THE VARIATION OF THE SPECTRUM OF A NORMAL MATRIX

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If A and B are two normal matrices, what can be said about the "distance" between their respective eigenvalues if the "distance" between the matrices is known? An answer is given in the following theorem (in what follows, all matrices considered are $n \times n$; the Frobenius norm $\| K \|$ of a matrix K is $(\sum_{ij} | k_{ij} |^2)^{1/2}$).

THEOREM 1. *If A and B are normal matrices with eigenvalues $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ respectively, then there exists a suitable numbering of the eigenvalues such that $\sum_i | \alpha_i - \beta_i |^2 \leq \| A - B \|^2$.*

Proof. Let A_0 and B_0 denote the diagonal matrices with diagonal elements $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ in arbitrarily fixed order. Since A and B are normal, there are unitary matrices U and V such that $A = UA_0U^*$ and $B = UVB_0V^*U^*$. Then we have $\| A - B \| = \| A_0 - VB_0V^* \|$; hence, Theorem 1 is equivalent to

(1) *The minimum of $\| A_0 - VB_0V^* \|^2$, where V ranges over the set of all unitary matrices, is attained for $V = P$, where P is an appropriate permutation matrix.*

To prove (1), observe that

$$\begin{aligned} \| A_0 - VB_0V^* \|^2 &= \text{Trace}\,(A_0 - VB_0V^*)(A_0^* - VB_0V^*) \\ &= \text{Trace}\,(A_0A_0^* + B_0B_0^*) + r(V), \end{aligned}$$

where $r(V) = \sum_{ij} d_{ij}w_{ij}$; $d_{ij} = -(\alpha_i\bar{\beta}_j + \bar{\alpha}_i\beta_j)$, $w_{ij} = v_{ij}\bar{v}_{ij}$, $V = (v_{ij})$. Hence, min $\| A_0 - VB_0V^* \|^2$ is attained at a V for which $r(V)$ is a minimum.

Let $\mathfrak{X}_n$ be the set of all matrices $X = (x_{ij})$ such that

$$(2) \qquad \sum_i x_{ij} = 1, \qquad \sum_j x_{ij} = 1, \qquad x_{ij} \geq 0 \qquad (i, j = 1, \cdots, n).$$

Let $\mathfrak{W}_n$ be the set of all matrices $W = (w_{ij}) = (v_{ij}\bar{v}_{ij})$, with $V = (v_{ij})$ a unitary matrix. Then $\mathfrak{W}_n$ is a subset of $\mathfrak{X}_n$ (indeed, $\mathfrak{W}_n$ is a proper subset, if $n \geq 3$, in view of

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$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \\ 0 & \frac{1}{2} & \frac{1}{2} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

which is in $\mathfrak{X}_n$, but not in $\mathfrak{W}_n$).

Consider each X as a point in n^2-dimensional affine space whose co-ordinates are its coefficients. Then (2) implies that $\mathfrak{X}_n$ is a closed, bounded, convex polyhedron, and we shall show that (1) is implied by the following lemma.

LEMMA. *The vertices of $\mathfrak{X}_n$ are the permutation matrices.*

Proof. Other proofs of this lemma or generalizations of it are in the literature (see, for example, [1], [2]), but for the reader's convenience we give a simple *ad hoc* demonstration.

The polyhedron $\mathfrak{X}_n$ is the intersection of the $2n - 1$ hyperplanes and n^2 half-spaces given in (2) (the $2n$ equations given by the first two relations in (2) are clearly dependent). Hence, every vertex of $\mathfrak{X}_n$ must lie on the bounding hyperplane of at least $n^2 - (2n - 1)$ of the half-spaces; that is, $x_{ij} = 0$ for at least $(n - 1)^2$ pairs i, j. This shows that at least one column of any vertex consists entirely of 0 except for one entry, which must be 1; and the same must be true for the row containing that 1. If we delete this row and column, we obtain a matrix of order $n - 1$ that also satisfies conditions (2) if n is replaced by $n - 1$, and must also be a vertex of $\mathfrak{X}_{n-1}$. Hence, by induction, every vertex of $\mathfrak{X}_n$ has the property that each column (row) consists entirely of 0 except for one entry which is 1, *i.e.* every vertex is a permutation matrix. Since it is trivial that every permutation matrix is a vertex, the lemma is proven.

The set of points at which a linear form defined on a convex body attains its minimum always includes a vertex. Hence $\sum_{ij} d_{ij}x_{ij}$ attains its minimum at some permutation matrix). But since $\mathfrak{W}_n$ is a subset of $\mathfrak{X}_n$

$$\min_{\mathfrak{W}_n} \sum_{ij} d_{ij}w_{ij} \geq \min_{\mathfrak{X}_n} \sum_{ij} d_{ij}x_{ij} .$$

Since P is in $\mathfrak{W}_n$ as well as $\mathfrak{X}_n$, $\min_{\mathfrak{W}_n} \sum_{ij} d_{ij} w_{ij}$ is attained for $W = P$, thus $r(V)$ reaches its minimum for $V = P$. This completes the proof of (1) and hence of Theorem 1.

Remarks. 1. It is clear that essentially the same proof, with obvious changes, will also show that it is possible to renumber the eigenvalues so that

$$\sum_i | \alpha_i - \beta_i |^2 \geq || A - B ||^2.$$

2. Although the arrangement of the eigenvalues mentioned in Theorem 1 is difficult, in general, to describe more explicitly, it is easy in the special case that A is Hermitian. Then a "best" arrangement is

$$\alpha_1 \geq \cdots \geq \alpha_n ; \qquad \mathrm{Re}\, \beta_1 \geq \cdots \geq \mathrm{Re}\, \beta_n \tag{3}$$

Proof. Assume the α_i are in the order given in (3) and the β_i are not; say $\mathrm{Re}\, \beta_1 \not\geq \mathrm{Re}\, \beta_2$. Because

$$| \alpha_1 - \beta_1 |^2 + | \alpha_2 - \beta_2 |^2 \geq | \alpha_1 - \beta_2 |^2 + | \alpha_2 - \beta_1 |^2,$$

$\sum_i | \alpha_i - \beta_i |^2$ is not increased by interchanging β_1 and β_2. Hence, by successive steps, each consisting of an interchange of two β_i, we can bring the β_i to the order in (3) without increasing $\sum_i | \alpha_i - \beta_i |^2$. If the original arrangement is "best", then so is (3).

3. Theorem 1 is false if we do not require both A and B to be normal. Let $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} -1 & -1 \\ +1 & +1 \end{smallmatrix}\right)$. Then A is normal but B is not, and $|| A - B ||^2 = 12$; $\sum_i | \alpha_i - \beta_i |^2 = 16$ for any ordering of the eigenvalues.

4. Let us make precise the notion of "distance" between spectra. If $\alpha = \{\alpha_1, \cdots, \alpha_n\}$, $\beta = \{\beta_1, \cdots, \beta_n\}$ are each a set of n complex numbers, we define

$$d(\alpha, \beta) = \min \left(\sum_i | \alpha_i - \beta_{\sigma i} |^2\right)^{1/2}$$

where $(\sigma 1, \cdots, \sigma n)$ runs through all permutations of $(1, \cdots, n)$. Using this concept, Theorem 1 essentially gives a complete solution to the question: If A is a normal matrix with spectrum α and k is a positive number, what spectrum β can occur for a normal matrix B such that $|| A - B || \leq k$?

THEOREM 2. *If A is a normal matrix with spectrum α and k is a positive number, then β is the spectrum of a normal matrix B with $|| A - B || \leq k$ if and only if $d(\alpha, \beta) \leq k$.*

Proof. The necessity is given by Theorem 1. The sufficiency is easily demonstrated by letting A_0 be the diagonal matrix with entries $\alpha_1, \cdots, \alpha_n$, B_0 be the diagonal matrix with entries $\beta_1, \cdots, \beta_n$, numbered so that $d(\alpha, \beta) = (\sum_i | \alpha_i - \beta_i |^2)^{1/2}$. We know there is a unitary U such that $A = UA_0U^*$. Then $B = UB_0U^*$ has the required property.

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