1 Multivariable Calc

Okay, so multivariable calc has always been a weak point for me. That means I'm going to have to write some nice notes for this bit.

1.1 Divergence

First of all, what is divergence? The first description is for vector fields in $\mathbb{R}^3$. Given $\mathbf{F}$ a continuously differentiable vector field in $\mathbb{R}^3$, we have a function

$$\mathrm{div}\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}$$

$\mathrm{div}\mathbf{F}(p)$ measures how much "stuff" is coming out of the point p. Specifically, you can get it via flow.

$$\mathrm{div}\mathbf{F}(p) = \lim_{V \to \{p\}} \int\int_{S(V)} \frac{\mathbf{F} \cdot \mathbf{n}}{|V|} dS.$$

Okay. So here V is a volume containing p that shrinks to exactly p. So a source will have positive divergence, and a sink will have negative.

The much more common version: write $\mathbf{F} = U\mathbf{i} + V\mathbf{j} + W\mathbf{k}$.

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}.$$

But of course, there's the similar thing for $\mathbb{R}^n$.

1.2 Laplacian from Divergence

This is actually kind of neat. So if $f : \mathbb{R}^n \to \mathbb{R}$ is a function, then ∇f is a vector field. We then define the laplacian as $\mathrm{div}\nabla f$. So we get that

$$\Delta f = \nabla \cdot \nabla f.$$

Here's my heuristic. $f(p)$ keeps track of amount of stuff at point p. ∇f tells us if we get more stuff or less stuff by moving in some direction starting from p. I still don't get what Δf really means, but I've been looking at it for a while and I think I have to move on. I do at least have some things to say about it. Think of back in harmonic functions, where we were so interested in $\Delta u = 0$. This gives the function the mean value property, which we always thought was so cool. Now think about the discretization of the 2D laplacian, with the 4 on the diagonal and the -1's on the banded parts.

$$\mathrm{DiscreteLaplacian} \cdot \mathrm{vector} = 0$$

means that my vector has this discrete average property.

2 Total Variation Norm

Here's the situation. We're working in graph land. $\mathcal{V}$ is $\mathbb{R}^{\text{num vertices}}$. $\mathcal{E}$ is $\mathbb{R}^{\text{num edges}}$. We're defining the norm we're going to minimize over. They call it the TV norm but it looks like L^1? I should check again to see if I understand.

$$\begin{aligned} TV_w(u) &= \max \left\{ \langle \mathrm{div}_w \phi, u \rangle \;:\; \phi \in \mathcal{E}, \|\phi\|_\infty \leq 1 \right\} && (1) \\ &= \frac{1}{2} \sum_{x,y} w(x,y) \left| u(x) - u(y) \right|. && (2) \end{aligned}$$

3 Ratio Cut

Recall Ratio cut is

$$\min_{S \subseteq V} cut(S, S^c)^2 \left(\frac{1}{|S|} + \frac{1}{|S^c|} \right).$$

Note that if χ_S is an indicator function for S, then we have

$$TV_w(\chi_S) = \mathrm{cut}\left(S, S^c\right).$$

So then the square root of the ratio cut is the same as

$$\min_{u:\, u(x) \in \{0,1\}} \frac{TV_w(u)}{\left\| u - mean(u) \right\|_{L^2}}.$$

Theorem from the paper: Even when we allow u to be arbitrary real-valued, the exact answer to the problem is still a binary partition. This is pretty cool. It means that we can relax to $u : \mathcal{V} \to \mathbb{R}$ without changing final results.

From here, we claim that the ginsburg landau thing approximates TV, and we go on to minimize

$$\min_{u} \frac{GL_\epsilon(u)}{\left\| u - mean(u) \right\|_{L^2}}.$$

Use gradient descent on this thing to get the answer.