Positive-definiteness of the Jacobian matrix of the Stokes system for homogeneous media

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The Stokes equation for incompressible fluid is stated as

$$\tau_{ij,j} - p_{,i} = f_i, \tag{1}$$

$$v_{i,i} = 0, \tag{2}$$

where τ_{ij} is the deviatoric stress tensor, p is pressure, v_i is velocity, and f_i stands for a body force. The weak form of the momentum conservative equation can be calculated by integrating the inner product of Eq. (1) and a virtual velocity δv_i across the computational domain Ω, which yields after integrating by part

$$\int_\Omega \delta\dot{\varepsilon}_{ij}\tau_{ij}\mathrm{d}\Omega - \int_\Omega \delta v_{i,i} p \mathrm{d}\Omega = \int_\Omega \delta v_i f_i \mathrm{d}\Omega, \tag{3}$$

where $\dot{\varepsilon}_{ij} \equiv (v_{i,j}+v_{j,i})/2$ denotes the strain rate tensor. Similarly, the weak form of mass conservative equation is obtained by multiplying Eq. (2) with a virtual pressure δp and integrating across Ω, which gives

$$\int_\Omega \delta p v_{i,i}\mathrm{d}\Omega = 0. \tag{4}$$

With proper finite element approximation

$$v_i \approx \sum_I V^I \varphi_i^I, \quad p \approx \sum_I P^I \vartheta^I, \tag{5}$$

Eqs. (3) and (4) can be discretized into the algebraic form

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}, \tag{6}$$

where the matrix and right-hand side blocks are defined by

$$A^{IJ} = \int_\Omega \varepsilon_{ij}^I \tau_{ij}^J \mathrm{d}\Omega, \quad B^{IJ} = -\int_\Omega \varphi_{i,i}^I \vartheta^J \mathrm{d}\Omega, \quad f^I = \int_\Omega \varphi_i^I f_i \mathrm{d}\Omega.$$

In the above expressions, $\varphi_{ij}^I \equiv (\varphi_{i,j}^I + \varphi_{j,i}^I)/2$, and τ_{ij}^I is a function of φ_{kl}^I and ϑ^I satisfying $\sum_I \tau_{ij}^I V^I = \tau_{ij}$. For instance, if the fluid is Newtonian, then we have $\tau_{ij}^I = 2\eta\varphi_{ij}^I$, hence $\sum_I \tau_{ij}^I V^I = 2\eta\dot{\varepsilon}_{ij} = \tau_{ij}$. However, in geodynamic problems the relationship between τ_{ij} and $\dot{\varepsilon}_{kl}$ is generally nonlinear.

We employ the Newton-Raphson method to resolve the nonlinearity in Eq. (6). In each Newton iteration, we need to solve the linearized system

$$\begin{bmatrix} \mathbf{J}_{vv} & \mathbf{J}_{vp} \\ \mathbf{J}_{pv} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathrm{d}\boldsymbol{V} \\ \mathrm{d}\boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{P} \end{bmatrix}. \tag{7}$$

The blocks of the Jacobian matrix have the following form:

$$J_{vv}^{IJ} = \int_\Omega \varphi_{ij}^I \frac{\partial \tau_{ij}}{\partial \dot{\varepsilon}_{kl}} \varphi_{kl}^J \mathrm{d}\Omega, \quad J_{vp}^{IJ} = \int_\Omega \varphi_{ij}^I \frac{\partial \tau}{\partial p} \vartheta^J \mathrm{d}\Omega + B^{JI}, \quad J_{pv}^{IJ} = B^{IJ}.$$

More specifically, we assume that the material is *homogeneous*, i.e. τ_{ij} is parallel with $\dot{\varepsilon}_{ij}$. In such cases, τ_{ij} can be expressed as

$$\tau_{ij} = 2\eta(\dot{\varepsilon}, p)\dot{\varepsilon}_{ij}, \tag{8}$$

where $\dot{\varepsilon} \equiv \sqrt{\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}$ is the norm of $\dot{\varepsilon}_{ij}$. Substitution of Eq. (8) in the expression of J_{vv}^{IJ} gives

$$J_{vv}^{IJ} = \int_\Omega 2\varphi_{ij}^I \left(\eta I_{ijkl}^s + \dot{\varepsilon}_{ij} \frac{\partial \eta}{\partial \dot{\varepsilon}_{kl}} \right) \varphi_{kl}^J \mathrm{d}\Omega, \tag{9}$$

where $I_{ijkl}^s \equiv (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$ is the symmetric projection tensor. Eq. (9) is equivalent to Eq. (14) in Fraters et al. [2019].

As illustrated by Fraters et al. [2019], to employ the Schur complement preconditioner to solve Eq. (7), we must guarantee that $\mathbf{J}_{vv}$ is symmetric and positive-definite, otherwise the top-left block of the preconditioner cannot be implicitly inversed with the CG solver. To analyze the properties of $\mathbf{J}_{vv}$, we shall expand $\dot{\varepsilon}_{ij} \frac{\partial \eta}{\partial \dot{\varepsilon}_{kl}}$ straightforwardly:

$$\dot{\varepsilon}_{ij} \frac{\partial \eta}{\partial \dot{\varepsilon}_{kl}} = \dot{\varepsilon}_{ij} \frac{\partial \eta}{\partial \dot{\varepsilon}} \frac{\partial \dot{\varepsilon}}{\partial \dot{\varepsilon}_{kl}} = \dot{\varepsilon}_{ij} \frac{\partial \eta}{\partial \dot{\varepsilon}} \frac{\dot{\varepsilon}_{kl}}{\dot{\varepsilon}} = \dot{\varepsilon} \frac{\partial \eta}{\partial \dot{\varepsilon}} \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}} \frac{\dot{\varepsilon}_{kl}}{\dot{\varepsilon}}. \tag{10}$$

Eq. (10) implies that J_{vv}^{IJ} is always symmetric. The positive-definiteness of J_{vv}^{IJ}, on the other hand, is not guaranteed since it depends on $\partial \eta / \partial \dot{\varepsilon}$. For simplicity, we define

$$E \equiv \dot{\varepsilon} \frac{\partial \eta}{\partial \dot{\varepsilon}} \quad \text{and} \quad n_{ij} \equiv \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}}.$$

As seen, E is a scalar quantity in units of Pa·s, and n_{ij} is the unit tensor parallel to $\dot{\varepsilon}_{ij}$. Then Eq. (10) can be rewritten as

$$\dot{\varepsilon}_{ij} \frac{\partial \eta}{\partial \dot{\varepsilon}_{kl}} = E n_{ij} n_{kl}. \tag{11}$$

It is easy to see from Eq. (11) that $\dot{\varepsilon}_{ij} \frac{\partial \eta}{\partial \dot{\varepsilon}_{kl}}$ has only one non-trivial eigenvalue E. In fact, we may consider n_{ij} as a 2-D unit vector on the π-plane of the principal strain rate space (the 3-D space with principal strain rate $\dot{\varepsilon}_1$, $\dot{\varepsilon}_2$ and $\dot{\varepsilon}_3$ as coordinate axes), which is equivalent to applying a rotational transformation to n_{ij}. Denote the unit vector by $\boldsymbol{n} \equiv [\cos\phi \ \ \sin\phi]^T$, then Eq. (11) can be explicitly expressed as

$$E\boldsymbol{n}\boldsymbol{n}^T = E \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}. \tag{12}$$

It is clear that the eigenvalues of $E\boldsymbol{n}\boldsymbol{n}^T$ are 0 and E. Consequently, *the Jacobian matrix $\boldsymbol{J}_{vv}$ is positive-definite as long as $E > -\eta$.*

1 Power-law creep

The power-law creep is generally stated as

$$\eta = \eta_0 \dot{\varepsilon}^{\frac{1}{n}-1}, \tag{13}$$

where η_0 is the prefactor, and n is a dimensionless parameter satisfying $n \geq 1$. Straightforward derivation gives

$$E = \eta_0 \left(\frac{1}{n} - 1 \right) \dot{\varepsilon}^{\frac{1}{n}-1} = \left(\frac{1}{n} - 1 \right) \eta. \tag{14}$$

Since $n \geq 1$, we always have $E > -\eta$, which implies that $\boldsymbol{J}_{vv}$ *for the power-law rheology is always positive-definite.*

2 The Drucker-Prager model

The yield function of the Drucker-Prager model is defined as

$$F = \frac{\tau}{\sqrt{2}} - \xi p - \zeta, \tag{15}$$

where $\tau \equiv \sqrt{\tau_{ij}\tau_{ij}}$, ξ and ζ are material parameters determined by the friction angle and the cohesion. When plastic yielding occurs, we have $F = 0$, which indicates that

$$\frac{\tau}{\sqrt{2}} = \xi p + \zeta. \tag{16}$$

Thus, we have

$$\eta = \frac{\tau}{2\dot{\varepsilon}} = \frac{\xi p + \zeta}{\sqrt{2}\dot{\varepsilon}}, \tag{17}$$

and

$$E = -\frac{\xi p + \zeta}{\sqrt{2}\dot{\varepsilon}} = -\eta. \tag{18}$$

The equation above implies that $\boldsymbol{J}_{vv}$ *for the Drucker-Prager model is non-negative-definite* (as plastic yielding does not occur everywhere). To restore the positive-definiteness of the top-left block, we should scale the Newton step with a factor $\alpha \in (0, 1)$.

3 Maxwell viscoplastic model

Simulations of lithosphere dynamics often employ a Maxwell-type viscoplastic model with an additive decomposition

$$\dot{\varepsilon} = \dot{\varepsilon}_v + \dot{\varepsilon}_p, \tag{19}$$

where the subscripts v and p stand for viscous and plastic contributions, respectively. Thus, the comprehensive viscosity can be expressed as

$$\eta = \left(\frac{1}{\eta_v} + \frac{1}{\eta_p} \right)^{-1}, \tag{20}$$

where η_v and η_p are defined by Eq. (13) and Eq. (17), respectively. Then we have

$$E = -\dot{\varepsilon}\eta^2 \left(-\frac{1}{\eta_v^2}\frac{\partial \eta_v}{\partial \dot{\varepsilon}} - \frac{1}{\eta_p^2}\frac{\partial \eta_p}{\partial \dot{\varepsilon}} \right) = \eta^2 \left(\frac{E_v}{\eta_v^2} + \frac{E_p}{\eta_p^2} \right) = \eta^2 \left(\frac{\frac{1}{n} - 1}{\eta_v} - \frac{1}{\eta_p} \right) = \left(\frac{\eta}{n\eta_v} - 1 \right) \eta. \tag{21}$$

As seen, $\boldsymbol{J}_{vv}$ *for Maxwell viscoplastic model is also positive-definite.* In practice, however, we often run into situations with $\eta_p \ll \eta_v$, which makes $\frac{\eta}{n\eta_v}$ close to 0. Therefore, to guarantee the convergence of the CG solver, we had better scale the Newton step for Maxwell viscoplastic models as well.

References

M R T Fraters, W Bangerth, C Thieulot, A C Glerum, and W Spakman. Efficient and practical Newton solvers for non-linear Stokes systems in geodynamic problems. *Geophysical Journal International*, 218(2):873–894, August 2019. ISSN 0956-540X. doi: 10.1093/gji/ggz183.