Note on Intensity Correlation Function of Light

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1 Classical Light

1.1 Definition of Correlation Function and Ergodic Assumption

Recall the definition of the second-order correlation function $g^{(2)}(\tau)$:

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}. \tag{1}$$

Typically, we think of $\langle ... \rangle$ as being a time average. Let us make two additional important assumptions.

1. We can measure intensity $I(t)$ instantaneously.
2. We take the time average to be over a time T that is greater than any other relevant time scale.

In this case, we can invoke the ergodic theorem and claim that time average $\langle ... \rangle$ is equivalent to statistical ensemble average. The reasoning is two-fold:

1. From classical and statistical mechanics, we know that a system is ergodic if it visits all available points in its phase space, and does so in a uniformly random trajectory.
2. Our assumptions above imply that we can accurately obtain all values of $I(t)$ that the system takes, and that our averaging time is long enough that the system probes all possible points in its phase space.

It then follows that taking time average $\langle ... \rangle$ will result in the same statistics that taking an ensemble average over all possible configurations would. Therefore, by assuming ergodicity throughout, we can proceed with explicitly computing $g^{(2)}(\tau)$.

1.2 Cauchy-Schwarz Inequality and Classical Bounds on $g^{(2)}(\tau)$

The Cauchy-Schwarz inequality holds for any vector space V endowed with an inner product, henceforth referred to as an inner product space. First we define what an inner product is.

Definition (Inner Product)**.** An inner product is a mapping $\langle \cdot, \cdot \rangle : V \times V \to F$, for vector space V over the field F, that satisfies the following three properties:

- Conjugate symmetry. $\forall \vec{x}, \vec{y} \in V$ and the overline denoting complex conjugate, the following is true:

$$\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}. \tag{2}$$

- Linearity in the first argument. $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and scalar $a, b \in F$, the following is true:

$$\langle a\vec{x} + b\vec{y}, \vec{z} \rangle = a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle. \tag{3}$$

- Positive definiteness. $\forall\, \vec{x} \in V$, the following is true:

$$\|\vec{x}\|^2 \equiv \langle \vec{x}, \vec{x} \rangle \geq 0, \tag{4}$$

with equality if and only if $\vec{x} = 0$.

The above definition is very general and applies to any vector space V and field F. Since classical intensities are always real-valued, we will only focus on $V = \mathbb{R}^n$ and $F = \mathbb{R}$. In this case, the inner product just becomes the scalar (dot) product in n dimensions.

Definition (Scalar Product in $\mathbb{R}^n$)**.** The scalar (dot) product in $\mathbb{R}^n$ is a generalization of the dot product in Euclidean space ($\mathbb{R}^3$), and is defined as follows:

$$\forall\, \vec{u} = (u_1, ...u_n), \vec{v} = (v_1, ..., v_n) \in \mathbb{R}^n, \qquad \langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i. \tag{5}$$

Theorem (Cauchy-Schwarz Inequality)**.** For any inner product space V with inner product $\langle , \rangle$ as defined above, the following inequality always holds:

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\| \qquad \forall\, \vec{u}, \vec{v} \in V \tag{6}$$

In $\mathbb{R}^2$, this theorem is obvious: $|\langle \vec{u}, \vec{v} \rangle| = \|\vec{u}\| \, \|\vec{v}\| \, |\cos\theta|$ by definition, and $|\cos\theta| \leq 1$ is also true by definition. However, the proof in $\mathbb{R}^n$ is less trivial. We state the corresponding inequality in $\mathbb{R}^n$ and its proof below for completeness.

Corollary (Cauchy-Schwarz Inequality in $\mathbb{R}^n$)**.** In $\mathbb{R}^n$, the Cauchy-Schwarz inequality can be written as:

$$\left(\sum_{i=1}^{n} u_i v_i \right)^2 \leq \left(\sum_{i=1}^{n} u_i^2 \right) \left(\sum_{i=1}^{n} v_i^2 \right), \qquad \forall\, \vec{u}, \vec{v} \in \mathbb{R}^n. \tag{7}$$

Proof. Consider the following real polynomial quadratic in x:

$$(u_1 x + v_1)^2 + ... + (u_n x + v_n)^2 \geq 0, \qquad \forall\, u_i, v_i \in \mathbb{R}. \tag{8}$$

The inequality is true because the LHS is a summation over individually square terms which are each non-negative. Expanding the LHS gives:

$$\left(\sum_{i=1}^{n} u_i^2 \right) x^2 + \left(2 \sum_{i=1}^{n} u_i v_i \right) x + \sum_{i=1}^{n} v_i^2 \geq 0. \tag{9}$$

Since the entire LHS is a non-negative quadratic, it can have at most 1 real root; in other words, it has a non-positive discriminant. Calculating the discriminant from equation (9) gives:

$$4 \left(\sum_{i=1}^{n} u_i v_i \right)^2 - 4 \left(\sum_{i=1}^{n} u_i^2 \right) \left(\sum_{i=1}^{n} v_i^2 \right) \leq 0. \tag{10}$$

Rearranging immediately gives us the desired result:

$$\left(\sum_{i=1}^{n} u_i v_i \right)^2 \leq \left(\sum_{i=1}^{n} u_i^2 \right) \left(\sum_{i=1}^{n} v_i^2 \right). \tag{11}$$

■

Now that we've proved the C-S inequality in $\mathbb{R}^n$, we can use it to obtain two useful results about $g^{(2)}(\tau)$ for classical light. First, from the ergodic assumption, we have that:

$$\langle I(t) \rangle = \frac{1}{N} \sum_{i=1}^{N} I(t_i); \qquad \langle I^2(t) \rangle = \frac{1}{N} \sum_{i=1}^{N} I^2(t_i); \qquad N \gg 1. \tag{12}$$

Next, suppose we take $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $u_i = I(t_i)$ and $v_i = 1$. Then, the C-S inequality gives:

$$\left(\sum_{i=1}^{N} I(t_i)\right)^2 \leq N \sum_{i=1}^{N} I^2(t_i). \tag{13}$$

Divide by N^2 to both sides, and we obtain:

$$\left(\frac{1}{N}\sum_{i=1}^{N} I(t_i)\right)^2 \leq \frac{1}{N}\sum_{i=1}^{N} I^2(t_i) \qquad \Longrightarrow \qquad \langle I(t)\rangle^2 \leq \langle I^2(t)\rangle. \tag{14}$$

Recalling the definition of $g^{(2)}(\tau)$, we thus have:

$$\boxed{\frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2} = g^{(2)}(0) \geq 1.} \tag{15}$$

Now, let's take two different vectors in $\mathbb{R}^n$. Suppose instead that we take $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $u_i = I(t_i)$ but $v_i = I(t_i + \tau)$. Then the C-S inequality gives:

$$\left(\sum_{i=1}^{N} I(t_i)I(t_i+\tau)\right)^2 \leq \left(\sum_{i=1}^{N} I(t_i)^2\right)\left(\sum_{i=1}^{N} I(t_i+\tau)^2\right). \tag{16}$$

Under the ergodic assumption, the two factors in the RHS product are equal, because the statistical distribution should be time-independent. Taking this into account, dividing both sides by N^2, and taking a square root gives:

$$\frac{1}{N}\sum_{i=1}^{N} (I(t_i)I(t_i+\tau)) \leq \frac{1}{N}\sum_{i=1}^{N} I(t_i)^2 \qquad \Longrightarrow \qquad \langle I(t)I(t+\tau)\rangle \leq \langle I^2(t)\rangle. \tag{17}$$

Finally, dividing both sides by $\langle I(t)\rangle^2$ and recalling the definitions of $g^{(2)}(0)$ and $g^{(2)}(\tau)$ gives:

$$\boxed{\frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} \leq \frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2} \qquad \Longleftrightarrow \qquad g^{(2)}(\tau) \leq g^{(2)}(0).} \tag{18}$$

The two inequalities we have derived for $g^{(2)}(\tau)$ are generically true for *any* classical statistical distribution of intensities. What does this mean? In this context, all it means is that as long as we can describe the light wave as a complex field $E = E_0(t)e^{i(\omega t+\phi(t))}$, then the intensity $I \sim |E|^2$ will have some statistical distribution over real number values. This is always true for so-called "classical light" but fails for "non-classical" light (which requires a quantum description).

1.3 $g^{(2)}(0)$ for Chaotic Light

An excellent question raised in a previous lecture was: Even though $g^2(0)$ classically has no upper bound, why does it seem (from quantum optics experiments) that there exists some sort of upper bound at $g^{(2)}(0) = 2$?

This question can be rigorously answered by considering the typical sources of light we encounter in Nature. Classically, there are two main types of sources: coherent and chaotic.

- Coherent light is what comes out of lasers, and can be simply modeled as a wave with perfect coherence for all time, i.e.no time-dependence in phase or amplitude. In this case, $g^{(2)}(\tau) = 1$ trivially.
- Chaotic light is typically due to the thermal motion of charged particles which radiate EM radiation (think light bulb, stars, basically any natural source of light). This can be modelled as a wave with non-trivial time-dependence in phase and/or amplitude.

Clearly, chaotic light is more interesting from the perspective of $g^{(2)}(\tau)$. The intensity fluctuations in time will create a non-trivial statistical distribution of intensities, for which we can calculate statistical quantities exactly. We do so below with the following simple model that captures all the physics.

The model we will consider is the **collisional broadening model**. Consider a thermal gas of ν charged particles where $\nu \gg 1$. Each individual particle emits radiation whose electric field can be described like:

$$E_i(t) = E_0 e^{-i\omega_0 t} e^{i\phi_i(t)}. \tag{19}$$

We then make the following assumptions:

- Each particle emits radiation at exact same frequency ω_0 and amplitude E_0 (with no time variation).
- Each particle has a different randomly chosen phase ϕ_i. The phase remains constant (i.e.coherent) until collisions occur between particles.
- Thermal motion causes collisions between particles. Each collision "resets" the phase of the participating particles to some random value.

With these assumptions, the total electric field emitted by the chaotic light source becomes:

$$E(t) = \sum_{i=1}^{\nu} E_i(t) = E_0 e^{-i\omega_0 t} \sum_{i=1}^{\nu} e^{i\phi_i(t)}. \tag{20}$$

People sometimes refer to chaotic light as "Gaussian". The terminology is confusing because there are also "Gaussian beams", as well as Gaussian vs Lorentizian frequency spectra. For the collisional broadening model considered here, we emphasize:

- The frequency spectrum (i.e.plot amplitude vs frequency) of collisionally-broadened chaotic light is **Lorentzian**.
- The total field is a sum over individual complex numbers of random phase and unit magnitude. This is equivalent to a complex number with phase $\varphi(t)$ and magnitude $a(t)$. The probability distribution of $a(t)e^{i\varphi(t)}$, when plotted in the complex plane, is **Gaussian**.

Now, we can proceed with calculating $g^{(2)}(0)$ for chaotic light. We need to compute $\langle I(t)\rangle$ and $\langle I^2(t)\rangle$ for this. As usual, we invoke the ergodic assumption so that we are taking statistical averages over all possible phase distributions. We find:

$$\langle I(t)\rangle = \frac{1}{2}\varepsilon_0 c E_0^2 \left\langle \left| \sum_{i=1}^{\nu} e^{i\phi_i(t)} \right|^2 \right\rangle \tag{21}$$

$$= \frac{1}{2}\varepsilon_0 c E_0^2 \nu. \tag{22}$$

In the second line, we used the following important reasoning: since all the different $\phi_i(t)$ independently randomly change phases, an ensemble average over $e^{i\phi_i(t)}$ for all such ϕ_i would be 0 (i.e.sinusoidal fluctuations will be washed out when averaging over all possible configurations). **Therefore, the only terms that survive in such "statistical average" expressions are terms that are not complex exponentials. Our strategy going forward is to identify all such terms.**

Proceeding, we find:

$$\langle I^2(t)\rangle = \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^2 \left\langle \left| \sum_{i=1}^{\nu} e^{i\phi_i(t)} \right|^4 \right\rangle. \tag{23}$$

All "cross-terms" where the complex exponential survives after taking 4 powers will average to 0. So there are only 2 "types" of terms that give non-zero values:

- "Self-terms", which are of the form $(e^{i\phi_i(t)} \cdot e^{-i\phi_i(t)})^2$. These were the only terms that survived in the $\langle I(t)\rangle$ calculation.
- "2-tuple cross-terms", which are of the form $(e^{i(\phi_i(t)+\phi_j(t))}) \cdot (e^{-i(\phi_i(t)+\phi_j(t))})$ for some 2-tuple (i,j) where $i \neq j$ and $i,j \in \{1,2,...,\nu\}$.

Keeping only these terms in the sum gives:

$$\langle I^2(t)\rangle = \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^2 \left\langle \left[\sum_{i=1}^{\nu} \left|e^{i\phi_i(t)}\right|^4 + \sum_{i>j}^{\nu} \left|2\, e^{i(\phi_i(t)+\phi_j(t))}\right|^2\right]\right\rangle, \tag{24}$$

where in the second summation we insert a factor of 2 to emphasize that the 2-tuple (i,j) gives the same result when swapping $i \leftrightarrow j$. Both summations are over complex numbers whose magnitude is 1, so we can drop the $\langle ...\rangle$ and count how many terms are in each sum.

- First sum is just from $i=1$ to $i=\nu$ so there are ν terms, each with value 1.
- Second sum is over all possible (i,j) pairs chosen from ν options, where order does not matter. There are $\binom{\nu}{2} = \frac{\nu(\nu-1)}{2}$ such options, each with value 1.

Hence we find:

$$\langle I^2(t)\rangle = \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^2 [\nu + 2\nu(\nu-1)] = \left(2 - \frac{1}{\nu}\right)\langle I(t)\rangle^2. \tag{25}$$

Since we work in the limit $\nu \gg 1$, we can ignore the fraction and thus conclude:

$$\boxed{g^{(2)}(0) = \frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2} = 2.} \qquad \text{[chaotic light.]} \tag{26}$$

1.4 $g^{(n)}(0)$ for Chaotic Light

The physical reasoning for obtaining $g^{(2)}(0)$ for chaotic light can be extended to higher order correlation functions, i.e.in principle up to the nth order (where we still assume $\nu \gg n$). To do so, we just need to compute $\langle I^n(t)\rangle$ where $n \in \mathbb{Z}^+$. We find:

$$\langle I^n(t)\rangle = \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^n \left\langle \left|\sum_{i=1}^{\nu} e^{i\phi_i(t)}\right|^{2n}\right\rangle. \tag{27}$$

As before, only two types of terms will survive:

- "Self-terms", which are now of the form $(e^{i\phi_i(t)} \cdot e^{-i\phi_i(t)})^n$. There will be ν of them.
- "n-tuple cross-terms", which are now of the form $(e^{i\sum_{(i_1,...,i_n)} \phi_i(t)}) \cdot (e^{-i\sum_{(i_1,...,i_n)} \phi_i(t)})$.

Keeping only these terms gives:

$$\langle I^n(t)\rangle = \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^n \left\langle \sum_{i=1}^{\nu} |e^{i\phi_i(t)}|^{2n} + \sum_{\forall(i_1,...,i_n)} \left|n!\, e^{i\sum_{(i_1,...,i_n)} \phi_i(t)}\right|^2\right\rangle, \tag{28}$$

where in the second summation we insert a factor of $n!$ to enforce that the n-tuple $(i_1,...,i_n)$ gives the same result under all possible permutations of the coordinates. Therefore, the second summation is only over all possible *combinations* (where order does not matter) of n indices drawn from ν options. Since the first summation will only scale like $\mathcal{O}(\nu)$, it is sub-dominant to the second summation and we drop it. Proceeding gives:

$$\langle I^n(t)\rangle = \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^n (n!)^2 \binom{\nu}{n} \approx \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^n \nu^n\, n!, \tag{29}$$

where in the last equality we used the fact that $\nu \gg n$. So it immediately follows that:

$$\boxed{g^{(n)}(0) = \frac{\langle I^n(t)\rangle}{\langle I(t)\rangle^n} = n!} \qquad \text{[chaotic light.]} \tag{30}$$

1.5 $g^{(2)}(\tau)$ for Chaotic Light

We can finally extend our collisional broadening model one step further to calculate $g^{(2)}(\tau)$ for any general τ for chaotic light. We start from the definition of $g^{(2)}(\tau)$. We note that, due to the ergodic assumption, statistical averages should be time-independent. Hence $\langle I(t)\rangle = \langle I(t+\tau)\rangle$. So we only need to calculate $\langle I(t)I(t+\tau)\rangle$.

We proceed by computing:

$$\langle I(t)I(t+\tau)\rangle = \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^2 \langle E^*(t)E^*(t+\tau)E(t)E(t+\tau)\rangle \tag{31}$$

$$= \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^2 \sum_{i,j,k,l}^{\nu} \left\langle e^{i(-\phi_i(t)-\phi_j(t+\tau)+\phi_k(t)+\phi_l(t+\tau))}\right\rangle, \tag{32}$$

where we used equation (20) to write the total field as the sum of individual fields from each emitter. Now, we must modify our reasoning slightly: whereas we're guaranteed that terms whose complex exponential argument is zero will survive, there is also a chance that correlation between the phase of a single emitter at times t and $t+\tau$ will lead to nonzero average even for a complex exponential with non-zero argument. We proceed by classifying the surviving terms:

- Self-terms correspond to $i = j = k = l$. As usual, there are ν of them.
- 2-tuple cross terms (i, j), where $i \neq j$, are classified into two types now.
  - $i = k$ and $j = l$. In this case the argument of the complex exponential cancels to 0, as before.
  - $i = l$ and $j = k$. In this case, it's not immediately obvious why it will survive averaging, so we write it out explicitly:

$$\left\langle e^{i(-\phi_i(t)-\phi_j(t+\tau)+\phi_j(t)+\phi_i(t+\tau))}\right\rangle = \left\langle e^{i(-\phi_i(t)+\phi_i(t+\tau))}\right\rangle \left\langle e^{i(\phi_j(t)-\phi_j(t+\tau))}\right\rangle, \tag{33}$$

    where we can break up the average of a product into a product of averages **solely because all emitters will have independent phase distributions, i.e. i should not affect j.**[1]

Keeping only these surviving terms gives:

$$\langle I(t)I(t+\tau)\rangle = \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^2 \left[\sum_{i=1}^{\nu} 1 + \sum_{i\neq j}^{\nu} 1 + \sum_{i\neq j}^{\nu} \left\langle e^{i(-\phi_i(t)+\phi_i(t+\tau))}\right\rangle \left\langle e^{i(\phi_j(t)-\phi_j(t+\tau))}\right\rangle\right]. \tag{34}$$

The first two sums give ν and $\nu(\nu-1)$, respectively. Putting everything together and recalling that $\langle I(t)\rangle^2 = \left(\frac{1}{2}\varepsilon_0 c E_0^2 \nu\right)^2$, we obtain:

$$\langle I(t)I(t+\tau)\rangle = \langle I(t)\rangle^2 + \left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^2 \sum_{i\neq j}^{\nu} \left\langle e^{i(-\phi_i(t)+\phi_i(t+\tau))}\right\rangle \left\langle e^{i(\phi_j(t)-\phi_j(t+\tau))}\right\rangle. \tag{35}$$

Now, we recall the definition of $g^{(1)}(\tau)$:

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle}. \tag{36}$$

[1] Thanks to Shaozhen for pointing this out, thereby enabling me to revisit the derivation from equation (32) to equation (45) and make it more rigorous.

As usual, writing $E(t)$ as a sum of ν independent random phase emitters, we find:

$$\langle E^*(t)E(t+\tau)\rangle = E_0^2 e^{i\omega_0\tau}\sum_{(i,j)}^{\nu}\left\langle e^{i(\phi_j(t+\tau)-\phi_i(t))}\right\rangle \tag{37}$$

$$= E_0^2 e^{i\omega_0\tau}\sum_{i=1}^{\nu}\left\langle e^{i(\phi_i(t+\tau)-\phi_i(t))}\right\rangle, \tag{38}$$

where we use the same reasoning that for $i \neq j$, the average of product equals product of averages. In that case, we obtain the product of average of two individual complex exponentials, each of which is zero.

Proceeding, we take the square modulus to obtain:

$$|\langle E^*(t)E(t+\tau)\rangle|^2 = E_0^4\left(\sum_{i=1}^{\nu}\left\langle e^{i(\phi_i(t+\tau)-\phi_i(t))}\right\rangle\right)\left(\sum_{j=1}^{\nu}\left\langle e^{i(\phi_j(t)-\phi_j(t+\tau))}\right\rangle\right) \tag{39}$$

$$= E_0^4\left(\nu + \sum_{i\neq j}^{\nu}\left\langle e^{i(-\phi_i(t)+\phi_i(t+\tau))}\right\rangle\left\langle e^{i(\phi_j(t)-\phi_j(t+\tau))}\right\rangle\right) \tag{40}$$

$$\approx E_0^4\sum_{i\neq j}^{\nu}\left\langle e^{i(-\phi_i(t)+\phi_i(t+\tau))}\right\rangle\left\langle e^{i(\phi_j(t)-\phi_j(t+\tau))}\right\rangle, \tag{41}$$

where in the second line we obtain ν from all $i = j$ self-terms. We justify the equality in the third line by noting that there are $\nu(\nu-1) \sim \mathcal{O}(\nu^2)$ terms in the sum (each with value of $\mathcal{O}(1)$), so for $\nu \gg 1$ it is valid to just look at leading order behavior in ν.

Finally, plugging equation (41) back into equation (35), we find:

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} = 1 + \frac{\left(\frac{1}{2}\varepsilon_0 c E_0^2\right)^2\sum_{i\neq j}^{\nu}\left\langle e^{i(-\phi_i(t)+\phi_i(t+\tau))}\right\rangle\left\langle e^{i(\phi_j(t)-\phi_j(t+\tau))}\right\rangle}{\langle I(t)\rangle^2} \tag{42}$$

$$= 1 + \frac{\left(\frac{1}{2}\varepsilon_0 c\right)^2}{\left(\frac{1}{2}\varepsilon_0 c\right)^2}\frac{E_0^4\sum_{i\neq j}^{\nu}\left\langle e^{i(-\phi_i(t)+\phi_i(t+\tau))}\right\rangle\left\langle e^{i(\phi_j(t)-\phi_j(t+\tau))}\right\rangle}{\langle E^*(t)E(t)\rangle^2} \tag{43}$$

$$= 1 + \frac{|\langle E^*(t)E(t+\tau)\rangle|^2}{\langle E^*(t)E(t)\rangle^2}. \tag{44}$$

So we conclude:

$$\boxed{g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2} \qquad \text{[chaotic light.]} \tag{45}$$

1.5.1 Collisional Broadening

We can use the formula derived in equation (45) to obtain closed-form expressions for several specific types of chaotic light. First, let's stick to our collisional model. To compute $g^{(1)}(\tau)$, we need to know $\langle E^*(t)E(t+\tau)\rangle$ from equation (38). We note that:

$$\left\langle e^{i(\phi_i(t+\tau)-\phi_i(t))}\right\rangle = \int_{\tau}^{\infty}\rho(t')\,dt', \tag{46}$$

where $\rho(t')\,dt'$ is the probability for the particle to freely fly without collisions from t' to $t'+dt'$. This makes sense because if the particle collides, its phase is reset to some random value and any correlation of phase at different points in time is lost.

The expression for $\rho(t')\,dt'$ comes from the kinetic theory of gases, and we will not derive it here. Instead, we quote the result:

$$\rho(t')dt' = \frac{1}{\tau_0}e^{-t'/\tau_0}dt', \qquad \text{where} \qquad \frac{1}{\tau_0} = \frac{4d^2N}{V}\sqrt{\frac{\pi k_B T}{M}}. \tag{47}$$

Here, N/V is the number density of the thermal gas of particles and d is the particle size. τ_0 can be thought of as the "coherence" time for reasons we will see shortly. By plugging this into the integral above, we obtain:

$$\left\langle e^{i(\phi_i(t+\tau)-\phi_i(t))}\right\rangle = \frac{1}{\tau_0}\int_\tau^\infty e^{-t'/\tau_0}\,dt' = e^{-\tau/\tau_0}. \tag{48}$$

Hence:

$$\langle E^*(t)E(t+\tau)\rangle = \nu E_0^2 e^{i\omega_0\tau}e^{-\tau/\tau_0}, \tag{49}$$

and:

$$g^{(1)}(\tau) = \frac{\nu E_0^2 e^{i\omega_0\tau}e^{-\tau/\tau_0}}{\nu E_0^2} = e^{i\omega_0\tau}e^{-\tau/\tau_0}. \tag{50}$$

Here we have been assuming $\tau > 0$, but if we allow for $\tau < 0$, we know the result has to be symmetric in time, so we replace τ with $|\tau|$. Then, from equation (45), we conclude:

$$\boxed{g^{(2)}(\tau) = 1 + e^{-2\gamma|\tau|}, \quad \gamma = \frac{1}{\tau_0}} \qquad \text{[collisionally-broadened chaotic light.]} \tag{51}$$

1.5.2 Doppler Broadening

In the Doppler broadened model, we assume the following:

- Each charged particle independently emits perfeclty coherent radiation (randomly distributed fixed phases).
- The emitted frequencies (as observed by us) are Doppler-shifted from resonance ω_0 based on the particle velocity (a Maxwell-Boltzmann thermal distribution).

In this case, we would find:

$$\langle E^*(t)E(t+\tau)\rangle = E_0^2\left\langle \sum_{i=1}^{\nu} e^{-i\omega_i\tau}\right\rangle, \tag{52}$$

where the ω_i are normally distributed about ω_0 and τ relates to the Doppler shift in frequency for each particle. Note that the sum is only over a single index i because, just like before, this correlation function is only "first-order" (product of two complex exponentials) - which corresponds to each charged particle contributing independently to the sum.

Due to statistical mechanics reasons that I will not dwell on here (see Maxwell-Boltzmann distribution), the statistical average gets converted into a Gaussian integral in the frequency ω_i:

$$\langle E^*(t)E(t+\tau)\rangle = \nu E_0^2(2\pi\delta^2)^{-1/2}\int_0^\infty e^{-i\omega_i\tau}e^{-\frac{(\omega_i-\omega_0)^2}{2\delta^2}}\,d\omega_i \tag{53}$$

$$= \nu E_0^2 e^{-i\omega_0\tau-\frac{1}{2}\delta^2\tau^2}, \tag{54}$$

where δ is the Doppler linewidth. So we conclude:

$$g^{(1)}(\tau) = e^{-i\omega_0\tau}e^{-\frac{1}{2}\delta^2\tau^2}, \tag{55}$$

and:

$$\boxed{g^{(2)}(\tau) = 1 + e^{-\delta^2\tau^2}, \qquad \delta = \omega_0\sqrt{\frac{k_BT}{Mc^2}}} \qquad \text{[Doppler-broadened chaotic light.]} \tag{56}$$

As a final sanity check, we can plot our results in Figure 1 to see what they look like.

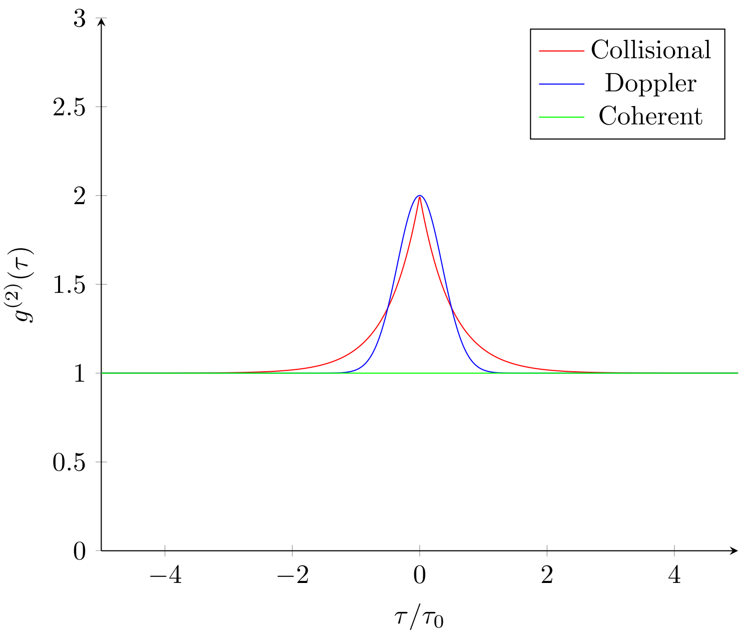

Figure 1: Plot of $g^{(2)}(\tau)$ vs τ/τ_0 for 2 types of chaotic light (Collisional and Doppler broadened) and for coherent light. Here we assume Doppler linewidth $\delta = 2/\tau_0$. As derived, both chaotic sources of light have maximal $g^{(2)}$ at $\tau = 0$, with a value of 2.

2 Non-Classical Light

Now, let's consider examples of non-classical light. We mentioned in a previous lecture that non-classical light satisfies $g^{(2)}(0) < 1$. The only way this is possible while remaining consistent with the C-S inequality is that non-classical light cannot be described by a statistical distribution of real-valued intensities. Instead, we have to go to a full quantum picture where the electric field and intensity are promoted to operators.

2.1 Quantum Picture of Intensity Measurement

A key tenet of quantum mechanics is that measurement of a system affects the state of the system. Naïvely, since we know that intensity is proportional to photon number, we would think quantities like $I^2(t) \sim \hat{n}^2$. This turns out not to be the case.

We view an intensity measurement as measuring the absorption of photons from the source light onto a photodetector. Simplistically, the photodetector is made up of many atoms which get ionized and produce photoelectrons (current). We can use quantum mechanics to model this. Recall the E-field operator:

$$\hat{\vec{E}}(\vec{r}, t) \sim \hat{a} - \hat{a}^\dagger \equiv \hat{\vec{E}}^{(+)} + \hat{\vec{E}}^{(-)}, \tag{57}$$

where we identify $\hat{a}$ ($\hat{a}^\dagger$) with absorption (emission) of a field photon. $\hat{\vec{E}}^{(+)}$ ($\hat{\vec{E}}^{(-)}$) is the part of the operator concerning absorption (emission).

Next, we take the initial state of the atom-field system to be $|I\rangle = |g\rangle |i\rangle$ where $|g\rangle$ is ground state of atom and $|i\rangle$ is initial state of source field. The final state of the atom-field system is $|F\rangle = |e\rangle |f\rangle$ where $|e\rangle$ is ionized state of atom and $|f\rangle$ is final state of source field. The interaction Hamiltonian is E1 term $\hat{H}^{(I)} = -\hat{\vec{d}} \cdot \hat{\vec{E}}$. Hence the transition probability for a single photon "detection" (i.e.absorption) to occur is:

$$|\langle F|\hat{H}^{(I)}|I\rangle|^2 = \left|\langle e|\hat{\vec{d}}|g\rangle\right|^2 \left|\langle f|\hat{\vec{E}}^{(+)}|i\rangle\right|^2. \tag{58}$$

We care about the field, whose transition probability is given by the second factor $\left|\langle f|\hat{\vec{E}}^{(+)}|i\rangle\right|^2$. To obtain the "final" state of the photodetector, we have to trace over all possible final states of the field. The corresponding transition probability would look like:

$$\sum_f \left|\langle f|\hat{\vec{E}}^{(+)}|i\rangle\right|^2 = \langle i|\hat{\vec{E}}^{(-)} \cdot \hat{\vec{E}}^{(+)}|i\rangle\,, \tag{59}$$

assuming that the final states of the field form a complete set. The conclusion we draw from this semi-sketchy derivation is that the operator relevant to single photon absorption on the photodetector (and hence the intensity measurement) is $\hat{\vec{E}}^{(-)} \cdot \hat{\vec{E}}^{(+)}$. Unsurprisingly, this is proportional to photon number operator $\hat{n} = \hat{a}^\dagger \hat{a}$.

Now, we can extend the analogy to derive $g^{(2)}(\tau)$. Since $g^{(2)}$ is a measure of spacetime correlation in "two-photon absorption", the transition probability for this to occur on our detectors would look like:

$$\langle i|\hat{\vec{E}}^{(-)}(\vec{r}_1,t_1) \cdot \hat{\vec{E}}^{(-)}(\vec{r}_2,t_2) \cdot \hat{\vec{E}}^{(+)}(\vec{r}_2,t_2) \cdot \hat{\vec{E}}^{(+)}(\vec{r}_1,t_1)|i\rangle\,. \tag{60}$$

This means the relevant operator is $\hat{\vec{E}}^{(-)}\hat{\vec{E}}^{(-)}\hat{\vec{E}}^{(+)}\hat{\vec{E}}^{(+)} \sim \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$. Using the ergodic assumption again (so that in the quantum picture, $\langle ... \rangle$ becomes an expectation value), we conclude:

$$\boxed{g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\rangle}{\langle \hat{a}^\dagger\hat{a}\rangle^2}} \qquad \text{[quantum picture.]} \tag{61}$$

Two things are worth pointing out here:

- The expression for $g^{(2)}(\tau)$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ is valid for all types of light (even coherent and chaotic, not just non-classical). The reason is that quantum mechanics is always true - it's not only valid in the non-classical regime.
- From our derivation above, it's clear our $g^{(2)}(\tau)$ does not even depend on time for stationary states (eigenstates) of the operators in the brackets.

2.2 Photon Number States

As mentioned in a previous lecture, the photon number states are non-classical. They have zero uncertainty in photon number by definition ($\Delta\hat{n} = 0$) and thus a "definite photon" emitter (like single-photon emitter) exhibits photon anti-bunching. We can compute $g^{(2)}(\tau)$ for photon number states as follows.

First, using the commutator $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$, we can write $\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} = \hat{n}^2 - \hat{n}$. Hence this gives:

$$g^{(2)}(\tau) = \frac{\langle \hat{n}(\hat{n} - \hat{\mathbf{1}})\rangle}{\langle \hat{n}\rangle^2}. \tag{62}$$

Recall the variance of photon number is given by $(\Delta\hat{n})^2 = \langle \hat{n}^2\rangle - \langle \hat{n}\rangle^2$. Hence we can restructure:

$$g^{(2)}(\tau) = 1 + \frac{(\Delta\hat{n})^2 - \langle \hat{n}\rangle}{\langle \hat{n}\rangle^2}. \tag{63}$$

Definite photon number state $|n\rangle$ have $\Delta\hat{n} = 0$, so we conclude:

$$\boxed{g^{(2)}(\tau) = g^{(2)}(0) = 1 - \frac{1}{n} < 1} \qquad \text{[photon number states.]} \tag{64}$$

$g^{(2)}(\tau) = 0$ (minimum value) when we have a single photon ($n = 1$), which corresponds to perfect antibunching. The formula is not valid for vacuum ($n = 0$) because of 0/0 division in the preceding equation (63). So we have found that $g^{(2)}(0) < 1$ can clearly be possible for non-classical states of light.