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5. Perturbations

5.1 Introduction to perturbation theory

[R05 7.1]

→ The point-mass two-body problem can be solved analytically, but when we have more than two bodies or if the point mass approximation is not justified (for instance, at least one of the two bodies is not spherical), we do not have analytic solutions.

→ In general the motion is governed by a (positive) potential function $U = U_0 + \mathcal{R}$, where U_0 is the point-mass two-body (positive) gravitational potential and $\mathcal{R}$ is the so-called disturbing function, which accounts for the presence of other bodies or for deviations from spherical symmetry in the mass distributions of the bodies.

→ Perturbation theory is a way to account for the presence of $\mathcal{R}$. We distinguish two kinds of perturbation methods: general perturbations and special perturbations.

→ In many applications the effect of $\mathcal{R}$ is at least an order of magnitude smaller than that of U_0, so both general perturbations and special perturbations can be used.

→ When the effects of $\mathcal{R}$ are comparable to those of U_0 it is not possible to use general perturbations, and special perturbation methods must be used.

→ *General perturbations:* this method exploits the fact that the orbit due to U_0 changes only slowly due to the effect of $\mathcal{R}$. So, at a given time the orbit is characterized by the osculating elements, which define the osculating ellipse (i.e. the "instantaneous" orbit due to U_0, which we assume here is an ellipse). Then equations for the variation of the elements with time are obtained and studied with analytic methods.

→ *Special perturbations (i.e. numerical integration of orbits):* given the masses of the bodies, starting from positions and velocities at a given time, positions and velocities at later times are obtained by numerical integration of the full equations of motion or of the perturbation equations (i.e. the equations for the variation of the elements).

- → General perturbation method is applicable only when the perturbation is small, but it allows to individuate the dominant perturbing terms and better understand the physical evolution. For instance, general perturbations can enable the sources of observed perturbations to be discovered, because the sources of the perturbations appear explicitly in the equations.

- → Special perturbation method is applicable to any system and over long timescales, but no attempt to isolate different perturbing terms. Fundamental tool, for instance, in studying the long-term evolution of planetary systems.

5.2 General perturbations

[R05 7.1]

- → Initial conditions: at time t_0 the osculating elements are a_0, e_0, i_0, Ω_0, ω_0 and τ_0. If $\mathcal{R} = 0$ (i.e. no perturbation) these elements are constant.

- → Due to $\mathcal{R} \neq 0$ the elements evolve and at a later time t_1 they will be a_1, e_1, i_1, Ω_1, ω_1 and τ_1.

- → The quantities $\Delta a = a_1 - a_0$ etc. are the perturbations in the time interval $\Delta t = t_1 - t_0$.

5.2.1 Lagrange's planetary equations

[R05 7.10]

Hamiltonian formulation

- → The equations that describe the evolution of the osculating elements are called Lagrange's planetary equations.

- → Here we derive Lagrange's planetary equations in the context of the Hamiltonian formulation of mechanics.

- → From the above derivation of the disturbing function (and from the equations of motion $\ddot{\mathbf{r}}_i = \nabla_i U_{0,i} + \nabla_i R_i$) we can infer the form of the corresponding Hamiltonian. Let's consider here the mass-normalized Hamiltonian $\tilde{\mathcal{H}}$ (see discussion in 2.4.2).

- → The Hamiltonian is

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_1,$$

where (dropping index i and using $\Phi_0 = -U_0$ as unperturbed gravitational potential)

$$\tilde{\mathcal{H}}_0 = \frac{1}{2}\left(\tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2\right) + \Phi_0 = \frac{1}{2}\left(\tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2\right) - U_0,$$

$$\tilde{\mathcal{H}}_1 = -\mathcal{R}$$

→ With this definition the canonic equations give the equations of motion derived above

$$\dot{x} = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_x} = \tilde{p}_x$$

$$\dot{\tilde{p}}_x = -\frac{\partial \tilde{\mathcal{H}}}{\partial x} \qquad \Longrightarrow \qquad \ddot{x} = \frac{\partial U_0}{\partial x} + \frac{\partial \mathcal{R}}{\partial x}$$

(here $\tilde{p}_x = \dot{x}$, because we have mass-normalized the Hamiltonian), and similarly for y and z.

Canonical coordinates

→ When considering the two-body problem (m_i and m) in Hamiltonian dynamics we have seen that it is possible to write the solution in terms of 6 canonic variables α_i, β_i with $i = 1, 2, 3$ such that they are all constant.

→ These constant (mass-normalized) canonical coordinates are

$$\alpha_1 = \tilde{E} = -\frac{GM}{2a}, \qquad \beta_1 = -\tau$$

$$\alpha_2 = \tilde{L} = \sqrt{GMa(1-e^2)}, \qquad \beta_2 = \omega$$

$$\alpha_3 = \tilde{L}_z = \sqrt{GMa(1-e^2)}\cos i, \qquad \beta_3 = \Omega,$$

where $M = m + m_i$, or, using $\mu = GM$,

$$\alpha_1 = -\frac{\mu}{2a}, \qquad \beta_1 = -\tau,$$

$$\alpha_2 = \sqrt{\mu a(1-e^2)}, \qquad \beta_2 = \omega,$$

$$\alpha_3 = \sqrt{\mu a(1-e^2)}\cos i, \qquad \beta_3 = \Omega.$$

→ The corresponding mass-normalized two-body Hamiltonian is null: $\tilde{\mathcal{H}}_0 = 0$. Clearly, the two-body (unperturbed) Hamiltonian $\tilde{\mathcal{H}}_0$ does not depend on α_i and β_i (α_i are the momenta and β_i are the coordinates), so

$$\dot{\alpha}_i = -\frac{\partial \tilde{\mathcal{H}}_0}{\partial \beta_i} = 0,$$

$$\dot{\beta}_i = \frac{\partial \tilde{\mathcal{H}}_0}{\partial \alpha_i} = 0.$$

→ Now, in our case the Hamiltonian is in the form $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_1 = \tilde{\mathcal{H}}_0 - \mathcal{R}$, so the canonic variables α_i and β_i are not constant, but they vary as

$$\dot{\alpha}_i = -\frac{\partial \tilde{\mathcal{H}}_1}{\partial \beta_i} = \frac{\partial \mathcal{R}}{\partial \beta_i},$$

$$\dot{\beta}_i = \frac{\partial \tilde{\mathcal{H}}_1}{\partial \alpha_i} = -\frac{\partial \mathcal{R}}{\partial \alpha_i}.$$

→ Now we combine the above Hamilton equations to obtain the perturbation equations, i.e. Lagrange planetary equations, i.e. the equations that describe the time-variation of the orbital elements of the osculating ellipse, due to the presence of the disturbing function $\mathcal{R}$.

Hamilton's equations in terms of orbital elements

→ Hamilton equation (I)

$$\dot{\alpha}_1 = \frac{\partial \mathcal{R}}{\partial \beta_1}$$

→ So we have

$$-\frac{\partial \mathcal{R}}{\partial \tau} = \frac{\partial \mathcal{R}}{\partial \beta_1} = \dot{\alpha}_1 = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\mu}{2a}\right) = \frac{\mu \dot{a}}{2a^2} = \frac{n^2 a \dot{a}}{2}$$

where we have used $\mu = n^2 a^3$ [we recall $\mu = GM = G(m + m_i)$], so

$$\dot{a} = -\frac{2}{n^2 a}\frac{\partial \mathcal{R}}{\partial \tau} \qquad (I)$$

→ Note that n appears in the expression of Lagrange's planetary equations. However, n is not an independent orbital element, and must be considered just a function of a: $n = n(a) = \sqrt{\mu/a^3}$. So $\mathrm{d}n/\mathrm{d}a = -(3/2)n/a$. Other useful relations: $\sqrt{\mu a} = na^2$ and $\sqrt{\mu/a} = na$.

→ Hamilton equation (II)

$$\dot{\alpha}_2 = \frac{\partial \mathcal{R}}{\partial \beta_2},$$

so

$$\frac{\partial \mathcal{R}}{\partial \omega} = \frac{\partial \mathcal{R}}{\partial \beta_2} = \dot{\alpha}_2 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{\mu a(1-e^2)}\right] = \frac{na\sqrt{1-e^2}}{2}\left[\dot{a} - \frac{2ae}{1-e^2}\dot{e}\right],$$

which can be written as

$$\dot{a} - \frac{2ae}{1-e^2}\dot{e} = \frac{2}{na\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial \omega} \qquad (II)$$

→ Hamilton equation (III)

$$\dot{\alpha}_3 = \frac{\partial \mathcal{R}}{\partial \beta_3},$$

so

$$\frac{\partial \mathcal{R}}{\partial \Omega} = \frac{\partial \mathcal{R}}{\partial \beta_3} = \dot{\alpha}_3 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{\mu a(1-e^2)}\cos i\right] = \frac{na\sqrt{1-e^2}}{2}\left[\dot{a} - \frac{2ae}{1-e^2}\dot{e}\right]\cos i - na^2\sqrt{1-e^2}\cos i \tan i \frac{\mathrm{d}i}{\mathrm{d}t} =$$

$$= \frac{na\sqrt{1-e^2}\cos i}{2}\left[\dot{a} - \frac{2ae}{(1-e^2)}\dot{e} - 2a\tan i\frac{\mathrm{d}i}{\mathrm{d}t}\right]$$

so

$$\dot{a} - \frac{2ae}{1-e^2}\dot{e} - 2a\tan i\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{2}{na\sqrt{1-e^2}\cos i}\frac{\partial \mathcal{R}}{\partial \Omega} \qquad (III)$$

→ Now we use combinations of the other three Hamilton equations

$$\dot{\beta}_1 = -\frac{\partial \mathcal{R}}{\partial \alpha_1},$$

a

$$\dot{\beta}_2 = -\frac{\partial \mathcal{R}}{\partial \alpha_2},$$

$$\dot{\beta}_3 = -\frac{\partial \mathcal{R}}{\partial \alpha_3}.$$

→ Equation (IV) for $\partial\mathcal{R}/\partial i$ (i appears only in α_3):

$$\frac{\partial\mathcal{R}}{\partial i} = \frac{\partial\mathcal{R}}{\partial\alpha_3}\frac{\partial\alpha_3}{\partial i} = -\dot{\beta}_3\left(-na^2\sqrt{1-e^2}\sin i\right) = \dot{\Omega}na^2\sqrt{1-e^2}\sin i$$

i.e.

$$\dot{\Omega} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial\mathcal{R}}{\partial i} \qquad (IV)$$

→ Equation (V) for $\partial\mathcal{R}/\partial e$ (e appears only in α_2 and α_3):

$$\frac{\partial\mathcal{R}}{\partial e} = \frac{\partial\mathcal{R}}{\partial\alpha_2}\frac{\partial\alpha_2}{\partial e} + \frac{\partial\mathcal{R}}{\partial\alpha_3}\frac{\partial\alpha_3}{\partial e} = -\dot{\beta}_2\frac{\partial\alpha_2}{\partial e} - \dot{\beta}_3\frac{\partial\alpha_3}{\partial e} = -\dot{\omega}\left(-\frac{ena^2}{(1-e^2)^{1/2}}\right) - \dot{\Omega}\left(-\frac{ena^2}{(1-e^2)^{1/2}}\cos i\right) =$$

$$= \frac{ena^2}{\sqrt{1-e^2}}\left(\dot{\omega} + \dot{\Omega}\cos i\right)$$

so

$$\dot{\omega} + \dot{\Omega}\cos i = \frac{(1-e^2)^{1/2}}{ena^2}\frac{\partial\mathcal{R}}{\partial e} \qquad (V)$$

→ Equation (VI) for $\partial\mathcal{R}/\partial a$ (a appears in α_1, α_2 and α_3):

$$\frac{\partial\mathcal{R}}{\partial a} = \frac{\partial\mathcal{R}}{\partial\alpha_1}\frac{\partial\alpha_1}{\partial a} + \frac{\partial\mathcal{R}}{\partial\alpha_2}\frac{\partial\alpha_2}{\partial a} + \frac{\partial\mathcal{R}}{\partial\alpha_3}\frac{\partial\alpha_3}{\partial a} = -\dot{\beta}_1\frac{\partial\alpha_1}{\partial a} - \dot{\beta}_2\frac{\partial\alpha_2}{\partial a} - \dot{\beta}_3\frac{\partial\alpha_3}{\partial a} =$$

$$= \frac{n^2a}{2}\dot{\tau} - \sqrt{1-e^2}\frac{na}{2}(\dot{\omega} + \dot{\Omega}\cos i)$$

so

$$\dot{\tau} - \frac{\sqrt{1-e^2}}{n}(\dot{\omega} + \dot{\Omega}\cos i) = \frac{2}{n^2a}\frac{\partial\mathcal{R}}{\partial a} \qquad (VI)$$

Equations for the variation of the elements

→ We now combine equations (I-VI) to obtain Lagrange's planetary equations in the form da/dt = ... etc.

(I):

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{2}{n^2a}\frac{\partial\mathcal{R}}{\partial\tau} \qquad (1)$$

(II+I):

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1-e^2}{2ae}\dot{a} - \frac{1-e^2}{2ae}\frac{2}{na\sqrt{1-e^2}}\frac{\partial\mathcal{R}}{\partial\omega}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{\sqrt{1-e^2}}{a^2en}\left[\frac{\sqrt{1-e^2}}{n}\frac{\partial\mathcal{R}}{\partial\tau} + \frac{\partial\mathcal{R}}{\partial\omega}\right] \qquad (2)$$

(III+II):

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{2a\tan i}\dot{a} - \frac{2ae}{2a\tan i(1-e^2)}\dot{e} - \frac{1}{a^2n\sin i\sqrt{1-e^2}}\frac{\partial\mathcal{R}}{\partial\Omega}$$

$$= \frac{1}{2a\tan i}\left[\dot{a} - \frac{2ae}{1-e^2}\dot{e}\right] - \frac{1}{a^2n\sin i\sqrt{1-e^2}}\frac{\partial\mathcal{R}}{\partial\Omega}$$

$$= \frac{1}{2a\tan i}\left[\frac{2}{na\sqrt{1-e^2}}\frac{\partial\mathcal{R}}{\partial\omega}\right] - \frac{1}{a^2n\sin i\sqrt{1-e^2}}\frac{\partial\mathcal{R}}{\partial\Omega}$$

$$= \frac{1}{a^2 n\sqrt{1-e^2}}\left[\frac{1}{\tan i}\frac{\partial \mathcal{R}}{\partial \omega} - \frac{1}{\sin i}\frac{\partial \mathcal{R}}{\partial \Omega}\right] \qquad (3)$$

(IV):

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathcal{R}}{\partial i} \qquad (4)$$

(V+IV):

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\dot{\Omega}\cos i + \frac{(1-e^2)^{1/2}}{ena^2}\frac{\partial \mathcal{R}}{\partial e} = -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathcal{R}}{\partial i} + \frac{(1-e^2)^{1/2}}{ena^2}\frac{\partial \mathcal{R}}{\partial e}$$

$$= \frac{1}{a^2 n\sqrt{1-e^2}}\left[\frac{1-e^2}{e}\frac{\partial \mathcal{R}}{\partial e} - \frac{1}{\tan i}\frac{\partial \mathcal{R}}{\partial i}\right] \qquad (5)$$

(VI+V):

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{n}(\dot{\omega} + \dot{\Omega}\cos i) + \frac{2}{n^2 a}\frac{\partial \mathcal{R}}{\partial a} =$$

$$= \frac{1-e^2}{en^2a^2}\frac{\partial \mathcal{R}}{\partial e} + \frac{2}{n^2 a}\frac{\partial \mathcal{R}}{\partial a}.$$

Summary of Lagrange's planetary equations

→ In summary Lagrange's planetary equations (1-6) are

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{2}{n^2 a}\frac{\partial \mathcal{R}}{\partial \tau} \qquad (1) \qquad (I)$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{1-e^2}{a^2en^2}\frac{\partial \mathcal{R}}{\partial \tau} - \frac{\sqrt{1-e^2}}{a^2en}\frac{\partial \mathcal{R}}{\partial \omega} \qquad (2) \qquad (II+I)$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{a^2 n\sqrt{1-e^2}}\left[\frac{1}{\tan i}\frac{\partial \mathcal{R}}{\partial \omega} - \frac{1}{\sin i}\frac{\partial \mathcal{R}}{\partial \Omega}\right] \qquad (3) \qquad (III+II)$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathcal{R}}{\partial i} \qquad (4) \qquad (V+IV)$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{a^2 n\sqrt{1-e^2}}\left[\frac{1-e^2}{e}\frac{\partial \mathcal{R}}{\partial e} - \frac{1}{\tan i}\frac{\partial \mathcal{R}}{\partial i}\right] \qquad (5) \qquad (IV+V)$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1-e^2}{ea^2n^2}\frac{\partial \mathcal{R}}{\partial e} + \frac{2}{n^2 a}\frac{\partial \mathcal{R}}{\partial a} \qquad (6') \qquad (VI+V)$$

→ The specific form of Lagrange's planetary equations depends on the choice of the elements.

→ Roy uses as orbital elements

a (semi-major axis)

e (eccentricity)

i (inclination)

Ω (longitude of the ascending node)

ω (argument of pericentre)

$\chi = -n\tau$ (mean anomaly at epoch, sometimes indicated with $\mathcal{M}_0$)

→ MD (see also Roy 7.7) use as orbital elements

a (semi-major axis)

e (eccentricity)

i (inclination)

Ω (longitude of the ascending node)

$\varpi = \Omega + \omega$ (longitude of pericentre)

$\epsilon = \varpi + \chi$ (mean longitude at epoch)

5.3 Computation of the precession of the perihelion

[C10]

→ The Delaunay variables for the 2D Kepler problems are

$$J_b = \sqrt{a\mu(1-e^2)}, \qquad J_c = \sqrt{a\mu},$$

and the corresponding Hamiltonian is $\tilde{\mathcal{H}}_0 = -\mu^2/2J_c^2$. The angles are $\theta_b = \omega$ (argument of perihelion) and $\theta_c = \mathcal{M}$ (mean anomaly)

→ Let us consider 3 bodies: Sun (1), Mercury (2), Jupiter (3). The equations of motion in the inertial frame are

$$\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{\mathrm{d}t^2} = \frac{Gm_2(\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)}{|\boldsymbol{\xi}_2-\boldsymbol{\xi}_1|^3} + \frac{Gm_3(\boldsymbol{\xi}_3-\boldsymbol{\xi}_1)}{|\boldsymbol{\xi}_3-\boldsymbol{\xi}_1|^3},$$
$$\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{\mathrm{d}t^2} = -\frac{Gm_1(\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)}{|\boldsymbol{\xi}_2-\boldsymbol{\xi}_1|^3} - \frac{Gm_3(\boldsymbol{\xi}_2-\boldsymbol{\xi}_3)}{|\boldsymbol{\xi}_2-\boldsymbol{\xi}_3|^3}.$$

→ Taking a heliocentric frame $\mathbf{r}_2 = \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1$ and $\mathbf{r}_3 = \boldsymbol{\xi}_3 - \boldsymbol{\xi}_1$, we get

$$\frac{\mathrm{d}^2\mathbf{r}_2}{\mathrm{d}t^2} = -\frac{G(m_1+m_2)\mathbf{r}_2}{|\mathbf{r}_2|^3} - \frac{Gm_3\mathbf{r}_3}{|\mathbf{r}_3|^3} + \frac{Gm_3(\mathbf{r}_3-\mathbf{r}_2)}{|\mathbf{r}_3-\mathbf{r}_2|^3}$$

→ Defining $\mu \equiv G(m_1+m_2)$ and $\epsilon = Gm_3 \ll \mu$, we have

$$\frac{\mathrm{d}^2\mathbf{r}_2}{\mathrm{d}t^2} + \frac{\mu\mathbf{r}_2}{|\mathbf{r}_2|^3} = -\epsilon\frac{\partial R}{\partial \mathbf{r}_2}$$

where

$$R = \frac{\mathbf{r}_2\cdot\mathbf{r}_3}{|\mathbf{r}_3|^3} - \frac{1}{|\mathbf{r}_3-\mathbf{r}_2|}$$

is the disturbing function.

→ Writing $\mathbf{r}_2$ and $\mathbf{r}_3$ as functions of the Delaunay variables, the perturbed Hamiltonian is

$$\tilde{\mathcal{H}}(J_c, J_b, \theta_c, \theta_b) = -\frac{\mu^2}{2J_c^2} + \epsilon R(J_c, J_b, \theta_c, \theta_b)$$

.

→ Normalizing time so that $n_3 = 2\pi/T_3 = 1$, we have $\mathbf{r}_3 = (r_3 \cos t, r_3 \sin t)$,

$$\mathbf{r}_2 \cdot \mathbf{r}_3 = r_2 r_3 \cos(\phi - t)$$

$$|\mathbf{r}_3 - \mathbf{r}_2| = \sqrt{r_2^2 + r_3^2 - 2r_2 r_3 \cos(\phi - t)}$$

$$R = \frac{r_2 \cos(\phi - t)}{r_3} - \frac{1}{\sqrt{r_2^2 + r_3^2 - 2r_2 r_3 \cos(\phi - t)}}$$

→ Expansion of the disturbing function. Legendre polynomials: $P_0(x) = 1$, $P_1(x) = x$,

$$P_{j+1}(x) = \frac{(2j+1)P_j(x) - jP_{j-1}(x)}{j+1} \qquad (j \geq 1).$$

→ So

$$R = -\frac{1}{r_3} \sum_{j=2}^{\infty} P_j[\cos(\phi - t)] \left(\frac{r_2}{r_3}\right)^j$$

→ Normalizing length so that $r_3 = 1$ (assuming circular orbit for Jupiter, we also have $a_J = 1$, $\mu = 1$, because $T_J = 2\pi$) we have

$$R = R_{00}(J_c, J_b) + \ldots,$$

$$R_{00} = -\frac{J_c^4}{4}\left(1 + \frac{9}{16}J_c^4 + \frac{3}{2}e^2\right) + \ldots.$$

where $e = \sqrt{1 - (J_b/J_c)^2}$, so

$$R_{00} = -\frac{J_c^4}{4}\left[1 + \frac{9}{16}J_c^4 + \frac{3}{2}\left(1 - \frac{J_b^2}{J_c^2}\right)\right] + \ldots.$$

→

$$\dot{\theta}_b = \frac{\partial \tilde{\mathcal{H}}}{\partial J_b} = \epsilon \frac{\partial R_{00}}{\partial J_b}$$

so, as $\theta_b = \omega$,

$$\dot{\omega} = \epsilon \frac{\partial R_{00}}{\partial J_b} = \epsilon \frac{3}{4} J_b J_c^2$$

→ Using $J_b = \sqrt{a\mu(1 - e^2)}$, $J_c = \sqrt{\mu a}$,

$$\dot{\omega} = \frac{3}{4}\epsilon \mu^{3/2} a^{3/2} \sqrt{1 - e^2}.$$

We are using units such that $a_J = 1$, $\mu = 1$, so

$$\dot{\omega} = \frac{3}{4}\frac{M_J}{M_\odot}\left(\frac{a_M}{a_J}\right)^{3/2} \sqrt{1 - e_M^2}.$$

with time unit $t_u = T_J/2\pi$.

Here $e = e_M$ and $a = a_M$ (Mercury). We have $M_J/M_\odot = 9.54 \times 10^{-4}$, $a_M/a_J = 0.0744$ and $e_M = 0.2056$, so $\dot{\omega} = 1.58 \times 10^{-5} radian/t_u$. As $radian/t_u = 2\pi \times 180 \times 3600 \times 100/(11.86\pi) = 1.09 \times 10^7$ arcsec/century, we get $\dot{\omega} \simeq 155$ arcsec/century.

→ We have computed only the effect of Jupiter. Similar effects from other planets: altogether $\simeq 532$ arcsec. In addition to this we have a precession of $\simeq 43$ arcsec due to the general relativity corrections.

5.4 Special perturbations

[R05 8.1-8.2]

→ The method of special perturbations consists in numerically integrating the equations of motion of the N bodies in any of their possible forms. Also known as method of numerical integration of orbits.

→ In celestial mechanics the number of bodies is small, so computing the force is not expensive. Main limitation is the rounding-off error, which affects the long-term evolution of given initial conditions.

→ There are several implementations of the special perturbation method, depending on the formulation of the equations of motion and on the numerical integration algorithm. The choice depends on several factors: type of orbit, required accuracy, length of time span, available computing facilities.

5.4.1 Numerical integration algorithms

[P92 chapter 16; B07]

→ In all approaches we have to integrate a system of ordinary differential equations (ODEs). These might be first-order (Lagrange's planetary equations) or second order (Encke's or Cowell's equations). But a second-order system can be always recast in the form of a first-order system. Note that in some cases it is more convenient to integrate directly second order: for instance Stoermer rule can be applied to the full N-body equations because the right-hand side does not depend on the first derivatives (force is conservative, i.e. it depends only on position; see P92 16.5). However, here we consider only the case of first-order systems.

→ A second-order ODE (or system of ODEs) can be transformed into a first-order ODE (or system of ODEs) as follows:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = F(t, y, y'),$$

where $y' \equiv \mathrm{d}y/\mathrm{d}t$.

$$v(t) \equiv \frac{\mathrm{d}y}{\mathrm{d}t}$$

so we get the system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = F(t, y, v)$$

with $v = v(t)$ and $y = y(t)$. Writing $\mathbf{w} = (y, v)$, the above system is clearly in the form

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{w}),$$

where $\mathbf{w} = \mathbf{w}(t)$.

→ When the problem is reduced to first-order system of ODE, we must just find a method to solve an equation in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y),$$

where $y = y(t)$. We recall that in the N-body case t is time and y is either a phase-space coordinate (Newton's equations of motion) or an orbital element (Lagrange's planetary equations). In all cases our problem is an initial value problem, so the initial values t_0 (i.e. initial time) and $y_0 = y(t_0)$ (initial coordinates or elements) are given.

→ Numerical methods to integrate a system of ODEs: several choices are possible. Well known algorithms are Runge-Kutta, Bulirsch-Stoer, symplectic integrators etc. Typically in celestial mechanics high accuracy is required. This is due to a combination of the chaotic nature of the orbits and the necessity of integrating over long time spans: if the integration is not accurate enough, relatively small integration errors can lead to completely wrong orbits over long timescales.

→ Here we present only the Bulirsch-Stoer method, which is a robust method, often used in applications of celestial mechanics.

5.4.2 Bulirsch-Stoer algorithm

[P92 16.1-16.4]

→ Bulirsch-Stoer algorithm is a method to integrate systems of ODE based on Euler method, Midpoint method and Modified midpoint method. Before describing the Bulirsch-Stoer algorithm, we briefly describe these simpler methods. We recall that we want to solve a first-order ODE in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y),$$

where $y = y(t)$, with initial conditions $y = y_0$ for $t = t_0$. In general, given t_i and $t_{i+1} = t_i + H$ (where H is the integration step), for known $y_i = y(t_i)$ we want to estimate $y(t_{i+1})$ numerically: the approximated result is called $y_{i+1} \approx y(t_{i+1})$. Such integration steps are repeated to go from the initial value t_0 to the final value $t_{\rm final}$ of the independent variable (i.e. time, in the N-body case).

→ *Euler method.*

$$y_{i+1} = y_i + Hf(t_i, y_i),$$

where H is increment (or step) and $f = y'$ is evaluated at (t_i, y_i). y_{i+1} is the estimated value of $y(t_{i+1})$ where $t_{i+1} = t_i + H$.

→ *Midpoint method.* The step H is divided in two steps of length $H/2$ and the slope is evaluated at $t + H/2$.

$$y_{i+1} = y_i + Hf(t_{i+1/2}, y_{i+1/2}),$$

where

$$y_{i+1/2} \equiv y_i + \frac{H}{2} f(t_i, y_i),$$

and $t_{i+1/2} \equiv t_i + (H/2)$.

→ *Modified midpoint method.* The step H is divided in n steps of length $h = H/n$. Now $y_i = y(t_i)$ and $y_{i+1} \approx y(t + H)$. The algorithm reads as follows:

$$z_0 = y_i = y(t_i)$$

$$z_1 = z_0 + hf_0 \approx y(t_i + h),$$

which is an estimate of $y(t_i + h)$ using Euler's method. Here we have introduced the following notation: $f_j \equiv f(t_i + jh, z_j)$, so, $f_0 = f(t_i, z_0)$.

$$z_2 = z_0 + 2hf_1 \approx y(t_i + 2h),$$

which is an estimate of $y(t_i + 2h)$ using the midpoint method. In general, for the j-th sub-step we have

$$z_j = z_{j-2} + 2hf_{j-1} \approx y(t_i + jh),$$

which is an estimate of $y(t_i + jh)$ using the midpoint method. Finally we define y_{i+1} by averaging between z_n and the average between z_{n+1} and z_{n-1}:

$$y_{i+1} = \frac{1}{2}\left(\frac{z_{n+1} + z_{n-1}}{2} + z_n\right),$$

i.e.

$$y_{i+1} = \frac{1}{4}\left(z_{n-1} + 2z_n + z_{n+1}\right),$$

$$= \frac{1}{4}\left(z_{n-1} + 2z_n + z_{n-1} + 2hf_n\right),$$

i.e.

$$y_{i+1} = \frac{1}{2}\left[z_n + z_{n-1} + hf_n\right].$$

→ *Bulirsch-Stoer method.* With this method each step goes from t to $t + H$, via several (n) modified-midpoint method sub-steps with $h = H/n$, which are extrapolated to $h \to 0$.

→ n is not fixed, but for each step we try first with $n = 2$, and then increase n iteratively up to a value which is estimated to be sufficient (i.e., such that the error is small enough).

→ n is not increased by one each time, but through a specific sequence. One of the optimal choices is

$$n = 2, 4, 6, 8, 10, ... \qquad \text{(i.e. } n_k = 2k\text{)},$$

where k is the index that represents the iteration step.

→ For given k, so for given n_k (and then for given $h_k = H/n_k$), the modified midpoint method gives us an estimate $y_{i+1}(h_k)$, depending on h_k. For each k, i.e. each n_k in the sequence, via polynomial extrapolation, we compute

$$y_{i+1,k} = \lim_{h \to 0} g_k(h),$$

where $g_k(h)$ is a polynomial function interpolating the k points $[h_k, y_{i+1}(h_k)]$. This method is known as Richardson extrapolation.

→ The extrapolation can be performed as follows. Given k estimates of $y_{i+1,k}$, corresponding to k different values of n, we define an interpolating function, a polynomial of order $k-1$

$$g_k(h) = a_0 + a_1 h + a_2 h^2 + ... + a_{k-1} h^{k-1}.$$

There is only an interpolating polynomial of order $k-1$ (obtained, for instance, with Lagrange formula or other interpolating algorithm). So we can compute the coefficients. Then we can compute the extrapolation to $h = 0$, which is simply $y_{i+1,k} \equiv g_k(0) = a_0$

→ We go on for increasing k. We stop for $k = k'$ when we meet a convergence criterion. For instance

$$\frac{|y_{i+1,k'} - y_{i+1,k'-1}|}{|y_{i+1,k'}|} < \epsilon,$$

where ϵ is a (small) dimensionless number, which is the accuracy (e.g. $\epsilon \sim 10^{-13}$).

→ Finally

$$y_{i+1} = y_{i+1,k'} \equiv g_{k'}(0).$$

See plot B07 Fig. 3.4 (FIG CM4.3) and P92 Fig. 16.4.1 (FIG CM4.4).

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