a) The basis in the reference frame, U, is:

$$\boldsymbol{u}_r = \frac{\boldsymbol{r}}{r}$$

$$\boldsymbol{u}_z = \boldsymbol{p}_z$$

$$\boldsymbol{u}_\nu = \boldsymbol{u}_z \times \boldsymbol{u}_r$$

This means that the position of the spacecraft expressed in this basis is:

$$\boldsymbol{r} = r\boldsymbol{u}_r$$

To find the velocity, the transport theorem must be applied:

$${}^I\boldsymbol{\omega}^U = \dot{\nu}\boldsymbol{u}_z$$

$${}^I\boldsymbol{v} = \frac{{}^I d\boldsymbol{r}}{dt} = \frac{{}^U d\boldsymbol{r}}{dt} + {}^I\boldsymbol{\omega}^U \times \boldsymbol{r} = \dot{r}\boldsymbol{u}_r + \dot{\nu}\boldsymbol{u}_z \times r\boldsymbol{u}_r$$

$${}^I\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\nu}\boldsymbol{u}_\nu$$

b) The two-body differential equation is:

$${}^I\boldsymbol{a} + \frac{\mu}{r^3}\boldsymbol{r} = 0$$

First ${}^I\boldsymbol{a}$ must be calculated. Applying the transport theorem to ${}^I\boldsymbol{v}$ gives us acceleration:

$${}^I\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\nu}\boldsymbol{u}_\nu$$

$${}^I\boldsymbol{a} = \frac{{}^I d\boldsymbol{v}}{dt} = \frac{{}^U d\boldsymbol{v}}{dt} + {}^I\boldsymbol{\omega}^U \times {}^I\boldsymbol{v}$$

$${}^I\boldsymbol{a} = \frac{{}^I d\boldsymbol{v}}{dt} = \ddot{r}\boldsymbol{u}_r + (r\ddot{\nu} + \dot{r}\dot{\nu})\boldsymbol{u}_\nu + \dot{\nu}\boldsymbol{u}_z \times (\dot{r}\boldsymbol{u}_r + r\dot{\nu}\boldsymbol{u}_\nu)$$

$${}^I\boldsymbol{a} = \frac{{}^I d\boldsymbol{v}}{dt} = \ddot{r}\boldsymbol{u}_r + (r\ddot{\nu} + \dot{r}\dot{\nu})\boldsymbol{u}_\nu + \dot{r}\dot{\nu}\boldsymbol{u}_\nu - r\dot{\nu}^2\boldsymbol{u}_r$$

$${}^I\boldsymbol{a} = (\ddot{r} - r\dot{\nu}^2)\boldsymbol{u}_r + (r\ddot{\nu} + 2\dot{r}\dot{\nu})\boldsymbol{u}_\nu$$

Substituting into the two-body differential equation:

$$(\ddot{r} - r\dot{\nu}^2)\boldsymbol{u}_r + (r\ddot{\nu} + 2\dot{r}\dot{\nu})\boldsymbol{u}_\nu + \frac{\mu}{r^3}r\boldsymbol{u}_r = 0$$

Separating the equation by vector gives two differential equations:

$$\ddot{r} - r\dot{\nu}^2 + \frac{\mu}{r^2} = 0$$

$$r\ddot{\nu} + 2\dot{r}\dot{\nu} = 0$$

c) Specific mechanical energy is given as:

$$\varepsilon = \frac{{}^{I}\boldsymbol{v} \cdot {}^{I}\boldsymbol{v}}{2} - \frac{\mu}{r}$$

Computing the dot product:

$$\varepsilon = \frac{\dot{r}^2 + r^2\dot{\nu}^2}{2} - \frac{\mu}{r}$$

To show that specific mechanical energy is constant, we must find the rate of change:

$$\frac{d\varepsilon}{dt} = \dot{r}\ddot{r} + r^2\dot{\nu}\ddot{\nu} + r\dot{r}\dot{\nu}^2 + \frac{\mu}{r^2}$$

Substituting the differential equations in for $\ddot{r}$ and $\ddot{\nu}$:

$$\frac{d\varepsilon}{dt} = \dot{r}\left(r\dot{\nu}^2 - \frac{\mu}{r^2}\right) + r^2\dot{\nu}\left(-\frac{2\dot{r}\dot{\nu}}{r}\right) + r\dot{r}\dot{\nu}^2 + \frac{\mu}{r^2}$$

$$\frac{d\varepsilon}{dt} = \dot{r}r\dot{\nu}^2 - \frac{\dot{r}\mu}{r^2} - 2\dot{r}r\dot{\nu}^2 + r\dot{r}\dot{\nu}^2 + \frac{\mu}{r^2} = 0$$

d) -

e) Specific angular momentum is defined as:

$${}^{I}\boldsymbol{h} = \boldsymbol{r} \times {}^{I}\boldsymbol{v}$$

Taking the cross product:

$${}^{I}\boldsymbol{h} = r\boldsymbol{u}_r \times (\dot{r}\boldsymbol{u}_r + r\dot{\nu}\boldsymbol{u}_\nu)$$

$${}^{I}\boldsymbol{h} = r^2\dot{\nu}\boldsymbol{u}_\nu$$

Taking the magnitude:

$$h = r^2\dot{\nu}$$

This can be rearranged to be:

$$\dot{\nu} = \frac{h}{r^2}$$

f) The change of variable for the second differential equation is:

$$\rho = 1/r$$

To find a second order differential equation with derivative of ρ with respect to ν, $\frac{d\rho}{d\nu}$ must be found.

$\frac{d\rho}{d\nu}$ can be found by multiplying:

$$\frac{d\rho}{dv} = \frac{d\rho}{dr} \cdot \frac{dr}{dv}$$

Taking another derivative gives:

$$\frac{d^2\rho}{dv^2} = \frac{d^2\rho}{dr^2}\left(\frac{dr}{dv}\right)^2 + \frac{d\rho}{dr}\frac{d^2 r}{dv^2}$$

Taking the derivative of ρ with respect to r:

$$\frac{d\rho}{dr} = -\frac{1}{r^2}$$

Using the second differential equation:

$$r\frac{d^2 v}{dt^2} + 2\frac{dr}{dt}\frac{dv}{dt} = 0$$

Solving for dr/dt:

$$\frac{dr}{dt}\frac{dv}{dt} = -\frac{r}{2}\frac{d^2 v}{dt^2}$$

$$\frac{dr}{dt} = -\frac{r}{2}\frac{dv}{dt}$$

Rearranging:

$$\frac{dr}{dv} = -\frac{r}{2}$$

Now $\frac{d\rho}{dv}$ can be calculated:

$$\frac{d\rho}{dv} = \frac{d\rho}{dr} \cdot \frac{dr}{dv} = \left(-\frac{1}{r^2}\right) \cdot \left(-\frac{r}{2}\right) = \frac{1}{2r} = \frac{1}{2}\rho$$

The second differential equation can be written as

$$\frac{d^2 v}{dt^2} = -2\rho\frac{dr}{dt}\frac{dv}{dt}$$