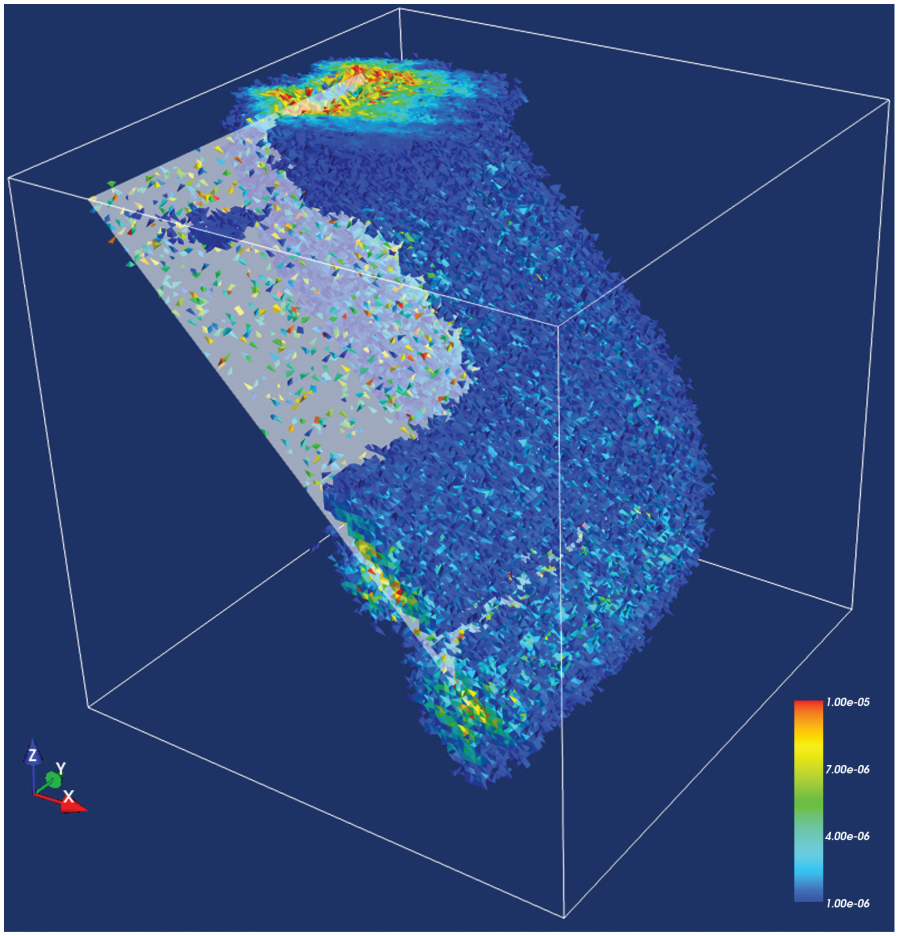
COMPUTATIONAL INFRASTRUCTURE FOR GEODYNAMICS (CIG)

Cigma

User Manual
Version 0.9.1



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Cigma

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Chapter 1

Introduction

1.1 About Cigma

The CIG Model Analyzer (Cigma) consists of a suite of tools for comparing numerical models. CIG has developed Cigma in response to demand from the short-term tectonics community for a simple tool that can perform rigorous error analysis on their finite element codes. The long-term goal for Cigma, however, is for it to be used for nearly all geodynamics modeling codes.

The current version of Cigma is intended for the calculation of L_2 residuals for finite element models. This software was designed for performing the following three main tasks: (1) error analysis, (2) benchmarking, and (3) code verification.

In error analysis, Cigma can calculate a global measure of the error between two solutions by performing an integration over a discretized version of the common domain. This comparison can take place even when the underlying discretizations do not overlap.

In benchmarking, Cigma can help the geodynamics community agree on a standard solution to specific problems by facilitating the process of comparing different numerical codes against each other.

Lastly, as an automated tool, Cigma can help application developers create regression tests to ensure that software changes do not affect the consistency of the results.

At its core, Cigma draws from a variety of libraries. The FInite element Automatic Tabulator (FIAT) (`www.fenics.org/wiki/FIAT`) Python Library supports generation of arbitrary order instances of the Lagrange elements on lines, triangles, and tetrahedra. It can also generate order instances of Jacobi-type quadrature rules on the same element shapes. The Approximate Nearest Neighbor (ANN) (`www.cs.umd.edu/~mount/ANN/`) Library, written in C++, supports data structures and algorithms for both exact and nearest-neighbor searching in arbitrarily high dimensions.

Both of these libraries particularly extend and generalize Cigma's functionality so it can handle other types of elements, and provide the ability to compare vector fields.

1.2 Citation

Computational Infrastructure for Geodynamics (CIG) is making this source code available to you in the hope that the software will enhance your research in geophysics. This is a brand-new code and at present no papers are published or at press for use as citations other than this manual, which is cited as follows:

- Armendariz, L., and S. Kientz. *Cigma User Manual.* Pasadena, CA: Computational Infrastructure of Geodynamics, 2007. URL: geodynamics.org/cig/software/cs/cigma/cigma.pdf

CIG requests that in your oral presentations and in your papers that you indicate your use of this code and acknowledge the author of the code and CIG (`geodynamics.org`).

1.3 Support

Cigma development is supported by a grant from the National Science Foundation to CIG, managed by the California Institute of Technology. The code is being released under the GNU General Public License.

Chapter 2

Installation and Getting Help

2.1 Getting Help

For help, send an e-mail to the CIG Computational Science Mailing List (`cig-cs@geodynamics.org`). You can subscribe to the `cig-cs` mailing list and view archived discussions at the CIG Mail Lists web page (`geodynamics.org/cig/lists`). If you encounter any bugs or have problems installing Cigma, please submit a report to the CIG Bug Tracker (`geodynamics.org/bugs`).

2.2 Installating from Source

To install Cigma, download the source package from the CIG Cigma web page (`geodynamics.org/cig/software/packages/cs/cigma`). You will need the GNU C++ compiler for this step. The HDF5 and VTK libraries, described later in this section, are also required.

After installing the necessary dependencies, simply unpack the source tar file and issue the following commands

```
$ tar xvfz cigma-0.9.1.tar.gz
$ cd cigma-0.9.1
$ ./configure --with-hdf5=$HDF5_PREFIX --with-vtk=$VTK_PREFIX
$ make
$ sudo make install
```

2.2.1 HDF5

The Hierarchical Data Format (HDF5) library is available for download from The HDF Group (`hdfgroup.org/HDF5`). Binaries can be obtained at hdfgroup.org/HDF5/release/obtain5.html (`hdfgroup.org/HDF5/release/obtain5.html`). To install from source, download the latest stable 1.6 version of this library (currently 1.6.5) and issue the following commands

```
$ tar xvfz hdf5-1.6.5
$ cd hdf5-1.6.5
$ ./configure
$ make
$ sudo make install
```

2.2.2 VTK

The Visualization Toolkit (VTK) library is a popular open source graphics library for scientific visualizations. We recommend that you obtain the binaries from a package manager, making sure to install the associated development headers along with the library. The source for the VTK library is available from Kitware,

Inc. (`www.vtk.org/get-software.php`). If you decide to compile VTK from source, you will also need the CMake build environment, available from CMake (`cmake.org`).

After downloading the source package, you can issue the following commands

```
$ tar xvfz vtk-5.0.4.tar.gz
$ cd VTK
$ mkdir build
$ cd build
$ ccmake ..
$ make
$ sudo make install
```

2.2.3 NumPy

NumPy is a Python module which adds support for large multi-dimensional arrays and matrices, and is available for download from the NumPy Home Page (`numpy.scipy.org`). To install this extension module from source, download it and issue the following command in the NumPy source directory

```
$ sudo python setup.py install
```

To verify the installation, the command '`import numpy`' should run successfully from within your standard Python interpreter shell.

2.2.4 PyTables

PyTables is a Python extension module that builds on top of the HDF5 library. It provides a convenient scripting interface to manipulate HDF5 files, which can be used to manipulate the input/output files created by Cigma. PyTables is available for download from PyTables (`www.pytables.org`).

To install this extension from source, download the latest stable version (currently 2.0.2) and issue the following commands

```
$ tar xvfz pytables-2.0.2
$ cd pytables-2.0.2
$ sudo python setup.py install
```

To verify the installation, the command '`import tables`' should run successfully from within your standard Python interpreter shell.

2.2.5 FIAT

The Finite Element Automatic Tabulator (FIAT) Python module supports generation of arbitrary order instances of the Lagrange elements, as well as arbitrary order instances of the Jacobi-type quadrature rules on the same element shapes. It is available from The FEniCS Project (`www.fenics.org/wiki/FIAT`).

To install this module, download it and issue the following command inside the FIAT source directory:

```
$ sudo python setup.py install
```

You should now be able to run '`import FIAT`' from within your standard Python interpreter shell.

2.2.6 HDFView (optional)

NCSA HDFView is a graphical user interface tool for accessing data in your HDF5 files. You can use it for viewing the internal file hierarchy in a tree structure, adding new datasets, and modifying or deleting existing datasets, or altering metadata on groups and datasets. You can download it from the HDFView home page (`hdf.ncsa.uiuc.edu/hdf-java-html/hdfview`).

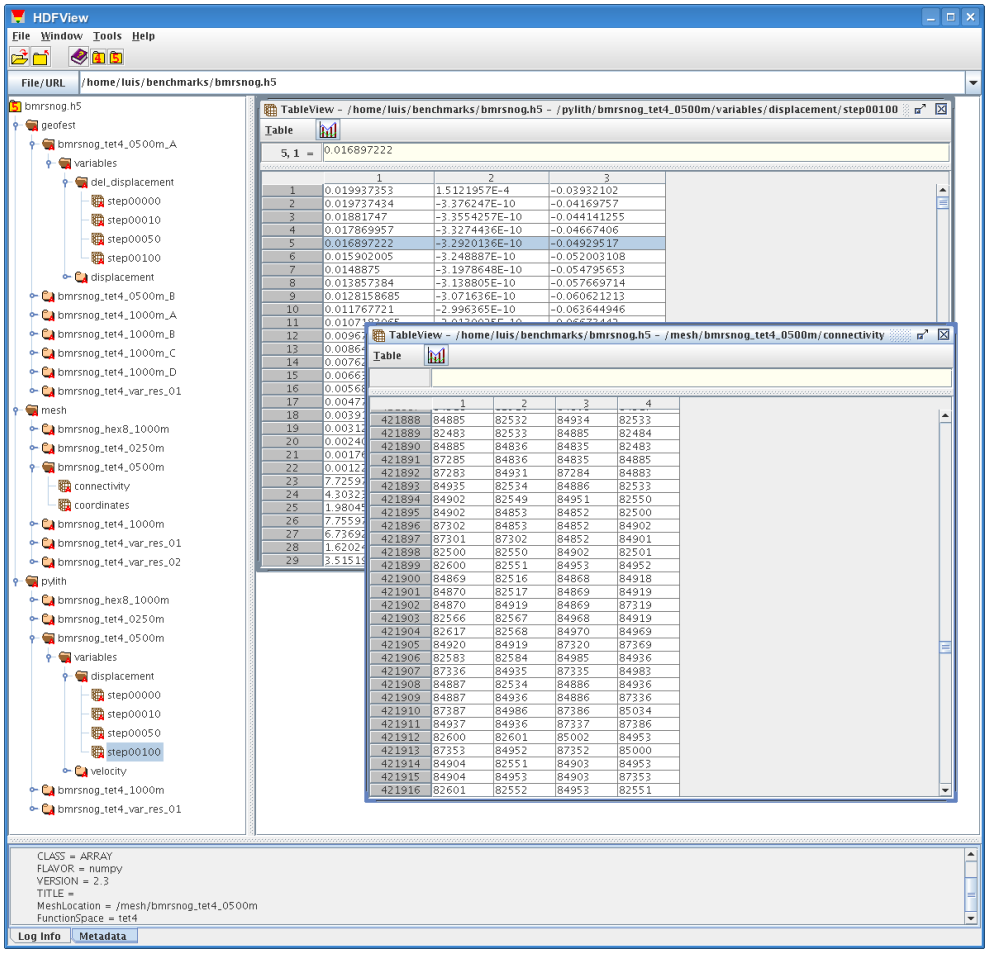

Figure 2.1: With HFDFView, you can view the internal file hierarchy in a tree structure, add new datasets, and modify or delete existing datasets

Chapter 3

Error Analysis

3.1 Introduction

When studying differential equations that represent physical systems we often obtain solutions by using a variety of techniques. Solving these equations using classical analytical methods is almost impossible, so we often resort to numerical algorithms such as the finite element method.

For assessing the quality of the solution to our equations, it is important to quantify the error in our numerical solutions. To calculate this error, we will have to compare tentative solutions against each other through the use of a distance measure between two functions.

3.2 Distance Measures

The simplest possible quantitative measure of the difference between two distinct fields you can make consists of taking the pointwise difference of both fields at a common set of points. While no finite sample of points can perfectly represent a continuum of values, valuable information can be inferred from a statistics analysis of the resulting set of differences.

Perhaps a more useful distance measure can be obtained by using the L_2 norm, defined by the following integral

$$\varepsilon = ||u - v||_{L_2} = \sqrt{\int_\Omega ||u(\vec{x}) - v(\vec{x})||^2 d\vec{x}}$$

This gives us a single global estimate ε representing the distance between the two fields $u(\vec{x})$ and $v(\vec{x})$. Alternatively, you may think of this as the size, or norm, of the residual field $\rho(\vec{x}) = u(\vec{x}) - v(\vec{x})$.

If we discretize the domain Ω into finite elements Ω_e, the above integral can be broken up into a sum over local contributions on each element. For efficiency, each contribution can be integrated over a reference element $\hat{\Omega}_e$ defined on a standard coordinate system.

$$\begin{aligned}
\varepsilon^2 &= \sum_{e=1}^{\text{nel}} \varepsilon_e^2 \\
&= \sum_{e=1}^{\text{nel}} \int_{\Omega_e} ||u(\vec{x}) - v(\vec{x})||^2 d\vec{x} \\
&= \sum_{e=1}^{\text{nel}} \int_{\hat{\Omega}_e} ||u(\vec{\xi}) - v(\vec{\xi})||^2 J(\vec{\xi}) d\vec{\xi}
\end{aligned}$$

In general, we won't be able to integrate each local contribution exactly since the two fields u and v may have a representation that is incompatible with the local domain Ω_e. However, we can approximate each ε_e^2 by applying an appropriate quadrature rule with a tolerable truncation error [2].

3.3 Global Error

Assuming we apply the same quadrature rule, with weights w_q and integration points $\vec{\xi}_q$, on every element,

$$\begin{aligned} \varepsilon_e^2 &= \sum_{q=1}^{\text{nq}} w_q ||\hat{\rho}(\vec{x}_q)||^2 \\ &= \sum_{q=1}^{\text{nq}} w_q ||u(\vec{\xi}_q) - v(\vec{\xi}_q)||^2 J(\vec{\xi}_q) \end{aligned}$$

thus we arrive at the final form

$$\varepsilon = \sqrt{\sum_{e=1}^{\text{nel}} \sum_{q=1}^{\text{nq}} w_q ||u(\vec{\xi}_q) - v(\vec{\xi}_q)||^2 J(\vec{\xi}_q)}$$

In calculating the norm of the residual field ρ, Cigma will output each of the local contributions ε_e^2 which by definition are scalar-valued cell quantities over each of their corresponding elements Ω_e.

3.4 Comparing Residuals

For comparing errors of different solutions, you can normalize the global error by the norm of the exact solution. This normalized error is given by

$$\begin{aligned} \varepsilon_{rel} &= \frac{||u - u^h||_{L_2}}{||u||_{L_2}} \\ &= \frac{\int_\Omega ||u(\vec{x}) - u^h(\vec{x})||^2 d\vec{x}}{\int_\Omega ||u(\vec{x})||^2 d\vec{x}} \end{aligned}$$

This normalized error can be interpreted as the average error in the physical quantity being evaluated, so that a value of 0.01 corresponds to a 1% averaged error. The norm in the denominator can be computed in Cigma by comparing against the built-in zero function (see Section 4.2.4).

Even if the exact solution is not currently known, this normalized error may be used to test the accuracy between two or more numerical solutions, defined on successively refined meshes.

3.5 Input and Output

The underlying data storage format for Cigma is the HDF5 format, due to its flexibility for storing and organizing large amounts of data.

The Hierarchical Data Format (HDF) is a portable file format developed at the National Center for Supercomputing Applications (NCSA) (`hdf.ncsa.uiuc.edu/HDF5`). It is designed for storing, retrieving, analyzing, visualizing, and converting scientific data. The current and most popular version is HDF5, which stores multi-dimensional arrays together with ancillary data in a portable self-describing format. It uses a hierarchical structure that provides application programmers with a host of options for organizing how data is stored in HDF5 files.

Using HDF5 datasets in Cigma allows us to avoid having to convert between too many distinct formats. Moreover, due to the amount of disk I/O, large finite element meshes can be handled more efficiently in binary format.

Cigma also supports the popular VTK format for certain kinds of input. A simple text format is also supported to aid in debugging. As described in Chapter 4, you can easily examine the structure of an input file by using the `cigma list` command, which will simply reveal the names and dimensions of all datasets inside the specified file.

3.6 Mesh

In Cigma, we define a finite element mesh simply by coordinates, (x_n, y_n, z_n) of its degrees of freedom, and the connectivity relations $\Omega_e = \{n_1, n_2, \ldots\}$ among them which define each individual element in the corresponding discretization.

Since the evaluation points on two distinct meshes will (TODO – will do what?), it is very important to index the location of each element into spatial data structure.

3.7 Fields

A field is a function which assigns a physical quantity to every point in space. This quantity may correspond to a scalar, a vector, or even a tensor. A finite element approximation to an unknown field $\phi(\vec{x})$ is simply the weighed sum over a fixed set of localized shape functions $\phi_n(\vec{x})$.

$$\phi(\vec{x}) = \sum_{n=1}^{N} d_n \phi_n(\vec{x})$$

Naturally, once the weights d_n, or degrees of freedom as they are also called, are known to us, we can evaluate $\phi(\vec{x})$ at any point $\vec{x}$ we desire. Thus, in Cigma a field is represented simply by a list of degrees of freedom d_n, which may be a scalar, vector or tensor quantity, depending on the nature of ϕ.

3.8 Elements

The most common element types used in the finite element method are given by the following shape functions, defined on their respective domains,

Function Space tet4

$$\begin{aligned} TN_a &= \frac{1}{2}(-1-x-y-z) \\ TN_b &= \frac{1}{2}(1+x) \\ TN_c &= \frac{1}{2}(1+y) \\ TN_d &= \frac{1}{2}(1+z) \end{aligned}$$

Function Space hex8

$$\begin{aligned} HN_a &= \frac{1}{8}(1-x)(1-y)(1-z) \\ HN_b &= \frac{1}{8}(1+x)(1-y)(1-z) \\ HN_c &= \frac{1}{8}(1-x)(1+y)(1-z) \\ HN_d &= \frac{1}{8}(1+x)(1+y)(1-z) \\ HN_e &= \frac{1}{8}(1-x)(1-y)(1+z) \\ HN_f &= \frac{1}{8}(1+x)(1-y)(1+z) \\ HN_g &= \frac{1}{8}(1-x)(1+y)(1+z) \\ HN_h &= \frac{1}{8}(1+x)(1+y)(1+z) \end{aligned}$$

These are built-in elements in Cigma, and you may refer to them by name in the FunctionSpace metadata attribute on your datasets.

Higher order elements can also be used by specifying an explicit tabulation matrix representing the values of your shape functions on the corresponding set of quadrature points.

Chapter 4

Running Cigma

Cigma is designed to be scriptable. Thus, all operations can be specified as command line arguments given to a single executable called `cigma`. A list of available commands can be obtained by typing

```
cigma --help
```

Further usage information can also be obtained with

```
cigma help <command>
```

Cigma is used for obtaining error estimates between arbitrary fields, so naturally its primary operation involves the `compare` command. You will need to provide two datasets describing each of the two fields, along with an integration rule and a mesh over which to integrate, although these last two will have reasonable defaults if they are not specified.

Specifying the complete path to a dataset consists of the special form `filepath:dataset`, a colon-delimited pair of file path and dataset path.

4.1 Listing Data

Because Cigma relies on the ability of the user to specify dataset paths, we have provided a command called `list` for viewing the structure of an input file. Its usage is very simple.

To view the structure of an HDF5 file:

```
$ cigma list file.h5
    /mesh/coordinates           Dataset {119827, 3}
    /mesh/connectivity          Dataset {661929, 4}
    /vars/displacement/step0    Dataset {119827, 3}
    /residuals/comparison0      Dataset {661929, 1}
```

If VTK support is enabled, you can view the structure of a VTK file with:

```
$ cigma list file.vtk
    Reading file.vtk
    Points = 119827
    Cells = 661929
    PointDataArray[0] = displacements_t0 (119827 x 3)
```

4.2 Comparing Two Fields

Comparing two arbitrary finite element fields can be accomplished with the `cigma compare` command-line utility. By default, the comparison will take place over the elements in the mesh of the first field. Finally, the square of each of the local residual values are written to the specified VTK output file as cell-based scalars.

A basic comparison can be as simple as specifying the following arguments:

```
cigma compare --output=squared-residuals.vtk
              --first=field1.h5:/field1/stepN
              --second=field2.h5:/field2/stepN
```

4.2.1 Specifying a Mesh

To override the mesh used in the integration, you can specify an extra argument providing the location of the mesh:

```
cigma compare [...]
  --mesh=mesh.h5:/model/mesh/
```

You can also specify the coordinates and connectivity arrays separately, in case they reside in separate files.

```
cigma compare [...]
  --mesh-coordinates=file1.h5:/model/mesh/coordinates
  --mesh-connectivity=file2.h5:/model/mesh/connectivity
```

4.2.2 Specifying a Quadrature Rule

To specify a quadrature rule, you will have to provide the quadrature weights and points in the appropriate reference element. This can be done with the following additional argument:

```
cigma compare [...]
  --rule=quadrature-rules.h5:/path/to/rule
```

You may also specify the location of the points and weights separately:

```
cigma compare [...]
  --rule-points=file.h5:/path/to/rule/points
  --rule-weights=file.h5:/path/to/rule/weights
```

4.2.3 Comparing against Known Values

A finite element description might not always be available for one of the fields. However, you can break the comparison into several steps if you have a means to compute that field on any of the required points.

First, extract the global coordinates of the integration points. This will result in an explicit list of points over which to evaluate your field.

```
cigma extract --mesh=field1.h5:/model/mesh/
              --output=points.h5:/points
```

Now you can evaluate your function at the designated points. You can provide the path to the explicit set of values with

```
cigma compare --output=squared-residuals.vtk
              --first=field1.h5:/field1/stepN
              --second=values.h5:/stepN_values
```

4.2.4 Comparing against a Known Function

If one of your fields is easily described by an analytic expression, then you also have the option to compile your analytic function into Cigma, which will enable the `compare` command to reference your function by name:

```
cigma compare --output=squared-residuals.vtk
              --first=field1.h5:/vars/displacement/step0
              --second=disloc3d
```

You may also interact with your analytic function by using the `cigma eval` command, and obtain a set of values which may then be passed back to the `compare` command.

```
cigma eval --function=disloc3d
           --points=points.h5:/points
           --values=values.h5:/disloc3d_values
cigma compare --output=squared-residuals.vtk
              --first=field1.h5:/vars/displacement/step0
              --second=values.h5:/disloc3d_values
```

Chapter 5

Examples

When the exact solution of your equations is known, the norm of the error for a finite element solution is easily computed through the procedure given in Chapter 3.

Another way of checking the accuracy of a solution is by rerunning the same problem with a finer mesh. Recall that in the finite element method, the weight functions and trial solutions are chosen so that the finite element method is convergent. In other words, as the element size h decreases, the solution tends to the correct solution. Thus, if the difference between the coarse and fine mesh solutions is small, we can conclude that the coarse mesh solution is quite accurate. However, if a solution changes noticeably with refinement of the mesh, the coarse mesh solution is inaccurate, and even the finer mesh may be inadequate as well.

5.1 Convergence

To demonstrate the convergence in the FEM method, in this example we will compare the error incurred in representing a continuous function by finite elements.

You can change the function by modifying the `TestFunction.cpp` source file. Once you recompile cigma, you will be able to refer to it by the name `test` when using the `cigma eval` command.

In the test directory you will find a series of successively refined meshes that you can use for testing. The original mesh files were created with the Gmsh, a three-dimensional finite element mesh generator, but you will be accessing the HDF5.

In the figure (TODO: regenerate plot), we show the log of the L_2 error norm as a function of the log of the element size h.

As can be seen from these results, the log of the error varies linearly with element size and the slope depends on the order of the element. Denoting the slope by α, then the error in the function can be expressed as

$$\log \varepsilon = \log C + \alpha \log h$$

where $\log C$ is an arbitrary constant, the y-intercept of the curve. The slope α is the rate of convergence of the element. We can rewrite this as

$$\varepsilon = Ch^{\alpha}$$

5.2 Different Element Types

For this example we will use sample datasets from the strike-slip benchmark case defined by the CIG Short-Term Tectonics working group. In this benchmark problem, we solve for the viscoelastic relaxation of stresses from a single finite earthquake while ignoring gravity. The problem is defined on a cube domain with sides of 24 km consisting of two layers of different material types. The top layer of the cube is nearly elastic, while the bottom layer is viscoelastic.

5.3 Comparing Two Codes

In this example, we will compare various codes

5.4 Analytic Comparison

(TODO: ask brad for his okada soln HDF5 file and link to it here)

When the analytic solution is known, we may use it directly when specifying a field argument to the `cigma compare` command, although this will require you to register your function as a built-in field. To demonstrate this capability, the current version of Cigma includes `disloc3d` as a built-in field.

(TODO: rerun these commands)

$cigma eval –function=disloc3d –output=tet4_1000m_qpts_displacement

Alternatively, you can also extract the required list of points on which you need to evaluate the exact solution. We can do this for the Pylith mesh in the previous section with the command

```
$ cigma extract
    --mesh=strikeslip.h5:/tet4_1000m/mesh/
    --output=tet4_1000m_qpts.h5:/tet4_1000m_qpts
```

You can now take the list of points in a global coordinate system, and apply your function. For demonstration purposes, we can use the disloc3d built-in function. The comparison can then take place

```
$ cigma compare
    --first=strikeslip.h5:/tet4_1000m/displacement/step0000
    --second=okada_solution.h5:/tet4_1000m_qpts_displacement
```

Here, we have taken the

Appendix A

File Formats

To differentiate between these formats, you will need to provide an appropriate extension for your input and output files. The supported extensions are currently `.h5` for HDF5, `.vtk` for VTK files, and `.txt` for simple text files.

A.1 Input File Format

There are about six different objects you will want to use as input to cigma, all of which can be represented as two-dimensional datasets. These are the

- mesh coordinates
- mesh connectivity
- field coefficients (degrees of freedom)
- quadrature rule points
- quadrature rule weights
- shape function tabulation matrix

A.1.1 HDF5

HDF5 files are organized in a hierarchical structure, similar to a UNIX file system. Two types of primary objects, groups and datasets, are stored in this structure. A group contains instances of zero or more groups or datasets, while a dataset stores a multi-dimensional array of data elements. Both kinds of objects are accompanied by supporting metadata known as attributes.

A dataset is physically stored in two parts: a header and a data array. The header contains miscellaneous metadata describing the dataset as well as information that is needed to interpret the array portion of the dataset. Essentially, it includes the name, datatype, dataspace, and storage layout of the dataset. The name is a text string identifying the dataset. The datatype describes the type of the data array elements. The dataspace defines the dimensionality of the dataset, i.e., the size and shape of the multi-dimensional array.

In Cigma you always provide an explicit path to every dataset, so you have a fair amount of flexibility for how you organize your datasets inside HDF5 files. For example, a typical Cigma HDF5 file could have the following structure.

```
model.h5
\__ model
    |__ mesh
    |   |__ coordinates    [nno x nsd]
    |   \__ connectivity   [nel x ndof]
```

```
\__ variables
    |__ temperature
    |   |__ step00000  [nno x 1]
    |   |__ step00010  [nno x 1]
    |   \...
    |__ displacement
    |   |__ step00000  [nno x 3]
    |   |__ step00010  [nno x 3]
    |   \...
    \__ velocity
        |__ step00000  [nno x 3]
        |__ step00010  [nno x 3]
        \...
```

Generally, Cigma will only require you to specify the path to a specific field dataset. If a small amount of metadata is present in your field dataset, the rest of the required information, such as which mesh and finite elements to use, will be deduced from that metadata.

MeshID an identifier assigned for use in linking child datasets to their parent mesh.

MeshLocation points to the HDF5 group which contains the appropriate coordinates and connectivity datasets.

FunctionSpace string identifier to determine which shape functions to use for interpolating values inside the element (e.g., tet4, hex8, quad4, tri3, ...).

DatasetType string identifier for classifying the type of data contained in the dataset (e.g., points, connectivity, degrees of freedom, quadrature rules, global quadrature points, global field values).

A.1.2 VTK

The current version of Cigma only supports VTK unstructured grid datasets of a single element type. You can still compare various unstructured grids with different element types to each other.

Note that while you typically provide a path (or name) for every dataset, this is not necessary when specifying a VTK mesh, since this data is taken from the special Points and Cells arrays, which you cannot rename. However, you will still need to provide a name when referring to the shape function coefficients, which are assumed to be stored as Point Data in the input VTK file.

For more information, you may want to refer to Visualization ToolKit's File Formats (`www.vtk.org/pdf/file-formats.pdf`) document.

A.1.3 Text

The text format is always in table form, with the dimensions of the table specified in the first line. For example, mesh coordinates can be specified in the following format

```
nno nsd
x1 y1 z1
x2 y2 z2
x3 y3 z3
...
```

Mesh connectivity with

```
nel ndofs
node_1 node_2 node_3 node_4 ...
node_1 node_2 node_3 node_4 ...
node_1 node_2 node_3 node_4 ...
...
```

A generic field with `ndim` components (i.e., scalar, vector, or tensor) is specified by

```
num ndim
f1 f2 f3 ..
f1 f2 f3 ..
...
```

In this last case, the number of rows could refer to either `nno` or `nel`, depending on whether the field is node-based or cell-based.

A.2 Output File Format

The output format for residuals consists simply of a list of scalars for each element in the discretization. If you specify a `.vtk` extension for your output file, this will result in an scalar dataset named epsilon stored using the legacy VTK file format in the Cell Data section of the output file. Note that this output consists of the squared values of the local residuals, so further post-processing will be necessary.

Appendix B

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