Reconstruction in high-dimensional spaces

David Salinas

PhD advisor : Dominique Attali

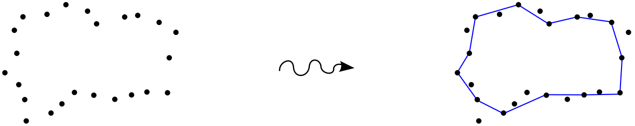
gipsa-lab

UNIVERSITÉ DE GRENOBLE

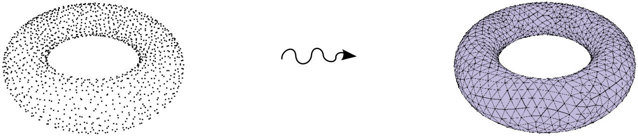
11 septembre 2013

Reconstruction problem

- Input : a point cloud that samples a shape
- Goal : connect the points



2D



3D

Reconstruction problem

Construct an approximation :

- efficiently
- that is “similar” to the sampled shape

Reconstruction problem

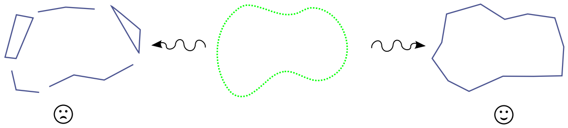
Construct an approximation :

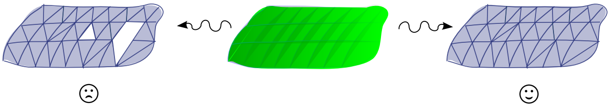
- efficiently
- that is "similar" to the sampled shape

Similar? Same topology.

Reconstruction problem

Why should we care about topology?

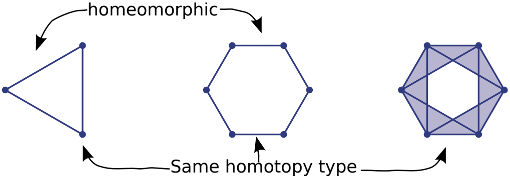




Two spaces A and B :

- are **homeomorphic** if there exists $f : A \to B$ bijective and bicontinuous
  $\to$ denoted by $A \approx B$
- have the same **homotopy type** if there exists an homotopy between them
  $\to$ denoted by $A \simeq B$

We say that A and B have **the same topology** if they have the same homotopy type.



Reconstruction

In low and high dimensional spaces

- Vast litterature when points are in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Less when points are in $\mathbb{R}^d$
- Point in $\mathbb{R}^d$?

Reconstruction

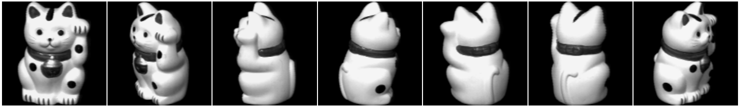
In low and high dimensional spaces

- Vast litterature when points are in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Less when points are in $\mathbb{R}^d$
- Point in $\mathbb{R}^d$?
  - Several measures (size, weight, age, ...) about a person

Reconstruction

In low and high dimensional spaces

- Vast litterature when points are in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Less when points are in $\mathbb{R}^d$
- Point in $\mathbb{R}^d$?
  - Several measures (size, weight, age, ...) about a person
  - An image : dimension = number of pixel

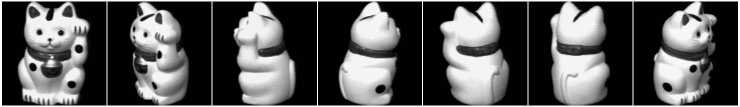


COIL DataBase

Reconstruction

In low and high dimensional spaces

- Vast litterature when points are in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Less when points are in $\mathbb{R}^d$
- Point in $\mathbb{R}^d$?
  - Several measures (size, weight, age, ...) about a person
  - An image : dimension = number of pixel



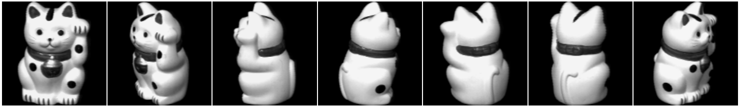
COIL DataBase

- Shape in $\mathbb{R}^d$?

Reconstruction

In low and high dimensional spaces

- Vast litterature when points are in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Less when points are in $\mathbb{R}^d$
- Point in $\mathbb{R}^d$?
  - Several measures (size, weight, age, ...) about a person
  - An image : dimension = number of pixel



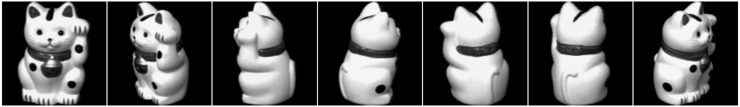
COIL DataBase

- Shape in $\mathbb{R}^d$?

Reconstruction

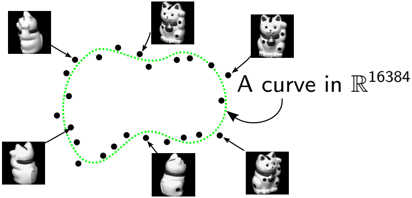
In low and high dimensional spaces

- Vast litterature when points are in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Less when points are in $\mathbb{R}^d$
- Point in $\mathbb{R}^d$?
  - Several measures (size, weight, age, ...) about a person
  - An image : dimension = number of pixel



COIL DataBase

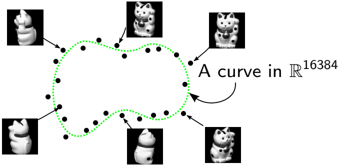
- Shape in $\mathbb{R}^d$?



Reconstruction in high-dimensional spaces

Efficiency

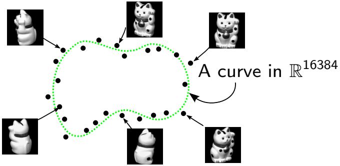
- Notation :
  - n : number of points
  - d : dimension of points
  - k : dimension of the shape



- $n = 22$
- $d = 16384$
- $k = 1$

Reconstruction in high-dimensional spaces
Efficiency

- Notation :
  - n : number of points
  - d : dimension of points
  - k : dimension of the shape



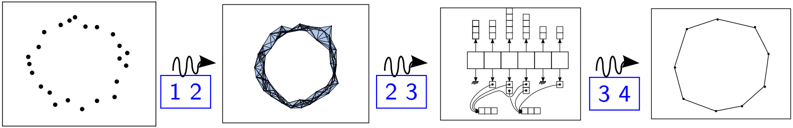
- $n = 22$
- $d = 16384$
- $k = 1$

- Fundamental hypothesis : $k << d$
- Efficient : $O(d)$ (n and k fixed)
  - ☹ $n^{d/2}$
  - ☺ dn^k

Reconstruction in high-dimensional spaces

Road map

1. Find conditions such that an approximation has the same topology as the shape
2. Compute (efficiently) the approximation
3. Data-structure to store this approximation
4. Simplification in this data-structure

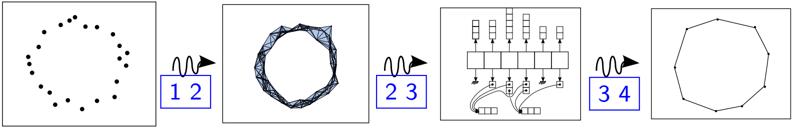


Point Cloud — Approximation — Data Structure — Simplified approximation

Reconstruction in high-dimensional spaces

Road map

1. Find conditions such that an approximation has the same topology as the shape
2. Compute (efficiently) the approximation
3. Data-structure to store this approximation
4. Simplification in this data-structure



Point Cloud | Approximation | Data Structure | Simplified approximation

Homotopy type

- Shape with Reach>0 [Niyogi Smale Weinberger 04]
- Compact with μ-Reach>0 [Chazal Cohen-Steiner Lieutier 06]

Reach$>$0 $\qquad$ μ-Reach$>$0

Homeomorphism

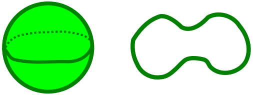
- First approach : using the Delaunay complex [Cheng Dey Ramos 05]
- With the witness complex [Boissonat Guibas Oudot 09]
- Tangential Delaunay complex [Boissonat Ghosh 10]

Reconstruction problem

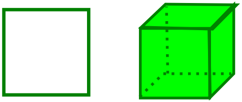
Previous results in high dimensions

Homotopy type

- Shape with Reach>0 [Niyogi Smale Weinberger 04]
- Compact with μ-Reach>0 [Chazal Cohen-Steiner Lieutier 06]



Reach$>$0

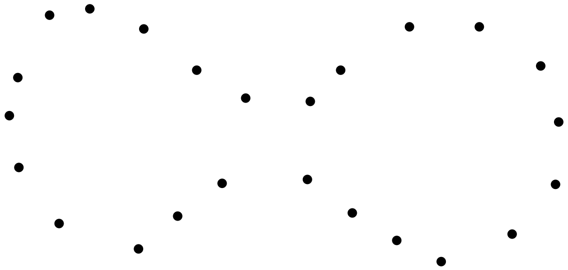


μ-Reach$>$0

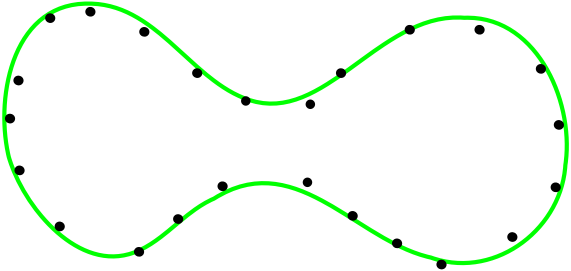
Homeomorphism

- First approach : using the Delaunay complex [Cheng Dey Ramos 05]
- With the witness complex [Boissonat Guibas Oudot 09]
- Tangential Delaunay complex [Boissonat Ghosh 10]

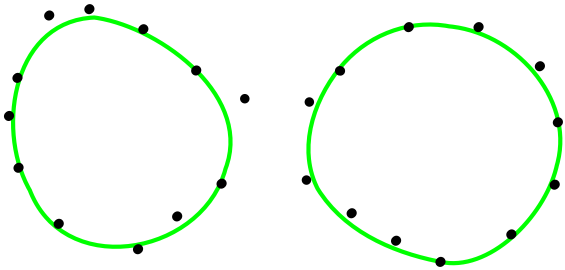
Reach



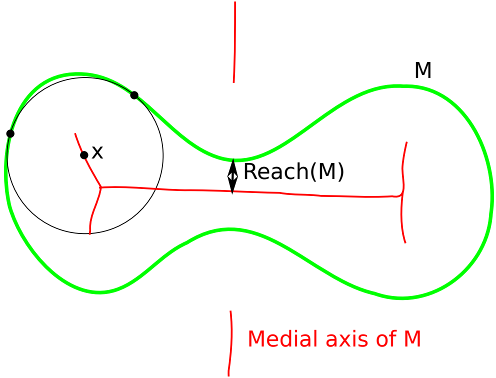
Reach



Reach

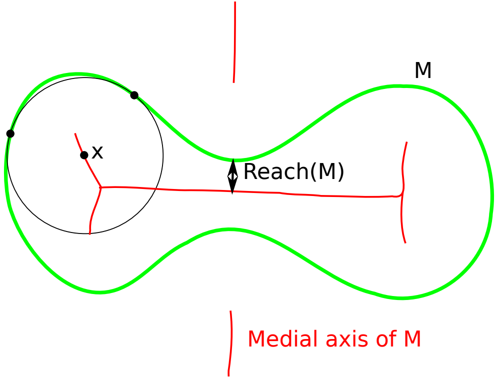


Reach



- $\text{MedialAxis}(M) = \{x \in \mathbb{R}^d \mid x \text{ has at least two closest points on } M\}$

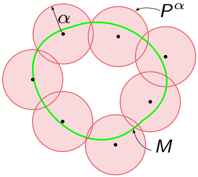
Reach

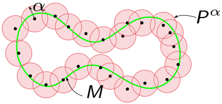


- $\text{MedialAxis}(M) = \{x \in \mathbb{R}^d \mid x \text{ has at least two closest points on } M\}$
- $\text{Reach}(M) = d(M, \text{MedialAxis}(M))$

Offset of points

Given $P \subset \mathbb{R}^d$ we denote $P^\alpha = \bigcup B(p, \alpha)$ the α-offset of P.





Reconstruction problem

Offset of points

Given $P \subset \mathbb{R}^d$ we denote $P^\alpha = \bigcup B(p, \alpha)$ the α-offset of P.

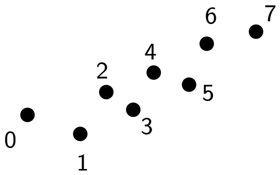
Theorem [Niyogi Smale Weinberger 08]

The α-offset P^α has the same homotopy type as M i.e. $P^\alpha \simeq M$ when

$$\begin{cases} d_H(P, M) < (3 - \sqrt{8}) \ \text{reach}(M) \\ d_H(P, M) \leq (1 - \frac{\sqrt{2}}{2})\alpha \leq (3 - \sqrt{8})\text{Reach}(M) \end{cases}$$

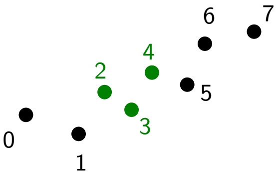
Simplicial complex

- P : a set of points in $\mathbb{R}^d$
- A simplex : a subset $\sigma \subset P$
- A simplicial complex K : a set of simplices with one rule
  $\rightarrow$ $\sigma \in K \implies \forall \tau \subset \sigma, \tau \in K$
- Geometry? Take the convex hull of simplices
  $\rightarrow$ $\mathrm{Shadow}(K) = \bigcup_{\sigma \in K} \mathrm{Hull}(\sigma)$



Simplicial complex

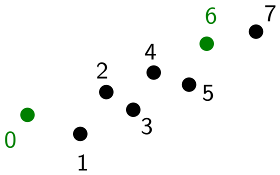
- P : a set of points in $\mathbb{R}^d$
- A simplex : a subset $\sigma \subset P$
- A simplicial complex K : a set of simplices with one rule
  $\rightarrow$ $\sigma \in K \implies \forall \tau \subset \sigma, \tau \in K$
- Geometry? Take the convex hull of simplices
  $\rightarrow$ $\text{Shadow}(K) = \bigcup_{\sigma \in K} \text{Hull}(\sigma)$



$\sigma = \{2, 3, 4\}$

Simplicial complex

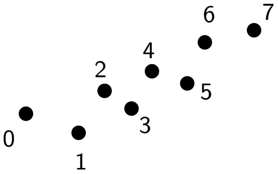
- P : a set of points in $\mathbb{R}^d$
- A simplex : a subset $\sigma \subset P$
- A simplicial complex K : a set of simplices with one rule
  $\rightarrow \sigma \in K \implies \forall \tau \subset \sigma, \tau \in K$
- Geometry? Take the convex hull of simplices
  $\rightarrow \text{Shadow}(K) = \bigcup_{\sigma \in K} \text{Hull}(\sigma)$



$\sigma = \{0, 6\}$

Simplicial complex

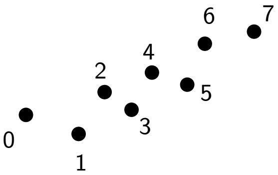
- P : a set of points in $\mathbb{R}^d$
- A simplex : a subset $\sigma \subset P$
- A simplicial complex K : a set of simplices with one rule
  $\rightarrow \sigma \in K \implies \forall \tau \subset \sigma, \tau \in K$
- Geometry? Take the convex hull of simplices
  $\rightarrow \text{Shadow}(K) = \bigcup_{\sigma \in K} \text{Hull}(\sigma)$



$K = \{\{0\}, \{1\}, \{0,1\}\} \rightarrow$ ☺

Simplicial complex

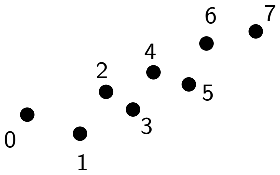
- P : a set of points in $\mathbb{R}^d$
- A simplex : a subset $\sigma \subset P$
- A simplicial complex K : a set of simplices with one rule
  $\rightarrow$ $\sigma \in K \implies \forall \tau \subset \sigma, \tau \in K$
- Geometry? Take the convex hull of simplices
  $\rightarrow$ $\text{Shadow}(K) = \bigcup_{\sigma \in K} \text{Hull}(\sigma)$



$K = \{\{0\}, \{0,1\}\} \rightarrow$ ☹

Simplicial complex

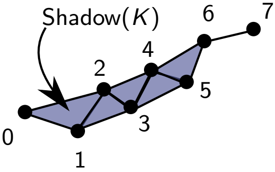
- P : a set of points in $\mathbb{R}^d$
- A simplex : a subset $\sigma \subset P$
- A simplicial complex K : a set of simplices with one rule
  $\rightarrow$ $\sigma \in K \implies \forall \tau \subset \sigma, \tau \in K$
- Geometry? Take the convex hull of simplices
  $\rightarrow$ $\text{Shadow}(K) = \bigcup_{\sigma \in K} \text{Hull}(\sigma)$



$$K = \{\{0,1,2\},\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5,6\},\{6,7\},\dots\}$$

Simplicial complex

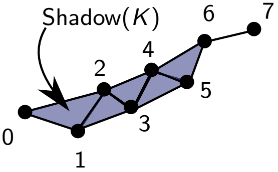
- P : a set of points in $\mathbb{R}^d$
- A simplex : a subset $\sigma \subset P$
- A simplicial complex K : a set of simplices with one rule
  $\rightarrow$ $\sigma \in K \implies \forall \tau \subset \sigma, \tau \in K$
- Geometry? Take the convex hull of simplices
  $\rightarrow$ $\text{Shadow}(K) = \bigcup_{\sigma \in K} \text{Hull}(\sigma)$



$$K = \{\{0,1,2\},\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5,6\},\{6,7\},\dots\}$$

Simplicial complex

- P : a set of points in $\mathbb{R}^d$
- A simplex : a subset $\sigma \subset P$
- A simplicial complex K : a set of simplices with one rule
  $\rightarrow \sigma \in K \implies \forall \tau \subset \sigma, \tau \in K$
- Geometry? Take the convex hull of simplices
  $\rightarrow \text{Shadow}(K) = \bigcup_{\sigma \in K} \text{Hull}(\sigma)$



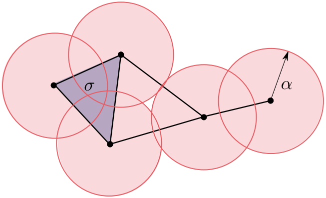
$K = \{\{0,1,2\},\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5,6\},\{6,7\},\dots\}$

$\rightarrow$ how do we construct a simplicial complex K?

The Cech complex

Cech complex

- Nerve of a family : $\operatorname{Nrv} F = \{\sigma \subset F \mid \bigcap \sigma \neq \emptyset\}$
- Cech complex $\mathcal{C}(P, \alpha) = \operatorname{Nrv}\{B(p, \alpha) \mid p \in P\}$

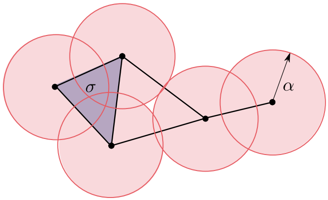


☺ Nerve theorem : $P^\alpha \simeq \mathcal{C}(P, \alpha)$

The Cech complex

Cech complex

- Nerve of a family : $\mathrm{Nrv}\, F = \{\sigma \subset F \mid \bigcap \sigma \neq \emptyset\}$
- Cech complex $\mathcal{C}(P, \alpha) = \mathrm{Nrv}\{B(p, \alpha) \mid p \in P\}$



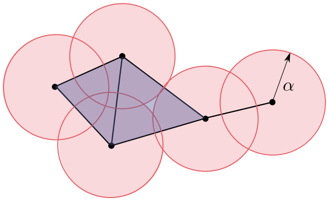
☺ Nerve theorem : $P^\alpha \simeq \mathcal{C}(P, \alpha)$

☹ Cannot be computed in $O(d)$

The Rips complex

Rips complex

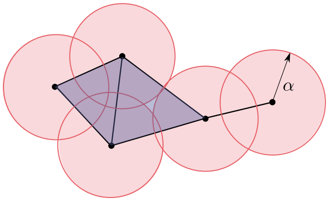
- Proximity graph $G(P, 2\alpha)$:
  $\rightarrow$ graph with edges whose length are smaller than 2α
- Simplices of $\mathcal{R}(P, \alpha)$:
  $\rightarrow$ cliques in the proximity graph $G(P, 2\alpha)$



The Rips complex

Rips complex

- Proximity graph $G(P, 2\alpha)$:
  $\rightarrow$ graph with edges whose length are smaller than 2α
- Simplices of $\mathcal{R}(P, \alpha)$:
  $\rightarrow$ cliques in the proximity graph $G(P, 2\alpha)$

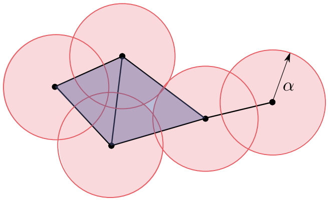


☹ $\mathcal{R}(P, \alpha)$ may not have the same topology as P^α

The Rips complex

Rips complex

- Proximity graph $G(P, 2\alpha)$:
  $\rightarrow$ graph with edges whose length are smaller than 2α
- Simplices of $\mathcal{R}(P, \alpha)$:
  $\rightarrow$ cliques in the proximity graph $G(P, 2\alpha)$



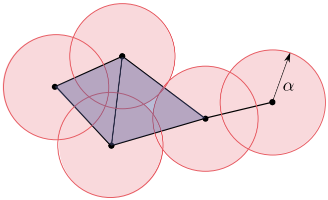
☹ $\mathcal{R}(P, \alpha)$ may not have the same topology as P^α

☺ Computation and storage in $O(n^2)$

The Rips complex

Rips complex

- Proximity graph $G(P, 2\alpha)$:
  $\rightarrow$ graph with edges whose length are smaller than 2α
- Simplices of $\mathcal{R}(P, \alpha)$:
  $\rightarrow$ cliques in the proximity graph $G(P, 2\alpha)$



☹ $\mathcal{R}(P, \alpha)$ may not have the same topology as P^{α}

☺ Computation and storage in $O(n^2)$

Flag complex

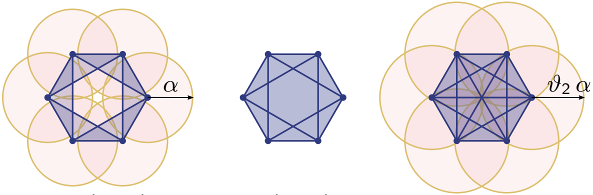
A complex whose simplices are cliques in its graph.

The Rips complex

Proximity with the Cech complex

Fundamental interleaving

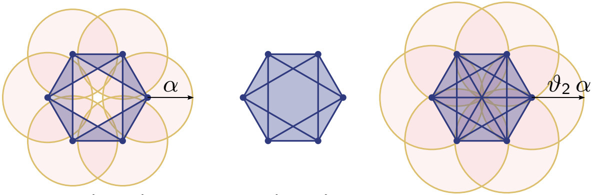
$\mathcal{C}(P,\alpha) \subset \mathcal{R}(P,\alpha) \subset \mathcal{C}(P,\theta_d\alpha)$ where $\theta_d = \sqrt{\frac{2d}{d+1}}$



$$\mathcal{C}(P,\alpha) \underset{\not\supset}{\subset} \mathcal{R}(P,\alpha) \underset{\not\supset}{\subset} \mathcal{C}(P,\theta_2\alpha)$$

Fundamental interleaving

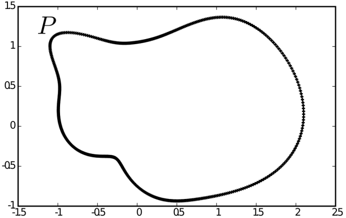
$\mathcal{C}(P,\alpha) \subset \mathcal{R}(P,\alpha) \subset \mathcal{C}(P,\theta_d\alpha)$ where $\theta_d = \sqrt{\frac{2d}{d+1}}$



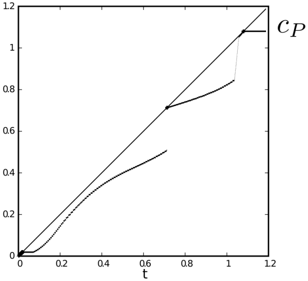
$\mathcal{C}(P,\alpha) \underset{\not\supset}{\subset} \mathcal{R}(P,\alpha) \underset{\not\supset}{\subset} \mathcal{C}(P,\theta_2\alpha)$

Question: Is it possible to find conditions on P such that $\mathcal{R}(P,\alpha) \simeq \mathcal{C}(P,\alpha)$?

Convexity defect function

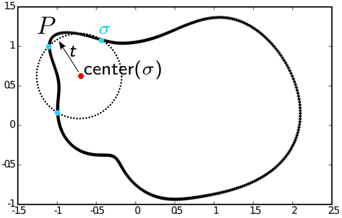


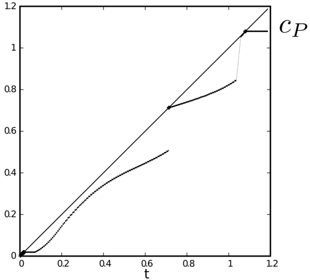
$$\mathrm{Centers}(P, t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \mathrm{Rad}\ \sigma \leq t}} \{\mathrm{Center}(\sigma)\}.$$



$$c_P(t) = d_H(\mathrm{Centers}(P, t), P)$$

Convexity defect function

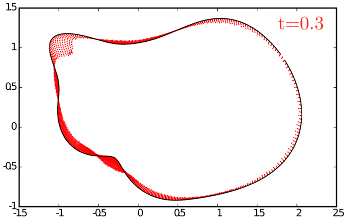




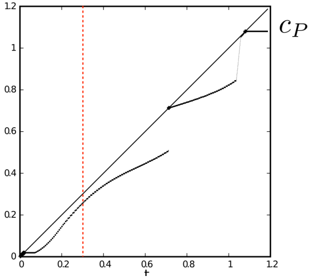
$$\mathrm{Centers}(P,t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \mathrm{Rad}\,\sigma \leq t}} \{\mathrm{Center}(\sigma)\}.$$

$$c_P(t) = d_H(\mathrm{Centers}(P,t), P)$$

Convexity defect function

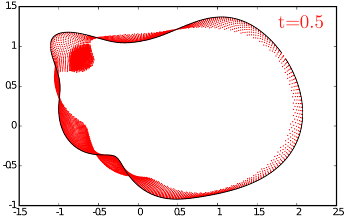


$$\text{Centers}(P, t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \text{Rad } \sigma \leq t}} \{\text{Center}(\sigma)\}.$$

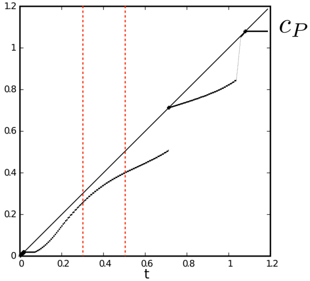


$$c_P(t) = d_H(\text{Centers}(P, t), P)$$

Convexity defect function

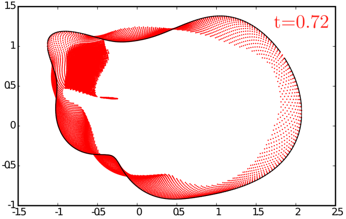


$$\mathrm{Centers}(P, t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \mathrm{Rad}\ \sigma \leq t}} \{\mathrm{Center}(\sigma)\}.$$

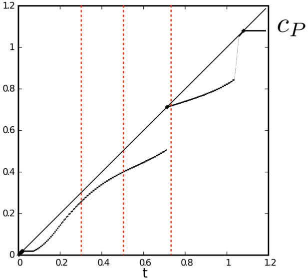


$$c_P(t) = d_H(\mathrm{Centers}(P, t), P)$$

Convexity defect function

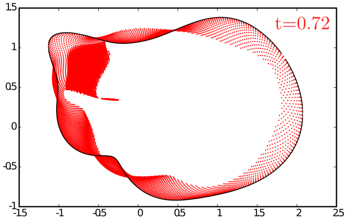


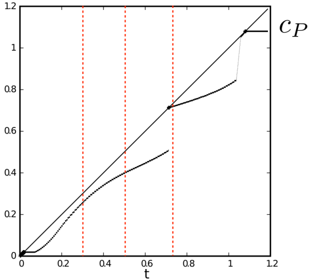
$$\text{Centers}(P, t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \text{Rad } \sigma \leq t}} \{\text{Center}(\sigma)\}.$$



$$c_P(t) = d_H(\text{Centers}(P, t), P)$$

Convexity defect function





$$\mathrm{Centers}(P, t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \mathrm{Rad}\ \sigma \leq t}} \{\mathrm{Center}(\sigma)\}.$$

$$c_P(t) = d_H(\mathrm{Centers}(P, t), P)$$

Proposition

$c_P(t) = t \Leftrightarrow$ the topology of P^α changes at t

Homotopy type of the Rips complex

Theorem [Attali Lieutier Salinas (SoCG 2011)]

If $c_P(\theta_d \alpha) < (2 - \theta_d)\alpha$ then $\mathcal{R}(P, \alpha) \simeq \mathcal{C}(P, \alpha)$.
The condition on c_P is optimal (at least in low dimension).

Homotopy type of the Rips complex

Theorem [Attali Lieutier Salinas (SoCG 2011)]

If $c_P(\theta_d \alpha) < (2 - \theta_d)\alpha$ then $\mathcal{R}(P, \alpha) \simeq \mathcal{C}(P, \alpha)$.
The condition on c_P is optimal (at least in low dimension).

Theorem [Attali Lieutier Salinas (SoCG 2011)]

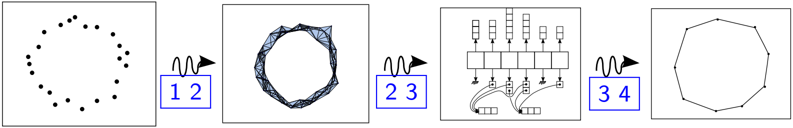
Assume that P samples well enough M i.e. :

$$\begin{cases} d_H(P, M) < \lambda \operatorname{reach}(M) \\ d_H(P, M) \le \rho\alpha \le \lambda \operatorname{Reach}(M) \end{cases}$$

then $\mathcal{R}(P, \alpha) \simeq M$.

- $\lambda \to \frac{2\sqrt{2-\sqrt{2}}-\sqrt{2}}{2+\sqrt{2}} \approx 0.0340$ when $d \to \infty$
- $\rho \to 0.13$ when $d \to \infty$

1. Find conditions such that an approximation has the same topology as the shape
2. Compute (efficiently) the approximation
3. Data-structure to store this approximation
4. Simplification in this data-structure

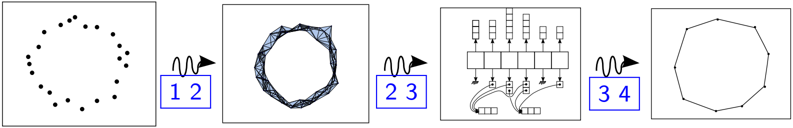


Point Cloud | Approximation | Data Structure | Simplified approximation

Reconstruction in high-dimensional spaces

Road map

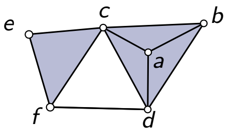
1. Find conditions such that an approximation has the same topology as the shape
2. Compute (efficiently) the approximation
3. Data-structure to store this approximation
4. Simplification in this data-structure



Point Cloud — Approximation — Data Structure — Simplified approximation

Data-structure for simplicial complex

- How can we store a simplicial complex?

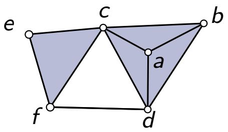


- Store all simplices?

☹ Many simplices in general → nice to avoid full representation for flag-complexes.

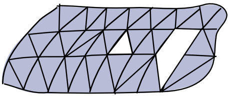
Data-structure for simplicial complex

- How can we store a simplicial complex?



- Store all simplices?

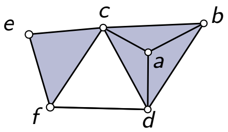
☹ Many simplices in general → nice to avoid full representation for flag-complexes.



Flag-complex nearly everywhere

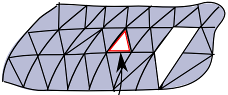
Data-structure for simplicial complex

- How can we store a simplicial complex?



- Store all simplices?

☹ Many simplices in general → nice to avoid full representation for flag-complexes.

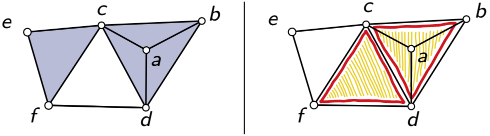


Flag-complex nearly everywhere
but here

Data-structure for simplicial complex

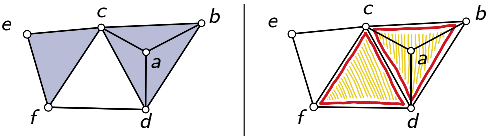
Graph and blockers

- Alternative representation :



- Graph(K) = $\{ab, ac, ad, bc, bd, dc, df, ec, fc, fe\}$
- Blockers(K)=$\{bcd, cdf\}$

- Alternative representation :



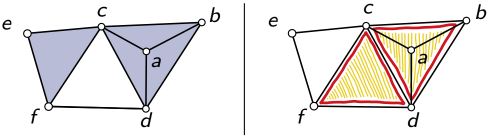
- Graph(K) = $\{ab, ac, ad, bc, bd, dc, df, ec, fc, fe\}$
- Blockers(K)=$\{bcd, cdf\}$

- Blockers(K)= $\{\sigma \subset P \mid \sigma \notin K \text{ and } \forall \tau \subsetneq \sigma, \tau \in K\}$
- The pair [Graph(K),Blockers(K)] is sufficient to encode entirely K!

Data-structure for simplicial complex

Graph and blockers

- Alternative representation :



- Graph(K) = $\{ab, ac, ad, bc, bd, dc, df, ec, fc, fe\}$
- Blockers(K)=$\{bcd, cdf\}$

- Blockers(K)= $\{\sigma \subset P \mid \sigma \notin K \text{ and } \forall \tau \subsetneq \sigma, \tau \in K\}$

- The pair [Graph(K),Blockers(K)] is sufficient to encode entirely K!

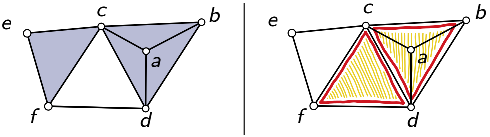
This data-structure :

☺ is compact if few blockers

Data-structure for simplicial complex

Graph and blockers

- Alternative representation :



- Graph(K) = $\{ab, ac, ad, bc, bd, dc, df, ec, fc, fe\}$
- Blockers(K)=$\{bcd, cdf\}$

- Blockers(K)= $\{\sigma \subset P \mid \sigma \notin K \text{ and } \forall \tau \subsetneq \sigma, \tau \in K\}$
- The pair [Graph(K),Blockers(K)] is sufficient to encode entirely K!

This data-structure :

☺ is compact if few blockers

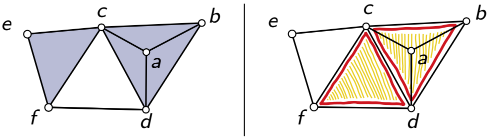
☺ handles efficiently many useful operations :

- contract an edge
- collapse a simplex

Data-structure for simplicial complex

Graph and blockers

- Alternative representation :



- Graph(K) = $\{ab, ac, ad, bc, bd, dc, df, ec, fc, fe\}$
- Blockers(K)=$\{bcd, cdf\}$

- Blockers(K)= $\{\sigma \subset P \mid \sigma \notin K \text{ and } \forall \tau \subsetneq \sigma, \tau \in K\}$
- The pair [Graph(K),Blockers(K)] is sufficient to encode entirely K!

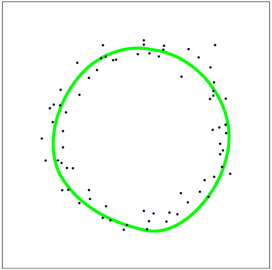
This data-structure :

☺ is compact if few blockers

☺ handles efficiently many useful operations :

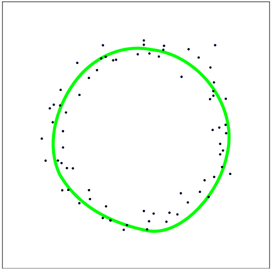
- contract an edge
- collapse a simplex

Edge contraction
Overview

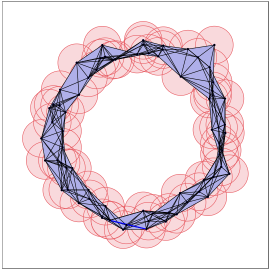


Point cloud $P \subset \mathbb{R}^d$ that approximates a manifold M

Edge contraction
Overview



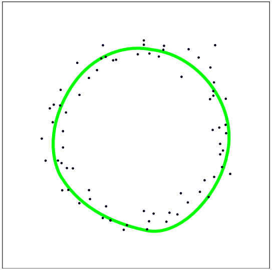
Point cloud $P \subset \mathbb{R}^d$ that approximates a manifold M



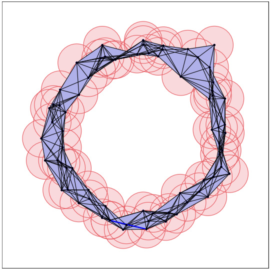
Under good sampling conditions $\mathcal{R}(P, \alpha) \simeq M$

Edge contraction

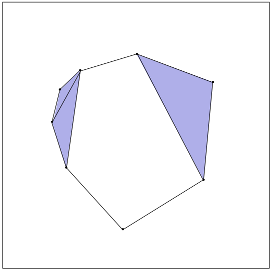
Overview



Point cloud $P \subset \mathbb{R}^d$ that approximates a manifold M



Under good sampling conditions $\mathcal{R}(P, \alpha) \simeq M$

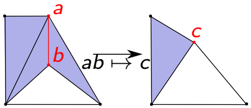


Is it possible to simplify $\mathcal{R}(P, \alpha)$ to a complex with few simplices?

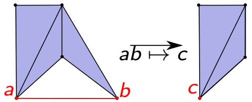
Topology-preserving edge contraction

A condition on the link

- Contracting an edge = identify two vertices in the complex
- May change the homotopy type



$ab \mapsto c$ preserves the homotopy type

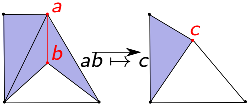


$ab \mapsto c$ changes the homotopy type

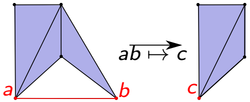
Topology-preserving edge contraction

A condition on the link

- Contracting an edge = identify two vertices in the complex
- May change the homotopy type



$ab \mapsto c$ preserves the homotopy type



$ab \mapsto c$ changes the homotopy type

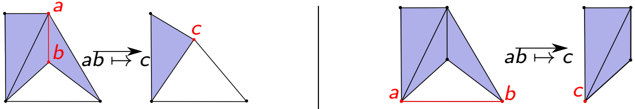
Theorem [Dey et al 99]

Let K be a simplicial complex homeomorphic to 2 or 3-manifold and ab an edge of K. If the link condition on ab is verified then the edge contraction ab preserves the homeomorphism.

Topology-preserving edge contraction

A condition on the link

- Contracting an edge = identify two vertices in the complex
- May change the homotopy type



$ab \mapsto c$ preserves the homotopy type | $ab \mapsto c$ changes the homotopy type

Theorem [Dey et al 99]

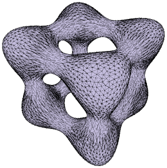
Let K be a simplicial complex homeomorphic to 2 or 3-manifold and ab an edge of K. If the link condition on ab is verified then the edge contraction ab preserves the homeomorphism.

Theorem [Attali Lieutier Salinas (SoCG 2011)]

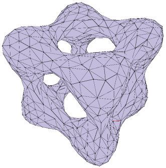
Let K be a simplicial complex and ab an edge of K.
If no blocker passes through ab then the edge contraction $ab \mapsto c$ preserves the homotopy type.

Topology-preserving edge contraction

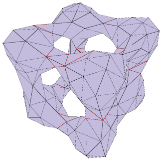
Experiment



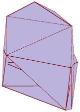
Rips complex



6000 contractions



6700 contractions

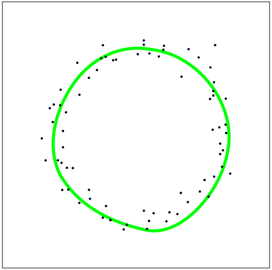


6787 contractions

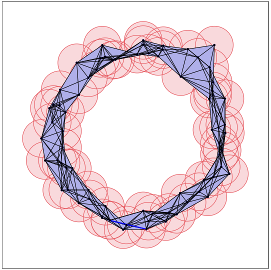
- Start with a Rips complex with 6806 vertices and 10^7 simplices and contract edges
- After contraction, the complex has only 19 vertices and 168 simplices
- Contractions takes only 10 seconds

Reconstruction with the Rips complex

Homeomorphism?



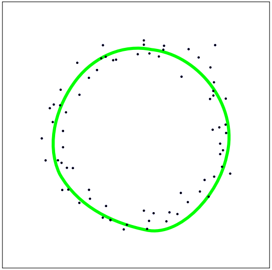
Point cloud $P \subset \mathbb{R}^d$ that approximates a manifold M



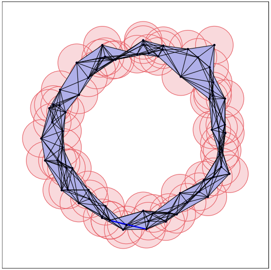
Under good sampling conditions $\mathcal{R}(P, \alpha) \simeq M$

Reconstruction with the Rips complex

Homeomorphism?



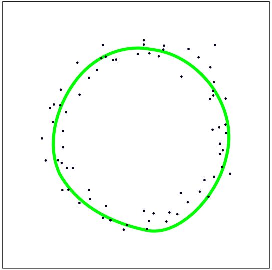
Point cloud $P \subset \mathbb{R}^d$ that approximates a manifold M



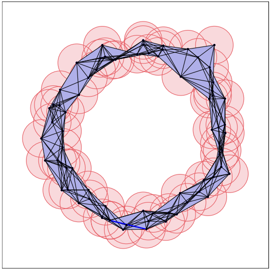
Under good sampling conditions $\mathcal{R}(P, \alpha) \simeq M$

Reconstruction with the Rips complex

Homeomorphism?



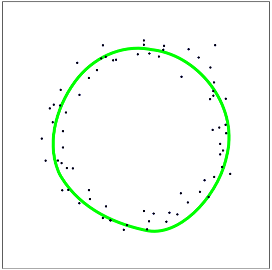
Point cloud $P \subset \mathbb{R}^d$ that approximates a manifold M



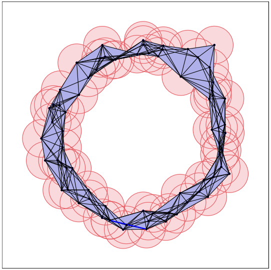
Under good sampling conditions $\mathcal{R}(P, \alpha) \simeq M$
But in general $\mathcal{R}(P, \alpha) \not\approx M$

Reconstruction with the Rips complex

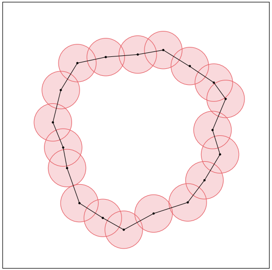
Homeomorphism?



Point cloud $P \subset \mathbb{R}^d$ that approximates a manifold M



Under good sampling conditions $\mathcal{R}(P, \alpha) \simeq M$
But in general $\mathcal{R}(P, \alpha) \not\approx M$



Is it possible to simplify $\mathcal{R}(P, \alpha)$ to a complex homeomorphic to the manifold?

Homeomorphic reconstruction

- Build a Rips complex such that $\mathcal{R}(P, \alpha) \simeq M$
- Keep removing the star of the largest edge whose link can be reduced to a point

Homeomorphic reconstruction

- Build a Rips complex such that $\mathcal{R}(P, \alpha) \simeq M$
- Keep removing the star of the largest edge whose link can be reduced to a point

Theorem

Let M be a 1-dimensional manifold and $P \subset M$ a finite point cloud.
If $d_H(P, M) < \alpha < \text{reach}(M)/2$ then this strategy returns a complex homeomorphic to M.

Homeomorphic reconstruction

- Build a Rips complex such that $\mathcal{R}(P, \alpha) \simeq M$
- Keep removing the star of the largest edge whose link can be reduced to a point

Theorem

Let M be a 1-dimensional manifold and $P \subset M$ a finite point cloud.
If $d_H(P, M) < \alpha < \text{reach}(M)/2$ then this strategy returns a complex homeomorphic to M.

☹ M is a 1-dimensional manifold

☹ $P \subset M$

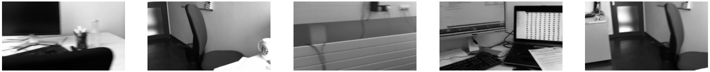
Collapse

An experiment

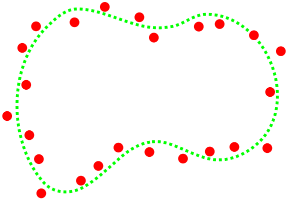
- A movie taken while turning with a rotating chair
- Data : 474 frames corresponding to points in $\mathbb{R}^{29056}$

Collapse
An experiment

- A movie taken while turning with a rotating chair
- Data : 474 frames corresponding to points in $\mathbb{R}^{29056}$



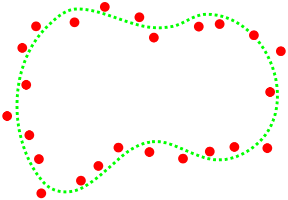
- A movie taken while turning with a rotating chair
- Data : 474 frames corresponding to points in $\mathbb{R}^{29056}$



- A sampling of a 1-manifold!

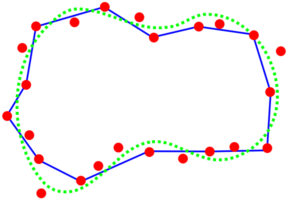
Collapse
An experiment

- A movie taken while turning with a rotating chair
- Data : 474 frames corresponding to points in $\mathbb{R}^{29056}$



- A sampling of a 1-manifold!
- Build a Rips complex (11 neighbors on average on its graph).

- A movie taken while turning with a rotating chair
- Data : 474 frames corresponding to points in $\mathbb{R}^{29056}$



- A sampling of a 1-manifold!
- Build a Rips complex (11 neighbors on average on its graph).
- After 2045 collapses, we get a complex homeomorphic to a 1-dimensional manifold

Conclusion

- The Rips complex has the same homotopy type as a sampled shape
- Complexes near flag-complexes can be stored efficiently
- Rips complexes can be reduced (drastically) with edge contractions
- Rips complexes can be simplified to a complex homeomorphic to the manifold (experimentally)

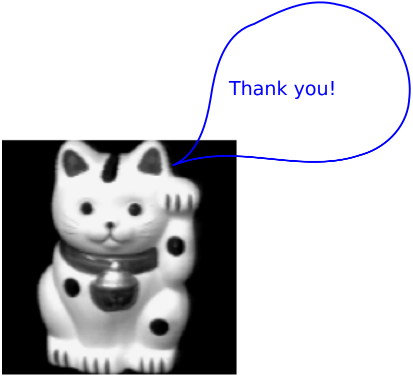
Perspectives and future works

Practical :

- implementation of the graph/blocker data-structure in a open-source project
- test this data-structure on others simplicial complexes

Theoretical :

- extend reconstruction results with weaker sampling conditions
- prove that the Rips complex can simplified efficiently to a complex homeomorphic to the manifold
- prove that edge contractions are *efficient*


Thank you!

Simplification operations that preserve the homotopy type

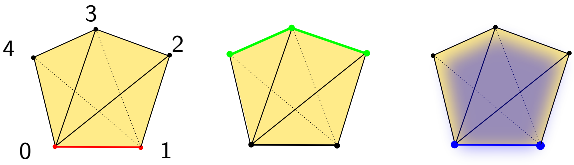
Ttwo simplification operations :

- the edge contraction of an edge σ
- the collapse of a simplex σ

These two operations preserve the homotopy type when a (local) condition is verified on the link of σ.

K : a simplicial complex
σ : a simplex of K

Link of σ : $\mathrm{Lk}_K(\sigma) = \{\tau \in K \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in K\}$
Star of σ : $\mathrm{St}_K(\sigma) = \{\tau \in K \mid \sigma \subset \tau\}$

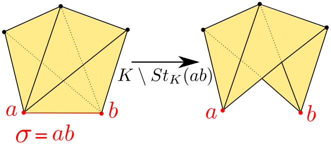


$\sigma = 01$ $\quad$ $\mathrm{Lk}_K(\sigma) = \{2, 3, 4, 23, 34\}$ $\quad$ $\mathrm{St}_K(\sigma) = \{01, 012, 0123, 01234, ...\}$

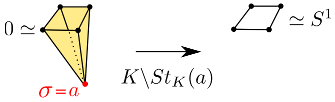
Collapse

Let K be a simplicial complex and σ a simplex of K.

- Removing the star of σ may change the homotopy type



Removing $\mathrm{St}_K(\sigma)$ preserves the homotopy type



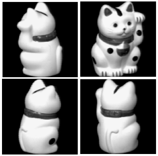
Removing $\mathrm{St}_K(\sigma)$ changes the homotopy type

- If the link of σ is a the closure of a simplex then removing the star of σ preserves the homotopy type
- In this case, we say that removing $\mathrm{St}_K(\sigma)$ from K is a **collapse**

Experimental results for collapses

Data-sets

- Data : a point cloud $P \in \{\texttt{Cat},\texttt{Ramses},\texttt{SO3}\}$ sampling a d-manifold M
  - `Cat`: 72 images of size 128x128
  - `Ramses`: A scan of a statue that consists in 200000 points in $\mathbb{R}^3$
  - `SO3`: 10000 points in $\mathbb{R}^9$ that samples rotational matrices



- Input of the simplification algorithm : $\mathcal{R}(P, \alpha)$ such that $\mathcal{R}(P, \alpha) \simeq M$

P	d	D	$\dim(\mathcal{R}(P, \alpha))$
`Cat`	1	16384	19
`Ramses`	2	3	14
`SO3`	3	9	16

- Output after simplification K_{out}

P	$\dim(K_{\text{out}})$	$K_{\text{out}} \approx M$	running time
`Cat`	1	YES	2 s
`Ramses`	2	YES	150 min
`SO3`	3	NO	7 min

Theorem

If $d_H(P, M) \leq \varepsilon$ then, $\forall t < \text{reach}(M) - \varepsilon$

$$c_P(t) \leq \text{reach}(M) - \sqrt{\text{reach}(M)^2 - (t + \varepsilon)^2} + 2\varepsilon$$