Weighted Graph Matching Problem (Umeyama 1988)

Given G, H two undirected weighted graphs, each with n nodes. Say that V_G, V_H are the nodes of the respective graphs, and W_G, W_H are the (symmetric) weight matrices). Want to create a bijection $V_G \leftrightarrow V_H$ that respects the weights. Can think of a bijection as a permutation on n letters. Represent it with a permutation matrix P. Then we define the energy of the matching as

$$J(P) = \left\| PW_G P^T - W_H \right\|^2 .$$

We try to minimize $J(P)$. In a perfect match, the weights would equal exactly and we would get $J(P) = 0$.

I think this problem is $NP-$complete. If not, it is at least unfeasible. So we introduced a relaxed version. Instead of minimizing $J(P)$ with P a permutation matrix, we minimize $J(Q)$ with Q and orthogonal matrix.

I'll skip the linear algebra background and just state the answer. Let

$$W_G = U_G \Lambda_G U_G^T$$

$$W_H = U_H \Lambda_H U_H^T$$

the eigendecompostions of the weight matrices. Then the matrices Q which minimize $J(Q)$ satisfy the formula

$$Q = U_H S U_G^T \tag{1}$$

where S is a diagonal matrix with any arrangements of $+1, -1$ on the diagonal.

This S is actually a problem. In the case where G, H are isomorphic, there exists an S such that Q is a permutation matrix, but not every S will work. We can't try all 2^n different choices for S, so we make one further approximation. In the case where we have a graph isomorphism given by P, and we choose the correct S, we have the following fact:

$$\begin{aligned} \operatorname{tr}(P^T U_H S U_G^T) &= \operatorname{tr}(P^T P) & (2)\\ &= n, & (3) \end{aligned}$$

and any other choice of P, S will result in a trace that is $\leq n$. Let $\bar{U}_G, \bar{U}_H$ be the matrices containing the absolute values of the elements in U_G, U_H. We get our approximate solution to the problem by choosing P a permutation that maximizes

$$\operatorname{tr}(P^T \bar{U}_H \bar{U}_G^T)$$

This last problem is tractable. The Hungarian Algorithm solves it exactly, and the runtime here is $O(n^3)$. See the page 4 of the Umeyama (1988) paper for a nice example.

Inexact Matching (Knossow 2010)

Choose $K < \min(|G|, |H|)$. Instead of U_G, U_H as above, restrict to the K eigenvectors of the Graph Laplacian corresponding to the smallest (nonzero) eigenvalues.