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Point pattern matching algorithm invariant to geometrical transformation and distortion

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Abstract

In this study, we propose a new point pattern matching algorithm based on minimum feature relations. The matching algorithm can be divided into two phases: relaxation process (RP) and select-match-pair process (SMPP). Some experiments are also performed to demonstrate the feasibility and accuracy of the proposed algorithm.

Keywords: Minimum feature relations; Relaxation process; Select-match-pair process; Pairwise error bound

1. Introduction

Image matching plays a prominent role in many applications, e.g., registration (Brown, 1992), navigation, change detection (Tsang et al., 1994), stereomapping (Trivedi, 1991) and character recognition (Cheng et al., 1993). For those methods based on the entire image, the cost of matching increases according to the image size. A more economical approach involves extracting a discrete set of features from the image, and using those features for matching. The cost of this process grows with the number of features rather than the image size. For a large size of images, the feature-based method is preferable.

A flexible vision system should be able to perform its tasks under two kinds of transformations: geometric transformation (e.g., translation, rotation and scale change) and distortion due to noise. An adequate approach of pattern matching must be able to perform matching under geometric transformations (called invariant matching) and matching under adding or suppressing points and distortion due to noise (called inexact matching). Therefore, a good matching algorithm must work well under the following five defective conditions:
(1) adding or suppressing points,
(2) location distortion,
(3) rotation,
(4) scaling,
(5) translation.
Point pattern matching algorithms can be addressed from the selected features and the selected matching algorithms. The conventional features for matching are invariant features (Hu, 1962; Li, 1992), interpoint distance (Simon et al., 1972; Lavine et al., 1983), point pairs (Yuen et al., 1994; Ogawa, 1984), convex hull (Goshtasby and Stockman, 1985) and neighbor relations (Seidl, 1982; Ahuja, 1982). The conventional matching algorithms include tree

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matching (Zahn, 1972; Tropf, 1980), graph matching (Ogawa, 1986), search method (Gennery, 1981; Baird, 1984), relaxation matching (David, 1979; Ranade and Rosenfeld, 1980), and neural network (Spirkovska and Reid, 1992). Most investigators can not solve the matching problem under all the above five conditions except the works of Li (1992) and Spirkovska and Reid (1992). Li (1992) used the invariant features such as angle and relative length between points and Spirkovska and Reid (1992) used a high-order neural network to solve the matching problem. Their works, although successful, are impractical owing to their complex computation.

In this study, we investigate a relatively simple and rapid approach of matching between a reference pattern and its distorted version which includes the above five defective conditions. The selected feature is merely the location of points. The selected matching algorithm is the relaxation technique. The matching algorithm includes two phases: relaxation process (RP) and select-match-pair process (SMPP). The first phase applies minimum feature relations and relaxation technique to determine the relative probability of each possible matching pair; the second phase applies the concept of pairwise error bound (PEB) to determine the exact number of matching pairs. Experimental results demonstrate that the matching algorithms are invariant to geometric transformation and distortion.

The rest of this paper is organized as follows. Before describing the matching algorithm, some definitions and problem description are stated in Section 2. Section 3 provides a detailed description of the proposed matching algorithm. Section 4 offers the experimental results along with a discussion. Finally concluding remarks are made.

2. Definitions and problem description

2.1. Definitions

Before discussing the point pattern matching algorithm, the algorithm can be described by defining the following terminologies; point pattern, matching M, affine registration R, pairwise error ε, optimal affine registration R_0, and matching error.

Definition 1 (*Point pattern*). Consider two sets of point patterns: $P=\{p_i \mid i=1,2,\ldots,m\}$ and $Q=\{q_j \mid j=1,2,\ldots,n\}$, where p_i and q_j are described with their coordinates x and y and denoted as $p_i=(x_i, y_i)$ and $q_j=(x_j, y_j)$.

Definition 2 (*Matching M*). A matching M of size k ($k \geqslant 2$) is a one-to-one mapping from a subset of P of size k into a subset of Q of size k, which is written as $M=(m_1, m_2, \ldots, m_k)$ and read as p_1 matches with q_{m1}, p_2 matches with q_{m2}, and so forth. The other notations can be represented by $q_{mi}=M(p_i)$, which means that p_i matches with q_{mi}. For instance, if $P=\{p_1, p_2, p_3, p_4\}$, $Q=\{q_1, q_2, q_3, q_4, q_5\}$ and $M=(2, 4, *, 5)$, which represents that p_1 matches with q_2, p_2 matches with q_4, p_3 matches with none and p_4 matches with q_5. Moreover, the total matching pair k is equal to 3.

Definition 3 (*Affine registration R*). An affine registration $R=(t_x, t_y, \theta, s)$ (Sprinzak and Werman, 1994) is a one-to-one mapping from the Euclidean plane $\mathbb{R}^2$ onto itself, consisting of a composition of translation, rotation and scaling (where t_x and t_y are the translations of x and y coordinate, θ is the rotation, and s is the scaling factor). The affine registration of point q_j can be represented by $q'_j=R(q_j)=(x'_j, y'_j)$, or

$$\begin{bmatrix} x'_j \\ y'_j \end{bmatrix} = R\begin{bmatrix} x_j \\ y_j \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} s\cos\theta & -s\sin\theta \\ s\sin\theta & s\cos\theta \end{bmatrix}\begin{bmatrix} x_j \\ y_j \end{bmatrix}. \tag{1}$$

Definition 4 (*Pairwise error ε*). Given a matching M and a registration R, the pairwise error for point p_i is defined as

$$\varepsilon = \| R(M(p_i)) - p_i \|^2 = \| R(q_{mi}) - p_i \|^2 = \| q_{mi} - p_i \|^2. \tag{2}$$

Definition 5 (*Optimal affine registration R_0*). Consider two sets of point patterns P and Q with a matching M. That is, $p_i=(x_i, y_i)$ in P matches with $q_{mi}=(x_{mi}, y_{mi})$ in Q, $i=1,2,\ldots,k$. The optimal affine registration R_0 is an affine registration R such that the average pairwise error is minimum in the least square sense (Umeyama, 1991). This means finding (t_x, t_y, θ, s) of R such that

$$\frac{1}{k}\sum_{i=1}^{k}\left[(x_i - x'_{mi})^2 + (y_i - y'_{mi})^2\right] \tag{3}$$

is minimum, where

$$\begin{bmatrix} x'_{mi} \\ y'_{mi} \end{bmatrix} = R\begin{bmatrix} x_{mi} \\ y_{mi} \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} s\cos\theta & -s\sin\theta \\ s\sin\theta & s\cos\theta \end{bmatrix}\begin{bmatrix} x_{mi} \\ y_{mi} \end{bmatrix} \tag{4}$$

and k is the number of matching pairs between P and Q.

Definition 6 (*Matching error*). Consider two sets of point patterns P and Q with the matching M and the optimal affine registration R_0. The matching error is defined as the total pairwise error under the matching M and registration R_0.

2.2. Problem description

After defining some terminologies in the previous section, the problem to be resolved is described here. Consider two sets of point patterns P and Q, where P is the reference pattern and Q is the distorted version of P which includes the five defective conditions mentioned in the first section. The solution involves finding the optimal matching M and the corresponding optimal affine registration R_0. The matching error can be used as the similarity measure between the reference pattern and its distorted version.

3. The proposed matching algorithm

In this section, we introduce the proposed point pattern matching algorithm. Before describing the proposed matching algorithm, we first express the concept of minimum feature relation to be used in the matching algorithm. Then, details of the matching algorithm are stated. Next, the proposed matching algorithms can be divided into two phases. The first phase is relaxation process (RP), and the other is select-match-pair process (SMPP). Fig. 1 displays the functional flowchart.

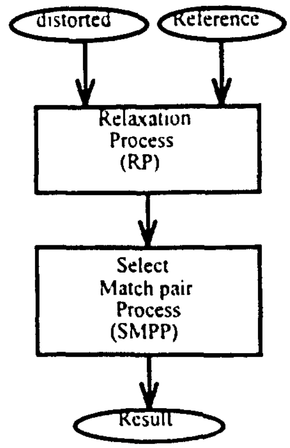

Fig. 1. The functional flowchart of the proposed algorithm.

3.1. Minimum feature relations

From the local feature point of view, how many points do we need to use their local relations of feature to solve the matching problem? Considering two sets of point pattern P and Q, how do we know whether p_i exactly matches with q_j? From the registration perspective, if p_i matches with q_j, there exists a registration R such that $p_i = R(q_j)$, or

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} s\cos\theta & -s\sin\theta \\ s\sin\theta & s\cos\theta \end{bmatrix}\begin{bmatrix} x_j \\ y_j \end{bmatrix}. \tag{5}$$

From Eq. (5), there are two equations with four unknown variables t_x, t_y, θ and s to be solved, so there exist infinitely many solutions. This means that there exist infinitely many R's that satisfy $p_i = R(q_j)$. This finding suggests that one matching pair cannot find the registration R. If the second matching pair is found, for example, p_h matches with q_l under the same registration R, then $p_h = R(q_l)$, or

$$\begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} s\cos\theta & -s\sin\theta \\ s\sin\theta & s\cos\theta \end{bmatrix}\begin{bmatrix} x_l \\ y_l \end{bmatrix}. \tag{6}$$

From Eqs. (5) and (6), there are four equations with four unknown variables t_x, t_y, θ and s to be solved, so there exists a unique solution. This means that every two matching pairs can exactly find a registration R. However, whether the registration R is a correct one can still not be determined. If the third matching pair exists, for example, p_w matches with q_t, then the registration R computed from $p_i = R(q_j)$ and $p_h = R(q_l)$ can be identified by p_w and q_t. If $p_w = R(q_t)$ is still holding (or within a preset bound), a local registration R can be assured. From the above analysis, we conclude that at least three matching pairs can identify a registration R and the minimum number of matching pairs is three. Therefore, the minimum feature relations to identify a registration R are the affine relations among three matching pairs.

3.2. Relaxation process (RP)

3.2.1. Initial probability

Each point pair from P and Q is initially assigned an initial matching probability 1. The probability of p_i matching with q_j at the rth iteration is assumed here to be expressed as $S^{(r)}(p_i, q_j)$, where r denotes the order of the iteration and $S^{(0)}(p_i, q_j)$ denotes the initial probability. The relaxation method iteratively updates the probability by an amount proportional to an estimate of its consistency.

3.2.2. Compatibility function C

From Definition 5, if three points $\{p_i, p_h, p_w\}$ match with $\{q_j, q_k, q_t\}$, an optimal affine registration R_0 can be found such that the average pairwise error is minimized in the least square sense. Let $p_i = (x_1, y_1)$, $p_h = (x_2, y_2)$, $p_w = (x_3, y_3)$ and $R_0(q_j) = (x'_1, y'_1)$, $R_0(q_k) = (x'_2, y'_2)$, $R_0(q_t) = (x'_3, y'_3)$. Then the minimum error E can be computed as follows:

$$E = \sum_{i=1}^{3} \left[(x_i - x'_i)^2 + (y_i - y'_i)^2 \right] = \rho^2. \tag{7}$$

The compatibility function $C(i, j \mid h, k; w, t)$ is then defined as

$$C = \frac{1}{1 + \mu E}, \tag{8}$$

where μ is a chosen positive constant. The above compatibility function denotes the support of p_i matching with q_j under the conditions of p_h matching with q_k and p_w matching with q_t.

3.2.3. The iteration scheme

The original iteration scheme is defined as

$$S^{(r)}(p_i, q_j) = \frac{1}{m} \sum_{h=1}^{m} \left[\max_{k=1}^{n} S^{(r-1)}(p_h, q_k) C(i, j \mid h, k) \right]. \tag{9}$$

The above equation denotes that the probability of the event "p_i matches with q_j" depends on the other event "p_h matches with q_k". In this paper, the probability of the event "p_i matches with q_j" depends on the other two events "p_h matches with q_k" and "p_w matches with q_t", so the iteration scheme may be modified as follows:

$$S^{(r)}(p_i, q_j) = \frac{1}{m^2} \sum_{h=1}^{m} \sum_{w=1}^{m} \left\{ \max_{k,t}^{n} \left[S^{(r-1)}(p_h, q_k) \times S^{(r-1)}(p_w, q_t) C(i, j \mid h, k; w, t) \right] \right\}. \tag{10}$$

After r iterations, a probability matrix $S^{(r)}(p_i, q_j)$ can be obtained.

3.3. Select-match-pair process (SMPP)

After the relaxation process, the optimum matching pairs from P and Q must be chosen from the probability matrix $S^{(r)}(p_i, q_j)$. Before introducing the SMPP, the constraints in selecting matching pairs should be first known. Because matching p_i with q_j is a one-to-one mapping, only one element in each column and row can be selected. According to this constraint, an algorithm called SMPP is developed to obtain the optimum matching pairs between P and Q. The algorithm is listed below.

SMPP algorithm

Given: Given two sets of points patterns P and Q and the probability matrix $S^{(r)}(p_i, q_j)$.
Goal: To find the number of matching pairs and the corresponding matching pairs.
Step 1: Set $k = \min(m, n)$.
Step 2: Set $no = k$, If $no < 3$ then $k = 0$ and stop.
Step 3: Load the probability matrix $S^{(r)}(p_i, q_j)$.

Step 4: Select the greatest value in the probability matrix, and clear all elements of its corresponding row and column.
Step 5: $no = no - 1$. If $no > 0$ then go to Step 4.
Step 6: Compute the total pairwise error and registration R under the selected matching M.
Step 7: If total pairwise error < threshold then stop; otherwise $k = k - 1$ and go to Step 2.

After the above process, the final number of matching pairs and the corresponding optimal matching M and the optimal affine registration R_0 can be found.

In Step 6, the total pairwise error is computed. If the error is less than a threshold, we can conclude that the matching is as good as desired and the final number of matching pairs is equal to k. Also, the threshold is determined by the number of matching pairs and the pairwise error bound (PEB). Moreover, the pairwise error bound is defined as the tolerance of each matching pair and the threshold is equal to $k * \text{PEB}$.

4. Experimental results and discussions

In order to verify the feasibility and accuracy of the proposed algorithm, we first generate a reference pattern as shown in Fig. 2(a) which is composed of fifteen random points. According to the five defective conditions mentioned in Section 1, we generate several distorted versions of the reference pattern. Fig. 2(b) denotes a subset of the reference pattern by discarding the points 10 to 15. Fig. 2(c) is obtained from the pattern of Fig. 2(b) such that the coordinates of points are randomly perturbed within two units and three extra points 16, 17 and 18 are added. Fig. 2(d) is a rotated (90 degree counter-clockwise) version of Fig. 2(c) and Fig. 2(e) is a scaled (doubly enlarged) and shifted version of Fig. 2(d). Fig. 2(f) is another set of point patterns which have no matching pairs between Fig. 2(a) and Fig. 2(f).

In the experiment, the pairwise error bound (PEB) is set to 4. According to Figs. 2(c) to 2(e), the maximum perturbation for x and y coordinates is two units. This means that the maximum difference of x and y coordinates for each matching point between the reference and its distorted version is

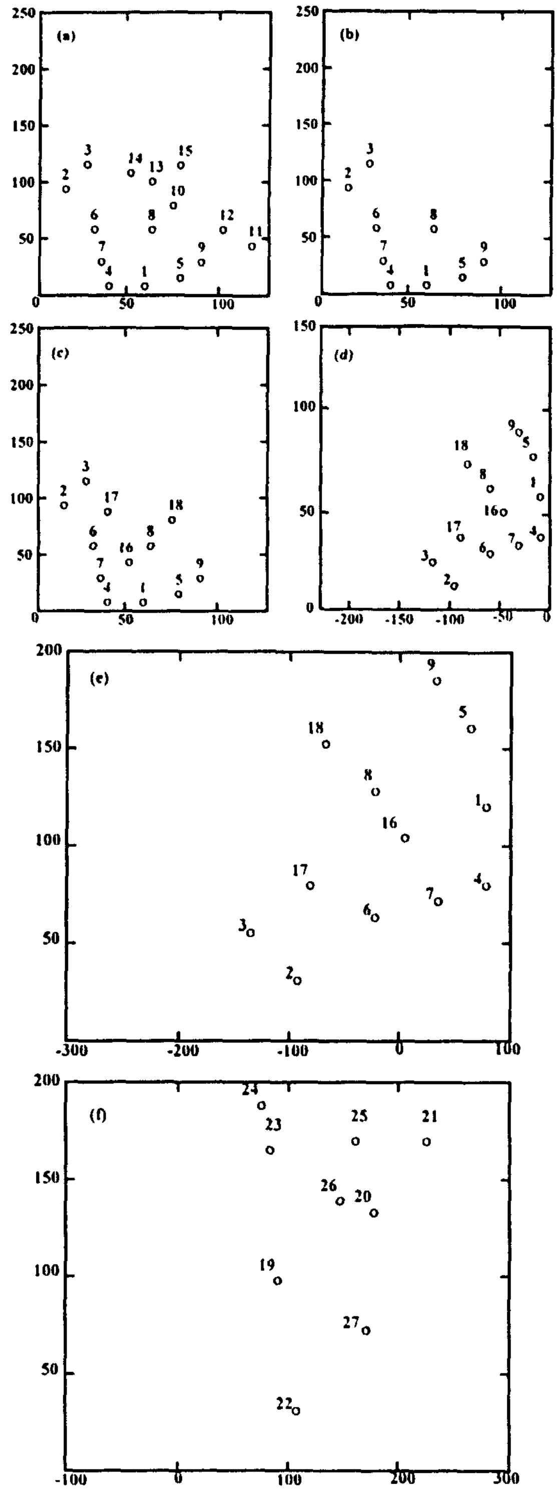

Fig. 2. An example of model point pattern and its geometric-transformed and distorted pattern.

Table 1
The matching result of Fig. 2(a) and Fig. 2(b)

Reference pattern	Fig. 2(a)
Distorted pattern	Fig. 2(b)
Reference point set	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
Distorted point set	{1, 2, 3, 4, 5, 6, 7, 8, 9}
Matching	$M=(1,2,3,4,5,6,7,8,9,*,*,*,*,*,*)$
Registration	$R=(0,0,0,1)$
Matching error	0
Average pairwise error	0
Pairwise error bound	4

Table 2
The matching result of Fig. 2(a) and Fig. 2(c)

Reference pattern	Fig. 2(a)
Distorted pattern	Fig. 2(c)
Reference point set	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
Distorted point set	{1, 2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 18}
Matching	$M=(1,2,3,4,5,6,7,8,9,*,*,*,*,*,*)$
Registration	$R=(-1.2,-1.1,0.26,1.0)$
Matching error	19.7
Average pairwise error	2.19
Pairwise error bound	4

within two units. Therefore, the maximum pairwise error is equal to 8 (i.e. $(\sqrt{2^2+2^2})^2=8$). Since the pairwise error is from 0 to 8, we choose 4 as the pairwise error bound. The threshold is equal to the

Table 3
The matching result of Fig. 2(a) and Fig. 2(d)

Reference pattern	Fig. 2(a)
Distorted pattern	Fig. 2(d)
Reference point set	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
Distorted point set	{1, 2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 18}
Matching	$M=(1,2,3,4,5,6,7,8,9,*,*,*,*,*,*)$
Registration	$R=(-1.2,-1.1,-89.74,1.0)$
Matching error	19.7
Average pairwise error	2.19
Pairwise error bound	4

Table 4
The matching result of Fig. 2(a) and Fig. 2(e)

Reference pattern	Fig. 2(a)
Distorted pattern	Fig. 2(e)
Reference point set	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
Distorted point set	{1, 2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 18}
Matching	$M=(1,2,3,4,5,6,7,8,9,*,*,*,*,*,*)$
Registration	$R=(-3.8,5.0,-89.74,0.5)$
Matching error	19.7
Average pairwise error	2.19
Pairwise error bound	4

pairwise error bound times the number of matching pairs.

Tables 1 to 4 summarize the matching results between Fig. 2(a) and Figs. 2(b) to 2(e). From the experimental results, we can actually find the optimal matching M and its corresponding registration R. From Table 4, for example, the matching M is $(1,2,3,4,5,6,7,8\ 9,*,*,*,*,*,*)$ and the registration is $(-3.8, 5.0, -89.74, 0.5)$. This result means that there are nine matching pairs between the distorted and reference pattern and the distorted pattern is doubly enlarged, counter-clockwise rotated and shifted by a vector of $(-3.8, 5.0)$. For Fig. 2(a) and Fig. 2(f), the matching algorithm finds that the number of matching pairs is less than 3, so there is no matching between them. From the experimental results, we conclude that the algorithm can work well under geometrical transformation and distortion of point patterns.

Because most investigators cannot solve the matching problem under all the five defective conditions as mentioned in Section 1 except Li (1992) and Spirkovska and Reid (1992), we only compare the computing speed between our algorithm and the works mentioned. Spirkovska and Reid used a high-order neural network to solve the matching problem, so its computing time is very slow. The computational complexity of our proposed algorithm is almost the same as that of the work of Li, but the actual computing speed of our algorithm is faster than that of Li. The reason is obvious that Li used complex features such as angle and relative length between points to solve the matching problem.

5. Conclusions

In this study, we have proposed a new point pattern matching algorithm based on minimum feature relations. The matching algorithm can be divided into two phases: relaxation process (RP) and select-match-pair process (SMPP). The RP applies the relaxation technique based on the concept of minimum feature relations to determine the relative probability of each possible matching pair. The geometrical registration obtained from the relaxation process enables the matching algorithm to be invariant to translation, rotation and scale change. The SMPP uses a pairwise error bound (PEB) not only to determine the exact number of matching pairs but also to compute their matching error. The ability of noise immunity is controlled by PEB.

Experiments are also performed to show the feasibility and accuracy of the proposed algorithm. The experimental results show that the proposed algorithms are not only invariant to translation, rotation and scaling but can also resist distortions. The computational complexity of the proposed algorithm is the same as that of the work of Li. However, the actual computing speed of the proposed algorithm is faster than that of Li because the selected feature of our algorithm is extremely simple.

The registration R in this paper is an affine transformation in the 2D case. The same concept can be applied to the 3D case. In addition, the planar features considered here are point features. The extension of the same concept may be applied to match line features or non-linear features such as curves or curved surfaces.

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References

Ahuja, N. (1982). Dot pattern processing using voronoi neighborhoods. *IEEE Trans. Pattern Anal. Mach. Intell.* 4 (3), 336–343.

Baird, H.S. (1984). Model based image matching using location. Ph.D. Dissertation, Princeton University.

Brown, L.G. (1992). A survey of image registration techniques. *ACM Comput. Surveys* 24 (4), 325–376.

Cheng, F.H., W.H. Hsu and M.C. Kuo (1993). Recognition of handprinted Chinese characters via stroke relaxation. *Pattern Recognition* 26 (4), 579–593.

David, L.S. (1979). Shape matching using relaxation techniques. *IEEE Trans. Pattern Anal. Mach. Intell.* 1 (1), 60–72.

Gennery, D.B. (1981). A feature-based scene matcher. *Proc. 7th Internat. Joint Conf. Artificial Intelligence*, 667–673.

Goshtasby, A. and G.C. Stockman (1985). Point pattern matching using convex hull edges. *IEEE Trans. Syst. Man Cybernet.* 15 (5), 631–637.

Hu, M.K. (1962). Visual pattern recognition by moment invariants. *IRE Trans. Inform. Theory* 8, 179–187.

Lavine, D., B.A. Lambird and L.N. Kanal (1983). Recognition of spatial point patterns. *Pattern Recognition* 16, 289–295.

Li, S.Z. (1992). Matching: Invariant to translations, rotations and scale changes. *Pattern Recognition* 25 (6), 583–594.

Ogawa, H. (1984). Labeled point pattern matching by fuzzy relaxation. *Pattern Recognition* 17 (5), 569–573.

Ogawa, H. (1986). Labeled point pattern matching by delaunay triangulation and maximal cliques. *Pattern Recognition* 19 (1), 35–40.

Ranade, S. and A. Rosenfeld (1980). Point pattern matching by relaxation. *Pattern Recognition* 12, 269–275.

Seidl, R.A. (1982). A theory of structure and encoding of visual patterns with applications to character recognition. Ph.D. Dissertation, Univ. Newcastle, Australia.

Simon, J.C., A. Checroun and C. Roche (1972). A method of comparing two patterns independent of possible transformations and small distortions. *Pattern Recognition* 4, 73–84.

Spirkovska, L. and M.B. Reid (1992). Robust position, scale and rotation invariant object recognition using high-order neural network. *Pattern Recognition* 25 (9), 975–985.

Sprinzak, J. and M. Werman (1994). Affine point matching. *Pattern Recognition Letters* 15 (4), 337–339.

Trivedi, H.P. (1991). Semi-analytic method for estimating stereo camera geometry from matched points. *Image and Vision Computing* 9 (2), 75–81.

Tropf, H. (1980). Analysis-by-synthesis search for semantic segmentation applied to workpiece recognition. *5th Internat. Conf. Pattern Recognition*, 241–244.

Tsang, P.W.M., P.C. Yuen and F.K. Lam (1994). Classification of partially occluded objects using 3-point matching and distance transformation. *Pattern Recognition* 27 (1), 27–40.

Umeyama, S. (1991). Least-squares estimation of transformation parameters between two point patterns. *IEEE Trans. Pattern Anal. Mach. Intell.* 13 (4), 376–380.

Yuen, P.C., P.W.M. Tsang and F.K. Lam (1994). Robust matching process: a dominant point approach. *Pattern Recognition Letters* 15 (12), 1223–1233.

Zahn, C. (1972). An algorithm for noisy template matching. *Proc. IFIP* 74, 727–732.