THE VIRTUAL STRUCTURE PROBLEM FOR HIGHER MODULAR GROUPS

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Dedicated to the 70^{th} birthday of Eric Jespers

Abstract. We classify the finite groups G such that the finitely presented group $\mathcal{U}(\mathbb{Z}G)$ has the good property. Furthermore we obtain several characterisations in terms of properties of the simple factors of $\mathcal{U}(\mathbb{Q}G)$. Ring theoretically it is shown that it coincides with having only low-dimensional $\mathbb{Q}G$-components (i.e. at most 1×1 and exceptional 2×2 components). In particular, we solve a new instance of the virtual structure problem, generalising the free-by-free work. Cohomologically this happens if and only if all simple factors have virtual cohomological dimension a divisor of 4. Geometrically it is proven to be equivalent to the components acting discontinuously on $\mathbb{H}^5$. The latter properties are investigated for general lattices in semisimple algebraic $\mathbb{Q}$-groups of (inner) type A where in general the properties are no longer equivalent.

Contents

2020 *Mathematics Subject Classification.* 20G25,20E06, 20C05.

Key words and phrases. linear groups, amalgamated products, group rings, profinite density, Virtual structure problem.

The first author is grateful for financial support from the FWO and the F.R.S.–FNRS under the Excellence of Science (EOS) program (project ID 40007542). The second author is grateful to Fonds Wetenschappelijk Onderzoek Vlaanderen - FWO (grant 88258), and le Fonds de la Recherche Scientifique - FNRS (grant 1.B.239.22) for financial support.

1. Introduction

Finally, we consider the the Virtual Structure Problem, which askes for a unit theorem. A very concrete idea of a unit theorem was given by Kleinert [27] in the context of orders:

> *A unit theorem for a finite dimensional semisimple rational algebra A consists of the definition, in purely group theoretical terms, of a class of groups $C(A)$ such that almost all generic unit groups of A are members of $C(A)$.*

Recall that a generic unit group of A is a subgroup of finite index in the group of reduced norm 1 elements of an order in A.

The Virtual structure problem for exceptional components. reformulate

Question 1.1 (Virtual Structure Problem)**.** Let $\mathcal{G}$ be a class of groups. Classify the finite groups G such that $\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index lying in $\mathcal{G}$.

Till recently, the finite groups G for which a unit theorem, in the sense of Kleinert, was known for $\mathcal{U}(\mathbb{Z}G)$ are those for which the class of groups considered are either finite groups (Higman), abelian groups (Higman), or direct products of free-by-free groups [21, 19, 33, 23]. Remarkably, the latter class can also be described in terms of the rational group algebra: every simple quotient of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or $\mathrm{M}_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$. Such type of unit theorem was also obtained recently in [4, 3] for several geometric properties such as property (T) and (HFA). To our knowledge this result covers all the known unit theorems on $\mathcal{U}(\mathbb{Z}G)$.

The aim of this article is to solve the above for the following class of groups

$$\mathcal{G}_{am} = \{\mathbb{Z}^n \times \prod_{i\in I} A_i \star_{C_i} B_i \mid [A_i : C_i], [B_i : C_i] \text{ are finite but not } 1\}.$$

The main bulk of the paper will be about classifying the finite groups G such that the only non-division algebra components of $\mathbb{Q}G$ are exceptional 2×2 components. Following result completes a line of research started more than 20 years ago with the papers ... .

Theorem 1.2. *Let G be a finite group. The following are equivalent*

(1) All simple components $\mathrm{M}_n(D)$, with $n \geq 2$, of $\mathbb{Q}G$ are exceptional matrix components, i.e. $n = 2$ and D isomorphic to $\{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right)\}$,
(2) $\mathrm{vcd}(\mathrm{SL}_1(\mathbb{Z}Ge)) \in \{0, 1, 2, 4\}$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$
(3) G is a quotient of one of the following families of groups : **blabla**

Moreover, in that case $\mathrm{SL}_1(\mathbb{Z}Ge)$ is discrete subgroup of $\mathrm{SL}_4(\mathbb{C})$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$. The converse holds iff **to complete**

Or PSL?

Remark 1.3.
- The fourth condition in Theorem 1.2 can also be equivalently stated that $\mathrm{SL}_1(\mathbb{Z}Ge)$ acts discontinuously on $\mathbb{H}_5$ or $\mathbb{H}_3$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$.
- one can be more precise in the actual components that appear. Namely **put final list**

Is het eerder $V(\mathbb{Z}G)$?

Denote by $\mathcal{C}(\mathbb{Q}G)$ the isomorphism types of the simple components of $\mathbb{Q}G$. Next, we record a list of geometric group theoretical properties of $\mathcal{U}(\mathbb{Z}G)$ that is equivalent to $\mathcal{C}(\mathbb{Q}G)$ to be as in Theorem 1.2.

Theorem 1.4. *Let G be a finite group. The following are equivalent*

(1) $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_{am}$
(2) $\mathcal{U}(\mathbb{Z}G)$ is good
(3) Stuff Angel-Zalesski paper.

As a corollary we get a statement which appeared without proof in ??. For this recall the following two classes of groups

$$\mathcal{G}_{pab} = \{\prod_i A_{i,1} \star \cdots \star A_{i,t_i} \mid A_{i,j} \text{ are finitely generated abelian } \}$$

and
$$\mathcal{G}_{\neq 1} := \{\prod_i \Gamma_i \mid e(\Gamma_i) \neq 1\}.$$

The following was announced without proof in [??].

Corollary 1.5. *The following classes are equal*
$$\{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{\neq 1}\} = \{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{pab}\}.$$

Theorem 1.2 and its predecessors suggest to investigate the following ring theoretical variant of the virtual structure problem for $\mathcal{U}(\mathbb{Z}G)$. **put the statement that is now later in the paper**

How to call the problem?

Problem. Let $\mathcal{P}$ be a set of isomorphism classes of finite dimensional simple algebras over $\mathbb{Q}$. Classify all finite groups G such that $\mathcal{C}(\mathbb{Q}G) \subseteq \mathcal{P}$.

Rational isomorphism problem. A classical problem is how much of the group structure of G is determined by the R-algebra structure the group ring RG. In this section we consider the rational isomorphism problem:

Problem 1. Let G and H be finite groups. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, is then $G \cong H$?

In general the answer to Problem 1 is negative as shown for the first time by the counterexample of Dade REF. A positive answer is known for abelian groups REF and among metayclic groups REF. The authors are only aware of these classes.

Our aim is to prove that if $\mathbb{Q}G$ has ($\mathrm{M}_{\mathrm{exc}}$) then Problem 1 has a positive answer.

The blockwise Zassenhaus property. In the 70's Zassenhaus formulated a set of conjectures which had to clarify the origin of the conjectural isomorphism between two group bases. The strongest of these, called the Third Zassenhaus conjecture, asserted that every finite subgroup H of $V(\mathbb{Z}G)$ is conjugated over $\mathbb{Q}G$ to a subgroup of G. This has been disproven in [??]. It nevertheless an important problem to determine which classes of groups satisfy the property asserted by the conjecture.

We define the Zassenhaus property, with respect to a type of subgroups, for any semisimple algebra. Subsequently we consider what we call the blockwise Zassenhaus property. For this no counterexample is yet known and in this paper we prove the following.

Theorem 1.6. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge$ is a field, totally definite quaternion algebra or an exceptional simple algebra. Then $(\mathbb{Q}G)e$ satisfies the Third Zassenhaus property.*

Corollary 1.7. *If G is a finite group such that $\mathbb{Q}G$ XX. Then it satisfies the blockwise Zassenhaus property.*

To finish, we would like to advertise the study of the block-wise version of the Isomorphism problem and the Zassenhaus conjectures. It can namely be verified that the known counterexamples to those conjectures are not counterexamples to the block-wise version!

Further questions. blabla

Acknowledgment. We thank Oberwolfach research in pairs (number) blabla. We are grateful to Angel del Río for sharing with us a proof of Lemma 6.2.

Conventions and notations. Throughout the full article all groups denotes by a latin letter will be a finite group. All orders will be understood to be $\mathbb{Z}$-orders. We also use the following notations:

- $\mathrm{PCI}(FG)$ for the set of primitive central idempotents of FG
- $\pi_e : \mathcal{U}(FG) \twoheadrightarrow FGe$ projection to a simple component
- $\mathcal{C}(FG) = \{FGe \mid e \in \mathrm{PCI}(FG)\}$ for the set of isomorphism types of the simple components of FG.

- Degree and index of a central simple algebra is **blabla**
- By $\phi(n)$ we denote Euler's phi function.
- The notation g^t will denote the conjugation $t^{-1}gt$.

2. Finite groups with only exceptional higher simple components

As explained in the introduction, one old approach to representation theory of finite groups G over the ring of integers R of a number field F is to understand torsion-free normal subgroups in $\mathcal{U}(RG)$ and their quotients. The properties (such as the index) of these normal subgroups depend on the form of the simple quotients of FG. For instance the presence of the following type of simple algebras often breaks the methods used (see [4, Section 6.1] for more details):

Definition 2.1 (Exceptional components)**.** A finite dimensional simple algebra B is called *exceptional* if it is isomorphic to one of the following:

(1) a non-commutative division algebra which is not a totally definite quaternion algebra over a number field,

(2) a matrix algebra $\mathrm{M}_2(D)$ with $D \in \{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right) \mid a, b, d \in \mathbb{N}_0\}$.

If B is of the latter form, then we speak of an *exceptional matrix* algebra and in the former case of a *exceptional division* algebra. If $B \cong FGe$ for some $e \in \mathrm{PCI}(FG)$, then B is called an exceptional component of FG.

The name was coined in [22] and the reason that in practice they are exceptional (i.e. require 'other methods') is different. The exceptional matrix algebras are exactly those $\mathrm{M}_n(D)$ for which the S-rank of $\mathrm{SL}_n(D)$ is 1, where S is the set of Archimedean places of $\mathcal{Z}(D)$ (see [4, Remark 6.7] for details and eq. (4.1) for the definition of SL over non-commutative rings). Consequently, they are exactly those having a maximal order[1] $\mathrm{M}_n(\mathcal{O})$ such that $\mathrm{SL}_n(\mathcal{O})$ does not satisfy the subgroup congruence problem. In particular there exist non-central normal subgroups which are not of finite index.

The non-exceptional division algebras D are exactly those having an order $\mathcal{O}$ such that $\mathrm{SL}_1(\mathcal{O})$ is finite. The others are problematic because $\mathrm{SL}_1(D)$ has no unipotent elements which is an ingredient for most generic constructions of units in $\mathcal{U}(RG)$.

The aim of this section is to characterise the finite groups G such that FG has the following property:

$$(\mathrm{M}_{\mathrm{exc}}) \quad \text{all } FGe \cong \mathrm{M}_n(D), \text{ with } n \geq 2, \text{ are of the form } \mathrm{M}_2(\mathbb{Q}(\sqrt{-d})) \text{ or } \mathrm{M}_2\left(\left(\frac{-a,-b}{\mathbb{Q}}\right)\right) \text{ with } a, b, d \in \mathbb{N}.$$

So with respect to the general theory, they represent the 'most degenerate groups'. The groups G such that $\mathbb{Q}G$ has the stronger property that (i) all matrix component are of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{N}$ *and* (ii) $\mathbb{Q}G$ has no exceptional division components were classified in [23]. So considering property $(\mathrm{M}_{\mathrm{exc}})$ generalises their considerations in two ways.

Considering the role of the ground field we directly note the following:

Lemma 2.2. *Let G be a finite group and $F \subseteq L$ fields of characteristic* 0*. If LG has* $(\mathrm{M}_{\mathrm{exc}})$*, then FG also has* $(\mathrm{M}_{\mathrm{exc}})$*.*

Proof. Recall that there is a bijection between the simple components of LG and the absolutely irreducible characters $\mathrm{Irr}(G)$ of G. For $\chi \in \mathrm{Irr}(G)$ denote the associated component of LG by $A_L(\chi)$. Now, see [20, Theorem 3.3.1], $A_L(\chi)$ and $L(\chi) \otimes_{F(\chi)} A_F(\chi)$ are isomorphic as $F(\chi)$-algebras.

If $L = F$ nothing needs to be proved, so let $F \subsetneq L$. Note that the aforementioned results imply that the reduced degree of $A_F(\chi)$ is smaller or equal to the one of $A_L(\chi)$. Hence if

[1] It is well-known that a maximal order in $\mathrm{M}_n(D)$ is of the form $\mathrm{M}_n(\mathcal{O})$ with $\mathcal{O}$ a maximal order in D. Furthermore in case of an exceptional matrix algebra, the division algebras have up to isomorphism a unique maximal order. Thus the property doesn't depend on the order chosen.

LG has only division algebra components, then so does FG. Furthermore if LG has $(\mathrm{M}_{\mathrm{exc}})$, then also all matrix components of FG have reduced degree two. Next suppose that $A_F(\chi)$ is a matrix component and hence also $A_L(\chi)$. Then by the above see that $L(\chi) = \mathcal{Z}(A_L(\chi))$ contains $F(\chi)$ in its centre. As LG has $(\mathrm{M}_{\mathrm{exc}})$, this means that $[F(\chi) : \mathbb{Q}] \mid [L(\chi) : \mathbb{Q}] \mid 2$. Consequently, either $F(\chi) = \mathbb{Q}$ or $F(\chi) = L(\chi)$ is an imaginary quadratic extension by $(\mathrm{M}_{\mathrm{exc}})$. $\square$

Remark 2.3. Finite groups G for which $\mathbb{Q}G$ has only division algebra components have been classified in [36, Theorem 3.5]. The latter result implies that LG has only division components exactly when $\mathbb{Q}G$ does and L is not a splitting field of $\left(\frac{-1,-1}{\mathbb{Q}}\right)$. Hence we may suppose that LG has a matrix component. First note that if LG has a matrix component, then, as $L \subseteq \mathcal{Z}(A_L(\chi))$ for all $\chi \in \mathrm{Irr}(G)$, the $(\mathrm{M}_{\mathrm{exc}})$ assumption implies that L is either the rationals or an imaginary quadratic extension of $\mathbb{Q}$. Consider $L = \mathbb{Q}(\sqrt{-d})$ for some square-free positive integer d.

Then the proof of Lemma 2.2 yields that LG can only have $(\mathrm{M}_{\mathrm{exc}})$ if $\mathbb{Q}G$ does and moreover $F(\chi, \sqrt{-d}) = F(\sqrt{-d})$ for all $\chi \in \mathrm{Irr}(G)$. In other words $F(\chi) \subseteq \mathbb{Q}(\sqrt{-d})$ for all χ. Note that the converse also holds. In summary:

$$\mathbb{Q}(\sqrt{-d})[G] \text{ has } (\mathrm{M}_{\mathrm{exc}}) \Leftrightarrow \mathbb{Q}G \text{ has } (\mathrm{M}_{\mathrm{exc}}) \text{ and } \mathcal{Z}(\mathbb{Q}Ge) \subseteq \mathbb{Q}(\sqrt{-d}) \text{ for all } e \in \mathrm{PCI}(\mathbb{Q}G).$$

This shows the importance of first understanding the case of rational group algebras. Most of the work will be dedicated to that. We start by recalling methods to describe the simple components of $\mathbb{Q}G$.

2.1. **Preliminaries on describing simple components and role coefficients.** We need to recall some methods to construct primitive central idempotents of $\mathbb{Q}G$. These methods were introduced by Olivieri-del Río-Simón [32], see [20, Chapter 3] for a good introduction. To start, recall that if $H \trianglelefteq G$, then

$$\widehat{H} := \frac{1}{|H|} \sum_{h \in H} h$$

is a central idempotent in $\mathbb{Q}G$. Now, set $\epsilon(H, H) = \widehat{H}$ and for a strict normal subgroup K of H define

$$\epsilon(H, K) = \prod_{M/K \in \mathcal{M}(H/K)} (\widehat{K} - \widehat{M}), \tag{2.1}$$

where $\mathcal{M}(H/K)$ denotes the set of the non-trivial minimal normal subgroups of H/K. The construction results in a central idempotent in $\mathbb{Q}H$. To obtain a central idempotent in $\mathbb{Q}G$ one associates to $K \trianglelefteq H$ the element

$$e(G, H, K) = \sum_{t \in \mathcal{T}} \epsilon(H, K)^t \tag{2.2}$$

where $\mathcal{T}$ is a right transversal of $\mathrm{Cen}_G(\epsilon(H, K))$ in G. The element $e(G, H, K)$ is central in $\mathbb{Q}G$ and is a primitive idempotent when (H, K) is a *Strong Shoda pair* of G, *SSP in short.* A tuple (H, K) is called a strong Shoda pair when $K \leq H \trianglelefteq N_G(K)$, H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$, and the G-conjugates of $\epsilon(H, K)$ are orthogonal.

To a central idempotent e there is also the associated epimorphism

$$\varphi_e : G \to Ge, \quad g \mapsto ge. \tag{2.3}$$

Note that Ge is a finite subgroup of $\mathcal{U}(\mathbb{Q}G)e$.

The following is a combination of [20, Proposition 3.4.1, Theorems 3.4.2 & 3.5.5 and Problem 3.5.1] and [17, Lemma 3.4].

Theorem 2.4 ([32])**.** *With notations as above, $e := e(G,H,K)$ is a primitive central idempotent of $\mathbb{Q}G$ if (H,K) is a strong Shoda pair. Moreover, in that case* $\mathrm{Cen}_G(\epsilon(H,K)) \cong N_G(K)$, $\ker(\varphi_e) = core_G(K) = \bigcap_{g\in G} K^g$ *and*

$$\dim_{\mathbb{Q}} \mathbb{Q}Ge = [G:H][G:N_G(K)]\phi([H:K]).$$

Furthermore, $\mathbb{Q}Ge \cong \mathrm{M}_{[G:N_G(K)]}\big(\mathbb{Q}(\zeta_{[H:K]}) * N_G(K)/H\big)$ *for some explicit crossing. In particular,* $\deg(\mathbb{Q}Ge) = [G:H]$.

The notation deg denotes *the degree* of the central simple algebra $\mathbb{Q}Ge(G,H,K))$, i.e. $\deg(A) = \sqrt{n}$ if $A \otimes_{\mathcal{Z}(A)} \mathbb{C} \cong \mathrm{M}_n(\mathbb{C})$.

Remark. For a SSP (H,K) it might happen that $H = G$. This occurs precisely when $G' \leq K$ which in turn is equivalent to $\mathbb{Q}Ge(G,H,K)$ being commutative, see [18, Lemma 2.4].

Next we recall the structure of the crossed product of $\mathbb{Q}(\zeta_{[H:K]}) * N_G(K)/H$ mentioned in Theorem 2.4, following [20, Remark 3.5.6]. For ease of notation, we let $N := N_G(K)$, $m := [H:K]$ and $H/K := \langle y\rangle$. The *twisting* is given by:

$$\alpha\colon N/H \to \mathrm{Aut}(\zeta_m), \quad \overline{x} \mapsto \alpha_{\overline{x}}(\zeta_m) = \zeta_m^j, \tag{2.4}$$

where j is determined by

$$(yK)^x = (yK)^j, \quad \text{with } xH = \overline{x}.$$

Moreover, the *crossing* is given by:

$$f\colon N/H \times N/H \to \mathbb{Q}(\zeta_m)^*, \quad (\overline{n_1}, \overline{n_2}) \mapsto \zeta_m^{\ell}, \tag{2.5}$$

with ℓ uniquely determined by

$$t_{\overline{n_1}} \cdot t_{\overline{n_2}} = y^{\ell} \cdot k \cdot t_{\overline{n_1 n_2}},$$

for $k \in K$ and $\overline{n_i} := t_{\overline{n_i}}H$ for a fixed left transversal $\mathcal{T} = \{t_{\overline{n_i}} \mid t_{\overline{n_i}}H \in N/H\}$ of H in N.

Finally, we will mainly use the theory of SSP for G metabelian, in which case the SSPs are easy to describe.

Theorem 2.5 ([20, Theorem 3.5.12])**.** *Let G be a finite metabelian group and let B be a maximal abelian subgroup of G containing G'. Let $K \leq G$ be such that $C' \leq K \leq C$ for some subgroup C of G with $B \leq C \leq G$. Then (H,K) is a strong Shoda pair if and only if the following hold:*

(1) H/K is cyclic,
(2) H is maximal in the set $\{C \leq G \mid B \leq C$ and $C' \leq K \leq C\}$.

2.2. **Restrictions on Division algebras and the Group.** In this section we first give several group theoretical properties on the finite groups G such that FG has $(\mathrm{M}_{\mathrm{exc}})$. Under that condition we also obtain which simple algebras can occur as simple component of FG. For instance the possible division algebras that can arise as $\mathbb{Q}Ge$ are still restricted. Thereafter we will show that there exist subgroups of index two such that the matrix components are as in [23].

Description components and first properties. To start we describe the possible simple algebras that can appear as a component of $\mathbb{Q}G$ with $(\mathrm{M}_{\mathrm{exc}})$. To do so, we will describe which finite groups can be isomorphic to Ge with $e \in \mathrm{PCI}(\mathbb{Q}G)$. This has the advantage to also yield a first set of interesting properties.

In [36, Theorem 3.5] it was proven that $\mathbb{Q}G$ has only division algebra components if and only if G is of the form $A \times C_2^n \times Q_8$ with A an odd abelian group such that $|A|$ and $o_{|A|}(2)$ are odd. Hence from now we may assume that $\mathbb{Q}G$ has at least one matrix component. **mention work Angel**

Theorem 2.6. *Let G be a finite group and suppose that $\mathbb{Q}G$ has a matrix component. If $\mathbb{Q}G$ has* $(\mathrm{M}_{\mathrm{exc}})$*, then*

(1) *the* 1×1 *components of* $\mathbb{Q}G$ *are either fields or quaternion algebras. More precisely, the non-commutative possibilities are*[2]*:*

$$\{\left(\frac{\zeta_{2^t}, -3}{\mathbb{Q}(\zeta_{2^t})}\right), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right) \mid t \in \mathbb{N}_{\geq 3}\}$$

(2) *The only exceptional matrix components are*

$$\{\mathrm{M}_2(\mathbb{Q}(\sqrt{-d})), \mathrm{M}_2(\left(\frac{-1,-1}{\mathbb{Q}}\right)), \mathrm{M}_2(\left(\frac{-1,-3}{\mathbb{Q}}\right)) \mid d = 0,1,2,3\}.$$

(3) G *has*[3] *an abelian normal subgroup* B *with* $\exp(G/B) \mid 2$.

(4) $\pi(G) \subseteq \{2,3\}$.

(5) $\exp(\mathcal{Z}(G) \cap G') \mid 2$ *and exp*$(G/\mathcal{Z}(G)) \mid 12$. **for now the 2nd is not in the table...**

(6) *if* $\mathbb{Q}G$ *has no exceptional division components, then*

$$\exp(\mathcal{Z}(G)) \mid 4 \text{ or } 6 \text{ and } \exp(G) \mid 24.$$

In particular, G *is metabelian with* $\mathrm{cd}(G) \subseteq \{1,2,4\}$*. Furthermore,* $\deg(\mathbb{Q}Ge) \mid 4$ *for every* $e \in \mathrm{PCI}(\mathbb{Q}G)$.

In Section 2.4 we will show that several of the above properties hold in a stronger form, e.g. G/B is an elementary abelian 2-group of rank at most 2.

Remark 2.7. In the proof we will obtain that for $C_3 \rtimes C_{2^n}$, where the action is by inversion, the rational group algebra has all non-division simple components of the form $\mathrm{M}_2(\mathbb{Q})$ and $\mathrm{M}_2(\mathbb{Q}(i))$. Furthermore, it has a division component of the form $\left(\frac{\zeta_{2^{n-1}}, -3}{\mathbb{Q}(\zeta_{2^{n-1}})}\right)$. Hence it gives an example of a group that has all matrix components of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{N}$ but which does not satisfy the stronger property considered in [23], where they require that $\mathbb{Q}G$ has no exceptional division components.

One interesting class of groups for which $\mathbb{Q}G$ has no exceptional division components is the class of *cut groups*, see [4, Proposition 6.12]. Recall that a group is called *cut* if $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite.

The proof of Theorem 2.6 requires a finer understanding of the quotient groups Ge, with $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Lemma 2.8. *For any finite group* G*, normal subgroup* $N \leqslant G$ *and* $e \in \mathrm{PCI}(\mathbb{Q}G)$ *holds:*

$$\mathcal{C}(\mathbb{Q}[G/N]) \subseteq \mathcal{C}(\mathbb{Q}G) \text{ and } \mathcal{C}(\mathbb{Q}[Ge]) \subseteq \mathcal{C}(\mathbb{Q}G).$$

Consequently,

$$\mathcal{C}(\mathbb{Q}G) = \bigcup_{e \in \mathrm{PCI}(\mathbb{Q}G)} \mathcal{C}(\mathbb{Q}[Ge]).$$

In particular, property $(\mathrm{M}_{\mathrm{exc}})$ *is inherited by quotients.*

Proof. The first claim follows from the fact that $\mathbb{Q}[G/N]$ is a semisimple subalgebra of $\mathbb{Q}G$. Indeed, it is immediately semisimple (since it is a group algebra), and a straightforward calculation shows that $\mathbb{Q}[G/N] \cong \mathbb{Q}G\widetilde{N} \leq \mathbb{Q}G$ with $\widetilde{N}$ the central idempotent $\frac{1}{|N|}\sum_{n \in N} n$. The second inclusion follows from the first since the group Ge is an epimorphic image of G. The rest is now also a direct consequence as every simple component of $\mathbb{Q}G$ corresponds to a primitive central idempotent $e \in \mathrm{PCI}(\mathbb{Q}G)$. □

Using Lemma 2.8, the proof of Theorem 2.6 reduces to a study of the fixed-point free groups classified by Amitsur [2] and the finite subgroups of exceptional components classified in [9]. In fact, the conclusion of Theorem 2.6 already holds under the weaker condition that each Ge is embedded in a division algebra or an exceptional matrix algebra.

[2]Note that by [9, Theorem 3.5] the only exceptional matrix component not appearing is $\mathrm{M}_2(\mathbb{H}_5)$.

[3]The proof will furthermore prove that $[Ge : Be] \mid 4$ for every $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Proof of Theorem 2.6. For a group G having ($\mathrm{M}_{\mathrm{exc}}$), the set $\mathrm{PCI}(\mathbb{Q}G)$ naturally decomposes into $\mathrm{PCI}_1 := \{e \mid \mathbb{Q}Ge \cong D\}$ and $\mathrm{PCI}_2 := \{e \mid \mathbb{Q}Ge \cong \mathrm{M}_2(D)\}$ where D always signifies a rational division algebra. Hence, with Lemma 2.8 in mind, for the first statement it suffices to analyse the components possibly appearing in $\mathbb{Q}[Ge]$ for $e \in \mathrm{PCI}_1$ or PCI_2.

We start with PCI_2. The finite subgroups $\mathcal{G}$ of $\mathrm{GL}_2(\mathbb{Q}(\sqrt{-d}))$ or $\mathrm{GL}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ with $a, b, d \in \mathbb{N}$ with the property that $\mathrm{Span}_{\mathbb{Q}}(\mathcal{G})$ is the respective $\mathrm{M}_2(\cdot)$ have been classified[4] in [9, Theorem 3.7]. This classification consists of 55 groups and in particular the groups Ge for $e \in \mathrm{PCI}_2$ are amongst these. The simple components may be computed using e.g. the Wedderga package in GAP, see the table in Appendix A for the result. This moreover shows that for groups Ge with $e \in \mathrm{PCI}_2$ the property ($\mathrm{M}_{\mathrm{exc}}$) is equivalent to the weaker property that each non-division component has reduced degree 2. A case-by-case verification also shows that each of these groups Ge contain an abelian normal subgroup of index a divisor of 4. Furthermore, as can be seen in the table, the only 1×1 components appearing are

$$\mathbb{Q}, \mathbb{Q}(\zeta_3), \mathbb{Q}(i), \mathbb{Q}(\zeta_8), \mathbb{Q}(\zeta_{12}), \left(\frac{-1,-1}{\mathbb{Q}}\right), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right).$$

Inspection of the table also shows that the remaining statements hold for such groups.

Next we consider the case that $e \in \mathrm{PCI}_1$. A similar reasoning applies. Indeed, the finite subgroups (such as Ge for $e \in \mathrm{PCI}_1$) of rational division algebras have been classified by Amitsur in [2]. We will use the rephrasing from [38, Theorem 2.1.4] which asserts that they are:

(a) a **Z**-group, i.e. having cyclic Sylow-subgroups, with restrictions listed later on in the proof.

(b) (i) the binary octahedral group O^* of order 48:

$$\left\{\pm 1, \pm i, \pm j, \pm ij, \frac{\pm 1 \pm i \pm j \pm ij}{2}\right\} \cup \left\{\frac{\pm a \pm b}{\sqrt{2}} \mid a, b \in \{1, i, j, ij\}\right\}.$$

(ii) $C_m \rtimes Q$, where m is odd, Q is quaternion of order 2^t for some $t \geq 3$, an element of order 2^{t-1} centralises C_m and an element of order 4 inverts C_m.

(iii) $M \times Q_8$, with M a **Z**-group of odd order m and the (multiplicative) order of 2 mod m is odd.

(iv) $M \times \mathrm{SL}_2(\mathbb{F}_3)$, where M is a **Z**-group of order m coprime to 6 and the (multiplicative) order of 2 mod m is odd.

(c) $\mathrm{SL}_2(\mathbb{F}_5)$.

None of the groups $O^*, \mathrm{SL}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_5)$ have ($\mathrm{M}_{\mathrm{exc}}$). Indeed, using[5] the Wedderga package in GAP one verifies that $\mathrm{M}_3(\mathbb{Q})$ is a simple component over $\mathbb{Q}$ of $O^* \cong \mathrm{SU}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_3)$, and moreover that $\mathrm{M}_5(\mathbb{Q})$ is a simple component for $\mathrm{SL}_2(\mathbb{F}_5)$. Consequently, they cannot be epimorphic images of the group G. Hence:

The cases (b)(i), (b)(iv) and (c) *do not* appear as groups Ge for $e \in \mathrm{PCI}_1$ with $\mathbb{Q}[G]$ ($\mathrm{M}_{\mathrm{exc}}$).

The groups in (b)(ii) are dicyclic groups of order $2^t m$, i.e. Dic_{4n} with $n = 2^{t-2}m$ and $t \geq 3$. When n odd, the dicylic groups coincide with case (B) in the family of **Z**-groups (see below). Therefore consider a general dicyclic group:

$$Dic_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle.$$

This is a metabelian group, and hence its Strong Shoda pairs are described by Theorem 2.5, which we apply now. Note that the commutator subgroup Dic'_{4n} is given by $\langle a^2 \rangle$, and $\langle a \rangle$ is the maximal abelian subgroup of Dic_{4n} containing it. For any SSP (H, K), one has that $\langle a \rangle \subseteq H$. In other words, $H = \langle a \rangle$ or $H = Dic_{4n}$. If $H = Dic_{4n}$ then the simple component associated to (H, K) is a field, by [18, Lemma 2.4]. Via Theorem 2.5, it is a direct verification that for $d \mid 2n$, the tuple $(\langle a \rangle, \langle a^d \rangle)$ is a SSP if and only if $d \neq 1, 2$. Note that K is normal

[4]In [9, Table 2] a group was missing, see [4, Appendix A] for a complete list.

[5]The SmallGroup ID of the three groups are respectively [48,28], [24,3] and [120,5].

in Dic_{4n}. Hence the associated primitive central idempotent is $\epsilon(\langle a\rangle, \langle a^d\rangle)$. Now, in [20, Example 3.5.7], it is noted that (for n not necessarily even)

$$\mathbb{Q}Dic_{4n}\epsilon(\langle a\rangle,\langle a^d\rangle) \cong \mathrm{M}_2(\mathbb{Q}(\zeta_d+\zeta_d^{-1})) \quad \text{if } d \mid n \text{ and } d \nmid 2, \tag{2.6}$$

where ζ_d denotes a complex primitive d-th root of unity. Since $\mathbb{Q}(\zeta_d+\zeta_d^{-1}) = \mathbb{Q}(\Re(\zeta_d))$ and $\Re(\zeta_d) = \cos\frac{2\pi}{d} \in \mathbb{R}\setminus\mathbb{Q}$, Niven's theorem implies that $\mathrm{M}_2(\mathbb{Q}(\zeta_d+\zeta_d^{-1}))$ is exceptional[6] if and only if $\phi(d) \leq 2$. The latter is equivalent to $d \in \{1,2,3,4,6\}$. In conclusion, if Dic_{4n} is non-abelian and has ($\mathrm{M}_{\mathrm{exc}}$), then it must be isomorphic to one of the groups

$$Dic_{4.2} = Q_8, \quad Dic_{4.4} = Q_{16}, \quad Dic_{4.6} = C_3 \rtimes Q_8 \quad \text{or} \quad Dic_{4.3} = C_3 \rtimes C_4.$$

The first three groups are in the family (b)(ii). Moreover, these groups indeed have ($\mathrm{M}_{\mathrm{exc}}$), since their Wedderburn-Artin components are:

$$\mathcal{C}(\mathbb{Q}Q_8) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right)\}, \qquad \mathcal{C}(\mathbb{Q}Q_{16}) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right), \mathrm{M}_2(\mathbb{Q})\}, \qquad \text{and}$$
$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes Q_8]) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right), \mathrm{M}_2(\mathbb{Q})\}.$$

Before we consider case (b)(iii) from above, we will discuss (a), the **Z**-groups. They have also been classified, see [38, Theorem 2.1.5]. They are the following:

(A) Finite cyclic groups,
(B) $C_m \rtimes C_4$, where m is odd and C_4 acts by inversion,
(C) $G_0 \times G_1 \times \ldots \times G_s$, with $s \geq 1$, $\gcd(|G_i|, |G_j|) = 1$ for all $0 \leq i \neq j \leq s$ and G_0 is the only cyclic subgroup amongst them. Furthermore each of the G_i, for $i \neq 0$, is of the form

$$C_{p^a} \rtimes \left(C_{q_1^{b_1}} \times \ldots \times C_{q_r^{b_r}}\right),$$

for $p, q_1, \ldots, q_r$ distinct primes. Moreover, each of the groups $C_{p^a} \rtimes C_{q_j^{b_j}}$ is non-cyclic (i.e. if $C_{q_j^{\alpha_j}}$ denotes the kernel of the action of $C_{q_j^{b_j}}$ on C_{p^a}, then $\alpha_j \neq b_j$) and satisfies the following properties:
  (i) $q_j o_{q_j^{\alpha_j}}(p) \nmid o_{\frac{|G|}{|G_i|}}(p)$.
  (ii) one of the following is true:
   - $q_j = 2$, $p \equiv -1 \mod 4$, and $\alpha_j = 1$,
   - $q_j = 2$, $p \equiv -1 \mod 4$, and $2^{\alpha_j+1} \nmid p^2-1$,
   - $q_j = 2$, $p \equiv 1 \mod 4$, and $2^{\alpha_j+1} \nmid p-1$,
   - $q_j > 2$, and $q_j^{\alpha_j+1} \nmid p-1$.

  Here $o_m(q)$ denotes the order of q modulo m.

It is clear the cyclic groups have ($\mathrm{M}_{\mathrm{exc}}$) since $\mathbb{Q}C_n$ is abelian. Moreover, by the theorem of Perlis-Walker [20, Theorem 3.3.6], $\mathcal{C}(\mathbb{Q}C_n) = \{\mathbb{Q}(\zeta_d) \mid d \text{ divides } n\}$.

Case (B), i.e Dic_{4n} with n odd, is treated in (2.6). The conclusion was that the only possible (non-abelian) such group having ($\mathrm{M}_{\mathrm{exc}}$) is $C_3 \rtimes C_4$. In this case,

$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_4])) = \{\mathbb{Q}, \mathbb{Q}(i), \left(\frac{-1,-3}{\mathbb{Q}}\right), \mathrm{M}_2(\mathbb{Q})\}.$$

Next we consider case (C). We first show that ($\mathrm{M}_{\mathrm{exc}}$) implies $2 \mid |G_i|$, for $1 \leq i \leq s$ and hence $s = 1$ by the coprimeness condition. For this, we consider $A_i = \prod_{j=1}^s C_{q^{\alpha_j}}$, the kernel of the action in the decomposition of G_i into a semidirect product. Then

$$B := G_i/A_i \cong C_{p^a} \rtimes C_{q_1^{k_1}\cdot\ldots\cdot q_s^{k_s}},$$

[6]Recall that $[\mathbb{Q}(\Re(\zeta_d)) : \mathbb{Q}] = \phi(d)/2$. In particular, in contrast to the case $e \in \mathrm{PCI}_2$, it can happen that all non-division components are exceptional without the group having ($\mathrm{M}_{\mathrm{exc}}$). Moreover, it can happen that all non-division components are of the form $\mathrm{M}_2(F)$ with F a quadratic extension of $\mathbb{Q}$. As shown by (2.6), this holds for Dic_{4n} with $n = 5, 10, 8, 12$.

where $k_j := b_j - \alpha_j > 0$ and the action is non-trivial and faithful. Denote by x and y the respective generators of the factors of B, i.e. $B = \langle x \rangle \rtimes \langle y \rangle$. By Lemma 2.8, the group B also has $(\mathrm{M_{exc}})$. Note that $C_{p^a} = \langle x \rangle$ is a maximal abelian subgroup of B containing B'. Now using Theorem 2.5, it is a direct verification that $(H, K) = (\langle x \rangle, 1)$ is a SSP of B. Moreover, $\mathbb{Q}Be(G, \langle x \rangle, 1) \cong \mathbb{Q}(\zeta_{o(x)}) * \langle y \rangle$ for some explicit crossing (see [20, Remark 3.5.6]) which implies that the component is non-division. Now using [17, Lemma 3.4], we compute that

$$\dim_{\mathbb{Q}} \mathbb{Q}Be(G, \langle x \rangle, 1) = [G : \langle x \rangle]\, \phi(o(x)) = q_1^{k_1} \cdot \ldots \cdot q_s^{k_s}\, p^{a-1}(p-1).$$

On the other hand, $\dim_{\mathbb{Q}} \mathbb{Q}Be(G, \langle x \rangle, 1) \mid 16$ since B has $(\mathrm{M_{exc}})$. Combining both facts with the fact that p and q_i are pairwise distinct primes, we obtain that $s = 1, q_1 = 2$ and $p^a = 3$. Thus $B \cong C_3 \rtimes C_{2^{k_1}}$. Furthermore, as the action is faithful, we obtain that $k_1 = 1$, i.e. $B \cong C_3 \rtimes C_2$ where the action is by inversion. Consequently $G_1 \cong C_3 \rtimes C_{2^{b_1}}$ with the action being inversion (as $\alpha_1 = b_1 - 1$). For such groups, using Theorem 2.5, one can verify that, for $b_1 \geq 4$,

$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_{2^n}]) = \Big\{ \mathbb{Q}(\zeta_{2^\ell}),\ \left(\tfrac{-1,-3}{\mathbb{Q}}\right), \left(\tfrac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right),\ \mathrm{M}_2(\mathbb{Q}),\ \mathrm{M}_2(\mathbb{Q}(i)) \mid 1 \leq \ell \leq n,\ 3 \leq t \leq n-1 \Big\}, \tag{2.7}$$

and, for $b_1 \leq 3$:

$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_2]) = \{\mathbb{Q}, \mathrm{M}_2(\mathbb{Q})\}, \qquad \mathcal{C}(\mathbb{Q}[C_3 \rtimes C_4]) = \{\mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \left(\frac{-1,-3}{\mathbb{Q}}\right),\ \mathrm{M}_2(\mathbb{Q})\},$$

$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_8]) = \{\mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\zeta_8), \left(\frac{-1,-3}{\mathbb{Q}}\right),\ \mathrm{M}_2(\mathbb{Q}),\ \mathrm{M}_2(\mathbb{Q}(\sqrt{-1}))\}.$$

Hence we see that such a $G_1 \cong C_3 \rtimes C_{2^{b_1}}$ has $(\mathrm{M_{exc}})$.

It remains to consider $G_0 \times G_1$ with $G_0 \cong C_m$ cyclic. From the Wedderburn-Artin decomposition of $\mathbb{Q}[G_1]$ above we see that $\mathbb{Q}[G_0 \times G_1]$ contains as a simple component

$$\mathbb{Q}(\zeta_m) \otimes_{\mathbb{Q}} \mathrm{M}_2(\mathbb{Q}) \cong \mathrm{M}_2(\mathbb{Q}(\zeta_m)) \oplus \mathrm{M}_2(\mathbb{Q}(\zeta_m)),$$

with $d \in \mathbb{N}$ (potentially zero). As $G_0 \times G_1$ is assumed to have $(\mathrm{M_{exc}})$ this implies that $[\mathbb{Q}(\zeta_m) : \mathbb{Q}] \leq 2$. The latter happens exactly when $m \in \{1, 2, 3, 4, 6\}$. Now recall that $|G_0| = m$ and $|G_1| = 3.2^{b_1}$ are relatively prime, yielding that $m = 1$, since $b_1 \geq 1$. Thus in conclusion[7],

$$\textbf{Z}\text{-groups of type (C) with } (\mathrm{M_{exc}}) \text{ are } C_3 \rtimes C_{2^n} \text{ with action by inversion.} \tag{2.8}$$

The last case to handle is (b)(iii), i.e. $M \times Q_8$ with M a **Z**-group of odd order m and the multiplicative order of 2 modulo m odd. By Lemma 2.8 the group M has $(\mathrm{M_{exc}})$. Looking at the possible **Z**-groups of odd order we see that M must be cyclic. Now recall that $\mathcal{C}(\mathbb{Q}Q_8) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right)\}$. If M is cyclic, then

$$\mathcal{C}(\mathbb{Q}[M \times Q_8]) = \{\mathbb{Q}(\zeta_d), \left(\frac{-1,-1}{\mathbb{Q}(\zeta_d)}\right) \mid d \text{ divides } m\}.$$

As m and $o_m(2)$ are odd, all the components are division algebras - a conclusion that also directly would have followed from [36]. In conclusion[8],

Groups of type (b)(iii) with $(\mathrm{M_{exc}})$ are given by $C_m \times Q_8$ with m and $o_m(2)$ odd.

In summary, with the analysis above we have shown part (1) and (2) from the statement by describing $\prod_{e \in \mathrm{PCI}(\mathbb{Q}G)} Ge$. Note that all allowed groups Ge have been highlighted in the

[7]To reach this conclusion we only used that every non-division component is of the form $\mathrm{M}_2(D)$ with $[\mathcal{Z}(D) : \mathbb{Q}] \leq 2$.

[8]This conclusion only requires that every non-division component is of the form $\mathrm{M}_2(D)$ with $[\mathcal{Z}(D) = \mathbb{Q}] \leq 2$ and not the full strength of $(\mathrm{M_{exc}})$.

proof. We see that they all have an abelian normal subgroup B_e with Ge/B_e isomorphic to C_2 or $C_2 \times C_2$. Hence

$$B := G \cap \prod_{e \in \mathrm{PCI}(\mathbb{Q}G)} B_e$$

is an abelian normal subgroup of G with $\exp(G/B) \mid 2$. Consequently, G is metabelian. Additionally, for the simple algebras $\mathrm{M}_n(D)$ allowed by ($\mathrm{M}_{\mathrm{exc}}$), we see that $\mathrm{M}_2(D) \otimes_{\mathcal{Z}(D)} \mathbb{C}$ is isomorphic to $\mathrm{M}_2(\mathbb{C})$ or $\mathrm{M}_4(\mathbb{C})$. By the first part of the proof, if $\mathbb{Q}Ge \cong D$, then $D \otimes_{\mathbb{Z}(D)} \mathbb{C}$ is either $\mathbb{C}$ or $\mathrm{M}_2(\mathbb{C})$. Hence indeed $\mathrm{cd}(G) \subseteq \{1, 2, 4\}$ and $\deg(\mathbb{Q}Ge) \mid 4$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Since $(\chi(1), q) = 1$ for any odd prime q and any $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ it follows from the Ito-Michler theorem [16, Corollary 12.34] that $G \cong A \rtimes Q$ with A an abelian subgroup of odd order and Q a Sylow 2-subgroup of G.

The detailed analysis above also shows that $\pi(Ge) \subseteq \{2, 3\}$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that Ge is non-abelian and not isomorphic to $C_m \times Q_8$. In the latter case case $\mathbb{Q}Ge \cong \left(\frac{-1,-1}{\mathbb{Q}(\zeta_m)}\right)$. Therefore, if we take $g \in G$ with $o(g)$ prime different from 2 or 3, then $ge \in \mathrm{Ker}(\varphi_e)$ except for primitive central idempotents e such that $\mathbb{Q}Ge$ is a field or a quaternion algebra with $ge \in \mathcal{Z}(\mathbb{Q}Ge)$. It follows that $g \in \mathcal{Z}(G)$. Now, as $G \cong A \rtimes Q$ with Q a 2-group, we can decompose G into $\langle g \rangle \times H$ for some $H \leq G$. However by the assumption that $\mathbb{Q}G$ has a matrix component, which is necessarily exceptional, we obtain hence that $o(g) \mid 4$ or 6, a contradiction.

Notice that the preceding argument proved that:

$$C_m \times Q_8 \text{ can only occur as quotient of } G \text{ if } m = 3. \tag{2.9}$$

However we also had the assumption that $o_m(2)$ is odd, which excludes the option $m = 3$.

Next we consider statement (5) and (6). First we note that the statements can be reduced to the same statement for the projections Ge such that $\mathbb{Q}Ge$ is not a field.

Indeed, take $1 \neq g \in G$ and let $\lambda : \langle g \rangle \to \mathbb{Q}(\zeta_{o(g)})$ be a faithful linear $\mathbb{Q}$-representation of $\langle g \rangle$. Subsequently, consider a decomposition into simple $\mathbb{Q}$-representations of the induction $\mathrm{Ind}_{\langle g \rangle}^{G}(\lambda) = \sigma_1 \oplus \cdots \oplus \sigma_\ell$. By Frobenius' reciprocity,

$$\mathrm{Hom}_{\mathbb{Q}\langle g \rangle}(\lambda, \mathrm{Res}_{\langle g \rangle}^{G}(\sigma)) \cong \mathrm{Hom}_{\mathbb{Q}G}(\mathrm{Ind}_{\langle g \rangle}^{G}(\lambda), \sigma)$$

for any $1 \leq i \leq \ell$. Hence $\mathrm{Res}_{\langle g \rangle}^{G}(\sigma)$ contains λ as simple sub-representation and thus $\mathrm{Res}_{\langle g \rangle}^{G}(\sigma)$ is also injective. Furthermore, recall that $\ker(\mathrm{Ind}_{\langle g \rangle}^{G}(\lambda)) = \mathrm{core}_G(\ker(\lambda))$ and therefore $\mathrm{Ind}_{\langle g \rangle}^{G}(\lambda)$ is a faithful representation of G. Now, one of these σ_i must be satisfy that $\sigma_i(G)$ is non-abelian as otherwise $G' \subset \ker(\mathrm{Ind}_{\langle g \rangle}^{G}(\lambda)) = 1$ and hence G abelian, contradicting the assumption that $\mathbb{Q}G$ has a matrix component. In conclusion, we have shown that there exists an irreducible representation σ_j such that $ker(\sigma_j) \cap \langle g \rangle = 1$ and $\mathrm{Span}_{\mathbb{Q}}\{\sigma_j(G)\}$ is not a field. Hence

$$o(g) \mid \max\{\exp(Ge) \mid \mathbb{Q}Ge \text{ is non-commutative}\}. \tag{2.10}$$

Also note that if $g \in \mathcal{Z}(G)$ (resp. in G'), then $ge \in \mathcal{Z}(Ge)$ (resp. in $(Ge)'$). Thus in that case one can lower the upper-bound in (2.10) with the exponent of the centres (resp. commutator subgroups).

Next let $\mathcal{E}_{nc} := \{e \in \mathrm{PCI}(\mathbb{Q}G) \mid \mathbb{Q}Ge \text{ non-commutative}\}$. Observe that the explanation above also shows that

$$i : G \hookrightarrow \prod_{e \in \mathcal{E}_{nc}} Ge : g \mapsto (ge)_e$$

is an embedding. On the other hand G maps surjectively on Ge. Combined these facts yield that $i(G) \cap \prod_{e \in \mathcal{E}_{nc}} \mathcal{Z}(Ge) = i(\mathcal{Z}(G)) \cong \mathcal{Z}(G)$. Therefore $\exp(G/\mathcal{Z}(G))$ divides $\mathrm{lcm}\{\exp(Ge/\mathcal{Z}(Ge)) \mid e \in \mathcal{E}_{nc}\}$.

Now consider the set

$$\mathcal{E} := \{e \in \mathrm{PCI}(\mathbb{Q}G) \mid \mathbb{Q}Ge \text{ is not a division component }\}.$$

As shown in the table of Appendix A: $\exp(\mathcal{Z}(Gfe) \mid 4$ or 6 for $e \in \mathcal{E}$. The same holds for $\exp(\mathcal{Z}Ge)$ when $\mathbb{Q}Ge$ is a non-commutative division component different from $\left(\frac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right)$. Indeed, verifying all Ge which are ($\mathrm{M}_{\mathrm{exc}}$) one notices that such components either occur from a Ge in the table or when $Ge \cong Q_{16}$ or $C_3 \rtimes Q_8$. In all those cases $\exp(\mathcal{Z}(Ge)) \mid 4$ or 6. In all the cases mentioned one can also manually verify the facts that $\exp(\mathcal{Z}(Ge) \cap (Ge)') \mid 2$, $\exp(Ge) \mid 24$. and $\exp(Ge/\mathcal{Z}(Ge)) \mid 12$. Finally, the division component $\left(\frac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right)$ only arises for $Ge \cong C_3 \rtimes C_{2^n}$ with the action by inversion. For that group $\exp(\mathcal{Z}(Ge) \cap (Ge)') = 1$ and $\exp(G/\mathcal{Z}(G)) = 6$.

In conclusion, these numerical facts combined with the observations made earlier yield both statements (5) and (6). ☐

2.3. **Groups with low character degrees.** This section and the next aim to give a precise description of the groups satisfying ($\mathrm{M}_{\mathrm{exc}}$). In light of Theorem 2.6, we first consider the more general class of groups having character degrees of the irreducible complex characters only divisors of 4. We denote by $\mathrm{cd}(G)$ the set of character degrees of irreducible complex representations of G. In this section we focus on the groups G with $\mathrm{cd}(G) = \{1, 4\}$.

More generally, whenever $\mathrm{cd}(G) \subseteq \{1, p^j, p^k\}$ then G is solvable of derived length at most 3 by [16, Theorem 12.15]. Furthermore, $(\chi(1), q) = 1$ for any prime q different from p and any $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. This allows to apply the Ito-Michler theorem [16, Corollary 12.34] which in this case yields that

$$G \cong A \rtimes Q, \tag{2.11}$$

where A is an abelian p'-subgroup and Q a Sylow p-subgroup of G. More precisely, A is the direct product of all Sylow q-subgroups of G, for all $q \neq p$, which are normal and abelian under the above assumption on $\mathrm{cd}(G)$.

Using character theory and [15], the following results will give further restrictions on the decomposition (2.11) in the case that $|\mathrm{cd}(G)| = 2$. Note that for a non-nilpotent group, one always has that $|\mathrm{cd}(G)| > 1$.

Proposition 2.9. *Let G be a non-nilpotent finite group such that $\mathrm{cd}(G) \subseteq \{1, p^i, p^k\}$ with $0 \leq i < k$ and $p^k = \max(\mathrm{cd}(G))$. Let $G \cong A \rtimes Q$ be the decomposition from eq. (2.11). If Q is abelian, then the following hold:*

1. *The Fitting subgroup $F(G)$ of G is the unique maximal abelian subgroup, and has index p^k,*
2. *$C_G(A) = F(G) = \mathcal{Z}(G) \times G'$,*
3. *G decomposes as a semidirect product*

$$G \cong N \rtimes \langle C, Z\rangle$$

with N an abelian subgroup of $F(G)$ containing A and with C and Z p-groups that are at most 2-generated, and such that Z is central in G. Moreover, $G/\langle N, Z\rangle$ is isomorphic to C_{p^k} or $C_{p^i} \times C_{p^{k-i}}$.

Furthermore, if $\mathrm{cd}(G) = \{1, p^k\}$ then Q is abelian **for** $k \neq 1$??. *Conversely, for every group G with $|\mathrm{cd}(G)| = 2$ which satisfies the properties (1) to (3), one has $\mathrm{cd}(G) = \{1, p^k\}$.*

In Lemma 2.14 we will prove that if $\mathbb{Q}G$ has ($\mathrm{M}_{\mathrm{exc}}$), then $[G : F(G)] = 2$. Hence, under the ($\mathrm{M}_{\mathrm{exc}}$) assumption, by Proposition 2.9 the Sylow 2-subgroup of G can only be abelian if $\mathrm{cd}(G) = \{1, 2\}$.

Proof. Suppose G is as in the statement. We prove (1) and (2) simultaneously. Since Q is assumed abelian, in particular G is solvable of derived length at most 2. In particular G' is abelian. It follows from the work of Taunt, [13, VI, Satz 14.7 b)] that $F(G) = \mathcal{Z}(G) \times G'$. By[9] [15, Theorem 2.2 (ii)], $C_G(A)$ is a normal abelian subgroup of G of index p^k. Additionally, $C_G(A)$ and $G/C_G(A)$ are abelian and hence it follows from [13, VI, Satz 14.7 a)] that

$$C_G(A) = \mathcal{Z}(G) \times (C_G(A) \cap G') = \mathcal{Z}(G) \times G' = F(G). \tag{2.12}$$

It now also immediately follows that $F(G)$ is the unique maximal *abelian* subgroup and that $[G : F(G)] = p^k$. Moreover, by [15, Theorem 2.2], $G/C_G(A)$ is either isomorphic to C_{p^k} or to $C_{p^i} \times C_{p^{k-i}}$.

We denote

$$F := C_G(A) = F(G).$$

Let $x, y \in Q$ be such that $\langle xF, yF \rangle = G/F$. More precisely, we choose $x, y \in Q$ as follows. If $G/F \cong C_{p^i} \times C_{p^{k-i}}$, we choose x such that xF has order p^i and yF has order p^{k-i}. If $G/F \cong C_{p^k}$, we choose x such that xF has order p^k, and we let y be an arbitrary power of x. Let then p^ℓ be the order of xF. Then $F \cap \langle x \rangle = \langle x^{p^\ell} \rangle$, and $F \cap \langle y \rangle = \langle y^{p^{k-i}} \rangle$.

Let $\langle z_1 \rangle$ be the cyclic subgroup of F containing x^{p^ℓ} and which is maximal amongst the cyclic subgroups of $Q \cap F$ for this property. Similarly let $\langle z_2 \rangle$ be the maximal cyclic $Q \cap F$ above $y^{p^{k-i}}$. Denote $Z = \langle z_1, z_2 \rangle$.

As Q is abelian and due to the maximality of $\langle z_1 \rangle$ and $\langle z_2 \rangle$, there exists a $M \leqslant Q$ such that $Q \cap F(G) \cong M \times Z$. Moreover, since $F(G)$ is abelian, eq. (2.11) implies that

$$F(G) = A \times (Q \cap F(G)),$$

and it follows that

$$F(G) = N \times Z, \text{ with } N = A \times M.$$

Denoting $C = \langle x, y \rangle$, we now show that

$$G \cong N \rtimes \langle C, Z \rangle.$$

Firstly, $N \cap \langle C, Z \rangle = \{1\}$. Indeed, $\langle C, Z \rangle$ is abelian, and since $F(G) = N \times Z$, we have that $N \cap \langle C, Z \rangle$ is at most C. So suppose $t \in N \cap \langle x, y \rangle \leqslant F$. Then $t = x^n y^m$ for some $n, m \in \mathbb{Z}$. But then $x^n = ty^{-m} \in F$ and since $F \cap \langle x \rangle = \langle x^{p^\ell} \rangle$, we obtain that $x^n = x^{kp^\ell} \in Z$ for some $k \in \mathbb{Z}$. But then $x^n \in Z \cap N$, and hence $x^n = 1$. Thus $t = y^m \in F$. Since $F \cap \langle y \rangle = \langle y^{p^{k-i}} \rangle \leqslant Z$, we obtain again that $t = y^m \in Z \cap N$ and hence $t = 1$.

Additionally, $N \cdot \langle C, Z \rangle = G$. Indeed,

$$\begin{aligned} |N||\langle C, Z \rangle| &= |N| \, [\langle C, Z \rangle : \langle x, Z \rangle] \, [\langle x, Z \rangle : Z] \\ &= \begin{cases} |N| \frac{|\langle y \rangle|}{|\langle y^{p^{k-i}} \rangle|} \frac{|\langle x \rangle|}{|\langle x^{p^i} \rangle|} |Z| & \text{if } G/F \cong C_{p^{k-i}} \times C_{p^i} \\ |N| \frac{|\langle x \rangle|}{|\langle x^{p^k} \rangle|} |Z| & \text{if } G/F \cong C_{p^k} \end{cases} \\ &= |N||Z| \cdot p^k = |F(G)|[G : F(G)] = |G|. \end{aligned}$$

Finally, we show that N is normal in G. Since $N = A \times M$, and A is normal in G by construction, it suffices to show that M is normal in G. We show that in fact $Z \times M$ is central in G. Indeed, if $t \in Z \times M \leqslant F$, it may be written as $t = t_1 t_2$ for unique $t_1 \in \mathcal{Z}(G)$ and $t_2 \in G'$, by eq. (2.12). Furthermore, since Q is an abelian Sylow p-subgroup (and hence $G' \leqslant A$), one has that the order of t_2 is coprime to p. But by definition $t \in Q$ has order a power of p, and it follows that $\langle t \rangle = \langle t^{o(t_2)} \rangle \subseteq \mathcal{Z}(G)$, as claimed.

Suppose $\mathrm{cd}(G) = \{1, p^k\}$. If Q is non-abelian, then [16, Exercise 12.6] implies that G is nilpotent, which is in contradiction with the hypothesis. Hence Q is abelian. For the converse, as $[G : F(G)] = p^k$ and $F(G)$ is a normal abelian subgroup, Ito's theorem [16, Theorem 6.15] yields that $\chi(1)$ divides p^k for all irreducible characters χ. Note also that $G' \leq N \leq F(G)$, hence by [14, Lemma 1] there exists some $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ with $\chi(1) = [G : F(G)] = p^k$. The assumption $|\,\mathrm{cd}(G)| = 2$ concludes the proof. ☐

[9] The fact that the Sylow p-subgroup Q is abelian yields that G is a $\{p\}$-character group as in [15, Definition 2.1.]

Remark 2.10. The proof of Proposition 2.9 yields extra restrictions on the structure of G. For example $F(G) = C_G(A) \cong Z(G) \times G'$ by (2.12). Additionally, denoting $C = \langle x, y \rangle$ and using [15, Theorem 3.1.(iii)], one can prove that the action of x^i on G' is free, for $1 \leq i \leq p^k$.

Example 2.11. Consider the group $G := C_5 \rtimes C_8 = \langle a, b \mid a^5, b^8, a^b = a^3 \rangle$. It is easily shown that $\mathrm{cd}(G) = \{1, 4\}$ and $F(G) = C_G(\langle a \rangle) = \langle a, b^4 \rangle$. Thus, in the notation of Proposition 2.9, $N = \langle a \rangle$ and $C = \langle b \rangle$. In this example, a complement for $F(G)$ itself does not exist. Hence the splitting in Proposition 2.9 is the finest possible in general.

Next we consider *nilpotent* groups G having ($\mathrm{M}_{\mathrm{exc}}$) and $\mathrm{cd}(G) = \{1, 4\}$. Then the decomposition in (2.11) is a direct product $G \cong A \times Q$. A precise classification in the nilpotent case remains elusive. Nevertheless note that one has $\{1, 4\} = \mathrm{cd}(G) = \mathrm{cd}(Q)$. Now applying [15, Theorem 3.10 & Lemma 5.4] yields the following.

Lemma 2.12. *Let G be a nilpotent group with $\mathrm{cd}(G) = \{1, 4\}$. Then $G \cong A \times Q$ with A an odd abelian group and Q a 2-group satisfying:*

(1) Q has nilpotency class 2,
(2) $[Q, Q]$ and $Q/\mathcal{Z}(Q)$ are elementary abelian 2-groups.

Remark 2.13. If $\mathrm{cd}(G) = \{1, 2, 4\}$, the statement of Lemma 2.12 does not hold in general. For example the group

$$C_8 \rtimes (C_2 \times C_2) := \langle a, b, c \mid a^8 = b^2 = c^2 = 1, bab = a^3, cac = a^5, bc = cb \rangle$$

has[10] $\mathrm{cd}(G) = \{1, 2, 4\}$ but is nilpotent of class 3 and $G' \cong C_4$.

2.4. **Strong Shoda pairs and main characteristics.** Recall from (2.11) that $G \cong A \rtimes Q$ with A an odd abelian group and Q a 2-group. Denote by

$$\varphi : Q \to \mathrm{Aut}(A)$$

the action of Q on A. In this section we give the main necessary properties on Q, A and the action.

In Theorem 2.6 we proved that $\mathrm{cd}(G) \subseteq \{1, 2, 4\}$. Moreover, in [23] the groups H such that the components $\mathbb{Q}H$ are of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{Z}_{\geq 0}$ and such that $\mathbb{Q}H$ no exceptional division component have been addressed. Our framework is more general. In particular we will not require the latter condition and note that the former condition yields the stronger restriction $\mathrm{cd}(G) \subseteq \{1, 2\}$. Now whether we are in the more general setting of $4 \in \mathrm{cd}(G)$, may be read off from both the Wedderburn-Artin decomposition of $\mathbb{Q}G$ and from G. If *G is non-nilpotent*:

$$\begin{aligned} 4 \in \mathrm{cd}(G) &\iff \mathrm{M}_2(\left(\frac{-a, -b}{\mathbb{Q}}\right)) \in \mathcal{C}(\mathbb{Q}G) \text{ for some a,b} > 0 \\ &\iff \text{The maximal abelian normal subgroup of } G \text{ has index } 4. \end{aligned}$$

The latter equivalence will follow from Lemma 2.19 below, and the former also holds for nilpotent groups. If *G is nilpotent*:

$$4 \in \mathrm{cd}(G) \iff \begin{cases} \text{The maximal abelian normal subgroup of } G \text{ has index } 4, & \text{if } cl(G) = 3 \\ G/\mathcal{Z}(G) \cong C_2^n \text{ with } n \geq 4, & \text{if } cl(G) = 2 \end{cases}$$

2.4.1. *The main necessary conditions.* The following result can be seen as describing the structure of G up to index 4.

Lemma 2.14. *Let G be a finite group such that $\mathbb{Q}G$ has ($\mathrm{M}_{\mathrm{exc}}$), with at least one matrix component. Then the following hold:*

(1) $G \cong A \rtimes Q$ with A an elementary abelian 3-group and Q a 2-group,
(2) $[G : F(G)] = [Q : \mathrm{Ker}(\varphi)]$ divides 2,

[10] This group has Small Group Id [32,43] and the stated information can be obtained via GAP or the GroupNames database.

(3) G contains an abelian normal subgroup B such that $G/B \cong C_2^k$ with $k \in \{1,2\}$,
(4) $|Q' \cap \mathcal{Z}(Q)| \leq 2$ if G is not nilpotent.

Remark 2.15. The fact that $[G : F(G)] \mid 2$, together with the form of G given by (1), can be interpreted very concretely. If $[G : F(G)] = 1$, then clearly G is nilpotent and $A \subseteq \mathcal{Z}(G)$, as A is abelian, and $G \cong A \times Q$. In general, in the proof of Lemma 2.14, (see (2.15)), we will obtain the following decomposition of A:

$$A = A_{-1} \times (A \cap \mathcal{Z}(G)), \tag{2.13}$$

with

$$A_{-1} := \left\{ a \in A \mid a^y = a^{-1} \text{ for all } y \in G \setminus F(G) \right\}.$$

In the non-nilpotent case, note that, as $F(G) = C_G(A)$ (by Theorem 2.6 and Proposition 2.9) and $[G : F(G)] = 2$, one has that $a \in A_{-1}$ if and only if $a^y = a^{-1}$ for a single element $y \in G \setminus F(G)$.

Combining the above decomposition of A with the precise decomposition of G' obtained in the proof of Lemma 2.14 (see eq. (2.16)), we obtain moreover that

$$G' = A_{-1} \times Q'. \tag{2.14}$$

Proof Lemma 2.14. As noted in (2.11) one has a decomposition $G \cong A \rtimes Q$ with Q a 2-group and A an abelian $2'$-group. Let $\varphi : Q \to \mathrm{Aut}(A)$ be the associated action. Note that $Q \cap F(G) = \mathrm{Ker}(\varphi)$ and thus in particular $F(G) = A \times \mathrm{Ker}(\varphi)$. By Theorem 2.6 (3), it then follows that $Q/\mathrm{Ker}(\varphi)$ is an elementary abelian 2-group, since $Q/\mathrm{Ker}(\varphi) \cong G/F(G)$, and $F(G)$ contains the group B given by Theorem 2.6 (3) since it is the maximal normal nilpotent subgroup of G. In particular $\exp(G/F(G)) \mid 2$ and $\exp(Q/(F(G) \cap Q)) = 2$.

From Theorem 2.6, it follows that the prime divisors of $|G|$ are given by $\{2, 3\}$. Hence A is a 3-group. To see (1), it remains to prove that $\exp(A) \mid 3$. We treat the cases G nilpotent and G non-nilpotent separately.

Suppose first that G is nilpotent. Then since $G \cong A \times Q$, we have $\mathbb{Q}G \cong \mathbb{Q}A \otimes_{\mathbb{Q}} \mathbb{Q}Q$. Moreover, since A is abelian,

$$\mathbb{Q}A \cong \bigoplus_{d \mid |A|} n_d \mathbb{Q}(\zeta_d),$$

with n_d the number of cyclic subgroups of A of order d. Since $\mathbb{Q}G$ has a matrix component $M_2(D)$ by assumption, it follows that this component appears in the decomposition of $\mathbb{Q}Q$. Hence $\mathbb{Q}G$ has a component $\mathrm{M}_2(D \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_d))$, for some division ring D and some $d \mid |A|$. Note that $\mathbb{Q}(\zeta_d)$ is contained in the centre of $D \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_d)$. However since d is odd, we see that the latter can only be an exceptional matrix component if $d = 3$ as $[\mathbb{Q}(\zeta_d) : \mathbb{Q}] = \varphi(d) > 2$ otherwise. Hence A is an elementary abelian 3-group.

Suppose now that G is non-nilpotent. We consider $\overline{G} := A \rtimes (Q/\mathrm{Ker}(\varphi))$, which has property ($\mathrm{M_{exc}}$) as well by Lemma 2.8. Moreover, the action of $Q/\mathrm{Ker}(\varphi)$ is faithful by construction. As $Q/\mathrm{Ker}(\varphi)$ is elementary abelian it follows from Proposition 2.9 that $[\overline{G} : F(\overline{G})] \mid 4$. Additionally, by construction $F(\overline{G}) = A$. Hence the statement will follow if we prove it for groups $A \rtimes C_2$ and $A \rtimes (C_2 \times C_2)$ with faithful action.

Let C be C_2 or $C_2 \times C_2$. In both cases it follows from [15, Lemma 1.2] that $A \cong (A \cap \mathcal{Z}(A \rtimes C)) \times P$ for some characteristic subgroup $P \leqslant A \rtimes C$. Additionally, P is non-trivial, since otherwise $A \leqslant \mathcal{Z}(A \rtimes C)$, a contradiction with the non-nilpotency assumption. Now,

$$P \rtimes C \cong G/\left(\mathcal{Z}(A \rtimes C) \cap A\right),$$

and in particular $P \rtimes C$ has ($\mathrm{M_{exc}}$) by Lemma 2.8. Let $x \in P$ have maximal order, say $o(x) = 3^m$. Then there is a subgroup $K \leqslant P$ such that $P \cong \langle x \rangle \times K$ (since $P \leqslant A$ is abelian).

There exists a SSP (H, K) for K as above. Indeed, the set

$$S := \{E \leqslant P \rtimes C_2 \mid P \leqslant E \text{ and } E' \leqslant K \leqslant E\}$$

is non-empty since it contains P, and it is finite by construction. Then (H,K) is given by a maximal element $H \in S$ by Theorem 2.5. From [17, Lemma 3.4], we have

$$\dim_{\mathbb{Q}}(\mathbb{Q}[P \rtimes C]e(P \rtimes C, H, K)) = [P \rtimes C : H] \cdot [H : N_H(K)] \cdot \phi([H : K]).$$

Moreover, since $P \rtimes C$ has $(\mathrm{M}_{\mathrm{exc}})$ by construction, the above $\mathbb{Q}$-dimension is an element of $\{4, 8, 16\}$ by Theorem 2.6. Since $[P : K] = |\langle x\rangle| = 3^m$, we have that

$$\phi([P : K]) = 3^{m-1}(3-1).$$

Additionally, $\phi([P : K])$ divides $\phi([H : K])$ and hence we also have

$$2 \cdot 3^{m-1} \mid \dim_{\mathbb{Q}}(\mathbb{Q}[P \rtimes C]e(P \rtimes C, H, K)) \in \{4, 8, 16\}.$$

It follows that $m = 1$. In particular, by definition of m, A is an elementary abelian 3-group. Statement (1) follows.

We now show (2). First note that $[Q : \mathrm{Ker}(\varphi)] = [G : F(G)]$ follows immediately since $A \leq F(G)$. Note also that $\mathrm{Ker}(\varphi)$ is a normal subgroup of G. We now have that

$$G/\mathrm{Ker}(\varphi) \cong A \rtimes Q/\mathrm{Ker}(\varphi) \cong C_3^n \rtimes C_2^k,$$

for some $n, k \in \mathbb{N}$ and where the action is faithful.

Since $A = C_3^n$ its automorphism group $\mathrm{Aut}(C_3^n)$ is isomorphic to $\mathrm{GL}_n(\mathbb{F}_3) = \mathrm{GL}(n, 3)$. Furthermore, a faithful action of C_2^k on A is equivalent to an injection $C_2^k \hookrightarrow \mathrm{GL}_n(\mathbb{F}_3)$.

Every involution in $\mathrm{GL}(n, 3)$ is diagonalisable with eigenvalues ± 1, and any commuting family of such involutions is simultaneously diagonalisable. Thus any elementary abelian 2-subgroup of $\mathrm{GL}(n, 3)$ is conjugate to a subgroup of the diagonal matrices

$$\{\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n) \mid \varepsilon_i = \pm 1\} \;\cong\; C_2^n.$$

Hence the largest possible value of k is n. Furthermore, for each possible value of $k \in \{0, \ldots, n\}$, one can choose the generators of A and $Q/\mathrm{Ker}(\varphi)$ in such a way that the action is given by:

$$\varphi \colon C_2^k \longrightarrow \mathrm{GL}_n(\mathbb{F}_3), \quad y_i \mapsto \mathrm{diag}(1, \ldots, 1, \underset{i\text{-th}}{-1}, 1, \ldots, 1),$$

for $i \in \{1, \ldots, k\}$, i.e. y_i acts by inversion on the i-th factor of A and trivially on the others.

Now as G has $(\mathrm{M}_{\mathrm{exc}})$, one has that $\mathrm{cd}(G) \subseteq \{1, 2, 4\}$ by Theorem 2.6. In particular $\mathrm{cd}(G/\mathrm{Ker}(\varphi)) \subseteq \{1, 2, 4\}$ as well. Moreover, from the above description it is clear that $F(G/\mathrm{Ker}(\varphi)) \cong A$. Thus Proposition 2.9 yields that

$$2^k = [G/\mathrm{Ker}(\varphi) : F(G/\mathrm{Ker}(\varphi))] = \max(\mathrm{cd}(G)).$$

Therefore $k \leq 2$. Since $2^k = [G : F(G)]$ as well, it suffices to discard the possibility $k = 2$ to obtain statement (2). By the above equality, that possibility only occurs if $4 \in \mathrm{cd}(G/\mathrm{Ker}(\varphi))$. This implies that $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right)) \in \mathcal{C}(G/\mathrm{Ker}(\varphi))$, since G has $(\mathrm{M}_{\mathrm{exc}})$. On the other hand, if $n \gneq k = 2$, then it follows from the description of the action φ that there are some elements in A which are centralised by $Q/\mathrm{Ker}(\varphi)$. In particular, $A \cap \mathcal{Z}(G/\mathrm{Ker}(\varphi)$ is non-trivial. However, this would imply that there exists a simple component $\mathbb{Q}(\zeta_3) \otimes \mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ of G, which contradicts the fact that G has $(\mathrm{M}_{\mathrm{exc}})$. Hence all that remains to understand is the case that $k = n = 2$. Note that in this case,

$$G/\mathrm{Ker}(\varphi) \cong C_3^2 \rtimes C_2^2 \cong D_6^2.$$

As $\mathrm{M}_2(\mathbb{Q}) \in \mathcal{C}(\mathbb{Q}[D_6])$, we have that $\mathrm{M}_2(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{M}_2(\mathbb{Q}) \cong \mathrm{M}_4(\mathbb{Q}) \in \mathcal{C}(\mathbb{Q}[D_6 \times D_6)]$. Thus $G/\mathrm{Ker}(\varphi)$ does not have $(\mathrm{M}_{\mathrm{exc}})$, a contradiction. It follows that $k = 1$, and in particular, $[G : F(G)] = 2$, as desired.

For statement (3), consider again the abelian normal subgroup B of G such that $\exp(G/B) \leq 2$, as provided by Theorem 2.6. Without loss of generality, we assume B to be maximal among such subgroups. Let $K \leqslant G$ be such that $C' \leqslant K \leqslant C$ for some subgroup $C \leqslant G$ containing B. By Theorem 2.5, a SSP (H, K) satisfies $B \leq H$. Furthermore, by Theorem 2.4, the degree of the associated simple algebra $\mathbb{Q}Ge(G, H, K)$ is $[G : H]$.

Consequently, since G has ($\mathrm{M}_{\mathrm{exc}}$), Theorem 2.6 implies that $[G : H] \mid 4$, and hence the following claim would imply statement (3).

Claim 1: there exists a SSP (H, K) of G with $H = B$.

Before proving the claim, we observe some consequences of the fact that $[G : F(G)] = 2$. Namely,

$$A = A_{-1} \times (A \cap \mathcal{Z}(G)), \tag{2.15}$$

where

$$A_{-1} := \{a \in A \mid a^y = a^{-1} \text{ for all } y \in G \setminus F(G)\}.$$

Indeed, we remark that for a given $a \in A$ and $y \notin C_G(\langle a \rangle)$, one has that $a^y = a^{-1}$ or $a^y \notin \langle a \rangle$, since $o(a) = 3$ and $y^2 \in F(G)$. Now the elements in A for which $\langle a \rangle$ is not normalised by y can be obtained from an element in A_{-1} and $A \cap \mathcal{Z}(G)$. Indeed, if $a^y \notin \langle a \rangle$ then $a^{-1}a^y \in A_{-1}$ and $aa^y \in A \cap \mathcal{Z}(G)$. Furthermore,

$$a = (a^{-1}a^y) \cdot (aa^y)^{-1},$$

which proves (2.15).

The decomposition (2.15) now allows to describe the commutator subgroup of G as follows:

$$G' = A_{-1} \times Q'. \tag{2.16}$$

Indeed, since $G \cong A \rtimes Q$, it follows that $G' = \langle [A, Q], Q' \rangle$. Hence to see (2.16), it suffices to show that $[A, Q] = A_{-1}$. If $a \in A_{-1}$ and $y \in Q \setminus (Q \cap F(G))$, then $[a, y] = a^2 = a^{-1}$, and hence $a \in [A, Q]$. Conversely, the only non-trivial generators of $[A, Q]$ are those of the form $[a, y]$ with $a \in A \setminus (A \cap \mathcal{Z}(G))$ and $y \in Q \setminus (Q \cap F(G))$. But for any such a and y, one has that $[a, y]^y = [a, y]^{-1}$, i.e. $[a, y] \in A_1$. Hence $[A, Q] = A_{-1}$.

Now we proceed with the proof of Claim 1. If $F(G)$ is abelian, then $B = F(G)$ and statement (3) follows from (2). So suppose that $F(G)$ is non-abelian.

Claim 2: there exists a $b \in B \cap Q$ which is maximal cyclic in $\mathrm{Ker}(\varphi)$ and such that $\langle b \rangle \cap Q' \neq 1$.

Let $b \in B \cap Q$ be an element such that $\langle b \rangle$ is maximal cyclic in $B \cap Q$. Then $\langle b \rangle$ is maximal cyclic in $\mathrm{Ker}(\varphi) = F(G) \cap Q$. Indeed, suppose this is not the case. Then there is an element $x \in \mathrm{Ker}(\varphi)$ such that $x^2 = b$, since $\exp(F(G)/B) = 2$. In particular, $x^2 e = be$, for every $e \in \mathrm{PCI}(\mathbb{Q}G)$. Now, consider a quotient Ge of G such that $F(Ge)$ is non-abelian. The latter exists since $F(G) \lesssim \prod_{e \in \mathrm{PCI}(\mathbb{Q}G)} F(Ge)$. A manual verification for the ($\mathrm{M}_{\mathrm{exc}}$) groups obtained in the proof of Theorem 2.6 shows that there Be contains a cyclic subgroup which is maximal cyclic in $F(Ge)$ and satisfying the commutator condition (such existence is somehow a consequence of the fact that B has the extra property that $[Ge : Be] \mid 4$, cf. footnote to Theorem 2.6). Thus b can be chosen as any pre-image of such element.

Using the b yielded by claim 2 we can now construct the desired SSP for Claim 1. If G is not-nilpotent, then $A_{-1} \neq 1$ and we take some $a \in A_{-1}$. In that case $K = \langle A \setminus \langle a \rangle \rangle \times \langle (B \cap Q) \setminus \langle b \rangle \rangle$ which satisfies that B/K is cyclic. Furthermore B is the maximal cyclic overgroup of K. Indeed, as the image of a in H/K is non-trivial we are not allowed to add an element $y \in Q \setminus \ker(\varphi)$. By construction of b we can also not add an element $x \in \ker(\varphi)$ as the image of $\langle x, b \rangle$ would not be cyclic.

If G is nilpotent, we consider $K = A \times \langle (B \cap Q) \setminus \langle b \rangle \rangle \}$. Now $\ker(\varphi) = Q$, but nevertheless the same argument on b applies.

Finally, we need to prove statement (4). **still need to think (maybe by proving that in that case $cd(Q) = \{1, 2\}$ and no exceptional component and hence info recoverable from free-by-free? That would also allow to have the isomorphism type of Q...**

$\square$

2.4.2. *Strong shoda pairs and* $(\mathrm{M}_{\mathrm{exc}})$ *groups.* **To do: make structure and bind zinnnen beter in this section** Suppose that $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$. In order to obtain more essential properties *we describe the tuples* (H,K) *that yield a strong Shoda pair for* $\mathbb{Q}G$ *such that* $\mathbb{Q}Ge(G,H,K)$ *is not commutative.* Recall that the latter, see [18, Lemma 2.4], means that $H \neq G$ or equivalently that $G' \nsubseteq K$.

By Lemma 2.14 and Theorem 2.5 the group H contains a maximal abelian normal subgroup B such that $G/B \cong C_2$ or $C_2 \times C_2$. Consequently, we need to describe for $H = B$ and $H = \langle B, t\rangle$, with $t \in Q \setminus (B \cap Q)$, what the possible subgroups K are such that (H,K) is a SSP. For this we need that H/K is cyclic. As $G \cong A \rtimes Q$ with A an elementary abelian 3-group and Q a 2-group, one is able to describe the possible groups K by saying which $a \in A$ and $c \in Q$ (and until which power) 'survive' in H/K. We now formalise this, which requires to define the following sets:

$$\mathcal{M}_{B_2} := \{x \in Q \cap B \mid \langle x\rangle \text{ is maximal cyclic in } Q \cap B\},$$

$$\mathcal{S}_B := \{(x,m) \in (Q \cap B) \cup \{1\} \times \mathbb{N} \mid 1 \neq x \in \mathcal{M}_{B_2} \text{ and } 0 \leq m \leq v_2(o(x))\}$$

and

$$\mathcal{S}_{tB} := \{(x,m) \in t(Q \cap B) \times \mathbb{N} \mid m = \begin{cases} 1, & \text{if } x^2 \notin \mathcal{M}_{B_2}, \\ \in [0, v_2(o(x))], & \text{else.} \end{cases} \}$$

To a triple $(a,x,m) \in A \times (\mathcal{S}_B \cup \mathcal{S}_{tB})$ we associate a subgroup of H:

$$K_{(a,x,m)} := \langle A \setminus \{a^{\pm 1}\}\rangle \rtimes \left\langle ((H \cap Q) \setminus \langle x\rangle) \cup \{x^{2^m}\} \right\rangle. \tag{2.17}$$

It is readily verified that any subgroup K in a SSP (H,K) with $H \neq G$ is of the form $K_{(a,c,m)}$. We now say which triples yield which overgroup H.

(I) SSP with $H = B$:

a tuple $(B, K_{(a,x,m)})$ is a SSP if and only if

- $(a,x,m) \in A \times \mathcal{S}_B$,
- $G' \nsubseteq K_{(a,x,m)}$,
- and there is no $t \in \mathrm{Cen}_Q(\langle a\rangle) \setminus (Q \cap B)$ such that $x = t^2$.

(II) SSP with $H = \langle B, t\rangle$*:*

Consider a fixed $t \in Q \setminus (Q \cap B)$. Then a tuple $(\langle B, t\rangle, K_{(a,x,m)})$ is a SSP if and only if either

- $(a,x,m) \in A \times \mathcal{S}_t$,
- $G' \nsubseteq K_{(a,c,m)}$,
- and $x \in \langle t^2\rangle$.

or

- $(a,x,m) \in A \times \mathcal{S}_{tB}$,
- and $G' \nsubseteq K_{(a,x,m)}$.

Next, we aim for a characrerisation of the $(\mathrm{M}_{\mathrm{exc}})$ property in terms of strong Shoda pairs.

To start we study $\mathrm{PCI}(\mathbb{Q}G)$. Recall that G is called *strongly monomial* if each primitive central idempotent e of $\mathbb{Q}G$ comes from a SSP, i.e. $e = e(G,H,K)$ for some SSP (H,K). For example all abelian-by-supersolvable groups are strongly monomial [20, Theorem 3.5.10]. In particular, by Theorem 2.6, $(\mathrm{M}_{\mathrm{exc}})$ implies strongly monomial. Therefore we now describe in terms of (H,K) when a simple component associated to a SSP yields an exceptional component. This consequently also yields a characterisation of $(\mathrm{M}_{\mathrm{exc}})$ in terms of the SSP.

We will use the terminology that "$\mathbb{Q}Ge$ *is* $(\mathrm{M}_{\mathrm{exc}})$*-exceptional* " to mean that $\mathbb{Q}Ge$ is an exceptional algebra that can occur as a simple component of a $\mathbb{Q}G$ with $(\mathrm{M}_{\mathrm{exc}})$. In other words if $\mathbb{Q}Ge$ is isomorphic to one of the simple algebras listed in Theorem 2.6. In particular if $\mathbb{Q}Ge$ is an exceptional division algebra, then it is isomorphic to $\left(\frac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right)$.

Lemma 2.16. *Let (H,K) be a SSP of G with $H \neq G$ and denote $e := e(G,H,K)$ the associated primitive central idempotent.*

(1) If $N_G(K) = H$, then:

$$\mathbb{Q}Ge \text{ is exceptional } \Leftrightarrow \phi([H:K]) = \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \leq 2 \text{ and } [G:H] = 2,$$

(2) If $H \lneq N_G(K) \lneq G$, then $\mathbb{Q}Ge$ is exceptional if and only if $[H:K] = 4$ or 6, $[G:H] = 4$, and

$$\text{for all } h \in H \setminus K,\ y \in N_G(K) \setminus H :\ h.h^y,\ y^2.h^{o(h)/2} \in K.$$

(3) For $N_G(K) = G$ we suppose $G/H \ncong C_4$. Then $\mathbb{Q}Ge$ is $(\mathrm{M}_{\mathrm{exc}})$-exceptional matrix algebra if and only if one of the following cases holds[11]*:*

(3.i) $[G:H] = 2$, $[H:K] = 4$ *or* 6 *and* $h.h^y, y^2 \in K$ *for all* $h \in H \setminus K, y \in G \setminus H$.

(3.ii) $[G:H] = 2$, $[H:K] = 8$ *or* 12 *and for some* $h \in H$ *such that* $H/K = \langle h \rangle$ *it holds*

- *either* $h^{-5}.h^y, y^2.h^{i.o(h)/4} \in K$ *for some* $0 \leq i \leq 3$,
- *or* $h^{-7}.h^y, y^2.h^{2j} \in K$ *for some* $0 \leq j \leq 4$ *(only if* $[H:K] = 12$*)*

for all $y \in G \setminus H$.

(3.iii) $[G:H] = 4$, $[H:K] = 8$ *or* 12. *Furthermore, for all* $h \in H \setminus K$ *such that* $H/K = \langle h \rangle$ *and* $y \neq t \in G \setminus H$ *one has that*

$$y^4 \in K,\ h^y.(h^{-1})^t \notin K$$

and $y^2 h^{o(h)/2} \in K$ *if* $h.h^y \in K$.

for a choice of y! Tell the grp they yield rather !

The value of $\phi([H:K])$ in case (3.i)-(3.ii) can be summarised in a similar way as in case (1). Namely if $N_G(K) = G$ and $[G:H] = 2$, then the proof yields that

$$\phi([H:K]) = 2 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \in \{2,4\}.$$

Also note that in case (3.iii) the condition that $h^y K \neq h^t K$ implies that for every $1 \neq d$ relatively prime to $o(h) = [H:K]$ there is a $t \in G \setminus H$ such that $h^t = h^d$.

Remark 2.17. The condition that $G/H \ncong C_4$ when K is normal in G is satisfied when $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$, hence sufficient for our purposes. However it could be removed by adding extra cases (whose description could be found via similar methods). To see that it is satisfied consider the abelian normal subgroup B from Theorem 2.6. As $\exp(G/B) = 2$ it contains the commutator subgroup G'. Consequently, Theorem 2.5 tells that $B \leq H$ which entails $\exp(G/H) = 2$ and hence $G/H \ncong C_4$.

Note also that verifying when $\mathbb{Q}Ge$ is $(\mathrm{M}_{\mathrm{exc}})$-exceptional hides the assumption that $5 \nmid [H:K]$. Indeed, as $\ker(\varphi_e \colon G \to Ge) = core_G(K) \leq K$ one has that $[H:K] = [\varphi_e(H) : \varphi_e(K)]$. Now Theorem 2.6 tells us that $\pi(G) \subseteq \{2,3\}$ and thus $5 \nmid [\varphi_e(H) : \varphi_e(K)]$. Also this case could be handled by adding extra cases.

Remark 2.18. The proof of Lemma 2.16 will furthermore tell the isomorphism type of $\mathbb{Q}Ge(G,H,K)$ for each case. More precisely, the following holds:

- $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q})$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(i))$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-2}))$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-3}))$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\left(\frac{-1,-1}{\mathbb{Q}}\right))$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\left(\frac{-1,-3}{\mathbb{Q}}\right))$ if and only if ...

[11] $\phi([H:K]) \leq 2$ means that it divides 4 or 6. Furthermore, when $5 \nmid [H:K]$, $\phi([H:K]) \mid 4$ means that it divides 8 or 12.

Proof of Lemma 2.16. For the remainder of the proof denote $N := N_G(K)$ and $e := e(G,H,K)$. Recall that by Theorem 2.4

$$\mathbb{Q}Ge \cong \mathrm{M}_{[G:N_G(K)]}(\mathbb{Q}(\zeta_{[H:K]}) * N_G(K)/H)$$

with the crossed product defined in (2.4) and (2.5). Furthermore, $\deg(\mathbb{Q}Ge) = [G:H]$ and hence if $\mathbb{Q}Ge$ is an exceptional matrix algebra one needs that $[G:H] \mid 4$.

To start suppose that $N = H$. Then $\mathbb{Q}Ge \cong \mathrm{M}_{[G:H]}(\mathbb{Q}(\zeta_{[H:K]}))$. As $G \neq H$, this is exceptional if and only if $[G:H] = 2$ and $[\mathbb{Q}(\zeta_{[H:K]}) : \mathbb{Q}] \leq 2$. Note that $\dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) = [\mathbb{Q}(\zeta_{[H:K]}) : \mathbb{Q}] = \phi([H:K])$, implying statement (1).

Next consider $H \lneq N \lneq G$. Since $N \lneq G$ the component $\mathbb{Q}Ge$ is not a division algebra. Hence, as noticed earlier, if exceptional the index $[G:H]$ divides 4. Since also $H \lneq N$, we obtain that $[G:N] = 2 = [N:H]$. Now note that these indices imply that $\mathbb{Q}Ge$ is exceptional if and only if it is isomorphic to $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some $a,b \in \mathbb{N}_0$. Using the dimension formula in Theorem 2.4 we obtain that

$$16 = \dim_{\mathbb{Q}} \mathbb{Q}Ge = 4.2.\phi([H:K]).$$

Thus $\phi([H:K]) = 2$ and hence $[H:K] = 3, 4$ or 6. For the other conditions we need to be more precise. First denote $m := [H:K], N/H = \langle \overline{y} \rangle$ and $H/K = \langle hK \rangle$. Now remark that

$$\mathbb{Q}(\zeta_m) * N/H = \mathrm{Span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\}.$$

By (2.4) one has that $\zeta_m^{\overline{y}} = \zeta_m^i$ with i such that $h^i.h^y \in K$. Furthermore, using that $[N:H] = 2$, the crossing (2.5) tells us that $\overline{y}^2 = \zeta_m^j$ with j such that $y^2h^{-j} \in K$. In summary, as an abstract group we obtained that

$$\langle \zeta_m, \overline{y} \rangle \cong \langle a, b \mid a^m = 1, b^2 = a^j, a^b = a^i \rangle.$$

Recall that, when $[G:N] = 2 = [N:H]$, we already knew that $\mathbb{Q}Ge$ will be exceptional exactly when $\mathbb{Q}(\zeta_m) * N/H \cong \left(\frac{-a,-b}{\mathbb{Q}}\right)$ for some $a,b \in \mathbb{N}_0$. This will yield the remaining conditions on (m,j,i). Indeed, looking at the centre $\mathbb{Q} = \mathcal{Z}(\mathbb{Q}(\zeta_m) * N/H)$, combined with $m \mid 4$ or 6, we see that necessarily $i = -1$. For the other restrictions the required observation is that one needs that $\mathcal{C}(\mathbb{Q}[\langle \zeta_m, \overline{y} \rangle])$ contains such a quaternion algebra. Now if $a^j = 1$ then $\langle \zeta_m, \overline{y} \rangle \cong D_{2m}$. However $\mathcal{C}(\mathbb{Q}D_{2m}) = \{\mathbb{Q}, \mathrm{M}_2(\mathbb{Q})\}$ for those m, so $a^j \neq 1$. If $a^j = a^{\pm 1}$, then $\langle \zeta_m, \overline{y} \rangle$ is cyclic and so its group algebra has only commutative components. In particular, for $m = 3$ there is no admissible j, yielding $[H:K] = 4$ or 6, as desired. And for $m = 4$ we necessarily have that $b^2 = a^2 = a^{o(a)/2}$. Finally, for $m = 6$ if $i = 2, 4$ then $\langle \zeta_m, \overline{y} \rangle \cong C_2 \times C_2$ abelian which yield $i = 3$ as desired. Altogether we obtained that $\langle \zeta_m, \overline{y} \rangle \cong Q_8$ if $m = 4$ and to $Dic_{4.3} := C_3 \rtimes C_4$ if $m = 6$. Inspecting their simple component, we conclude that the former case ($m = 4$) yields $\mathrm{Span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\} = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ and the latter ($m = 6$) that $\mathrm{Span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\} = \left(\frac{-1,-3}{\mathbb{Q}}\right)$ (to conclude that $\mathrm{Span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\} \neq \mathrm{M}_2(\mathbb{Q})$, which is another simple component of $\mathbb{Q}[Dic_{12}]$ we use that $\langle \zeta_m, \overline{y} \rangle \cong Dic_{12}$ must be embedded in the component). Hence the conditions written are also enough to conclude that $\mathbb{Q}Ge$ is exceptional.

Finally suppose that K is normal in G. Then $\mathbb{Q}Ge \cong \mathbb{Q}(\zeta_{[H:K]}) * G/H$. We will obtain the desired statement via the same methods as in the case $H \lneq N \lneq G$. Recall $\deg(\mathbb{Q}Ge) = [G:H]$ which need to divide 4 in order for $\mathbb{Q}Ge$ to be an exceptional matrix algebra.

First suppose that $[G:H] = 2$. Hence $\mathbb{Q}Ge$ is exceptional matrix component if and only if $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{N}$. Therefore, $4 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) = \dim_{\mathbb{Q}} \mathbb{Q}Ge = 2.1.\phi([H:K])$ and consequently

$$\phi([H:K]) = 2 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \in \{2, 4\}. \tag{2.18}$$

The value $\phi([H:K]) = 4$ is equivalent to $[H:K] \in \{5, 8, 10, 12\}$. As we assumed that $5 \nmid [H:K]$ we conclude that $[H:K] = 8$ or 12 when $\phi([H:K]) = 4$. We consider both values in (2.18) separately.

Firstly *suppose that* $\phi([H:K]) = 2$, i.e. $[H:K] = 3, 4$ or 6 and $\mathcal{Z}(\mathbb{Q}Ge) = \mathbb{Q}$. In this case $\mathbb{Q}Ge$ is an exceptional matrix component if and only if $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q})$. For this we need to describe $\mathbb{Q}Ge \cong \mathbb{Q}(\zeta_{[H:K]}) * N/H$. Note that $\phi([H:K]) = 2 = [N:H]$ as in case (2). Therefore the computations are literally the same as there. The only difference is that now we need to understand which of the groups $\langle \zeta, \overline{y} \rangle$ yield that $\mathrm{Span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\} = \mathrm{M}_2(\mathbb{Q})$. Looking back to the cases we see that this exactly happens when $\langle \zeta, \overline{y} \rangle \cong D_{2m}$, whose presentation represents the relations mentioned in case (3.i).

Next *suppose that* $\phi([H:K]) = 4$, i.e. $[H:K] = 8$ or 12 and $[\mathcal{Z}(\mathbb{Q}Ge) : \mathbb{Q}] = 2$. **to write**

It remains to consider *the case that* $[G:H] = 4$. Then $\mathbb{Q}Ge$ is an exceptional matrix component if and only if $\mathbb{Q}Ge \cong \mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some $a, b \in \mathbb{N}_0$. The dimension formula now gives $16 = 4.1.\phi([H:K])$, i.e. $\phi([H:K]) = 4$. As noticed earlier, as $5 \nmid [H:K]$, this means that $[H:K] = 8$ or 12. For the stated relation we again follow the same strategy. **part proof still need to write**

$\square$

2.4.3. *On the Sylow* 2*-subgroup of a* $(\mathrm{M}_{\mathrm{exc}})$ *group.* With the description of the SSP just obtained we now have the necessary tools to obtain the remaining essential necessary conditions for $\mathbb{Q}G$ to have $(\mathrm{M}_{\mathrm{exc}})$. For one of the properties the following two groups will appear:

$$(C_2 \times C_2) \rtimes C_4 := \langle a, b, c \mid a^2 = b^2 = c^4 = 1, a^c = ab, b^c = b, ab = ba \rangle, \tag{2.19}$$

$$(C_2 \times C_2) \wr C_2 := \langle x_1, \ldots, x_4, y \mid \forall i, j : y^2 = x_i^2 = 1,\ [x_i, x_j] = 1,\ x_1^y = x_3, x_2^y = x_4 \rangle. \tag{2.20}$$

These groups have respectively SmallGroupId $[16, 3]$ and $[32, 27]$

Lemma 2.19. *Let G be a finite group and Q a Sylow 2-subgroup. Suppose that $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ and has a matrix component. Then the following holds:*

1. *The nilpotency class of Q is at most 3,*
2. *if Q is non-abelian, then $\mathbb{Q}G$ has no exceptional division components,*
3. *if Q is class 2, then $\exp(Q/\mathcal{Z}(Q)) = 2$,*
4. *Q has class at most 2 if and only if $x.x^y \in \mathcal{Z}(Q)$ for all $x \in F(G) \cap Q$ and $y \notin C_Q(x)$,*
5. *if Q is class 3 then $Q/\mathcal{Z}(Q)$ is isomorphic to $C_2^n \times K$ with K one of the following groups*
$$D_8, \quad (C_2 \times C_2) \rtimes C_4, \quad \text{or } (C_2 \times C_2) \wr C_2.$$

Furthermore, the latter two groups only occur for G nilpotent with $4 \in \mathrm{cd}(G)$. If G is moreover non-nilpotent then

6. *$\mathrm{cd}(Q) \subseteq \{1, 2\}$,*
7. *$F(G)$ is abelian if and only if $\mathrm{cd}(G) = \{1, 2\}$ and else it has class 2,*

In the next section we will give necessary and sufficient conditions for $\mathbb{Q}G$ to have $(\mathrm{M}_{\mathrm{exc}})$ depending on nilpotency class of a Sylow 2-subgroup.

Proof of Lemma 2.19. Note that G maps onto Ge. If we choose a Sylow 2-subgroup Q of G, then Qe will also be a Sylow 2-subgroup of Qe. Therefore $cl(Qe) \leq cl(Q)$. Furthermore $Q \leq \prod_e Qe$. This yields that

$$cl(Q) = \max\{cl(Qe) \mid e \in \mathrm{PCI}(\mathbb{Q}G)\}. \tag{2.21}$$

Now verifying the possible $(\mathrm{M}_{\mathrm{exc}})$ groups classified in the proof of Theorem 2.6 we get (1).

For (2): The proof of Theorem 2.6 shows that the only way that $\mathbb{Q}G$ can have exceptional division components is if G maps onto on either (I) a group $C_3 \rtimes C_{2^m}$ with $m \geq 4$ and the action by inversion or (II) a group of the form $C_m \times Q_8$ in the family b.iii) with m odd. In Equation (2.9) it was obtained that case (II) can't occur as we assumed that $\mathbb{Q}G$ has a matrix component. Suppose that it would map on a group of the form $C_3 \rtimes C_{2^m}$, then there is an element $x \in Q$ with $v_2(o(x)) \geq m$.

Next, if $\mathbb{Q}G$ has ($\mathrm{M_{exc}}$), then also the quotient $\mathbb{Q}Q$. Since Q is a non-abelian nilpotent 2-group, it has no exceptional division components. Indeed, it can not map onto one of the non-nilpotent groups of type (I). Further it can only map on Q_8 in type (II) whose rational group algebra however has no exceptional division components. Now Theorem 2.6 yields that $\exp(Q) \mid 24$ and hence a divisor of 8, being a 2-group. This shows that $m \leq v_2(o(x)) \leq 3$ and hence G does not map onto a group of type (I) either, yielding (2).

For (3) note that $\mathcal{Z}(Q) = Q \cap \prod_{e \in \mathrm{PCI}(\mathbb{Q}G)} \mathcal{Z}(Qe)$ and hence $\exp(Q/\mathcal{Z}(Q)) = \exp(Q/Q \cap \prod_e \mathcal{Z}(Qe)) \mid \exp(\prod_e Qe/\mathcal{Z}(Qe))$. Furthermore by (2.21) we have that $cl(Qe) \leq 2$ for all e. Also $\mathbb{Q}G$ has no exceptional components by (2), thus again looking at the groups obtained in the proof of Theorem 2.6, combined with this extra restriction, yields that $\exp(Qe/\mathcal{Z}(Qe)) \mid 2$ for all e, yielding (3).

Next we consider statement (4).

For (6): The Sylow 2-subgroup Q, being also a quotient of G, has ($\mathrm{M_{exc}}$). Hence by Theorem 2.6 $\mathrm{cd}(Q) \subseteq \{1,2,4\}$ and every simple component of $\mathbb{Q}Q$ is associated to a SSP. Furthermore $4 \in \mathrm{cd}(Q)$ if and only if $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ is a simple component of $\mathbb{Q}G$.

Let B be the maximal abelian normal subgroup of G from Lemma 2.14. Recall that $B = A \rtimes (B \cap Q)$ and $[G:B] = [Q : B \cap Q]$ divides 4. Now consider a SSP (H,K) of Q such that $\mathbb{Q}Qe(Q,H,K) \cong \mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$. This implies that $[Q:H] = 4$ and thus $H = B \cap Q$ by Theorem 2.4. Moreover, $K = K_{(1,x,m)}$ for some $(x,m) \in \mathcal{S}_B$, see (2.17). From the description of the SSP with H equal to the maximal abelian normal subgroup given in Section 2.4.2 we obtain that $x \in Q'$ and maximal cyclic in Q.

Next, take $1 \neq a \in A_{-1} \cup (\mathcal{Z}(G) \cap A)$ and observe that $(B, K_{(a,x,m)})$ is a SSP of G such that $[G:B] = [Q: B\cap Q]$, $[G : N_G(K_{(a,x,m)})] = [Q : N_Q(K_{(1,x,m)})]$ and $\phi([B : K_{(a,x,m)}]) = 3^0(3-1)\phi([B \cap Q : K_{(1,x,m)}])$. Therefore the dimension formula in Theorem 2.4 yields that

$$\dim_{\mathbb{Q}} \mathbb{Q}Ge(G,B,K_{(a,x,m)}) = 2 \dim_{\mathbb{Q}} \mathbb{Q}Qe(Q, B\cap Q, K_{(1,x,m)}) = 32,$$

which is a contradiction to $\mathbb{Q}G$ having ($\mathrm{M_{exc}}$).

Finally we consider (7). Since G is metabelian by Theorem 2.6 we can use [16, Lemma 12.25] which says that

$$\max\{\chi(1) \mid \chi \in \mathrm{Irr}_{\mathbb{C}}(F(G))\} \leq \frac{\max\{\psi(1) \mid \psi \in \mathrm{Irr}_{\mathbb{C}}(G)\}}{[G:F(G)]}. \tag{2.22}$$

If $\mathrm{cd}(G) = \{1,2\}$ the right hand side of (2.22) equals 1 by Lemma 2.14 and hence $F(G)$ is abelian. Conversely if $F(G)$ is abelian, then $\mathrm{cd}(G) = \{1,2\}$ as $[G:F(G)] = 2$ by Lemma 2.14.

If $4 \in \mathrm{cd}(G)$, then by the above $F(G)$ is not abelian. Furthermore Lemma 2.14 tells that $[Q:B] = 4 = 2.[F(G):B]$ where B is an abelian normal subgroup of smallest index in G. This allows to combine [16, Theorem 12.11] and part (2) in order to obtain that $[Q:Z(Q)] = 8$. If $Z(Q)$ is contained in $F(G)$, then we would have that $[F(G):Z(Q)] = 4$ which entails that $F(G)/Z(Q)$ is abelian and hence $F(G)$ class 2. If $Z(Q)$ is not contained in $F(G)$, then there exists left coset representatives $\{1,z\}$ of $F(G) \cap Q$ in Q with $z \in Z(Q)$. In particular, $\mathcal{Z}(F(G)) = \mathcal{Z}(Q) \cap F(G)$ and $F(G)/\mathcal{Z}(F(G))$ is a group of order 8. If it is abelian, then $F(G)$ would have class 2. Thus suppose that it is D_8 or Q_8.

□

All finite groups such that $\mathrm{SL}_1(\mathbb{Q}Ge)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ have been classified in [23]. In loc.cit. presentations for such groups were even given. Furthermore, they showed that the aforementioned property is equivalent to saying that all simple components of $\mathbb{Q}G$

are either fields, totally definite quaternion algebras or of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \geq 0$. The following result shows that groups with $(\mathrm{M}_{\mathrm{exc}})$ are index 2 overgroups of the groups classified in [23].

Corollary 2.20. *If $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ then G has an index two subgroup H whose non-division components are all of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \geq 0$.*

Example 2.21. The converse of Corollary 2.20 is not true in general. For example consider the following extraspecial group of order 2^5 (whose SmallGroup ID is [32,49]):

$$D_8 \circ D_8 := \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, c^2 = a^2, a^b = a^{-1}, c^d = a^2 c,$$
$$[a, c] = [a, d] = [b, c] = [b, d] = 1\rangle.$$

It can be verified that $\mathcal{C}(\mathbb{Q}[D_8 \circ D_8]) = \{\mathbb{Q}, \mathrm{M}_4(\mathbb{Q})\}$ (e.g. via the well-known description of its complex irreducible representations). On the other, e.g. to be proven via a manual check (via GAP), all the 2-groups until order 16, except D_{16}, have only matrix components of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \geq 0$.

2.5. **Characterisation of groups having** $(\mathrm{M}_{\mathrm{exc}})$**.** In this section we give a complete characters

2.5.1. *Groups with* $(\mathrm{M}_{\mathrm{exc}})$ *and Q abelian.*

Theorem 2.22. *Let G be a non-abelian finite group with abelian Sylow 2-subgroup. Then $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ if and only if*

$$G \cong D \times (C_3^k \rtimes \langle y : y^{2^t} = 1\rangle)$$

with $a^y = a^{-1}$ for all $a \in C_3^k$, D an abelian group with $\exp(D) \mid 4, 6$ and $t = 1$ or 2 if $3 \mid |D|$.

Note that the condition on D written means that either $D \cong C_3^{n_3} \times C_2^{n_2}$ or $D \cong C_2^{n_2} \times C_4^{n_4}$.

Proof. Suppose that $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ and hence G is of the form $A \rtimes Q$ given by (2.11) with action denoted $\varphi : Q \to \mathrm{Aut}(A)$. As we assume that G is non-abelian and Q is abelian, it implies that G is not nilpotent. Therefore it must have a matrix component by [36, Theorem 3.5]. Now Lemma 2.14 yields that $A \cong C_3^n$ and $[Q : \ker(\varphi)] = 2$.

Let $y \in Q$ be an element of maximal order in $Q \backslash \ker(\varphi)$. As Q is abelian we can decompose $Q \cong \langle y\rangle \times E$. As $[Q : \ker(\varphi)] = 2$ we have that $y^2 \in \ker(\varphi)$ and $E \leq \ker(\varphi)$. In particular, E is a central subgroup of G.

Next recall Proposition 2.9 saying that $F(G) = \mathcal{Z}(G) \times G'$. Furthermore $G' = A_{-1}$ by (2.16) and $\mathcal{Z}(G) = \langle E, (A \cap \mathcal{Z}(G)), y^2\rangle$. Note that we can decompose

$$G = (E \times (A \cap \mathcal{Z}(G)) \times (A_{-1} \rtimes \langle y\rangle).$$

Denote $D := E \times (A \cap \mathcal{Z}(G))$. From the allowed form of the centre of a matrix component ones deduces that $\exp(D)$ divides 4 or 6.

It only remains to show that $o(y) = 2$ or 4 if $3 \mid |D|$. Denote $o(y) = 2^t$ and decompose $A_{-1} = \langle x_1\rangle \times \cdots \times \langle x_\ell\rangle$. Next consider the quotient $\overline{G} := G/\langle x_2, \ldots, x_\ell, D\rangle \cong \langle x_1\rangle \rtimes \langle y : y^{2^t} = 1\rangle$ which again has $(\mathrm{M}_{\mathrm{exc}})$ by Lemma 2.8. The Wedderburn-Artin decomposition of $\overline{G}$ was given in (2.7). From this we see that $\mathrm{M}_2(\mathbb{Q}(i)) \in \mathcal{C}(\mathbb{Q}\overline{G}) \subset \mathcal{C}(\mathbb{Q}G)$ if $t \geq 3$. However, if $3 \mid D$, then $\mathbb{Q}G$ would have a component $\mathrm{M}_2(\mathbb{Q}(i, \zeta_3))$, which contradicts $(\mathrm{M}_{\mathrm{exc}})$.

We will now show that conversely all groups from the statement have $(\mathrm{M}_{\mathrm{exc}})$. Using the notations from the statement, note that $B := \langle D, C_3^k, y^2\rangle$ is the maximal abelian normal subgroup of G. In the sequel of the proof we denote B_2 and D_2 for the Sylow 2-subgroup of B, respectively D.

Consider a SSP (H, K). As $[G : B] = 2$ and we do not need to describe the field components, we have that $H = B$. The possible K's are of the form $K_{(a,x,m)}$ given in (2.17). Hereby $1 \neq a \in C_3^k$ arbitrary and (x, m) any tuple in $\mathcal{S}_B$. Note that in this case x is central and $\langle a\rangle$ is normal in G. Hence $N_G(K_{(a,x,m)}) = G$. Therefore, if we consider

$$\pi_{(a,x,m)} := \pi_{e(G,B,K_{(a,x,m)})} : G \to Ge(G, B, K_{(a,x,m)})$$

we have that $\pi_{(a,x,m)}(G) \cong \langle a, \pi(x), \pi(y)\rangle$. Since $B_2 = \bigcup_{j=0}^{o(y^2)} y^{2j} D_2$ there is two type of choices for x:

(i) $x \in y^{2j} D_2$ for $j \neq 0$, say $x = y^2 d$. Since $d \in \ker(\pi_{(a,x,m)})$ we obtain that

$$\pi_{(a,x,m)}(G) \cong \langle a\rangle \rtimes \langle \pi(y)\rangle \cong C_3 \rtimes C_{2^{m+1}}$$

with action given by inversion and $0 \leq m \leq v_2(o(y^{2j}))$.

(ii) $x \in D_2$. Then

$$\pi_{(a,x,m)}(G) \cong \langle x\rangle \times (\langle a\rangle \rtimes \langle \pi(y)\rangle \cong C_{2^m} \times (C_3 \rtimes C_2)$$

where $0 \leq m \leq 2$.

All the groups obtained in (i) and (ii) are in the list of ($\mathrm{M_{exc}}$) groups with a faithful irreducible $\mathbb{Q}$-representation listed in the proof of Theorem 2.6. For instance the decomposition of $\mathbb{Q}[\pi_{(a,x,m)}(G)]$ as in (i) is given in (2.7). Furthermore the groups in (ii) are D_6, D_{12} or $C_4 \times D_6$, depending on the value m, which are all in the table of Appendix A. Hence by Lemma 2.8 the groups from the statement indeed have ($\mathrm{M_{exc}}$). ☐

2.5.2. *Groups with* ($\mathrm{M_{exc}}$) *and* Q *class* 2. Next,

Theorem 2.23. *Let G be a non-nilpotent group with Sylow 2-subgroup of nilpotency class 2. Then $\mathbb{Q}G$ is ($\mathrm{M_{exc}}$) if and only if the following holds:*

(1) $G \cong A \rtimes Q$ *with A an elementary abelian 3-group and Q a 2-group,*
(2) $[G : F(G)] = 2$,
(3) there exists an abelian normal subgroup B such that G/B is isomorphic to C_2 or $C_2 \times C_2$,
(4) $\exp(Q/\mathcal{Z}(Q)) = 2$,
(5) $\exp(\mathcal{Z}(Q)) \mid 4$,
(6) $x.x^y \in \mathcal{Z}(Q)$ *for all* $x \in F(G)$ *and* $y \notin C_Q(x)$.

3. The rational isomorphism problem for ($\mathrm{M_{exc}}$) groups

The aim of this section is to prove the following instance of the rational isomorphism problem.

Theorem 3.1. *Let G and H be finite groups such that $\mathbb{Q}G \cong \mathbb{Q}H$. If $\mathbb{Q}G$ has ($\mathrm{M_{exc}}$), then $G \cong H$.*

3.1. **On the multiplicities and invariants of the rational group algebra when ($\mathrm{M_{exc}}$).** Suppose that $\mathbb{Q}G$ has ($\mathrm{M_{exc}}$). Then Theorem 2.6 tells us what are the possible simple components of $\mathbb{Q}G$. We will now count how many of each component arises in terms of various structural properties of G obtained in Section 2.5.

For this recall that, when $\mathbb{Q}G$ has ($\mathrm{M_{exc}}$), every simple component of $\mathbb{Q}G$ is of the form $\mathbb{Q}Ge(G,H,K)$ for (H,K) a strong Shoda pair. Recall that the latter were described: $G \cong A \rtimes Q$ contains a maximal abelian subgroup B such that $G/B \cong C_2$ or $C_2 \times C_2$. The group H is either B or $\langle B, t\rangle$ for some $t \in Q \setminus (Q \cap B)$. Furthermore K is of the form $K_{(a,x,m)}$ defined in (2.17).

Proposition 3.2. *Let G be a finite group such that $\mathbb{Q}G$ has ($\mathrm{M_{exc}}$). Then the following properties are determined by $\mathbb{Q}G$:*

(1) The isomorphism type of G^{ab} and $\mathcal{Z}(G)$,
(2) $|G|$ and $|G'|$,
(3) the set of character degrees $\mathrm{cd}(G)$,
(4) the nilpotency class of the Sylow 2-subgroups,
(5) nilpotency class of G (in particular whether it is nilpotent).

The first three parts of Proposition 3.2 in fact hold for any G and is well-known, but we could not point out an explicit reference, hence for ease of reader we add a brief proof.

Proof. If we decompose $\mathbb{Q}G \cong (\mathbb{Q}G)_{com} \times (\mathbb{Q}G)_{nc}$ where $(\mathbb{Q}G)_{com}$ is the product of the commutative simple components and $(\mathbb{Q}G)_{nc}$ the rest, then it is well-known that $(\mathbb{Q}G)_{com} \cong \mathbb{Q}[G^{ab}]$ and $(\mathbb{Q}G)_{nc} = \mathbb{Q}G(1-\hat{G})$. Clearly $(\mathbb{Q}G)_{com}$ is determined by $\mathbb{Q}G$. Now by the theorem of Perlis-Walker $\mathbb{Q}[G^{ab}] \cong \bigoplus_{d||G^{ab}|} m_d\mathbb{Q}(\zeta_d)$ where m_d is the number of cyclic subgroups of G^{ab} of order d. Therefore, by the fundamental theorem of abelian groups, $\mathbb{Q}[G^{ab}]$ determines the isomorphism type of the abelian group G^{ab}.

Next, the subalgebra $\mathcal{Z}(\mathbb{Q}G)$ is clearly determined. By Berman-Higman's theorem all central units are trivial and hence also $\mathcal{Z}(G)$ is determined.

The cardinality $|G| = \dim_{\mathbb{Q}} \mathbb{Q}G$ and hence is determined. Subsequently, $|G'|$ is determined as we also obtained $|G^{ab}|$ in (1). Next, the set $\mathrm{cd}(G)$ is determined by the Wedderburn-Artin decomposition of $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}G$.

Finally, we consider the nilpotency class. As $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ we know that $G \cong A \rtimes Q$ with A an elementary abelain 3-groups and Q a 2-group. Furthermore, $A = A_{-1} \times (A \cap \mathcal{Z}(G))$ by (2.15) and $G' = A_{-1} \rtimes Q'$ by (2.16). From this we see that G is nilpotent if and only if G' is a 2-group. The latter can be read off $|G'|$ and hence is determined by (2). When G is nilpotent, we have that $G \cong A \times Q$ and its nilpotency class is equal to the one of Q.

Thus it remains to show that the nilpotency class of Q is determined. Since Q has at most class 3, it is enough to explain how to recognise when Q is abelian or of class 2. For the former, note that (2.16) implies that Q is abelian if and only $|G'|$ is a 3-group. This fact is determined by point (2). Next, Q has nilpotency class 2 can be reduced to Q abelian via (1). □

Now consider G and H such that $\mathbb{Q}G \cong \mathbb{Q}H$. By definition if $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$, then also $\mathbb{Q}H$. Hence they are both of the form $C_3^n \rtimes Q$ with Q some 2-group. Furthermore by Proposition 3.2 their Sylow 2-subgroups have the same nilpotency class which is at most 3 by Lemma 2.19. This allows us to consider each class separately and use the corresponding characterisations obtained in Section 2.5. Distinguishing G and H will go through a careful counting of the multiplicities of the simple components.

Two strong Shoda pairs (H_1, K_1) and (H_2, K_2) yield the same component if and only if the associated primitive central idempotents are equal. By [20, Exercise 3.4.3], *if* H_1 *is normal*, this can be group theoretically determined as following:

$$e(G, H_1, K_1) = e(G, H_2, K_2) \Leftrightarrow \exists\, g \in G : H_1 \cap K_2 = H_2 \cap K_1^g. \tag{3.1}$$

Furthermore if $H_1 = H_2$, then (3.1) holds if and only if K_1 and K_2 are conjugated inside G.

Lemma 3.3. *Suppose* $\mathbb{Q}G$ *has* $(\mathrm{M}_{\mathrm{exc}})$ *and with notations as above, we have that:*

1. *the number of SSP* (B, K) *with* K *normal in* G *is ...*
2. *the number of SSP* $(\langle B, t\rangle, K)$ *with* K *normal in* G *and* $t \in Q \setminus (Q \cap B)$ *is ...*

3.2. **The Sylow 2-subgroup abelian case.** We start by describing the full Wedderburn-Artin decomposition of $\mathbb{Q}G$ for G as in Theorem 2.22 and we use the notations from that statement. For such G the commutator subgroup $G' = C_3^k$ and $G^{ab} \cong D \times C_{2^t}$. Thus it remains to understand the simple components coming from a SSP with $H \neq G$. Then, as $B := \langle D, C_3^k, y^2\rangle$ is the maximal abelian normal subgroup which is of index 2, we have that $H = B$. In the proof of Theorem 2.22 it was obtained that K is also normal. Therefore the number of simple components of $\mathbb{Q}G$ is the number of possible K's up to equality inside G.

Now consider a SSP $(B, K_{(a,x,m)})$. In the proof of Theorem 2.22 we also described the triples (a, x, m) and the associated projections $Ge(G, B, K_{(a,x,m)})$ (which we denote by $\pi_{(a,x,m)}$ or simply π). Concretely, we obtained that a is an arbitrary non-trivial element C_3^k and (x, m) any tuple in $\mathcal{S}_B$. We have that $\pi(G) \cong \langle a, \pi(x), \pi(y)\rangle$. Furthermore, there was two type of situations for x:

(i) $x \in y^{2j}D_2$ for $j \neq 0$, say $x = y^2 d$. Since $d \in \ker(\pi_{(a,x,m)})$ we obtain that

$$\pi_{(a,x,m)}(G) \cong \langle a\rangle \rtimes \langle \pi(y)\rangle \cong C_3 \rtimes C_{2^{m+1}}$$

with action given by inversion and $0 \leq m \leq v_2(o(y^{2j}))$.

(ii) $x \in D_2$. Then

$$\pi_{(a,x,m)}(G) \cong \langle x \rangle \times (\langle a \rangle \rtimes \langle \pi(y) \rangle \cong C_{2^m} \times (C_3 \rtimes C_2)$$

where $0 \leq m \leq 2$.

Concerning the equality of K's, note that $K_{(a,x_1,m_1)} = K_{(a,x_2,m_2)}$ if and only if the subgroups $\langle (\langle D, y^2 \rangle \setminus \langle x_i \rangle) \cup \{x_i^{2^{m_i}}\} \rangle$ are equal.

For every type of component of $\mathbb{Q}G$ a copy is contributed by the following $\pi_{(a,x,m)}(G)$:

- A copy of $\left(\frac{-1,-3}{\mathbb{Q}}\right)$: by type (i) if $m+1=2$.
- A copy of $\left(\frac{\zeta_{2^{m+1}},-3}{\mathbb{Q}(\zeta_{2^{m+1}})}\right)$ by type (i) for each $4 \leq m+1 \leq t$.
- A copy of $\mathrm{M}_2(\mathbb{Q})$: by type (i) if $m+1=1$ and by type (ii) for $m=0,1$.
- A copy of $\mathrm{M}_2(\mathbb{Q}(i))$: by type (i) if $m+1=3$ and by type (ii) for $m=2$.

Since $\exp(D_2) \mid 4$ we can write $D_2 = C_2^{n_1} \times C_4^{n_2}$. Counting now all the options yield the following decomposition:

$$\begin{aligned} \mathbb{Q}G \quad &\cong \quad \mathbb{Q}[G^{ab}] \oplus (3^k-1).\Big(|D_2|\left(\tfrac{-1,-3}{\mathbb{Q}}\right) \oplus \textstyle\bigoplus_{\ell=4}^{t} |D_2| \left(\tfrac{\zeta_{2^\ell},-3}{\mathbb{Q}(\zeta_{2^\ell})}\right) \\ &\oplus 2|D_2|\,\mathrm{M}_2(\mathbb{Q}) \oplus (|D_2| + \#\text{el. order 4 in } D_2)\,\mathrm{M}_2(\mathbb{Q}(i))\Big) \end{aligned} \tag{3.2}$$

with $G^{ab} \cong D \times C_{2^t}$ as $G' = C_3^k$ by (2.16).

Now the rational isomorphism problem follows readily. Indeed, consider two groups $G_i = D(i) \times (C_3^{k_i} \rtimes \langle y_i \rangle)$ as in Theorem 2.22. Then G_1 and G_2 are isomorphic if and only if $k_1 = k_2$, $D(1) \cong D(2)$ and $o(y_1) = o(y_2)$. The equality $k_1 = k_2$ is equivalent to $|G_1'| = |G_2'|$ which is determined by Proposition 3.2. By considering the abelianisation, and using Proposition 3.2, we also get that $D(1) \times C_{2^{t_1}} \cong D(2) \times C_{2^{t_2}}$. Hence if we know that $t_1 = t_2$, then we would be finished. If the components $\left(\frac{\zeta_{2^\ell},-3}{\mathbb{Q}(\zeta_{2^\ell})}\right)$ appears, it can be read from the centre with highest dimension (or the number of such components). Otherwise, we first inspect in (3.2) the multiplicity of $\left(\frac{-1,-3}{\mathbb{Q}}\right)$, recovering the cardinality of the 2-part $D(i)_2$ of $D(i)$. However we also had the isomorphism type of the 2-part of $D(i) \times C_{2^{t_i}}$, thus combining the informations yield indeed that $o(y_1) = |C_{2^{t_1}}| = |C_{2^{t_2}}| = o(y_2)$. Subsequently from the abelianisation we now also obtain $D(1) \cong D(2)$, obtaining all the required invariants. Hence such groups are indeed determined by their rational group algebra.

4. The block Virtual Structure Problem

We now recall the concept of reduced norm. First, let A be a finite dimensional central simple algebra over a field K of characteristic 0 and E be a splitting field of A (i.e. $A \otimes_K E \cong \mathrm{M}_n(E)$ for some n). Then the *reduced norm* of $a \in A$ is defined as

$$\mathrm{RNr}_{A/K}(a) = \det(1_E \otimes_K a).$$

Note that $\mathrm{RNr}_{A/K}(\cdot)$ is a multiplicative map, $\mathrm{RNr}_{A/K}(A) \subseteq K$ and $\mathrm{RNr}_{A/K}(a)$ only depends on K and $a \in A$ (and not on the chosen splitting field E and isomorphism $E \otimes_K A \cong \mathrm{M}_n(E)$), see [20, page 51]. For a subring R of A, define

$$\mathrm{SL}_1(R) = \{\ a \in \mathcal{U}(R) \mid \mathrm{RNr}_{A/K}(a) = 1\ \}, \tag{4.1}$$

which is a (multiplicative) group. If $A = \mathrm{M}_n(A')$ and $R = \mathrm{M}_n(R')$ with A' a finite dimensional central simple algebra over K and R' a subring of A', then we also write $\mathrm{SL}_1(A) = \mathrm{SL}_n(A')$ and $\mathrm{SL}_1(R) = \mathrm{SL}_n(R')$. Next, if $A = \prod \mathrm{M}_{n_i}(D_i)$ is semisimple and h_i is the projection onto the i-th component, then

$$\mathrm{SL}_1(R) := \{\ a \in R \mid \forall\ i\colon\ \mathrm{RNr}_{\mathrm{M}_{n_i}(D_i)/\mathcal{Z}(D_i)}(h_i(a)) = 1\ \}.$$

The overarching spirit of this paper is to determine properties of a finite group that are determined by its irreducible representations over a number field. For example, one could wonder which groups are fully determined by certain interesting predescribed condition on them (such as property ($\mathrm{M}_{\mathrm{exc}}$)):

Question 4.1 (block Virtual Structure problem)**.** Let $\mathcal{P}$ be a property. Classify the group algebras FG such that for every $e \in \mathrm{PCI}(FG)$

- either FGe has property $\mathcal{P}$
- or $\mathrm{SL}_1(\mathcal{O}_e)$ has property $\mathcal{P}$ for any order $\mathcal{O}_e$ in FGe.

Note that, in the case that $\mathcal{P}$ is a property that behaves well with direct products and is constant on commensurability classes, Question 4.1 is equivalent to the classical Virtual Structure Problem. This thanks to the general fact that, for any order $\mathcal{O}$ in FG, $\mathrm{SL}_1(\mathcal{O})$ is a finite index subgroup of $\prod_{e\in \mathrm{PCI}(FG)} \mathrm{SL}_1(\mathcal{O}e)$.

4.1. **The case of properties defined on commensurability classes.** Let $\mathcal{P}$ be a group theoretical property such that

- $\mathcal{P}$ implies *not* FA or not SCP
- SL_1 of all exceptional 2×2 components satisfy it
- it is a property of commensurability classes.

By $\prod \mathcal{P}$ we mean that the group is a direct product of groups satisfying $\mathcal{P}$ *and* an abelian group. Combining the methods in the proof of [23, Theorem 2.1.] with the results of [26] we obtain the following.

Proposition 4.2. *Let A be a finite dimensional semisimple F-algebra with F a number field and $\mathcal{O}$ an order in A. If $\mathcal{U}(\mathcal{O})$ is virtually-$\prod \mathcal{P}$, then for every $e \in \mathrm{PCI}(A)$:*

(1) $\mathrm{SL}_1(\mathcal{O}e)$ is either virtually-$\mathbb{Z}$ or virtually-$\mathcal{P}$ (Is general!!!!)
(2) The degree of Ae, as CSA, is at most 4

Proof. Denote $S_j := \mathrm{SL}_{n_j}(\mathcal{O}_j)\cap H$ which is of finite index in $\mathrm{SL}_{n_j}(\mathcal{O}_j)$, hence it is enough to proof that $e(S_j)=\infty$. Let p_k be the projection of H on H_k. Fix some j as in the claim. The condition is equivalent with saying that $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is infinite [25]. In particular there exists some k such that $p_k(S_j)$ is infinite[12]. For such k we will now prove that $|p_k(\prod_{i\neq j} S_i)| < \infty$. For this consider $S := S_j \times \prod_{i\neq j} S_i$ which by the first claim is of finite index in H. Therefore $p_k(S)$ is of finite index in H_k and hence $e(p_k(S)) = \infty$. However, $p_k(S_j)$ and $p_k(\prod_{i\neq j} S_i)$ are subgroups as in the second claim[13], yielding the desired. Indeed, the two subgroups clearly commute, are normal in $\pi_k(S)$ and $p_k(S_j)\cap p_k(\prod_{i\neq j} S_i) \subseteq \mathcal{Z}(p_k(S))$ which is finite since $p_k(S)$ has infinitely many ends.

Now consider the set $\mathcal{I}_j := \{k \mid |p_k(S_j)| < \infty\}$. From the previous it follows that if $k \in \{1,\dots,q\}\setminus \mathcal{I}_j$, then $p_k(S_j)$ is of finite index in H_k. Hence $S_j/\big(S_j \cap \prod\limits_{i\in\mathcal{I}_j} H_i\big)$ is a subgroup of finite index in $\prod_{k\notin \mathcal{I}_j} H_k$. As the quotient was with a finite subgroup, we obtain that S_j is virtually-$\mathcal{G}_\infty$ and hence $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ also. However under the conditions above SL_1 is virtual indecomposable [26, Theorem 1]. Therefore $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ in fact is even virtually a group with infinitely many ends and so in fact $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$, as claimed.

Part 2: This is more generally the case for a property which imply not FA, cf. work of Kleinert-Del Rio where they deduce this by considering the associated semisimple Lie group and do rank computations. ☐

4.2. **Groups of virtual cohomological dimension 4.** Let Γ be a discrete group. Then the *cohomological dimension* of Γ over the ring R is

$$\mathrm{hdim}_R\,\Gamma := \min\{n \mid H^k(G,M)=0 \text{ for all } k>n \text{ and } M\in \mathrm{mod}(RG)\}.$$

[12] Otherwise S_j would be finite and hence also the overgroup of finite index $\mathrm{SL}_{n_j}(\mathcal{O}_j)$.

[13] Instead of claim 2 one could have used the well-known result [**?**, 4.A.6.3.] saying that infinite finitely generated normal subgroups of a group with infinitely many ends have finite index.

If no such n exists one says that $\operatorname{hdim}_R \Gamma = \infty$. A usual obstruction to have a finite cohomological dimension is torsion in Γ. However, if Γ has a torsion-free subgroup of finite index (e.g. when Γ is linear), then each of such finite index subgroups has the same cohomological dimension. Hence

$$\operatorname{vcd}(\Gamma) := \{\operatorname{hdim}_{\mathbb{Z}} \Gamma' \mid [\Gamma : \Gamma'] < \infty \text{ and } \Gamma' \text{ torsion-free}\}.$$

In particular, as the unit group of two orders $\mathcal{O}_1$ and $\mathcal{O}_2$ in A are commensurable, one has that $\operatorname{vcd}(\mathcal{O}_1) = \operatorname{vcd}(\mathcal{O}_2)$. Furthermore, as shown by eq. (4.2) their virtual cohomological dimension can be expressed purely in terms of number theoretical properties of A. In [23, Proposition 3.3] the finite dimensional simple F-algebras A, over a number field F, such that $\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) \leq 2$ for some order $\mathcal{O}$ in A (and hence for all orders) have been classified. We extend this result to virtual cohomological dimension 4. See Remark 4.4 below for $\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) = 3$.

Proposition 4.3. *Let A be a finite dimensional simple F-algebra with F a number field. If $\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) = 4$ for an order $\mathcal{O}$ in A, then A is isomorphic as an F-algebra to:*

(1) $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ *with $a, b \in \mathbb{N}_0$,*

(2) $\mathrm{M}_2(F)$, *with F a cubic field with precisely one real embedding and one pair of complex embeddings,*

(3) or to $\left(\frac{-a,-b}{F}\right)$ such that it is non-ramified at exactly two real places and F is totally real.

Proof. Let $A = M_n(D)$, $F = \mathcal{Z}(D)$ for an integer $n \geq 1$ and D a division ring of degree d. We make use of the following formula, as stated in [23, Eq. (1)].

$$\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_1 \frac{(nd-2)(nd+1)}{2} + r_2 \frac{(nd+2)(nd-1)}{2} + s(n^2d^2 - 1) - n + 1, \tag{4.2}$$

where s is the number of pairs of non-real complex embeddings of F, r_1 is the number of real embeddings of F at which A is ramified, and r_2 the number of real embeddings of F at which A is not ramified. We may assume that $nd > 1$, since when $nd = 1$, A is a field, which implies that $\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) = 0$ by [23, Proposition 3.3]. Note that for any choice of $nd \geq 2$, the first two terms of eq. (4.2) are non-negative. Additionally, when d is odd, it is well-known that $r_1 = 0$.

Suppose $s \geq 2$. Then for any choice of n or d such that $nd = 2$,

$$\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) \geq s(n^2d^2 - 1) - n + 1 > 4,$$

and this expression is strictly increasing in both n and d. Hence $\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) > 4$ for any $s \geq 2$, and in particular F has at most one pair of complex embeddings.

Suppose $s = 1$. Then

$$\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) \geq s(n^2d^2 - 1) - n + 1 = n^2d^2 - n > 4,$$

when $n \geq 3$. Hence n is at most 2 in this case. Suppose first $n = 1$. Then

$$\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) \geq d^2 - 1 > 4 \text{ when } d > 2.$$

Thus $d = 2$ because $nd \geq 2$ by assumption. Then we find

$$\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_2 \frac{4.1}{2} + 4 - 1 = 4 \iff r_2 = -\frac{1}{2},$$

which is a contradiction since $r_2 \in \mathbb{N}$. Hence we cannot have $n = 1$. The only other option is $n = 2$. Then $n^2d^2 - n = 4d^2 - 2 \leq 4$ if and only if $d = 1$, in which case

$$\operatorname{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_2 \frac{4.1}{2} + 2 = 4 \iff r_2 = 1.$$

In this case we find that $A = M_2(F)$ with F a cubic number field with precisely one real embedding and one pair of complex embeddings.

Suppose now $s = 0$. We examine the case $r_2 = 0$ first. Suppose $r_2 = 0$. Then $r_1 \geq 1$ and

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_1 \frac{(nd-2)(nd+1)}{2} - n + 1,$$

and hence $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 4$ necessarily implies that $nd \geq 3$. Suppose first that $n = 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_1 \frac{(d-2)(d+1)}{2} = 4 \iff r_1 \leq 2, \text{ since } d \geq 3.$$

If $r_1 = 2$, then $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = (d-2)(d+1) = 4$ implies that $d = 3$, but we have already remarked that $r_1 = 0$ when d is odd, a contradiction. If $r_1 = 1$, then $(d-2)(d+1) = 8$, which implies that $d = \frac{2 \pm \sqrt{32}}{2}$, which is not an integer, a contradiction. We conclude that if $s = 0$ and $r_2 = 0$, then $n > 1$. Suppose $n = 2$. Then necessarily $d \geq 2$ since $nd \geq 3$. It follows that

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = \frac{r_1}{2}(2d-2)(2d+1) - 1 \geq 4,$$

with equality if and only if $r_1 = 1$ and $d = 2$. We find that in this case, $A = M_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some positive integers a, b. Now the expression $\frac{r_1}{2}(nd-2)(nd+1) - n + 1$ is strictly increasing in n if and only if $n > \frac{d-1}{r_1 d^2}$, which in the case at hand is satisfied since $d \geq 1$ and $r_1 \geq 1$ by assumption. It follows that $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) > 4$ whenever $n \geq 3$.

Still under the assumption that $s = 0$, we now turn our attention to the case $r_2 \neq 0$, meaning $r_2 \geq 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = \frac{r_2}{2}(nd+2)(nd-1) + \frac{r_1}{2}(nd-2)(nd+1) - n + 1.$$

If $r_1 \geq 1$, then d is even, meaning that $nd \geq 4$ when $n \geq 2$. But when $n = 2$, then

$$\frac{r_2}{2}(2d+2)(2d-1) + \frac{r_1}{2}(2d-2)(2d+1) - 1 \geq 9r_2 + 5r_1 - 1 > 4,$$

and again the expression above for $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O}))$ is strictly increasing in n. If $r_1 = 0$, then $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 4$ if and only if $\frac{r_2}{2}(nd+2)(nd-1) - n = 3$. This expression is strictly increasing in n if and only if $n > \frac{1-d}{r_2 d^2}$, which is always satisfied by assumption on r_2 and d. But when $n \geq 2$ and $d \geq 2$, meaning in particular that $nd \geq 4$, one finds that

$$\frac{r_2}{2}(nd+2)(nd-1) - n \geq 7,$$

implying that only the cases $n = 1$, and $(n, d) = (2, 1)$ need to be investigated. If $n = 2$ and $d = 1$, then

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = \frac{r_2}{2}(nd+2)(nd-1) - n + 1 = 2r_2 - 1,$$

which equals 4 if and only if $r_2 = \frac{5}{2}$, a contradiction.

In particular, for any value of r_1 only the case $n = 1$ remains. Assuming $n = 1$, we have in particular that $d \geq 2$, and

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = \frac{r_2}{2}(d+2)(d-1) + \frac{r_1}{2}(d-2)(d+1) \geq \frac{r_2}{2}(d+2)(d-1) \geq 4,$$

where the last inequality becomes an equality if and only if $d = 2 = r_2$. In that case one also finds that $\frac{r_1}{2}(d-2)(d+1) = 0$, and hence $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 4$ if $s = 0$, $r_2 = d = 2$ and r_1 takes any arbitrary integer value. We obtain that $A = \left(\frac{-a,-b}{F}\right)$ with F a totally real number field and A is non-ramified at precisely two places. If $d \geq 3$, it now immediately follows that $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) \geq 9r_2 + 4r_1 > 4$, and this concludes our analysis. $\square$

Remark 4.4. With a similar proof one can verify that $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 3$ if and only if A is isomorphic to one of the following simple algebras:

- $\mathrm{M}_3(\mathbb{Q})$,
- $\mathrm{M}_2(\mathbb{Q}(\sqrt{d}))$ with $d \in \mathbb{N}$ square-free,
- $\left(\frac{-a,-b}{F}\right)$ such that F has one pair of non-real complex embeddings and is ramified at all real places.

Concerning $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) \leq 2$, [23, Proposition 3.3] tells that

- $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 0$ if and only if A is a field or a totally definite quaternion algebra,
- $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 1$ if and only if $A \cong \mathrm{M}_2(\mathbb{Q})$
- $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 2$ if and only if $A \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ or a quaternion algebra with a totally real centre and which is non-ramified at exactly one infinite place.

We now have all ingredients for our following main theorem.

Theorem 4.5. *Let G be a finite group, F a number field with $[F : \mathbb{Q}] \leq 2$ and R its ring of integers. Then the following are equivalent:*

(1) FG has $(\mathrm{M}_{\mathrm{exc}})$,
(2) $\mathrm{vcd}(\mathrm{SL}_1(RGe)) \mid 4$ for every $e \in \mathrm{PCI}(FG)$ such that FGe is a non-division simple component.

Furthermore, if FG has $(\mathrm{M}_{\mathrm{exc}})$ and FGe is an exceptional division component, then $\mathrm{vcd}(RGe) > 4$.

Remark 4.6. That (1) implies (2) in Theorem 4.5 is true for any field F. However for the converse the condition that F is a quadratic number field is required. For example for any field $F \supseteq \mathbb{Q}$ one has that $FD_8 \cong F \times F \times \mathrm{M}_2(F)$. In particular if F is a cubic number field with one real embedding and one pair of complex embeddings (e.g. $F = \mathbb{Q}(\sqrt[3]{d})$), then $\mathrm{vcd}(\mathrm{SL}_2(R))) = 4$, but FD_8 has not $(\mathrm{M}_{\mathrm{exc}})$.

The proof of Theorem 4.5 will follow quickly out of the results obtained in earlier sections together with following fact which is of independent interest.

Lemma 4.7. *Let G be a finite group. Let F be a cubic number field. Suppose that the simple algebra $\mathrm{M}_2(F)$ is a quotient of $\mathbb{Q}G$, say $\mathrm{M}_2(F) \cong \mathbb{Q}Ge$ with $e \in \mathrm{PCI}(\mathbb{Q}G)$. Then $\pi(Ge) \subseteq \{2, 3, 7\}$, and $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ or $F = \mathbb{Q}(\zeta_9 + \zeta_9^{-1})$.*

Example 4.8. Recall that from eq. (2.6) it follows that when $G = Dic_{4n}$ with $7 \mid n$ (or $9 \mid n$), then $\mathbb{Q}Dic_{4n}$ has a component which is isomorphic to $\mathrm{M}_2(F)$, where $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ (respectively $F = \mathbb{Q}(\zeta_9 + \zeta_9^{-1})$), is a cubic extension of $\mathbb{Q}$.

Proof. Let $\lambda \in F$ be a torsion element, say of order n. Then it is a primitive n^{th} root of unity, denoted ζ_n, and

$$3 = |F : \mathbb{Q}| = |F : \mathbb{Q}(\zeta_n)||\mathbb{Q}(\zeta_n) : \mathbb{Q}| \geq \phi(n). \tag{4.3}$$

It follows that $n \in \{1, 2, 3, 4, 6\}$.

Let $g \in G$. By fixing a $\mathbb{Q}$-basis of F as a 3-dimensional $\mathbb{Q}$-space, one can realise g as a 6-by-6 matrix over $\mathbb{Q}$. We denote by $\chi_{F,g}$ and $\chi_{\mathbb{Q},g}$ respectively the characteristic polynomials of g over F and over $\mathbb{Q}$. Similarly, we write $\mu_{F,g}$ and $\mu_{\mathbb{Q},g}$ for the minimal polynomials of g over respectively F and $\mathbb{Q}$. By definition, any minimal polynomial $\mu_{\mathbb{Q},g}$ has degree at most 6, and any $\mu_{F,g}$ has degree at most 2. Remark that $\mu_{F,g}$ has degree 1 if and only if g is a scalar matrix over F.

From [28, Page 147], it follows that for any $g \in G$ of prime power order p^k, the $p^{k\mathrm{th}}$ cyclotomic polynomial Φ_{p^k} equals $\mu_{\mathbb{Q},g}$. In particular, $\mathbb{Q}(\zeta_{p^k}) \cong \frac{\mathbb{Q}[X]}{(\mu_{\mathbb{Q},g})}$. The latter also holds over F, when $p^k > 4$. Indeed, $\mu_{F,g}$ is given as the unique monic polynomial generating the ideal $I_g := \{P \in F[X] \mid P(g) = 0\}$, and the minimal polynomial of ζ_{p^k} over F is given by the unique monic polynomial generating the ideal $I_{\zeta_{p^k}} := \{Q \in F[X] \mid Q(\zeta_{p^k}) = 0\}$. We claim that $I_g = I_{\zeta_{p^k}}$. Indeed, by [28, Page 146], there is some matrix $B \in \mathrm{GL}_2(\mathbb{C})$ such that $BgB^{-1} = \mathrm{diag}(\lambda_1, \lambda_2)$, with each λ_i a $p^{k\mathrm{th}}$ root of unity, amongst which at least one primitive (otherwise BgB^{-1} and hence g would have order strictly smaller than p^k). Without loss of generality, assume $\lambda_1 = \zeta_{p^k}$. Remark that since conjugation by an invertible matrix induces an algebra automorphism of $\mathrm{M}_2(\mathbb{C})$, it follows that $P \in I_g$ if and only if $P(BgB^{-1}) = 0$. In particular, $P \in I_g$ if and only if $P(\zeta_{p^k}) = 0 = P(\lambda_2)$. We conclude that

$I_g \subseteq I_{\zeta_{p^k}}$. But since $\deg(\mu_{F,g}) = 2$, and $p^k > 4$ (meaning that $\zeta_{p^k} \notin F$), it follows that I_g is a maximal ideal of $F[X]$, and hence $I_g = I_{\zeta_{p^k}}$. In particular, $F(\zeta_{p^k}) \cong \frac{F[X]}{(\mu_{F,g})}$.

Since Φ_p divides $\mu_{\mathbb{Q},g}$ when $g \in G$ is an element of prime order p, and the degree of $\mu_{\mathbb{Q},g}$ is at most 6 as remarked earlier, it follows that $p \in \{2,3,5,7\}$, and in particular $\pi(G) \subseteq \{2,3,5,7\}$. Suppose $g \in G$ has order $p \in \{5,7\}$. Remark that $\deg(\chi_{F,g}) = 2$, since otherwise g would be a scalar matrix with a p^{th} root of unity on the diagonal, which is a contradiction with the description of torsion elements in F as given in eq. (4.3). Over $F(\zeta_p)$, $\chi_{F,g}$ splits as $(X - \zeta_p^i)(X - \zeta_p^k)$ for some $0 \leq i \neq k \leq p-1$. Now $\zeta_p^{i+k} \neq 1$ would imply that F has a p^{th} root of unity, a contradiction by eq. (4.3). Thus, $k = -i$, and the degree 1 coefficient in $\chi_{F,g}$ is equal to $-(\zeta_p^i + \zeta_p^{-i})$. In particular $\mathbb{Q}(\zeta_p^i + \zeta_p^{-i}) \subseteq F$. If $p = 5$, then since $|\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) : \mathbb{Q}| = 2$ does not divide $|F : \mathbb{Q}| = 3$, we obtain a contradiction. When $p = 7$, $|\mathbb{Q}(\zeta_7 + \zeta_7^{-1}) : \mathbb{Q}| = 3$. It follows that $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.

Let now $\pi(G) \subseteq \{2,3\}$. We bound the exponent of G. Let $g \in G$, say of order $2^i 3^j$ for some non-negative integers i, j. If $i = 0$ or $j = 0$, then $o(g) = p^n$ since Φ_{p^n} divides $\mu_{\mathbb{Q},g}$ for each $p \in \{2,3\}$ and $\deg(\mu_{\mathbb{Q},g}) \leq 6$, it follows that $o(g) \in \{p, p^2\}$. If $i \neq 0 \neq j$, then considering a realisation of g as an element in $\mathrm{GL}_6(\mathbb{Q})$, [28, Page 147] implies that there exist $m_1, \ldots, m_r$ such that $o(g) = \mathrm{lcm}\{m_1, \ldots, m_r\}$, Φ_{m_i} divides $\mu_{\mathbb{Q},g}$, and $6 = \sum_{i=1}^r d_i \phi(m_i)$, for some $d_i \geq 1$. In particular each $\phi(m_i) \leq 6$. Since $\pi(G) \subseteq \{2,3\}$, from a case-by-case analysis it follows that $m_i \in \{1,2,3,4,6,9,18\}$. In particular, $\exp(G) \mid 36$.

Suppose that G contains an element g of order 9. Then since $F(\zeta_9) \cong \frac{F[X]}{(\mu_{F,g})}$, and $\deg(\mu_{F,g}) = 2$,

$$|F(\zeta_9) : \mathbb{Q}| = |F(\zeta_9) : F||F : \mathbb{Q}| = 6,$$

and $|\mathbb{Q}(\zeta_9) : \mathbb{Q}| = 6$, it follows that $F(\zeta_9) = \mathbb{Q}(\zeta_9)$, and in particular $F \subseteq \mathbb{Q}(\zeta_9)$. The only subfields contained in $\mathbb{Q}(\zeta_9)$ are $\mathbb{Q}$, $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$, a contradiction, and of these only $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$ is of degree 3 over $\mathbb{Q}$. In particular, we conclude that if $\pi(G) \subseteq \{2,3\}$ and elements of order 3^n necessarily have order 3, then $\exp(G) \mid 12$. Suppose we are in this case, and let $e \in \mathrm{PCI}(G)$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2(F)$. Then by Brauer's splitting field theorem, $\mathbb{Q}(\zeta_{12})$ is a splitting field for G, since G has exponent a divisor of 12. In particular, $F \subseteq \mathbb{Q}(\zeta_{12})$. However, $\mathrm{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$, which has order coprime to 3, which together with the fundamental theorem of Galois theory implies a contradiction. □

We are now able to prove Theorem 4.5.

Proof of Theorem 4.5. First suppose that FG has ($\mathrm{M}_{\mathrm{exc}}$) and let $e \in \mathrm{PCI}(FG)$. If FGe is an exceptional matrix component then $\mathrm{vcd}(RGe) = 2$ or 4 by Proposition 4.3 and Remark 4.4. Next, if FGe is a division algebra, then by Theorem 2.6 either it is a field or one of the following quaternion algebras:

$$\{\left(\frac{-1,-1}{\mathbb{Q}(\zeta_m)}\right), \left(\frac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right) \mid m \in 2\mathbb{N}+1,\ t \in \mathbb{N}_{\geq 3}\}.$$

Of these only those with a non totally-real centre (i.e. the first two) are exceptional. Inspecting the possible quaternions algebras with virtual cohomological dimension smaller than 4, we see that they are not part of the list.

It remains to prove the converse, so suppose that $\mathrm{vcd}(\mathrm{SL}_1(RGe)) \mid 4$ for every $e \in \mathrm{PCI}(FG)$ such that FGe is a non-division simple component. By the results referred to above, the only simple algebras not allowed by the property ($\mathrm{M}_{\mathrm{exc}}$) are those of the form $\mathrm{M}_2(K)$, with K a cubic number field with one real embedding and one pair of complex embeddings. Now as $F \subseteq \mathcal{Z}(FGe) = K$ and $[F : \mathbb{Q}] \mid [K : \mathbb{Q}] = 3$, one has that $F = \mathbb{Q}$. However by Lemma 4.7 the algebra $\mathrm{M}_2(K)$ can't be the simple component of FG, finishing the proof. □

4.3. **Higher Kleinian groups: discrete subgroups of** $\mathrm{SL}_4(\mathbb{C})$**.** Another interesting property is a kind of higher Kleinian property:

Definition 4.9. A group Γ is said to *have property* Di_n if it is a discrete subgroup of $\mathrm{SL}_n(\mathbb{C})$, but not of $\mathrm{SL}_{n-1}(\mathbb{C})$.

We will be interested in the case that Γ has Di_n with n a divisor of 4. An alternative way to look at this is via the 5-dimensional hyperbolic space, as one has the following isomorphism:

$$\mathrm{Iso}^+(\mathbb{H}_5) \cong \mathrm{PGL}_2(\left(\frac{-1,-1}{\mathbb{Q}}\right)).$$

In particular a group Γ acts discontinuously on[14] $\mathbb{H}_5$ if and only if Γ has Di_4 .

The finite dimensional simple algebras A such that $\mathrm{SL}_1(\mathcal{O})$ is Kleinian, i.e. is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$, for an order $\mathcal{O}$ in A were classified in [23, Proposition 3.2]. Note that if $\mathrm{SL}_1(\mathcal{O})$ has property Di_n, then so does $\mathrm{SL}_1(\mathcal{O}')$ for any other order $\mathcal{O}'$ in A (as both groups are commensurable, see Lemma 4.14).

Proposition 4.10. *Let A be a finite dimensional simple F-algebra with F a number field and $\mathcal{O}$ an order in A. If $\mathrm{SL}_1(\mathcal{O})$ has property Di_4, then A has one of the following forms:*

(1) $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ *with $a, b \in \mathbb{N}_0$,*
(2) $\mathrm{M}_4(\mathbb{Q})$*,*
(3) $\left(\frac{-a,-b}{\mathbb{Q}(\sqrt{-d})}\right)$ *with $a, b \in \mathbb{N}_0$ and $d \in \mathbb{N}_{>1}$ square-free,*
(4) a division algebra of degree 4 which is non-ramified at at most one infinite place.

Proof. Write $A = \mathrm{M}_n(D)$ with D a finite dimensional division algebra and denote $K = \mathcal{Z}(D)$. As mentioned earlier, property Di_n doesn't depend on the order chosen. For ease we choose one of the form $\mathrm{M}_n(\mathcal{O})$ with $\mathcal{O}$ an order in D. We will consider the set of infinite non-compact places:

$$V^{nc} := V_\infty(K) \setminus \{v \in V_\infty(K) \mid \mathrm{SL}_1(\mathrm{M}_n(D \otimes_K K_v)) \text{ is compact }\}.$$

The places in V^{nc} are exactly those at which we can use strong approximation. Note that if $n \geq 2$, then $V^{nc} = V_\infty(K)$.

Now suppose that $\mathrm{SL}_1(\mathcal{O})$ is discrete in $\mathrm{SL}_4(\mathbb{C})$ and take $v_0 \in V_\infty(K)$ such that $\mathrm{SL}_n(\mathcal{O}) \leq \mathrm{SL}_n(D \otimes_K K_{v_0})$ embeds discretely in $\mathrm{SL}_4(\mathbb{C})$.

Next consider another place $v_0 \neq v_1 \in V^{nc}$ and consider the diagonal embedding

$$\Delta\colon \mathrm{SL}_1(\mathrm{M}_n(D)) \hookrightarrow \prod_{v \in V^{nc} \setminus \{v_1\}} \mathrm{SL}_1(\mathrm{M}_n(D \otimes_K K_v)).$$

By strong approximation, see [34, Theorem 7.12, pg 418], $\mathrm{Im}(\Delta)$ is dense. However, $v_0 \in V^{nc} \setminus \{v_1\}$ yielding a combination of dense and discrete and hence a contradiction (as $\mathrm{SL}_n(\mathcal{O})$ being discrete in $\mathrm{SL}_4(\mathbb{C})$ can't be finite). Consequently,

$$|V^{nc}| \leq 1 \tag{4.4}$$

Now, notice that being discrete in $\mathrm{SL}_4(\mathbb{C})$ implies that $\dim_{\mathbb{C}} \mathrm{M}_n(D \otimes_K \mathbb{C}) \leq 16$. Furthermore if it is not discrete in some $\mathrm{SL}_m(\mathbb{C})$ for $m \leq 4$, then $\dim_{\mathbb{C}} \mathrm{M}_n(D \otimes_K \mathbb{C}) = 16$. The latter implies that $A = \mathrm{M}_4(F), \mathrm{M}_2(D)$ or D' with D a quaternion algebra and D' a division algebra of degree 4. This combined with eq. (4.4), yields the stated possibilities. Indeed, simply recall that $\mathrm{SL}_1(\mathrm{M}_n(D \otimes_K K_v))$ is compact if and only if $n = 1$ an A is ramified at v. $\square$

Remark 4.11. The proof of Proposition 4.10 also yields that $\mathrm{SL}_1(\mathcal{O})$ has property Di_3 if and only if A is isomorphic to $\mathrm{M}_3(\mathbb{Q}), \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ or a division algebra of degree 3 which is non-ramified at at most one infinite place.

Concerning Di_2, [23, Remark 3.5] says that $\mathrm{SL}_1(\mathcal{O})$ has Di_2 (i.e. is Kleinian) if and only if $\mathrm{vcd}(A) \leq 2$ or A is quaternion division algebra which is ramified at all its infinite places and having exactly one pair of complex embeddings.

[14]Here one assumes that the action on $\mathbb{H}_5$ does not come from the embedding of an action on $\mathbb{H}_4$.

Theorem 4.12. *Let G be a finite group. Then the following are equivalent:*

(1) $\mathbb{Q}G$ *has* $(\mathrm{M}_{\mathrm{exc}})$ *(resp. and also no exceptional division algebra components)*
(2) $[G : F(G)] = 2$ *and* $\mathrm{SL}_1(\mathbb{Z}Ge)$ *has* Di_n *for* $n \mid 4$ *and* $e \in \mathrm{PCI}(\mathbb{Q}G)$ *such that* $\mathbb{Q}Ge$ *a non-division algebra (resp. for each* $e \in \mathrm{PCI}(\mathbb{Q}G)$*)*

Remark 4.13. The condition that there are no exceptional division algebra components is an important one. Indeed, the groups $C_3 \rtimes C_{2^n}$ and $C_m \times Q_8$ have $(\mathrm{M}_{\mathrm{exc}})$ but not Di_n with $n \mid 4$ due to bad division components. In particular we see that in Free-by-free, i.e. when all components are even nicer, then also the condition is required.

4.4. **The good property.** For this section we fix for each $e \in \mathrm{PCI}(G)$ a maximal order $\mathrm{M}_{n_e}(\mathcal{O}_e)$ in $\mathbb{Q}Ge$. The arguments will however be independent of this choice. Further denote by $\widehat{\Gamma}$ the profinite completion of a group Γ. Recall that Γ is called *good* if the map

$$H^j(\widehat{\Gamma}, M) \to H^j(\Gamma, M),$$

induced by the inclusion of Γ in its completion, is an isomorphism for any j and any finite Γ-module M. Note that any finite group has the good property, since finite groups are isomorphic to their profinite completion. Moreover, free groups have the good property (see [11, Proposition 3.7]).

Recall the following folklore result.

Lemma 4.14. *Let A be a finite-dimensional semisimple F-algebra with F a field of characteristic* 0. *If* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are orders in A, then* $\mathrm{SL}_1(\mathcal{O}_1)$ *and* $\mathrm{SL}_1(\mathcal{O}_2)$ *are commensurable.*

By [11, Lemma 3.2, Proposition 3.4] the good property is preserved under commensurability and is closed under direct products. Writing $FG = \prod_{i=1}^m \mathrm{M}_{n_i}(D_i)$, R an order in F and $\mathcal{O}_i$ an order in D_i, Lemma 4.14 implies that $\mathrm{SL}_1(RG)$ is good exactly when $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ is good for all $1 \leq i \leq m$. Concerning the latter, one has the following statement which follows directly by assembling results in the literature and is in fact implicit in the proof of [7, Theorem 1.5]. For convenience of the reader, we record it here explicitly and provide a proof.

Proposition 4.15. *Let G be a finite group and F a field of characteristic* 0. *Further let* $\mathcal{O}$ *be an order in FGe with* $e \in \mathrm{PCI}(FG)$ *such that FGe is not a division algebra. Then* $\mathrm{SL}_1(\mathcal{O})$ *is good if and only if FGe is an exceptional matrix algebra.*

Proof. Let $FGe = \mathrm{M}_n(D)$. Then, thanks to Lemma 4.14 and the fact that the good property is preserved under commensurability, we may assume that $\mathcal{O}$ is of the form $\mathrm{M}_n(\mathcal{O}')$ with $\mathcal{O}'$ a maximal order in D.

Suppose FGe is an exceptional matrix algebra, i.e. $n = 2$ and $\mathcal{O}'$ is $\mathcal{I}_d$, the ring of integers in $\mathbb{Q}(\sqrt{-d})$ for $d \geq 0$, or $\mathcal{O}'$ is an order in a totally definite quaternion algebra. In [11, Theorem 1.1], it is shown that $\mathrm{PSL}_2(\mathcal{I}_d)$ is good, for all $d \geq 1$. Since the centre of $\mathrm{SL}_2(\mathcal{I}_d)$ is finite, it follows moreover from [11, Lemma 3.3] that $\mathrm{SL}_2(\mathcal{I}_d)$ is good. Additionally, $\mathrm{SL}_2(\mathbb{Z})$ is good since it is commensurable to a free group.

As explained in the proof of [7, Remark 3.5], $\mathrm{SL}_2(\mathcal{O}')$ with $\mathcal{O}'$ an order in a totally definite quaternion algebra is commensurable to a standard arithmetic subgroup of $\mathrm{SO}(1,5)$. Such a subgroup of $\mathrm{SO}(1,5)$ has the good property by [11, Theorem 3.9]. We conclude that $\mathrm{SL}_1(\mathcal{O})$ is good when FGe is an exceptional matrix algebra.

Conversely, in [11, Proposition 5.1] it was proven that $\mathrm{SL}_n(\mathcal{O}')$ is not good if it enjoys the subgroup congruence property. In particular if $n \geq 3$ or $n = 2$ and $\mathcal{U}(\mathcal{O}')$ is infinite, then it is not good, see [6] for the case $n \geq 3$ and [37] for the case $n = 2$. Suppose $n = 2$. Now $\mathcal{U}(\mathcal{O}')$ is finite if and only if D is either isomorphic to $\mathbb{Q}$, to an imaginary quadratic field or to a totally definite quaternion algebra, see [4, Theorem 2.10], i.e. in the case at hand if and only if FGe is an exceptional matrix algebra. It follows that when FGe is a non-exceptional matrix component, $\mathrm{SL}_1(\mathcal{O})$ does not have the good property, as desired. ∎

We now readily obtain following characterisation.

Corollary 4.16. *Let G be a finite group with a non-abelian Sylow 2-subgroup. Then the following are equivalent:*

(1) $\mathrm{SL}_1(\mathbb{Z}G)$ *is good,*
(2) $\mathcal{U}(\mathbb{Z}G)$ *is good,*
(3) $\mathbb{Q}G$ *has* $(\mathrm{M}_{\mathrm{exc}})$.

Proof. For the equivalence between (1) and (2), it follows by [20, Corollary 5.5.3] that $\mathcal{U}(\mathbb{Z}G)$ is commensurable with $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \times \mathrm{SL}_1(\mathbb{Z}G)$. (Note that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$.) Moreover, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finitely generated, since $\mathcal{Z}(\mathbb{Z}G)$ is an order in $\mathcal{Z}(\mathbb{Q}G)$ and hence $\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$ is finitely generated by [20, Theorem 5.3.1]. Because $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is a finitely generated abelian group, it is good (since $\mathbb{Z}$ is good and finite groups are good). Hence $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \times \mathrm{SL}_1(\mathbb{Z}G)$ is good if and only if $\mathrm{SL}_1(\mathbb{Z}G)$ is good[15]. This finishes the equivalence between (1) and (2).

Next, the condition on the Sylow 2-subgroup, combined with Lemma 2.19, yields that $\mathbb{Q}G$ has no exceptional division components. Recall that SL_1 over a non-exceptional division component is finite, e.g. see [4, Theorem 2.10]. Hence they are good. We obtain the equivalence between (1) and (3) from Proposition 4.15. ☐

5. Congruence subgroups of rank 1 have virtually free quotient

Theorem 5.1. *Let $\left(\frac{u,v}{\mathbb{Q}}\right)$ be a totally definite quaternion algebra with centre $\mathbb{Q}$ and let $\mathcal{L}_{u,v}$* recall def *be the $\mathbb{Z}$-order with basis $\{1, i, j, k\}$. Then every principal congruence subgroup of $\mathrm{SL}_2(\mathcal{L}_{u,v})$ has a quotient which is virtually free. In particular, it is has a finite index subgroup with infinite abelianisation.*

This shows, in particular, that $\mathrm{SL}_2(\mathcal{O})$, for $\mathcal{O}$ any order in such a quaternion algebra, does not have property (FAb). Recall that a group is said to have property (FAb) if every finite index subgroup has finite abelianisation. This is a property of commensurability classes.

In order to obtain Theorem 5.1 we need to relate the groups $\mathrm{SL}_2(\mathcal{L}_{u,v})$ to certain groups of isometries of the hyperbolic 5-space. This is done in Section 5.2. The groups acting on hyperbolic 5-space will be special linear groups of Clifford groups, all of which are introduced in Section 5.1.

Finally in Section 5.3, we use this to obtain another characterisation of property $(\mathrm{M}_{\mathrm{exc}})$.

5.1. Background on SL_2 over quaternionic orders.

To prove the main result, we introduce some new notation. We define, for u and v strictly negative integers, the subset of $\left(\frac{-1,-1}{\mathbb{R}}\right)$:

$$\mathcal{H}_{u,v}(\mathbb{Q}) := \Big\{ a_0 + a_1\sqrt{|u|}i + a_2\sqrt{|v|}j + a_3\sqrt{|uv|}k \mid a_0, a_1, a_2, a_3 \in \mathbb{Q} \\ \text{and } i^2 = j^2 = -1, ij = -ji = k \Big\}.$$

It is not hard to see that

Lemma 5.2 ([24, Lemma 6.1])**.** *$\mathcal{H}_{u,v}(\mathbb{Q})$ is a subalgebra of $\left(\frac{-1,-1}{\mathbb{R}}\right)$ isomorphic to $\left(\frac{u,v}{\mathbb{Q}}\right)$. The isomorphism is explicitly given by*

$$\Lambda \colon \mathcal{H}_{u,v}(\mathbb{Q}) \to \left(\frac{u,v}{\mathbb{Q}}\right) \tag{5.1}$$

$$a_0 + a_1\sqrt{|u|}i + a_2\sqrt{|v|}j + a_3\sqrt{|uv|}k \mapsto a_0 + a_1 i + a_2 j + a_3 k.$$

We will be using the well-established Clifford algebras $\mathrm{Cliff}_n(\mathbb{R})$, the $\mathbb{R}$-algebras generated abstractly (as an algebra) by $n-1$ elements $i_1, \ldots, i_{n-1}$ satisfying the relations

$$i_h i_k = -i_k i_h, \text{ for } h \neq k \quad \text{ and } \quad i_h^2 = -1 \text{ for } 1 \leq h \leq n-1.$$

[15]The argument in the proof of [11, Proposition 3.4] using the Künneth theorem may be adapted to show that if $G_1 \times G_2$ is good, and G_1 is good, then necessarily G_2 is good. This follows by considering the 0^{th} cohomology groups for N_1 in the proof, which coincide with $\mathbb{F}_p$.

E.g.

$$\mathrm{Cliff}_1(\mathbb{R}) \cong \mathbb{R}, \quad \mathrm{Cliff}_2(\mathbb{R}) \cong \mathbb{C} \quad \text{and} \quad \mathrm{Cliff}_3(\mathbb{R}) \cong \left(\frac{-1,-1}{\mathbb{R}}\right).$$

The following definitions are also well-established, for example [1]. Recall that the main conjugation $.'$ on $\mathrm{Cliff}_n(\mathbb{R})$ is defined as the $\mathbb{R}$-linear automorphism determined by $i_h \mapsto -i_h$ for every $1 \leq h \leq n-1$, and that there is an anti-involution $.^*$ defined as the $\mathbb{R}$-linear automorphism defined by reversing the order of an arbitrary basis element $i_{h_1} i_{h_2} \dots i_{h_m}$. The composition of these two commuting automorphisms yields a new anti-involution denoted by $\overline{a} := a'^*$.

The vectorspace $\mathrm{Cliff}_n(\mathbb{R})$ is endowed with a Euclidean norm

$$|a| := \sqrt{\sum a_I^2},$$

where $a = \sum a_I I \in \mathrm{Cliff}_n(\mathbb{R})$, $I = i_{j_1} \cdots i_{j_r}$, $1 \leq j_1 < \dots < j_r \leq n-1$ (notice that such elements I form a basis for $\mathrm{Cliff}_n(\mathbb{R})$).

Moreover, for a, an element of the subvectorspace $\mathrm{V}^n(\mathbb{R}) \leq \mathrm{Cliff}_n(\mathbb{R})$ generated by the basis $\{i_0 := 1, i_1, \dots, i_{n-1}\}$, one readily computes that $a\overline{a} = |a|^2$, showing that every non-zero vector of $\mathrm{V}^n(\mathbb{R})$ is invertible. The group *generated* by all such vectors is denoted by $\Gamma_n(\mathbb{R})$ and is called the Clifford group.

This Clifford group also gives rise to general and special linear groups:

$$\begin{aligned}
\mathrm{GL}(\Gamma_n(\mathbb{R})) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \Gamma_n(\mathbb{R}) \cup \{0\},\ ad^* - bc^* \in \mathbb{R}^*, \right. \\
&\qquad\qquad \left. ab^*,\ cd^*,\ c^*a,\ d^*b \in \mathrm{V}^n(\mathbb{R}) \right\}, \\
\mathrm{SL}_+(\Gamma_n(\mathbb{R})) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\Gamma_n(\mathbb{R})) \mid ad^* - bc^* = 1 \right\},
\end{aligned}$$

for which it can be shown that the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad^* - bc^*$ is multiplicative. Lastly, $\mathrm{Cliff}_n(\mathbb{Z})$ denotes the $\mathbb{Z}$-subalgebra of $\mathrm{Cliff}_n(\mathbb{R})$ generated by $i_0, \dots, i_{n-1}$ and $\Gamma_n(\mathbb{Z}) =: \mathrm{Cliff}_n(\mathbb{Z}) \cap \Gamma_n(\mathbb{R})$, the submonoid of products of vectors from $\mathrm{Cliff}_n(\mathbb{Z})$ which are themselves invertible in $\mathrm{Cliff}_n(\mathbb{R})$.

Remark 5.3. The groups $\mathrm{SL}_+(\Gamma_4(\mathbb{R}))$ and $\mathrm{SL}\left(\left(\frac{-1,-1}{\mathbb{R}}\right)\right)$ are isomorphic. This non-obvious fact is proven in [10, Section 6], though the explicit isomorphism, as we will use later, can be found in [24, Section 5]. For completeness sake, we mention it here. Note that the elements

$$\begin{aligned}
\varepsilon_1 &= \tfrac{1+i_1i_2i_3}{2}, \\
\varepsilon_2 &= \tfrac{1-i_1i_2i_3}{2}
\end{aligned}$$

form a pair of central, orthogonal idempotents of $\mathrm{Cliff}_4(\mathbb{R})$. Indeed, one readily checks they commute with all generators of $\mathrm{Cliff}_4(\mathbb{R})$,

$$\varepsilon_i^2 = \varepsilon_i, \quad \varepsilon_1\varepsilon_2 = 0, \quad \text{and} \quad \varepsilon_1 + \varepsilon_2 = 1.$$

Using this, we obtain that every element $\alpha \in \mathrm{Cliff}_4(\mathbb{R})$ can be uniquely expressed as

$$\alpha = a\varepsilon_1 + b\varepsilon_2,$$

with $a, b \in \mathrm{Cliff}_3(\mathbb{R})$. Indeed, writing $\alpha = \sum \alpha_I I$ for $I \in \{1, i_1, i_2, i_3, i_1i_2, i_1i_3, i_2i_3, i_1i_2i_3\}$, and $a = \sum a_I I, b = \sum b_I I$ for $I \in \{1, i_1, i_2, i_1i_2\}$ one straightforwardly calculates that

$$\begin{aligned}
a_1 &= \alpha_1 + \alpha_{i_1i_2i_3}, & b_1 &= \alpha_1 - \alpha_{i_1i_2i_3}, \\
a_{i_1} &= \alpha_{i_1} - \alpha_{i_2i_3}, & b_{i_1} &= \alpha_{i_1} + \alpha_{i_2i_3}, \\
a_{i_2} &= \alpha_{i_2} + \alpha_{i_1i_3}, & b_{i_2} &= \alpha_{i_2} - \alpha_{i_1i_3}, \\
a_{i_1i_2} &= \alpha_{i_1i_2} - \alpha_{i_3}, & b_{i_1i_2} &= \alpha_{i_1i_2} + \alpha_{i_3}.
\end{aligned}$$

Because of the fact that ε_1 and ε_2 are central, orthogonal idempotents, one can make an algebra homomorphism $\chi' : \mathrm{Cliff}_4(\mathbb{R}) \to \mathrm{Cliff}_3(\mathbb{R}) : \alpha \mapsto a$. Using the isomorphism

$\mathrm{Cliff}_3(\mathbb{R}) \cong \left(\frac{-1,-1}{\mathbb{R}}\right)$ allows us to extend it to an algebra homomorphism $\chi : \mathrm{Cliff}_4(\mathbb{R}) \to \left(\frac{-1,-1}{\mathbb{R}}\right)$, which can be explicitly expressed as

$$\chi(\alpha) = \alpha_1 + \alpha_{i_1i_2i_3} + (\alpha_{i_1} - \alpha_{i_2i_3})i + (\alpha_{i_2} + \alpha_{i_1i_3})j + (\alpha_{i_1i_2} - \alpha_{i_3})k.$$

It is this homomorphism (which is obviously not an isomorphism) that extends, by entrywise application, to a group isomorphism between $\mathrm{SL}_+(\Gamma_4(\mathbb{R}))$ and $\mathrm{SL}\left(\left(\frac{-1,-1}{\mathbb{R}}\right)\right)$.

We now define a transformation σ of $\mathrm{Cliff}_n(\mathbb{R})$ to be

$$\sigma(z) = -z'.$$

Clearly $\sigma^2 = Id$.

We also define the subset of $\mathrm{Cliff}_4(\mathbb{R})$

$$\mathrm{Cliff}_4^{u,v}(\mathbb{Q}) := \Big\{\alpha_0 + \alpha_1\sqrt{|u|}i_1 + \alpha_2\sqrt{|v|}i_2 + \alpha_3\sqrt{|uv|}i_3 + \alpha_4\sqrt{|uv|}i_1i_2 \\ +\alpha_5\sqrt{|v|}i_1i_3 + \alpha_6\sqrt{|u|}i_2i_3 + \alpha_7 i_1i_2i_3 \mid \alpha_i \in \mathbb{Q}\Big\},$$

and one straightforwardly checks that it is a subalgebra of $\mathrm{Cliff}_4(\mathbb{R})$. Analogously, we define $\mathrm{Cliff}_4^{u,v}(\mathbb{Z})$, a $\mathbb{Z}$-order of $\mathrm{Cliff}_4^{u,v}(\mathbb{Q})$, $\Gamma_4^{u,v}(\mathbb{Z})$ and $\mathrm{SL}_+(\Gamma_4^{u,v}(\mathbb{Z}))$ (which is a subgroup of $\mathrm{SL}_+(\Gamma_4(\mathbb{R}))$). We consider the congruence subgroups of level $n \geq 1$ of the latter group, i.e.

$$Con_n := \left\{\begin{pmatrix} 1+na & nb \\ nc & 1+nd\end{pmatrix} \in \mathrm{SL}_+(\Gamma_4^{u,v}(\mathbb{Z})) \mid a,b,c,d \in \mathrm{Cliff}_4^{u,v}(\mathbb{Z})\right\}.$$

5.2. **Congruence subgroups SL_2 over quaternion algebras.** In order to prove Theorem 5.1, we will be considering some torsion free principal congruence subgroup. That such a subgroup exists requires the following lemma.

Lemma 5.4. *Let $\mathcal{L}_{u,v}$ be as above. There exists a prime p such that for every larger prime q, with u not a square modulo q, the principal congruence subgroup of level q of $\mathrm{GL}_2(\mathcal{L}_{u,v}) = \mathrm{SL}_2(\mathcal{L}_{u,v})$ is torsion free.*

Proof. Consider the map

$$\varphi : \mathrm{GL}_2(\mathcal{L}_{u,v}) \to \mathrm{GL}_2(\mathcal{L}_{u,v}/q\mathcal{L}_{u,v}).$$

We want to find a prime p and show that for q described in the statement, the kernel of φ is torsion free.

It is not hard to check that

$$i : \left(\frac{u,v}{\mathbb{Q}}\right) \to \mathrm{M}_2(\mathbb{Q}(\sqrt{u}))$$
$$a + bi + cj + dk \mapsto \begin{pmatrix} a+b\sqrt{u} & c+d\sqrt{u} \\ cv - dv\sqrt{u} & a - b\sqrt{u}\end{pmatrix},$$

is an injective homomorphism, such that $i(\mathcal{L}_{u,v}) \subseteq \mathrm{M}_2(\mathcal{I}_u)$. We may thus extend i to an injection $i : \mathrm{GL}_2(\mathcal{L}_{u,v}) \to \mathrm{GL}_4(\mathcal{I}_u)$.

For this larger group, we may also consider the principal congruence subgroup of level q, namely the kernel of the canonical morphism

$$\psi : \mathrm{GL}_4(\mathcal{I}_u) \to \mathrm{GL}_4(\mathcal{I}_u/q\mathcal{I}_u).$$

From the expression of i, it is not hard to see that $i(\ker\varphi) \leq \ker\psi$. From the assumptions on q, namely that u is not a square modulo q, it follows that $q\mathcal{I}_u$ is a prime ideal. This is well-known, for example see [35, § 5.4 Proposition 1]. As such, from [12, Lemma 9] we know that every torsion element of the kernel has order some power of q. However, on the other hand, by a result of Schur which is also stated in [12] as Theorem 14, there exists a natural number s such that the orders of the torsion elements of $\mathrm{GL}_4(\mathcal{I}_u)$ divide s.

So, if we let p be a prime larger than s and q a prime larger than p such that u is not a square modulo p, then all torsion elements of $\ker\varphi$ have to have an order which is a power of q and divides s. This is only possible when the order is 1. This proves that $\ker\varphi$ is torsion free. $\square$

Lemma 5.5. *If the group Con_n is torsion free, then it has a quotient which is virtually free.*

Proof. Consider the hyperbolic space of dimension 5

$$\mathbb{H}^5 = \{z_0 + z_1 i_1 + z_2 i_2 + z_3 i_3 + z_4 i_4 \mid z_i \in \mathbb{R}, z_4 \geq 0\} \subseteq \mathrm{V}^5(\mathbb{R}).$$

It is not hard to see that σ restricted to $\mathbb{H}^5$ is simply a reflection along the plane with equation $z_0 = 0$. It is well-known that $\mathrm{PSL}_+(\Gamma_4(\mathbb{R}))$ acts faithfully via Möbius transformation on $\mathbb{H}^5$:

$$Mz = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} z = (\alpha z + \beta)(\gamma z + \delta)^{-1}, \quad M \in \mathrm{PSL}_+(\Gamma_4(\mathbb{R})), z \in \mathbb{H}^5,$$

where we abuse the notation by representing an element of $\mathrm{PSL}_+(\Gamma_4(\mathbb{R}))$ by a matrix. However, it is clear this does not impact the way it acts via Möbius transformation.

Let now $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be in the subgroup Con_n, i.e. $\alpha = 1 + na$, $\beta = nb$, $\gamma = nc$ and $\delta = 1 + nd$ for some $a, b, c, d \in \mathrm{Cliff}_4^{u,v}(\mathbb{Z})$. Denote $\mathrm{P}\,Con_n$ by its image in $\mathrm{PSL}_+(\Gamma_4(\mathbb{R}))$ and again abusively say that $M \in \mathrm{P}\,Con_n$ (indeed, since Con_n is torsion-free, we may assume $M \in \mathrm{P}\,Con_n$). We claim that $\sigma M \sigma$ is again in $\mathrm{P}\,Con_n$. Indeed, for every $z \in \mathbb{H}^5$ we have

$$(\sigma M \sigma)\, z = -\left((-\alpha z' + \beta)(-\gamma z' + \delta)^{-1}\right)' = (\alpha' z - \beta')(-\gamma' z + \delta')^{-1}.$$

Hence, $\sigma M \sigma$ corresponds to the Möbius transformation given by the matrix $\begin{pmatrix} \alpha' & -\beta' \\ -\gamma' & \delta' \end{pmatrix}$, which is an element of $\mathrm{P}\,Con_n$.

Hence, the subgroup $\mathrm{P}\,Con_n$ is normalised by a reflection. This shows, by [30, Corollary 3.6], that $\mathrm{P}\,Con_n$ maps to a virtually free group. $\square$

These lemmas provide us with enough results to construct a proof for Theorem 5.1.

Proof of Theorem 5.1. Let G_n be the principal congruence subgroup of level n of $\mathrm{SL}_2(\mathcal{L}_{u,v})$. To prove the result, we will assume G_n to be a torsion-free principal congruence subgroup. That such an index n exists follows from Lemma 5.4. This will indeed suffice since congruence subgroups which are contained in one-another are of finite index and $G_n \cap G_m = G_{lcm(n,m)}$.

We expand the isomorphism Λ of (5.1) to an isomorphism Λ between $\mathrm{SL}_2(\mathcal{H}_{u,v}(\mathbb{Q}))$ and $\mathrm{SL}_2\left(\left(\frac{u,v}{\mathbb{Q}}\right)\right)$. Setting $\mathcal{G}_n = \Lambda^{-1}(G_n)$, the n^{th} principal congruence subgroup of $\mathrm{SL}_2\left(\Lambda^{-1}(\mathcal{L}_{u,v})\right)$, it suffices to prove $\mathcal{G}_n$ has a finite index subgroup with infinite abelianisation. Remark that also $\mathcal{G}_n$ is torsion-free.

By Remark 5.3, one is able to deduce that for the explicit isomorphism $\chi : \mathrm{SL}_+(\Gamma_4(\mathbb{R})) \to \mathrm{SL}_2\left(\left(\frac{-1,-1}{\mathbb{R}}\right)\right)$, $\chi(Con_n) \leq \mathcal{G}_n$ and thus that Con_n is torsion free.

Even more so, $\chi\left(\mathrm{SL}_+\left(\Gamma_4^{u,v}(\mathbb{Z})\right)\right) \leq \mathrm{SL}_2\left(\Lambda^{-1}(\mathcal{L}_{u,v})\right)$, and, using the classical argument that the units of orders are commensurable, this index is finite. However, since Con_n is a finite index subgroup of $\mathrm{SL}_+\left(\Gamma_4^{u,v}(\mathbb{Z})\right)$ and the same holds for $\mathcal{G}_n$ in $\mathrm{SL}_2\left(\Lambda^{-1}(\mathcal{L}_{u,v})\right)$, we obtain that $[\mathcal{G}_n : \chi(Con_n)] < \infty$. Lemma 5.5 now finishes the proof. $\square$

6. The blockwise Zassenhaus property

An interesting corollary of **??** is that the finite groups such that $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$ satisfy a "component-wise" third Zassenhaus conjecture.**to update to new notations**

Corollary 6.1. *Let G be a finite group such that $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$. If H is a finite subgroup of $V(\mathbb{Z}G)$, then for all $e \in \mathrm{PCI}(\mathbb{Q}G)$ the group He is conjugated by an element of $\mathcal{U}(\mathbb{Z}Ge)$ to a finite subgroup of Ge.*

Recall that $\mathcal{U}(\mathbb{Z}G) = \pm 1 \cdot V(\mathbb{Z}G)$ where $V(\mathbb{Z}G)$ is the set of units in $\mathcal{U}(\mathbb{Z}G)$ of augmentation 1. Recall moreover that $\mathcal{U}(\mathbb{Z}G)$ satisfies the third Zassenhaus conjecture if any finite $H \leq V(\mathbb{Z}G)$ is conjugated inside $\mathbb{Q}G$ to a subgroup of G. From the recent survey [31] it follows that the 12 families of groups which are virtually-$\mathcal{G}_\infty$ (see [29, Theorem 1]) are not

See if this paragraph fits better after the definitions of section 6.1

all amongst the known cases of examples satisfying the third Zassenhaus conjecture. In the case of the Zassenhaus conjectures, the conjugation is expected to be by an element of $\mathcal{U}(\mathbb{Q}G)$, hence in particular it is remarkable that for this class of groups the groups are G and H are conjugate by an element of an *order of* $\mathcal{U}(\mathbb{Q}Ge)$.

The proof of Corollary 6.1 is a corollary of Lemma 6.2 below and the investigation of independent interest of the "Strong Zassenhaus property", which we introduce in Section 6.1, for exceptional components.

6.1. **On (blockwise) Zassenhaus property for semisimple algebras.** We now introduce a weaker version.

To prove Corollary 6.1 we first record the following lemma.

Lemma 6.2. *Let G be a finite group and $H \leq V(\mathbb{Z}G)$ a finite subgroup. Then the following hold, for every primitive central idempotent $e \in \mathrm{PCI}(\mathbb{Q}G)$:*

(i) $|He|$ *divides* $|Ge|$*,*
(ii) $\exp(He)$ *divides* $\exp(Ge)$*,*

Proof. Fix a primitive central idempotent e of $\mathbb{Q}G$ and consider the associated epimorphism $\varphi : G \to Ge$. We denote by $\Phi : \mathbb{Z}[G] \to \mathbb{Z}[Ge]$ the ring epimorphism obtained by $\mathbb{Z}$-linearly extending φ. Denote moreover $N := \ker(\varphi) = \{g \in G \mid ge = e\}$. Note that $\ker(\Phi) = \omega(G, N)$, the relative augmentation ideal. Also, by definition of the map, $\Phi(V(\mathbb{Z}[G])) \subseteq V(\mathbb{Z}[Ge])$. Hence $\Phi(H)$ is a finite subgroup of $V(\mathbb{Z}[Ge])$ and it follows that $|\Phi(H)| \mid |Ge|$ (e.g. see [8, Corollary 2.7]). To prove (i), we show that $|He| \mid |\Phi(H)|$. For this, define the ring epimorphism

$$\pi \colon \mathbb{Z}[G] \to \mathbb{Z}[G]e, \quad x \mapsto xe.$$

Since $\pi(n-1) = ne - e = 0$ for all $n \in N$, one has that $\omega(G, N) \subseteq \ker(\pi)$. It follows that π factorises through $\mathbb{Z}[Ge]$ i.e. there exists a unique morphism $\sigma \colon \mathbb{Z}[Ge] \to \mathbb{Z}[G]e$ such that $\pi = \sigma \circ \Phi$. In particular $He = \pi(H) = \sigma(\Phi(H))$ is an epimorphic image of $\Phi(H)$ and hence $|He|$ divides $|\Phi(H)|$, as desired. Since $V(\mathbb{Z}[Ge])$ has exponent $\exp(Ge)$, it follows moreover that $\exp(He) \mid \exp(Ge)$, which finishes (ii). $\square$

Let A be a finite-dimensional semisimple algebra over a field F, with $\mathrm{char}(F) = 0$. Let $\mathcal{O}$ be a maximal order in A. We introduce the following definition.

Definition 6.3. The *set of spanning subgroups relative to* $\mathcal{O}$ is the set of finite subgroups of $\mathcal{U}(\mathcal{O})$ such that their F-linear span equals A:

$$\mathcal{S}_F(\mathcal{O}) := \{\Gamma \leqslant \mathcal{U}(\mathcal{O}) \mid |\Gamma| < \infty,\ \mathrm{Span}_F\{g \in \Gamma\} = A\}\,. \tag{6.1}$$

Note that the set $\mathcal{S}_F(R)$ may more generally be defined for any subring $R \leqslant A$, but is in general empty.

Definition 6.4. Let $\mathcal{G}$ be a set of finite subgroups of $\mathcal{U}(\mathcal{O})$, and let $B \subseteq \S_F(\mathcal{O})$ be a set of spanning subgroups. Then $\mathcal{O}$ is said to *have the Zassenhaus property relative to $\mathcal{G}$ and B* if for all $\Gamma \in B$ and for all $H \in \mathcal{G}$ such that $|H|$ divides $|\Gamma|$ and $\exp(H)$ divides $\exp(\Gamma)$, there exists an element $\alpha_H \in \mathcal{U}(A)$ such that

$$H^{\alpha_H} \leqslant \Gamma.$$

Remark 6.5. With the above terminology, the Zassenhaus conjectures may be reformulated as follows. Let $A = \mathbb{Q}G$ and $\mathcal{O} = \mathbb{Z}G$. Then $\mathcal{O}$ satisfies the i^{th} Zassenhaus conjecture if it has the Zassenhaus property relative to $\mathcal{G}_i$ and $B = \{G\}$, where

$$\begin{aligned}
\mathcal{G}_1 &:= \{H \leqslant G \mid H \text{ cyclic}\}\,,\\
\mathcal{G}_2 &:= \{H \leqslant G \mid |H| = |G|\}\,,\\
\mathcal{G}_3 &:= \{H \leqslant G\}\,.
\end{aligned}$$

We then introduce the following new terminology.

Definition 6.6. Let $\mathcal{G}$ be a set of finite subgroups of $\mathcal{U}(\mathcal{O})$, and $B \subseteq \mathcal{S}_F(\mathcal{O})$. We say that *$A$ has the blockwise Zassenhaus property relative to $\mathcal{G}$ and B* if for all $e \in \mathrm{PCI}(A)$, Ae has the blockwise Zassenhaus property relative to $\mathcal{G}e$ and Be.

In the next section, we shamm show that if $\mathbb{Q}G$ has ($\mathrm{M}_{\mathrm{exc}}$), then it has the blockwise Zassenhaus property relative to $\mathcal{G} := \{H \leqslant V(\mathbb{Z}G)\}$ and $B := \{G\}$.

6.2. **Strong Zassenhaus property for some simple algebras.**

6.2.1. *Fields and quaternion algebras.*

Proposition 6.7. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge$ is some quaternion algebra or a field. Then for any $H \leq \mathrm{V}(\mathbb{Z}G)$ the groups He and Ge are conjugated in $\mathbb{Q}Ge$.*

Proof. **TO COMPLETE to all quaternions** Suppose $\mathbb{Q}G_e$ is isomorphic to a field F. The unit group of its unique maximal order (i.e. its rings of integers) is a finitely generated abelian group. Thus He and Ge are subgroups of the torsion group which is cyclic. Hence the dividing orders yield that $He \leq Ge$, as desired. Suppose $\mathbb{Q}Ge \cong \left(\frac{-a,-b}{\mathbb{Q}}\right)$, then by [39, Theorem 11.5.14] $\mathcal{U}(\mathbb{Z}Ge)$ is cyclic except if $(a,b) = (-1,-1)$ or $(a,b) = (-1,-3)$ and $\mathbb{Z}Ge$ is the Lipschitz order, the Hurwitz order or the maximal order[16] of $\left(\frac{-1,-3}{\mathbb{Q}}\right)$. Recall that the last two cases are the unique maximal order, thus we already have that $He \leq Ge$ if $\mathbb{Z}Ge$ is not the Lipschitz order in $\left(\frac{-1,-1}{\mathbb{Q}}\right)$. In the remaining case $Ge \cong Q_8$ and He is some subgroup of the unit group of the Hurwitz quaternions (i.e. a subgroup of $\mathrm{SL}(2,3) \cong Q_8 \rtimes C_3$). Since $|He| \mid |Ge|$ we see that in fact $He \leq Ge$, finishing all possible cases. ☐

6.2.2. *The exceptional* $\mathrm{GL}_2(\mathcal{O})$ *case.* We now collect some results surrounding finite subgroups of the group of invertible 2×2 matrices over a division ring.

Lemma 6.8. *Let G be a finite subgroup of some $\mathrm{GL}_n(D)$ with D a finite dimensional division F-algebra with F a field with $\mathrm{char}(F) = 0$. Then $\mathrm{Span}_F\{g \in G\} = \mathrm{M}_n(D)$ if and only if $\mathrm{M}_n(D) \in \mathcal{C}(FG)$.*

Proof. Consider the canonical representation $\varphi : FG \to \mathrm{M}_n(D)$ associated to the embedding of G in $\mathrm{GL}_n(D)$. Would φ be irreducible, then $\mathrm{M}_n(D) \in \mathcal{C}(FG)$ by the correspondence between irreducible F-representations of G and simple component of FG. Now the irreducibility follows from the spanning condition. Indeed, write $\mathrm{M}_n(D) = \mathrm{End}_{FG}(V)$. As $\mathrm{Span}_F\{g \in G\} = \mathrm{M}_n(D)$, we have that $\mathrm{Span}_F\{g.v \mid g \in G\} = V$ for any non-zero $v \in V$. This shows that V can not have a G-invarant subspace, as claimed.

Conversely suppose that $\mathrm{M}_n(D) \in \mathcal{C}(\mathbb{Q}G)$. In other words, there exists an irreducible F-representation $\rho : FG \to \mathrm{M}_n(D)$. It is a classical theorem by burnside that $\mathrm{Span}_F\{\rho(g) \mid g \in G\} = \mathrm{M}_n(D)$. ☐

Now consider an exceptional matrix algbera $\mathrm{M}_2(D)$ and an order $\mathcal{O}$ in $\mathrm{M}_2(D)$. Then, by the above lemma and [9], one obtains that $\mathcal{S}_{\mathbb{Q}}(\mathcal{O})$ is empty except if $D = \mathbb{Q}(\sqrt{-d})$ with $d = 0, 1, 2, 3$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with $(a,b) \in \{(-1,-1), (-1,-3), (-2,-5)\}$. Furthermore for those values from [9, Table 2] one can read off the isomorphism type of finite groups G that have a faithful irreducible representation in $\mathrm{M}_2(D)$, and hence whose image in $\mathrm{M}_2(D)$ spans $\mathrm{M}_2(D)$.

Next recall the following definition (see [5, section 2.1]).

Definition 6.9. Let V be the 2-dimensional D-module given by column vectors in D. Let $G \leqslant \mathrm{GL}_2(D)$. Then G has a natural action on V by left multiplication. The group G is said to be *imprimitive* if and only if V decomposes as a direct sum

$$V = V_1 \oplus V_2$$

[16] A short concrete summary of the above orders can also be found before and after Theorem 3.14. of [4].

of 1-dimensional D-modules such that for any $g \in G$, one has $gV_i = V_{\sigma(i)}$, for $i \in \{1,2\}$ and $\sigma \in \mathrm{Sym}(\{1,2\})$). A group which is not imprimitive is said to be *primitive*.

Note that in particular for a primitive group $G \leqslant \mathrm{GL}_2(D)$, the action of G on V is irreducible. For the following statement, we follow the techniques from the proof of [5, Lemma 2.2].

Lemma 6.10. *Suppose $G \leqslant \mathrm{GL}_2(D)$ is imprimitive. Let $V = V_1 \oplus V_2$ be the associated decomposition into 1-dimensional D-modules. If there exists some $g \in G$ such that $gV_1 = V_2$, then*

$$G \cong (\Gamma \times \Gamma) \rtimes C_2,$$

where $\Gamma \leqslant \mathcal{U}(D)$ and where C_2 acts by exchanging the factors. Otherwise, $G \cong \Gamma_1 \times \Gamma_2$ for some subgroups $\Gamma_1, \Gamma_2 \leqslant \mathcal{U}(D)$.

Proof. Suppose $gV_i = V_i$ for all $g \in G$. Then up to conjugation by a base change matrix $A \in \mathrm{GL}_2(D)$, G consists of diagonal matrices. It follows that $G \cong \Gamma_1 \times \Gamma_2$ for some finite subgroups $\Gamma_1, \Gamma_2 \leqslant \mathcal{U}(D)$.

Suppose then that there exists some $g \in G$ such that $gV_1 = V_2$. Let $H \leqslant G$ be the subgroup of G such that $hV_i = V_i$ for all $h \in H$ and $i \in \{1,2\}$. Again up to conjugation of G by a base change matrix $A \in \mathrm{GL}_2(D)$, g is of the form $\left(\begin{smallmatrix} 0 & g_1 \\ g_2 & 0 \end{smallmatrix}\right)$, while H consists precisely of diagonal matrices of G. Moreover, H has index 2 in G. Indeed, $G = H \sqcup gH$, since if $x := \left(\begin{smallmatrix} 0 & x_1 \\ x_2 & 0 \end{smallmatrix}\right)$ is another antidiagonal matrix in G, then

$$h := g^{-1}x = \begin{pmatrix} g_2^{-1}x_2 & 0 \\ 0 & g_1^{-1}x_1 \end{pmatrix} \in H$$

is a matrix such that $gh = x$, and by the fact that G is imprimitive, any $x \in G$ is either diagonal or antidiagonal. In particular, H is a normal subgroup, and $G/H = \langle gH \rangle$ acts on H by exchanging the diagonal entries. It follows that $G \cong (\Gamma \times \Gamma) \rtimes C_2$, for some finite subgroup $\Gamma \leqslant \mathcal{U}(D)$. $\square$

Hence in particular the finite subgroups of $\mathrm{GL}_2(D)$ of imprimitive type are easy to describe in terms of finite subgroups of $\mathcal{U}(D)$.

Thus it is relevant to recall $\mathcal{U}(R)$ where R is the maximal order in D with $D \in \{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right)\}$ and $d = 1,2,3$ and $(a,b) = (-1,-1), (-1,-3), (-2,-5)$. **ik ben beetje verward over D versus zijn maximal order. Maar alvast voor de velden zijn dat ook de enige finite subgroups. Misschien iets van finite order is integral?**

The maximal order of $\mathbb{Q}$ is $\mathbb{Z}$ with unit group $C_2 = \{\pm 1\}$. For an order $\mathcal{O}$ in an imaginary quadratic number field $\mathcal{U}(\mathcal{O}) = \langle -1 \rangle$, unless $\mathcal{O} \in \{\mathcal{I}_1, \mathcal{I}_3\}$ (i.e. $d = -1$ or -3), in which case $\mathcal{U}(\mathcal{I}_1) = \langle i \rangle$ and $\mathcal{U}(\mathcal{I}_3) = \langle -\zeta_3 \rangle \cong C_6$, respectively.

Next, denote

$$\mathbb{H}_2 = \left(\frac{-1,-1}{\mathbb{Q}}\right), \quad \mathbb{H}_3 = \left(\frac{-1,-3}{\mathbb{Q}}\right) \quad \text{and} \quad \mathbb{H}_5 = \left(\frac{-2,-5}{\mathbb{Q}}\right).$$

They have up to conjugation a unique maximal order, which we respectively denote by $\mathcal{O}_2, \mathcal{O}_3$ and $\mathcal{O}_5$. In the table below specific $\mathbb{Z}$-bases $\{b_1, b_2, b_3, b_4\}$ of these orders **al dit heb gekopieerd van onze abelianisatie paper, we zien bij einde wel wat we hier behouden**

(6.2)

	b_1	b_2	b_3	b_4
$\mathcal{O}_2$	1	i	j	$\omega_2 = \frac{1+i+j+k}{2}$
$\mathcal{O}_3$	1	i	$\omega_3 = \frac{1+j}{2}$	$\frac{i+k}{2}$
$\mathcal{O}_5$	1	$\frac{1+i+j}{2}$	$\omega_5 = \frac{2+i-k}{4}$	$\frac{2+3i+k}{4}$

Furthermore, if we set $\omega_2 = \frac{1+i+j+k}{2} \in \mathcal{O}_2$, $\omega_3 = \frac{1+j}{2} \in \mathcal{O}_3$ and $\omega_5 = \frac{2+i-k}{4} \in \mathcal{O}_5$, then we have

$$\begin{aligned} \mathcal{U}(\mathcal{O}_2) &= \langle i, \omega_2 \rangle \cong \mathrm{SL}(2,3) \cong Q_8 \rtimes C_3, \\ \mathcal{U}(\mathcal{O}_3) &= \langle i, \omega_3 \rangle \cong C_3 \rtimes C_4, \\ \mathcal{U}(\mathcal{O}_5) &= \langle \omega_5 \rangle \cong C_6. \end{aligned} \tag{6.3}$$

Theorem 6.11. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2(D)$ with*[17] *$D \in \{\mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right) \mid a,b,d \in \mathbb{N}\}$. Then for any finite $H \leq \mathrm{V}(\mathbb{Z}G)$ the groups He and Ge are conjugated in $\mathbb{Q}Ge$.*

plan: component per component werken via expliciete amalgam van 'grote' finite index deelgroepen. Probleem is dat niet steeds de hele component zo'n decompositie heeft, maar mischien $\mathbb{Z}Ge$ desondanks toch steeds in eentje?

Proof. If it is $\mathrm{M}_2(\mathbb{Q})$, then $\mathcal{U}(\mathbb{Z}Ge) = \mathrm{GL}_2(\mathbb{Z}) \cong D_8 *_{C_2 \times C_2} D_{12}$. Therefore there exists some $\alpha_e \in \mathcal{U}(\mathbb{Z}Ge)$ such that $\alpha_e^{-1} Ge \alpha_e$ is a subgroup of D_8 or D_{12}. Since the $\mathbb{Q}$-span of Ge is $\mathrm{M}_2(\mathbb{Q})$ we get that $\alpha_e^{-1} Ge \alpha_e = D_6, D_{12}$ or D_8. In particular $|Ge|$ determines uniquely its isomorphism type. The same holds for He if He is non-abelian. Therefore, as $|He| \mid |Ge|$ by Lemma 6.2, we have the desired statement in that case. Suppose now that $|He| = 4$. If $He \cong C_4$, then He is up to conjugation a subgroup of D_8 and due to the dividing of the orders, again $He \leq Ge$ after conjugation. If it is an elementary abelian 2-group, then it is uniquely defined in both D_{12} and D_8. As this subgroup is amalgamated we are done. ☐

Proof of Corollary 6.1. By Lemma 6.2 $|He| \mid |Ge|$ for every primitive central idempotent e of $\mathbb{Q}G$. By Theorem 2.6 the simple algebra $\mathbb{Q}Ge$ is either a field, a specific type of quaternion algebra or some exceptional simple algebras. The strong Zassenhaus property of the former is proven in ?? and for the latter in Theorem 6.11. **TO UPDATE**. ☐

[17] In other words $\mathbb{Q}Ge$ is an exceptional component of $\mathbb{Q}G$.

Appendix A. Table of groups with a faithful exceptional 2×2 embedding

In this appendix we reproduce [9, Table 2] listing those finite groups G that have a faithful exceptional component of type (2) (see Definition 2.1) in the Wedderburn-Artin decomposition of the rational group algebra $\mathbb{Q}G$[18]. For all these groups, we have added extra information which is relevant to the property ($\mathrm{M_{exc}}$):

($\mathrm{M_{exc}}$):	indicates whether the group satisfies ($\mathrm{M_{exc}}$)
SmallGroup ID:	the identifier of the group G in the small group library
Structure:	the structure description of the group.
cℓ:	the nilpotency class of the group; ∞ indicates that the group is not nilpotent (omitted for non-solvable groups)
dℓ:	derived length of the group; ∞ for non-solvable groups
$\exp(G), \exp(Z(G))$:	the exponent of the group G (resp $Z(G)$, the centre of G), i.e. the lowest common multiple of the order of the elements
$Z(G) \cap G'$:	the structure of the intersetion of the centre of G with the derived subgroup of G
$Q/Z(Q)$:	the structure of the quotient of Q (a Sylow 2-subgroup of G) by its centre, which is independent of the chosen Q
Smallest $[G : B]$:	the smallest index of a maximal abelian, normal subgroup B in G
1×1 components:	the division-algebra components appearing in the Wedderburn-Artin decomposition of $\mathbb{Q}G$ (with multiplicity); omitted for groups without ($\mathrm{M_{exc}}$).

In the structure descriptions, we use the following conventions: 1 indicates the trivial group, colons indicate split extensions, a period an extension that is (possibly) non-split and the n^{th} power of a group indicates its n-fold direct product.

Recall that we use the following shorthands for quaternion algebras appearing:

$$\mathbb{H}_2 = \left(\frac{-1,-1}{\mathbb{Q}}\right), \quad \text{and} \quad \mathbb{H}_3 = \left(\frac{-1,-3}{\mathbb{Q}}\right),$$

and that ζ_k represents a complex, primitive k-root of unity.

[18] including the group with SmallGroupID `[24, 1]` that was accidentally omitted in the original table

($\mathrm{M_{exc}}$)	SmallGroup ID	Structure	cl	dℓ	$exp(G)$	$exp(Z(G))$	$Z(G) \cap G'$	$Q/Z(Q)$	Smallest $[G : B]$	1×1 components
✓	[6, 1]	S_3	∞	2	6,	1,	1,	1,	2,	$2{\times}\mathbb{Q}$,
✓	[8, 3]	D_8	2	2	4,	2,	C_2,	C_2^2,	2,	$4{\times}\mathbb{Q}$,
✓	[12, 4]	D_{12}	∞	2	6,	2,	1,	1,	2,	$4{\times}\mathbb{Q}$,
✓	[16, 6]	$C_8 : C_2$	2	2	8,	4,	C_2,	C_2^2,	2,	$4{\times}\mathbb{Q}$, $2{\times}\mathbb{Q}(i)$,
✓	[16, 8]	QD_{16}	3	2	8,	2,	C_2,	D_8,	2,	$4{\times}\mathbb{Q}$,
✓	[16, 13]	$(C_4 \times C_2) : C_2$	2	2	4,	4,	C_2,	C_2^2,	2,	$8{\times}\mathbb{Q}$,
✓	[18, 3]	$C_3 \times S_3$	∞	2	6,	3,	1,	1,	2,	$2{\times}\mathbb{Q}$, $2{\times}\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$,
✓	[24, 1]	$C_3 : C_8$	∞	2	24,	4,	1,	1,	2,	$2{\times}\mathbb{Q}$, $1{\times}\mathbb{Q}(i)$, $1{\times}\mathbb{Q}(\zeta_8)$, $1{\times}\mathbb{H}_3$,
×	[24, 3]	$\mathrm{SL}(2,3)$	∞	3	12,	2,	C_2,	C_2^2,	12,	
✓	[24, 5]	$C_4 \times S_3$	∞	2	12,	4,	1,	1,	2,	$4{\times}\mathbb{Q}$, $2{\times}\mathbb{Q}(i)$,
✓	[24, 8]	$(C_6 \times C_2) : C_2$	∞	2	12,	2,	C_2,	C_2^2,	2,	$4{\times}\mathbb{Q}$,
✓	[24, 10]	$C_3 \times D_8$	2	2	12,	6,	C_2,	C_2^2,	2,	$4{\times}\mathbb{Q}$, $4{\times}\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$,
✓	[24, 11]	$C_3 \times Q_8$	2	2	12,	6,	C_2,	C_2^2,	2,	$4{\times}\mathbb{Q}$, $4{\times}\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $1{\times}\mathbb{H}_2$,
✓	[32, 8]	$C_2^2.(C_4 \times C_2)$	3	2	8,	2,	C_2,	$(C_4 \times C_2) : C_2$,	4,	$4{\times}\mathbb{Q}$, $2{\times}\mathbb{Q}(i)$,
✓	[32, 11]	$C_4^2 : C_2$	3	2	8,	4,	C_2,	D_8,	2,	$4{\times}\mathbb{Q}$, $2{\times}\mathbb{Q}(i)$,
✓	[32, 44]	$(C_2 \times Q_8) : C_2$	3	2	8,	2,	C_2,	$C_2 \times D_8$,	4,	$8{\times}\mathbb{Q}$,
✓	[32, 50]	$(C_2 \times Q_8) : C_2$	2	2	4,	2,	C_2,	C_2^4,	4,	$16{\times}\mathbb{Q}$,
✓	[36, 6]	$C_3 \times (C_3 : C_4)$	∞	2	12,	6,	1,	1,	2,	$2{\times}\mathbb{Q}$, $1{\times}\mathbb{Q}(i)$, $2{\times}\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $1{\times}\mathbb{Q}(\zeta_{12})$, $1{\times}\mathbb{H}_3$,
✓	[36, 12]	$C_6 \times S_3$	∞	2	6,	6,	1,	1,	2,	$4{\times}\mathbb{Q}$, $4{\times}\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$,
×	[40, 3]	$C_5 : C_8$	∞	2	40,	2,	1,	1,	4,	
✓	[48, 16]	$(C_3 : Q_8) : C_2$	∞	2	24,	2,	C_2,	D_8,	4,	$4{\times}\mathbb{Q}$,
✓	[48, 18]	$C_3 : Q_{16}$	∞	2	24,	2,	C_2,	D_8,	4,	$4{\times}\mathbb{Q}$, $1{\times}\left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right)$,
×	[48, 28]	$\mathrm{SL}(2,3).C_2$	∞	4	24,	2,	C_2,	D_8,	24,	
×	[48, 29]	$\mathrm{GL}(2,3)$	∞	4	24,	2,	C_2,	D_8,	24,	
×	[48, 33]	$((C_4 \times C_2) : C_2) : C_3$	∞	3	12,	4,	C_2,	C_2^2,	12,	
✓	[48, 39]	$(C_4 \times S_3) : C_2$	∞	2	12,	2,	C_2,	C_2^2,	4,	$8{\times}\mathbb{Q}$,
✓	[48, 40]	$Q_8 \times S_3$	∞	2	12,	2,	C_2,	C_2^2,	4,	$8{\times}\mathbb{Q}$, $2{\times}\mathbb{H}_2$,
×	[64, 37]	$(C_4 \times C_2).(C_4 \times C_2)$	4	2	8,	2,	C_2,	$C_2^3 : C_4$,	4,	
✓	[64, 137]	$(C_4 : Q_8) : C_2$	3	2	8,	2,	C_2,	$C_2^4 : C_2$,	4,	$8{\times}\mathbb{Q}$,
×	[72, 19]	$C_3^2 : C_8$	∞	2	24,	2,	1,	1,	4,	
×	[72, 20]	$(C_3 : C_4) \times S_3$	∞	2	12,	2,	1,	1,	4,	
×	[72, 22]	$(C_6 \times S_3) : C_2$	∞	2	12,	2,	C_2,	C_2^2,	4,	
×	[72, 24]	$C_3^2 : Q_8$	∞	2	12,	2,	C_2,	C_2^2,	4,	
×	[72, 25]	$C_3 \times \mathrm{SL}(2,3)$	∞	3	12,	6,	C_2,	C_2^2,	12,	
✓	[72, 30]	$C_3 \times ((C_6 \times C_2) : C_2)$	∞	2	12,	6,	C_2,	C_2^2,	2,	$4{\times}\mathbb{Q}$, $4{\times}\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$,
×	[96, 67]	$\mathrm{SL}(2,3) : C_4$	∞	4	24,	4,	C_2,	D_8,	24,	
×	[96, 190]	$(C_2 \times \mathrm{SL}(2,3)) : C_2$	∞	4	24,	2,	C_2,	$C_2 \times D_8$,	24,	
×	[96, 191]	$\mathrm{SL}(2,3).C_2) : C_2$	∞	4	24,	2,	C_2,	$C_2 \times D_8$,	24,	
×	[96, 202]	$((C_2 \times Q_8) : C_2) : C_3$	∞	3	12,	2,	C_2,	C_2^4,	24,	
×	[120, 5]	$\mathrm{SL}(2,5)$		∞	60,	2,	C_2,	C_2^2,	60,	
×	[128, 937]	$(Q_8 \times Q_8) : C_2$	4	3	8,	2,	C_2,	$(C_2^4 : C_2) : C_2$,	8,	
×	[144, 124]	$\mathrm{SL}(2,3).C_2)$	∞	4	24,	2,	C_2,	D_8,	24,	
×	[144, 128]	$S_3 \times \mathrm{SL}(2,3)$	∞	3	12,	2,	C_2,	C_2^2,	24,	
×	[144, 135]	$C_3^2 : (C_8 : C_2)$	∞	2	24,	2,	C_2,	C_2^2,	4,	
×	[144, 148]	$C_3^2 : ((C_4 \times C_2) : C_2)$	∞	2	12,	2,	C_2,	C_2^2,	4,	
×	[160, 199]	$((C_2 \times Q_8) : C_2) : C_5$	∞	3	20,	2,	C_2,	C_2^4,	80,	
×	[192, 989]	$(\mathrm{SL}(2,3) : C_4) : C_2$	∞	4	24,	2,	C_2,	$C_2^4 : C_2$,	48,	
×	[240, 89]	$\mathrm{SL}(2,5).C_2$		∞	120,	2,	C_2,	D_8,	120,	
×	[240, 90]	$\mathrm{SL}(2,5) : C_2$		∞	120,	2,	C_2,	D_8,	120,	
×	[288, 389]	$C_3^2 : (C_4^2 : C_2)$	∞	3	24,	2,	C_2,	D_8,	8,	
×	[320, 1581]	$(((C_2 \times Q_8) : C_2) : C_5).C_2$	∞	4	40,	2,	C_2,	$C_2^4 : C_2$,	160,	
×	[384, 618]	$((Q_8 \times Q_8) : C_2) : C_3$	∞	3	24,	2,	C_2,	$(C_2^4 : C_2) : C_2$,	96,	
×	[384, 18130]	$((Q_8 \times Q_8) : C_3) : C_2$	∞	4	24,	2,	C_2,	$(C_2^4 : C_2) : C_2$,	96,	
×	[720, 409]	$\mathrm{SL}(2,9)$		∞	120,	2,	C_2,	D_8,	360,	
×	[1152, 155468]	$(((Q_8 \times Q_8) : C_3) : C_2) : C_3$	∞	4	24,	2,	C_2,	$(C_2^4 : C_2) : C_2$,	288,	
×	[1920, 241003]	$C_2.(C_2^4 : A_5)$		∞	120,	2,	C_2,	$(C_2^4 : C_2) : C_2$,	960,	

References

[1] Lars V. Ahlfors. Möbius transformations and Clifford numbers. In *Differential geometry and complex analysis*, pages 65–73. Springer, Berlin, 1985. 35

[2] S. A. Amitsur. Finite subgroups of division rings. *Trans. Amer. Math. Soc.*, 80:361–386, 1955. 7, 8

[3] Andreas Bächle, Geoffrey Janssens, Eric Jespers, Ann Kiefer, and Doryan Temmerman. A dichotomy for integral group rings via higher modular groups as amalgamated products. *J. Algebra*, 604:185–223, 2022. 2

[4] Andreas Bächle, Geoffrey Janssens, Eric Jespers, Ann Kiefer, and Doryan Temmerman. Abelianization and fixed point properties of units in integral group rings. *Math. Nachr.*, 296(1):8–56, 2023. 2, 4, 7, 8, 33, 34, 39

[5] Behnam Banieqbal. Classification of finite subgroups of 2×2 matrices over a division algebra of characteristic zero. *J. Algebra*, 119(2):449–512, 1988. 40

[6] H. Bass, J. Milnor, and J.-P. Serre. Solution of the congruence subgroup problem for SL_n $(n \geq 3)$ and Sp_{2n} $(n \geq 2)$. *Inst. Hautes Études Sci. Publ. Math.*, (33):59–137, 1967. 33

[7] Sheila Chagas, Ángel del Rio, and Pavel A. Zalesskii. Aritmethic lattices of $\mathrm{SO}(1,n)$ and units of group rings. *J. Pure Appl. Algebra*, 227(11):Paper No. 107405, 17, 2023. 33

[8] Ángel del Río. Finite groups in integral group rings. Lecture notes, arXiv:1805.06996v9, 2022. 38

[9] Florian Eisele, Ann Kiefer, and Inneke Van Gelder. Describing units of integral group rings up to commensurability. *J. Pure Appl. Algebra*, 219(7):2901–2916, 2015. 7, 8, 40, 42

[10] J. Elstrodt, F. Grunewald, and J. Mennicke. Vahlen's group of Clifford matrices and spin-groups. *Math. Z.*, 196(3):369–390, 1987. 35

[11] F. Grunewald, A. Jaikin-Zapirain, and P. A. Zalesskii. Cohomological goodness and the profinite completion of Bianchi groups. *Duke Math. J.*, 144(1):53–72, 2008. 33, 34

[12] Robert M. Guralnick and Martin Lorenz. Orders of finite groups of matrices. In *Groups, rings and algebras*, volume 420 of *Contemp. Math.*, pages 141–161. Amer. Math. Soc., Providence, RI, 2006. 36

[13] B. Huppert. *Endliche Gruppen. I*, volume 134 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1967. 13

[14] Bertram Huppert and Olaf Manz. Degree-problems. I. Squarefree character degrees. *Arch. Math. (Basel)*, 45(2):125–132, 1985. 13

[15] I. M. Isaacs and D. S. Passman. A characterization of groups in terms of the degrees of their characters. II. *Pacific J. Math.*, 24:467–510, 1968. 12, 13, 14, 15

[16] I. Martin Isaacs. *Character theory of finite groups*. Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)]. 11, 12, 13, 22

[17] Geoffrey Janssens and Leo Margolis. On integral decomposition of unipotent elements in integral group rings, 2023. 5, 10, 16

[18] E. Jespers and W. Sun. Nilpotent decomposition in integral group rings. *J. Algebra*, 575:127–158, 2021. 6, 8, 18

[19] Eric Jespers and Ángel del Río. A structure theorem for the unit group of the integral group ring of some finite groups. *J. Reine Angew. Math.*, 521:99–117, 2000. 2

[20] Eric Jespers and Ángel del Río. *Group ring groups. Vol. 1. Orders and generic constructions of units*. De Gruyter Graduate. De Gruyter, Berlin, 2016. 4, 5, 6, 9, 10, 18, 25, 26, 34

[21] Eric Jespers, Guilherme Leal, and Ángel del Río. Products of free groups in the unit group of integral group rings. *J. Algebra*, 180(1):22–40, 1996. 2

[22] Eric Jespers, Gabriela Olteanu, Ángel del Río, and Inneke Van Gelder. Central units of integral group rings. *Proc. Amer. Math. Soc.*, 142(7):2193–2209, 2014. 4

[23] Eric Jespers, Antonio Pita, Ángel del Río, Manuel Ruiz, and Pavel Zalesskii. Groups of units of integral group rings commensurable with direct products of free-by-free groups. *Adv. Math.*, 212(2):692–722, 2007. 2, 4, 6, 7, 14, 22, 23, 27, 28, 30, 32

[24] Ann Kiefer. On units in orders in 2-by-2 matrices over quaternion algebras with rational center. *Groups Geom. Dyn.*, 14(1):213–242, 2020. 34, 35

[25] E. Kleinert. Two theorems on units of orders. *Abh. Math. Sem. Univ. Hamburg*, 70:355–358, 2000. 27

[26] E. Kleinert and Á. del Río. On the indecomposability of unit groups. *Abh. Math. Sem. Univ. Hamburg*, 71:291–295, 2001. 27

[27] Ernst Kleinert. Units of classical orders: a survey. *Enseign. Math. (2)*, 40(3-4):205–248, 1994. 2

[28] Reginald Koo. A Classification of Matrices of Finite Order over $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$. *Math. Mag.*, 76(2):143–148, 2003. 30, 31

[29] Guilherme Leal and Ángel del Río. Products of free groups in the unit group of integral group rings. II. *J. Algebra*, 191(1):240–251, 1997. 38

[30] Alexander Lubotzky. Free quotients and the first Betti number of some hyperbolic manifolds. *Transform. Groups*, 1(1-2):71–82, 1996. 37

[31] Leo Margolis and Ángel del Río. Finite subgroups of group rings: a survey. *Adv. Group Theory Appl.*, 8:1–37, 2019. 38

[32] A. Olivieri, Á. del Río, and J. J. Simón. On monomial characters and central idempotents of rational group algebras. *Commun. Algebra*, 32(4):1531–1550, 2004. 5, 6

[33] Antonio Pita, Ángel Del Río, and Manuel Ruiz. Groups of units of integral group rings of Kleinian type. *Trans. Amer. Math. Soc.*, 357(8):3215–3237, 2005. 2

[34] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. 32

[35] Pierre Samuel. Algebraic theory of numbers. (Translated from the French by Allan J. Silberger). Boston, Mass.: Houghton Mifflin Co., 109 pp. $ 7.95 (1970)., 1970. 36

[36] Sudarshan K. Sehgal. Nilpotent elements in group rings. *Manuscripta Math.*, 15:65–80, 1975. 5, 6, 10, 23

[37] Jean-Pierre Serre. Le problème des groupes de congruence pour SL2. *Ann. of Math. (2)*, 92:489–527, 1970. 33

[38] M. Shirvani and B. A. F. Wehrfritz. *Skew linear groups*, volume 118 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1986. 8, 9

[39] John Voight. *Quaternion algebras, v.0.9.13*. June 10, 2018. https://math.dartmouth.edu/~jvoight/quat/quat-book-v0.9.13.pdf. 39

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