Collapsing Rips complexes*

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Abstract

Given a finite set of points that samples a shape in Euclidean space, the Rips complex of the points provides an approximation of the shape often used in manifold learning. Indeed, it suffices to compute the proximity graph of the points to encode the whole Rips complex as the latter is an example of flag completion. Recently, it has been proved that the Rips complex reflects the homotopy type of the shape when sufficiently densely sampled by the points. Unfortunately, the Rips complex is generally high-dimensional.

In this paper, we simplify the Rips complex by iteratively applying elementary operations that preserve both the homotopy type and the property of the complex to be a flag completion. We first prove that if the data points sample a convex set sufficiently densely, the Rips complex is collapsible. We provide two proofs of this result. One of them uses only vertex and edge collapses thus producing a sequence of flag complexes. Building upon this result, we then turn our attention to the simplification of Rips complexes that approximate manifolds. We propose to simplify them by iteratively removing vertices and edges (together with all their cofaces), assuming their link can be reduced to a point by a sequence of elementary operations. We propose and compare different heuristics for finding such a sequence. We then show how our heuristics perform on real datasets.

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1 Introduction

Manifold learning aims at recovering low-dimensional structures hidden in high-dimensional data [19, 17, 8]. An example of data might be a collection of m by m images of a rigid body taken under different poses. Each image can be thought of as a point in $\mathbb{R}^{m\times m}$ and the collection of images as a point cloud in $\mathbb{R}^{m\times m}$. Assuming the space of images is equipped with a reasonable metric (possibly the Euclidean metric), we expect the points to be distributed over a 6-dimensional manifold corresponding to the group of rigid displacements (e.g. rotations and translations). A manifold learning algorithm should be able, given as input the points, to output a topologically consistent representation of that manifold. Typically, the representation can encode a simplicial complex which, in the ideal case, is homeomorphic to the underlying manifold.

Given a finite set of points that samples a shape in Euclidean space, a classical approach for building an approximation of the shape consists in returning the Rips complex of the points [6, 13, 12]. Formally, the *Rips complex* of a set of points P at scale α is the simplicial complex whose simplices are subsets of points in P with diameter at most 2α. Recently, it has been proved that the Rips complex reflects the homotopy type of the shape, assuming the shape has a positive reach and is sufficiently densely sampled by the points [9, 1, 4]. In theory, other complexes could be used as well to achieve this reconstruction task such as the Čech complex, a closely related construction [7, 16, 11, 10, 4]. The *Čech complex* of P at scale α is the simplicial complex whose simplices fit in a ball of radius α. However, the Rips complex has the computational advantage over the Čech complex to be a flag completion: it suffices to compute its 1-skeleton to encode the whole complex. Unfortunately, the Rips complex is generally high-dimensional so that the true dimension of the shape remains elusive in the representation.

In order to turn the Rips complex into a complex homeomorphic to the underlying shape, we proposed in [3] to simplify the Rips complex by repeatedly applying edge contractions. We proved that if an edge satisfies the *link condition* introduced in [14] its contraction preserves the homotopy type. We also proposed a data structure for representing in a compact way simplicial complexes that do not differ too much from flag complexes. Motivated by the good behavior of the simplification process observed in practice, we have sought to better understand it. Besides edge contractions, another kind of homotopy-preserving elementary operations turns out to exhibit a comparable behavior and appears to be more accessible to theoretical investigations. This is the *face collapse*. As a first step towards a better understanding of what happens in practice, some of the authors have shown that, under some mild conditions, the Rips complex can be transformed into a complex homeomorphic to the underlying manifold by a sequence of collapses [2]. This theoretical result sheds light on the role of geometry and convexity. However the proof is not constructive and considers arbitrary collapses while, in practice, we would like to restrict ourselves to vertex and edge collapses, thereby preserving the property of the complex to be a flag completion.

In this paper, we first give conditions under which there exists a sequence of collapses reducing the Rips complex to a vertex, in the restricted situation of points that sample a convex set. We propose three different proofs: the first one transforms the Rips complex into a Čech complex first; the second one maintains a flag complex; the third one maintains a Rips complex but at the price of increasing first the size of the complex before reducing it. Unlike results in [2], all our proofs are constructive. Same as in [2], convexity plays a key role in the arguments. Using this theoretical insight, we then propose to simplify Rips complexes by repeatedly applying vertex and edge collapses. We introduce a *generalized collapse* as one that allows to remove a simplex and all its cofaces, assuming its link can be reduced to a vertex by a sequence of homotopy-preserving elementary operations (including possibly generalized collapses). We propose and compare several heuristics for finding such a sequence. It seems that reducing the link by edge contractions is the strategy which, in practice, has the highest rate of success for deciding whether a simplex can be collapsed.

2 Preliminaries

In this section we review the necessary background.

Simplicial complexes. An *abstract simplex* is any finite non-empty set. The *dimension* of a simplex σ is one less than its cardinality. A k*-simplex* designates a simplex of dimension k. If $\tau \subset \sigma$ is a non-empty subset, we call τ a *face* of σ and σ a *coface* of τ. If in addition $\tau \subsetneq \sigma$, we say that τ is a *proper face* and σ is a *proper coface*. Given a set of simplices Δ and a simplex $\sigma \in \Delta$, we say that σ is *inclusion-maximal* in Δ if it has no proper coface in Δ. Similarly, we say that σ is *inclusion-minimal* if it has no proper face in Δ. An *abstract simplicial complex* is a collection of simplices K, that contains, with every simplex, the faces of that simplex. The vertex set of the abstract simplicial complex K is the union of its elements, $\mathrm{Vert}(K) = \bigcup_{\sigma \in K} \sigma$. A *subcomplex* of K is a simplicial complex $L \subset K$. The *star* of σ in K, denoted $\mathrm{St}_K(\sigma)$, is the set of cofaces of σ. The *link* of σ in K, denoted $\mathrm{Lk}_K(\sigma)$, is the set of simplices τ in K such that $\tau \cup \sigma \in K$ and $\tau \cap \sigma = \emptyset$. Both the star and the link are subcomplexes of K. Another particular subcomplex is the i-skeleton consisting of all simplices of dimension i or less, which we denote by $K^{(i)}$.

Homotopy-preserving operations. Let $\pi : \mathrm{Vert}(K) \to \mathbb{R}^n$ be an injective map that sends the n vertices of K to n affinely independent points of $\mathbb{R}^n$, such as for instance the n vectors of the standard basis of $\mathbb{R}^n$. Let $\mathrm{Conv}(X)$ denote the convex hull of $X \subset \mathbb{R}^n$. The *underlying space* of K is the point set $|K| = \bigcup_{\sigma \in K} \mathrm{Conv}(\pi(\sigma))$ and is defined up to a homeomorphism. We shall say that an operation preserves the homotopy-type of K if the result is a simplicial complex K' whose underlying space is homotopy equivalent to that of K. For simplicity, we shall omit the phrase "the underlying space of" and use K instead of $|K|$. This will not be ambiguous. We will thus write $K \simeq K'$ to indicate that K and K' share the same homotopy-type. In this paper we are interested in simplifying a simplicial complex K by iteratively applying homotopy-preserving operations.

Collapses. An *elementary collapse* is the operation that removes a pair of simplices $(\sigma_{\min}, \sigma_{\max})$ assuming $\sigma_{\max}$ is the unique proper coface of $\sigma_{\min}$. The result is a simplicial complex $K \setminus \{\sigma_{\min}, \sigma_{\max}\}$ to which K deformation retracts. The reverse operation, which adds back the two simplices $\sigma_{\min}$ and $\sigma_{\max}$ to K is called an *elementary anti-collapse* and is also a homotopy-preserving operation. A simplicial complex is said to be *collapsible* if it can be reduced to a single vertex by a finite sequence of elementary collapses. For instance, the closure of a simplex σ, $\mathrm{Cl}(\sigma) = \bigcup_{\emptyset \neq \tau \subset \sigma} \{\tau\}$, is collapsible. $\mathrm{Cl}(\sigma)$ is an example of cone. A *cone* is a simplicial complex K which contains a vertex o such that the following implication holds: $\sigma \in K \implies \sigma \cup \{o\} \in K$. Cones are also collapsible. Another elementary operation that we shall use is the *edge contraction*. The edge contraction $ab \mapsto c$ is the operation that identifies the two vertices $a \in K$ and $b \in K$ to the vertex c. It preserves the homotopy-type whenever $\mathrm{Lk}_K(ab) = \mathrm{Lk}_K(a) \cap \mathrm{Lk}_K(b)$ [3].

We now list several possible generalizations of elementary collapses. We call the operation that removes $\mathrm{St}_K(\sigma_{\min})$ from K:

- a *(classical) collapse* if the star of $\sigma_{\min}$ has a unique inclusion-maximal element $\sigma_{\max} \neq \sigma_{\min}$ [15]; the reverse operation is called a *(classical) anti-collapse*.
- an *(extended) collapse* if the link of $\sigma_{\min}$ is a cone [3]; the reverse operation is called an *(extended) anti-collapse*;
- a *(generalized) collapse* if the link of $\sigma_{\min}$ can be reduced to a point by a sequence of collapses, anti-collapses and homotopy-preserving edge contractions; the reverse operation is called a *(generalized) anti-collapse*.

All three collapses (classical, extended, generalized) preserve the homotopy-type. The first two can be expressed as a composition of elementary collapses. They are thus convenient to apply several elementary collapses at once. Deciding whether the operation that removes $\mathrm{St}_K(\sigma_{\min})$ from K is a classical or extended collapse can be done efficient (*i.e. in polynomial time*) using the data structure described in [3]. Deciding whether the operation that removes $\mathrm{St}_K(\sigma_{\min})$ from K is a generalized collapse is computationally more involved. Even if we restrict ourselves to 3-dimensional simplicial complexes and use only elementary collapses in the reduction sequence, Martin Tancer established recently that the problem is NP-complete [18]. Here, we will focus on Rips complexes (defined below) which will allow us to design specific reduction sequences when the vertices sample a convex set.

Čech complexes and Rips complexes. We now review two standard ways of building a simplicial complex, given as input a finite set of n points P in $\mathbb{R}^d$ and a scale parameter $\alpha \geq 0$. The *Čech complex* is the abstract simplicial complex whose k-simplices correspond to subsets of $k+1$ points that can be enclosed in a ball of radius α. Equivalently, a k-simplex σ belongs to the Čech complex if and only if $\bigcap_{v \in \sigma} B(v, \alpha) \neq \emptyset$. In other words, the Čech complex is the nerve of the collection of balls centered at P with radius α. Recall that the nerve of the family $\mathcal{C} = \{C_p \mid p \in P\}$ is the simplicial complex $\mathrm{Nrv}\, \mathcal{C} = \{\sigma \subset P \mid \sigma \neq \emptyset \text{ and } \bigcap_{v \in \sigma} C_v \neq \emptyset\}$. We have:

$$\mathrm{Cech}(P, \alpha) = \mathrm{Nrv}\{B(p, \alpha) \mid p \in P\}.$$

The Nerve Lemma implies that $\mathrm{Cech}(P, \alpha)$ is homotopy equivalent to the α-offset of the points, $P^{\oplus \alpha} = \bigcup_{p \in P} B(p, \alpha)$.

The *Rips complex* is a variant of the Čech complex which is easier to compute. The Rips complex is an example of flag completion. Recall that the *flag complex* of a graph G, denoted $\mathrm{Flag}(G)$, is the maximal simplicial complex whose 1-skeleton is G. The Rips complex is the largest simplicial complex sharing with the Čech complex the same 1-skeleton. Formally, let $G(P, \alpha)$ denote the 1-skeleton of $\mathrm{Cech}(P, \alpha)$. Equivalently, $G(P, \alpha)$ is the graph whose vertices are the points P and whose edges connect all pairs of points within distance 2α. We have

$$\mathrm{Rips}(P, \alpha) = \mathrm{Flag}(G(p, \alpha)).$$

Čech complexes and the Rips complexes form two intertwined filtrations, related by the chain of inclusions:

$$\mathrm{Cech}(P, \alpha) \quad \subset \quad \mathrm{Rips}(P, \alpha) \quad \subset \quad \mathrm{Cech}(P, \vartheta_d\, \alpha) \qquad \text{where } \vartheta_d = \sqrt{\frac{2d}{d+1}}.$$

Hereafter, as we simplify the Rips complex, we will intensively use the fact that the link of any of its simplices is again a Rips complex. Precisely, writing $\mathcal{B}(\sigma, \alpha) = \bigcap_{z \in \sigma} B(z, \alpha)$ for the common intersection of balls with radius α centered at σ, the link of σ in $\mathrm{Rips}(P, \alpha)$ is $\mathrm{Rips}(P \cap \mathcal{B}(\sigma, \alpha) \setminus \sigma, \alpha)$. Another useful observation is the following. If we collapse a vertex or an edge in a flag complex, the result is still a flag complex.

3 Collapsing Rips complexes whose vertices approximate convex sets

Consider a finite set of points $P \subset \mathbb{R}^d$ and assume that $\mathrm{Conv}(P) \subset P^{\oplus \varepsilon}$ for some $\varepsilon \geq 0$. This condition is fulfilled as soon as P samples a convex set C in such a way that $P \subset C$ and $C \subset P^{\oplus \varepsilon}$. In this section, we present three different ways of reducing $\mathrm{Rips}(P, \alpha)$ to a point by a sequence of collapses and anti-collapses, requiring more or less restrictive conditions on ε. The proofs are all constructive and will help us design reduction sequences in the next section. For simplicity, we will assume throughout the section that the points in P is in general position. This condition is not too restrictive as we can get rid of it using arguments similar to those in [5].

3.1 Passing through the Čech complex

The first strategy relies on a result in [5] which formulates conditions under which there exists a sequence of elementary collapses turning $\mathrm{Rips}(P,\alpha)$ into $\mathrm{Cech}(P,\alpha)$. The sequence is obtained by considering the family of complexes $\{\mathrm{Cech}(P,t) \cap \mathrm{Rips}(P,\alpha)\}_{t\geq 0}$ and monitoring the changes that occur as t continuously decreases from $\vartheta_d \alpha$ to α. We show below (Theorem 1) that once $\mathrm{Cech}(P,\alpha)$ is obtained, it can further be reduced to a point by a sequence of elementary collapses. Theorem 1 can be seen as a variant of Theorem 3 in [2]. Letting $\mathrm{B}^\circ(p,r)$ designate the open ball centered at p with radius r and setting $P^{\odot r} = \bigcup_{p\in P} \mathrm{B}^\circ(p,r)$, we have:

Theorem 1. *Let $P \neq \emptyset$ be a finite set of points in $\mathbb{R}^d$ and $\alpha > 0$. If $\mathrm{Conv}(P) \subset P^{\odot\alpha}$, then $\mathrm{Cech}(P,\alpha)$ is collapsible.*

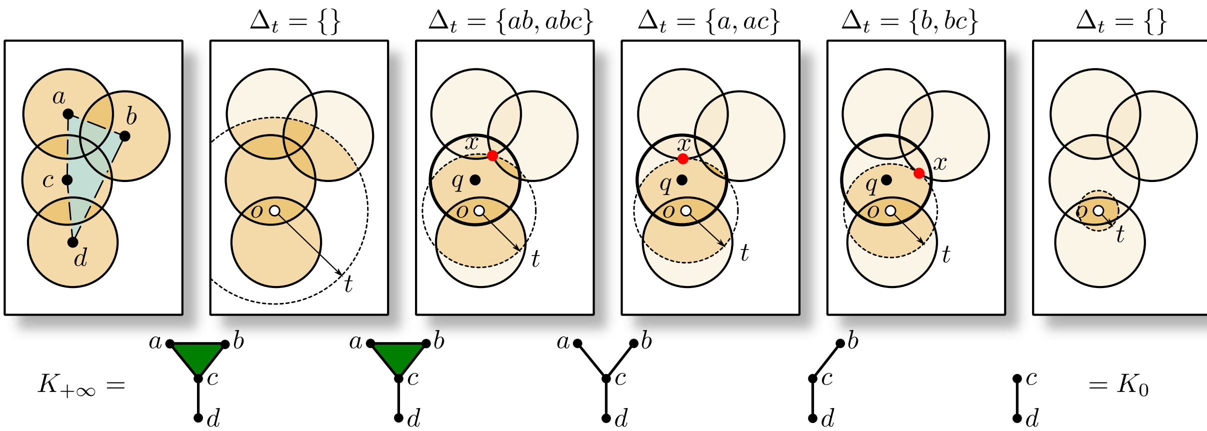

Figure 1: Notation for the proof of Theorem 1. Left: The four unit open disks contain the convex hull of the four centers a, b, c and d. From left to right: Our proof technique consists in sweeping space with a sphere centered at $o \in \mathrm{Conv}(\{a,b,c,d\})$ and whose radius t continuously decreases from $+\infty$ to 0. We deduce from the sweep a sequence of collapses reducing $\mathrm{Cech}(P,\alpha)$ to a vertex.

Proof. It suffices to establish the theorem for $\alpha = 1$. Let us fix an arbitrary point $o \in \mathrm{Conv}(P)$. Writing B_x for the unit closed ball centered at x and letting $t \geq 0$, we define the simplicial complex $K_t = \mathrm{Nrv}\{B(o,t) \cap B_p \mid p \in P\}$; see Figure 1. Notice that $K_{+\infty} = \mathrm{Cech}(P,1)$. We claim that K_0 is collapsible. Indeed, the vertex set of K_0 is the set of points $\tau_o = \{p \in P \mid o \in B_p\} = P \cap B_o$ which is non-empty since $o \in \mathrm{Conv}(P) \subset P^{\odot 1}$. It follows that $K_0 = \mathrm{Cl}(\tau_0)$ and is collapsible.

We prove that as t decreases continuously from $+\infty$ to 0, the only changes that occur in K_t are classical collapses. Specifically, let Δ_t be the set of simplices that disappear at time t, that is

$$\Delta_t = \{\sigma \subset P \mid \sigma \neq \emptyset \text{ and } d(o, \bigcap_{p\in\sigma} B_p) = t\}.$$

Suppose $\Delta_t \neq \emptyset$ and let us prove that the deletion of simplices Δ_t from K_t is a collapse for all $t \in (0,+\infty)$. Generically, we may assume that the set of simplices Δ_t has a unique inclusion-minimal element $\sigma_{\min}$. By construction, the intersection $B(o,t) \cap \bigcap_{p\in\sigma_{\min}} B_p$ is reduced to a single point x. It is easy to see that Δ_t has a unique inclusion-maximal element $\sigma_{\max} = \{p \in P \mid x \in B_p\}$. Hence, Δ_t consists of all cofaces of $\sigma_{\min}$ and these cofaces are faces of $\sigma_{\max}$. To prove that removing Δ_t from K_t is a collapse, it suffices to establish that $\sigma_{\min} \neq \sigma_{\max}$. By Lemma 7, x lies on the boundary of B_p for all $p \in \sigma_{\min}$. Notice that x lies in the convex hull of the points $\{o\} \cup \sigma_{\min}$ for otherwise, we could project x onto $\mathrm{Conv}(\{o\} \cup \sigma_{\min})$ and get a point y in the common intersection $B(o,t) \cap \bigcap_{p\in\sigma_{\min}} B_p$

but whose distance to o is smaller than that of x, $\|y - o\| < \|x - o\|$, reaching a contraction. Since $\mathrm{Conv}(P) \subset P^{\odot 1}$, there is a point $q \in P$ such that $\|q - x\| < 1$. Equivalently, x lies in the interior of B_q. Thus, q belongs $\sigma_{\max}$ but not to $\sigma_{\min}$. Hence, $\sigma_{\min} \neq \sigma_{\max}$ as desired. □

Theorem 2. *Let $P \neq \emptyset$ be a finite set of points in $\mathbb{R}^d$ and consider a real number $\alpha > 0$. If $\mathrm{Conv}(P) \subset P^{\odot(2-\vartheta_d)\alpha}$, then $\mathrm{Rips}(P, \alpha)$ is collapsible.*

The proof is in the appendix.

3.2 Maintaining a flag complex at all time

The reduction sequence we obtained in the previous section passes through the Čech complex and is quite involved to compute. By strengthening the condition on P, we shall see in this section how to simplify the Rips complex while preserving its property to be a flag completion at all time.

Theorem 3. *Let $P \neq \emptyset$ be a finite set of points in $\mathbb{R}^d$ and $\alpha > 0$. If $\mathrm{Conv}(P) \subset P^{\oplus(2-\sqrt{3})\alpha}$, then there exists a sequence of extended collapses reducing $\mathrm{Rips}(P, \alpha)$ to a vertex in such a way that the result of each extended collapse is a flag complex.*

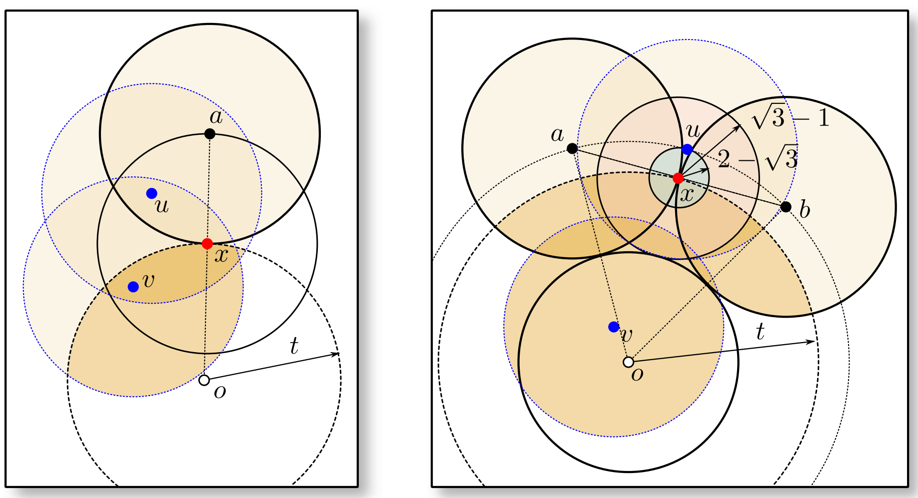

Figure 2: Notation for the proof of Theorem 3. Two kinds of events may occur: either a vertex collapse (on the left) or an edge collapse (on the right). The edge collapse is illustrated when triangle oab is equilateral.

Proof. As for the proof of Theorem 1, set $\alpha = 1$ and write B_x for the closed unit ball centered at x. Fix a point o in the convex hull of P. We construct a sequence of collapses by sweeping the space with a sphere centered at o and whose radius $t \geq 0$ continuously decreases from $+\infty$ to 0. Specifically, let G_t be the graph whose vertices are points $a \in P$ such that $B(o, t) \cap B_a \neq \emptyset$ and whose edges connect all pair of points $a, b \in P$ such that $B(o, t) \cap B_a \cap B_b \neq \emptyset$. Let $K_t = \mathrm{Flag}(G_t)$. Clearly, $K_{+\infty} = \mathrm{Rips}(P, 1)$ and K_0 is collapsible, using exactly the same argument as in the proof of Theorem 1. As we continuously decreases t from $+\infty$ to 0, changes in the simplicial complex K_t occur whenever a vertex or an edge disappears from the graph G_t. Generically, we may assume that these events do not happen simultaneously.

When a vertex a disappears from K_t at time t, the intersection $B(o, t) \cap B_a$ reduces to a single point x; see Figure 2, left. In this situation, we claim that the link of a in K_t is the closure of the

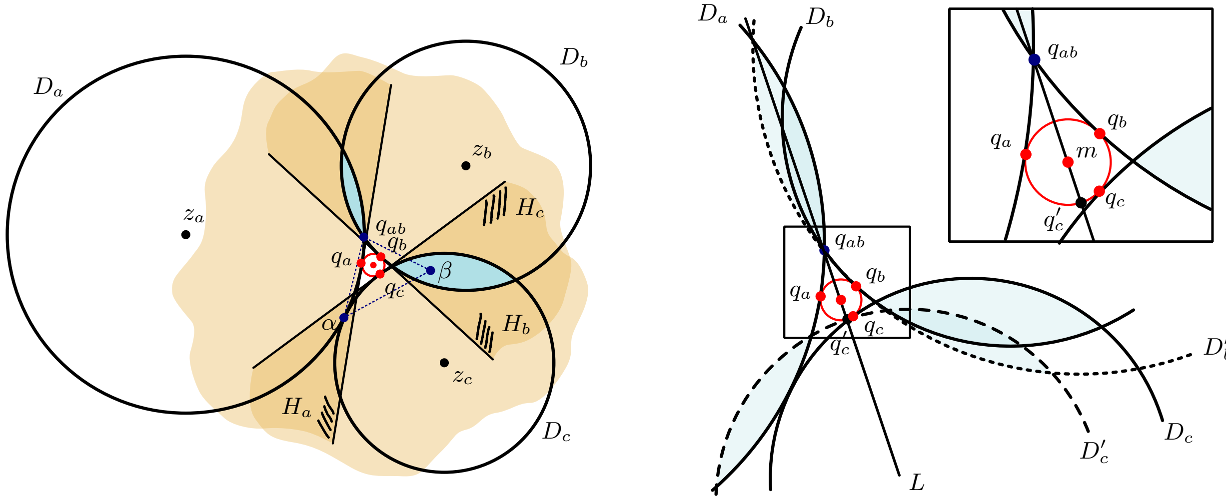

Figure 3: Notation for the proof of Lemma 4.

simplex $\tau_x = \{p \in P \setminus \{a\} \mid x \in B_p\} = P \cap B_x \setminus \{a\}$. First, note that τ_x is non-empty since x lies on the segment connecting o to a and therefore belongs to the convex hull of P which is contained in $P^{\oplus 2-\sqrt{3}} \subset P^{\odot 1}$. Furthermore, τ_x is precisely the vertex set of the link since an edge au belongs to K_t if and only if $B(o,t) \cap B_a \cap B_u \neq \emptyset$ with $u \in P \setminus \{a\}$ which can reformulated as $u \in P \cap B_x \setminus \{a\}$. Finally, any two vertices u and v in the link are connected by an edge since clearly $u, v \in \tau_x$ implies $B(o,t) \cap B_u \cap B_v \supset \{x\}$.

When an edge ab disappears from K_t at time t, there exists a point x such that $\{x\} = B(o,t) \cap B_a \cap B_b$; see Figure 2, right. Note that x lies in the convex hull of $\{a, b, o\}$ and therefore lies in the convex hull of P. Since $\mathrm{Conv}(P) \subset P^{\oplus 2-\sqrt{3}}$, there exists $u \in P$ such that $\|u - x\| \leq 2 - \sqrt{3}$. In particular, $x \in B_u$ and therefore u belongs to the link of ab in K_t. We claim that the link of ab is a cone with apex u. Consider a point $v \in P$ which belongs to the link of ab in K_t. Equivalently, both $B(o,t) \cap B_a \cap B_v \neq \emptyset$ and $B(o,t) \cap B_b \cap B_v \neq \emptyset$ and Lemma 5 implies that $B(o,t) \cap B(x, \sqrt{3}-1) \cap B_v \neq \emptyset$. Since $B(x, \sqrt{3}-1) \subset B_u$, we also have $B(o,t) \cap B_u \cap B_v \neq \emptyset$, showing that uv also belongs to the link of ab in K_t. We have just proved that the link of ab in K_t is a cone. Thus, removing the star of ab from K_t corresponds to a sequence of collapses. $\square$

3.3 Two geometric lemmas

The proof of Theorem 3 relies on two geometric lemmas. The first one states facts about three disks in the plane that intersect pairwise but have no common intersection (Lemma 4). It will allow us to deduce facts about the way four balls intersect in $\mathbb{R}^d$ (Lemma 5). As before, B_x denotes the unit closed ball centered at x.

Lemma 4. *Let D_a, D_b and D_c be three disks with radius equal to or less than one and such that any two disks have a non-empty intersection while the three together have no common intersection. Let q_{ab} be the point of $D_a \cap D_b$ closest to the center of D_c. There exists a point $q_c \in D_c$ such that:*

- *for all points $\alpha \in D_a \cap D_c$ and $\beta \in D_b \cap D_c$, the point q_c is in the convex hull of α, β and q_{ab};*
- $\|q_c - q_{ab}\| \leq \sqrt{3} - 1$.

Proof. Consider the disk D_m whose boundary is tangent to the boundaries of the three disks D_a, D_b and D_c and whose interior intersects none of the three disks D_a, D_b and D_c. For $x \in \{a, b, c\}$, the two disks D_x and D_m intersect in a single point q_x; see Figure 3, left. Let $\alpha \in D_a \cap D_c$ and $\beta \in D_b \cap D_c$. We claim that q_c belongs to the convex hull of α, β and q_{ab}. Indeed, for $x \in \{a, b, c\}$, let H_x be the half-plane that contains D_x and avoids the interior of D_m. We have $\alpha \in H_a \cap H_c$, $\beta \in H_b \cap H_c$, and $q_{ab} \in H_a \cap H_b$. The triangle $\alpha\beta q_{ab}$ covers the closure of $\mathbb{R}^2 \setminus (H_a \cup H_b \cup H_c)$ and therefore q_c.

Let us prove that $\|q_c - q_{ab}\| \leq \sqrt{3} - 1$. For $x \in \{a, b, c\}$, we denote the center of D_x by z_x and its radius by r_x. We are going to transform the three disks D_a, D_b and D_c in such a way that after the transformation:

(i) the three disks intersect pairwise but have no common intersection;

(ii) the distance between q_c and q_{ab} is at least as large as it was before the transformation;

(iii) $r_x \leq 1$ for $x \in \{a, b, c\}$;

(iv) the centers z_a, z_b, and z_c form an equilateral triangle of side length two.

Let q'_c be the point on the boundary of D_m that is farthest away from q_{ab}; see Figure 3, right. Clearly, $\|q'_c - q_{ab}\| \geq \|q_c - q_{ab}\|$. The two tangency points q_a and q_b decompose the boundary of D_m in two arcs and it is not difficult to see that one of them contains both q_c and q'_c. Consider the disk D'_c obtained by rotating D_c around m until it meets q'_c. As we do so, the rotated disk maintains a contact with at least one of the two disks D_a or D_b. Without loss of generality, we may assume that $D_a \cap D'_c \neq \emptyset$. Let L be the straight-line passing through q_{ab}, q'_c and m. Let D'_b be the symmetric of D_a with respect to L. We have $D'_b \cap D'_c \neq \emptyset$. The two boundaries of D_a and D'_b meet in two points, one of them being q_{ab}. If we replace D_b by D'_b and D_c by D'_c, it is easy to check that now the three disks D_a, D_b and D_c satisfies (i), (ii) and (iii) and their centers form an isosceles triangle. We can then further transform the three disks in such a way that after the transformation, they satisfy in addition (iv). When this is the case, we clearly have $\|q_c - q_{ab}\| = \sqrt{3} - 1$. $\square$

Lemma 5. *Let a and b be two points such that B_a and B_b have a non-empty intersection. Let o be a point such that $d(o, B_a \cap B_b) = t > 0$. Let x be the (unique) point of $B_a \cap B_b$ closest to o. Any unit ball which has a non-empty intersection with both $B_a \cap B(o,t)$ and $B_b \cap B(o,t)$ has a non-empty intersection with $B(x, \sqrt{3} - 1) \cap B(o,t)$.*

Proof. Consider c such that $B_c \cap B_a \cap B(o,t) \neq \emptyset$ and $B_c \cap B_b \cap B(o,t) \neq \emptyset$. If $x \in B_c$, then the claim holds trivially since $x \in B(x, \sqrt{3} - 1) \cap B(o,t)$. Let us assume from now on that $x \notin B_c$. Take $\alpha \in B_c \cap B_a \cap B(o,t)$ and $\beta \in B_c \cap B_b \cap B(o,t)$ and consider a 2-plane Π that contains the three points x, α and β. This 2-plane intersects the four balls B_a, B_b, B_c and $B(o,t)$ in four disks that we denote respectively D_a, D_b, D_c and D_o; see Figure 4, left. The three disks D_a, D_b and D_o have a non-empty intersection reduced to point x. We have $\alpha \in D_c \cap D_a \cap D_o \neq \emptyset$, $\beta \in D_c \cap D_b \cap D_o \neq \emptyset$ and $x \notin D_c$.

Let c' be the center of D_c. We claim that x is the point of $D_a \cap D_b$ closest to c'. Define the *outer cone* of $D_a \cap D_o$ at x as the set of points:

$$\mathcal{K}_{ao}(x) = \{y \in \Pi \mid \forall z \in D_a \cap D_o,\ \langle y - x, z - x \rangle \leq 0\}.$$

Equivalently, $K_{ao}(x)$ is the set of points whose distance from x is less than or equal to the distance from any other point of $D_a \cap D_o$. The fact that $x \notin D_c$ while $\alpha \in D_c$ implies that $\|c' - \alpha\| \leq \|c' - x\|$. Thus, α is a point in the intersection $D_a \cap D_o$ closer to c' than x. Equivalently, $c' \notin \mathcal{K}_{ao}(x)$. Similarly, $c' \notin \mathcal{K}_{bo}(x)$. Since $\mathcal{K}_{ao}(x) \cup \mathcal{K}_{bo}(x) \cup \mathcal{K}_{ab}(x) = \Pi$, it follows that $c' \in \mathcal{K}_{ab}(x)$. In other words, x is the point of $D_a \cap D_b$ closest to c' as claimed.

Therefore, Lemma 4 applies and shows the existence of a point $x' \in D_c$ in the convex hull of α, β and x such that $x' \in B(x, \sqrt{3} - 1)$. Since all three points α, β and x belong to $B(o,t)$, it follows that $x' \in B(o,t)$, yielding the result. $\square$

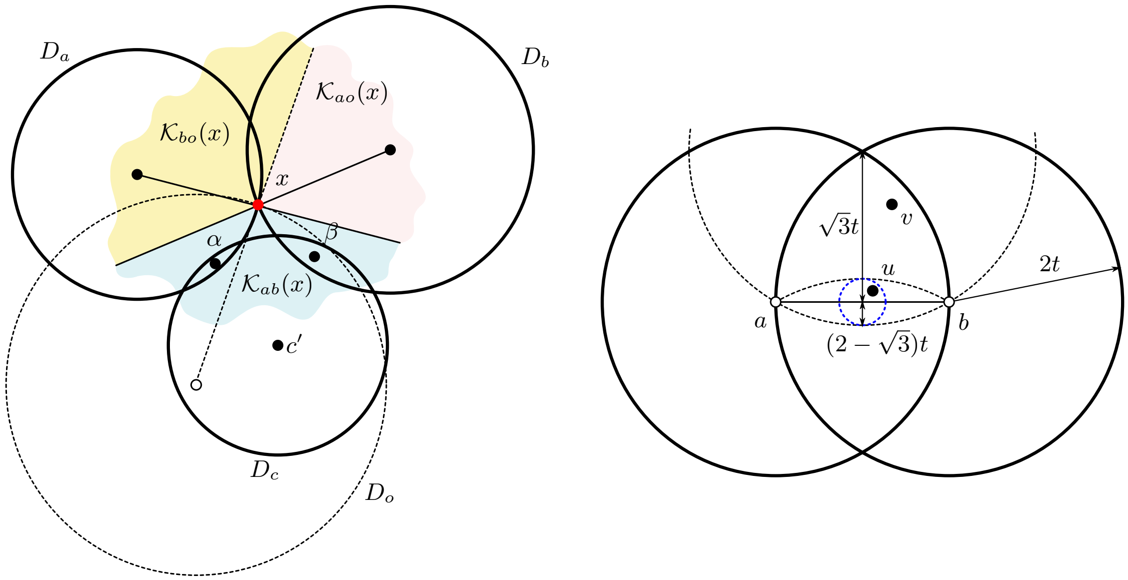

Figure 4: Notation for the proofs of Lemma 5 (left) and Theorem 6 (right).

3.4 Increasing first the scale parameter

Our third strategy for reducing $\mathrm{Rips}(P, \alpha)$ consists in first increasing the scale parameter α, adding edges to the 1-skeleton until we get the complete graph of P. When this happens, the result is $\mathrm{Cl}(P) = \mathrm{Rips}(P, +\infty)$ which is collapsible. The following theorem gives us conditions under which this strategy is guaranteed to succeed.

Theorem 6. *Let $P \neq \emptyset$ be a finite set of points in $\mathbb{R}^d$ and $\alpha > 0$ such that $\mathrm{Conv}(P) \subset P^{\oplus(2-\sqrt{3})\alpha}$. As t decreases continuously from $+\infty$ to α, the only changes that occur in $\mathrm{Rips}(P, t)$ are extended edge collapses.*

Proof. Recall that the Rips complex of P at scale t can be defined as the set of simplices with diameter at most $2t$, $\mathrm{Rips}(P, t) = \{\sigma \subset P \mid \sigma \neq \emptyset \text{ and } \mathrm{Diam}(\sigma) \leq 2t\}$. Using this definition, it is easy to see that the set of simplices that disappear at time t is

$$\Delta_t = \{\sigma \subset P \mid \sigma \neq \emptyset \text{ and } \mathrm{Diam}(\sigma) = 2t\}.$$

Suppose $\Delta_t \neq \emptyset$ and let us prove that the deletion of simplices Δ_t from $\mathrm{Rips}(P, t)$ is an extended collapse for all $t \in (\alpha, +\infty)$. Generically, all simplices in Δ_t are cofaces of an edge ab such that $\|a - b\| = 2t$. To prove the result, it suffices to show that the link of ab in $\mathrm{Rips}(P, t)$ is a cone. Setting $L = (B(a, 2t) \cap B(b, 2t)) \setminus \{a, b\}$, we have that the link of ab in $\mathrm{Rips}(P, t)$ is $\mathrm{Rips}(P \cap L, t)$. Let $c = \frac{a+b}{2}$. Since $c \in \mathrm{Conv}(P) \subset P^{\oplus(2-\sqrt{3})\alpha}$, we can find a point $u \in P$ such that $\|c - u\| \leq (2 - \sqrt{3})\alpha$; see Figure 4, right. Clearly, $u \in L$ and therefore u is a vertex of the link of ab. Furthermore, for every point $v \in P \cap L$, we have $\|v - u\| \leq \|v - c\| + \|c - u\| \leq \sqrt{3}t + (2 - \sqrt{3})\alpha \leq 2t$ and therefore uv is an edge of the link of ab. In particular, this implies that the link of ab is a cone with apex u. □

4 Simplifying Rips complexes whose vertices approximate a manifold

In the previous section, we gave theoretical guarantees that ensure the Rips complex can be reduced to a point by a sequence of collapses when its vertex set approximates a convex set. It results that if we sample a 0-manifold, we are able to find a sequence of collapses that transforms the Rips complex to a point. The goal of this section is to present and compare strategies for simplifying a Rips complex whose vertex set samples a higher-dimensional manifold. In the ideal case, we would like to get a complex homeomorphic to the manifold or at least whose dimension is as close as possible to that of the manifold. Throughout the section, we will assume that P is a point cloud that samples a d-dimensional manifold A embedded in D-dimensional Euclidean space and suppose that we can find a value of α such that $\mathrm{Rips}(P, \alpha) \simeq A$.

Overview of the simplification. The Rips complex is simplified in two stages: the first stage iteratively collapses vertices and the second stage iteratively collapses edges; see Algorithm 1 in the appendix for the pseudo-code. During the simplification, the complex remains a flag completion, since this property is not altered by collapsing vertices and edges. For $k \in \{1, 2\}$, stage k proceeds as follows. Initially, all $(k-1)$-dimensional simplices are stored in a priority queue Q. Each k-simplex receives as priority its diameter. Thus, all vertices receive the same priority and the largest edge receives the highest priority. During stage k, we iteratively take the k-simplex σ with highest priority and remove it from the current complex K together with all its cofaces whenever REDUCIBLE$(\mathrm{Lk}_K(\sigma))$ returns true; see Algorithm 2 for the pseudo-code. Ideally, we would like the function REDUCIBLE$(\mathrm{Lk}_K(\sigma))$ to be true if and only if the operation that removes σ and all its cofaces is a generalized collapse as described in Section 2. This means that ideally, we would like the function REDUCIBLE to take as input a simplicial complex L and returns true whenever there exists a sequence of homotopy-preserving elementary operations (collapses, anti-collapses and edge-contractions) that goes from L to a point and false otherwise. Unfortunately, the problem of deciding whether a complex L is reducible to a point by a sequence of elementary operations is NP-complete, even when we limit ourselves to elementary collapses and 3-dimensional complexes [18]. Instead, we will propose four more or less sophisticated heuristics to find such a sequence, drawing inspiration from the constructive proofs of the previous section and sometimes taking advantage of the fact that L is a flag complex.

Finding reduction sequences. We present four possible procedures that can be used in place of REDUCIBLE in Algorithm 2. The pseudo-code for each procedure can be found in the appendix except for the first one whose pseudo-code can be found in [3]. Each procedure either tests whether the complex has a simple form which makes it clearly collapsible (for instance, L is a cone) or tries to find a reduction sequence. For later reference, strategies are numbered from (S1) to (S4):

- (S1) ISCONE: takes as input a simplicial complex L and returns true if and only if L is a cone.
- (S2) REDUCIBLE_BY_SWEEP: takes as input a simplicial complex L and tries to apply a sequence of vertex and edge collapses in the order induced by sweeping space with a sphere centered at one of the vertex as described in the proof of Theorem 3. Returns true iff it manages to do so.
- (S3) REDUCIBLE_BY_COMPLETION: takes as input a flag complex L and tries to apply a sequence of edge anti-collapses in the order of increasing length until the 1-skeleton is the complete graph as describe in the proof of Theorem 6. Returns true iff it manages to do so.
- (S4) REDUCIBLE_BY_EDGE_CONTRACTIONS: takes as input a simplicial complex L and applies a sequence of edge contractions $ab \mapsto \frac{a+b}{2}$ assuming $\mathrm{Lk}_L(ab) = \mathrm{Lk}_L(a) \cap \mathrm{Lk}_L(b)$ in the order of increasing length as explained in [3]. Returns true iff the simplex L after simplification consists of a single vertex.

If we assume that P is initially a dense sampling of A, the vertices in the link of a simplex are likely to be close to a convex (at least at the beginning of the simplification). In this situation, Theorem 3 and Theorem 6 show that the complex can be reduced to a point when using the procedure REDUCIBLE_BY_SWEEP or the procedure REDUCIBLE_BY_COMPLETION. We now describe various computational experiments we performed and the results we obtained.

Datasets. The first dataset, referred to as `Lucky_cat`, is a collection of 72 images of a toy cat placed on a turntable and observed by a fixed camera. Each image has size $128^2 = 16384$ and can be thought of as a point-cloud that samples a curve in $\mathbb{R}^{16384}$. For this dataset, $d = 1$ and $D = 16384$.

The second dataset, referred to as `Noisy_sphere`, is obtained by sampling a 2-sphere with 384 points and adding noise. In this case, $d = 2$ and $D = 3$.

The third dataset, referred to as `Ramses`, is a 3D scan data consisting of 193252 points measured on the surface of a statue representing Ramses II. The surface of the statue is homeomorphic to $\mathbb{S}^2$. For this dataset, $d = 2$ and $D = 3$.

The fourth dataset is obtained by sampling the special orthogonal group using the method described in [20]. We get a point set `SO3` $\subset \mathbb{R}^9$ with size 10000. Recall that the special orthogonal group SO_3 is diffeomorphic to the real projective space $\mathbb{RP}^3$ which is a 3-dimensional manifold that can be embedded in $\mathbb{R}^9$ by representing each rotation in 3D by a 3×3 matrix. We have $d = 3$ and $D = 9$.

In the first three cases, we chose α such that $\mathrm{Rips}(P, \alpha) \simeq A$. In the last case, we just checked that $\mathrm{Rips}(P, \alpha)$ has the same homology groups as SO_3. Table 1 gives for each dataset the number of points, the number of simplices in $\mathrm{Rips}(P, \alpha)$, the size of its 1-skeleton, the intrinsic dimension (d), the ambient dimension (D), and dimension of $\mathrm{Rips}(P, \alpha)$. Figure 6 pictures the first 3 datasets.

Experimental results and discussion. To compare our four strategies, we iteratively simplify the complex $\mathrm{Rips}(P, \alpha)$ by applying a sequence of collapses. For each strategy (Sk), we maintain a counter s_k which initially is set to 0. After the ith collapse, we increment s_k by 1 if strategy (Sk) fails to find a reduction sequence while another strategy succeeds to find one. In Figure 5, we plot s_k for our four strategies when $P =$ `Noisy_sphere` and during the first stage, that is, when collapsing vertices. During the second stage, that is, when collapsing edges, all strategies give the same answer.

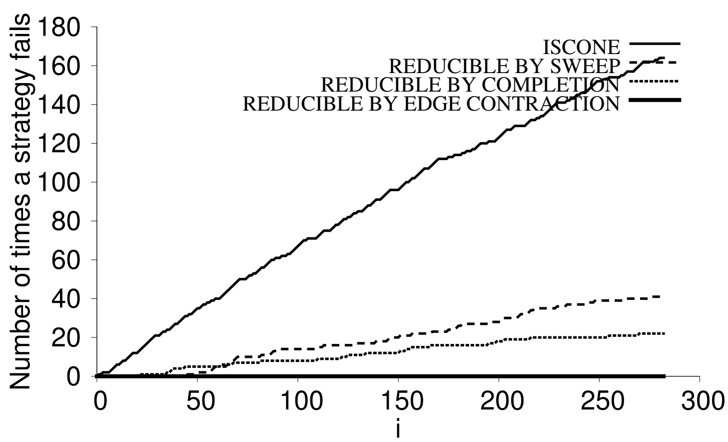

Figure 5: Number of times a strategy fails to find a reduction sequence when another succeeds to find one, during the first i vertex collapses of `Noisy_sphere`.

In Table 2, we describe the complex obtained after simplification using each of the 4 strategies on our 4 datasets. Not surprisingly as suggested by the previous experiment, the best results are obtained when using (S4), that is when calling function REDUCIBLE_BY_EDGE_CONTRACTIONS. Indeed, using (S4), the result of the simplification is a flag complex homeomorphic to the sampled manifold, except for the dataset `SO3`. Still, in that case, we get a complex with the correct dimension.

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A One missing proof and an additional lemma

Proof of Theorem 2. We proceed in two steps. First, we prove that there exists a sequence of elementary collapses which transform $\mathrm{Rips}(P, \alpha)$ into $\mathrm{Cech}(P, \alpha)$. Applying Theorem 1, we then deduce that $\mathrm{Cech}(P, \alpha)$ is collapsible and therefore so is $\mathrm{Rips}(P, \alpha)$.

For the first step, we apply Theorem 7 in [5] which gives conditions on P under which $\mathrm{Rips}(P, \alpha)$ can be reduced to $\mathrm{Cech}(P, \alpha)$ by a sequence of elementary collapses. Thus, all we need to do is check that P satisfies the hypothesis of Theorem 7 in [5]. Define the set of centers of P at scale t as the subset

$$\mathrm{Centers}(P, t) \quad = \quad \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \mathrm{Rad}(\sigma) \leq t}} \{\mathrm{Center}(\sigma)\},$$

where $\mathrm{Center}(X)$ and $\mathrm{Rad}(X)$ designate respectively the center and radius of the smallest ball enclosing $X \subset \mathbb{R}^d$. The hypothesis is that $d_H(\mathrm{Centers}(P, \vartheta_d \alpha) \,|\, P) < (2 - \vartheta_d)\alpha$ and is satisfied by P since for every $\emptyset \neq \sigma \subset P$, we have the sequence of inclusions $\mathrm{Center}(\sigma) \subset \mathrm{Conv}(\sigma) \subset \mathrm{Conv}(P) \subset P^{\odot(2-\vartheta_d)\alpha}$. □

Recall that B_p designates the closed unit ball centered at p.

Lemma 7. *Consider a point $o \in \mathbb{R}^d$ and a finite set $\sigma \subset \mathbb{R}^d$ such that $\bigcap_{p \in \sigma} B_p \neq \emptyset$. Suppose $d(o, \bigcap_{p \in \sigma} B_p) = t$ for some $t \in \mathbb{R}$ and let x be the unique point in $\bigcap_{p \in \sigma} B_p$ whose distance to o is t. Suppose $o \neq x$. If x lies in the interior of some B_u for $u \in \sigma$, then x is also the point in $\bigcap_{p \in \sigma \setminus \{u\}} B_p$ closest to o.*

Proof. Suppose that x lies in the interior of some B_u for $u \in \sigma$. Because $o \neq x$, we cannot have $\sigma = \{u\}$ and therefore $\sigma' = \sigma \setminus \{u\}$ is non-empty. Let us prove that x is the point of $\bigcap_{p \in \sigma'} B_p$ closest to o. Suppose for a contradiction that there exists a point x' in $\bigcap_{p \in \sigma'} B_p$ closer to o than x. Since the map $x \mapsto \|x - o\|$ is convex, the distance to o would be decreasing along the segment $[xx']$ in the vicinity of x and since this segment, in the vicinity of x, is contained in $\bigcap_{p \in \sigma} B_p$ this would contradict the fact that x is the closest point to o in $\bigcap_{p \in \sigma} B_p$. □

B Missing figures and tables

In this section, we provide additional tables and illustrations.

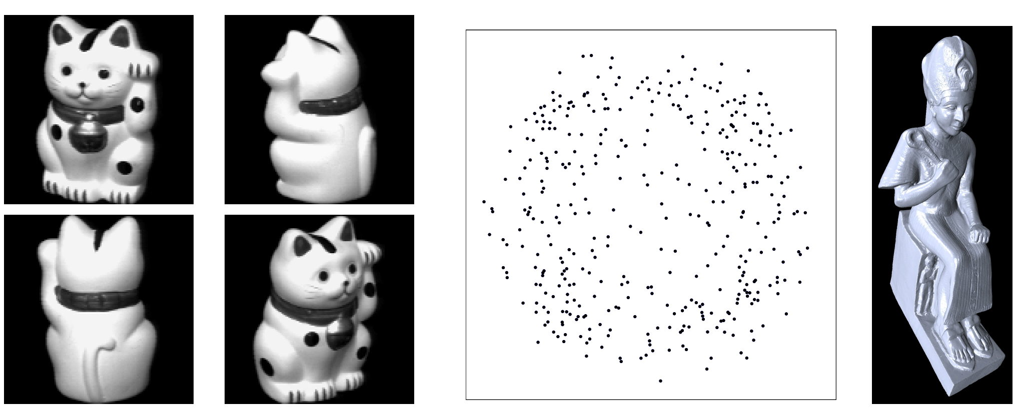

Figure 6: Illustration of datasets. Left : 4 points of `Lucky_cat` represented as images. Middle: The `Noisy_sphere` point cloud. Right: Statue sampled by the `Ramses` point cloud.

P	$\sharp P$	size of $\mathrm{Rips}(P,\alpha)$	size of $\mathrm{Rips}(P,\alpha)^{(1)}$	intrinsic dimension	ambient dimension	dimension of $\mathrm{Rips}(P,\alpha)$
Lucky_cat	72	$1,275,452$	645	1	$16,384$	19
Noisy_sphere	384	$1,331,676$	$5,192$	2	3	15
Ramses	$193,252$	$6,596,418$	$1,402,921$	2	3	14
SO3	$10,000$	$289,478,608$	$305,000$	3	9	16

Table 1: For each of the four datasets P considered in our experiments, we indicate the number of points in P, the number of simplices in $\mathrm{Rips}(P,\alpha)$, the size of the 1-skeleton, the intrinsic dimension d of the underlying manifold A, the ambient dimension D and the dimension of $\mathrm{Rips}(P,\alpha)$.

(S1)	dimension of output	homeomorphic	running time
Lucky_cat	1	YES	1 s
Noisy_sphere	3	NO	1 s
Ramses	3	NO	25 min
SO3	6	NO	94 s
(S2)	dimension of output	homeomorphic	running time
Lucky_cat	2	NO	3 min
Noisy_sphere	2	YES	3 s
Ramses	3	NO	180 min
SO3	4	NO	10 min
(S3)	dimension of output	homeomorphic	running time
Lucky_cat	1	YES	1 s
Noisy_sphere	2	YES	8 min
Ramses	X	X	X
SO3	X	X	X
(S4)	dimension of output	homeomorphic	running time
Lucky_cat	1	YES	2 s
Noisy_sphere	2	YES	1 s
Ramses	2	YES	150 min
SO3	3	NO	7 min

Table 2: Description of the simplicial complex output by our simplification algorithm when using each of the four strategies (S1), (S2), (S3) and (S4) for finding sequences of reductions. We marked with a cross computations that we interrupted because they were taking too much time.

C Algorithms

Algorithm 1 SIMPLIFY(Simplicial complex K) { Simplify the complex K }

SIMPLIFY(K,0,**true**) {Collapse the vertices of K}
SIMPLIFY(K,1,**false**) {Collapse the edges of K}

Algorithm 2 SIMPLIFY(Simplicial complex K, integer k, boolean reinsert)

```
Q = K^(k) \ K^(k-1)
while Q ≠ ∅ do
  Remove the simplex σ from Q with highest diameter
  if REDUCIBLE (Lk_K(σ)) then
    K = K \ St_K(σ)
    if reinsert then
      Insert in Q the k-dimensional simplices of Lk_K(σ) that are not already present in Q
    end if
  end if
end while
```

We chose not to reinsert edges (unlike vertices) whose link may have become reducible because we observed that, in practice, reinserting edges in the priority queue does not improve significantly the number of times that we get a complex either homeomorphic to or with the same dimension as the manifold A.

To describe algorithm REDUCIBLE_BY_SWEEP which reproduces the sequence of collapses used in the proof of theorem 3, we need to introduce a function INITIALIZE_SWEEP(L, o) that outputs a priority list of vertices and edges. To each vertex a of L different from o we associate a priority $\varphi(a) = ||o - a|| - \alpha$ and to each edge ab of L, we associate a priority $\varphi(ab) = d(o, B(a, \alpha) \cap B(b, \alpha))$. Using priority φ, INITIALIZE_SWEEP(L, o) then sorts the list of simplices containing all vertices different from o and all edges ab such that $\varphi(ab) < \min(\varphi(a), \varphi(b))$.

Algorithm 3 REDUCIBLE_BY_SWEEP(Simplicial complex L)

```
if L = ∅ then
  return false
end if
o = a vertex of L
Q.INITIALIZE_SWEEP(L, o) {See above for an explanation of what this function does.}
while |Q| ≠ ∅ do
  Remove simplex σ with highest priority from Q
  if not (Lk_L(σ) is a cone) then
    return false
  else
    L = L \ St_L(σ)
  end if
end while
return true
```

Algorithm 4 REDUCIBLE_BY_COMPLETION(Flag complex L)

```
if L = ∅ then
   return false
end if
if L is a cone then
   return true
end if
ab = Shortest edge not in L connecting two vertices of L
L = Flag(Sk^(1)(L) ∪ {ab})
if REDUCIBLE_BY_COMPLETION( Lk_L(ab)) then
   return REDUCIBLE_BY_COMPLETION(L)
else
   return false
end if
```

Algorithm 5 REDUCIBLE_BY_EDGE_CONTRACTIONS(Simplicial complex L)

```
if L = ∅ then
   return false
end if
Q = Edges(L)
while Q ≠ ∅ do
   ab = Shortest edge of Q
   Remove ab from Q
   if Lk_L(a) ∩ Lk_L(b) = Lk_L(ab) then
      contract in L the edge ab to the new vertex c = (a+b)/2
   end if
end while
return |Vert(L)| = 1
```