Homework 2

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Theorem 1: Suppose that $f(n) = O(g(n))$. Then $g(n) = \Omega(f(n))$.
Proof: We have that $f(n) = O(g(n))$. Then there exists constants $n_0 \geq 0$ and $c > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. Since $c > 0$, it follows that $\frac{1}{c} > 0$ and $\frac{1}{c} \cdot f(n) \leq g(n)$. Let $\varepsilon = \frac{1}{c}$ and let $n_0' = n_0$. Then there exists constants $\varepsilon > 0$ and $n_0' \geq 0$ such that for all $n \geq n_0'$, $g(n) \geq \varepsilon \cdot f(n)$ which is the definition of $g(n) = \Omega(f(n))$.

1. $\ln(n!), (\ln n)!$
   Since $(\ln n)! > \ln(n!)$ for all $n > 1500$, we have that $\ln(n!) = O((\ln n)!)$.

   I can't figure out how to show that $(\ln n)! \neq O(\ln(n!))$.

2. $3n^2 + 17n, \frac{1}{3}n^3$
   $3^2 + 17n = O(\frac{1}{3}n^3)$
   Proof:

$$\begin{aligned} 3n^2 + 17n &\leq 3n^3 + 17n^3 \quad \text{for } n \geq 1 \\ &= 20n^3 \\ &= 60 \cdot \frac{1}{3}n^3. \end{aligned}$$

   Thus, $n_0 = 1$ and $c = 60$.

   We need to show that $\frac{1}{3}n^3 \neq O(3n^2 + 17n)$. Thus we want to show that for all $c > 0$ and $n_0 \geq 0$, there exists an $n \geq n_0$ such that $\frac{1}{3}n^3 > c(3n^2 + 17n)$. Solving this inequality for n, we find that $n > \frac{1}{2}\left(9c + \sqrt{3}\sqrt{68c + 27c^2}\right)$. Thus, we can let $n = \frac{1}{2}\left(9c + \sqrt{3}\sqrt{68c + 27c^2}\right) + n_0 + 1 > n_0$ and we are done.

3. $n^{\frac{1}{\ln n}}, n^2$

   $n^{\frac{1}{\ln n}} = O(n^2)$
   Proof: We have that $2 > \frac{1}{\ln 2} \approx 1.4427$. Thus $2 > \frac{1}{\ln n}$ for $n \geq 2$. Thus $n^2 \geq n^{\frac{1}{\ln n}}$ for $n \geq 2$, with $n_0 \geq 2$ and $c = 1$.

   We want to show that $n^2 \neq O(n^{\frac{1}{\ln n}})$. Thus we want to show that for all $c > 0$ and $n_0 \geq 0$, there exists an $n \geq n_0$ such that $n^2 > c \cdot n^{\frac{1}{\ln n}}$. Taking the logarithm of both sides, we obtain $2 \ln n > \ln c + 1$. Thus whenever $n > e^{\frac{\ln c + 1}{2}}$ the inequality holds true. Thus, we can let $n = \max(e^{\frac{\ln c+1}{2}}, n_0) + 1$ and we are done.

4. $(\ln n)^{\ln n}, n^{\ln \ln n}$.

Claim: $(\ln n)^{\ln n} = n^{\ln \ln n}$.
Proof: Let $f(n) = (\ln n)^{\ln n}$ and $g(n) = n^{\ln \ln n}$. It is sufficient to show that $\ln f(n) = \ln g(n)$. Since $\ln f(n) = (\ln n)(\ln \ln n)$ and $\ln g(n) = (\ln \ln n)(\ln n)$, we are done.

Claim: $(\ln n)^{\ln n} = O(n^{\ln \ln n})$ and $n^{\ln \ln n} = \Omega((\ln n)^{\ln n})$.
Proof: Since we know that the two functions we are concerned with are equal, let $f(n) = (\ln n)^{\ln n}$. Note that $f(n) = n^{\ln \ln n}$ as well. Thus we have that:

$$\begin{aligned} f(n) = (\ln n)^{\ln n} &\leq f(n) \\ &= n^{\ln \ln n} \\ f(n) = n^{\ln \ln n} &\leq f(n) \\ &= (\ln n)^{\ln n}, \end{aligned}$$

and we are done. This also shows that they are Θ of each other since the two original functions are equaliviant.

5. $\ln n, \dfrac{n}{\ln n}$.
Claim: $\ln n = O(\frac{n}{\ln n})$.
Proof: We need to show that there exists $c > 0$ and $n_0 \geq 0$ such that $c \cdot \frac{n}{\ln n} \geq \ln n$. Consider $\frac{n}{\ln n} > \ln n$. Since this is true for all $n \geq e$, we can let $n_0 = e$ and $c = 1$.

We want to show that $\frac{n}{\ln n} \neq O(\ln n)$. Thus we want to show that for all $c > 0$ and $n_0 \geq 0$, there exists and $n \geq n_0$ such that $\frac{n}{\ln n} > c \ln n$. This simplifies to $\frac{n}{(\ln n)^2} > c$. Since

$$\begin{aligned} \lim_{n \to \infty} \frac{n}{(\ln n)^2} &= \lim_{n \to \infty} \frac{n}{2 \ln n} \\ &= \lim_{n \to \infty} \frac{n}{2} = \infty, \end{aligned}$$

we can choose an $n' \in \mathbb{Z}^+$ such that $\frac{n'}{(\ln n')^2} > c$. Then let $n = \max(n', n_0) + 1$ and we are done.