1a. Solving: $\ddot{x} + {\omega_n}^2 x = 0$

Substituting $\ddot{x}$ for r^2 : $r^2 + {\omega_n}^2 = 0$

Solve for r: r = $\pm\, \omega_n i$

The solution takes the form: $x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$

Solve for C_1 and C_2 using x(0) = x_0 and $\dot{x}$ = v_0:

$$C_1 = x_0$$

$$C_2 = \frac{v_0}{\omega_n}$$

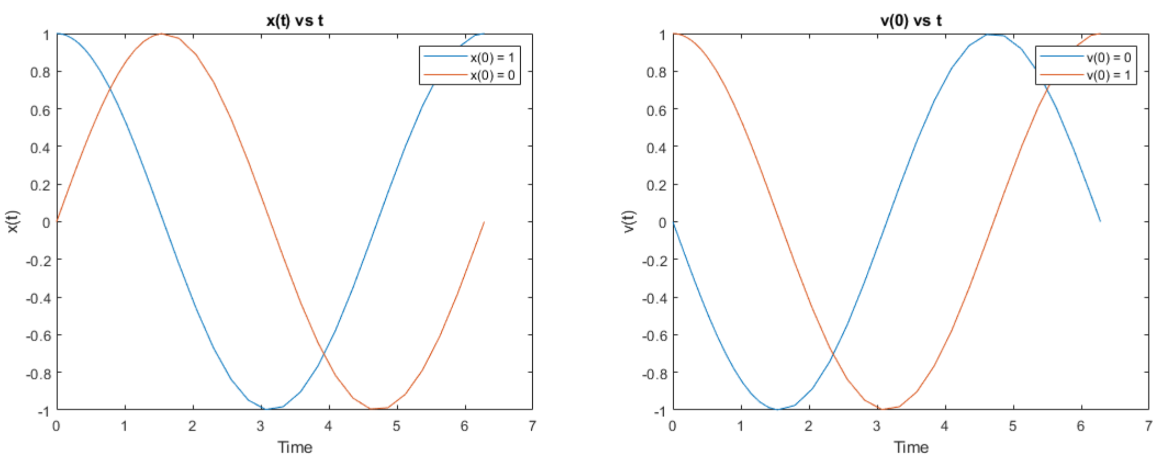
Substituting the constants back in and simplifying gives the solution:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} sin(\omega_n t)$$

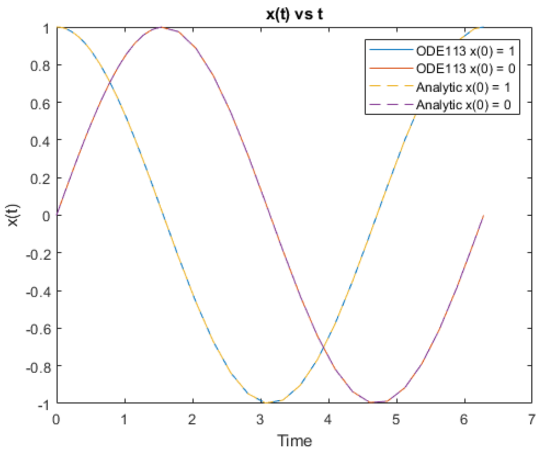
1b. The system is:

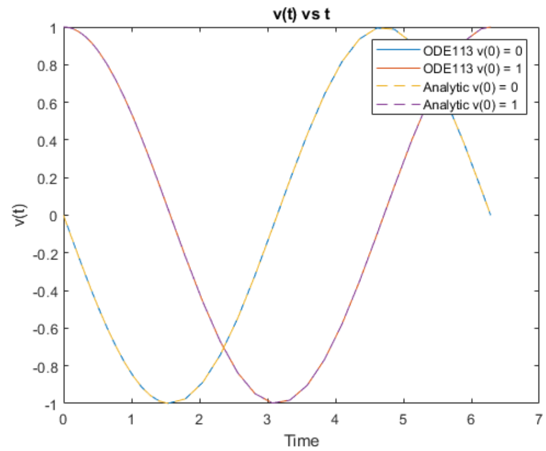
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -{\omega_n}^2 x_1$$

1c.



1d.





1 Code:

```
%% Defines all of the variables used in the differential equation
clc;clear;close all;
omegan = 1;
trange = [0 2*pi/omegan];
conditions = [1 0 0 1];
options = odeset('RelTol',1e-8);

%% Loops through both sets of inital conditions and gets a solution and plots
it
for i=1:2:length(conditions)
    po = [conditions(i) conditions(i+1)];
    [t p] = ode113(@harmonicOscillator,trange,po,options,omegan);
    x(:,i) = p(:,1);
    x(:,i+1) = p(:,2);
end

%% Creates Plots for part c
figure
plot(t,x(:,1),t,x(:,3))
xlabel('Time')
ylabel('x(t)')
title("x(t) vs t")
legend('x(0) = 1','x(0) = 0')
figure
plot(t,x(:,2),t,x(:,4))
xlabel('Time')
ylabel('v(t)')
title("v(0) vs t")
legend('v(0) = 0','v(0) = 1')

%% Defines analytic solution and its derivative
analytic = @(t,x0,omegan) x0+2*(sin(omegan*t).^2);
danalytic= @(t,x0,omegan) 4*omegan*x0.*cos(omegan*t).*sin(omegan*t);

%% Plots the analytic solution and the matlab solution
%the postion plot
```

```
figure
plot(t,x(:,1),t,x(:,3),t,analytic(t,conditions(1),omegan),t,analytic(t,condit
ions(3),omegan))
xlabel('Time')
ylabel('x(t)')
title("x(t) vs t")
legend('ODE113 x(0) = 1','ODE113 x(0) = 0','Analytic x(0) = 1','Analytic x(0)
= 0')
%the velocity plot
figure
plot(t,x(:,2),t,x(:,4),t,danalytic(t,conditions(1),omegan),t,danalytic(t,cond
itions(3),omegan))
xlabel('Time')
ylabel('v(t)')
title("v(t) vs t")
legend('ODE113 v(0) = 0','ODE113 v(0) = 1','Analytic v(0) = 0','Analytic v(0)
= 1')

%% Funtion to turn the second order ODE into first order system
function pdot = harmonicOscillator(t,p,omegan)
    x1=p(1);
    x2=p(2);
    pdot=zeros(size(p));
    x1dot=x2;
    x2dot=-omegan^2*x1;
    pdot(1) = x1dot;
    pdot(2) = x2dot;
end
```

2. Proving that:

$$\ddot{x} = \dot{x}\frac{d\dot{x}}{dx}$$

First define these derivatives:

$$\ddot{x} = \frac{d\dot{x}}{dt} \quad \text{and} \quad \dot{x} = \frac{dx}{dt}$$

We can substitute these into the original equation:

$$\frac{d\dot{x}}{dt} = \frac{dx}{dt} * \frac{d\dot{x}}{dx}$$

The dx on the right-side cancel simplifying to:

$$\frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dt}$$

Substituting into the original equation:

$$\dot{x}\frac{d\dot{x}}{dx} + {\omega_n}^2 x = 0$$

Subtracting and multiplying:

$$\dot{x}d\dot{x} = -{\omega_n}^2 x dx$$

Integrating both sides:

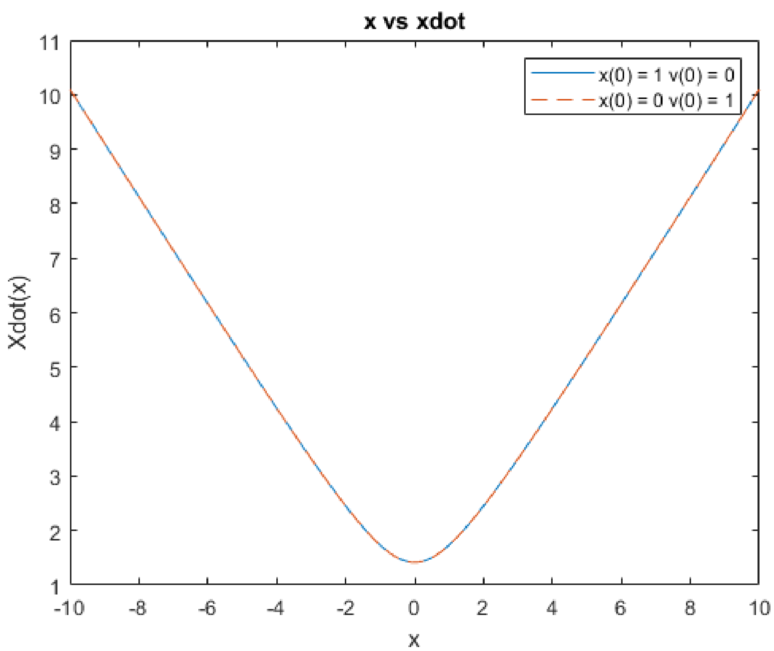
$$\frac{\dot{x}^2}{2} = -{\omega_n}^2\frac{x^2}{2} + C$$

Solving for C using initial conditions, x(0) = x_0 and $\dot{x}(0)$ = v_0

$$C = x_0\omega_n + v_0$$

Substituting back in for C and solving for $\dot{x}$ results in the equation:

$$\dot{x} = \sqrt{{\omega_n}^2 x^2 + 2v_0 + 2x_0\omega_n}$$




```
%% Defines variables and range for x values
clc;clear;close all;
omegan = 1;
conditions = [1 0 0 1];
x = -10:0.01:10;
%% defines the function
xdot = @(x,x0,v0,omegan) sqrt((omegan)^2*x.^2+2*v0+2*omegan*x0);
%% Computes the equation for each set of initial conditions and x
for i=1:2:length(conditions)
    x0 = conditions(i);
    v0 = conditions(i+1);
    v(:,i) = xdot(x,x0,v0,omegan);
end
%% plots the function for both sets of initial conditions over x
plot((x),(v(:,1)),x,v(:,3),'--')
xlabel('x')
ylabel('Xdot(x)')
title("x vs xdot")
legend('x(0) = 1 v(0) = 0','x(0) = 0 v(0) = 1')
```

3.

--------Newton's Method------		---------Fixed Point-----------
Initial Guess: 0.000000	Initial Guess: 3.141593	Initial Guess: 0.000000
0.000000 6.613186	3.141593 2.885608	0.000000 2.735510
6.613186 -1.668447	2.885608 2.884565	2.735510 2.967128
-1.668447 1.945357	2.884565 2.884565	2.967128 2.837290
1.945357 3.045255	2.884565 2.884565	2.837290 2.911198
3.045255 2.885278	2.884565 2.884565	2.911198 2.869411
2.885278 2.884565	2.884565 2.884565	2.869411 2.893142
2.884565 2.884565	2.884565 2.884565	2.893142 2.879696
2.884565 2.884565	2.884565 2.884565	2.879696 2.887325
2.884565 2.884565	2.884565 2.884565	2.887325 2.883000
2.884565 2.884565	2.884565 2.884565	2.883000 2.885453
Converges to 2.884565	Converges to 2.884565	Converges to 2.885453
Initial Guess: 1.570796	Initial Guess: 4.712389	Initial Guess: 1.570796
1.570796 3.321865	4.712389 2.149154	1.570796 3.321865
3.321865 2.883341	2.149154 2.964998	3.321865 2.630378
2.883341 2.884565	2.964998 2.884808	2.630378 3.022376
2.884565 2.884565	2.884808 2.884565	3.022376 2.805247
2.884565 2.884565	2.884565 2.884565	2.805247 2.929030
2.884565 2.884565	2.884565 2.884565	2.929030 2.859210
2.884565 2.884565	2.884565 2.884565	2.859210 2.898894
2.884565 2.884565	2.884565 2.884565	2.898894 2.876424
2.884565 2.884565	2.884565 2.884565	2.876424 2.889177
2.884565 2.884565	2.884565 2.884565	2.889177 2.881948
Converges to 2.884565	Converges to 2.884565	Converges to 2.881948

Initial Guess: 3.141593

3.141593	2.735510
2.735510	2.967128
2.967128	2.837290
2.837290	2.911198
2.911198	2.869411
2.869411	2.893142
2.893142	2.879696
2.879696	2.887325
2.887325	2.883000
2.883000	2.885453

Converges to 2.885453

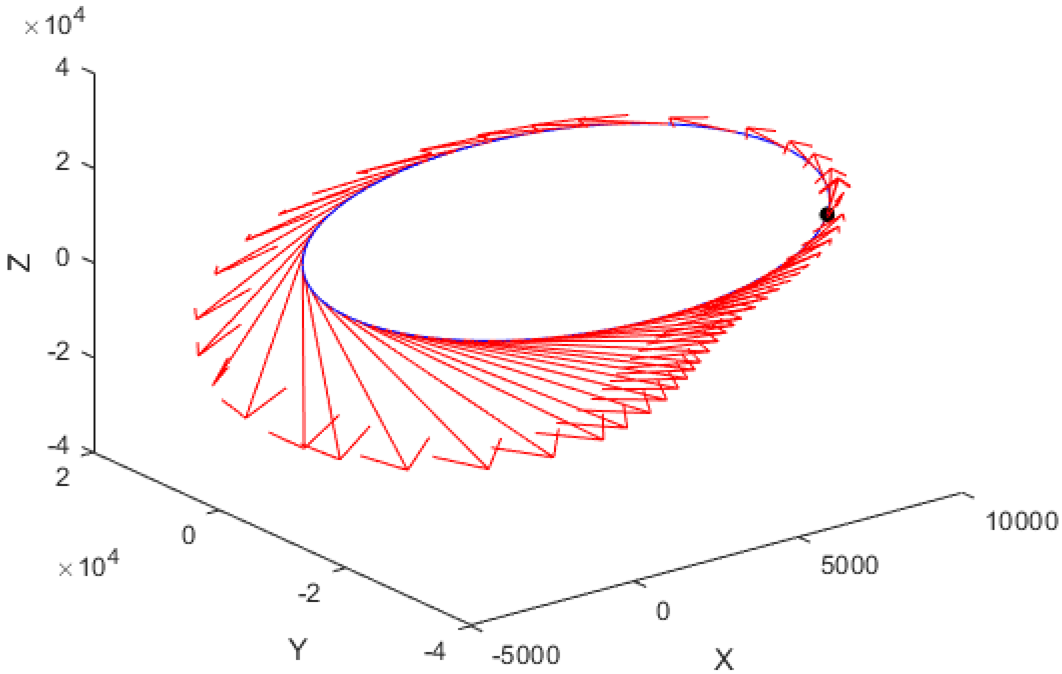
Initial Guess: 4.712389

4.712389	2.149154
2.149154	3.226501
3.226501	2.685783
2.685783	2.993617
2.993617	2.821960
2.821960	2.919753
2.919753	2.864522
2.864522	2.895901
2.895901	2.878127
2.878127	2.888213

Converges to 2.888213

```
%% Defines variables used in the equaions and the four initial guesses
clc;clear;
a = 0.5863552428728520;
C = -2.735509517401657;
x0 = [0 pi/2 pi 3*pi/2];
%% Defines the three functions used, orginal, derivative, and solved for x
f = @(x) x-a*sin(x)+C;
g = @(x) 1-a*cos(x);
b = @(x) a*sin(x)-C;
fprintf('--------Newton''s Method---------- \n');
%% loops through initial conditions and applies newton's method for 10
iterations
for j=1:length(x0)
    guess = x0(j);
    fprintf('Initial Guess: %f\n',guess);
    for i=1:10
        nextg = guess - (f(guess)/g(guess));
        fprintf('%f   %f \n',guess,nextg);
        guess = nextg;
    end
    fprintf('\n Converges to %f \n\n\n',guess);
end
%% loops through initial conditions and applies fixed point method for 10
iterations
fprintf('---------Fixed Point----------- \n');
for j=1:length(x0)
    guess = x0(j);
    fprintf('Initial Guess: %f\n',guess);
    for i=1:10
        nextg = b(guess);
        fprintf('%f   %f \n',guess,nextg);
        guess = nextg;
    end
    fprintf('\nConverges to %f \n\n\n',guess);
end
```

4.




```
%% Defines variables and initial conditions
clc;clear;close all;
mu = 398600;
trange = [0 38032];
xconditions = [6997.56; -34108; 20765.49];
vconditions = [0.1559; 0.25517; 1.80763];
po = [xconditions vconditions];
options = odeset('RelTol',1e-8);

%% Solves and plots the solution to the ODE, Includes velocity vectors and
point at t = 0
[t p] = ode113(@twoBodyOde,trange,po,options,mu);
figure
plot3(p(:,1),p(:,2),p(:,3),'b')
hold on
plot3(p(1,1),p(1,2),p(1,3),'.k','MarkerSize',20)
hold on
quiver3(p(1:3:153,1),p(1:3:153,2),p(1:3:153,3),p(1:3:153,4),p(1:3:153,5),p(1:
3:153,6),4,'r')
xlabel('X')
ylabel('Y')
zlabel('Z')

%% Function which defines the ODE
function pdot = twoBodyOde(t,p,mu)
    pos = p(1:3);
```

```
    vel = p(4:6);
    rad = norm(pos,2);
    posdot = vel;
    veldot = -mu/(rad^3)*pos;
    pdot = [posdot; veldot];
end
```

5a. r is the distance from O to P. The direction, $\mathbf{e}_r$, is in the direction of P from O. Therefore, the position of P relative to O can be expressed as r $\mathbf{e}_r$

5b. To take the derivative of r $\mathbf{e}_r$ in reference frame I, the transport theorem must be used since the vector is expressed in a different reference frame.

$$\frac{{}^{I}d}{dt}(\mathrm{r}\,\boldsymbol{e}_r) = \frac{d}{dt}(\mathrm{r}\,\boldsymbol{e}_r) + {}^{I}\omega^{P} x\,(\mathrm{r}\,\boldsymbol{e}_r)$$

$${}^{I}\omega^{P} = \dot{\theta}\boldsymbol{e}_z$$

Computing the cross product and simplifying results in

$${}^{I}\boldsymbol{v}_p = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$$

5c. Using the substitutions in the previous equation:

$$\dot{r} = v_r \quad \text{and} \quad \dot{\theta} = {}^{v_\theta}/_{r}$$

$${}^{I}\boldsymbol{v}_p = v_r\boldsymbol{e}_r + r\frac{v_\theta}{r}\boldsymbol{e}_\theta$$

$${}^{I}\boldsymbol{v}_p = v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta$$

5d. Differentiating to find acceleration of P in I using the transport theorem results in:

$${}^{I}\boldsymbol{a}_p = \dot{v_r}\boldsymbol{e}_r + \dot{v_\theta}\boldsymbol{e}_\theta + \frac{v_\theta v_r}{r}\boldsymbol{e}_\theta - \frac{{v_\theta}^2}{r}\boldsymbol{e}_r$$

5e. Using Newton's second law:

$$\frac{-\mu}{r^2}\boldsymbol{e}_r = \dot{v_r}\boldsymbol{e}_r + \dot{v_\theta}\boldsymbol{e}_\theta + \frac{v_\theta v_r}{r}\boldsymbol{e}_\theta - \frac{{v_\theta}^2}{r}\boldsymbol{e}_r$$

Separating each vector:

$$\dot{v_r} = \frac{{v_\theta}^2}{r} - \frac{\mu}{r^2}$$

$$\dot{v_\theta} = -\frac{v_\theta v_r}{r}$$

5f. The final system is:

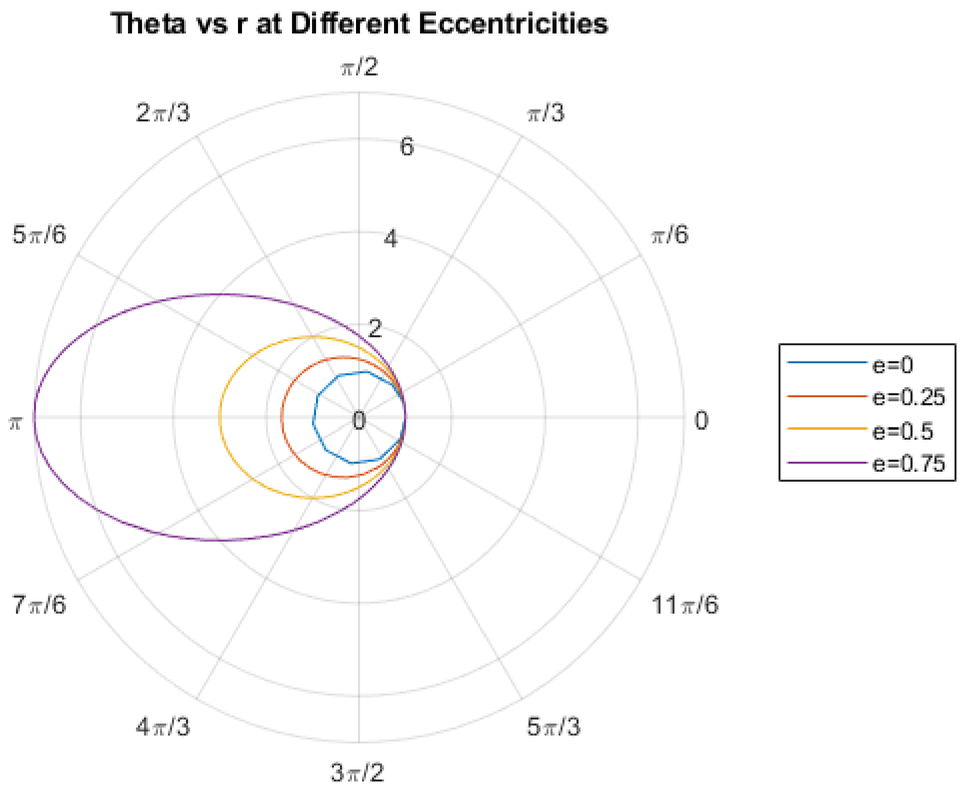
$$\dot{r} = v_r$$

$$\dot{\theta} = {}^{v_\theta}/_{r}$$

$$\dot{v_r} = \frac{{v_\theta}^2}{r} - \frac{\mu}{r^2}$$

$$\dot{v_\theta} = -\frac{v_\theta v_r}{r}$$

5g-f.




```
%% Defines the initial conditions and variables needed
clc;clear;close all;
mu = 1;
options = odeset('RelTol',1e-10);
e = [0 1/4 1/2 3/4];
conditions = [1 0 0];
j=1;
%% Loops through all values of e. Calculates the new vtheta0 each time.
Solves the ODE and stores the results in a cell array
for i=1:length(e)
    r0 = conditions(1);
    theta0=conditions(2);
    vr0=conditions(3);
    eval = e(i);
    pval=r0*(1+eval);
    ra = pval/(1-eval);
    a = (r0+ra)/2;
    vtheta0= sqrt(mu*pval)/r0;
    trange=[0 2*pi*sqrt(a^3/mu)];
    p0 = [r0 theta0 vr0 vtheta0];
    [t p] = ode113(@twoBody,trange,p0,options,mu);
    z{j}=p(:,1);
    z{j+1}=p(:,2);
    j=j+2;
end
%% Plots theta vs r and adds labels
```

```
polarplot(z{2},z{1},z{4},z{3},z{6},z{5},z{8},z{7})
legend('e=0','e=0.25','e=0.5','e=0.75')
polaraxis = gca;
polaraxis.ThetaAxisUnits = 'radians';
title('Theta vs r at Different Eccentricities')
%% the function which defines the ODE
function pdot = twoBody(t,p,mu)
    r = p(1);
    theta = p(2);
    vr = p(3);
    vtheta = p(4);
    rdot = vr;
    thetadot = vtheta/r;
    vrdot = (vtheta^2)/r-mu/(r^2);
    vthetadot = -vr*vtheta/r;
    pdot=zeros(size(p));
    pdot(1)=rdot;
    pdot(2)=thetadot;
    pdot(3)=vrdot;
    pdot(4)=vthetadot;
end
```