Simplification of simplicial complexes

Towards the intrinsic dimension

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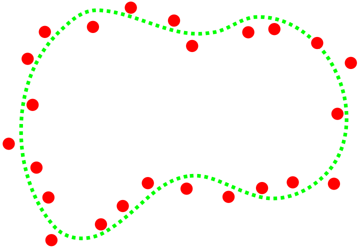
[2]Dassault-système, Aix en provence

Workshop on Computational Topology at SoCG 2012

Reconstruction problem

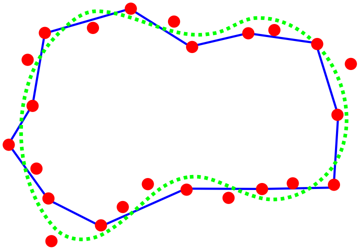
Data A finite point cloud P of a d-dimensional manifold M



Reconstruction problem

Data A finite point cloud P of a d-dimensional manifold M

Goal Find a simplicial complex K that approximates M (such that $K \simeq M$ or $K \approx M$)



Reconstruction problem

Data A finite point cloud $P \subset \mathbb{R}^D$ of a d-dimensional manifold M

- D : ambient dimension
- d : intrinsic dimension

Hypothesis $D >> d$

Goal Find a simplicial complex K that approximates M in $O(D)$

Previous work

- α-shape [Edelsbrunner et al.]
- Tangential Delaunay Complexes [Boissonat et al.]
- Witness Complexes [Silva et al.]

Simplification of simplicial complexes

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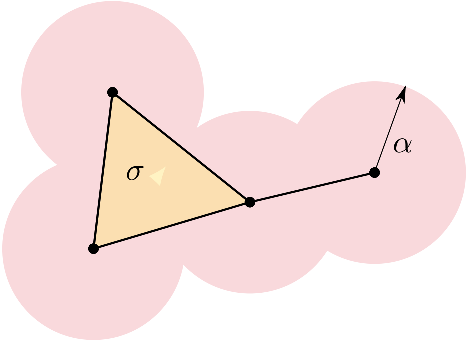
Starting point : the Rips complex

$$Rips(P, \alpha) = \{\sigma \mid \emptyset \neq \sigma \subset P, diam(\sigma) \leq 2\alpha\}$$



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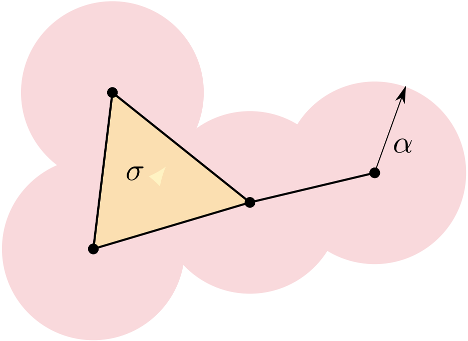
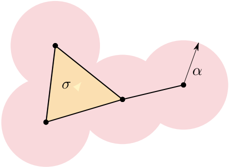


Property

The Rips complex is a flag complex

Rips complex

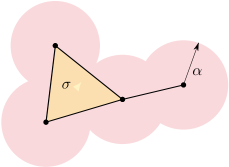
Rips complex



- ☺ Compact data structure in high dimension [SOCG'11][IJCGA 12]

Rips complex

Rips complex



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- ☺ Correct homotopy type if the point cloud is "dense" enough [SOCG'10][SOCG'11]

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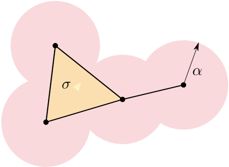
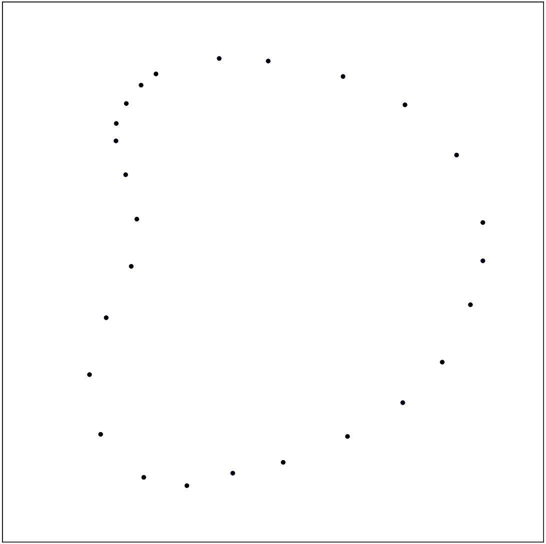


- ☺ Compact data structure in high dimension [SOCG'11][IJCGA 12]
- ☺ Correct homotopy type if the point cloud is "dense" enough [SOCG'10][SOCG'11]
- ☹ May have large simplicial dimension

Rips complex



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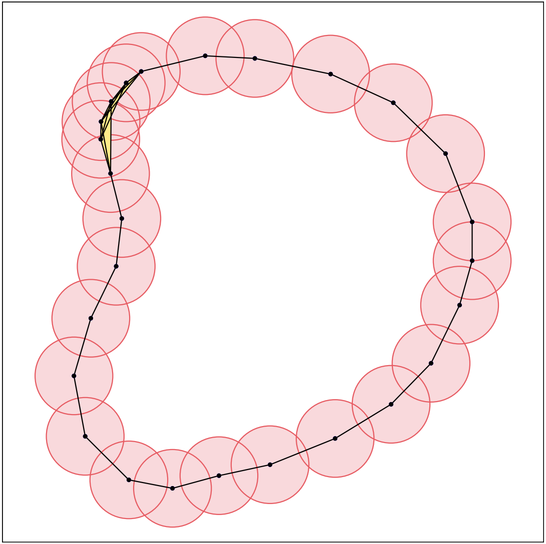

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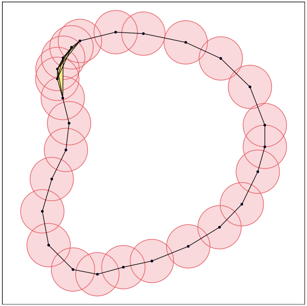
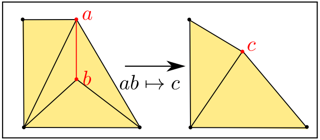


- Goal : find a sequence of simplification that will "crush the thickness"
- More formally, get a complex K such that :

$$K \approx M \text{ or } dim(K) = d$$

Simplification of simplicial complexes

- Edge contractions



[SOCG'11][IJCGA 12]

Simplification of simplicial complexes

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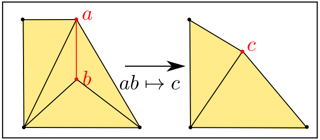
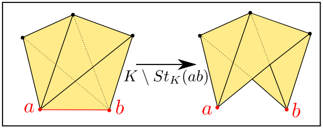


[SOCG'11][IJCGA 12]

- Extended collapses



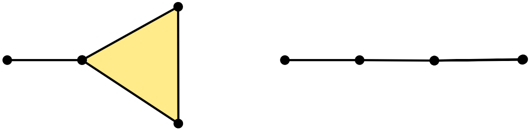
Extended Collapse - cone

Definition

K is a **cone** if there exists $o \in Vert(K)$ such that:

$$\forall \sigma \in K, o \cup \sigma \in K$$

In this case, K is a cone with apex o.



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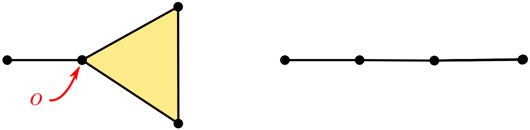

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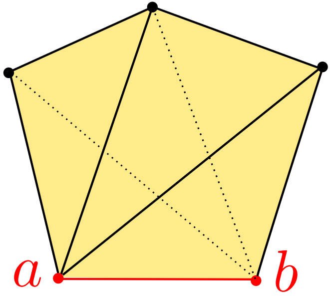

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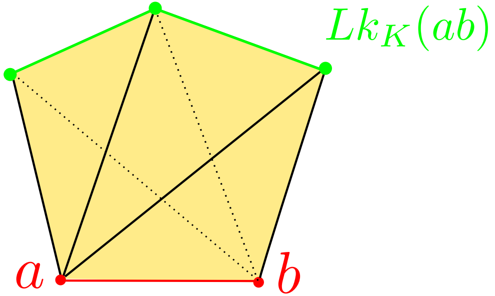

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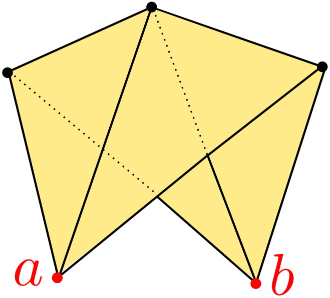

$K \setminus St_K(ab)$

Extended Collapse

Lemma

Let $\sigma \in K$

If $\mathrm{Lk}_K(\sigma)$ is a cone then

- There exists a sequence of collapses from K to $K \setminus \mathrm{St}_K(\sigma)$

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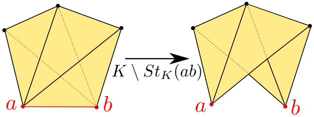
Definition

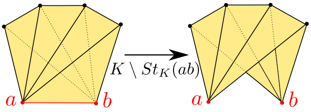
If $\mathrm{Lk}_K(\sigma)$ is a cone then the operation $K \to K \setminus \mathrm{St}_K(\sigma)$ is called an **extended collapse**

A less restrictive condition for the extended collapse operation

- The condition that allows one to make extended collapse is simple but quite restrictive



$K \to K \setminus \mathrm{St}_K(ab)$
is an extended collapse



$K \to K \setminus \mathrm{St}_K(ab)$
is not an extended collapse
but $K \simeq K \setminus \mathrm{St}_K(ab)$

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(new) Definition

If $\mathrm{Lk}_K(\sigma)$ is collapsible then the operation $K \rightarrow K \setminus \mathrm{St}_K(\sigma)$ is called an **extended collapse**

The collapsibility problem

Collapsibility problem :
Given a finite simplicial complex K, decide if there is a sequence of collapses from K to a point

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Collapsibility problem :
Given a finite simplicial complex K, decide if there is a sequence of collapses from K to a point

- Hard problem in general (for example, deciding whether a 3-dimensional complex collapses to a 1-complex is NP-complete [Malgouyres])
- Some geometric conditions known [Chillingworth et al] [Adiprasito et al]
- Here we are looking for a condition for collapsibility on Rips complexes

A result for 0-dimensional manifold

Theorem

Let $\emptyset \neq P \subset \mathbb{R}^D$. If $\mathrm{Hull}\, P \subset P^\varepsilon$ then

(i) $K = \mathcal{R}(P, \alpha)$ is collapsible for $\alpha \geq (2+\sqrt{3})\varepsilon$

A result for 0-dimensional manifold

Theorem

Let $\emptyset \neq P \subset \mathbb{R}^D$. If Hull $P \subset P^\varepsilon$ then

(i) $K = \mathcal{R}(P, \alpha)$ is collapsible for $\alpha \geq (2 + \sqrt{3})\varepsilon$

(ii) We can compute a sequence of extended collapse (vertices and edges) from K to a single point

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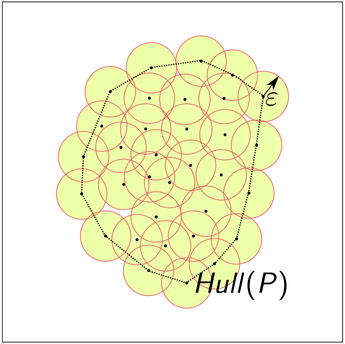
$Hull(P) \not\subset P^\varepsilon$



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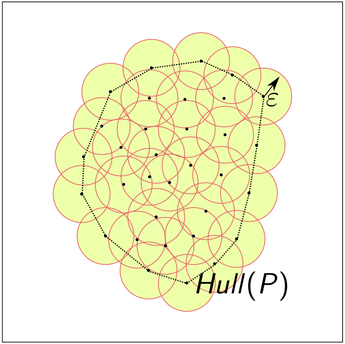

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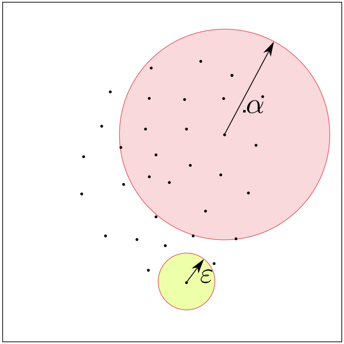

A simplification algorithm for noiseless sample

$\rightarrow$ Keep collapsing largest edge whose link is collapsible.

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H:=$\mathit{Edges}(\mathcal{R}(P,\alpha))$

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e:= extract the largest edge from H

If ($\mathrm{Lk}_K(e)$ is collapsible) **Then**

$K := K \setminus \mathrm{St}_K(e)$

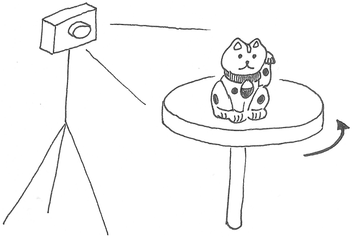
A result for 1-dimensional manifold and noiseless sample

Theorem

Let M be a 1-dimensional manifold and $P \subset M$ a finite point cloud.
If $d_H(P, M) < \alpha < Reach(M)$ then the previous algorithm returns a complex homeomorphic to M

An experiment on a 1-dimensional manifold

Lucky Cat



- A cat statue is placed on a motorized turntable

An experiment on a 1-dimensional manifold

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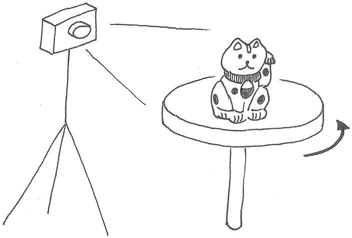

- A cat statue is placed on a motorized turntable
- Images of the statue are taken at pose interval of 5 degrees

An experiment on a 1-dimensional manifold

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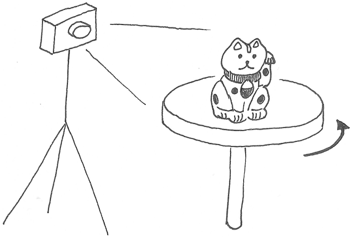

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- We get 72 images of size 128x128

(drawing from Dominique Attali)

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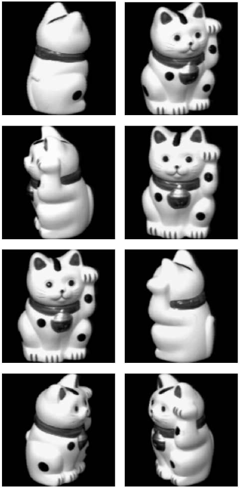

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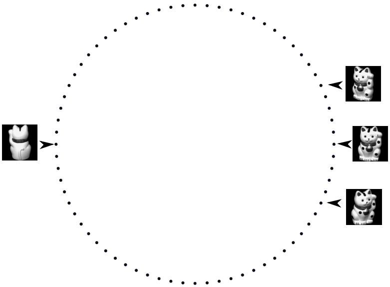
- Data : 72 images of the cat statue



An experiment on a 1-dimensional manifold

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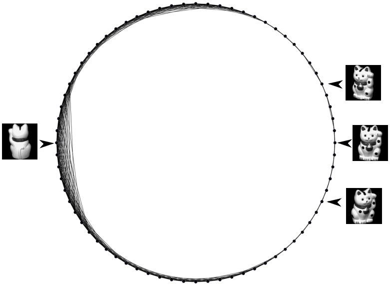
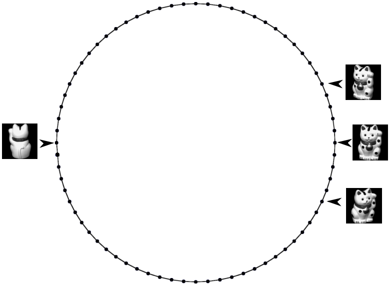
- Data : 72 images of the cat statue
- We build the rips complex of these points



An experiment on a 1-dimensional manifold

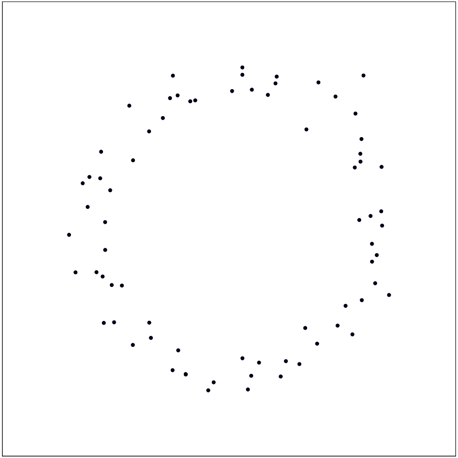
Lucky Cat

- Data : 72 images of the cat statue
- We build the rips complex of these points
- We get a complex of dimension 1 which is a discrete 1-dimensional manifold



Noisy sample

- Now let just suppose that $d_H(P, M) \leq \varepsilon$
- Previous algorithm does not work anymore



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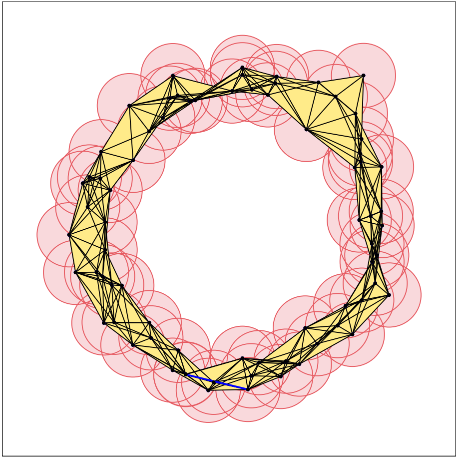

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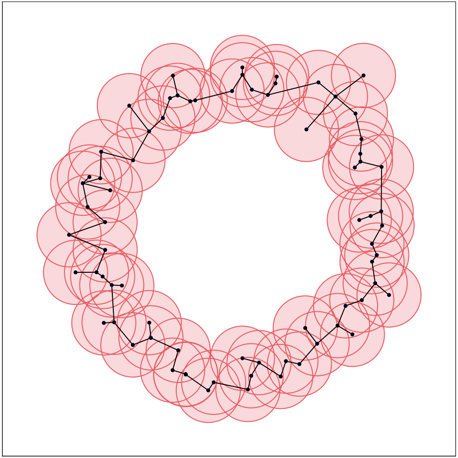
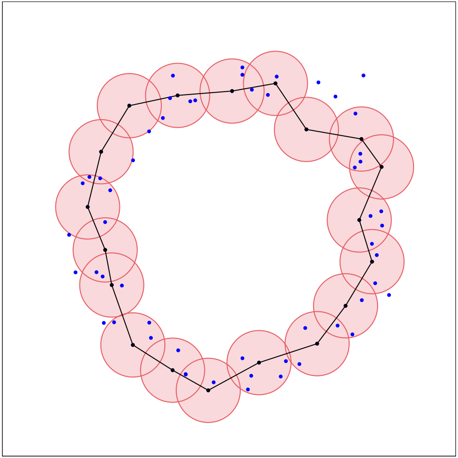


Noisy sample

- Now let just suppose that $d_H(P, M) \leq \varepsilon$
- Previous algorithm does not work anymore
- But it seems to work with vertex extended-collapses



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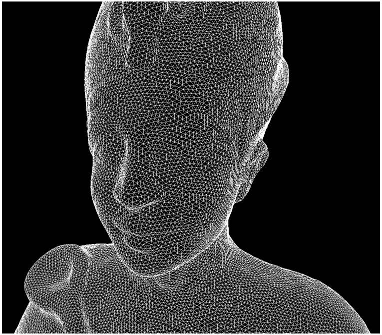
An experiment on a 2-dimensional manifold

A 2-sphere



- Data : 200000 points on a Ramesses statue

An experiment on a 2-dimensional manifold

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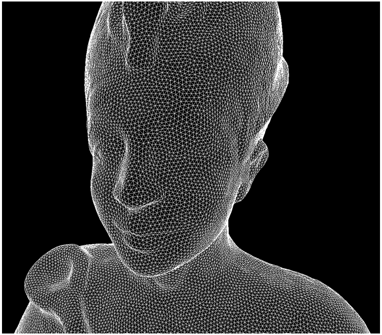

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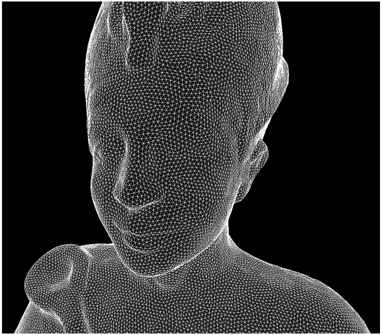

- Data : 200000 points on a Ramesses statue
- We build the Rips complex of these points (1804581 tetrahedrons)
- We get a 2-dimensional simplicial complex that is homeomorphic to the 2-sphere

An experiment on a 3-dimensional manifold

$SO(3)$

- We sample $SO(3)$ with 10000 points in $\mathbb{R}^9$

An experiment on a 3-dimensional manifold

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  10000 vertices
  195664 edges
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  3000682 tetrahedrons
  4642250 4-simplices
  ...

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  0 4-simplices

Summary

Collapses seem efficient for simplifying Rips complexes

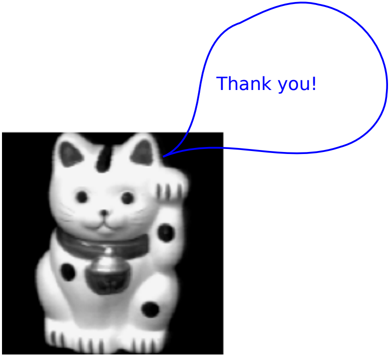
0-dimensional Possible to collapse rips complex of points close from a convex

1-dimensional We can simplify a Rips complex to a complex homeomorphic to the original manifold (without noise)

2-dimensional Good behavior in practice, we often get a complex homeomorphic to the original manifold

Future work:

- Prove that under density conditions, our algorithm always return a complex homeomorphic to the original manifold


Thank you!

A result for 0-dimensional manifold

Idea of the proof

Let $o \in P$ and $G(t) = Flag(Nerve(\{B(p, \alpha) \cap B(o, t), p \in P\}))$

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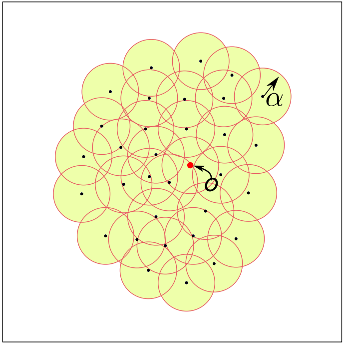

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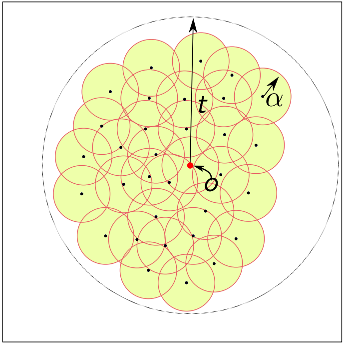
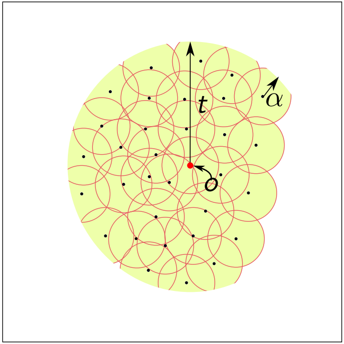


A result for 0-dimensional manifold

Idea of the proof

Let $o \in P$ and $G(t) = Flag(Nerve(\{B(p,\alpha) \cap B(o,t), p \in P\}))$



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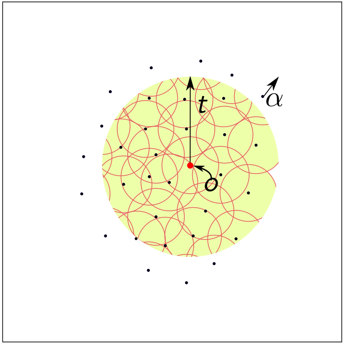

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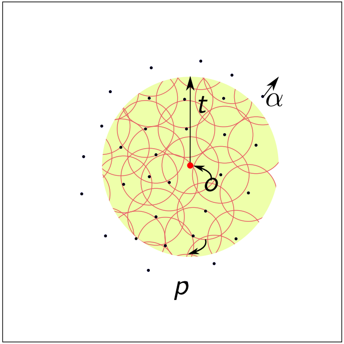

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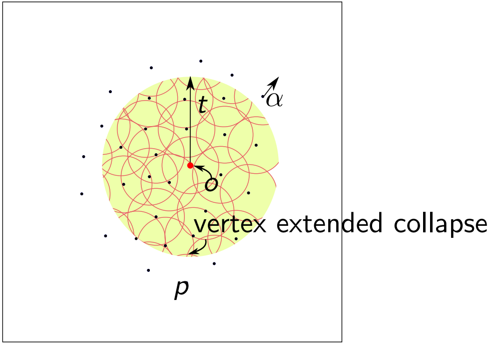

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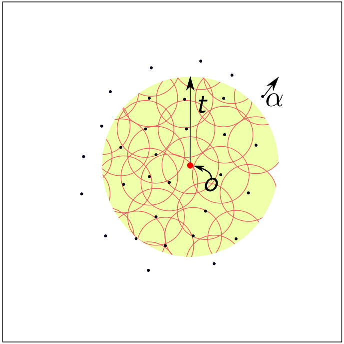

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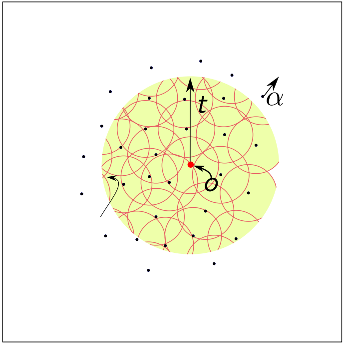

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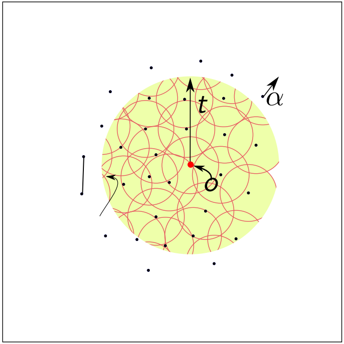

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