# Chapter 1

# Differentiation - Test 1 Content

## Differentiation Rules

**Formulae Bank.** This is it!!

$$\frac{d}{dx}ax^n = anx^{n-1} \qquad \text{(Power Rule)}$$

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x) \qquad \text{(Product Rule)}$$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \qquad \text{(Quotient Rule)}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x) \qquad \text{(Chain Rule)}$$

### Explanation of Concepts

**Prerequisites:** What is a derivative? The derivative can be cast in one of three ways... but they're all one and the same.

- The derivative of the function is the **rate of change** of a function at a point. (Example: The growth rate of mice at time = 2 years)
- The derivative of a function is the 'limiting' **slope of the function**. (First Principles) [To insert diagram]
- The derivative of a function is the **gradient/slope of the tangent** at a point. [To insert diagram]

The 2nd definition is the following 1st principles definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

#### Power Rule

To memorise this, we just take the power, and put it out in front, then reduce the power by one.

$$y = x^3 \implies y' = 3x^{3-1} = 3x^2$$

When a coefficient is in front, we keep it outside, so

$$y = ax^3 \implies y' = a(3x^{3-1}) = 3ax^2$$

Recalling that $\sqrt{x}$, $\sqrt[3]{x}$, $\frac{1}{x}$, $\frac{1}{x^2}$ correspond to $x^{\frac{1}{2}}, x^{\frac{1}{3}}, x^{-1}, x^{-2}$ we can apply power rule on anything.

### Test-Style Examples

**Test-Example 1.** Find the gradient of the curve $y = x^3 + 2\sqrt{x} - 3$ at $(4, 69)$.

**Breakdown of Question.** Here, when we ask for the gradient of the curve, we're asking what is the derivative of $y = x^3 + 2\sqrt{x} - 3$ at $x = 4$.

**Solution.** Find $\frac{dy}{dx}$, at $x = 4$. We now use the power rule,

$$\frac{dy}{dx} = 3x^2 + 2 \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

Substituting $x = 4$, we have $\frac{dy}{dx} = 3(4^2) + \frac{1}{2} = \frac{99}{2}$.
$\therefore$ The gradient of the curve is $\frac{99}{2}$ at $(4, 69)$.

**Test-Example 2.** Let $f(x) = 3x^3 + 2x + 5$. Determine the equation of the tangent to the curve at $(2, 33)$.

**Breakdown of Question.** Recall the derivative can be considered the **gradient/slope** of the tangent at a point. First, we need to find the gradient of the tangent at $x = 2$ and then we can solve for the linear equation of the line because it **passes through the point** $(2, 33)$.

**Solution.** Find $f'(x)$ at $x = 2$. By power rule,

$$f'(x) = 3(3x^{3-1}) + 2$$

Substituting, $x = 1$, $f'(1) = 3(3) + 2 = 11$. The equation of the tangent line must have a gradient of 11; i.e. $y = 11x + c$. Now, $33 = 11(2) + c \Longrightarrow c = 11$. Thus, $\boxed{y = 11x + 11}$ is the equation of tangent line.

## 1.1 Stationary Points & Rectillinear Motion

## 1.2 Optimisation & Marginal Rates of Change

**Formula Bank.**

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

We being by recalling the 1st principles, but replacing $h$ with $\delta x$[1]

$$\frac{dy}{dx} \approx \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \dots (\dagger)$$

Remember when we shrink $\delta x$ towards 0, we get better and better approximations for $\frac{dy}{dx}$.

Incremental change is just the flip side of this coin; instead of calculating the fraction

$$\frac{f(x + \delta x) - f(x)}{\delta x}$$

[1] Recall $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

In producing items, this comes in the form of marginal cost, marginal revenue and marginal profit. If we have a cost function, say $C(x) = x^3 + 3x$ and we have produced 20 items then we can approximate the cost of producing one more item with the derivative:

$$\frac{\delta C}{1} \approx \frac{dC}{dx}$$

**For Curious People**

This leaves a fundamental question unanswered. If I want to find $f(x + \delta x)$ why can't I just compute it straight up? The course leaves this unexplained, but in terms of practical applications, sometimes it is hard to compute the exact values of this function but we can approximate their values.

**Example.** *Find an approximation of $\sqrt{5}$ using the idea of incremental change on $f(x) = \sqrt{x}$.*

**Solution.** Without a calculator, this is a difficult task, but incremental change gives us a way. With power rule, we know

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

In our formula, we use $y = \sqrt{x}$. $\sqrt{5}$ will likely be close to $4 = \sqrt{2}$ so we start at $x = 4$ and have $\delta x = 1$. Thus, at $x = 4$,

$$\frac{\delta y}{1} \approx \frac{dy}{dx} = \frac{1}{2}(4)^{-\frac{1}{2}} = \frac{1}{4}$$

Therefore, our approximate answer is $\sqrt{5} = y + \delta y = 2 + \frac{1}{4}$.

### 1.2.1 Test-Style Questions

**How to Approach:** Here are da following steps.