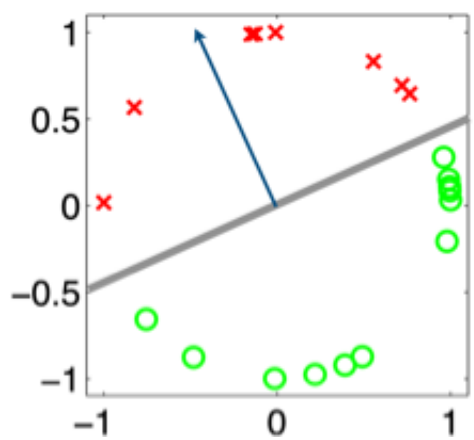
The perceptron algorithm

The perceptron algorithm is a simple method for learning a linear classifier. It works on a stream of examples (x_t, y_t) where x_t is in some vector space V and $y_t \in \{-1, 1\}$.

The state of the perceptron algorithm is a vector $w_t \in V$ that represents our linear classifier: we predict according to whether w_t has positive or negative inner product with the next example,

$$\hat{y}_t = \mathrm{sgn}(w_t \cdot x_t)\,.$$

(We can break ties arbitrarily if $w_t \cdot x_t = 0$.)



The perceptron algorithm initializes w_1 to 0, and updates w_t as it processes the stream of examples. The perceptron update has three cases:

- If we predict correctly for example t (that is, if $y_t = \mathrm{sgn}(x_t \cdot w_t)$), then we keep $w_{t+1} = w_t$.
- If we make a mistake on a positive example, we update $w_{t+1} = w_t + x_t$.
- If we make a mistake on a negative example, we update $w_{t+1} = w_t - x_t$.

The perceptron update makes sense since it moves us toward a correct prediction. For example, on a positive example $x_t \neq 0$, if we see the same x_t again, our dot product increases:

$$w_{t+1} \cdot x_t = w_t \cdot x_t + x_t \cdot x_t > w_t \cdot x_t\,.$$

Mistake bound

The perceptron algorithm satisfies many nice properties. Here we'll prove a simple one, called a mistake bound: if there exists an optimal parameter vector w^* that can classify all of our examples correctly, then the perceptron algorithm will make at most a bounded number of mistakes before discovering some optimal parameter vector.

In more detail, suppose that $w^* \cdot x_t \geq \epsilon$ for all positive examples, and $w^* \cdot x_t \leq -\epsilon$ for all negative ones. Also assume that our examples are bounded: there is a constant U such that $\|x_t\| \leq U$ for all t.

Then, we will show that the perceptron algorithm will make at most

$$\frac{U^2 \|w^*\|^2}{\epsilon^2}$$

mistakes in total. For example, if our examples have norm at most 2, if our optimal parameter vector has norm $\|w^*\| = 3$, and if $\epsilon = \frac{1}{2}$, then the number of mistakes M satisfies

$$M \leq 144\,.$$

Tools

In our proof we'll use some properties of inner product spaces. One of the key ones is Hölder's inequality: for any two vectors u, v, we have

$$u \cdot v \leq \|u\| \, \|v\|$$

We'll also use the usual axioms for addition, scalar multiplication, norm, and inner product, such as the fact that inner product distributes over addition and the fact that $\|u\|^2 = u \cdot u$.

Proof I:

First we show a lower bound on $w_t \cdot w^*$. We'll use induction and proof by cases.

After a mistake on a positive example, we have

$$\begin{aligned} w_{t+1} \cdot w^* &= w_t \cdot w^* + x_t \cdot w^* \\ &\geq w_t \cdot w^* + \epsilon \end{aligned}$$

since, by assumption, $x_t \cdot w^* \geq \epsilon$. Similarly, on a negative example, we have

$$\begin{aligned} w_{t+1} \cdot w^* &= w_t \cdot w^* - x_t \cdot w^* \\ &\geq w_t \cdot w^* + \epsilon \end{aligned}$$

since, by assumption, $x_t \cdot w^* \leq -\epsilon$. So, after M mistakes, we have

$$w_t \cdot w^* \geq \epsilon M$$

by induction: the LHS starts at 0, doesn't change when we predict correctly, and increases by at least ϵ with each mistake.

Proof II:

Next we show an upper bound on $\|w^t\|$. Again we use induction and proof by cases.

After a mistake on a positive example, we have

$$\begin{aligned} w_{t+1} \cdot w_{t+1} &= w_t \cdot w_t + 2w_t \cdot x_t + x_t \cdot x_t \\ &\leq w_t \cdot w_t + 0 + U^2 \end{aligned}$$

To see why, note that $w_t \cdot x_t \leq 0$, since we (mistakenly) classified this example as negative. And, $x_t \cdot x_t = \|x_t\|^2 \leq U^2$ by assumption.

Similarly, after a mistake on a negative example, we have

$$\begin{aligned} w_{t+1} \cdot w_{t+1} &= w_t \cdot w_t - 2w_t \cdot x_t + x_t \cdot x_t \\ &\leq w_t \cdot w_t + 0 + U^2 \end{aligned}$$

In this case, $w_t \cdot x_t \geq 0$, since we (mistakenly) classified this example as positive. So, after M mistakes, we have

$$w_t \cdot w_t \leq MU^2$$

since $w_t \cdot w_t$ starts at zero, doesn't change unless we make a mistake, and increases by at most U^2 on each mistake.

Equivalently, we have

$$\|w_t\|^2 \leq MU^2$$

for all t.

Proof III:

If we divide the conclusion of part I by ϵ and then square both sides, we get

$$M^2 \leq \left(\frac{w_t \cdot w^*}{\epsilon}\right)^2$$

(We are implicitly using $M \geq 0$ so that squaring preserves order.) By Hölder's inequality, we therefore have

$$M^2 \leq \left(\frac{\|w_t\|\|w^*\|}{\epsilon}\right)^2$$

Substituting in the conclusion of part II, we get

$$M^2 \leq \frac{MU^2\|w^*\|^2}{\epsilon^2}$$

and dividing through by M we get

$$M \leq \frac{U^2\|w^*\|^2}{\epsilon^2}$$

as claimed.