# Final Project - Radiation of Gluon Jets

Georgios Billis

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## 1

For this project I ignore the masses of quarks and electron whereas I assume that gluons do have a small non-zero mass $\mu$ and in the end of calculation it will be taken to zero.

The diagram cotributing to $e^- e^+ \to q\bar{q}$ is at the end of this report (to save time I did it by hand). This is the only one contributing since we do take into consideration only the one particle irreducible diagrams (in contrast to the diagrams that a gluon-quark loop appears at each of the final state external legs). For this diagram the matrix element is given below:

$$
\begin{aligned}
i\mathcal{M}_{g.loop} = Q_f \bar{u}(p')\Big[\int \frac{d^4k}{(2\pi)^4}(-ig\gamma_\rho T^\alpha)\frac{i}{\not{k}' + i\epsilon}(-ie\gamma_\nu)\frac{i}{\not{k} + i\epsilon}(-ig\gamma_\sigma T^\beta)\frac{-ig^{\rho\sigma}\delta^{\alpha\beta}}{(p-k)^2 - \mu^2 + i\epsilon}\Big]\upsilon(p)\times \\
\times \frac{-ig^{\mu\nu}}{q^2}\bar{\upsilon}(q_2)(-ie\gamma_\mu)u(q_1) = \\
= -ig^2 Q_f tr(T^\alpha T^\alpha)\bar{u}(p')\Big[\int \frac{d^4k}{(2\pi)^4}\gamma_\rho \frac{1}{\not{k}' + i\epsilon}\gamma^\nu \frac{1}{\not{k} + i\epsilon}\gamma^\rho \frac{1}{(p-k)^2 - \mu^2 + i\epsilon}\Big]\upsilon(p)\times \\
\times \bar{\upsilon}(q_2)\Big[(-ie)^2\gamma_\nu \frac{-i}{q^2}\Big]u(q_1) = \\
= -ig^2 \bar{u}(p')\Big[\int \frac{d^4k}{(2\pi)^4}\frac{\gamma_\rho \not{k}'\gamma^\nu \not{k}\gamma^\rho}{(k'^2 + i\epsilon)(k^2 + i\epsilon)((p-k)^2 - \mu^2 + i\epsilon)}\Big]\upsilon(p)\times i\mathcal{M}_{lep\nu}
\end{aligned}
$$

where

$$
i\mathcal{M}_{\nu lep} = \bar{\upsilon}(q_2)\Big[(-ie)^2 Q_f \gamma_\nu \frac{-i}{q^2}\Big]u(q_1)
$$

and the one loop correction to the quark-gluon vertex is given by the other part, i.e.:

$$
\bar{u}(p')\delta\Gamma^\nu \upsilon(p) = -ig^2 \bar{u}(p')\Big[\int \frac{d^4k}{(2\pi)^4}\frac{\gamma_\rho \not{k}'\gamma^\nu \not{k}\gamma^\rho}{(k'^2 + i\epsilon)(k^2 + i\epsilon)((p-k)^2 - \mu^2 + i\epsilon)}\Big]\upsilon(p)
$$

with $\Gamma^\nu = \gamma^\nu + \delta\Gamma^\nu$

In order to work out the loop momentum integral the method of Feynman parameters has to be introduced which simply combines the three propagators into a single quadratic polynomial in k, (for our case) raised in the third power. Working only the denominator part we have:

$$\frac{1}{(k'^2+i\epsilon)(k^2+i\epsilon)((p-k)^2-\mu^2+i\epsilon)} =$$
$$= \int dxdydz\delta(x+y+z-1)\frac{2!}{[y((p-k)^2-\mu^2+i\epsilon)+x(k^2+i\epsilon)+z(k'^2+i\epsilon)]^3}$$

Having in mind that $k' = k+q$ and that $x+y+z=1$ we have that

$$\int dxdydz\delta(x+y+z-1)\frac{2!}{[y(p-k)^2-y\mu^2+xk^2+z(q-k)^2+i\epsilon]^3}$$

and having in mind that quarks are considered to be massless i.e. $p'^2 = p^2 = 0$, I obtain:

$$\int dxdydz\delta(x+y+z-1)\frac{2}{[k^2-2k\cdot(yp-zq)-y\mu^2+zq^2+i\epsilon]^3} =$$
$$= \int dxdydz\delta(x+y+z-1)\frac{2}{[(k-(yp-zq))^2-(yp-zq)^2-y\mu^2+zq^2+i\epsilon]^3} =$$
$$= \int dxdydz\delta(x+y+z-1)\frac{2}{[l^2+zq^2(1-z)+2yzp\cdot q-y\mu^2+i\epsilon]^3} =$$
$$= \int dxdydz\delta(x+y+z-1)\frac{2}{[l^2+zq^2(1-z)+yz(p+q)^2-yzq^2-y\mu^2+i\epsilon]^3} =$$
$$= \int dxdydz\delta(x+y+z-1)\frac{2}{[l^2+zq^2(1-z)+yzp'^2-yzq^2-y\mu^2+i\epsilon]^3} =$$
$$= \int dxdydz\delta(x+y+z-1)\frac{2}{[l^2+zxq^2-y\mu^2+i\epsilon]^3}$$

thus the denominator can be written as $D = l^2-\Delta+i\epsilon$ with $\Delta = -zxq^2+y\mu^2$. Now the numerator:

$$N^\nu = \bar{u}(p')\left[\gamma_\rho \not{k}'\gamma^\nu \not{k}\gamma^\rho\right]\upsilon(p) =$$
$$= (-2)\bar{u}(p')\left[\not{k}\gamma^\nu \not{k}'\right]\upsilon(p) = (-2)\bar{u}(p')\left[\not{k}\gamma^\nu(\not{k}+\not{q})\right]\upsilon(p)$$

but $k = l - zq - yp$, leading to:

$$N^\nu = (-2)\bar{u}(p')\left[(\not{l}-z\not{q}+y\not{p})\gamma^\nu(\not{q}+\not{l}-z\not{q}+y\not{p})\right]\upsilon(p)$$

once again the fact that quarks are massless implies that: $\bar{u}(p')\not{p}' = 0$ and $\not{p}u(p) = 0$. Now the goal is to bring it into a form similar to the Gordon Identity, so:

$$
\begin{aligned}
N^\nu &= (-2)\bar{u}(p')\Big[(\not{l}\gamma^\nu\not{l} + (y\not{p} - z\not{q})\gamma^\nu((1-z)\not{q} + yp)\Big]\upsilon(p) = \\
&= (-2)\bar{u}(p')\Big[\frac{-l^2}{2}\gamma^\nu + y\not{p}\gamma_\nu(1-z)\not{q} - z\not{q}\gamma^\nu(1-z)\not{q}\Big]\upsilon(p) = \\
&= (-2)\bar{u}(p')\Big[\frac{-l^2}{2}\gamma^\nu - y(\not{p}' - \not{p})\gamma_\nu(1-z)\not{q} - z\not{q}\gamma^\nu(1-z)\not{q})\Big]\upsilon(p) = \\
&= (-2)\bar{u}(p')\Big[\frac{-l^2}{2}\gamma^\nu - (y+z)(1-z)\not{q}\gamma^\nu\not{q}\Big]\upsilon(p) = \\
&= (2)\bar{u}(p')\Big[\frac{l^2}{2}\gamma^\nu - (y+z)(1-z)q^2\gamma^\nu\Big]\upsilon(p)
\end{aligned}
$$

where we have used properties of the gamma matrices such as $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and that $\not{a}\not{a} = a^2$.
The Gordon identity enables us to write the vertex in terms of the form factors $F_1(q^2)$ and $F_2(q^2)$ but as it can be seen here, $\delta\Gamma^\nu$ is only dependent in $\gamma^\nu$ concluding that I already have in hand the form factor $\delta F_1(q^2)$. Concretely:

$$
\bar{u}(p')\delta\Gamma^\nu\upsilon(p) = (-4ig^2)\bar{u}(p')\Big[\int\frac{d^4k}{(2\pi)^4}\int dxdydz\delta(x+y+z-1)\frac{\frac{l^2}{2} - (y+z)(1-z)q^2}{D^3}\gamma^\nu\Big]\upsilon(p)
$$

Now in order to perform the momentum integral we are going do a Wick rotation. This consists of a counter-clockwise rotation in the $l^0$ plane such that we avoid the poles that appear. The Euclidian variables are defined as: $l^0 = il_E^0$ and $\vec{l}_E = \vec{l}$. After the rotation, the $k$ integral will be performed in the four dimensional sperical coordinates. I will avoid restating the formulas of the integrals but I will mention the ones that apply in our case:

$$
\int\frac{d^4l}{(2\pi)^4}\frac{1}{[l^2-\Delta]^n} = \frac{i(-1)^n}{(4\pi)^2}\frac{1}{(n-1)(n-2)}\frac{1}{\Delta^{n-2}}
$$

and

$$
\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{[l^2-\Delta]^n} = \frac{i(-1)^{n-1}}{(4\pi)^2}\frac{1}{(n-1)(n-2)(n-3)}\frac{1}{\Delta^{n-3}}
$$

It can be seen that the second integral for this case ($n = 3$) is divergent. In order to address this issue, the Pauli-Villars regularization will be employed. This consists in introducing a ficticious, heavy, particle whose contribution is subtracted from that of the gluon. This results in a substitution in the gluon propagator:

$$
\frac{1}{(p-k)^2 - \mu^2 + i\epsilon} \to \frac{1}{(p-k)^2 - \mu^2 + i\epsilon} - \frac{1}{(p-k)^2 - \Lambda^2 + i\epsilon}
$$

Thus:

$$
\int\frac{d^4l}{(2\pi)^4}\left(\frac{l^2}{[l^2-\Delta]^3} - \frac{l^2}{[l^2-\Delta_\Lambda]^3}\right) = \frac{i}{(4\pi)^2}log\Big(\frac{\Delta_\Lambda}{\Delta}\Big)
$$

where $\Delta_\Lambda = -zxq^2 + y\Lambda^2$, and for $\Lambda >>$ we have

$$\approx \frac{i}{(4\pi)^2} log\Big(\frac{y\Lambda^2}{\Delta}\Big)$$

As indicated by the exercise, the renormalisation will be realised by subtraction at $q^2 = 0$ where the following substitution will be made:
$\delta F_1(q^2) \to \delta F_1(q^2) - \delta F_1(0)$ such that the condition $F_1(q^2) = 1$ is fullfilled. For the integral that is divergent the subtraction yields the following term:

$$\frac{i}{(4\pi)^2}\Bigg[log\Big(\frac{y\Lambda^2}{\Delta}\Big) - log\Big(\frac{y\Lambda^2}{y\mu^2}\Big)\Bigg] = \frac{i}{(4\pi)^2} log\Big(\frac{y\mu^2}{\Delta}\Big)$$

and the non divergent integral is:

$$\int \frac{d^4l}{(2\pi)^4}\frac{1}{[l^2-\Delta]^3} = \frac{-i}{(4\pi)^2}\frac{1}{2\Delta}$$

To sum up:

$$\begin{aligned} i\mathcal{M}_{g.loop} &= \frac{2g^2}{(4\pi)^2}\bar{u}(p')\Bigg[\int dxdydz\delta(x+y+z-1)\Bigg(log\Big(\frac{y\mu^2}{\Delta}\Big) + \frac{(1-x)(1-z)q^2}{\Delta}\Bigg)\gamma^\nu\Bigg]\upsilon(p)i\mathcal{M}_{\nu lep} = \\ &= \frac{\alpha_g}{(2\pi)}\bar{u}(p')\Bigg[\int dxdydz\delta(x+y+z-1)\Bigg(log\Big(\frac{y\mu^2}{\Delta}\Big) + \frac{(1-x)(1-z)q^2}{\Delta}\Bigg)\gamma^\nu\Bigg]\upsilon(p)i\mathcal{M}_{\nu lep} \end{aligned}$$

leading to the form factor that includes the zero order corrections:

$$F_1(q^2) = \Bigg[Q_f + \frac{\alpha_g Q_f}{(2\pi)}\int dxdydz\delta(x+y+z-1)\Bigg(log\Big(\frac{y\mu^2}{\Delta}\Big) + \frac{(1-x)(1-z)q^2}{\Delta}\Bigg)\Bigg]$$

satisfying $F_1(q^2) = 0$. Therefore the cross section for the production of $q\bar{q}$ can be written as:

$$\sigma(e^-e^+ \to q\bar{q}) = \frac{4\pi\alpha^2}{3s}\cdot 3\left|F_1(q^2)\right|^2$$

with $F_1(q^2)$ containing the first order quark gluon corrections.