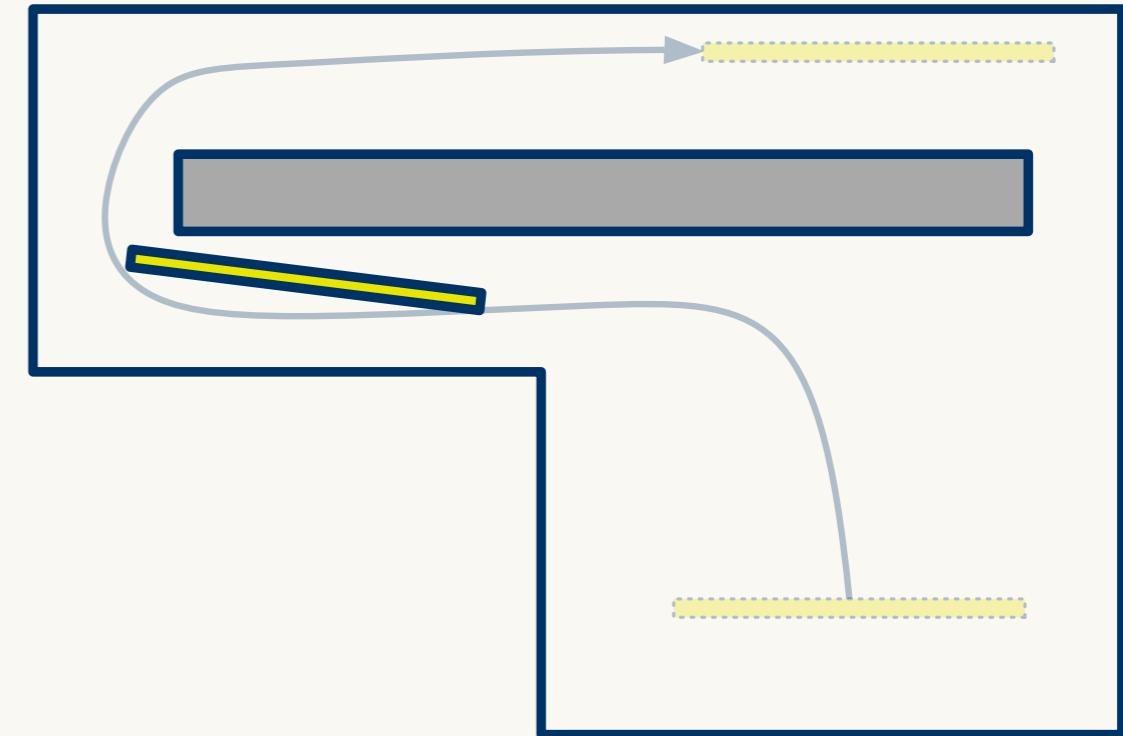
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

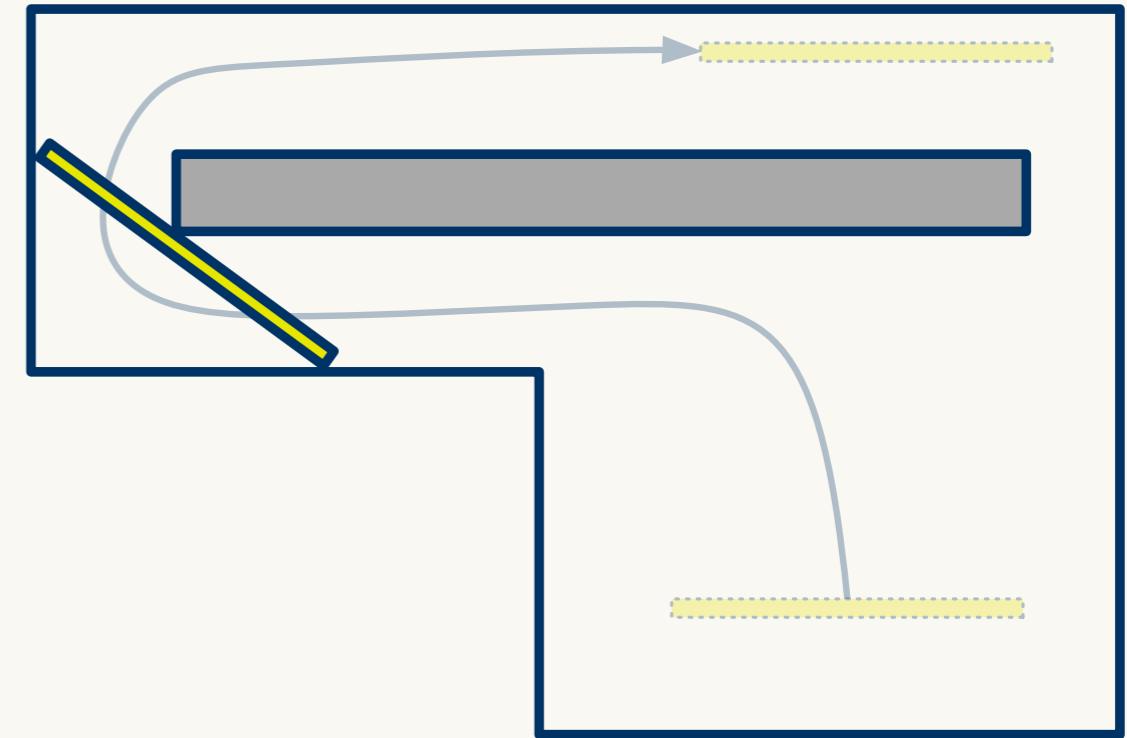
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

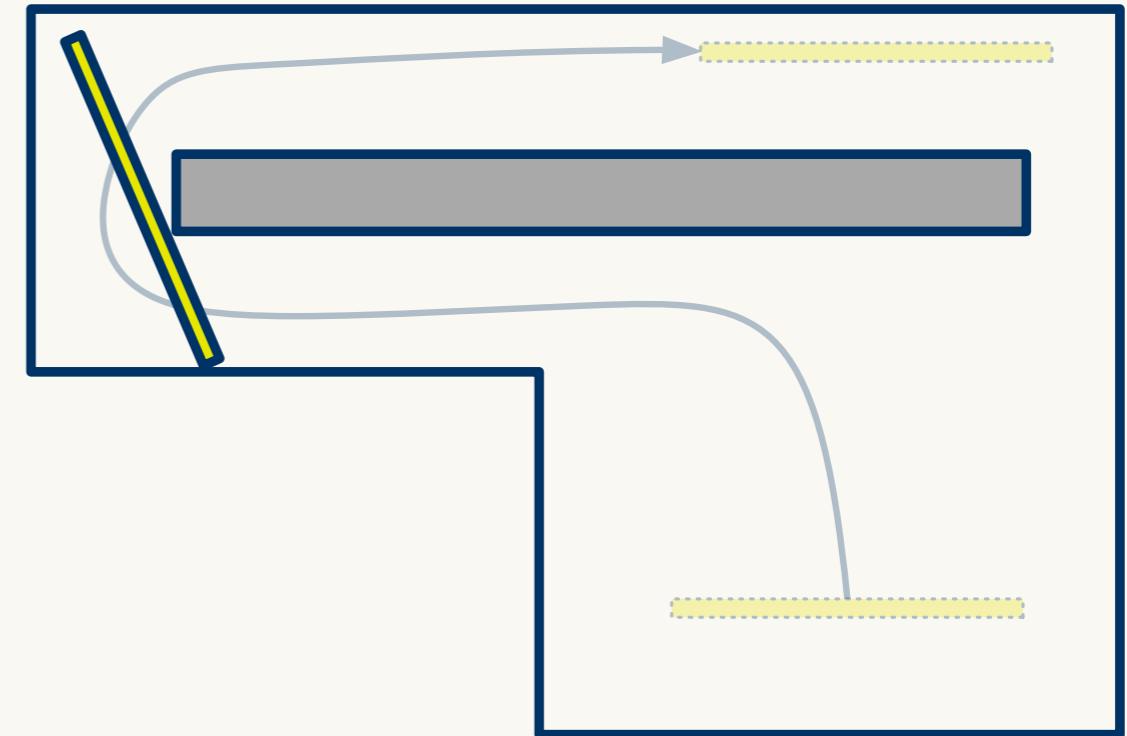
NP

vs

ER



Travelling Salesperson  
finite possibilities



Motion Planning

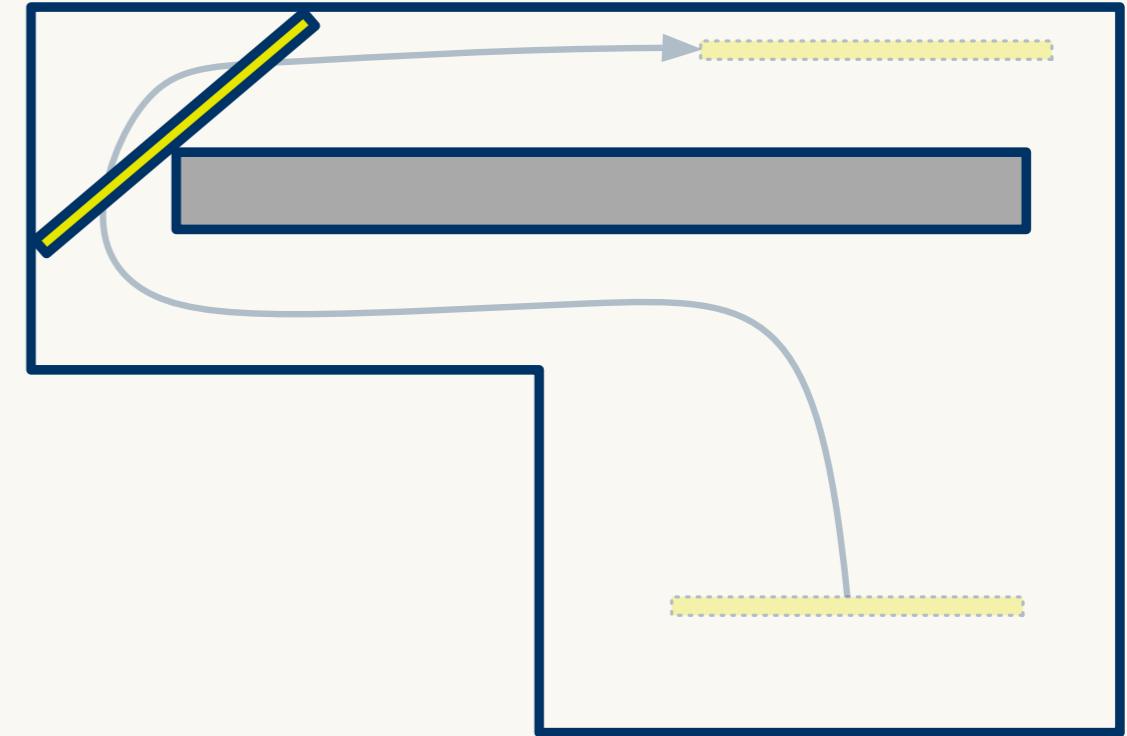
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

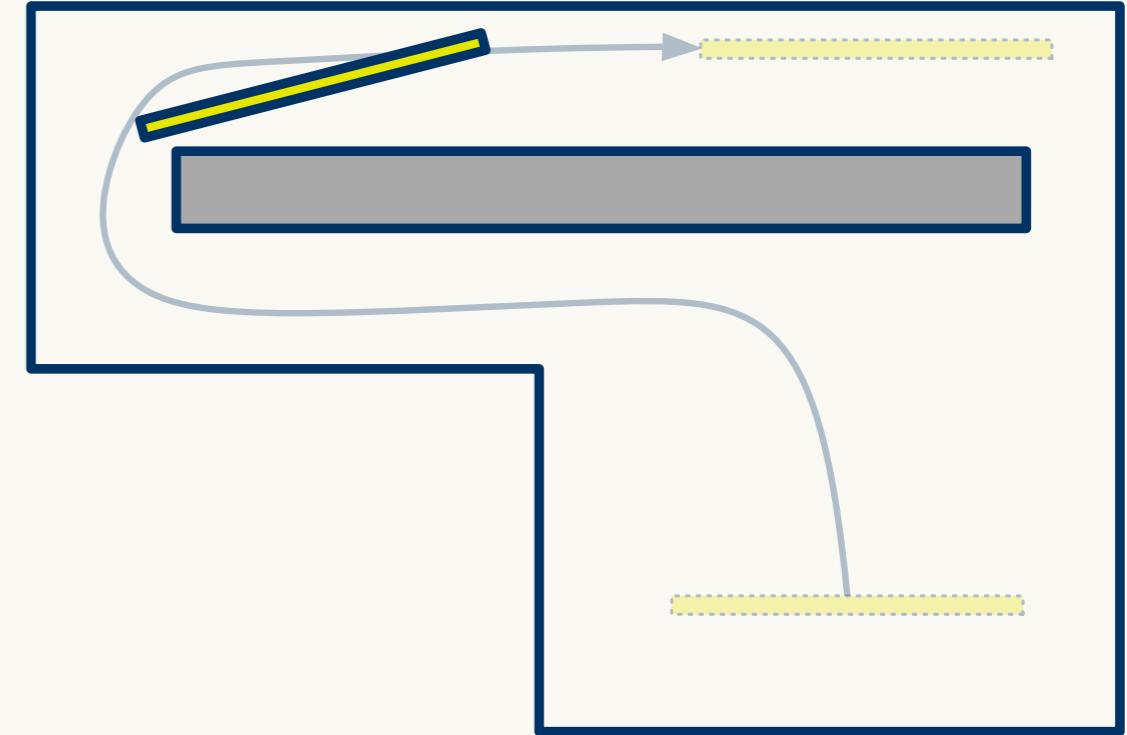
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

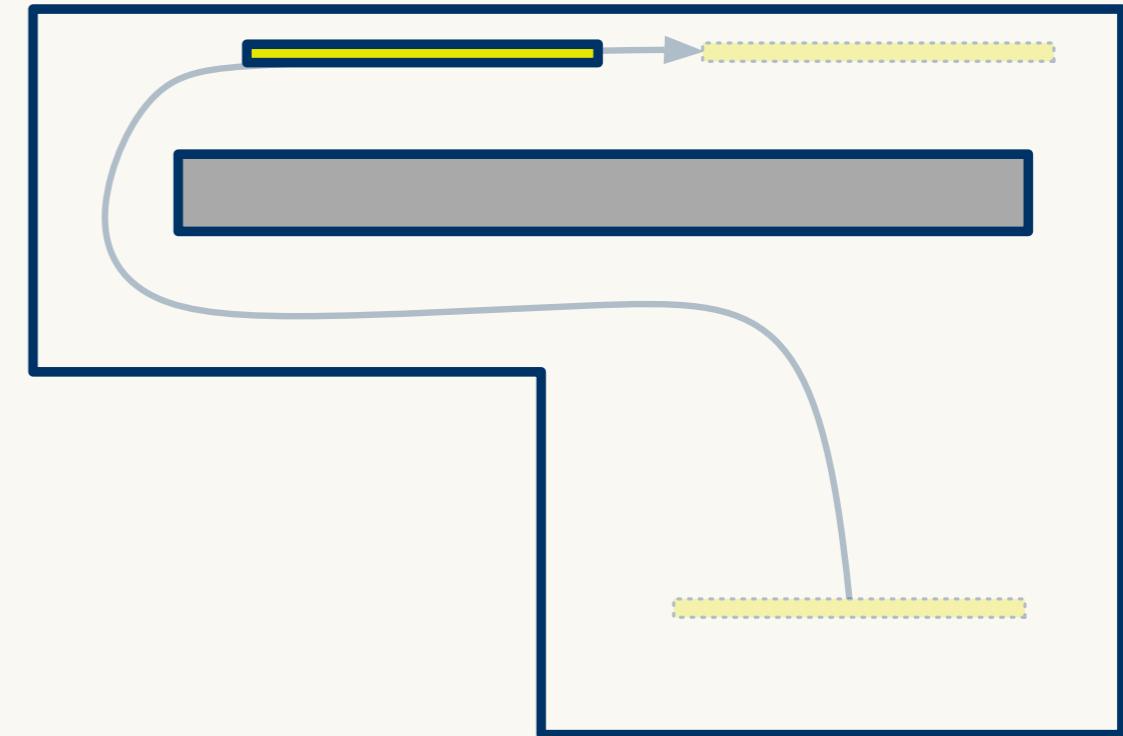
NP

VS

ER



Travelling Salesperson  
finite possibilities



Motion Planning

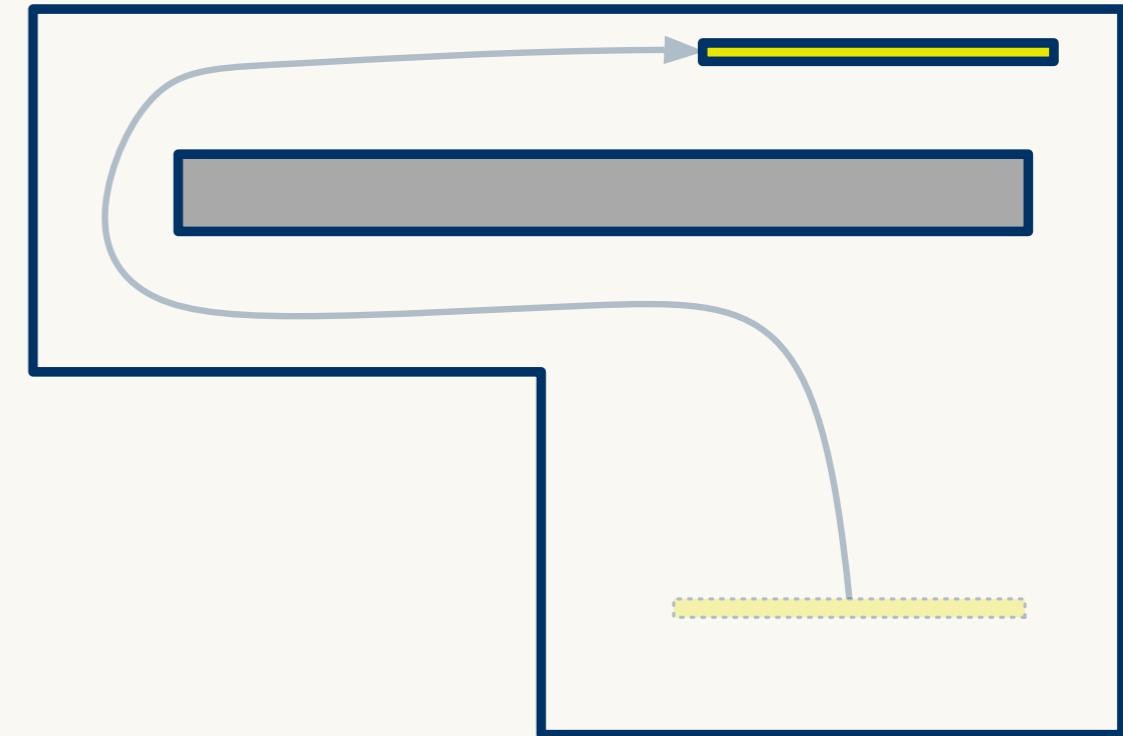
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Travelling Salesperson  
finite possibilities



Motion Planning

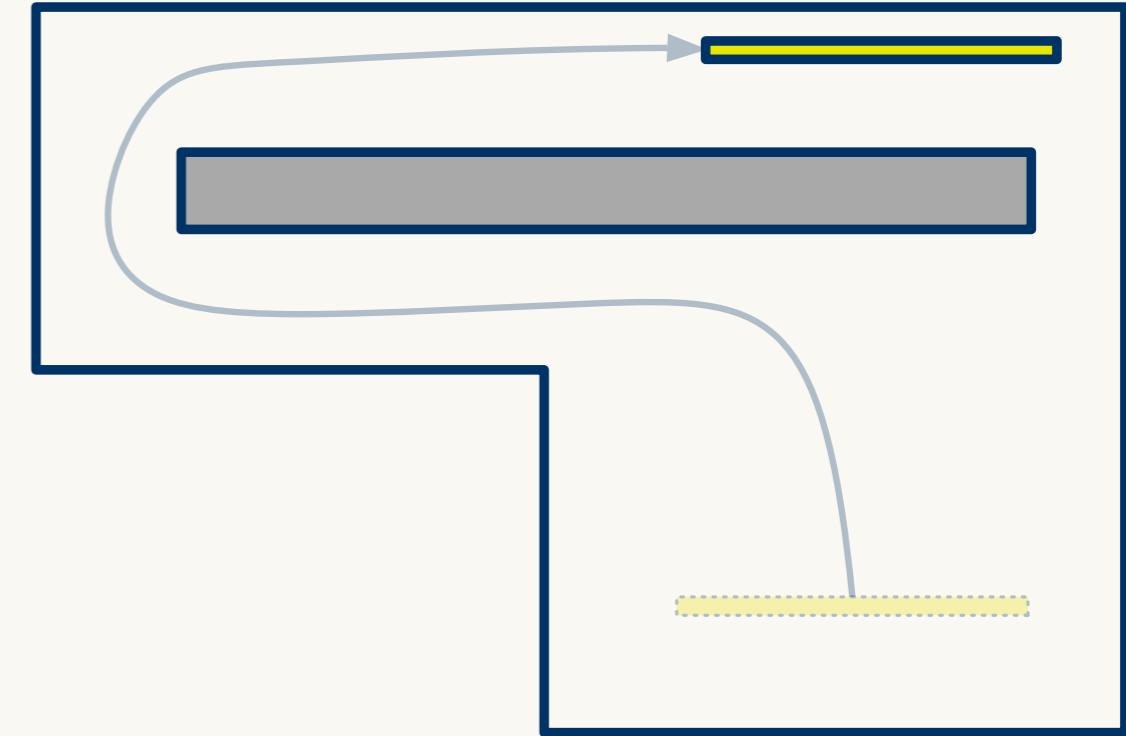
NP

VS

ER



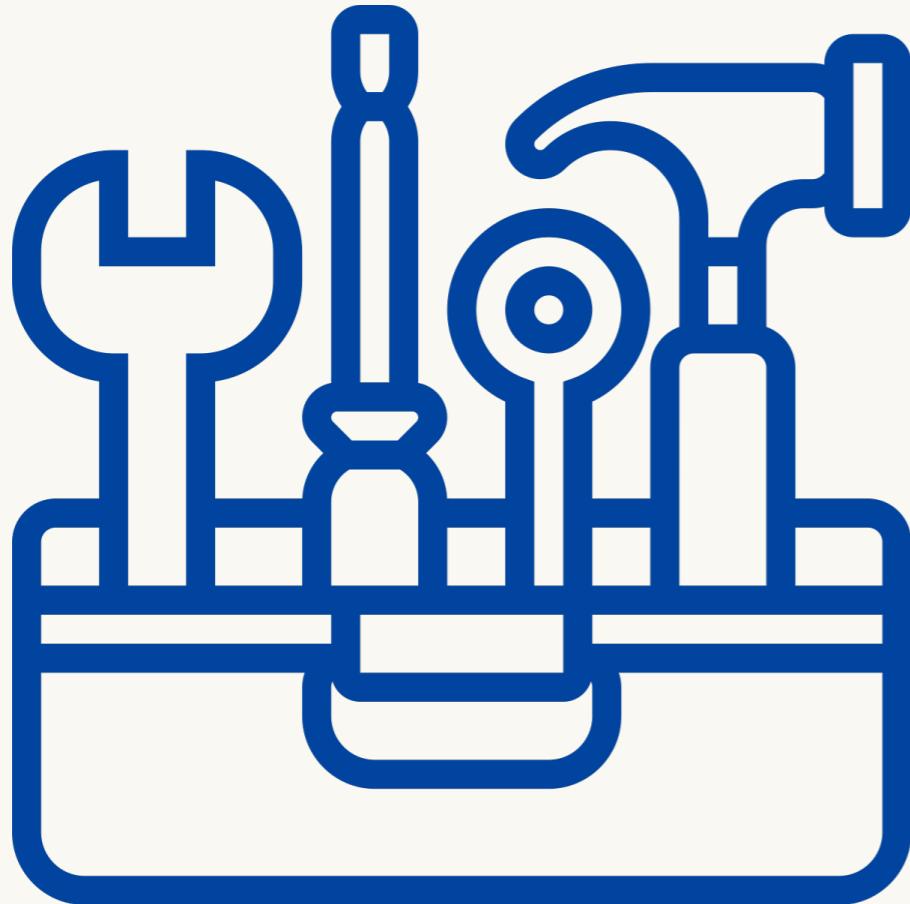
Travelling Salesperson  
finite possibilities



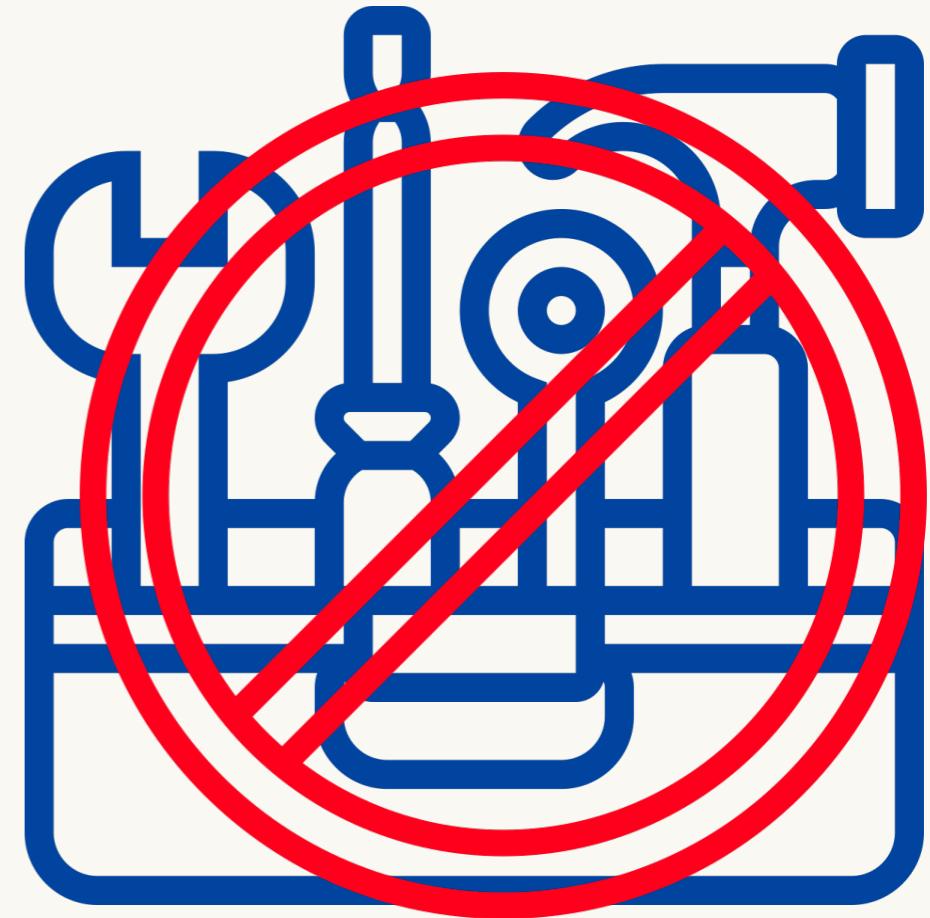
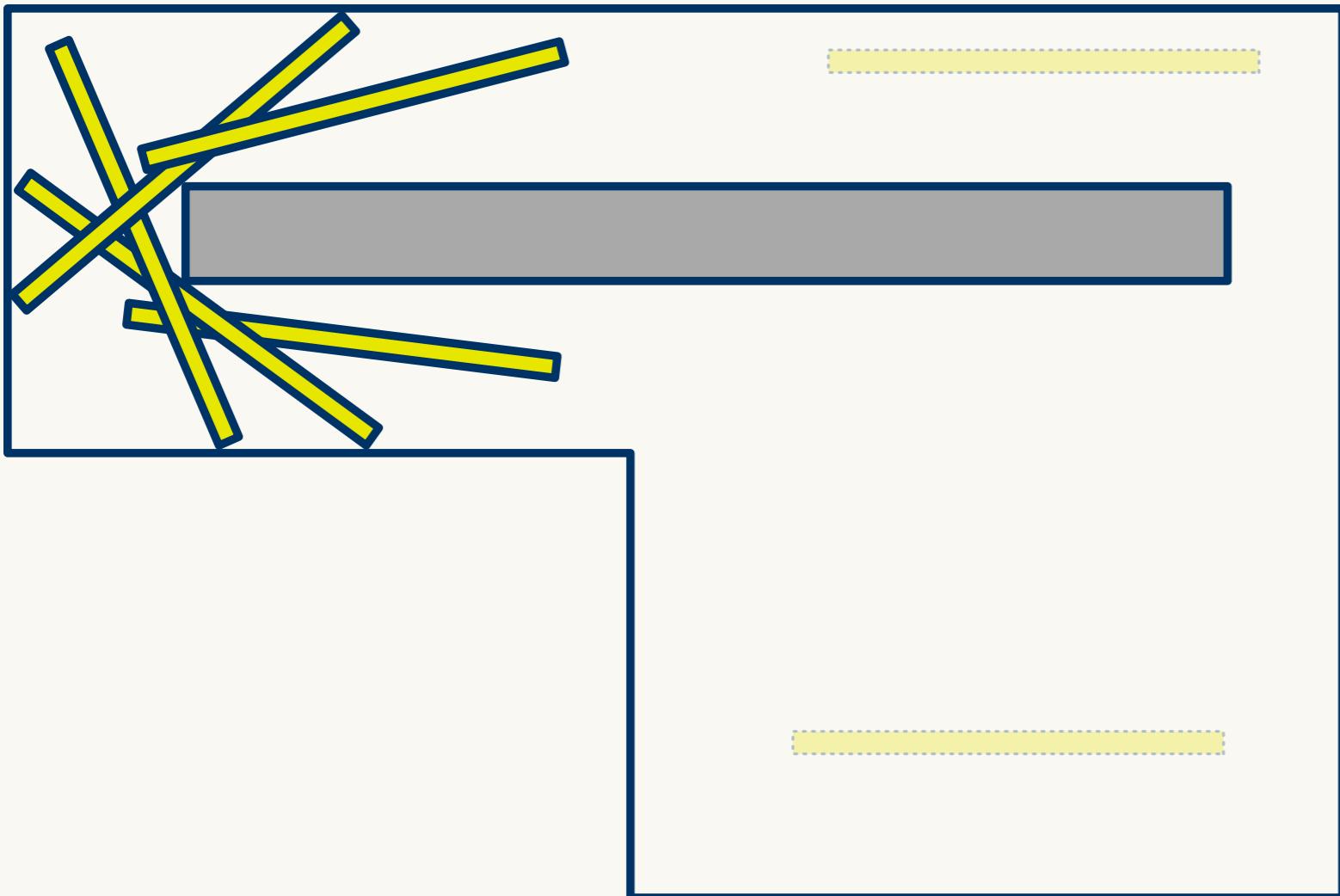
Motion Planning  
continuous

# How to solve it?

# How to solve it?

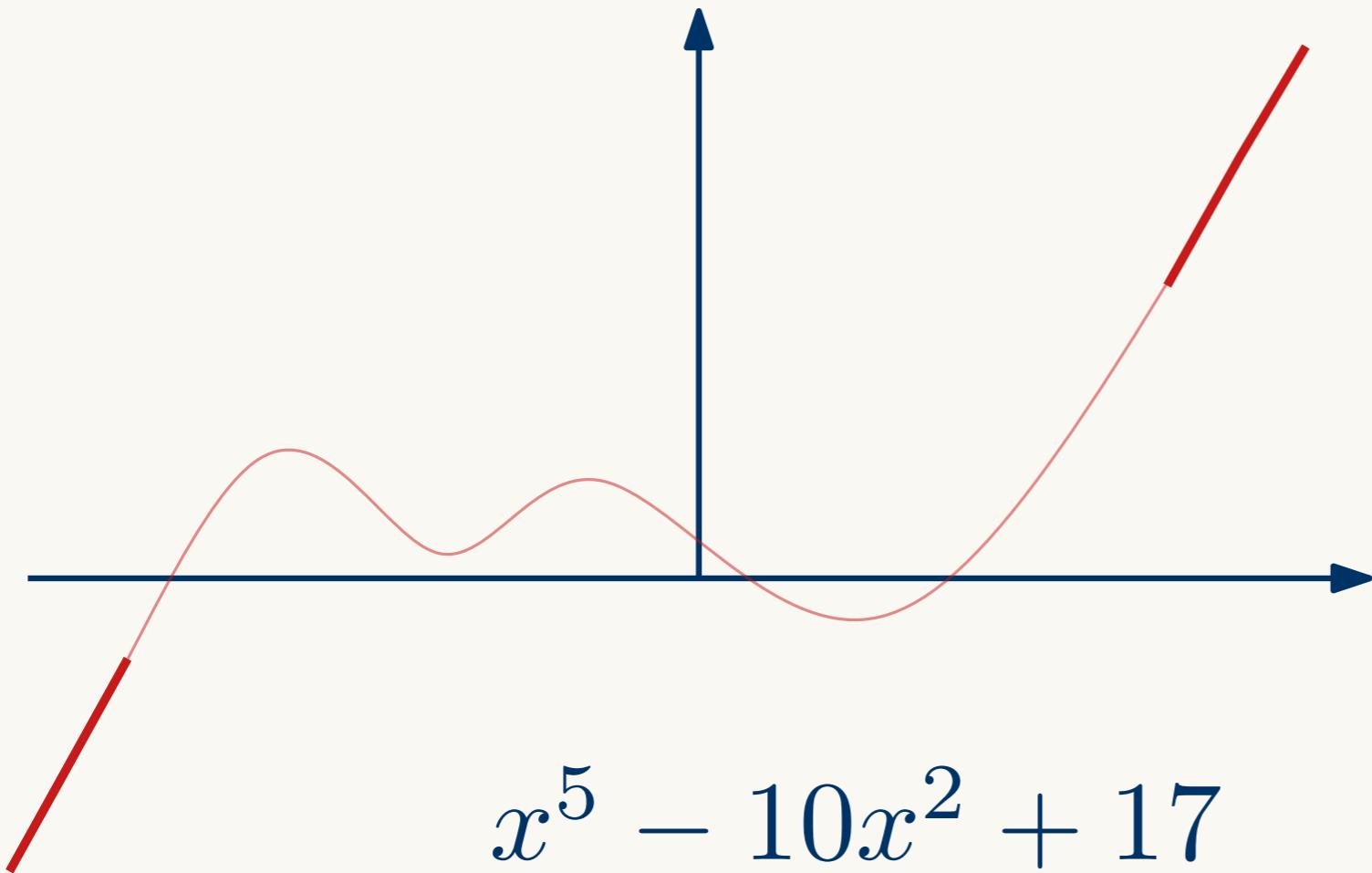


# How to solve it?

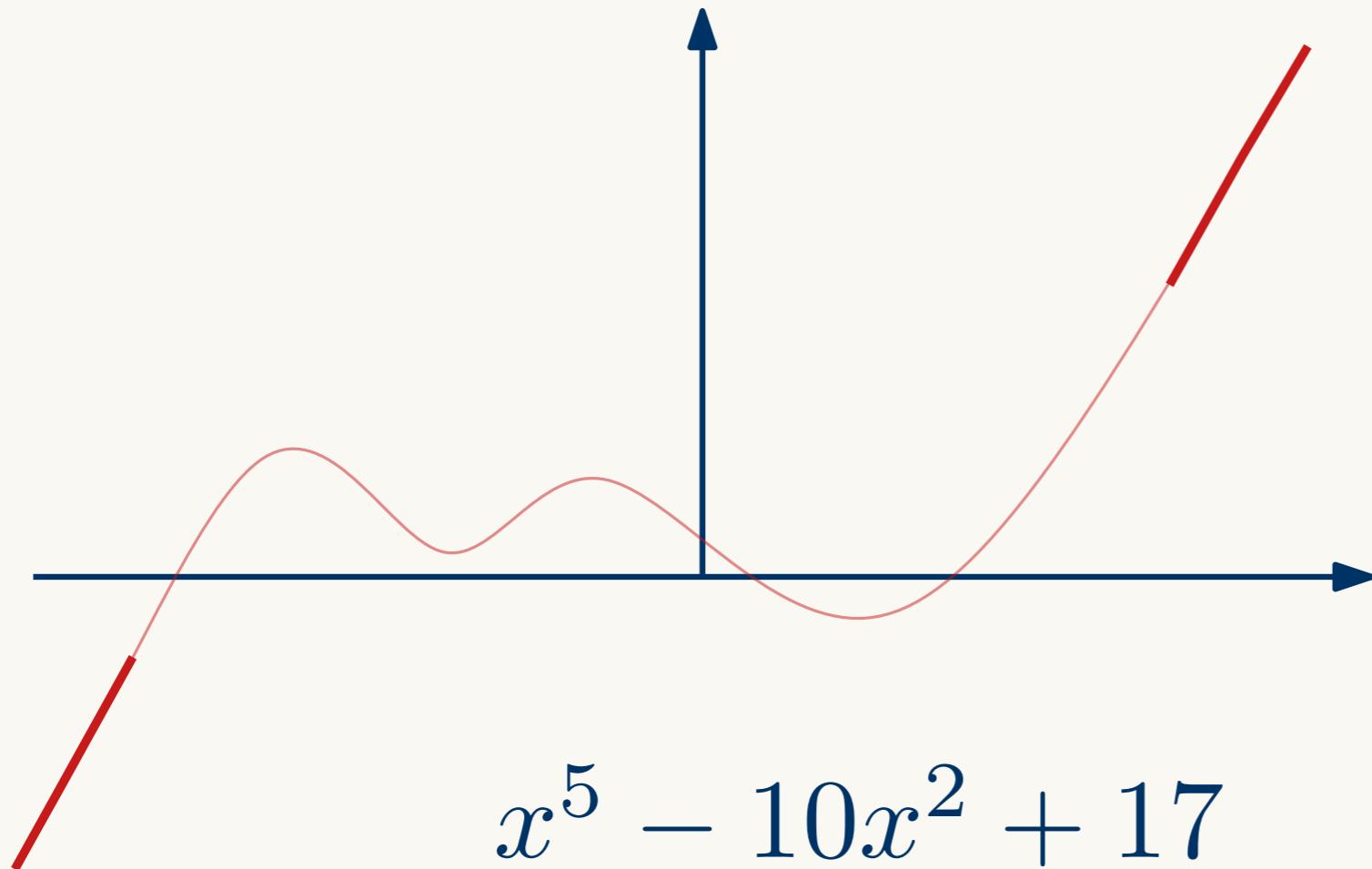


# Symbolic

# Symbolic



# Symbolic



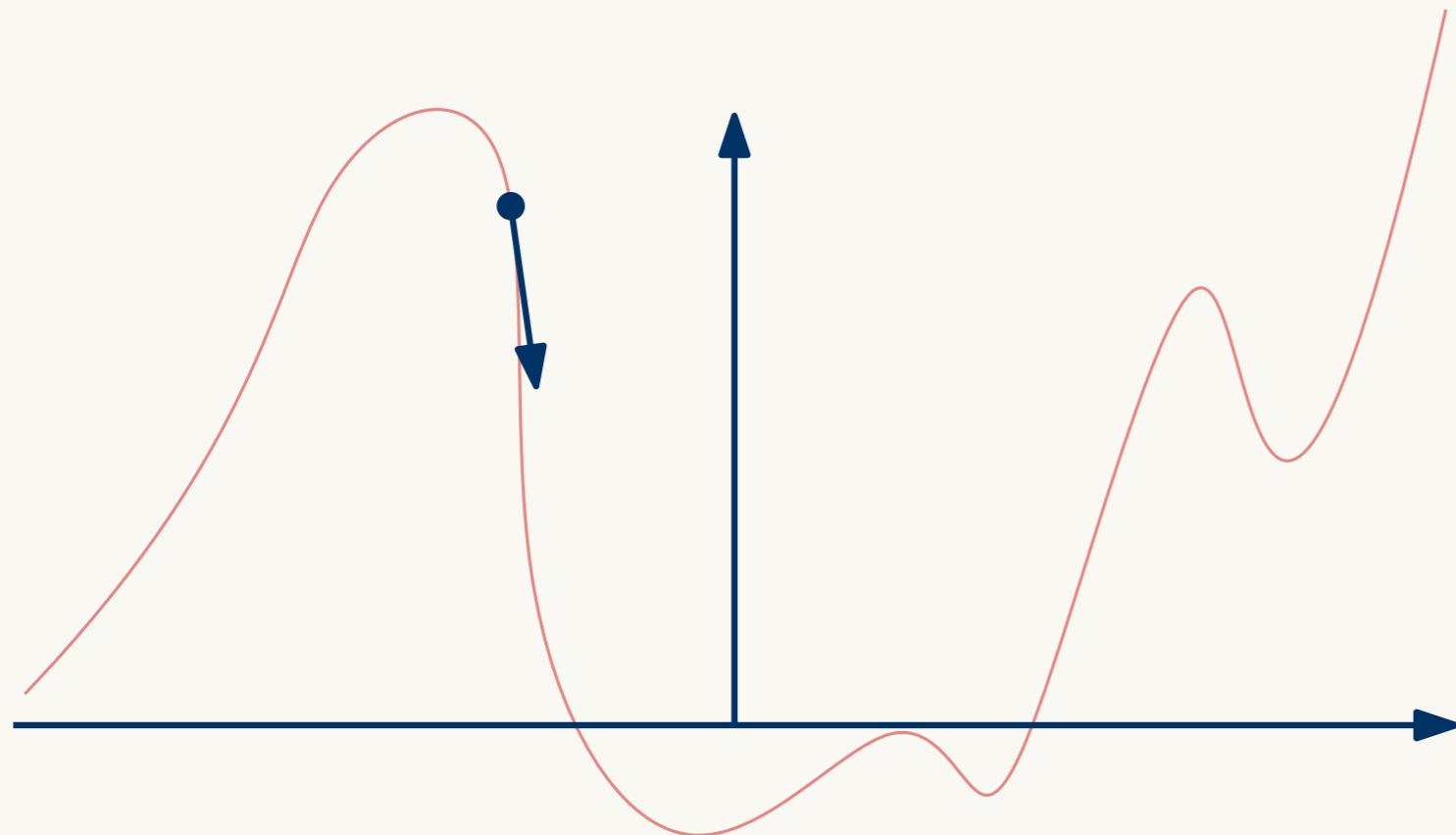
guaranteed  
optimal  
solution

provable  
time  
bounds

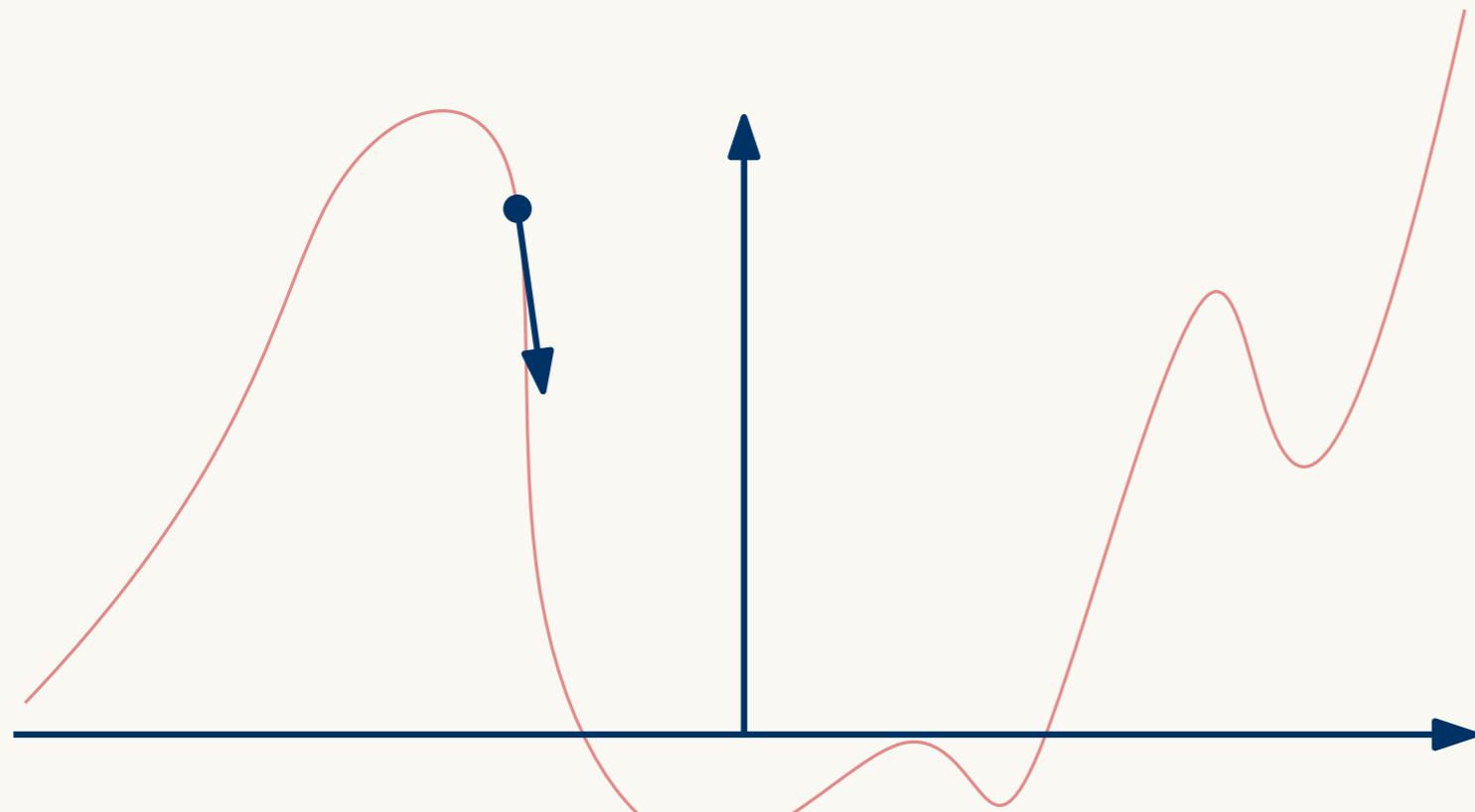
efficient  
in  
practice

# Gradient

# Gradient



# Gradient

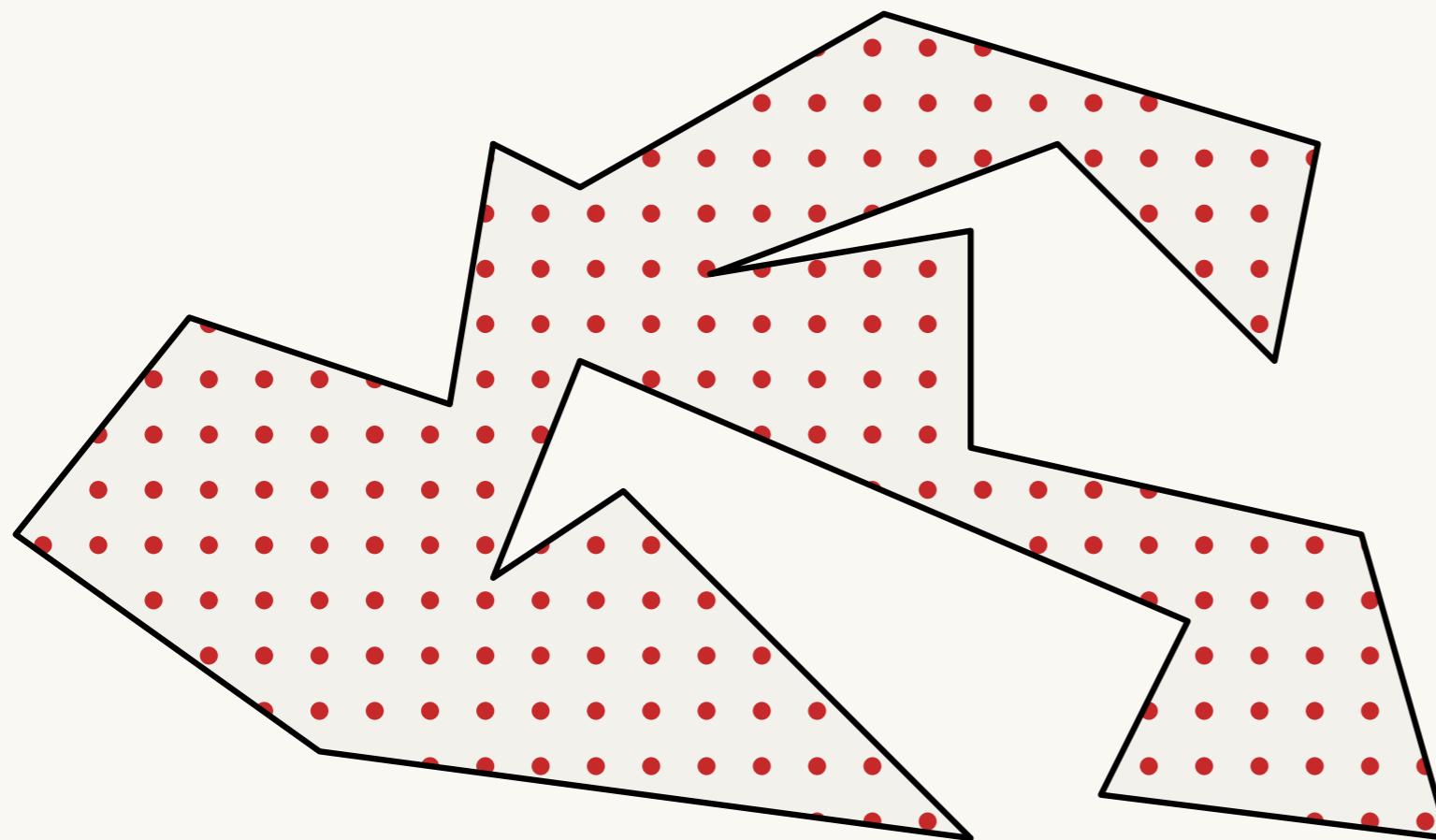


guaranteed  
optimal  
solution

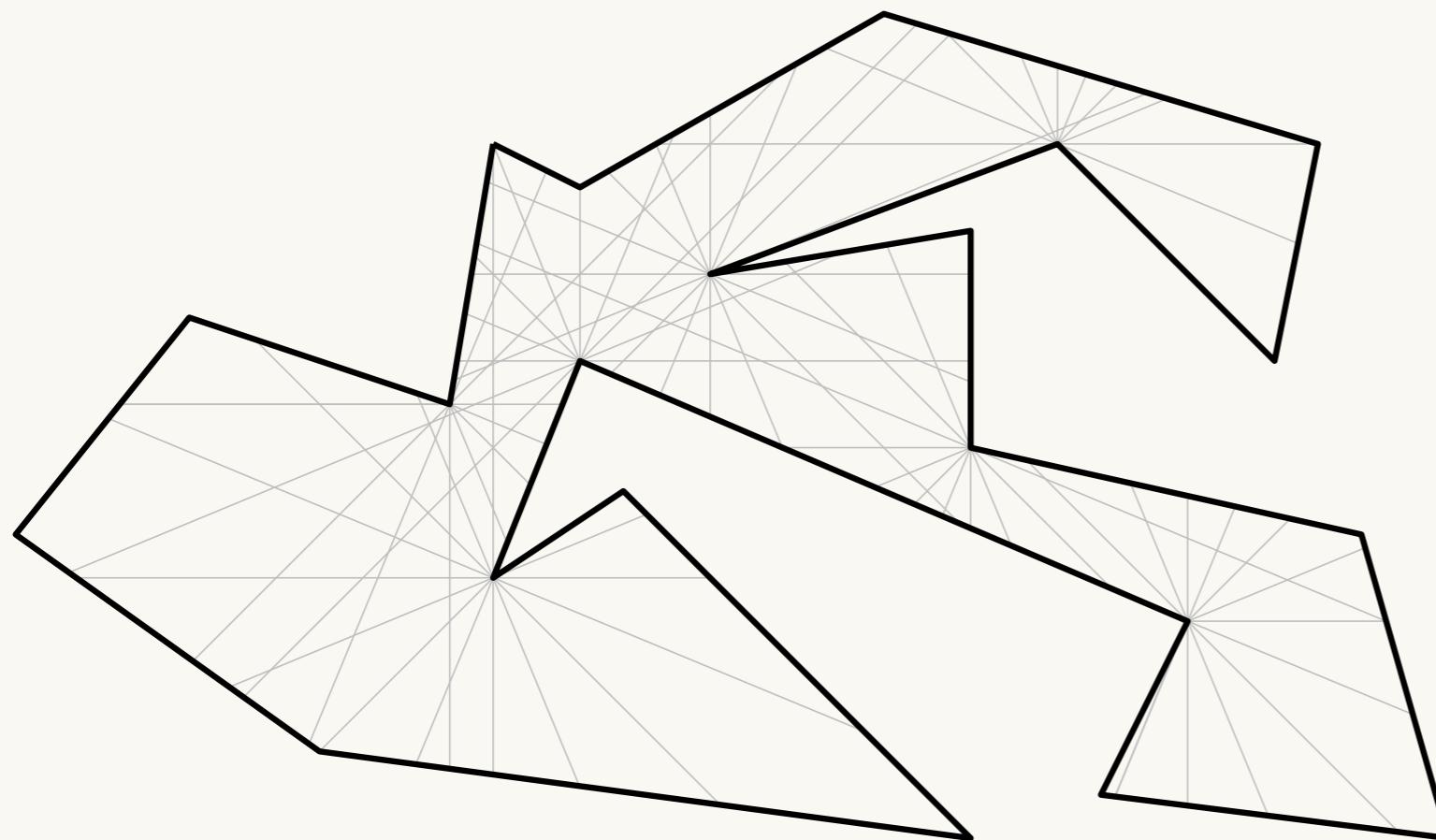
provable  
time  
bounds

efficient  
in  
practice

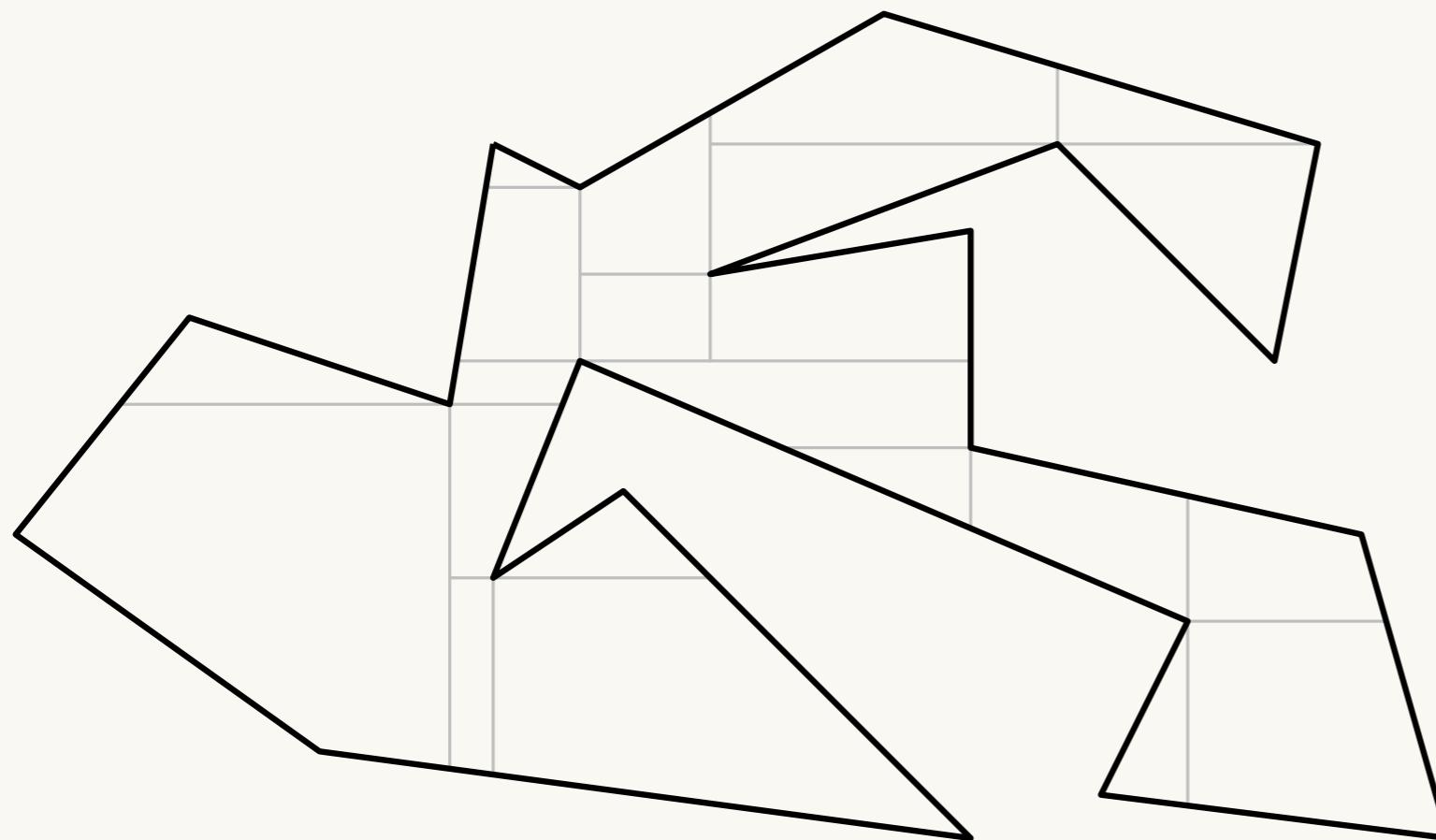
# Discretization



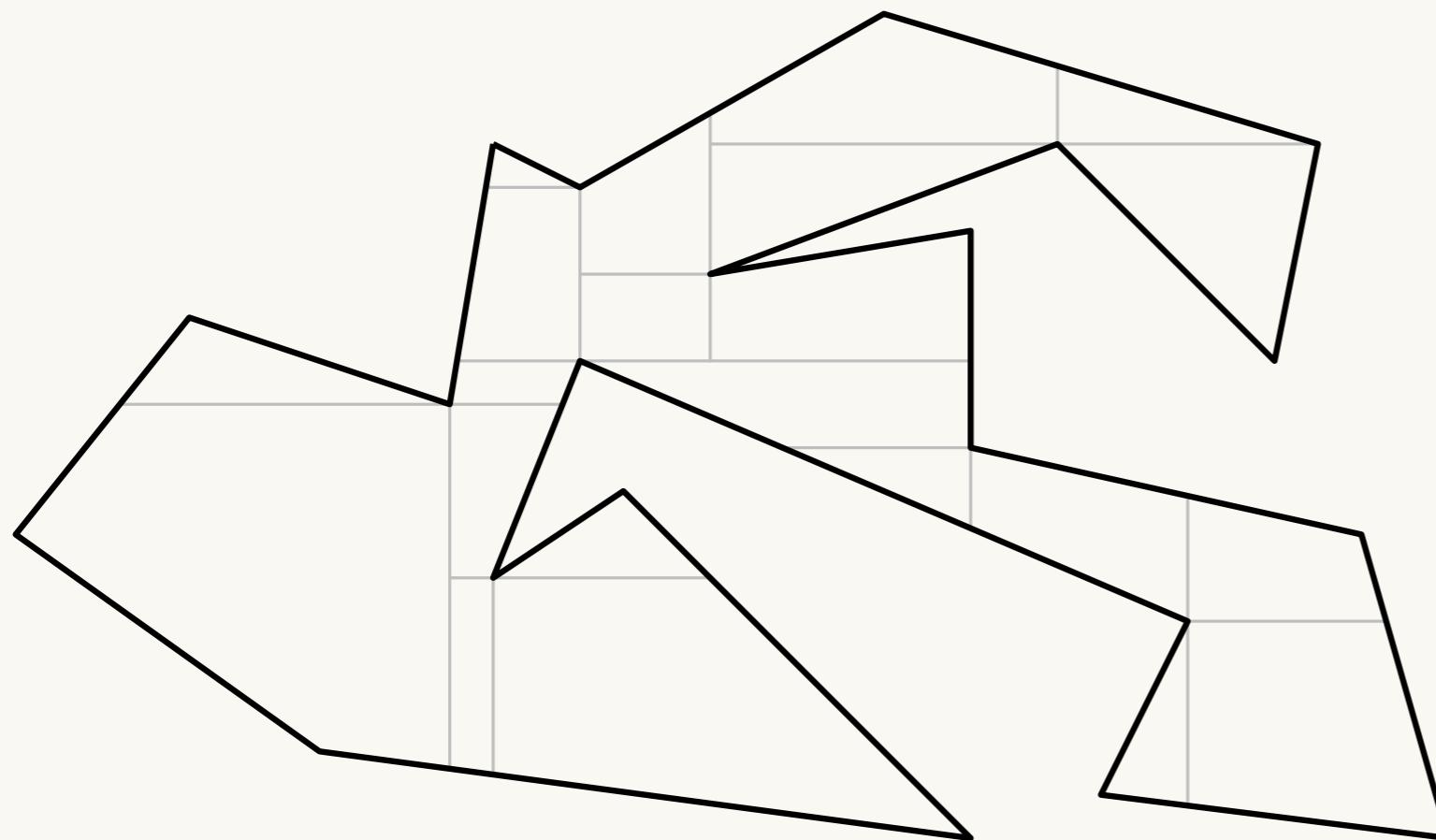
# Discretization



# Discretization



# Discretization



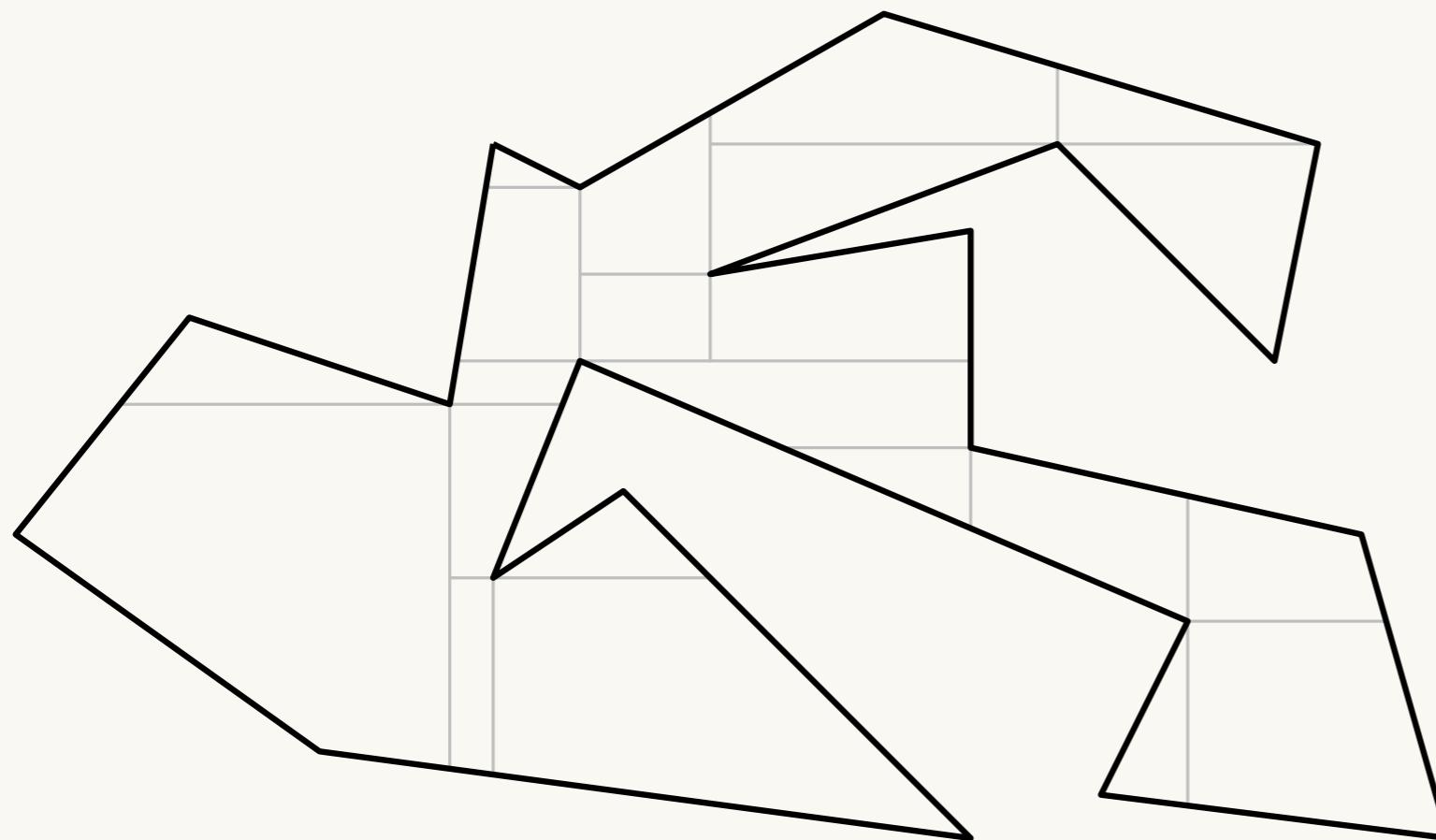
guaranteed  
optimal  
solution

provable  
time  
bounds

efficient  
in  
practice

# Discretization

Difficult



guaranteed  
optimal  
solution

provable  
time  
bounds

efficient  
in  
practice

# Develop Algorithmic Theory

guaranteed  
optimal  
solution

provable  
time  
bounds

efficient  
in  
practice

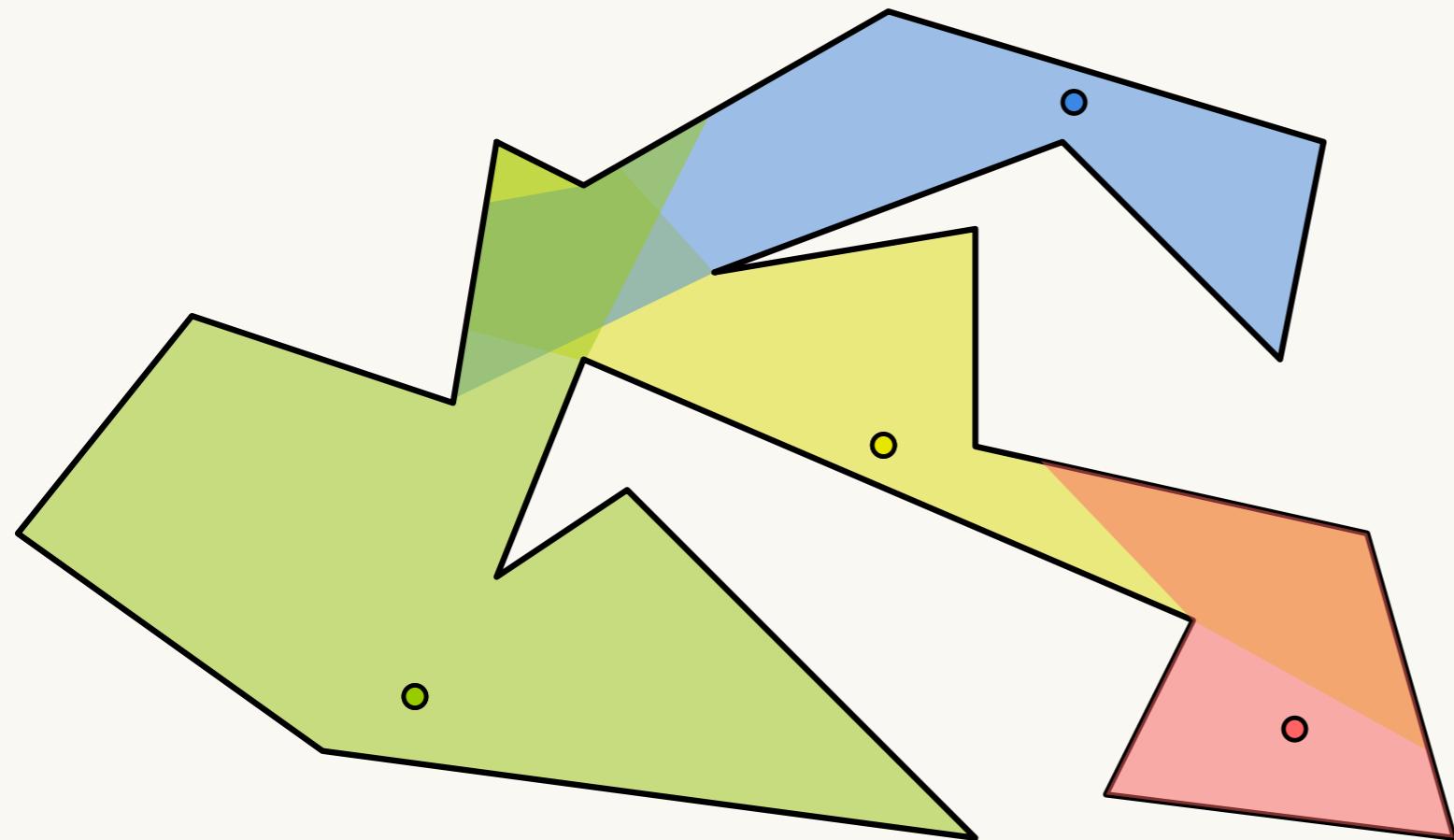
Assumptions

# Task 2: Develop Algorithmic Theory

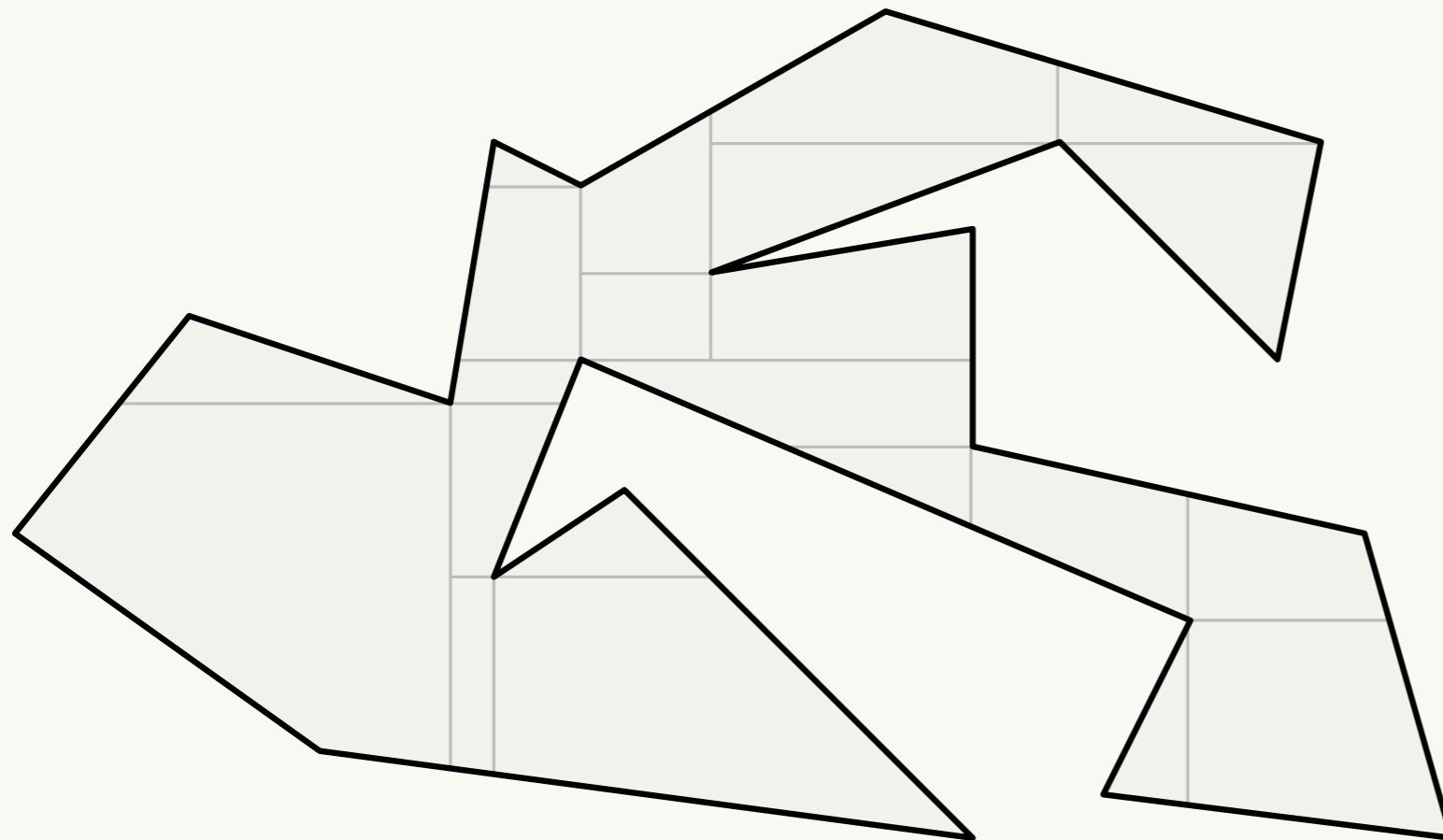
# Audience Participation

What algorithmic techniques do you use?

# Example

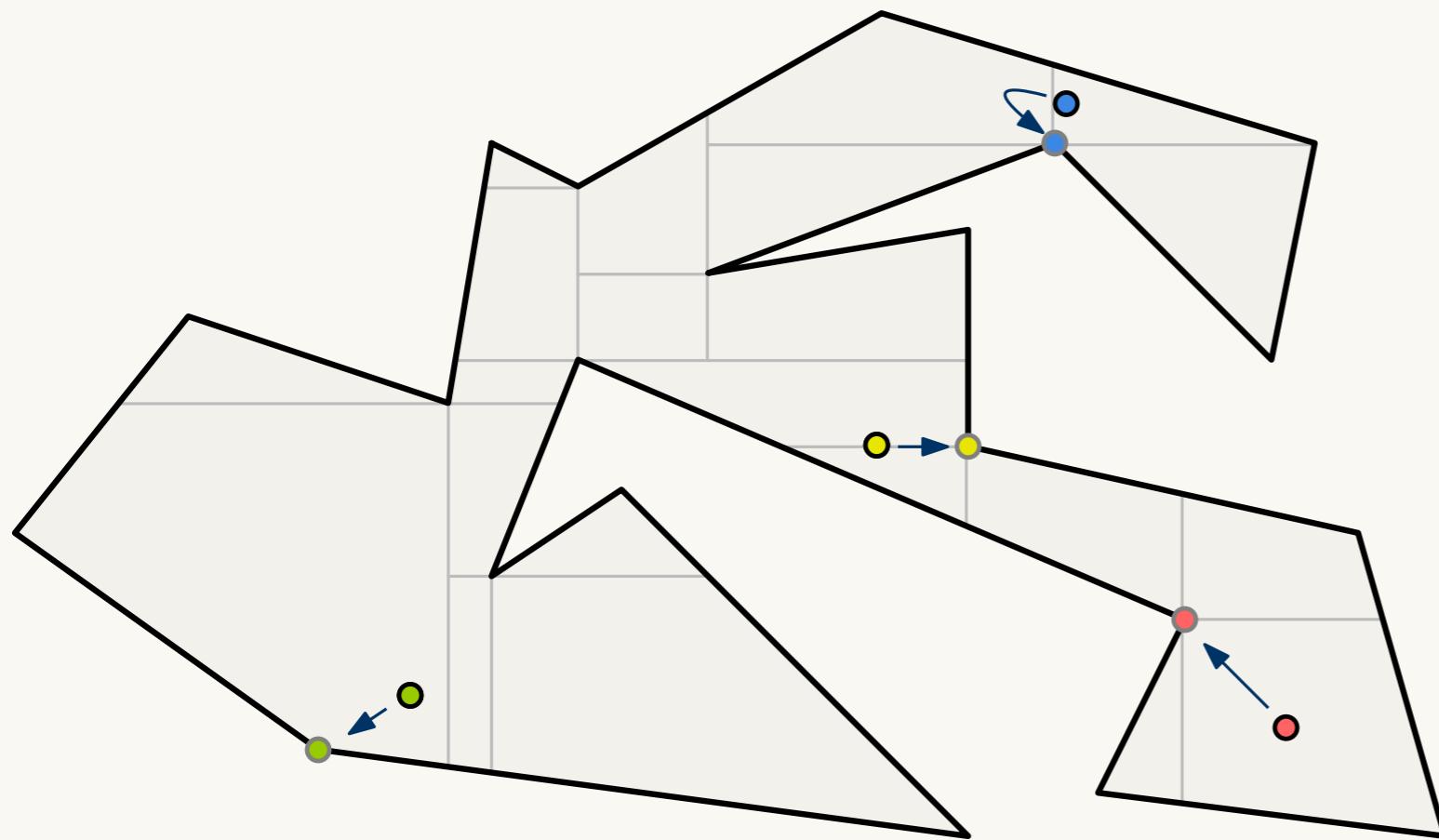


# Example



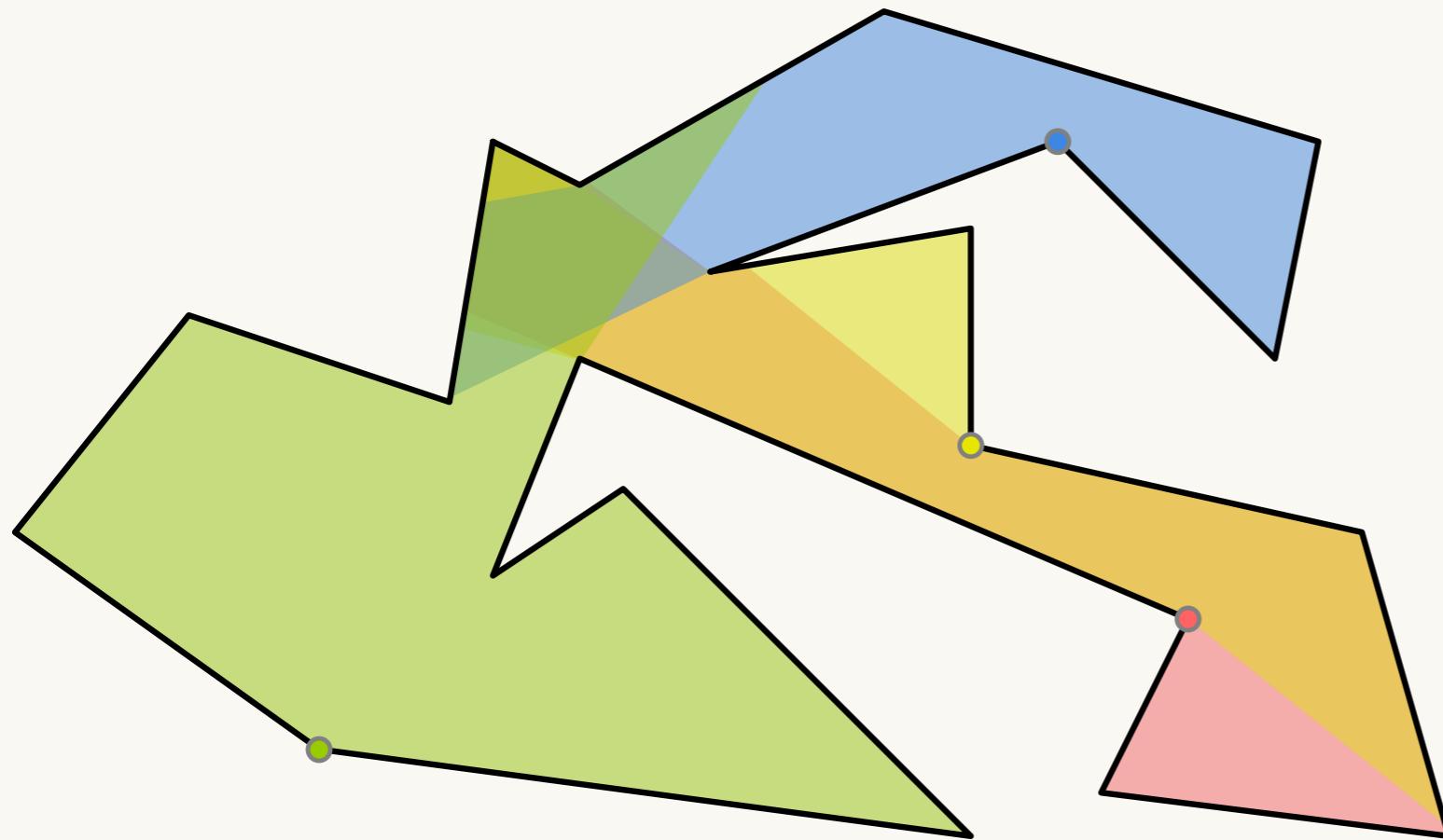
1 - Formulate Discretization

# Example



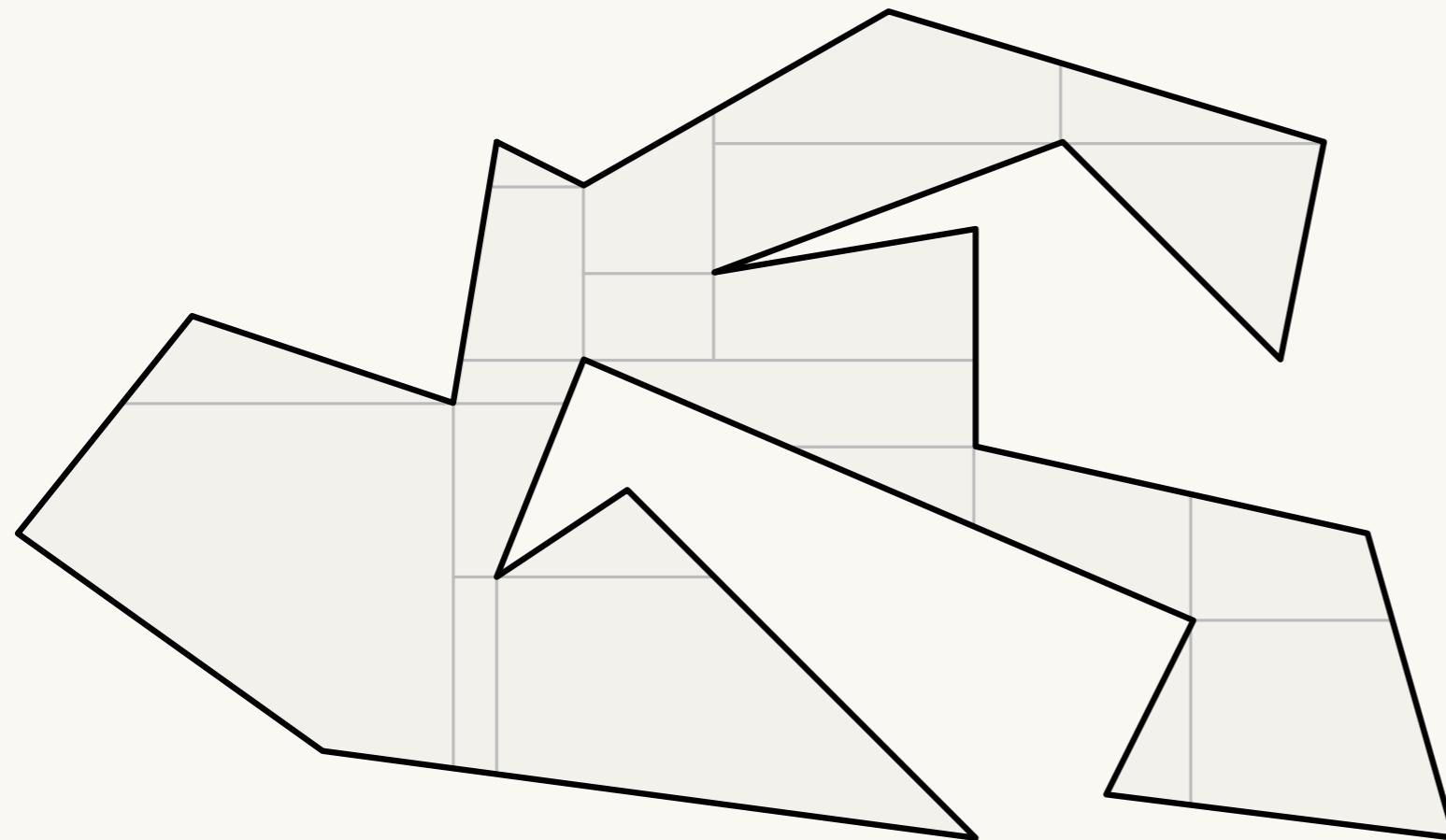
2 - Prove Optimality

# Example



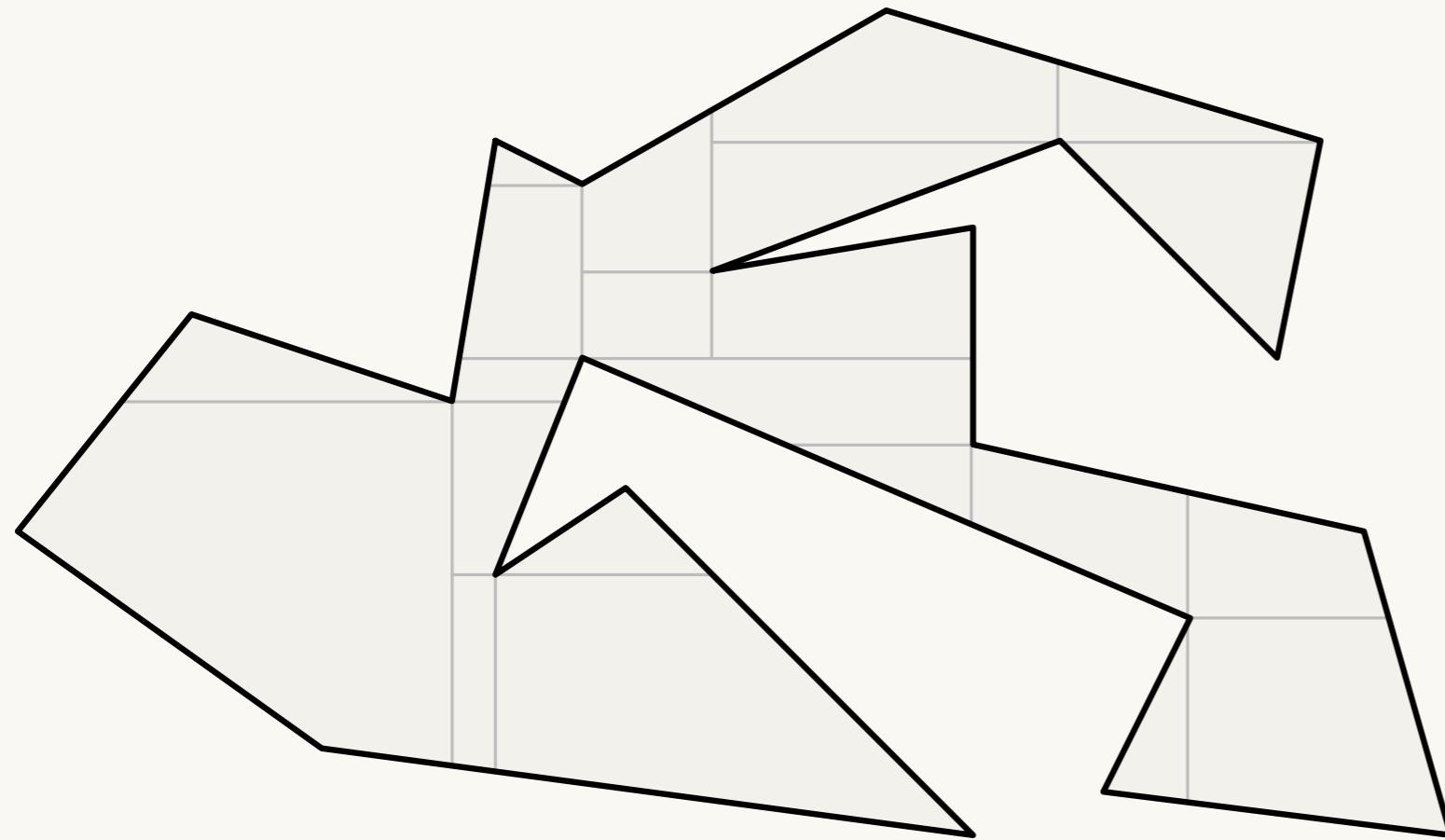
2 - Prove Optimality

# Example



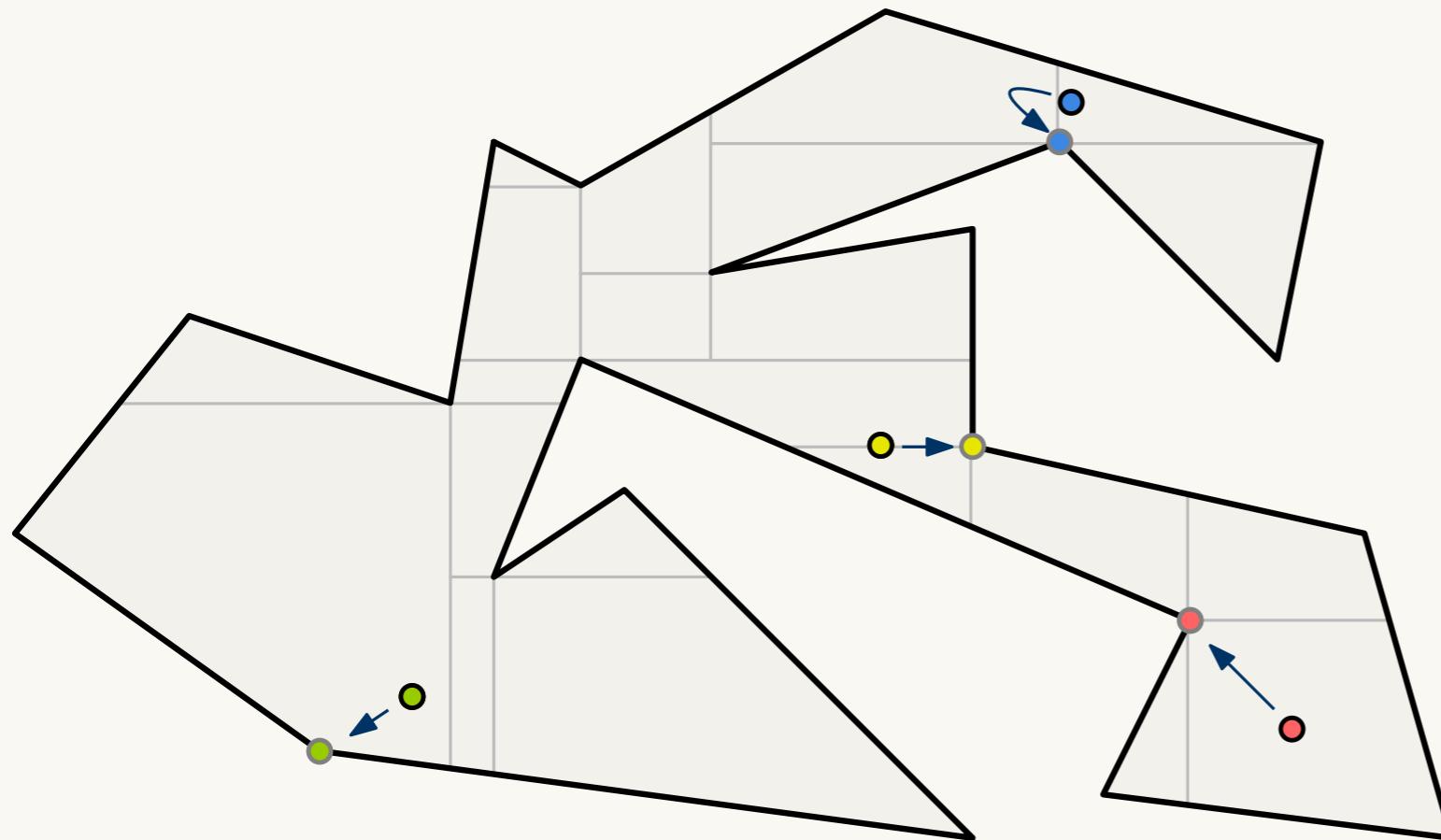
3 - Bound Size

# Example



4 - Implement & Test

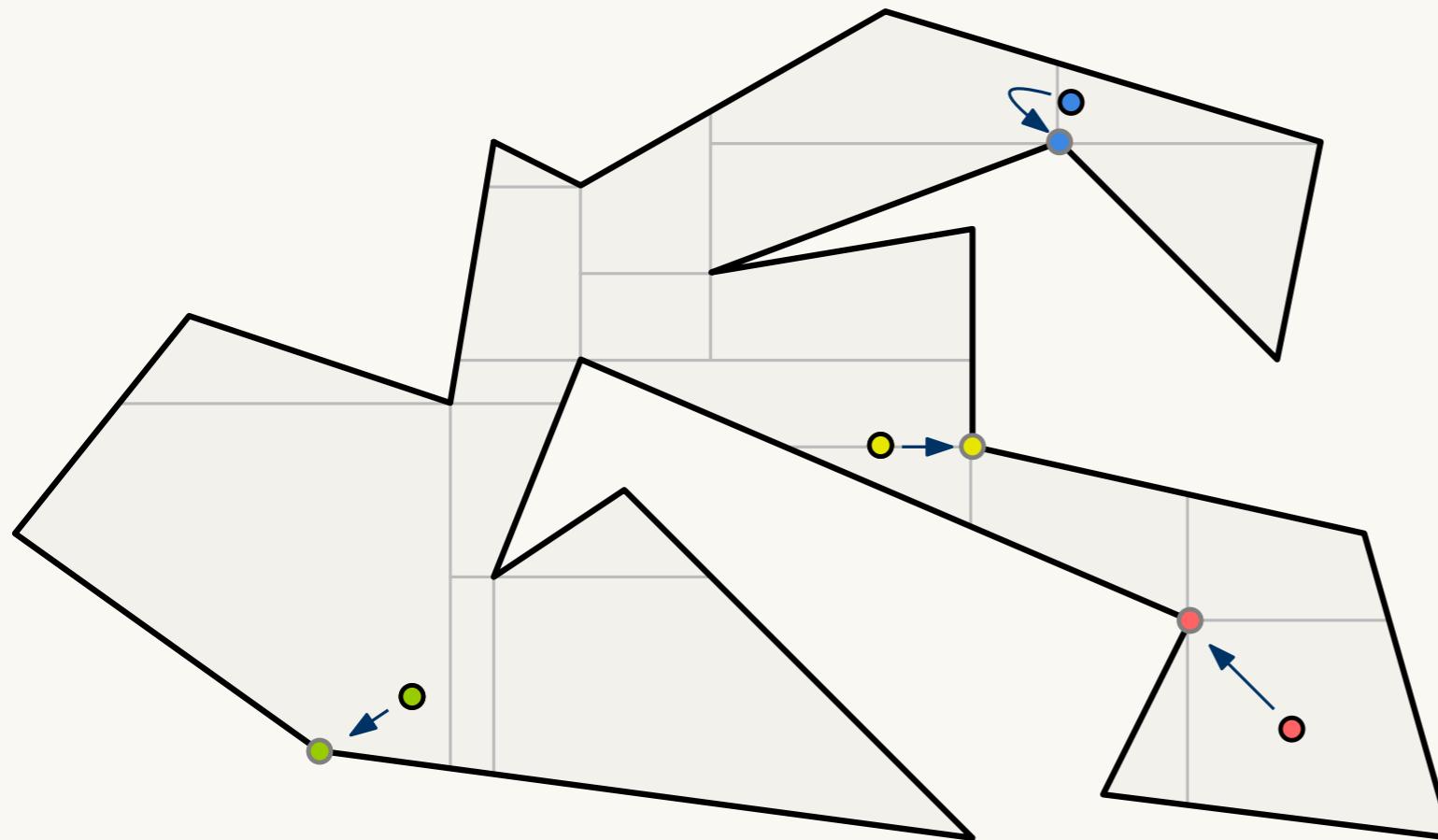
# Example



5 - New Assumption

Solution Stability

# Example



5 - New Assumption

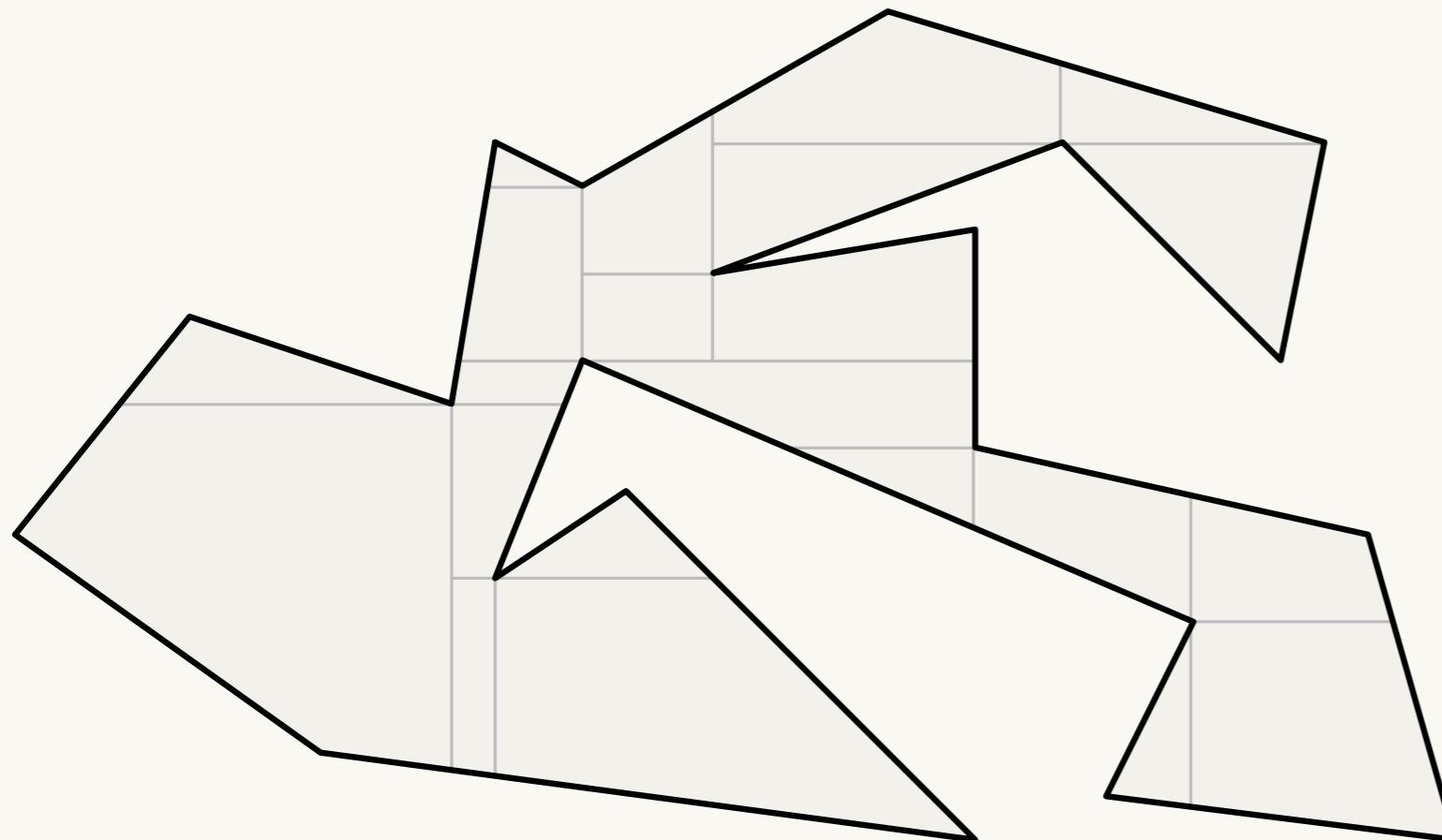
Solution Stability

guaranteed  
optimal  
solution

provable  
time  
bounds

efficient  
in  
practice

# Example



Challenge:

More Sparse   Faster

guaranteed  
optimal  
solution

provable  
time  
bounds

efficient  
in  
practice

# Definition

# Definition

# Definition

SAT = Satisfiability

$$\exists x_1, \dots, x_n \in \{0, 1\} : \Phi(x_1, \dots, x_n)$$

$\Phi$  formula with symbols:  $\wedge, \vee, \neg$

$$\begin{aligned} & \exists u, v, w, x, y, \in \{0, 1\} : \\ & (x \vee y \vee \neg u) \wedge (\neg u \vee w \vee \neg x) \end{aligned}$$

# Definition

ETR = Existential Theory of the Reals

$$\exists x_1, \dots, x_n \in \mathbb{R} : \Phi(x_1, \dots, x_n)$$

$\Phi$  formula with symbols:

$$0, 1, x_1, \dots, x_n, =, <, \leq, +, \cdot, \wedge, \vee, \neg$$

$$\begin{aligned} & \exists x, y, z \in \mathbb{R} : \\ & (x^2 + (y - x)(z + 12) = 5) \wedge \neg(y^5 + (3 + x) > 5) \end{aligned}$$

# Definition

SAT



NP

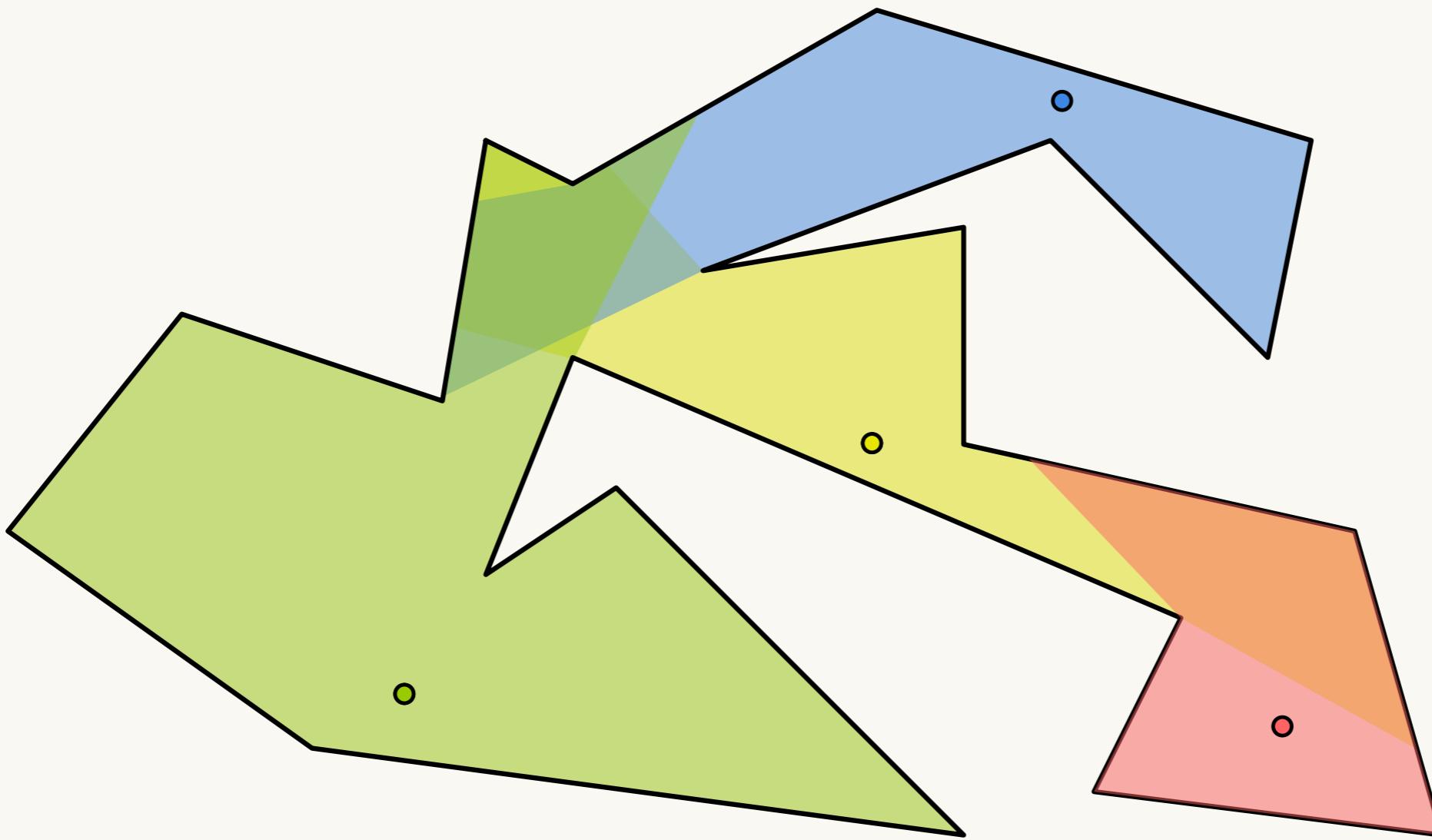
ETR



$\exists R$

# Art Gallery - $\exists \mathbb{R}$

# Art Gallery - $\exists \mathbb{R}$



# Art Gallery - $\exists \mathbb{R}$

Encode

Real Variables

Addition

Multiplication

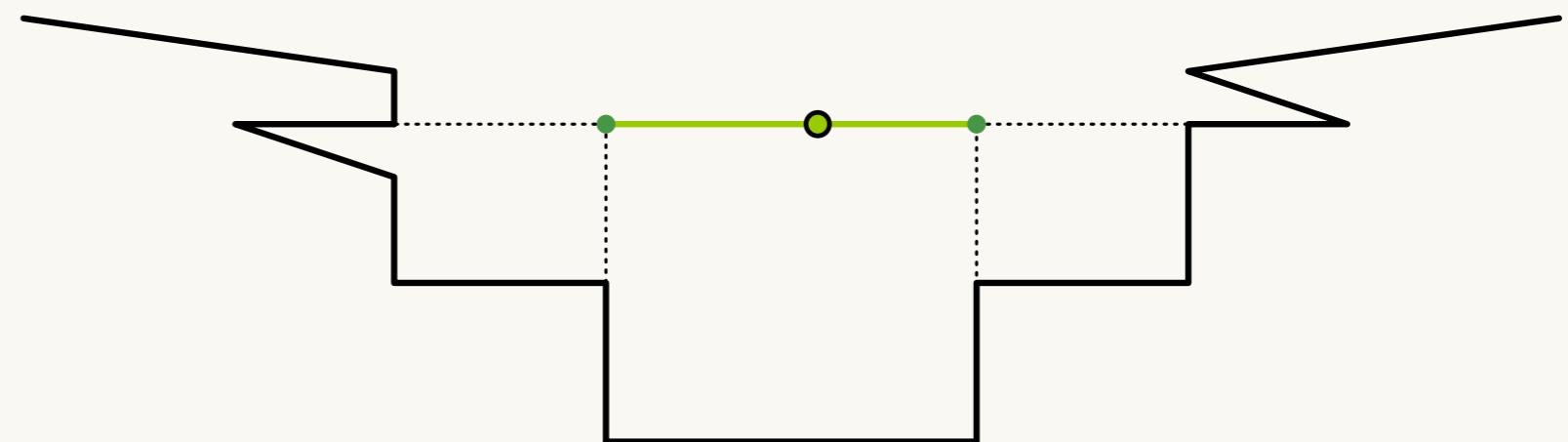
# Art Gallery - $\exists \mathbb{R}$

## Encode

Real Variables

Addition

Multiplication

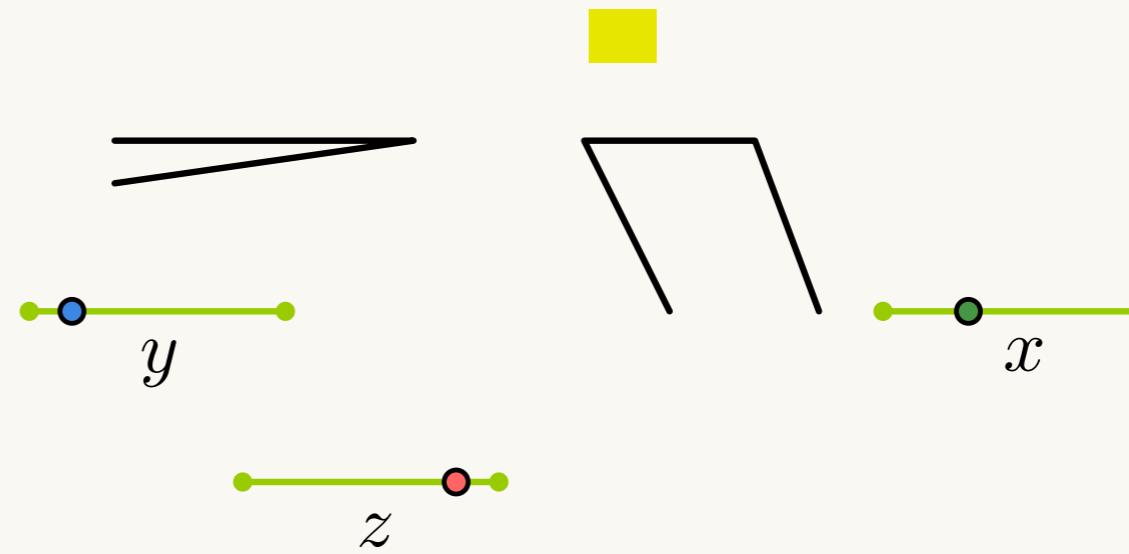


# Art Gallery - $\exists \mathbb{R}$

## Encode Real Variables

Addition

Multiplication

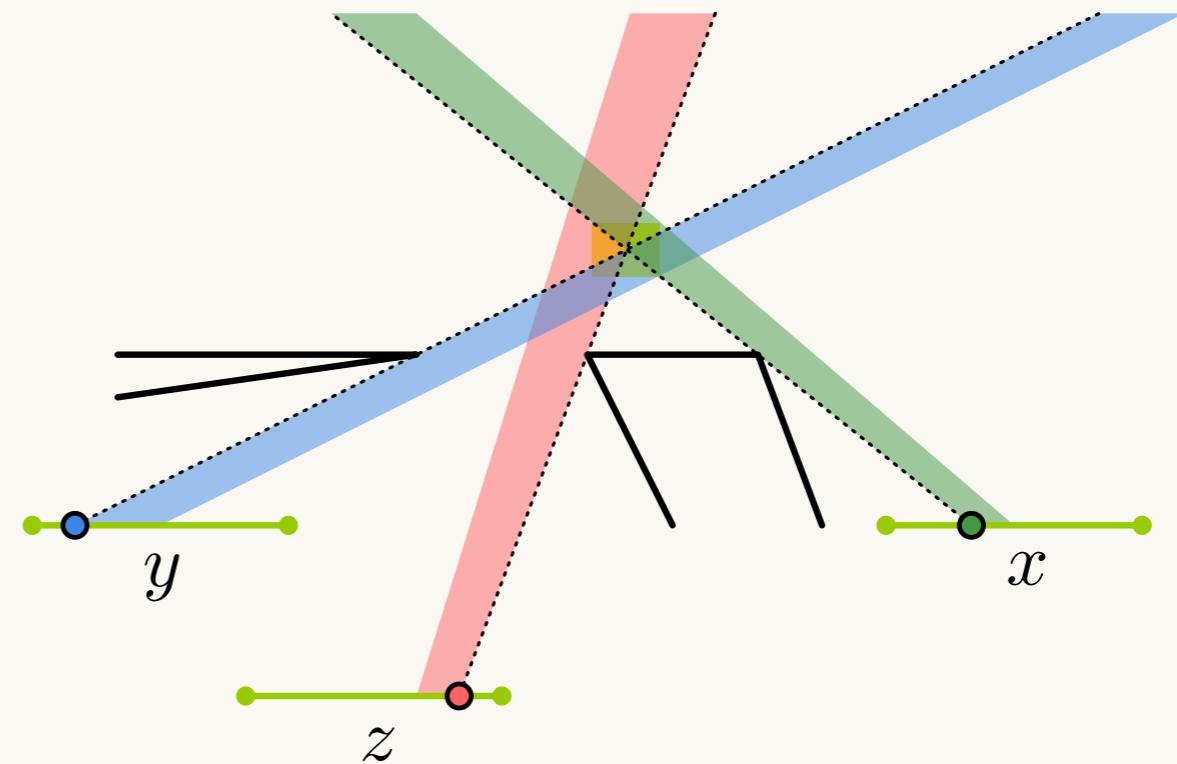


# Art Gallery - $\exists \mathbb{R}$

Encode  
Real Variables

Addition

Multiplication



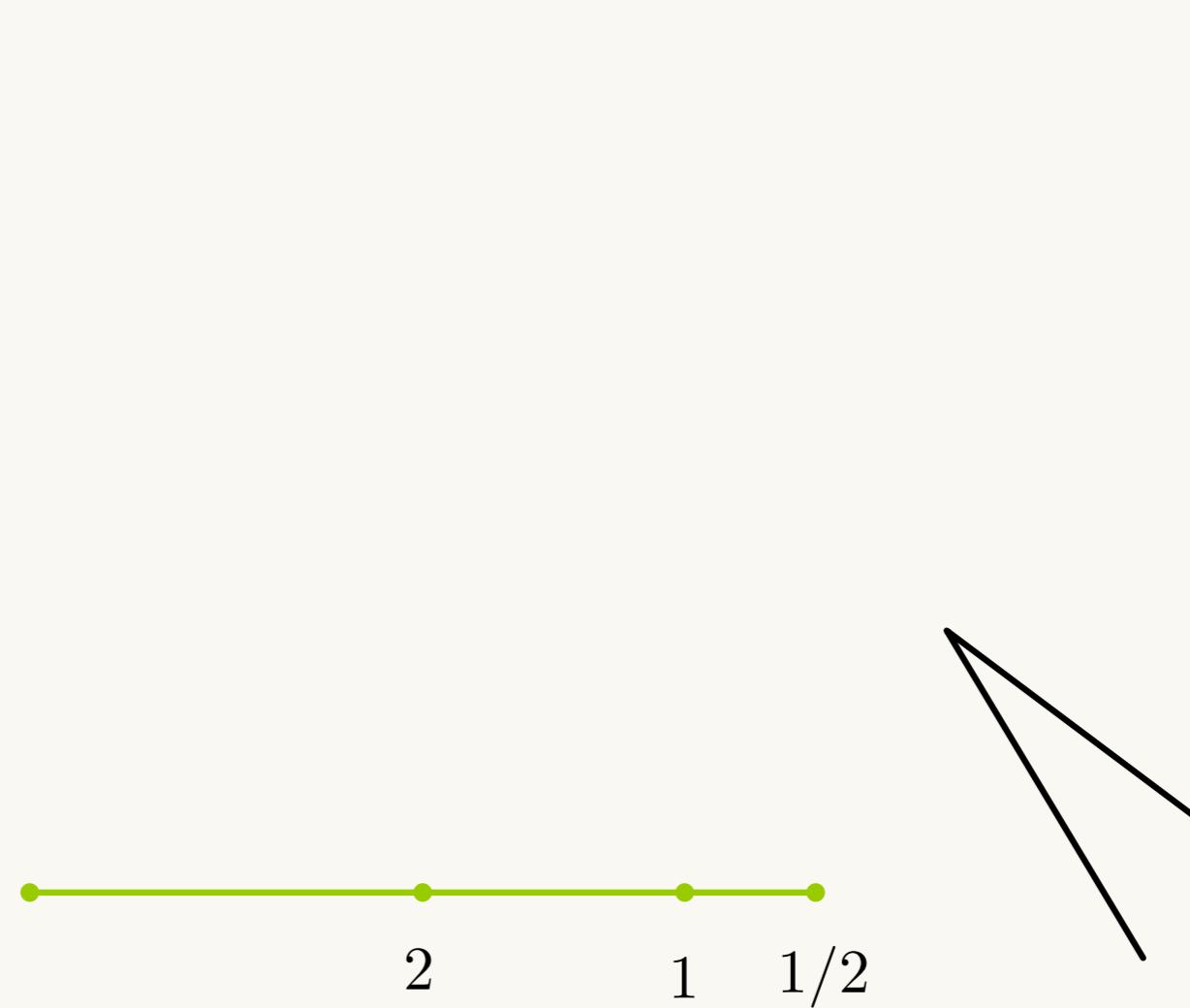
# Art Gallery - $\exists \mathbb{R}$

Encode

Real Variables

Addition

Multiplication



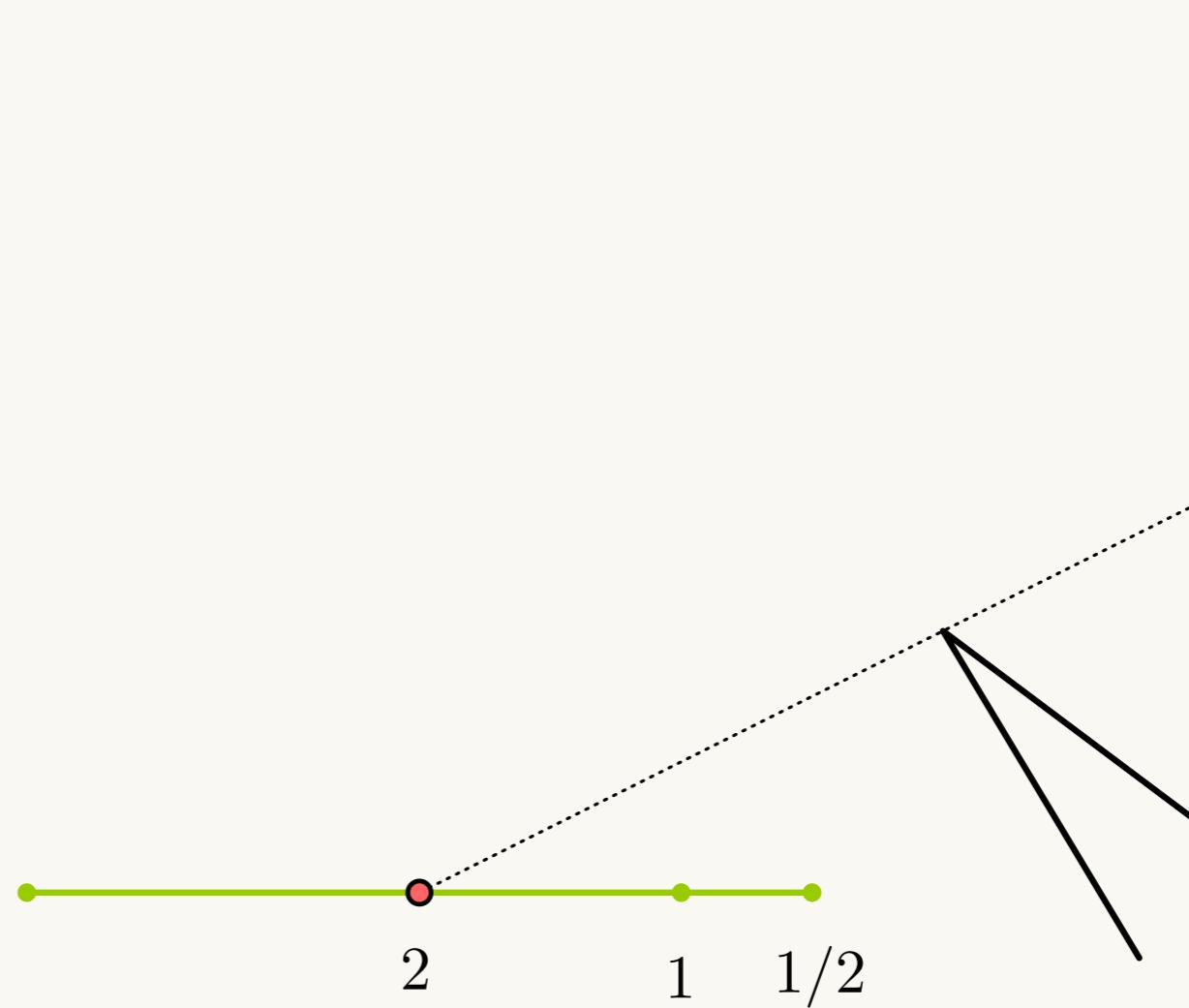
# Art Gallery - $\exists \mathbb{R}$

Encode

Real Variables

Addition

Multiplication



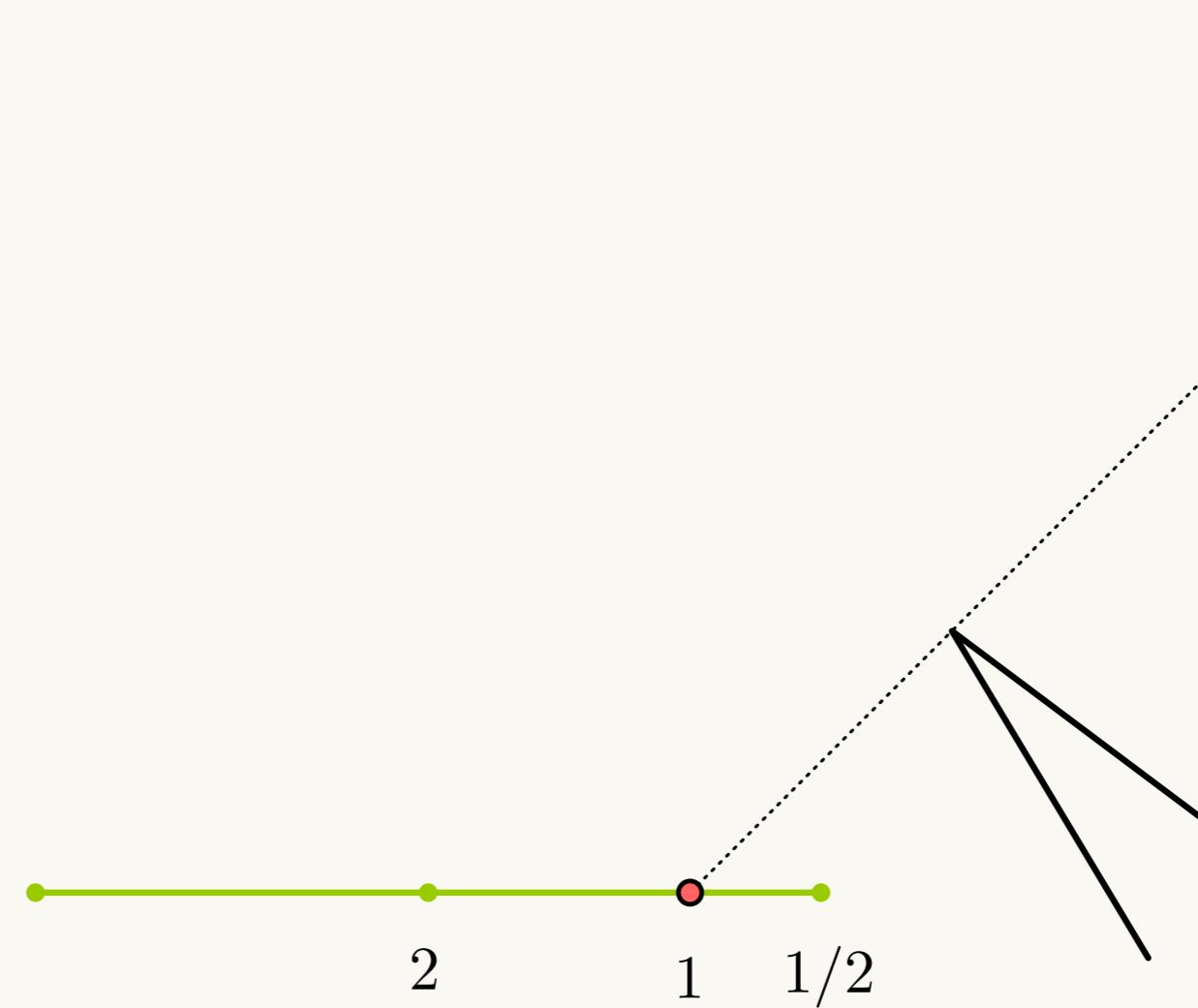
# Art Gallery - $\exists \mathbb{R}$

Encode

Real Variables

Addition

Multiplication



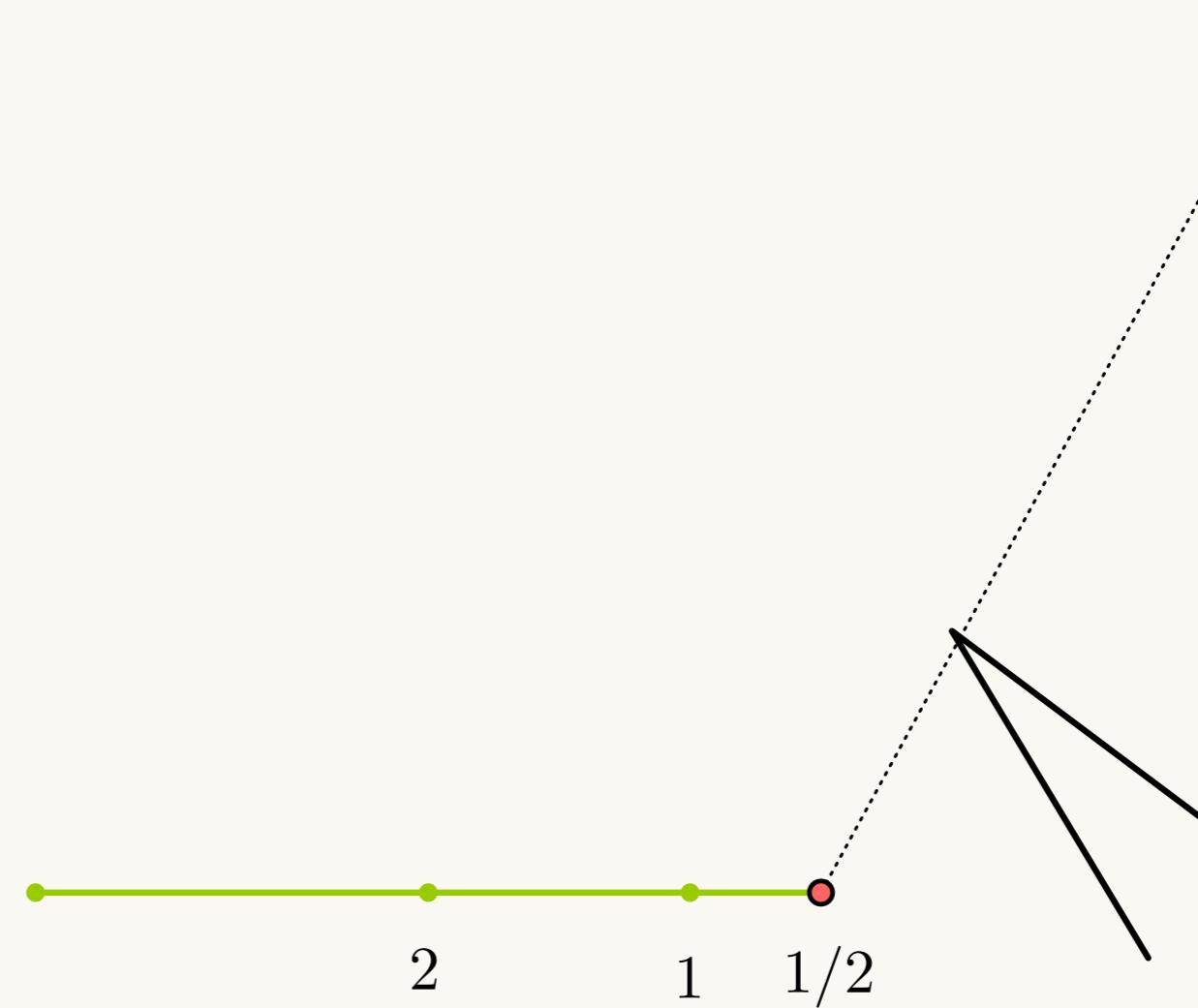
# Art Gallery - $\exists \mathbb{R}$

Encode

Real Variables

Addition

Multiplication



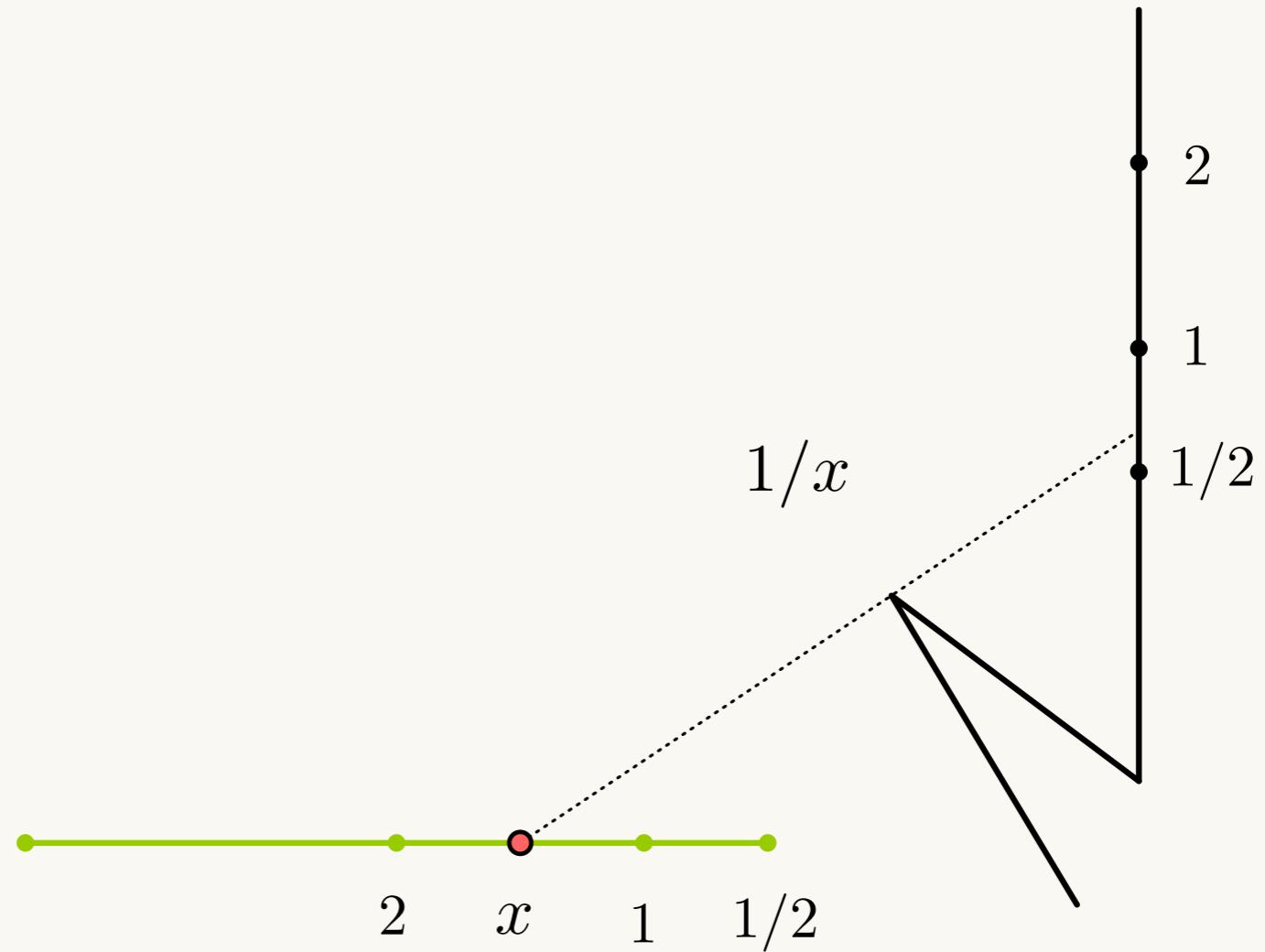
# Art Gallery - $\exists \mathbb{R}$

Encode

Real Variables

Addition

Multiplication



# Art Gallery - $\exists \mathbb{R}$

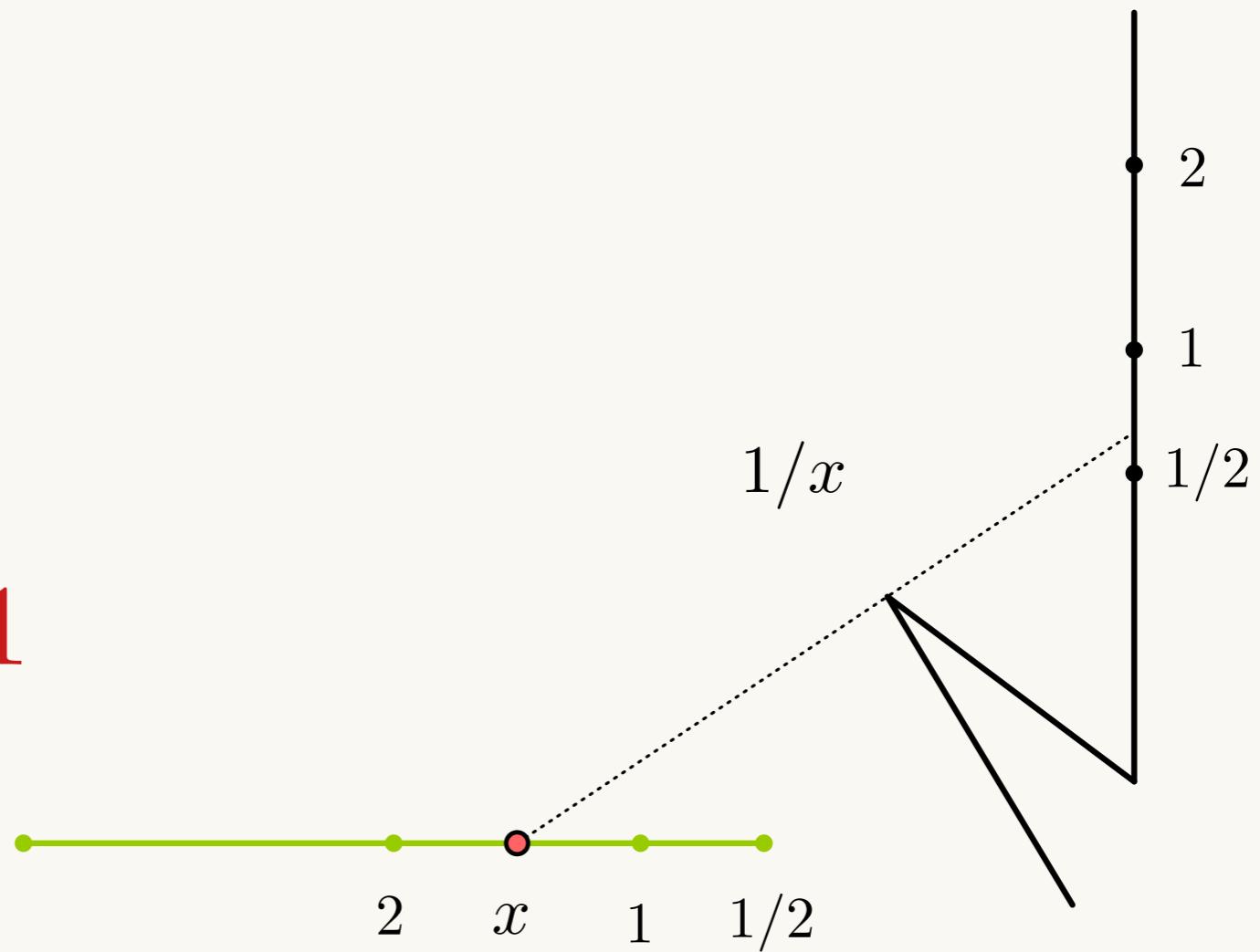
Encode

Real Variables

Addition

Multiplication

Inversion  $xy = 1$



# Inversion

$$xy = \frac{(x+y)^2 - (x-y)^2}{4}$$

# Inversion

$$xy = \frac{(x+y)^2 - (x-y)^2}{4}$$

$$x^2 = \frac{1}{\frac{1}{x} - \frac{1}{x+1}} - x$$

# Classification

Encode

Real Variables

Addition

$$f(x,y) = 0$$

# Classification

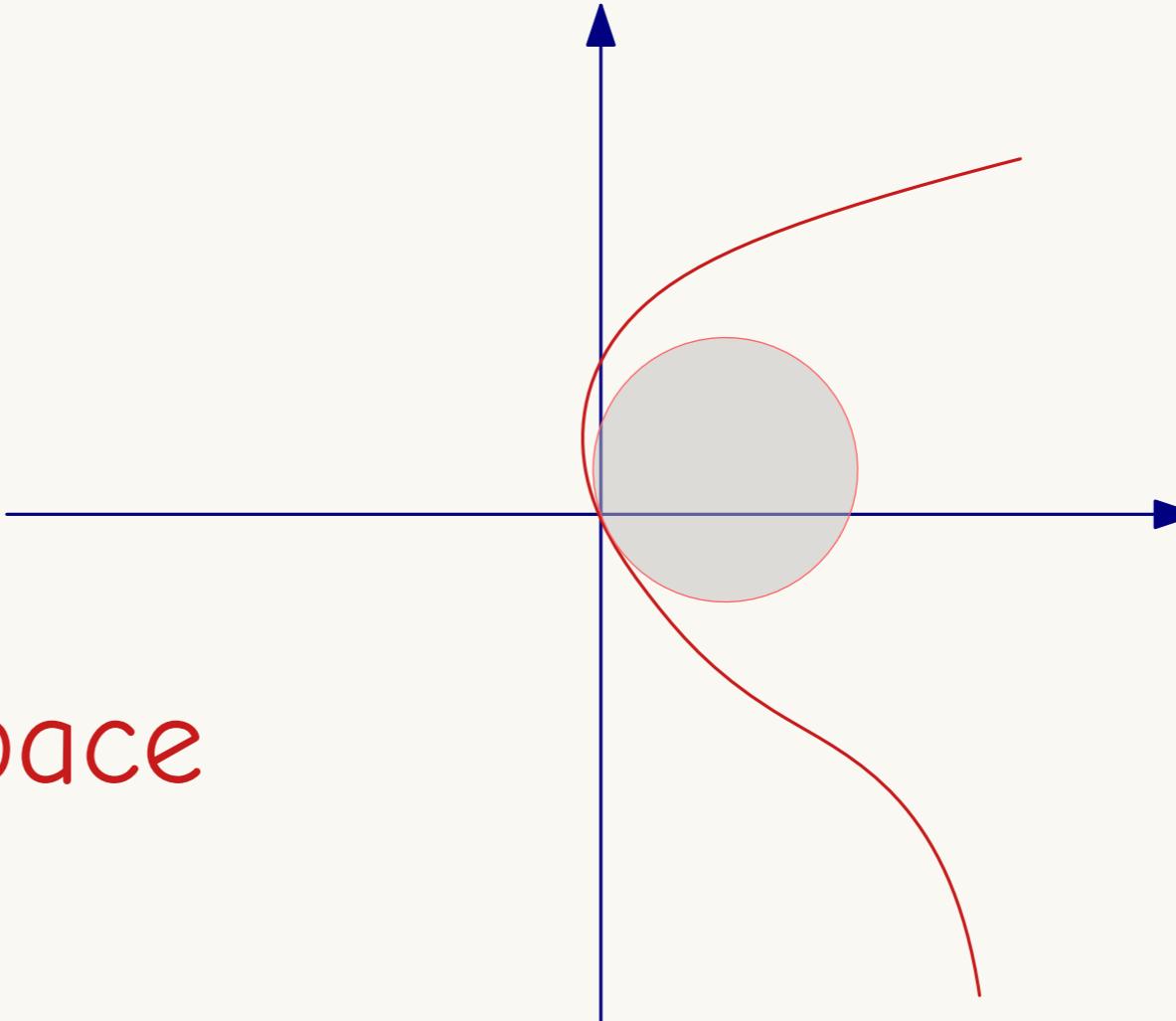
Encode

Real Variables

Addition

$f(x,y) = 0$

curved solution space



# Task 3: Attain Structural Insight



Unveil ER  
Develop Algorithmic Theory  
Attain Structural Insight