SIMULTANEOUS PING-PONG FOR FINITE SUBGROUPS OF REDUCTIVE GROUPS

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Dedicated to the memory of Jacques Tits with great admiration for his legacy.

Abstract. Let Γ be a Zariski-dense subgroup of a reductive group $\mathbf{G}$ defined over a field F. Given a finite collection of finite subgroups H_i ($i \in I$) of $\mathbf{G}(F)$ avoiding the center, we establish a criterion to ensure that the set of elements of Γ that form a free product with every H_i (so-called ping-pong partners for H_i) is both Zariski- and profinitely dense in Γ. This criterion applies to direct products of inner $\mathbb{R}$-forms of GL_n, and implies a particular case (the case of torsion elements in such products) of a 1994 question of Bekka-Cowling-de la Harpe. We also point out an obstruction to the question. Subsequently, for such $\mathbf{G}$ we give constructive methods to obtain free products between two given finite subgroups.

Next, we investigate the case where $\mathbf{G}(F) = \mathcal{U}(FG)$ for G a finite group and $\Gamma = \mathcal{U}(RG)$ for R an order in F. Hereby our main theorem is that the set of bicylic unit ping-pong partners of a given shifted bicyclic unit is profinitely dense, answering a long standing belief in the field. This is deduced as an application of the above with some new existence results of especially nice irreducible representations of G that are (centrally) faithful on a given subgroup. Finally, we answer Kleinert's virtual structure problem for the property to have an amalgam or HNN spliting over a finite group.

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1. Introduction

1.1. **Background.** The construction and study of free products in linear groups is a classical topic going back to the early days of group theory. A groundbreaking step was Tits' celebrated alternative [83], establishing existence of free subgroups in linear groups which a re not virtually solvable. In fact he proved the stronger statement that if Γ is a finitely generated Zariski-connected linear group over a field F, then either Γ is solvable, or it contains a Zariski-dense free subgroup of rank 2. A version of this statement for non-Zariski-connected groups was given by Breuillard and Gelander in [13], where they prove an analogous (but stronger) theorem for the topology induced by a local field. The speed at which a given finite set produces two elements generating a free subgroup, has mostly been determined, see [1, 14, 16, 15, 17, 3] for some recent results.

In the present article, we are interested in constructing free subgroups of linear groups, one of whose generators is prescribed. More generally, given a finite subset S of a linear group Γ, the question of interest is: does there exists $\gamma \in \Gamma$ such that for all $h \in S$, the subgroup $\langle h, \gamma\rangle$ is *freely* generated by h and γ? Such an element γ is called a *simultaneous ping-pong partner (in* Γ*) for the set* S.

Following [7], a discrete group Γ is said to have *property* ($P_{naï}$) if any finite subset S of Γ admits a simultaneous ping-pong partner. In 1995, Bekka, Cowling and de la Harpe proved [7, Theorem 3] that Zariski-dense subgroups of connected simple real Lie groups of real rank 1 with trivial center have property ($\mathrm{P}_{\mathrm{naï}}$), and asked in [7, Remark 3] whether the same holds for semisimple groups of arbitrary rank. This question was again highlighted in 2007 by de la Harpe [23, Question 17], and we record it here under the following form.

Question 1.1 (Bekka–Cowling–de la Harpe)**.** Let G be a connected adjoint semisimple real Lie group without compact factors, and let Γ be a Zariski-dense subgroup of G. Let S be a finite subset of Γ. Does there exist an element $\gamma \in \Gamma$ such that for every $h \in S$, the subgroup $\langle h, \gamma\rangle$ is canonically isomorphic to the free product $\langle h\rangle * \langle\gamma\rangle$?

It is well known (see [7, Lemmas 2.1 & 2.2]) that property ($\mathrm{P}_{\mathrm{naï}}$) for Γ implies that the reduced C*-algebra $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ of Γ is simple and has a unique tracial state; this is, in fact, one of the historical reasons for the interest in property ($\mathrm{P}_{\mathrm{naï}}$). Over the years, the simplicity and unique trace property of $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ was established for large classes of groups (see namely [66, 23, 67, 18]). The stronger property ($\mathrm{P}_{\mathrm{naï}}$) however remains poorly understood.

Besides [7, Theorem 3] just mentioned, we are aware of the work of Soifer and Vishkautsan [78, Theorem 1.3], which gives a positive answer for $\Gamma = \mathrm{PSL}_n(\mathbb{Z})$ and S only containing elements whose semisimple part is either biproximal[1] or torsion. Contemporary to this work, a positive answer was claimed by Poznansky in his thesis [67, Theorem 6.5], for arbitrary finite subsets S of a semisimple algebraic group $\mathbf{G}$ containing no factor of type $\mathrm{A}_n, \mathrm{D}_{2n+1}$ or E_6. Unfortunately the proof of this theorem contains an error, as it relies on [67, Proposition 2.11] whose statement is not true. Nonetheless, if one assumes that S consists of elements satisfying the conclusion of [67, Proposition 2.11] (that is, of elements whose conjugacy class intersect the big Bruhat cell, see Remark 3.15 for more details) and the elements have an almost-embedding (see later for definition), then the proof of [67, Theorem 6.5] given by Poznansky goes through to the best of our knowledge, and was an instructive source for our work.

1.2. **Outline.** This article consists of essentially two parts.

In the first part, we consider the variant of Bekka, Cowling and de la Harpe's question in which S is actually a finite set of *finite subgroups* in a reductive algebraic group $\mathbf{G}$. In this setting, the statement of Theorem 3.2 gives two conditions jointly implying the

[1] [78] uses for *biproximal* the term 'hyperbolic', whereas [67] uses 'very proximal'; see Definition 3.7 for the terminology used here.

existence of simultaneous ping-pong partners for S inside a fixed Zariski-dense subgroup Γ of $\mathbf{G}$. When these conditions are satisfied, we show that the set of simultaneous ping-pong partners for S is both Zariski- and profinitely-dense in Γ.

Subsequently, in Theorem 3.23, we establish these conditions for $\mathbf{G}$ a product of inner $\mathbb{R}$-forms of GL_n or SL_n. As a corollary, we answer in the affirmative (a slightly stronger version of) Bekka, Cowling and de la Harpe's question for subsets S consisting of torsion elements.

In fact one needs to be careful, because unexpectedly Question 1.1 is ill-posed when $\mathbf{G}$ is semisimple, but not simple. Indeed the elements in the set S must all (almost) embeds in a simple quotient of $\mathbf{G}$ (see Proposition 2.7). Theorem 3.23 shows that this is in fact sufficient.

A particular instance of this setting is when $\mathbf{G}$ is the unit group of a finite-dimensional semisimple algebra A over a number field. In that instance, we show in Theorem 4.1 that ping-pong partners exists for a finite subgroup H if and only if H has an almost-embedding in a "good" factor of A, that is, in a factor that is neither a field nor a totally definite quaternion algebra. When such almost-embedding exists, Section 4 aims at providing a constructive method to construct a simultaneous partner. We introduce there the convenient formalism of first-order deformations of a subgroup (see Definition 4.6), which will be used to deform finite subgroups H_1 and H_2 of the unit group $\mathcal{U}(A)$ of A to put them in ping-pong position (Theorem 4.12). In passing, we record that such deformations are in fact obtained by specific conjugation inside A (Theorem 4.9), a consequence of Hochschild's cohomology theory.

In the second part of the paper, we will delve into the case of the unit group $\mathcal{U}(FG)$ of the group algebra of a finite group G. For these groups, interesting interplay with the representation theory of G arises. As an illustration, it follows from Theorem 5.1 (see Corollary 5.4) that if a subgroup H of G embeds in a simple factor of FG, then there is also an almost embedding in a "good" simple factor.

The goal of the second part is to obtain answers to old open problems in the field of group rings. There has been for long an active interest in this field for a particular kind of unipotent elements, the so-called *bicyclic units*, which arise naturally in the study of the group ring RG where R is the rings of integers of F. For several decades it has been a long standing belief that two bicyclic units should generically generate a free group, a claim that we substantiate in Theorem 5.27.

This is done by first proving that the group of bicyclic units is always Zariski-dense in the group of unimodular elements of an "appropriate part" of the group ring. At that stage, using the main results of the first part of the paper, the aforementioned belief is reduced to show that any cyclic subgroup $\langle h \rangle$ of G has an almost-embedding into that "appropriate part". Concretely, our next main result is Theorem 5.1 where we obtain that any finite subgroup H having a faithful irreducible F-representation (e.g. nilpotent with cyclic center) has an almost embedding in a simple component of FG where the projection $p(G)$ of G is not fixed point free, a result of independent interest. That $p(G)$ is not fixed point free for instance imply that $\mathrm{span}_F\{p(G)\}$ is not a division algebra.

As a final application of the methods, we study the virtual structure problem, which asks to classify all finite groups G for which the unit group of every order in FG satisfies some prescribed structural property (see below for a precise question). In Theorem 6.2, we establish an explicit description of these groups, for the property of admitting an amalgamated or HNN splitting over a finite group.

We now give a more detailed account of the main results.

1.3. **Simultaneous ping-pong with finite subgroups.** After a short recollection on the structure of free amalgamated products in Section 2, we will consider in Section 3 a slightly more general version of Question 1.1, as we allow the finite set S to consist of subgroups (not just elements). To this end, we study the dynamics of linear transformations on projective spaces over division algebra; the details are contained in Section 3.2, and are mostly a rework of classical results of Tits, themselves already revisited by several authors. The main result of that section is Proposition 3.11, stating the abundance of simultaneously biproximal elements (when they exist).

These developments are necessary to prove the main result of Section 3:

Theorem 3.2. *Let $\mathbf{G}$ be a connected algebraic F-group with center $\mathbf{Z}$. Let Γ be a Zariski-connected subgroup of $\mathbf{G}(F)$. Let $(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(F)$, and set $C_i = H_i \cap \mathbf{Z}(F)$. Assume that for each $i \in I$ there exists a local field K_i containing F and a projective K_i-representation $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$, where V_i is a finite-dimensional module over a finite division K_i-algebra D_i, with the following properties:*

(Proximality) $\rho_i(\Gamma)$ *contains a proximal element;*

(Transversality) *For every $h \in H_i \backslash C_i$ and every $p \in \mathbb{P}(V_i)$, the span of the set $\{\rho_i(xhx^{-1})p \mid x \in \Gamma\}$ is the whole of $\mathbb{P}(V_i)$.*

Let S be the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order, such that for all $i \in I$, the canonical map

$$(\langle\gamma\rangle \times C_i) *_{C_i} H_i \to \langle\gamma, H_i\rangle \leq \mathbf{G}(F)$$

is an isomorphism. Then S is dense in Γ for the join of the profinite topology and the Zariski topology.

The transversality condition gets its name for its role in Lemma 3.14. As hinted in Lemma 3.14 and Remark 3.4, transversality is a kind of "higher irreducibility condition".

Next, in Section 3.4 we verify the proximality and transversality conditions for finite subgroups in products of inner forms of SL_n and GL_n. As a consequence of Theorem 3.2, this proves the abundance of simultaneous ping-pong partners in Zariski-dense subgroups in this setting:

Theorem 3.23. *Let $\mathbf{G}$ be a reductive $\mathbb{R}$-group, and let Γ be a subgroup of $\mathbf{G}(\mathbb{R})$ whose image in $\mathrm{Ad}\,\mathbf{G}$ is Zariski-dense. Let $(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(\mathbb{R})$.*

Suppose that for each $i \in I$, the subgroup H_i almost embeds in a simple quotient $\mathbf{Q}_i$ of $\mathbf{G}$ isogenous to $\mathrm{PGL}_{D_i^{n_i}}$, for D_i some finite dimensional division $\mathbb{R}$-algebra and $n_i > 1$. Then the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order such that for all $i \in I$, the canonical map

$$(\langle\gamma\rangle \times C_i) *_{C_i} H_i \to \langle\gamma, H_i\rangle \leq \mathbf{G}(\mathbb{R})$$

is an isomorphism, is dense in Γ for the join of the profinite topology and the Zariski topology.

Given a reductive F-group $\mathbf{G}$ with center $\mathbf{Z}$ and a subgroup $H \leq \mathbf{G}(F)$, we say that H *almost embeds in a (simple) quotient* $\mathbf{Q}$ *of* $\mathbf{G}$ if there exists a (simple) quotient $\mathbf{Q}$ of $\mathbf{G}$ for which the kernel of the restriction $H \to \mathbf{Q}(F)$ is contained in $\mathbf{Z}(F)$. As mentioned in Remark 3.24, this is a necessary condition for the subgroup H to admit a ping-pong partner in $\mathbf{G}(F)$.

Despite the abundance of simultaneous ping-pong partners, the above theorem does not immediately give an explicit construction of such partners. In Section 4, we will provide an explicit construction for certain Zariski-dense subgroups of algebraic groups $\mathbf{G}$ which are a direct product of inner forms of SL_n for $n \geq 2$. We also investigate whether two given finite subgroups H and K of $\mathbf{G}(F)$ can appear as the factors of a free product $H * K$ inside $\mathbf{G}(F)$. To this end, we introduce *first-order deformations* of H in Section 4.2, which are linear deformations of H suitable to obtain appropriate dynamics. The main result of that section is Theorem 4.12.

1.4. **The case of semisimple algebras and the unit group of a group ring.** Let M be a finite dimensional semisimple algebra over F. By the Artin–Wedderburn theorem,

$$M \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_m}(D_m).$$

as F-algebra, for D_i some finite-dimensional division algebras over F. In particular, the F-group $\mathcal{U}(A)$ of units of A is the reductive group

$$\mathrm{GL}_{D_1^{n_1}} \times \cdots \times \mathrm{GL}_{D_m^{n_m}} .$$

Furthermore, if $\mathcal{O}$ is an order in A, then by a classical result of Borel and Harish-Chandra $\Gamma = \mathcal{U}(\mathcal{O})$ is an arithmetic subgroup of $\mathcal{U}(A)$, placing us in the setting of Theorem 3.23.

Existence of almost-embeddings and ping-pong partners. We deduce from the aforementioned the following criterion for the existence of simultaneous ping-pong partners for a finite subgroup of $\mathcal{U}(A)$.

Theorem 4.1. *Let F be a number field, A be a finite semisimple F-algebra, and $\mathcal{O}$ be an order in A. Let Γ be a Zariski-dense subgroup of $\mathcal{U}(\mathcal{O})$. Let H be a finite subgroup of $\mathcal{U}(A)$, and C be its intersection with the center of A.*

There exists $\gamma \in \Gamma$ of infinite order with the property that the canonical map

$$(\langle \gamma \rangle \times C) *_C H \to \langle \gamma, H \rangle$$

is an isomorphism, if and only if H almost embeds in a simple factor of A which is neither a field nor a totally definite quaternion algebra. Moreover, in the affirmative, the set of such elements γ is dense in the join of the Zariski and the profinite topology.

In the specific case where $A = FG$ is a group algebra, the simple components $\mathrm{M}_{n_i}(D_i)$ of A are not arbitrary. Indeed, each $\mathrm{M}_{n_i}(D_i)$ is precisely the F-span of the projection of G in the ith factor of this decomposition of FG. Thanks to this the simple factors are related to each other via the representation theory of one common finite group G. For instance, as shown in Corollary 5.7, the condition in Theorem 4.1 to have an almost embedding in an appropriate simple factor simplifies to having an almost embedding in any type of simple factor. This is a consequence of our main almost-embedding theorem:

Theorem 5.1. *Let F be a field of characteristic 0, G a finite group and $H \leq G$ such that it admit a faithful irreducible F-representation $\psi \in \mathrm{Irr}(H)$. Then there exists an irreducible F-representation $\rho \in \mathrm{Irr}(G)$ such that*

(1) $H \cap \ker(\rho) \subseteq \mathcal{Z}(G)$,

(2) $\rho(G)$ is not *a Frobenius complement.*

if and only if G is not a Dedekind group. In particular, $\rho(FG)$ is not a division algebra.

The work above also allows us to give for most subgroups H necessary and sufficient conditions on when an amalgamated product of the form $H *_{H \cap \mathcal{Z}(G)} H$ exists in $\mathcal{U}(RG)$.

Corollary 5.4. *Let F be a number field and R its ring of integers. Further let G be a finite group and $H \leq G$ such that H embeds in a simple factor of FG. Then, there exists some $t \in \mathcal{U}(RG)$ such that*

$$\langle H, t \rangle \cong H *_C \langle t, C \rangle \cong H *_C (\mathbb{Z} \times C).$$

where $C = \langle h \rangle \cap \mathcal{Z}(G)$ if and only if $G \not\cong Q_8 \times C_2^n$ for some n. In particular, $H *_C H$ exists in $\mathcal{U}(RG)$ in that case.

When $G \cong Q_8 \times C_2^n$, then $\mathcal{U}(RG)$ consists only of trivial units and hence contain no non-trivial amalgamated products.

Bicyclic units yield generically free products. Next, we focus on the construction of free products with certain specific units in $\mathcal{U}(RG)$. More precisely, consider in RG the unipotent elements of the form

$$b_{\tilde{h},x} = 1 + (1-h)x\widetilde{h} \quad \text{and} \quad b_{x,\tilde{h}} = 1 + \widetilde{h}x(1-h), \quad \text{for } x \in RG, h \in G, \text{ and } \widetilde{h} := \textstyle\sum_{i=1}^{o(h)} h^i,$$

called *bicyclic units.* The group they generate is denoted

$$\mathrm{Bic}_R(G) = \langle b_{\tilde{h},x}, b_{x,\tilde{h}} \mid h \in G, x \in RG \rangle.$$

These bicyclic elements constitute one of the few known generic constructions of units in RG. We record in passing the new construction given in [41].

For almost 30 years it has been conjectured in the field of group rings that two *generically chosen* bicyclic units generate a free group. That being said, the meaning to give to 'generic' has, to our knowledge, never been made precise. See [37] for a quit complete survey until 2013 and also see [34, 36, 35, 38, 48, 70] (and the references therein). This conjecture, for which we obtain an affirmative answer, was the original motivation for this work.

To start we propose a new point of view on bicyclic units, as being first-order deformations of torsion units. These deformations are maps of the form

$$\Delta : H \to \mathcal{U}(FG) : h \mapsto \Delta(h) = h + \delta_h$$

which furthermore are group morphisms. The latter is encoded in two properties on $\delta = \Delta - 1$, see Definition 4.6. In the case of bicyclic units $\delta_h = (1-h)x\widetilde{h}$ (or $\widetilde{h}x(1-h)$). In particular, the image of Δ lies inside $\mathcal{U}(RG)$, despite Theorem 4.9 saying that first order deformations are given by conjugation in FG.

In the literature it has often been an obstacle that $\mathrm{Bic}_R(G)$ might be of infinite index in $\mathrm{SL}_1(RG)$ if FG has certain low rank simple factors. Lemma 5.29 forms a next important observation: $\mathrm{Bic}_R(G)$ is nevertheless Zariski dense in $SL_1(FGf)$ with f the sum of all primitive central idempotents e such that Ge is not fixed point free. The subalgebra FGf is the 'appropriate part" referred to earlier. This restriction arises due to the fact that Ge is fixed point free if and only if $\widetilde{h}e$ is central for all $1 \neq h \in G$. Hence bicyclic units project to the identity in such simple components. Due to this, Proposition 2.7 tells that a necessary condition for $\langle \alpha h \rangle$ to have a bicyclic ping-pong partner is that $\langle h \rangle$ embeds in a component where Ge is not fixed point free.

The main result states that if one replaces the given bicyclic unit $b_{\tilde{h},x}$ by its closely related variant $b_{\tilde{h},x}h = h + (1-h)x\widetilde{h}$ (called *shifted bicyclic units* in the literature), this long-standing conjecture holds true for a profinitely-dense and Zariski-dense subset of units.

Theorem 5.27. *Let F be a number field and R be its rings of integers. Further let G be finite group and $\alpha = 1 + (1-h)x\widetilde{h}$ be a non-trivial bicyclic unit for some $h \in G$ and $x \in RG$. Denote $C = \langle h \rangle \cap \mathcal{Z}(G)$. Then*

$$S_\alpha = \{\beta \in \mathrm{Bic}_R(G) \mid \langle \alpha h, \beta \rangle \cong \langle \alpha h \rangle *_C (\langle \beta \rangle \times C)\}$$

is dense in $\mathrm{Bic}_R(G)$ *for the join of the profinite and Zariski topologies.*

Theorem 5.27 follows from a combination of two interesting results on its own. The first is a general statement, Theorem 5.25, about the existence of bicyclic ping-pong partners for arbitrary first-order deformations of a finite group H in RG. Namely, there exists one (and subsequently densely many) if and only if H has an almost-embedding in a simple component where the projection of G is not fixed point free. This theorem is an application of the Zariski density result and of Theorem 3.23.

The second ingredient for Theorem 5.27 is the almost-embedding result, i.e. Theorem 5.1 mentioned earlier. It namely shows the existence of such nice almost-embedding for every subgroup H of G that embeds in a simple factor of FG. This situation is shown to occur when H has a faithful irreducible F-representation (a condition well understood in the literature).

Finally, note that such embedding result does not hold for any (cyclic) subgroup of FG and hence Theorem 5.1 is truly a representation theoretical result. As such Theorem 5.27 turned out to be a combination of a result in algebraic groups and one in representation theory.

1.5. **The virtual structure problem for amalgams or HNN extensions over finite groups.** Lastly, we consider the the Virtual Structure Problem, which loosely asks for a structure theorem for unit groups of orders. The latter is also called a 'unit theorem' and a concrete meaning was formulated in Kleinert's 1994 survey [54]:

> *"A unit theorem for a finite-dimensional semisimple rational algebra A consists of the definition, in purely group-theoretical terms, of a class of groups $C(A)$ such that almost all generic unit groups of A are members of $C(A)$."*

Recall that a generic unit group of A is a subgroup of finite index in the group of elements of reduced norm 1 of an order[2] in A.

One instance of a unit theorem in the sense of Kleinert is when the class $C(A)$ consists of all groups satisfying some prescribed group theoretical property $\mathcal{P}$. For instance when A is chosen to be a group algebra FG, then such interpretation reformulates to:

Classify all finite groups G such that (almost) all generic unit groups of FG satisfy property $\mathcal{P}$.

Up to our knowledge, the only properties $\mathcal{P}$ for which such a unit theorem is known are the following:

- $\mathcal{P} = \{$ finite groups$\}$, [45, Corollary 5.5.8]
- $\mathcal{P} = \{$ abelian groups$\}$, [52]
- $\mathcal{P} = \{$ solvable groups$\}$, [52, Theorem 2]
- $\mathcal{P} = \{$ direct product of free-by-free groups $\}$ ([47, 44, 63, 49]),
- $\mathcal{P}$ are groups satisfying some fixed point property such as property (T) or HFA ([5, 4]).

Remarkably, in all these case the property can also be described in terms of the rational group algebra. For example in case that $\mathcal{P}_{free} := \{$ direct product of free-by-free groups $\}$, one has that all generic unit groups of $\mathbb{Q}G$ have $\mathcal{P}_{free}$ if and only if every simple factor of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or $\mathrm{M}_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$.

In this article, we address the problem above for the class of groups

$$\mathcal{P}_\infty := \{\prod_i \Gamma_i \mid \Gamma_i \text{ has infinitely many ends}\}.$$

By Stallings' theorem [80, 79], a group has infinitely many ends if and only if it can be decomposed as a non-trivial amalgamated product or HNN extension over a finite group. (In fact, we will mostly work with this characterization.)

Theorem 6.2. *Let G be a finite group and F a number field. The following are equivalent:*

(i) *$H \leq \mathcal{U}(\mathcal{O})$ has $\mathcal{P}_\infty$ for all orders $\mathcal{O}$ in FG and finite index subgroup $H \leq \mathcal{U}(\mathcal{O})$;*

(ii) *$F = \mathbb{Q}$ and $\mathcal{U}(\mathbb{Z}G)$ has $\mathcal{P}_\infty$;*

(iii) *$F = \mathbb{Q}$ and $\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of non-abelian free groups;*

[2] Here, and in the remaining of this article, an order refers to a $\mathbb{Z}$-order, but the question also makes sense for more general orders.

(iv) *all the simple components of FG are of the form $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{N}$, $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with non-zero $a, b \in \mathbb{N}$ or $\mathrm{M}_2(\mathbb{Q})$, and the latter needs to occur.*

Moreover, only the parameters $(-1,-1)$ and $(-1,-3)$ can occur for $(-a,-b)$. Also, $\mathcal{U}(\mathbb{Z}G)$ itself has infinitely many ends if and only if it is virtually free, if and only if G is isomorphic to D_6, D_8, Dic_3, or $C_4 \rtimes C_4$.

The finite groups satisfying assertion (iii) in Theorem 6.2 have been classified in [44], so the theorem does indeed answer the Virtual Structure problem for the class $\mathcal{P}_\infty$.

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2. Amalgams in almost-direct products

In this section, we recall the classical ping-pong lemma for amalgamated products. Thereafter we exhibit a necessary condition for a subgroup of an almost-direct product to be an amalgamated product.

Given a subgroup C of a group G, we will denote by T_C^G a set of representatives of the left cosets of C in G, containing the identity element.

The ping-pong lemma for amalgams and its variant for HNN extensions can be found in [56, Propositions 12.4 & 12.5]. For the convenience of the reader, we provide a proof as it will be instrumental in the rest of this paper.

Lemma 2.1 (Ping-pong for amalgams). *Let A, B be subgroups of a group G and suppose $C = A \cap B$ satisfies $|A : C| > 2$. Let G act on a set X. If $P, Q \subset X$ are two subsets with $P \not\subset Q$, such that for all elements $a \in T_C^A \setminus \{e\}$, $b \in T_C^B \setminus \{e\}$ and $c \in C$, we have*

$$aP \subset Q, \quad bQ \subset P, \quad cP \subset P, \quad \text{and} \quad cQ \subset Q,$$

then the canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism.

As in the case of free products, the proof of Lemma 2.1 is straightforward once one knows the normal form for elements in an amalgamated product. The normal form also allows us to unambiguously speak of *words starting with A* and *words starting with B*. In the next lemma, these are the elements for which $\dot{a}_1 \notin C$, resp. for which $\dot{a}_1 \in C$.

Lemma 2.2 (Normal form in amalgams). *Let $A, B \leq G$ be groups and $C \leq A \cap B$. The following are equivalent.*

(i) *The canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism.*

(ii) *Every element in $\langle A, B\rangle$ has a unique decomposition of the form $\dot{a}_1 b_1 \cdots a_n \dot{b}_n c$, where $a_i \in T_C^A \setminus \{e\}$, $b_i \in T_C^B \setminus \{e\}$, $\dot{a}_1 \in T_C^A$, $\dot{b}_n \in T_C^B$, and $c \in C$.*

(iii) *Given $a_i \in A \setminus C$, $b_i \in B \setminus C$, $\dot{a}_1 \in A$, and $\dot{b}_n \in B$, the product $\dot{a}_1 b_1 \cdots a_n \dot{b}_n$ belongs to C only if $n = 1$ and $\dot{a}_1, \dot{b}_n \in C$.*

In consequence of the affirmative, $C = A \cap B$.

Sketch of proof. The implication (i) $\Longrightarrow$ (ii) is the existence and uniqueness of a normal form (see for instance [75, Theorem 1]), and its converse amounts to checking the injectivity of the canonical map, which follows from the uniqueness of the decomposition in $\langle A, B\rangle$.

After replacing $\dot{b}_n, a_n, \dots, b_1, \dot{a}_1$ by the appropriate coset representatives, (ii) $\Longrightarrow$ (iii) becomes obvious. For the contrapositive of its converse, note that two different decompositions of an element in $\langle A, B\rangle$ result in a non-trivial expression of the form $\dot{a}_1 b_1 \cdots a_n \dot{b}_n$ in C. □

Proof of Lemma 2.1. Note that the assumptions imply that $aP_1 \subset P_2$ for all $a \in A \setminus C$, $bP_2 \subset P_1$ for all $b \in B \setminus C$, and $cP_1 = P_1$, $cP_2 = P_2$ for every $c \in C$.

Suppose that given $a_i \in A \setminus C$, $b_i \in B \setminus C$, $\dot{a}_1 \in A$ and $\dot{b}_n \in B$, the non-empty word $c = \dot{a}_1 b_1 \cdots a_n \dot{b}_n$ lies in C. The possible cases for $\dot{a}_1$ and $\dot{b}_n$ to belong to C are:

- $\dot{a}_1 \notin C$, $\dot{b}_n \in C$. We have $\dot{b}_n P_1 = P_1$, $a_n \dot{b}_n P_1 \subset P_2$, $b_{n-1} a_n \dot{b}_n P_1 \subset P_1$, etc., so that eventually, $cP_1 = \dot{a}_1 b_1 \cdots a_n P_1 \subset P_2$. Since $cP_1 = P_1$ and $P_1 \not\subset P_2$, this case cannot occur.
- $\dot{a}_1 \in C$, $\dot{b}_n \notin C$. Pick $a \in A \setminus C$, and let $a' \in A$ and $c' \in C$ be such that $a^{-1}ca = a'c'$. We have $aa' \notin C$, hence the word $c' = (aa')^{-1} b_1 \cdots a_n \dot{b}_n a$ starts and ends with an element of $A \setminus C$. This case thus reduces to the first one.
- $\dot{a}_1 \notin C$, $\dot{b}_n \notin C$. As $|A : C| > 2$, we may pick $a \in A \setminus (C \cup \dot{a}_1 C)$, so that $a^{-1}\dot{a}_1 P_1 \subset P_2$ hence $\dot{a}_1 P_1 \subset aP_2$. As in the first case, we have $cP_2 \subset \dot{a}_1 P_1$. Since $cP_2 = P_2$, this would imply $aP_1 \subset P_2 \subset aP_2$, hence this case does not occur either.
- $\dot{a}_1 \in C$, $\dot{b}_n \in C$. If $n > 1$, replacing c by c^{-1} reduces to the third case. The only remaining possibility is thus $n = 1$ and $\dot{a}_1, \dot{b}_n \in C$, as expected.

We conclude from Lemma 2.2 that the canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism. □

Lemma 2.3. *Let $A *_C B$ be a free amalgamated product. If f is a surjective morphism from a group Γ to $A *_C B$, then Γ is the free product with amalgamation $f^{-1}(A) *_{f^{-1}(C)} f^{-1}(B)$.*

If moreover Γ is generated by two subgroups Γ_1, Γ_2 with the properties $f(\Gamma_1) \subseteq A$, $f(\Gamma_2) \subseteq B$, the induced map $\Gamma_1 \to A/C$ is injective, and $\Gamma_1(\Gamma_2 \cap f^{-1}(C))$ is a subgroup, then $f^{-1}(B) = \Gamma_2$ and $\Gamma \cong (\Gamma_1 f^{-1}(C)) *_{f^{-1}(C)} \Gamma_2$.

Proof. The first part of the lemma is standard (see for instance [88, Lemma 3.2]). For the second part, let $g = b_0 a_1 b_1 \cdots a_n b_n$ with $a_i \in \Gamma_1 \setminus \{e\}$ and $b_i \in \Gamma_2$, be an element of $f^{-1}(B)$. Since $(\Gamma_2 \cap f^{-1}(C))\Gamma_1 = \Gamma_1(\Gamma_2 \cap f^{-1}(C))$, after perhaps reducing the expression for g, we may assume that $b_i \notin f^{-1}(C)$ for $0 < i < n$. Because $f(\Gamma) = A *_C B$ and $f(b_i) \in B \setminus C$, Lemma 2.2 implies that $n = 0$, hence $g = b_0 \in \Gamma_2$. Thus $f^{-1}(B) = \Gamma_2$, and in consequence $f^{-1}(C) \leq \Gamma_2$. On the other hand, if $g = b_0 a_1 b_1 \cdots a_n b_n \in f^{-1}(A)$, we may assume as before that $a_i \neq e$ for $1 \leq i \leq n$ and $b_i \notin f^{-1}(C)$ for $0 < i < n$. Applying f again then shows that $n \leq 1$ and $b_i \in f^{-1}(C)$ for $i \leq n$, so that $g \in \Gamma_1 f^{-1}(C)$. □

The following folkloric terminology is inspired by Lemma 2.1.

Definition 2.4. Let A and B be subgroups of a group G. We say that A is *a ping-pong partner for B*, or that *A and B play ping-pong*, if the subgroup $\langle A, B\rangle$ is freely generated by A and B, or in other words if the canonical map $A * B \to \langle A, B\rangle$ is an isomorphism. Similarly, we say that $a \in A$ is *a ping-pong partner for B in A*, or that *a and B play ping-pong*, if the subgroup $\langle a, B\rangle$ is freely generated by $\langle a\rangle$ and B; same for $a \in A$ and $b \in B$ generating $\langle a, b\rangle$ freely. We will also use this terminology for free amalgamated products when the subgroup being amalgamated is unambiguous. Sets P_1 and P_2 to which one can apply Lemma 2.1 are sometimes called a *ping-pong table for A and B*.

In the subsequent sections, we will look to play ping-pong inside a group $G = \prod_{i=1}^n G_i$ which decomposes into a direct product of subgroups G_i. Using some simple facts about free (amalgamated) products, the next proposition will show that this requires an embedding of the ping-pong partners in one of the factors G_i.

Given subgroups $H_1, \ldots, H_n$ of a group G, let $[H_1, \ldots, H_n] = [H_1, [H_2, \ldots, H_n]]$ denote the *left-iterated (or right-normed) commutator subgroup* of the H_i.

Lemma 2.5. *Let N, N_1, ..., N_n be normal subgroups of $A *_C B$, where $|A : C| > 2$.*

(i) *Either $N \subset C$, or N contains a non-abelian free group.*

(ii) *If $[N_1, N_2] \subset C$, then either $N_1 \subset C$ or $N_2 \subset C$.*

In consequence, if $[N_1, \ldots, N_n]$ admits no non-abelian free subgroups, there exists $i \in \{1, \ldots, n\}$ for which $N_i \subset C$.

Proof. First, suppose that N is a normal subgroup of $A *_C B$ not contained in C. Pick $x \in N \setminus C$; by Lemma 2.2, we may assume after conjugation that x either belongs to $B \setminus C$, belongs to $A \setminus C$, or is cyclically reduced starting with $a_1 \in A \setminus C$.

- If $x \in B \setminus C$, pick $a, a' \in A \setminus C$ such that $a \notin a'C$. Using Lemma 2.2, one readily checks that the cyclically reduced words $w = [a, x]$ and $w' = [a', x]$ generate a free group, as every non-empty word in w and w' remains a non-empty word alternating in elements of $A \setminus C$ and $B \setminus C$. (Only simplifications of the form $[a, x][a', x]^{-1} = ax(a^{-1}a')x^{-1}a'^{-1}$ occur, and the condition on a and a' ensures no further cancellations arise.)
- If $x \in A \setminus C$, pick $b \in B \setminus C$ and $a, a' \in A \setminus C$ such that $a \notin a'C$, and consider $w = [x, bab^{-1}]$ and $w' = [x, ba'b^{-1}]$ instead.
- In the last case, write $x = a_1 b_1 \cdots, a_n b_n$ with $n \geq 1$ and $a_i \in A \setminus C$, $b_i \in B \setminus C$. Pick $b \in B \setminus C$ and $a \in A \setminus C$ such that $a \notin a_1 C$. Then the words $w = x$ and $w' = aba^{-1}xab^{-1}a^{-1}$ generate a free group.

This proves part (i).

Second, suppose that there exist elements $x \in N_1 \setminus C$ and $x' \in N_2 \setminus C$. By Lemma 2.2, we may assume after conjugation that x, x' either belong to $A \setminus C$, belongs to $B \setminus C$, or is cyclically reduced starting with A. We exhibit in each case a commutator in $[N_1, N_2] \setminus C$.

- If $x = a_1$ and $x' = b'_1$, then $[x, x'] \notin C$.
- If x is cyclically reduced starting with a_1 and $x' = a'_1$, then $[x, bx'b^{-1}] \notin C$ for any $b \in B \setminus (C \cup b_n^{-1}C)$.
- If x is cyclically reduced starting with a_1 and $x' = b'_1$, then $[a^{-1}xa, x'] \notin C$ for any $a \in A \setminus (C \cup a_1 C)$.
- If $x = a_1$ and $x' = a'_1$, then $[x, bx'b^{-1}] \notin C$ for any $b \in B \setminus C$.
- If $x = b_1$ and $x' = b'_1$, then $[axa^{-1}, x'] \notin C$ for any $a \in A \setminus C$.
- If x, x' are both cyclically reduced starting with a_1, and ending with $b'_{n'}$ respectively, then $[a^{-1}xa, b^{-1}x'b] \notin C$ for any $a \in A \setminus (C \cup a_1 C)$ and $b \in B \setminus (C \cup b'^{-1}_{n'} C)$.

This proves part (ii).

Lastly, if $[N_1, \ldots, N_n]$ admits no non-abelian free subgroups, we deduce from part (i) that $[N_1, \ldots, N_n] \subset C$. Part (ii) then implies that either $N_1 \subset C$, or $[N_2, \ldots, N_n] \subset C$, and recursively, that eventually $N_i \subset C$ for some $i \in \{1, \ldots, n\}$. □

Definition 2.6. Let $\mathcal{S}$ be a class of groups closed under subquotients and extensions. For the purposes of the following proposition, we will say that G is an *$\mathcal{S}$-almost direct product of* $G_1, \ldots, G_n$ if G has a normal subgroup $K \in \mathcal{S}$ such that G/K is the direct product $G_1 \times \cdots \times G_n$.

Equivalently, if there are normal subgroups $M_1, \dots, M_n$ of G such that $\bigcap_{i=1}^n M_i \in \mathcal{S}$ and $M_i(M_{i+1} \cap \dots \cap M_n) = G$ for $i = 1, \dots, n-1$, then G is the $\mathcal{S}$-almost direct product of $G/M_1, \dots, G/M_n$. Indeed, the second condition ensures that the canonical map $G/\bigcap_{i=1}^n M_i \to G/M_1 \times \dots \times G/M_n$ is surjective; conversely, writing M_i for the kernel of $G \to G_i$, it is obvious that $K = \bigcap_{i=1}^n M_i$ and $M_j(\bigcap_{i \neq j} M_i) = G$.

Almost direct products with respect to the class containing only the trivial group are just direct products. In the literature, almost direct products appear most often for $\mathcal{S}$ the class of finite groups. Here are a few straightforward observations:

(i) If $\mathcal{S} \subset \mathcal{S}'$, then any $\mathcal{S}$-almost direct product is an $\mathcal{S}'$-almost direct product (of the same groups).

(ii) Any group in $\mathcal{S}$ is an empty $\mathcal{S}$-almost direct product; so of course the notion is meaningful only for groups outside of $\mathcal{S}$.

(iii) An $\mathcal{S}$-almost direct product of groups $G_1, \dots, G_n$ themselves $\mathcal{S}$-almost direct products of respectively $H_{i1}, \dots, H_{in_j}$ $(i = 1, \dots, n)$, is an $\mathcal{S}$-almost direct product of the H_{ij}, $i = 1, \dots, n$, $j = 1, \dots, n_j$.

(iv) Any quotient or extension of an $\mathcal{S}$-almost direct product by a group in $\mathcal{S}$ is again an $\mathcal{S}$-almost direct product.

Sometimes, almost direct products are defined by the following variant: G is the quotient of a direct product $G_1 \times \dots \times G_n$ by a normal subgroup $H \in \mathcal{S}$. An almost direct product in this second sense is also an $\mathcal{S}$-almost direct product in the sense of Definition 2.6. Indeed, if $G = (G_1 \times \dots \times G_n)/H$, denoting π_i the projection onto G_i and $K = \pi_1(H) \times \dots \times \pi_n(H)$, we see that $G/(K/H) \cong (G_1 \times \dots \times G_n)/K = G_1/\pi_1(H) \times \dots \times G_n/\pi_n(H)$. The converse however does not always hold, as the images of the factors G_i in $(G_1 \times \dots \times G_n)/H$ are commuting normal subgroups, and this may not happen in G even if G/K is a direct product.

Proposition 2.7 (Free subgroups in almost direct products)**.** *Let $\mathcal{S}$ be the class of groups not containing a non-abelian free group. Let G be the $\mathcal{S}$-almost direct product of groups $G_1, \dots, G_m$, and suppose that $G_{n+1}, \dots, G_m$ belong to $\mathcal{S}$. If A and B are subgroups of G whose intersection C satisfies $|A : C| > 2$, and are such that the canonical map $A *_C B \to \langle A, B \rangle$ is an isomorphism, then there exists $i \in \{1, \dots, n\}$ for which the kernel of the projection $\langle A, B \rangle \to G_i$ is contained in C.*

Proof. Since $G_{n+1}, \dots, G_m$ belong to $\mathcal{S}$, it is clear that G is also the $\mathcal{S}$-almost direct product of $G_1, \dots, G_n$. Let π_i denote the projection $G \to G_i$ and set $M_i = \ker \pi_i$. Identify $\langle A, B \rangle$ with $A *_C B$ and set $N_i = M_i \cap (A *_C B)$.

By assumption, $\bigcap_{i=1}^n M_i$ does not contain a non-abelian free group. The same then holds for $[N_1, \dots, N_n] \subset [M_1, \dots, M_n] \subset \bigcap_{i=1}^n M_i$, and Lemma 2.5 implies the existence of an index $i \in \{1, \dots, n\}$ for which $N_i \subset C$. □

There are versions of Lemma 2.5 and Proposition 2.7 for HNN extensions. We leave their statement and proof to the reader.

3. Simultaneous ping-pong partners for finite subgroups of reductive groups

Let F be a field. Let $\mathbf{G}$ be a reductive[3] algebraic F-group, Γ a Zariski-dense subgroup of $\mathbf{G}(F)$, and H a finite subgroup of $\mathbf{G}(F)$. This section is concerned with finding elements γ of Γ which are ping-pong partners for H.

[3]In this paper, all reductive (in particular, all semisimple) algebraic groups are connected by definition. This convention sometimes differs in the literature. We also call *simple* a non-commutative algebraic group whose proper normal subgroups are finite (sometimes called 'almost simple' in the literature).

3.1. **Existence of simultaneous ping-pong partners in linear groups.** The construction and study of free products in linear groups is a classical topic, going back way beyond Tits' celebrated work [83] establishing existence of free subgroups in linear groups which are not virtually solvable. Given a subset F of a linear group G, the existence of *simultaneous* ping-pong partners for elements of F (that is, elements which are ping-pong partners for every $h \in F$) has also been studied, see namely the works of Poznansky [67, Theorem 6.5] and Soifer & Vishkautsan [78, Theorem 1.3]. We also recall from the introduction the open question going back to Bekka, Cowling and de la Harpe, cases of which are answered in the two works just cited.

Question 3.1 (see [7, Remark 3] and [23, Question 17])**.** Let G be a connected semisimple adjoint real Lie group without compact factors, and let Γ be a Zariski-dense subgroup of G. Let F be a finite set of non-trivial elements of Γ. Does there exists an element $\gamma \in \Gamma$ of infinite order such that $\langle h, \gamma\rangle \cong \langle h\rangle * \langle \gamma\rangle$ for every $h \in F$?

Of course, if F is a subgroup, the condition that $\langle h, \gamma\rangle$ be freely generated for every element $h \in F$ does not imply that the subgroup $\langle F, \gamma\rangle$ is freely generated by F and γ. For instance, even when the subgroup $\langle (h_1, h_2), (\gamma_1, \gamma_2)\rangle$ of $G_1 \times G_2$ is freely generated for every pair $(h_1, h_2) \in F_1 \times F_2$, the group $\langle F_1 \times F_2, (\gamma_1, \gamma_2)\rangle$ is never freely generated, as $(\gamma_1 h_1 \gamma_1^{-1}, 1)$ commutes with $(1, h_2)$.

Similarly, even if the projections $\langle h_i, \gamma_i\rangle$ $(i = 1, 2)$ of a subgroup $\langle (h_1, h_2), (\gamma_1, \gamma_2)\rangle$ of $G_1 \times G_2$ are freely generated subgroups, it may be that $\langle (h_1, h_2), (\gamma_1, \gamma_2)\rangle$ is itself not freely generated: if h_1, h_2 have distinct finite orders n_1 and n_2, then again $(\gamma_1, \gamma_2)(h_1, h_2)^{n_2}(\gamma_1, \gamma_2)^{-1}$ and $(h_1, h_2)^{n_1}$ commute. Proposition 2.7 shows in fact that for Question 1.1 to possibly have a positive answer, one must at the very least require that each $h \in F$ generates a subgroup $\langle h\rangle$ that embeds into one of the factors of G (cf. Remark 3.24 in general).

For these reasons and others, we cannot directly use the works mentioned above; but we will use similar techniques to prove the following.

Theorem 3.2. *Let F be a field. Let $\mathbf{G}$ be a connected algebraic F-group with center $\mathbf{Z}$. Let Γ be a Zariski-connected subgroup of $\mathbf{G}(F)$. Let $(A_i, B_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(F)$. Suppose that for each $i \in I$ there exists a local field K_i containing F and a projective K_i-representation $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$, where V_i is a finite-dimensional module over a finite division K_i-algebra D_i, with the following two properties:*

(Proximality) *$\rho_i(\Gamma)$ contains a proximal element;*

(Transversality) *For every $h \in (A_i \cup B_i) \setminus \mathbf{Z}(F)$ and every $p \in \mathbf{P}(V_i)$, the span of the set $\{\rho_i(xhx^{-1})p \mid x \in \Gamma\}$ is the whole of $\mathbf{P}(V_i)$.*

Let S be the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order, such that for all $i \in I$, the canonical maps

$$\langle \gamma, C_{A_i}\rangle *_{C_{A_i}} A_i \to \langle \gamma, A_i\rangle \qquad \text{where } C_{A_i} = A_i \cap \mathbf{Z}(F),$$

$$\langle \gamma, C_{B_i}\rangle *_{C_{B_i}} B_i \to \langle \gamma, B_i\rangle \qquad \text{where } C_{B_i} = B_i \cap \mathbf{Z}(F),$$

and provided $|A_i : C_{A_i}| > 2$ or $|B_i : C_{B_i}| > 2$, also

$$\langle A_i, C_i\rangle *_{C_i} \langle B_i, C_i\rangle \to \langle A_i, \gamma B_i \gamma^{-1}\rangle \qquad \text{where } C_i = C_{A_i} \cdot C_{B_i},$$

are all isomorphisms. Then S is dense in Γ for the join of the profinite topology and the Zariski topology.

Remark 3.3. The conditions defining S in the statement of the theorem amount to the kernel of the canonical maps

$$\langle \gamma\rangle * A_i \to \langle \gamma, A_i\rangle, \quad \langle \gamma\rangle * B_i \to \langle \gamma, B_i\rangle, \quad A_i * B_i \to \langle A_i, B_i\rangle$$

being respectively $\langle\langle [\gamma, C_{A_i}]\rangle\rangle$, $\langle\langle [\gamma, C_{B_i}]\rangle\rangle$, and $\langle\langle [C_{A_i}, B_i], [A_i, C_{B_i}]\rangle\rangle$.

When $\mathbf{Z}(F)$ is trivial, the theorem states that for any $\gamma \in S$ and for all $i \in I$, the subgroups $\langle \gamma, A_i \rangle$, $\langle \gamma, B_i \rangle$, and $\langle A_i, \gamma B_i \gamma^{-1} \rangle$ of $\mathbf{G}(F)$ are freely generated.

Remark 3.4. Note that the transversality condition implies that every ρ_i is irreducible. Moreover, the transversality condition holds equivalently for Γ or for its Zariski closure (it is a *Zariski-closed* condition). Thus, if Γ happens to be Zariski-dense (as is most common), this condition can be replaced by the analogue for $\mathbf{G}(K_i)$:

(Transversality′) *For every $h \in H_i \backslash C_i$ and every $p \in \mathbf{P}(V_i)$, the span of the set $\{\rho_i(xhx^{-1})p \mid x \in \mathbf{G}(K_i)\}$ is the whole of $\mathbf{P}(V_i)$.*

Remark 3.5. Theorem 3.2 is only meaningful for pseudo-reductive groups. Indeed, the F-unipotent radical $\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ must acts trivially under ρ_i, as the fixed-point set of $\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ is non-empty by the Lie–Kolchin theorem, hence is the whole of V_i. Thus each ρ_i factors through the pseudo-reductive quotient $\mathbf{G}/\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ of $\mathbf{G}$. We remind the reader that if $\operatorname{char} F = 0$, the full unipotent radical $\mathrm{R}_{\mathrm{u}}(\mathbf{G})$ of $\mathbf{G}$ is defined over F, hence pseudo-reductive groups are reductive (the converse always holding).

In subsequent sections, we will mostly be concerned with number fields and their archimedean completions, leaving aside the usual complications arising in positive characteristic.

Remark 3.6. There is no obvious analogue of Theorem 3.2 for HNN extensions. Indeed, $\mathbf{G}(F)$ may admit finite subgroups H containing a proper subgroup H_1 whose centralizer in $\mathbf{G}(F)$ is trivial. For instance, $\mathrm{PGL}_2(\mathbb{C})$ contains a copy of the symmetric group on 4 letters, whose alternating subgroup has trivial centralizer (see for instance [6, Proposition 1.1]). In such a situation, there is no HNN extension in $\mathbf{G}(F)$ of H with respect to the identity $H_1 \to H_1$, as any $g \in \mathbf{G}(F)$ centralizing H_1 is trivial, but $H*_{H_1}$ is not.

3.2. **Proximal dynamics in projective spaces.** Before proving Theorem 3.2, we need to extend a few known facts about the dynamics of the action of $\mathrm{GL}(V)$ on $\mathbf{P}(V)$ to projective spaces over division algebras. Foremost, we will need the contents of [83, §3] over a division algebra, but the proofs given by Tits are valid with minor adaptations to keep track of the D-structure and the fact D is not necessarily commutative. All of this is straightforward, so we will not rewrite arguments whenever they apply in the same way.

In this subsection, let K be a local field, D a division algebra of dimension d over K, and V a finite-dimensional right D-module. Recall that the absolute value $|\cdot|$ of K extends uniquely to an absolute value on D which will also denote by $|\cdot|$; we have the formula $|x| = |\mathrm{N}(x)|^{1/d}$ for $x \in D$. For each K-variety $\mathbf{V}$, the topology of K induces a locally compact topology on $\mathbf{V}(K)$; this topology is often called the *local topology*, to distinguish it from the Zariski topology when needed.

With little deviation, we will follow the notations and conventions of [82] and [83], which the reader may consult along with [10] for background material on the representation theory of algebraic groups (including over division algebras).

Recall that GL_V is the algebraic K-group of automorphisms of the D-module V, so that for any F-algebra A, the group $\mathrm{GL}_V(A)$ is the group of automorphisms of the right $(D \otimes_K A)$-module $V \otimes_K A$. Provided $\dim V \geq 2$, the K-group PGL_V is the quotient of GL_V by its center (which is the multiplicative group of the center of D). The *projective general linear group* PGL_V acts on the *projective space* $\mathbf{P}(V)$ of V, which is the space of right D-submodules of V of dimension 1. The D-submodules of V and their images in $\mathbf{P}(V)$ are both called *(D-linear) subspaces*. A projective representation $\rho : \mathbf{G} \to \mathrm{PGL}_V$ of a K-group $\mathbf{G}$ is called *irreducible* if there are no proper non-trivial linear subspaces of $\mathbf{P}(V)$ stable under $\rho(\mathbf{G})$. A representation $\mathbf{G} \to \mathrm{GL}_V$ is then irreducible if and only if its projectivization is.

Given two subspaces X, Y of $\mathbf{P}(V)$, we denote their span by $X \vee Y$. If $X \cap Y = \emptyset$ and $X \vee Y = \mathbf{P}(V)$, we denote by $\mathrm{proj}(X, Y)$ the mapping $\pi : X \to Y$ defined by

$\{\pi(p)\} = (X \vee \{p\}) \cap Y$. We will denote by $\mathring{C}$ the interior (for the local topology) of a subset C of $\mathbf{P}(V)$.

When it is needed to view V as a K-module instead of a D-module, we will add the corresponding subscript to the notation.

Definition 3.7. Let g be an element of $\mathrm{GL}_V(K)$ or $\mathrm{PGL}_V(K)$.

(1) Momentarily view V as a vector K-space, so as to identify GL_V with the subgroup of $\mathrm{GL}_{V,K}$ centralizing the right action of D on V, and likewise for PGL_V. The *attracting subspace of* g is the subspace $\mathrm{A}(g)$ of V which is the direct sum of the generalized eigenspaces (over some algebraic closure) associated to the eigenvalues of maximal absolute value of (any lift to GL_V of) g. The complementary set $\mathrm{A}'(g)$ is defined to be the direct sum of the remaining generalized eigenspaces of g. By construction, $V = \mathrm{A}(g) \oplus \mathrm{A}'(g)$.

Note that since the Galois group of any extension of K preserves the absolute value, it permutes the generalized eigenspaces of maximal absolute value, hence $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ are stable under the Galois group and are indeed defined over K. Moreover, if g commutes with the action of D, then D preserves the generalized eigenspaces of g (after perhaps extending scalars). In this case, $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ are themselves stable under D, i.e. they are D-subspaces of V.

The subspaces $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ only depend on the image of g in PGL_V. In what follows, we will often omit projectivization from the notation as long as it causes no confusion between V and $\mathbf{P}(V)$.

(2) We call g *proximal* if $\dim_D \mathrm{A}(g) = 1$, in other words if $\mathrm{A}(g)$ is a point in $\mathbf{P}(V)$. In case $D = K$, this means that g has a unique eigenvalue (counting with multiplicity) of maximal absolute value. In general, this means that g has d (possibly different) eigenvalues of maximal absolute value. If both $\mathrm{A}(g)$ and $\mathrm{A}(g^{-1})$ are one-dimensional, we call g *biproximal*[4]. We call a (projective) representation $\rho : \Gamma \to \mathrm{(P)GL}_V(K)$ *proximal* if $\rho(\Gamma)$ contains a proximal element.

Proximal elements have contractive dynamics on $\mathbf{P}(V)$: if g is proximal, then for any $p \in \mathbf{P}(V) \setminus \mathrm{A}'(g)$ the sequence $(g^n \cdot p)_{n\in\mathbb{N}}$ converges to the point $\mathrm{A}(g)$ (see Lemma 3.8).

The complement $\mathbf{P}(V) \setminus X$ of a hyperplane $X \subset \mathbf{P}(V)$ can be identified with an affine space over D by choosing for V a system of coordinate functions $\xi = (\xi_0, \dots, \xi_{\dim \mathbf{P}(V)})$, $\xi_i \in V^*$, such that $X = \ker \xi_0$. The functions $\xi_i\xi_0^{-1}$ $(i = 1, \dots, \dim \mathbf{P}(V))$ then define affine coordinates on $\mathbf{P}(V) \setminus X$. If $g \in \mathrm{PGL}_V(K)$ stabilizes X, its restriction to $\mathbf{P}(V) \setminus X$ need not be an affine map in these coordinates, but will be semiaffine (with respect to conjugation by the factor by which g scales ξ_0). In particular, if $\mathbf{P}(V) \setminus X$ is seen as an affine space over K, then the restriction of g is K-affine.

For the rest of this section, we fix an *admissible* distance d on $\mathbf{P}(V)$, that is, a distance function $d : \mathbf{P}(V) \times \mathbf{P}(V) \to \mathbf{P}(V)$ inducing the local topology on $\mathbf{P}(V)$ and satisfying the property that for every compact subset C contained in an affine subspace of $\mathbf{P}(V)$, there exist constants $M, M' \in \mathbb{R}$ such that

$$M \cdot d_\xi|_{C\times C} \leq d|_{C\times C} \leq M' \cdot d_\xi|_{C\times C}.$$

Here d_ξ is the supremum distance with respect to the affine coordinates $(\xi_i\xi_0^{-1})_{i=1}^{\dim \mathbf{P}(V)}$ described above. Note that two different coordinate systems on the same affine subspace A of $\mathbf{P}(V)$ define comparable distance functions on this affine subspace. Moreover, if instead of using D-coordinates one views A as an affine K-space, the supremum distance in any set of affine K-coordinates will again be comparable to d_ξ.

[4]Biproximal elements are sometimes called 'very proximal' or 'hyperbolic' in the literature.

As indicated by Tits, when K is an archimedean local field, any elliptic metric on $\mathbf{P}(V)$ is admissible. Tits also indicates in [83, §3.3] how to construct an admissible metric in the non-archimedean case by patching together different d_ξ's; this construction works identically over a division algebra.

Having fixed an (admissible) distance d on $\mathbf{P}(V)$, the *norm* of a mapping $f : X \to \mathbf{P}(V)$ defined on some subset $X \subset \mathbf{P}(V)$ is the quantity

$$\|f\| = \sup_{\substack{p,q\in X\\ p\neq q}} \frac{d(f(p), f(q))}{d(p,q)}.$$

Note that the norm is submultiplicative: given mappings $f : X \to \mathbf{P}(V)$ and $g : Y \to X$, we have $\|f \circ g\| \leq \|f\|\cdot\|g\|$. Projective transformations always have finite norm [83, Lemma 3.5]. Indeed, given $g \in \mathrm{PGL}_V(K)$, the distance function d^g defined by $d^g(p,q) = d(gp, gq)$ is again admissible. Since $\mathbf{P}(V)$ is compact, it can be covered by finitely many compact sets contained in affine subspaces, on which the ratio between d^g and d is uniformly bounded, by admissibility.

We can now state the needed results from [83, §3] in our setting. The following two lemmas describe the dynamics of D-linear transformations.

Lemma 3.8 (Lemma 3.8 in [83]). *Let $g \in \mathrm{PGL}_V(K)$, let C be a compact subset of $\mathbf{P}(V)$ and let $r \in \mathbb{R}_{>0}$.*

(i) *Suppose that g is proximal and that $C \cap \mathrm{A}'(g) = \emptyset$. Then there exists an integer N such that $\|g^n|_C\| < r$ for all $n > N$; and for any neighborhood U of $\mathrm{A}(g)$, there exists an integer N' such that $g^nC \subset U$ for all $n > N'$.*

(ii) *Assume that, for some $m \in \mathbb{N}$, one has $\|g^m|_C\| < 1$ and $g^mC \subset \mathring{C}$. Then $\mathrm{A}(g)$ is a point contained in $\mathring{C}$.*

Note that in loc. cit. Tits assumes the existence of a semisimple proximal element; but as he indicates in the footnotes, this assumption is superfluous and the proof of the lemma is identical with an arbitrary proximal element.

Proof. The argument given by Tits applies, taking into account the following adaptations.

In part (i), the transformation g restricted to $\mathbf{P}(V) \setminus A'(g)$ is not necessarily D-linear, as was already mentioned. It is nevertheless K-linear, with eigenvalues of absolute value strictly smaller than 1 by assumption. So one can apply [83, Lemma 3.7 (i)] over K and use that the norms defined over D or K are comparable to conclude.

In part (ii), one cannot pick a representative of g in GL_V whose eigenvalues corresponding to the fixed point $p \in \mathbf{P}(V)$ equal one (as g may have different eigenvalues on the D-line p). Nevertheless, they are all of the same absolute value, which we can assume to be 1. If there is another eigenvalue of the same absolute value (i.e. if $\mathrm{A}(g) \neq \{p\}$), then the restriction of g to $\mathrm{A}(g)$ is a block-upper-triangular matrix in a well-chosen basis. Since the compact set C has non-empty interior, this contradicts the hypothesis of (ii). □

Lemma 3.9 (Lemma 3.9 in [83]). *Let $g \in \mathrm{PGL}_V(K)$ be semisimple, let $\bar{g} \in \mathrm{GL}_V(K)$ be a representative of g, let Ω be the set of eigenvalues of $\bar{g}$ (over an appropriate field extension of K) whose absolute value is maximum, let C be a compact subset of $\mathbf{P}(V) \setminus \mathrm{A}'(g)$, set $\pi = \mathrm{proj}(\mathrm{A}'(g), \mathrm{A}(g))$, and let U be a neighborhood of $\pi(C)$ in $\mathbf{P}(V)$.*

(i) *There exists an infinite set $N \subset \mathbb{N}$ such that $\lim_{\substack{n\in N\\ n\to\infty}} (\lambda^{-1}\mu)^n = 1$ for all $\lambda, \mu \in \Omega$.*

(ii) *The set $\{\|g^n|_C\| \mid n \in \mathbb{N}\}$ is bounded.*

(iii) *If N is as in (i), $g^nC \subset U$ for almost all $n \in N$.*

Proof. The easiest way to obtain this lemma over the division algebra D is to take a representative of g in GL_V, see it as an K-linear transformation in $\mathrm{GL}_{V,K}$ and apply Tits' original lemma [83, Lemma 3.9]. Part (i) is then immediate.

For part (ii) and (iii), denote $\mathbf{P}_K(V)$ the projective space of V seen as a vector K-space. Since the canonical GL_V-equivariant map $q : \mathbf{P}_K(V) \to \mathbf{P}(V)$ is proper and continuous, $C' = q^{-1}(C)$ is compact, and $U' = q^{-1}(U)$ is open. Thus [83, Lemma 3.9] applies with C' and U' over K, and in turn yields the same conclusions over D, since the norms of g restricted to C and C' bound each-other. □

We will also make use of a version of part (i) of Lemma 3.9 for multiple representations, due to Margulis and Soifer. They initially stated it for multiple vector spaces over the same local field, but as already observed in [67, Lemma 3.1], the proof is identical.

Lemma 3.10 (Lemma 3 in [60])**.** *Let $\{K_i\}_{i\in I}$, be a finite collection of local fields and V_i be a finite-dimensional vector K_i-space. Let g_i be a semisimple element of $\mathrm{GL}_{V_i}(K)$, and let $\Omega(g_i)$ be the set of eigenvalues of g_i whose absolute value is maximum. There exists an infinite subset $N \subset \mathbb{N}$ such that $\lim_{\substack{n\in N\\ n\to\infty}} (\lambda^{-1}\mu)^n = 1$ for all $i \in I$ and $\lambda, \mu \in \Omega(g_i)$.*

We are now ready to prove the following slight generalization of [67, Corollary 3.7], which is itself a refinement of both [83, Proposition 3.11] and [60, Lemma 8]. This proposition is a crucial piece of the proof of Theorem 3.2: it will be used to find enough biproximal elements in Γ.

Proposition 3.11 (Abundance of simultaneously biproximal elements)**.** *Let $\mathbf{G}$ be a connected algebraic F-group and let Γ be a Zariski-dense subgroup of $\mathbf{G}(F)$. Let $\{K_i\}_{i\in I}$ be a finite collection of local fields each containing F. For each $i \in I$, let $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$ be an irreducible projective K_i-representation, where V_i is a finite-dimensional module over a finite division K_i-algebra D_i.*

Suppose that for each $i \in I$, $\rho_i(\Gamma)$ contains a proximal element. Then the set of regular semisimple elements $\gamma \in \Gamma$ such that $\rho_i(\gamma)$ is biproximal for every $i \in I$, is dense in Γ for the join of the Zariski topology and the profinite topology.

Proof. We follow the line of arguments given in [83, 60, 67], keeping track of the different representations, and using the extension of Tits' work to projective representations over a division algebra laid out above.

Given an arbitrary element $g \in \mathbf{G}(F)$, let us abbreviate $\rho_i(g)$ by g_i.

<u>Step 1:</u> The set of simultaneously proximal elements in Γ is Zariski-dense if it is non-empty.

Let $g \in \Gamma$ be such that g_i is proximal for all $i \in I$. Since ρ_i is irreducible, for each $i \in I$ the set of elements x of $\mathbf{G}(F)$ such that $x_i\mathrm{A}(g_i) \not\in \mathrm{A}'(g_i)$ is non-empty and Zariski-open. Because $\mathbf{G}$ is Zariski-connected, the intersection of these sets remains non-empty (and Zariski-open). Let us then pick $x \in \Gamma$ satisfying $x_i\mathrm{A}(g_i) \not\in \mathrm{A}'(g_i)$ for every $i \in I$.

By construction of x, we can pick a compact neighborhood C_i of $\mathrm{A}(g_i)$ in $\mathbf{P}(V_i)$ such that x_iC_i is disjoint from $A'(g)$. Since projective transformations have finite norm, we have $\max_{i\in I} \|x_i|_{C_i}\| < r$ for some $r \in \mathbb{R}$. By Lemma 3.8.(i), for each $i \in I$ there exists an integer N_i such that

$$\|g_i^n|_{x_iC_i}\| < r^{-1} \qquad \text{and} \qquad g_i^n(x_iC_i) \subset \mathring{C}_i \qquad \text{for } n > N_i.$$

Set $N_x = \max_{i\in I} N_i$. Then for any $i \in I$, we have that

$$\|g_i^n x_i|_{C_i}\| < 1 \qquad \text{and} \qquad (g_i^n x_i)C_i \subset \mathring{C}_i \qquad \text{for } n > N_x.$$

We deduce from Lemma 3.8.(ii) that $g_i^n x_i = \rho_i(gx)$ is proximal for every $n > N_x$.

Observe that the Zariski closure Z of $\{g^n \mid n > N_x\}$ in Γ has the property that $gZ \subset Z$. Since the Zariski topology is Noetherian, we deduce that $g^{m+1}Z = g^m Z$ for some $m \in \mathbb{N}$. This implies that $g^n Z = Z$ for every $n \in \mathbb{Z}$, and in particular that $g \in Z$. Let now $\overline{S}$ denote the Zariski closure in Γ of the set S of elements of Γ which are proximal under every ρ_i. We have shown that S contains $g^n x$ for each $x \in \Gamma$ chosen as above and $n > N_x$. By our last observation, $\overline{S}x^{-1}$ contains g, hence $gx \in \overline{S}$. As this holds for every x in a Zariski-dense (open) subset of Γ, we conclude that $\overline{S}$ contains $g\Gamma = \Gamma$, as claimed.

Step 2: Γ contains a semisimple element that is simultaneously proximal.

We argue by induction on $\#I$. Fix $j \in I$, and suppose that there are elements $g, h \in \Gamma$ such that $\rho_j(h)$ is proximal and $\rho_i(g)$ is proximal for $i \in I \setminus \{j\}$. By Step 1, we may in addition assume that g and h are semisimple. Write $\pi_i = \operatorname{proj}(\mathrm{A}'(h_i), \mathrm{A}(h_i))$ for $i \neq j$, and $\pi_j = \operatorname{proj}(\mathrm{A}'(g_j), \mathrm{A}(g_j))$.

Let $N \subset \mathbb{N}$ be an infinite set such as afforded by Lemma 3.10 applied to the elements h_i for $i \neq j$ and g_j for $i = j$, so that we have $\lim_{\substack{n \in N \\ n \to \infty}} (\lambda^{-1}\mu)^n = 1$ for $\lambda, \mu \in \Omega(h_i)$ if $i \neq j$, and for $\lambda, \mu \in \Omega(g_j)$.

Since ρ_i is irreducible and Γ is Zariski-dense, as before we can fix $x \in \Gamma$ such that

$$x_i \mathrm{A}(g_i) \not\subset \mathrm{A}'(h_i) \quad \text{for every } i \in I.$$

Similarly, the elements $y \in \mathbf{G}(F)$ satisfying

$$\begin{aligned} y_i \cdot \pi_i(x_i \mathrm{A}(g_i)) &\notin \mathrm{A}'(g_i) \quad \text{for } i \in I \setminus \{j\}, \\ \text{and} \quad y_j \mathrm{A}(h_j) &\notin \Big(x_j^{-1} \mathrm{A}'(h_j) \cap \mathrm{A}(g_j) \Big) \vee \mathrm{A}'(g_j), \end{aligned}$$

form a non-empty Zariski-open subset of $\mathbf{G}(F)$. Let us then fix y such an element in Γ.

For $i \neq j$, let B_i be a compact neighborhood of $y_i \cdot \pi_i(x_i \mathrm{A}(g_i))$ disjoint from $\mathrm{A}'(g_i)$, and let B_j be a compact neighborhood of $x_j \cdot \pi_j(y_j \mathrm{A}(h_j))$ disjoint from $\mathrm{A}'(h_j)$. The latter exists because $\pi_j^{-1}(x_j^{-1}\mathrm{A}'(h_j)) \subset (x_j^{-1}\mathrm{A}'(h_j) \cap \mathrm{A}(g_j)) \vee \mathrm{A}'(g_j)$ does not contain $y_j \mathrm{A}(h_j)$. We also choose for $i \neq j$ a compact neighborhood C_i of $\mathrm{A}(g_i)$ disjoint from $x_i^{-1}\mathrm{A}'(h_i)$ and small enough to satisfy $y_i \cdot \pi_i(x_i C_i) \subset \mathring{B}_i$; and choose a compact neighborhood C_j of $\mathrm{A}(h_j)$ disjoint from $y_j^{-1}\mathrm{A}'(g_j)$ and satisfying $x_j \cdot \pi_j(y_j C_j) \subset \mathring{B}_j$.

The careful choice of B_i, C_i and N sets us up for the following applications of Lemmas 3.8 and 3.9. By Lemma 3.9, for each $i \neq j$ there exists $r_i \in \mathbb{R}$ and $N_i \in \mathbb{N}$ such that

$$\|h_i^n|_{x_i C_i}\| < r_i \text{ for } n \in \mathbb{N} \quad \text{and} \qquad y_i h_i^n x_i C_i \subset \mathring{B}_i \quad \text{for } n \in N,\, n > N_i.$$

Similarly, there exists $N_j \in \mathbb{N}$ and $r_j \in \mathbb{R}$ such that

$$\|g_j^n|_{y_j C_j}\| < r_j \text{ for } n \in \mathbb{N} \quad \text{and} \qquad x_j g_j^n y_j C_j \subset \mathring{B}_j \quad \text{for } n \in N,\, n > N_j.$$

By Lemma 3.8(i), for each $i \neq j$ there exists $N_i' \in \mathbb{N}$ such that

$$\|g_i^n|_{B_i}\| < \big(\|y_i|_{y_i^{-1}B_i}\| \cdot r_i \cdot \|x_i|_{C_i}\| \big)^{-1} \quad \text{and} \qquad g_i^n B_i \subset \mathring{C}_i \quad \text{for } n > N_i'.$$

Similarly, there exists $N_j' \in \mathbb{N}$ such that

$$\|h_j^n|_{B_j}\| < \big(\|x_j|_{x_j^{-1}B_j}\| \cdot r_j \cdot \|y_j|_{C_j}\| \big)^{-1} \quad \text{and} \qquad h_j^n B_j \subset \mathring{C}_j \quad \text{for } n > N_j'.$$

Set $N' = \{n \in N \mid n > N_i \text{ and } n > N_i' \text{ for all } i \in I\}$. For $i \neq j$, we have by construction that

$$\|g_i^m y_i h_i^n x_i|_{C_i}\| < 1 \quad \text{and} \quad g_i^m y_i h_i^n x_i C_i \subset \mathring{C}_i \quad \text{for } m, n \in N'.$$

Similarly, we have that

$$\|h_j^n x_j g_j^m y_j|_{C_j}\| < 1 \quad \text{and} \quad h_j^n x_j g_j^m y_j C_j \subset \mathring{C}_j \quad \text{for } m, n \in N'.$$

We conclude from Lemma 3.8.(ii) that for all $m, n \in N'$, the element $g_i^m y_i h_i^n x_i$ is proximal for $i \neq j$, and so is $h_j^n x_j g_j^m y_j$. But $h_j^n x_j g_j^m y_j$ and $g_j^m y_j h_j^n x_j$ are conjugate, so $g^m y h^n x \in \Gamma$ is proximal under ρ_i for every $i \in I$.

In view of Step 1, the set of simultaneously proximal elements in Γ is Zariski-dense, so there is also a semisimple one as claimed.

Step 3: Γ contains an element which is simultaneously biproximal.

By Steps 1–2, there is a semisimple element $g \in \Gamma$ such that $\rho_i(g^{-1})$ is proximal for every $i \in I$. Let N be an infinite set such as afforded by Lemma 3.10. Replacing N by an appropriate subset, we may assume that the set $g^N = \{g^n \mid n \in N\}$ is Zariski-connected.

Since ρ_i is irreducible and Γ is Zariski-dense, the elements $x \in \mathbf{G}(F)$ such that

$$x_i \mathrm{A}(g_i) \not\subset \mathrm{A}'(g_i^{-1}) \quad \text{and} \quad x_i^{-1} \mathrm{A}(g_i) \not\subset \mathrm{A}'(g_i^{-1}) \quad \text{for every } i \in I$$

form a non-empty Zariski-open subset. Fix such an element $x \in \Gamma$. For the same reasons, the set U of elements $y \in \mathbf{G}(F)$ satisfying

$$\begin{aligned} & y_i \mathrm{A}(g_i^{-1}) \notin x_i \mathrm{A}'(g_i) \vee (x_i \mathrm{A}(g_i) \cap \mathrm{A}'(g_i^{-1})), \\ \text{and} \quad & y_i^{-1} x_i \mathrm{A}(g_i^{-1}) \notin \mathrm{A}'(g_i) \vee (\mathrm{A}(g_i) \cap x_i \mathrm{A}'(g_i^{-1})) \qquad \text{for every } i \in I \end{aligned}$$

is also non-empty and Zariski-open; fix $y \in U \cap \Gamma$.

Write $\pi_i = \mathrm{proj}(\mathrm{A}'(g_i), \mathrm{A}(g_i))$ and $\pi_i' = \mathrm{proj}(x_i \mathrm{A}'(g_i), x_i \mathrm{A}(g_i))$. For each $i \in I$, let B_i be a compact neighborhood of $\pi_i'(y_i \mathrm{A}(g_i^{-1}))$ disjoint from $\mathrm{A}'(g_i^{-1})$, and let B_i' be a compact neighborhood of $\pi_i(y_i^{-1} x_i \mathrm{A}(g_i^{-1}))$ disjoint from $x_i \mathrm{A}'(g_i^{-1})$. We also choose a compact neighborhood C_i of $\mathrm{A}(g_i^{-1})$ disjoint from $y_i^{-1} x_i \mathrm{A}'(g_i)$ satisfying $\pi_i'(y_i C_i) \subset \mathring{B}_i$, and a compact neighborhood C_i' of $y_i^{-1} x_i \mathrm{A}(g_i^{-1})$ disjoint from $\mathrm{A}'(g_i)$ satisfying $\pi_i(C_i') \subset \mathring{B}_i'$.

By Lemma 3.9 (ii), for each $i \in I$ there exist $N_i, N_i' \in \mathbb{N}$ and $r_i, r_i' \in \mathbb{R}$ such that

$$\begin{aligned} \|x_i g_i^n x_i^{-1}|_{y_i C_i}\| < r_i \text{ for } n \in \mathbb{N} \quad & \text{and} \quad & x_i g_i^n x_i^{-1} y_i C_i \subset \mathring{B}_i \quad & \text{for } n \in N,\, n > N_i, \\ \|g_i^n|_{C_i'}\| < r_i' \text{ for } n \in \mathbb{N} \quad & \text{and} \quad & g_i^n C_i' \subset \mathring{B}_i' \quad & \text{for } n \in N,\, n > N_i'. \end{aligned}$$

By Lemma 3.8.(i), for each $i \in I$ there exist $M_i, M_i' \in \mathbb{N}$ such that

$$\begin{aligned} \|g_i^{-n}|_{B_i}\| < \left(r_i \cdot \|y_i|_{C_i}\|\right)^{-1} \quad & \text{and} \quad & g_i^{-n} B_i \subset \mathring{C}_i \quad & \text{for } n > M_i, \\ \|x_i g_i^{-n} x_i^{-1}|_{B_i'}\| < \left(\|y_i^{-1}|_{y_i C_i'}\| \cdot r_i'\right)^{-1} \quad & \text{and} \quad & x_i g_i^{-n} x_i^{-1} B_i' \subset y_i \mathring{C}_i' \quad & \text{for } n > M_i'. \end{aligned}$$

Set $N_{x,y} = \{n \in N \mid n > \max \bigcup_{i \in I} \{N_i, N_i', M_i, M_i'\}\}$. We then have by construction that

$$\begin{aligned} \|g_i^{-n} x_i g_i^n x_i^{-1} y_i|_{C_i}\| < 1 \quad & \text{and} \quad & g_i^{-n} x_i g_i^n x_i^{-1} y_i C_i \subset \mathring{C}_i \quad & \text{for } n \in N_{x,y}, \\ \|y_i^{-1} x_i g_i^{-n} x_i^{-1} g_i^n|_{C_i'}\| < 1 \quad & \text{and} \quad & y_i^{-1} x_i g_i^{-n} x_i^{-1} g_i^n C_i' \subset \mathring{C}_i' \quad & \text{for } n \in N_{x,y}. \end{aligned}$$

We conclude from Lemma 3.8.(ii) that for all $n \in N_{x,y}$ and for each $i \in I$, the element $g^{-n} x g^n x^{-1} y$ is biproximal under ρ_i.

Step 4: The set of regular semisimple simultaneously biproximal elements is dense.

Let S denote the set of elements in Γ which are biproximal under every ρ_i. Let Λ be a normal subgroup of finite index in Γ, and let $\gamma \in \Gamma$. Because the set of regular semisimple elements is Zariski-open, it suffices to show that $S \cap \Lambda\gamma$ is Zariski-dense to prove the proposition.

Since Γ is Zariksi-connected and Λ has finite index in Γ, every coset of Λ is Zariski-dense. Moreover, if $h \in \Gamma$ is such that h_i is proximal, then $h^{|\Gamma:\Lambda|}$ is also proximal under ρ_i, and belongs to Λ. We can thus apply Steps 1–3 to Λ, to find an element $g \in \Lambda$ such that g_i is biproximal for every $i \in I$.

As before, the set U of elements $x \in \mathbf{G}(F)$ such that

$$x_i\gamma_i\mathrm{A}(g_i) \notin \mathrm{A}'(g_i) \quad \text{and} \quad \gamma_i^{-1}x_i^{-1}\mathrm{A}(g_i^{-1}) \notin \mathrm{A}'(g_i^{-1}) \quad \text{for every } i \in I$$

is Zariski-open and non-empty. In particular, $\Lambda \cap U$ is Zariski-dense in Γ; pick $x \in \Lambda \cap U$.

Let $C_i^{\pm}$ be a compact neighborhood of $\mathrm{A}(g_i^{\pm 1})$ such that $(x\gamma)_i^{\pm 1}C_i^{\pm}$ is disjoint from $\mathrm{A}'(g_i^{\pm 1})$. Since projective transformations have finite norm, we have that $\max_{i\in I} \|(x\gamma)_i^{\pm 1}|_{C_i^{\pm}}\| < r$ for some $r \in \mathbb{R}$. By Lemma 3.8.(i), there exist integers N_i^+ and N_i^- such that

$$\|g_i^n|_{x_i\gamma_iC_i^+}\| < r^{-1} \qquad \text{and} \qquad g_i^n x_i\gamma_i C_i^+ \subset \mathring{C}_i^+ \qquad \text{for } n > N_i^+.$$

$$\|g_i^{-n}|_{(x\gamma)_i^{-1}C_i^-}\| < r^{-1} \qquad \text{and} \qquad g_i^{-n}(x\gamma)_i^{-1}C_i^- \subset \mathring{C}_i^- \qquad \text{for } n > N_i^-.$$

For $N_x = \max \bigcup_{i\in I}\{N_i^+, N_i^-\}$, we then have for every $i \in I$ that

$$\|g_i^n x_i\gamma_i|_{C_i^+}\| < 1 \qquad \text{and} \qquad g_i^n x_i\gamma_i C_i^+ \subset \mathring{C}_i^+ \qquad \text{for } n > N_x.$$

$$\|g_i^{-n}\gamma_i^{-1}x_i^{-1}|_{C_i^-}\| < 1 \qquad \text{and} \qquad g_i^{-n}\gamma_i^{-1}x_i^{-1}C_i^- \subset \mathring{C}_i^- \qquad \text{for } n > N_x.$$

We deduce from Lemma 3.8.(ii) that $g_i^n x_i\gamma_i$ and $g_i^{-n}\gamma_i^{-1}x_i^{-1}$ are proximal for every $i \in I$ and for $n > N_x$. But $g_i^{-n}\gamma_i^{-1}x_i^{-1}$ and $\gamma_i^{-1}x_i^{-1}g_i^{-n}$ are conjugate, so $g_i^n x_i\gamma_i$ is in fact biproximal for every $i \in I$. Of course $g^n x\gamma \in \Lambda\gamma$, so we have shown that $S \cap \Lambda\gamma$ contains $g^n x\gamma$ for every $x \in \Lambda \cap U$ and $n > N_x$.

As was observed in Step 1, the Zariski closure of $\{g^n \mid n > N_x\}$ in Γ contains g. Thus the Zariski closure of $S \cap \Lambda\gamma$ contains $gx\gamma$ for every $x \in \Lambda \cap U$. As $\Lambda \cap U$ is Zariski-dense, so is $S \cap \Lambda\gamma$. This concludes the proof of the proposition. ☐

3.3. **Towards the proof of Theorem 3.2.** Before starting the proof of Theorem 3.2, we record the following lemmas.

Lemma 3.12. *Let K, D and V be as in §3.2. Let $\mathbf{G}$ be a connected K-subgroup of PGL_V, acting irreducibly on $\mathbf{P}(V)$. Suppose that $\mathbf{G}(K)$ contains a proximal element g_0. Then the set*

$$X = \{\mathrm{A}(g) \mid g \in \mathbf{G}(K) \text{ is proximal}\} \subseteq \mathbf{P}(V)$$

coincides with the orbit $\mathbf{G}(K) \cdot \mathrm{A}(g_0)$ and constitutes the unique irreducible projective subvariety of $\mathbf{P}(V)$ stable under $\mathbf{G}(K)$. In consequence, $\mathrm{Stab}_{\mathbf{G}}(\mathrm{A}(g_0))$ is a parabolic subgroup of $\mathbf{G}$.

Proof. By a theorem of Chevalley, there is a Zariski-closed $\mathbf{G}(K)$-orbit $Y \subseteq \mathbf{P}(V)$. Let $g \in \mathbf{G}(K)$ be proximal. Because $\mathbf{G}$ acts irreducibly on $\mathbf{P}(V)$, there exists $y \in Y \setminus \mathrm{A}'(g)$. We then have $g^n \cdot y \xrightarrow{n\to\infty} \mathrm{A}(g)$, thus $\mathrm{A}(g)$ lies in the closure of Y in the local hence in the Zariski topology. As Y was Zariski-closed, $\mathrm{A}(g) \in Y$. Since this happens for any proximal element g, we deduce that $X \subseteq Y$. As X is $\mathbf{G}(K)$-stable and Y is a single orbit, equality holds. It is now clear that X is the set of K-points of a projective variety $\mathbf{X}$, which is irreducible because $\mathbf{G}$ is.

Let $\mathbf{P} = \mathrm{Stab}_{\mathbf{G}}(\mathrm{A}(g))$ denote the stabilizer of $\mathrm{A}(g)$ in $\mathbf{G}$. The above shows that orbit map yields an isomorphism $\mathbf{G}/\mathbf{P} \to \mathbf{X}$, hence $\mathbf{G}/\mathbf{P}$ is a complete variety, meaning that $\mathbf{P}$ is parabolic. The same holds for every other proximal element. ☐

Remark 3.13. Lemma 3.12 can also be proven by arguing that if g_0 is proximal, $\mathrm{A}(g_0)$ must be a highest weight line.

Lemma 3.14 (Transversality)**.** *Let* $\mathbf{G}$ *be as in Lemma 3.12, and suppose that* $\mathbf{G}(K)$ *contains a proximal element* g*. For any* $h \in \mathbf{G}(K)$*, the set*

$$U_{h,g} = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g) \cup \mathrm{A}'(g^{-1})\}$$

is Zariski-open in $\mathbf{G}(K)$*. If* $h \in \mathbf{G}(K)$ *is such that the span of* $\{xhx^{-1}\mathrm{A}(g) \mid x \in \mathbf{G}(K)\}$ *is the whole of* $\mathbf{P}(V)$*, then* $U_{h,g}$ *is non-empty.*

Proof. The two sets

$$U_1 = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g)\}$$
$$U_2 = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g^{-1})\}$$

are Zariski-open by a standard argument: for any subspaces $W_1, W_2 \subseteq V$, the set $\{x \in \mathbf{G}(K) \mid x \cdot W_1 \subseteq W_2\}$ is Zariski-closed. We have to show they are both non-empty.

There is a minimal parabolic K-subgroup $\mathbf{B}$ of $\mathbf{G}$ that contains h. By Lemma 3.12, there is a conjugate $x\mathbf{B}x^{-1}$ of $\mathbf{B}$ which fixes $\mathrm{A}(g)$. But then for this choice of x, we surely have $xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g)$. This shows that U_1 is not empty.

Finally, U_2 is non-empty because of the assumption made on h. Indeed, U_2 being empty means $xhx^{-1}\mathrm{A}(g) \in \mathrm{A}'(g^{-1})$ for every $x \in \mathbf{G}(K)$, but the latter is a proper subspace of $\mathbf{P}(V)$. □

Remark 3.15*.* At first glance, Lemma 3.14 above may seem to be weaker than [67, Proposition 2.17]. Unfortunately, the proof of [67, Proposition 2.17] relies on [67, Proposition 2.11], whose statement is erroneous. The set of elements whose conjugacy class intersects a big Bruhat cell is in fact smaller than stated there (see for instance [27, 28, 20] for a description in the case of SL_n). In consequence, the results of [67] are only valid under the additional assumption that the conjugacy classes of the elements h under consideration intersect a big Bruhat cell. Note that there are non-central torsion elements whose conjugacy class does not intersect the big Bruhat cell. We will address this in the next section by arranging for the transversality assumption of Lemma 3.14 and Theorem 3.2 to hold.

We note in addition that the proof of [67, Theorem 6.5] overlooks the possibility that the subgroup generated by a given torsion element h may not embed in any simple quotient of $\mathbf{G}$. As will be emphasized in Remark 3.24, this condition is necessary for constructing a ping-pong partner for h.

Proof of Theorem 3.2. For an arbitrary element $g \in \mathbf{G}(F)$, let us abbreviate $\rho_i(g)$ by g_i. For simplicity, we also write $A_i^* = A_i \setminus \mathbf{Z}(F)$ and $B_i^* = B_i \setminus \mathbf{Z}(F)$. Recall that $C_{A_i} = A_i \cap \mathbf{Z}(F)$, $C_{B_i} = B_i \cap \mathbf{Z}(F)$, and $C_i = C_{A_i} \cdot C_{B_i}$.

Fix a normal subgroup Λ of finite index in Γ, and fix $\gamma_0 \in \Gamma$. First, because of the proximality hypothesis, Proposition 3.11 applied to the Zariski-closure $\mathbf{H}$ of Γ in $\mathbf{G}$ states that the set S' of regular semisimple elements $\gamma' \in \Lambda\gamma_0$ such that $\gamma'_i = \rho_i(\gamma')$ is biproximal for every $i \in I$, is Zariski-dense in Γ. Pick $\gamma' \in S'$.

Second, using the transversality hypothesis on ρ_i, we exhibit a simultaneously biproximal element in $\Lambda\gamma_0$ acting transversely to A_i^* and B_i^* for all $i \in I$. By Lemma 3.14, for every $i \in I$ and every $h \in A_i^* \cup B_i^*$ the sets

$$U_{i,h,\gamma'^{\pm 1}} = \{x \in \mathbf{H}(F) \mid x_i h_i x_i^{-1}\mathrm{A}(\gamma_i'^{\pm 1}) \notin \mathrm{A}'(\gamma'_i) \cup \mathrm{A}'(\gamma_i'^{-1})\}$$

are Zariski-open and non-empty. In consequence, we can pick an element λ in the Zariski-dense set $\Lambda \cap U_{\gamma'}$, where $U_{\gamma'} = \bigcap_{i \in I} \bigcap_{h \in A_i^* \cup B_i^*} (U_{i,h,\gamma'} \cap U_{i,h,\gamma'^{-1}})$. Setting $\gamma = \lambda^{-1}\gamma'\lambda$, we see that $\gamma \in S'$, while for any $h \in A_i^* \cup B_i^*$,

$$h_i\mathrm{A}(\gamma_i) \notin \mathrm{A}'(\gamma_i) \cup \mathrm{A}'(\gamma_i{}^{-1}) \quad \text{and} \quad h_i\mathrm{A}(\gamma_i{}^{-1}) \notin \mathrm{A}'(\gamma_i) \cup \mathrm{A}'(\gamma_i{}^{-1}).$$

Next, we construct the sets that will allow us to apply Lemma 2.1. Given $i \in I$, let $P_i^{\pm}$ be a compact neighborhood of $\mathrm{A}({\gamma_i}^{\pm 1})$ in $\mathbf{P}(V_i)$ small enough to achieve

$$((A_i^* \cup B_i^*) \cdot P_i^{\pm}) \cap (\mathrm{A}'(\gamma_i) \cup \mathrm{A}'({\gamma_i}^{-1})) = \emptyset.$$

Such a set exists by construction of γ: by local compactness, the complement of the closed set $(A_i^* \cup B_i^*) \cdot \big(\mathrm{A}'(\gamma_i) \cup \mathrm{A}'({\gamma_i}^{-1})\big)$ contains a compact neighborhood of $\mathrm{A}({\gamma_i}^{\pm 1})$. In the same way, we can arrange that also

$$((A_i^* \cup B_i^*) \cdot P_i^{\pm}) \cap (P_i^+ \cup P_i^-) = \emptyset.$$

Note that $\mathbf{Z}(F)$ fixes $\mathrm{A}(\gamma_i)$ and $\mathrm{A}({\gamma_i}^{-1})$. The finite intersection $\bigcap_{c \in C_i}(c \cdot P_i^{\pm})$ is thus again a compact neighborhood of $\mathrm{A}({\gamma_i}^{\pm 1})$. Replacing $P_i^{\pm}$ by this intersection, we will further assume that $P_i^{\pm}$ is stable under C_i, hence under C_{A_i} and C_{B_i}.

Set $P_i = P_i^+ \cup P_i^-$ and set

$$Q_i = (A_i^* \cup B_i^*) \cdot P_i;$$

these two subsets of $\mathbf{P}(V_i)$ are compact, disjoint, and preserved by C_i. As Q_i is disjoint from $\mathrm{A}'(\gamma_i) \cup \mathrm{A}'({\gamma_i}^{-1})$, Lemma 3.8.(i) shows that there exists $N \in \mathbb{N}$ such that for any $n > N$,

$$\gamma_i^n Q_i \subset P_i^+ \quad \text{and} \quad \gamma_i^{-n} Q_i \subset P_i^- \quad \text{for each } i \in I.$$

Pick $N_1 > N$ with $N_1 = 1 \mod |\Gamma : \Lambda|$, so that $\gamma^{N_1 + n|\Gamma:\Lambda|} \in \Lambda\gamma_0$ for every $n \in \mathbb{Z}$. For $n \in \mathbb{N}$, put $[n] = N_1 + n\,|\Gamma : \Lambda|$ and note that $[n] > N$ and $[n] = 1 \mod |\Gamma : \Lambda|$.

For each $i \in I$, Lemma 2.1 now applies to the following triples of subgroups of $\mathbf{G}(F)$:

$$(*) \qquad \begin{gathered} \langle \gamma^{[n]}, C_{A_i} \rangle \text{ and } A_i \text{ along } C_{A_i}, \quad \langle \gamma^{[n]}, C_{B_i} \rangle \text{ and } B_i \text{ along } C_{B_i}, \\ \langle A_i, C_i \rangle \text{ and } \langle \gamma^{[n]} B_i \gamma^{-[n]}, C_i \rangle \text{ along } C_i, \end{gathered}$$

with the same sets P_i and Q_i constructed above! Indeed, by construction, for all $m \in \mathbb{Z}\backslash\{0\}$ we have

$$\gamma_i^{m[n]} Q_i \subset P_i, \quad A_i^* \cdot P_i \subset Q_i, \quad B_i^* \cdot P_i \subset Q_i,$$
$$\gamma_i^{[n]} B_i^* \gamma_i^{-[n]} \cdot Q_i \subset \gamma_i^{[n]} B_i^* \cdot P_i^- \subset \gamma_i^{[n]} \cdot Q_i \subset P_i^+ \subset P_i.$$

But $\{\gamma^{m[n]} \mid m \neq 0\}$ clearly represents every non-trivial coset of C_{A_i} (resp. of C_{B_i}) in the group they generate, and the same holds for A_i^* and $\gamma^{[n]} B_i^* \gamma^{-[n]}$ with respect to C_i. We conclude that for every $i \in I$ and all $n \in \mathbb{N}$, the subgroups $\langle \gamma^{[n]}, A_i \rangle$, $\langle \gamma^{[n]}, B_i \rangle$, and $\langle A_i, \gamma^{[n]} B_i \gamma^{-[n]} \rangle$ are the free amalgamated products of the triples given in $(*)$ above.

This establishes that $S \cap \Lambda\gamma_0$ contains $\gamma^{[n]}$ for every $n \in \mathbb{N}$; it remains to show that $S \cap \Lambda\gamma_0$ is Zariski-dense.

The Zariski closure Z of $\{\gamma^{[n]} \mid n \in \mathbb{N}\}$ satisfies $\gamma^{|\Gamma:\Lambda|} Z \subset Z$. Since the Zariski topology is Noetherian, it follows that $\gamma^{(m+1)|\Gamma:\Lambda|} Z = \gamma^{m|\Gamma:\Lambda|} Z$ for some $m \in \mathbb{N}$, and in turn that $\gamma \in Z$.

We have seen that S' is Zariski-dense, and that for each $\gamma' \in S'$, the set $\Lambda \cap U_{\gamma'}$ is Zariski-dense. In consequence, the set $S'' = \{(\gamma', \lambda) \in \Gamma \times \Gamma \mid \gamma' \in S', \lambda \in \Lambda \cap U_{\gamma'}\}$ is Zariski-dense in $\Gamma \times \Gamma$. Indeed, its closure contains $\overline{\{\gamma'\} \times \Lambda \cap U_{\gamma'}} = \{\gamma'\} \times \Gamma$ for each $\gamma' \in S'$, therefore contains $\overline{S' \times \{\gamma\}} = \Gamma \times \{\gamma\}$ for each $\gamma \in \Gamma$.

Since the conjugation map $\mathbf{H} \times \mathbf{H} \to \mathbf{H} : (x, y) \mapsto y^{-1}xy$ is dominant, it sends S'' to a Zariski-dense subset of Γ. Following the thread of the argument, we see that the Zariski closure of $S \cap \Lambda\gamma_0$ contains the image of S''. This proves the theorem. □

Remark 3.16. Each of the two properties assumed in Theorem 3.2 can be satisfied individually. Given a finitely generated Zariski-dense subgroup of a (connected) semisimple algebraic group, the existence of a local field and a representation satisfying the proximality property was first shown by Tits (see the proof of [83, Proposition 4.3]). A refinement to non-connected simple groups can also be found in [60, Theorem 1].

The second property, transversality, can be established for a given non-central element using representation-theoretic techniques. However, it is not always possible to find a representation that works for all $h \in A_i \cup B_i$ at the same time.

Even so, it may not always be possible to find a single representation which satisfies both properties of Theorem 3.2 simultaneously. Our next task will be to construct for real inner forms of SL_n and $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_n)$ a representation which does. This will be sufficient for the applications appearing in §4 & §5.

3.4. **Constructing a proximal and transverse representation for inner $\mathbb{R}$-forms of SL_n and GL_n.** Let D be a finite division $\mathbb{R}$-algebra and set $d = \dim_{\mathbb{R}} D$. Let $n \geq 2$ and let $\mathbf{H}$ be any algebraic $\mathbb{R}$-group in the isogeny class of SL_{D^n} or GL_{D^n}, viewing D^n as a right D-module. For example, if $D = \mathbb{C}$ this means that $\mathbf{H}$ is a quotient of the $\mathbb{R}$-group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_n)$ or $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_n)$ by a (finite) central subgroup. The *standard projective representation* of $\mathbf{H}$ is the canonical morphism $\rho_{\mathrm{st}} : \mathbf{H} \to \mathrm{PGL}_{D^n}$. This is the projective representation which will exhibit both proximal and transverse elements.

First, we recall that an element $g \in \mathbf{G}(\mathbb{R})$, in some reductive $\mathbb{R}$-group $\mathbf{G}$, is called $\mathbb{R}$-*regular* if the number of eigenvalues (counted with multiplicity) of $\mathrm{Ad}(g)$ of absolute value 1 is minimal. Any $\mathbb{R}$-*regular* element is semisimple (see [69, Remark 1.6.1]), and when $\mathbf{G}$ is split, every $\mathbb{R}$-regular element is regular.

With $\mathbf{H}$ as specified above, an element $g \in \mathbf{H}(\mathbb{R})$ is $\mathbb{R}$-regular if and only if some (any) representative of $\rho_{\mathrm{st}}(g)$ in $\mathrm{GL}_{D^n}(\mathbb{R})$ is conjugate to a diagonal n-by-n matrix with entries in D of distinct absolute values. Indeed, if $\rho_{\mathrm{st}}(g)$ is represented by $\mathrm{diag}(a_1, \ldots, a_n)$ with $|a_i| \neq |a_j|$ for $i \neq j$, the absolute values of the eigenvalues of $\mathrm{Ad}(g)$ are $\{|a_i a_j^{-1}|\}_{1 \leq i,j \leq 1}$ (with the correct multiplicities) and are equal to 1 only for $i = j$, which are the least possible occurrences. Conversely, if g is $\mathbb{R}$-regular, the centralizer of the $\mathbb{R}$-regular element $\rho_{\mathrm{st}}(g)$ contains a unique maximal $\mathbb{R}$-split torus $\mathbf{S}$ of PGL_{D^n} (see [69, Lemma 1.5]). Thus $\rho_{\mathrm{st}}(g)$ belongs to the centralizer of $\mathbf{S}(\mathbb{R})$, which, up to conjugation, is the subgroup of (classes of) diagonal n-by-n matrices with entries in D; say $\rho_{\mathrm{st}}(g)$ is represented by $\mathrm{diag}(a_1, \ldots, a_n)$. The absolute values of the eigenvalues of $\mathrm{Ad}(g)$ are again $\{|a_i a_j^{-1}|\}_{1 \leq i,j \leq 1}$. From the $\mathbb{R}$-regularity of $\rho_{\mathrm{st}}(g)$, we deduce that each value $|a_i a_j^{-1}|$ with $i \neq j$ must differ from 1, as claimed.

It follows from this description that if $\ell_{\max}$ (resp. $\ell_{\min}$) denotes the D-line in D^n on which a $\mathbb{R}$-regular element $g \in \mathbf{H}(\mathbb{R})$ acts by multiplication by an element of $D^\times$ of largest (resp. smallest) absolute value, then $\ell_{\max} = \mathrm{A}(g)$ is the attracting subspace of g (resp. $\ell_{\min} = \mathrm{A}(g^{-1})$), so that g is biproximal.[5] We record this here.

Lemma 3.17. *Let $\mathbf{H}$ and ρ_{st} be as above. Any $\mathbb{R}$-regular element $g \in \mathbf{H}(\mathbb{R})$ is biproximal under ρ_{st}.*

So, in order to exhibit proximal elements in $\rho_{\mathrm{st}}(\Gamma)$ for $\Gamma \leq \mathbf{H}(\mathbb{R})$ a Zariski-dense subgroup, it suffices to show Γ admits a $\mathbb{R}$-regular element. This is the content of the following theorem, due to Benoist and Labourie [8, A.1 Théorème]. We also refer the reader to the direct proof given by Prasad in [68].

Theorem 3.18 (Abundance of $\mathbb{R}$-regular elements, A.1 Théorème in [8])**.** *Let $\mathbf{G}$ be a reductive $\mathbb{R}$-group. Let Γ be a Zariski-dense subgroup of $\mathbf{G}(\mathbb{R})$. The subset of $\mathbb{R}$-regular elements in Γ is Zariski-dense.*

[5] Conversely, there exists a representation under which any proximal element is $\mathbb{R}$-regular, see [69, Lemma 3.4].

Corollary 3.19. *Let* $\mathbf{H}$ *and* ρ_{st} *be as above. Let* Γ *be a Zariski-dense subgroup of* $\mathbf{H}(\mathbb{R})$*. The elements* $g \in \Gamma$ *such that* $\rho_{\mathrm{st}}(g)$ *is biproximal, form a Zariski-dense subset of* Γ*.*

Remark 3.20. The existence of elements proximal under ρ_{st} in any Zariski-dense sub(semi)group can also be established using the results of Goldsheid and Margulis [32, Theorem 6.3] (see also [1, 3.12–14]). This approach is more tedious, as the standard representation of GL_{D^n} does not admit proximal elements if D^n is seen as a vector $\mathbb{R}$-space (which is in fact one of the motivations to extend the framework of [83] to division algebras). Instead, one should embed $\mathbf{P}_D(D^n)$ inside $\mathbf{P}_{\mathbb{R}}(\bigwedge_{\mathbb{R}}^d D^n)$ via the Plücker embedding, and exhibit proximal elements in that projective representation.

Next, we move on to the question of transversality. It turns out that under ρ_{st}, every non-central element $h \in \mathbf{H}(\mathbb{R})$ satisfies the transversality condition of Theorem 3.2.

Proposition 3.21. *Let* $\mathbf{H}$ *and* ρ_{st} *be as above. Let* $h \in \mathbf{H}(\mathbb{R})$ *be non-central. For every* $p \in \mathbf{P}(D^n)$*, the span of* $\{\rho_{\mathrm{st}}(xhx^{-1})p \mid x \in \mathbf{H}(\mathbb{R})\}$ *is the whole of* $\mathbf{P}(D^n)$*.*

Proof. Taking preimages in GL_{D^n}, we may without loss of generality work with the action of GL_{D^n} on D^n instead of $\rho_{\mathrm{st}}(\mathbf{H}) = \mathrm{PGL}_{D^n}$ on $\mathbf{P}(D^n)$. We will show in this setting that, for every non-zero $v \in D^n$ and every non-central $h \in \mathrm{GL}_{D^n}(\mathbb{R})$, the $\mathbb{R}$-span of $\{xhx^{-1} \cdot v \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ is the whole of D^n. The statement of the proposition then follows immediately by projectivization.

Viewing $\mathrm{End}_D(D^n)$ as a vector $\mathbb{R}$-space, the conjugation action defines a linear representation of SL_{D^n} on $\mathrm{End}_D(D^n)$. This representation decomposes into two irreducible components: a copy of the trivial representation given by the action of SL_{D^n} on the center of $\mathrm{End}_D D^n$, and a copy of the adjoint representation given by the action of SL_{D^n} on the subspace $\mathfrak{sl}_n(D)$ of traceless endomorphisms.

When h is not central, it admits a distinct conjugate xhx^{-1} of the same trace, hence the $\mathbb{R}$-span W_h of $\{xhx^{-1} \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ contains for some $g \in \mathrm{SL}_{D^n}(\mathbb{R})$ the nonzero traceless element $h' = h - ghg^{-1}$. In turn, W_h contains the $\mathbb{R}$-span $W_{h'}$ of $\{xh'x^{-1} \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$, a SL_{D^n}-stable subspace of $\mathfrak{sl}_n(D)$ which must equal $\mathfrak{sl}_n(D)$, as the latter is irreducible for the adjoint action. Thus, either $W_h = \mathfrak{sl}_n(D)$ if $\mathrm{Tr}(h) = 0$, or $W_h = \mathrm{End}_D(D^n)$ if $\mathrm{Tr}(h) \neq 0$.

Finally, for any non-zero $v \in D^n$ we have that $\mathfrak{sl}_n(D) \cdot v = D^n$, from which we conclude that the $\mathbb{R}$-span of $\{xhx^{-1} \cdot v \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ contains $W_h \cdot v = D^n$. ☐

Definition 3.22. Given a reductive F-group $\mathbf{G}$ with center $\mathbf{Z}$ and a subgroup $H \leq \mathbf{G}(F)$, for the purposes of this paper, we will say that H *almost embeds in a (simple, adjoint) quotient* $\mathbf{Q}$ *of* $\mathbf{G}$ if there exists a (simple, adjoint) quotient $\mathbf{Q}$ of $\mathbf{G}$ for which the kernel of the restriction $H \to \mathbf{Q}(F)$ is contained in $\mathbf{Z}(F)$.

It is clear that if $\mathbf{Q}$ is a simple factor of $\mathbf{G}$ and H is a subgroup of $\mathbf{Q}(F)$, then H almost embeds in $\mathbf{Q}$. In particular, if $\mathbf{G}$ is itself simple, every subgroup almost embeds in a simple quotient.

With this, we are ready to prove the following application of Theorem 3.2, establishing the abundance of simultaneous ping-pong partners for finite subgroups in products of inner forms of SL_n and GL_n which almost embed in a factor.

Theorem 3.23. *Let* $\mathbf{G}$ *be a reductive* $\mathbb{R}$*-group with center* $\mathbf{Z}$*. Let* Γ *be a subgroup of* $\mathbf{G}(\mathbb{R})$ *whose image in* $\mathrm{Ad}\,\mathbf{G}$ *is Zariski-dense. Let* $(A_i, B_i)_{i \in I}$ *be a finite collection of finite subgroups of* $\mathbf{G}(\mathbb{R})$*.*

Suppose that for each $i \in I$*, there exists a quotient of* $\mathbf{G}$ *of the form* $\mathrm{PGL}_{D_i^{n_i}}$ *for* D_i *some finite division* $\mathbb{R}$*-algebra and* $n_i \geq 2$*, for which the kernels of the restrictions* $A_i, B_i \to \mathrm{PGL}_{D_i^{n_i}}(\mathbb{R})$ *are contained in* $\mathbf{Z}(\mathbb{R})$*. Then the set of regular semisimple elements*

$\gamma \in \Gamma$ *of infinite order such that for all* $i \in I$*, the canonical maps*

$$\langle \gamma, C_{A_i} \rangle *_{C_{A_i}} A_i \to \langle \gamma, A_i \rangle \qquad \text{where } C_{A_i} = A_i \cap \mathbf{Z}(F),$$
$$\langle \gamma, C_{B_i} \rangle *_{C_{B_i}} B_i \to \langle \gamma, B_i \rangle \qquad \text{where } C_{B_i} = B_i \cap \mathbf{Z}(F),$$

and provided $|A_i : C_{A_i}| > 2$ *or* $|B_i : C_{B_i}| > 2$*, also*

$$\langle A_i, C_i \rangle *_{C_i} \langle B_i, C_i \rangle \to \langle A_i, \gamma B_i \gamma^{-1} \rangle \qquad \text{where } C_i = C_{A_i} \cdot C_{B_i},$$

are all isomorphisms, is dense in Γ *for the join of the profinite topology and the Zariski topology.*

Proof. For $i \in I$, let ρ_i denote the quotient map $\mathbf{G} \to \mathrm{PGL}_{D_i^{n_i}}$ afforded by the statement. The morphism ρ_i is an irreducible projective representation of $\mathbf{G}$ over D_i. Note that ρ_i factorizes via $\mathbf{G} \to \mathrm{Ad}\,\mathbf{G} \to \mathrm{PGL}_{D_i^{n_i}}$. In particular the image of Γ in $\mathrm{PGL}_{D_i^{n_i}}$ is Zariski-dense.

Corollary 3.19 shows that the set of elements in $\rho_i(\Gamma)$ which are biproximal is Zariski-dense in $\mathrm{PGL}_{D_i^{n_i}}$; a fortiori, $\rho_i(\Gamma)$ contains a proximal element.

Moreover, since by assumption the kernel of the restriction of ρ_i to A_i (resp. B_i) is contained in $\mathbf{Z}(\mathbb{R})$, every $h \in (A_i \cup B_i) \setminus \mathbf{Z}(\mathbb{R})$ maps to a non-trivial element in $\mathrm{PGL}_{D_i^{n_i}}(\mathbb{R})$. Proposition 3.21 then precisely states that ρ_i satisfies the transversality condition of Theorem 3.2 (see Remark 3.4). We are thus at liberty to apply Theorem 3.2 to $\Gamma \leq \mathbf{G}(\mathbb{R})$ and the collection $(A_i, B_i)_{i \in I}$, deducing this theorem. □

Remark 3.24. Let F be any field, and let $\mathbf{G}$ be a reductive F-group with center $\mathbf{Z}$. In order for a subgroup $H \leq \mathbf{G}(F)$ to admit a ping-pong partner in $\mathbf{G}(F)$, it is necessary that H embeds in an adjoint simple factor. In fact, if the subgroup $\langle \gamma, H \rangle$ is the free amalgamated product of $\langle \gamma, C \rangle$ and H over $C = H \cap \mathbf{Z}(F)$, then in the quotient $\mathbf{G}/\mathbf{Z}$, the image of $\langle \gamma, H \rangle$ is the free product of the images of $\langle \gamma \rangle$ and H. Indeed, the kernel of the quotient map being central in $\langle \gamma, C \rangle *_C H$, it lies in C, hence coincides with C. But $\mathbf{G}/\mathbf{Z}$ is the direct product of adjoint simple quotients of $\mathbf{G}$, so by Proposition 2.7, H/C embeds in (the F-points of) one of these factors.

Similarly, if A and B are such that $\langle A, B \rangle \leq \mathbf{G}(F)$ is the free amalgamated product of $\langle A, C \rangle$ and $\langle B, C \rangle$ over $C = \langle A \cap \mathbf{Z}(F), B \cap \mathbf{Z}(F) \rangle$, then in the quotient $\mathbf{G}/\mathbf{Z}$, the image of $\langle A, B \rangle$ is again the free product of the images of A and B. By Proposition 2.7 once more, A/C and B/C both embed in (the F-points of) a common adjoint simple factor.

In other words, Theorem 3.23 states that a collection of finite subgroups $(H_i)_{i \in I}$ in an $\mathbb{R}$-group $\mathbf{G}$ whose simple quotients are each isogenous to some $\mathrm{PGL}_n(D)$ with $n \geq 2$, admits simultaneous ping-pong partners in Γ *if and only if* each H_i embeds in an adjoint simple factor. Similarly, two finite subgroups A, B of $\mathbf{G}(F)$ admit conjugates by Γ which play ping-pong *if and only if* A and B embed in the same adjoint simple factor.

Remark 3.25. There are versions of Theorem 3.23 for semisimple $\mathbb{R}$-groups of other types, but proving them requires a more careful study of the representation theory of $\mathbf{G}$ to exhibit a representation playing the role of ρ_{st}. However, as indicated in Remark 3.16, there are also cases where one needs additional information on the H_i to get a representation satisfying the transversality assumption of Theorem 3.2.

There are also versions of the theorem for other local fields. However, to prove those one needs additional information on Γ. Indeed, over a local field different from $\mathbb{R}$, bounded Zariski-dense subgroups exist, and a bounded subgroup obviously never admits proximal elements. Nevertheless, in the extreme case where $\mathbf{G}$ is defined over a number field F and $\Gamma = \mathbf{G}(F)$, we prove below a version of Theorem 3.23 using the same method, which only requires to assume $n_i \geq 2$ when D_i is a field.

Theorem 3.26. *Let $\mathbf{G}$ be a reductive group defined over a number field F with center $\mathbf{Z}$. Let $(A_i, B_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(F)$.*

Suppose that for each $i \in I$, there exists a (non-trivial) simple quotient of $\mathbf{G}$ of the form $\mathrm{PGL}_{D_i^{n_i}}$ for D_i some finite division F-algebra, for which the kernels of the restrictions $A_i, B_i \to \mathrm{PGL}_{D_i^{n_i}}(F)$ are contained in $\mathbf{Z}(F)$. Then the set of regular semisimple elements $\gamma \in \mathbf{G}(F)$ of infinite order such that for all $i \in I$, the canonical maps

$$\langle \gamma, C_{A_i}\rangle *_{C_{A_i}} A_i \to \langle \gamma, A_i\rangle \qquad \text{where } C_{A_i} = A_i \cap \mathbf{Z}(F),$$
$$\langle \gamma, C_{B_i}\rangle *_{C_{B_i}} B_i \to \langle \gamma, B_i\rangle \qquad \text{where } C_{B_i} = B_i \cap \mathbf{Z}(F),$$

and provided $|A_i : C_{A_i}| > 2$ or $|B_i : C_{B_i}| > 2$, also

$$\langle A_i, C_i\rangle *_{C_i} \langle B_i, C_i\rangle \to \langle A_i, \gamma B_i \gamma^{-1}\rangle \qquad \text{where } C_i = C_{A_i} \cdot C_{B_i},$$

are all isomorphisms, is dense in $\mathbf{G}(F)$ for the join of the profinite topology and the Zariski topology.

Proof. For $i \in I$, let ρ_i denote the quotient map $\mathbf{G} \to \mathrm{PGL}_{D_i^{n_i}}$ afforded by the statement.

Let S_0 be the finite (possibly empty) set of places of F where the weak approximation property fails for $\mathbf{G}$ (see [64, §7.3 & Theorem 7.7]). Pick for each $i \in I$ a completion K_i of F, not belonging to S_0, and over which the division algebra D_i splits; this means that $\mathrm{PGL}_{D_i^{n_i}}$ becomes isomorphic to $\mathrm{PGL}_{n_i d_i}$ over K_i, for $d_i = \deg_F(D_i)$.

For each $i \in I$, the set R of K_i-regular elements in $\mathrm{PGL}_{n_i d_i}(K_i)$ (that is, of elements g for which the number of eigenvalues of $\mathrm{Ad}(g)$ of absolute value 1 is minimal), is open in the local topology. Moreover, R obviously intersects the image of the simply connected group $\mathrm{SL}_{n_i d_i}(K_i)$, as witnessed by any diagonal matrix of determinant 1 whose entries have pairwise distinct absolute values. In consequence, the preimage $\rho_i^{-1}(R)$ is a non-empty, open subset of $\mathbf{G}(K_i)$ (for the local topology). By weak approximation, $\mathbf{G}(F)$ is dense in $\mathbf{G}(K_i)$, hence we can find an element $\gamma_i \in \mathbf{G}(F) \cap \rho_i^{-1}(R)$. By construction, γ_i is K_i-regular, hence biproximal, under ρ_i.

In order to apply Theorem 3.2 to $\Gamma = \mathbf{G}(F)$, the finite groups $(A_i, B_i)_{i\in I}$, and the pairs $(K_i, \rho_i)_{i\in I}$, it only remains to check the transversality condition. Taking into account Remark 3.4, the latter is a direct consequence of the homologue of Proposition 3.21 over a general local field, whose proof is identical (having replaced $\mathbb{R}$ by said local field). ☐

As highlighted in Remark 3.24, the existence of a free amalgamated product $A *_C B$ in $\mathbf{G}$ is in general subject to the existence of almost embeddings of A and B in an appropriate common factor of $\mathrm{Ad}\,\mathbf{G}$. Of course, if $\mathrm{Ad}\,\mathbf{G}$ is a simple group to begin with, this embedding condition is void.

In this regard, Theorem 3.26 implies at once that if $\mathbf{G}$ is isogenous to PGL_{D^n} with D a finite division F-algebra, and A, B are finite subgroups of $\mathbf{G}(F)$, then there exists $\gamma \in \mathbf{G}(F)$ for which $\langle A, \gamma B \gamma^{-1}\rangle$ is freely amalgamated along $C = (A \cap \mathbf{Z}(F)) \cdot (B \cap \mathbf{Z}(F))$.

One can wonder whether in the same setting, there is a conjugate of B for which the subgroup $\langle A, \gamma B\gamma^{-1}\rangle$ is freely amalgamated along $A \cap B$ instead. This fairly natural question currently eludes us.

4. Free products between finite subgroups of units in a semisimple algebra

Conventions: throughout the remainder of this article, all orders will be understood to be $\mathbb{Z}$-orders. We also use the following notations: Update this at the end.

- Whenever we say that a given F-algebra M is a finite algebra we mean that M is finite dimensional over F.

- $\mathrm{PCI}(M)$ is the set of primitive central idempotents of a finite (semisimple) algebra M. Each $e \in \mathrm{PCI}(M)$ corresponds to an irreducible factor algebra Me of M, and $\pi_e : M \twoheadrightarrow Me$ denotes the projection of M onto Me.
- $\mathcal{U}(M) = M^\times = \mathrm{GL}_1(M) = \mathrm{GL}_M(F)$ all denote the group of units of the F-algebra M.
- RG denotes the group ring of G with coefficients in a ring R and $\mathcal{V}(RG)$ the group of units in RG whose augmentation equals 1.

4.1. **Simultaneous partners in the unit group of an order.** By Wedderburn's theorem, every semisimple F-algebra M factors as

$$M = \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_m),$$

for V_i an n_i-dimensional right module over some finite division F-algebra D_i, $i = 1, \dots m$. In consequence, the F-group of units of M identifies with the reductive group

$$\mathbf{G} = \mathrm{GL}_M = \mathrm{GL}_{D_1^{n_1}} \times \cdots \times \mathrm{GL}_{D_m^{n_m}},$$

whose adjoint is

$$\mathrm{Ad}\,\mathbf{G} = \mathrm{PGL}_M = \mathrm{PGL}_{D_1^{n_1}} \times \cdots \times \mathrm{PGL}_{D_m^{n_m}}.$$

The factors of $\mathbf{G}$ are in one-to-one correspondence with those of M, while the non-trivial factors of $\mathrm{Ad}\,\mathbf{G}$ correspond to the factors of M which are not fields.

The original motivation for this project was the study of free amalgamated products inside $\mathcal{U}(\mathcal{O})$, the unit group of an order $\mathcal{O}$ in M; more precisely, the aim was to answer Conjecture 5.24. In this context, the following neat necessary and sufficient condition for a given finite subgroup of $\mathcal{U}(\mathcal{O})$ to admit a ping-pong partner, is a direct consequence of Theorem 3.23.

Theorem 4.1. *Let F be a number field, M be a finite semisimple F-algebra, and $\mathcal{O}$ be an order in M. Let Γ be a subgroup of $\mathcal{U}(\mathcal{O})$ for which the Zariski closure of the image of Γ in PGL_M contains the Zariski-connected component of the image of $\mathcal{U}(\mathcal{O})$.*[6] *Let A and B be finite subgroups of $\mathcal{U}(\mathcal{O})$, and assume that $|A : A \cap \mathcal{Z}(M)| > 2$.*

There exists $\gamma \in \Gamma$ of infinite order with the property that the canonical maps

$$\begin{aligned} \langle \gamma, C_A \rangle *_{C_A} A &\to \langle \gamma, A \rangle && \text{where } C_A = A \cap \mathcal{Z}(M), \\ \langle \gamma, C_B \rangle *_{C_B} B &\to \langle \gamma, B \rangle && \text{where } C_B = B \cap \mathcal{Z}(M), \\ \langle A, C \rangle *_C \langle B, C \rangle &\to \langle A, \gamma B \gamma^{-1} \rangle && \text{where } C = C_A \cdot C_B, \end{aligned}$$

are all isomorphisms, if and only if A and B almost embeds in PGL_{Me} for some common $e \in \mathrm{PCI}(M)$ for which Me is neither a field nor a totally definite quaternion algebra.

Moreover, in the affirmative, the set of such elements γ which are regular semisimple, is dense in Γ for the join of the Zariski and the profinite topology.

Proof. We can base-change $\mathbf{G} = \mathrm{GL}_M$ to the $\mathbb{R}$-group $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$, whose $\mathbb{R}$-points $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ are a product of groups of the form $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, or $\mathrm{GL}_n(\mathbb{H})$, for various $n \geq 1$.

Any subgroup H of $\mathcal{U}(M) = \mathbf{G}(F)$ embeds in $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$. In fact, H (almost) embeds in a F-simple factor of $\mathbf{G}$ if and only if it does so in any $\mathbb{R}$-simple factor of $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$. More precisely, let $K_1, \dots, K_s$ denote the summands of the étale $\mathbb{R}$-algebra $F \otimes_{\mathbb{Q}} \mathbb{R}$; they are precisely the different archimedean completions of F. Given a finite division algebra D over F, let D_{ij} be the division $\mathbb{R}$-algebras such that $D \otimes_F K_i \cong \prod_{j=1}^{m_i} \mathrm{M}_{r_{ij}}(D_{ij})$ as $\mathbb{R}$-algebras. The group $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_{D^n} \times_{\mathbb{Q}} \mathbb{R}$ then factors into the product $\prod_{i=1}^{s} \prod_{j=1}^{m_i} \mathrm{GL}_{D_{ij}^{nr_{ij}}}$.

[6] This condition is satisfied if Γ is Zariski-dense in $\mathcal{U}(\mathcal{O}) = \mathrm{GL}_1(\mathcal{O})$ or in $\mathrm{SL}_1(\mathcal{O})$, e.g. Γ is a finite-index subgroup of $\mathcal{U}(\mathcal{O})$.

The image of $\mathrm{GL}_{D^n}(F)$ in this product is obtained by embedding it diagonally using the canonical maps $\mathrm{GL}_{D^n}(F) \to \mathrm{GL}_{D^n}(K_i) \to \mathrm{GL}_{D_{ij}^{nr_{ij}}}(\mathbb{R})$. Thus if H (almost) embeds in a factor PGL_{D^n} over F, then it does so in any of the $\mathrm{PGL}_{D_{ij}^{nr_{ij}}}$ over $\mathbb{R}$, and the converse is obvious.

Now, an adjoint simple quotient $\mathrm{PGL}_{D_{ij}^{nr_{ij}}}$ over $\mathbb{R}$ of a given factor GL_{D^n} of $\mathbf{G}$ satisfies $nr_{ij} = 1$, if and only if the jth factor in $Me \otimes_F K_i$ is a division algebra, where e is the projection onto the factor of M corresponding to GL_{D^n}. In other words, the factor GL_{D^n} has a simple quotient $\mathrm{PGL}_{D_{ij}^{nr_{ij}}}$ with $nr_{ij} \geq 2$ for some i, j, if and only if Me is not a division algebra that remains so under every archimedean completion of its center. This amounts in turn to Me not being a field nor a totally definite quaternion algebra.

Next, let $\mathbf{G}_{\mathrm{is}}$ denote the $\mathbb{R}$-subgroup of $\mathrm{Res}_{F/\mathbb{Q}}\, \mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$ which is the direct product of those subgroups $\mathrm{GL}_{D_{ij}^{nr_{ij}}}$ for which $nr_{ij} \geq 2$. Since $\mathcal{U}(\mathcal{O})$ is an arithmetic subgroup of $\mathcal{U}(M) = \mathbf{G}(F)$, a classical theorem of Borel and Harish-Chandra [9, Theorem 7.8] attests that the connected component of $\mathcal{U}(\mathcal{O})$ in $\mathrm{Res}_{F/\mathbb{Q}}\, \mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$ is a lattice in the derived subgroup $\mathcal{D}\mathbf{G}_{\mathrm{is}}$ of $\mathbf{G}_{\mathrm{is}}$. In consequence, the image of $\mathcal{U}(\mathcal{O})$ hence of Γ in $\mathrm{Ad}\, \mathbf{G}_{\mathrm{is}}$ is Zariski-dense. Let f denote the canonical map $\mathbf{G}(\mathbb{R}) \to \mathbf{G}_{\mathrm{is}}(\mathbb{R})$, whose kernel is the product of the compact factors of $\mathbf{G}(\mathbb{R})$. Note that $\ker f$ commutes with $\mathbf{G}_{\mathrm{is}}(\mathbb{R})$, and that $\ker f \cap \Gamma$ is finite.

In view of all the above, given that the subgroups A and B satisfy the embedding condition stated in the theorem, we deduce from Theorem 3.23 applied to $\mathbf{G}_{\mathrm{is}}$ the existence of a dense set $S \subset f(\Gamma)$ of ping-pong partners for $f(A)$ and $f(B)$. By Lemma 2.3, the preimage $f^{-1}(S) \cap \Gamma$ consists of elements $\gamma \in \Gamma$ for which the canonical maps

$$\langle \gamma, C_A \rangle *_{C_A} A \to \langle \gamma, A \rangle, \quad \langle \gamma, C_B \rangle *_{C_B} B \to \langle \gamma, B \rangle, \quad \langle A, C \rangle *_C \langle B, C \rangle \to \langle A, \gamma B \gamma^{-1} \rangle$$

are isomorphisms.

As S is dense in the join of the Zariski and the profinite topology, the same holds for $f^{-1}(S) \cap \Gamma$. Indeed, if $\Lambda\gamma_0$ is a coset of finite index in Γ, and U is a Zariski-open subset of Γ intersecting it, perhaps after shrinking and translating by $\ker f \cap \Gamma$, we can arrange that $\Lambda\gamma_0$ and U are contained in the connected component Γ° of Γ, and that $(\ker f \cap \Gamma^\circ) \cdot U = U$. Then $f(\Lambda\gamma_0 \cap U)$ equals the open set $f(\Lambda\gamma_0) \cap f(U)$. We may thus pick $x \in S \cap f(\Lambda\gamma_0 \cap U)$, implying that $f^{-1}(S) \cap \Lambda\gamma_0 \cap U$ is non-empty.

It remains to verify that the embedding condition is necessary. Suppose $\gamma \in \Gamma$ is such that the canonical map $A *_C B \to \langle A, \gamma B \gamma^{-1} \rangle$ is an isomorphism. Let $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) denote the product of the factors of $\mathbf{G}$ over F for which the corresponding factor Me of M is not (resp. is) a field or a totally definite quaternion algebra. Because this product decomposition is defined over F, the projections of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_1(F \otimes_{\mathbb{Q}} \mathbb{R})$ and $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ are discrete. Since $\mathcal{D}\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact, the image of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ is in fact finite.

As $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$, Proposition 2.7 shows that one of the kernels N_1, N_2 of the respective projections $\pi_i : \langle A, \gamma B \gamma^{-1} \rangle \to \mathbf{G}_i(F \otimes_{\mathbb{Q}} \mathbb{R})$, is contained in C. Of course, N_2 can not be contained in C, otherwise the image of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ would contain the infinite group $(A/N_2) *_{C/N_2} (B/N_2)$. We deduce that $N_1 \subset C$, that is, $\langle A, \gamma B \gamma^{-1} \rangle$ almost embeds in $\mathbf{G}_1$. Another application of Proposition 2.7 then shows that $\langle A, \gamma B \gamma^{-1} \rangle$ almost embeds in some factor of $\mathrm{Ad}\, \mathbf{G}_1$ over F, associated with a factor of M which is not a field nor a totally definite quaternion algebra as claimed. □

Remark 4.2. In the unfortunate event that $|A : A \cap \mathcal{Z}(M)| = |B : B \cap \mathcal{Z}(M)| = 2$, one can draw no conclusion about the map $\langle A, C \rangle *_C \langle B, C \rangle \to \langle A, \gamma B \gamma^{-1} \rangle$. However, having removed this line from the statement, the remainder of Theorem 4.1 still holds (with the same proof). Indeed, in the same manner as in the proof, Proposition 2.7 can be used to show that the existence of a free product $\langle\gamma\rangle *_{C_A} A$ inside $\mathcal{U}(\mathcal{O})$ implies that A almost embeds inside PGL_{Me} for some factor Me which is neither a field nor a totally definite quaternion algebra.

Thus, formally speaking, if there were a free product of the form $\langle\gamma\rangle * A$ in $\mathcal{U}(\mathcal{O})$, one could infer that A embeds in an appropriate quotient PGL_{Me} of GL_M, and then apply Theorem 4.1 with $A = B$ to deduce the existence of $A * A$ in $\mathcal{U}(\mathcal{O})$. Similarly, if $A * B$ occurs in $\mathcal{U}(\mathcal{O})$, one could infer that A and B embed in an appropriate quotient PGL_{Me}, and then apply Theorem 4.1 to deduce that $\langle\gamma\rangle * A$ and $\langle\gamma\rangle * B$ also occur. But of course both these deductions are self-evident, as $\langle A, \gamma A \gamma^{-1}\rangle$ is a freely generated subgroup of $\langle\gamma\rangle * A$, and the same holds for $\langle ab, A\rangle$ and $\langle ab, B\rangle$ in $A * B$ (for non-trivial $a \in A$, $b \in B$).

We should probably remove the following remark, as its conclusion is not a straightforward generalization. There are some subtleties with Zariski-dense subgroups in non-archimedean fields, already outlined in Remark 3.25. I'm leaving it up for the moment, so that you know the extent of what we can prove.

Remark 4.3. There is an S-arithmetic version of Theorem 4.1, in which a ping-pong partner γ for A and B exists in a Zariski-dense subgroup Γ of an S-arithmetic order $\mathcal{O}[S^{-1}]$, if and only if over some completion F_v with $v \in S$, A and B almost embed in a common isotropic factor of $\mathrm{PGL}_M(F_v)$ in which Γ admits proximal elements. The proof is essentially the same, but requires an S-arithmetic version of Theorem 3.26, which in turn requires a more careful analysis of the S-arithmetic properties of Γ.

Example 4.4. Update this for $A * B$? If $A = FG$ and $\mathcal{O} = RG$ for some order R in the number field F and G a finite group, then by a theorem of Berman and Higman [50, Theorem 2.3] the only torsion central units are the trivial ones (i.e. the elements of $\mathcal{U}(R) \cdot \mathcal{Z}(G)$). Thus if we take $H \leq \mathcal{V}(RG)$, then $C = H \cap \mathcal{Z}(G)$. In particular, $G * \mathbb{Z}$ exists if and only if G itself embeds in a simple factor and intersects the center of $\mathcal{U}(R)$ trivially (this happens when G is simple, for instance).

Although Theorem 3.23 and Theorem 4.1 are nice existence results, they leave open the following two questions.

Questions 4.5. With the notation of Theorem 4.1:

(i) When do A and B (almost) embed in a simple factor of PGL_M? If so, do they (almost) embed in a common simple factor of PGL_M?

(ii) How can we construct a ping-pong partner γ concretely?

In the remainder of this section, we present a method to approach question (ii), which will reduce the problem to constructing certain *deformations* of A or B (see Definition 4.6); the main result is Theorem 4.12. In Section 5.2, we will propose a construction of such deformations when M is a group algebra. Question (i) will be addressed in Section 5.1.

4.2. **Deforming finite subgroups and subalgebras.** Keeping the notation of Section 4.1, the aim of this section is to introduce an explicit linear method that allows to replace a finite subgroup $H \leq \mathcal{U}(M)$ by an isomorphic copy which has the necessary ping-pong dynamics. Concretely, we want to construct a group morphism of the form

$$\Delta : H \to \mathcal{U}(M) : h \mapsto \Delta(h) = h + \delta_h.$$

This is possible when the map $\delta = \Delta - 1$ satisfies the following conditions.

Definition 4.6. Let H be a subgroup of $\mathcal{U}(M)$. We call an F-linear map $\Delta : H \to M$ *a first-order deformation of* H if the map $\delta = \Delta - 1$ satisfies the following conditions:

(Derivation) $\delta_{hk} = \delta_h k + h\delta_k$ for all $h, k \in H$;

(Order 1) $\delta_h \delta_k = 0$ for all $h, k \in H$.

A straightforward calculation shows that if Δ is a first-order deformation of H, then the maps $\Delta_t(h) = h + t\delta_h$ for $t \in F$ are group morphisms from H to $\mathcal{U}(M)$, interpolating between the identity Δ_0 and $\Delta_1 = \Delta$. In fact, if a linear map $\Delta : H \to M$ is a group morphism and $\delta = \Delta - 1$ satisfies either (Derivation) or (Order 1), then δ also satisfies the remaining property. Moreover, since Δ_t is assumed to be linear, it extends uniquely to an algebra morphism $FH \to M$. We define a *first-order deformation* of a subalgebra N of M analogously, so that first-order deformations of subalgebras are algebra morphisms, and the linear extension of a deformation of H is a deformation of FH. We say that Δ is an *inner (first-order) deformation* when the derivation δ is inner over M, that is, when $\delta_h = [n, h]$ for some $n \in M$.

Examples 4.7. Let $H \leq \mathcal{U}(M)$.

(i) If n is an element of M satisfying $nhn = 0$ for all $h \in H$, then the assignment

$$\delta_h = [n, h]$$

defines a first-order deformation of H, which is actually given by the conjugation

$$\Delta(h) = (1 + n)h(1 + n)^{-1} = h + [n, h].$$

This deformation is inner by construction.

(ii) If $m \in M$ satisfies $mh = m$ for all $h \in H$, then the assignment

$$\delta_h = (1 - h)m$$

defines a first-order deformation of H. (The assignment $\delta_h = m(1-h) = 0$ defines the trivial deformation.)

Assume for a moment that H is finite and that $\operatorname{char} F$ does not divide $|H|$; set $e = \frac{1}{|H|}\sum_{h\in H} h$. Then this deformation is in fact of the first kind with $n = (1-e)m$, as $(1 - e)m \cdot h \cdot (1 - e)m = 0$ since $m(1 - e) = 0$, and

$$\delta_h = [(1 - e)m, h] = (1 - e)m - (h - e)m = (1 - h)m.$$

Note that under this additional assumption on H, the condition $mh = m$ for all $h \in H$ is equivalent to $me = m$. This deformation might be trivial, for instance when $m = e$, but if H is F-linearly independent[7] and not central, one can find some m for which it is not.

(iii) Any example of the second kind obviously satisfies $\delta_g h = \delta_g$ for all $g, h \in H$. The converse holds under the assumption that H is finite and that $\operatorname{char} F$ does not divide $|H|$. Set $e = \frac{1}{|H|}\sum_{h\in H} h$ as above. If $\Delta = 1 + \delta$ is a first-order deformation which happens to satisfy $\delta_g h = \delta_g$ for all $g, h \in H$, then the equation $\delta_e = \delta_{he} = \delta_h e + h\delta_e = \delta_h + h\delta_e$ implies that

$$\delta_h = (1 - h)\delta_e.$$

Thus, this deformation is of the second kind with $m = \delta_e$. Since $\delta_e = e\delta_e + \delta_e e$ implies $n = (1 - e)\delta_e = \delta_e$, we deduce as in the first two examples that

$$\Delta(h) = (1 + \delta_e)h(1 + \delta_e)^{-1}.$$

Note that when Δ is an inner first-order deformation, $\Delta(h) = h$ for every $h \in H \cap \mathcal{Z}(M)$. Also, examples (ii) and (iii) above can only occur if $H \cap F = \{1\}$.

Lemma 4.8. *The kernel of a first-order deformation $\Delta : H \to M$ consists of unipotent elements. In consequence, if H is finite and* $\operatorname{char} F$ *does not divide $|H|$, every first-order deformation $H \to M$ is injective.*

[7]This condition is natural for the applications later on. Indeed, it is well-known that if H is a finite subgroup of $\mathcal{V}(RG)$, with R a $|G|$-adapted ring (i.e. $|G|$ is not invertible in R), then H is F-linearly independent in FG for $F = \operatorname{Frac}(R)$.

Proof. An element h lies in the kernel of Δ if and only if $\delta_h = 1 - h$. By assumption, $(\delta_h)^2 = (1-h)^2 = 0$, showing that h is a unipotent element of $M^\times$.

Over a field of characteristic p (resp. 0), any non-trivial unipotent element has order p (resp. infinite order); so if H has no elements of order p nor ∞, the map Δ is injective. □

If H is infinite, or if $\operatorname{char} F$ divides $|H|$, there are first-order deformations which are not injective. For instance, assuming $\operatorname{char} F \neq 2$, the trivial map

$$H = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \middle| \, x \in F \right\} \to \mathrm{M}_2(F) : h \to 1$$

is an inner first-order deformation, associated with $\delta_h = [n, h] = 1 - h$ for $n = \left[\begin{smallmatrix} 1/2 & 0 \\ 0 & -1/2 \end{smallmatrix}\right]$.

In more generality, we have the following description of first-order deformations of semisimple subalgebras, which may be of independent interest.

Theorem 4.9. *Let M be an F-algebra, and let N be a separable subalgebra of M. Then every first-order deformation $\Delta : N \to M$ is given by conjugation, that is, there exists $a \in M$ such that $\Delta(b) = aba^{-1}$ for every $b \in N$. In particular, any such deformation extends to an automorphism of M and fixes $N \cap \mathcal{Z}(M)$.*

Remark 4.10. In Section 5, the F-algebra M will be taken finite-dimensional and semisimple, and the first-order deformations $\Delta : N \to M$ under consideration will restrict to ring morphisms between an order $\mathcal{O}_N$ of N and an order $\mathcal{O}_M$ of M. In that setting, it will be convenient to use the formalism of first-order deformations, despite the fact that in the end, they are given by conjugation in the ambient algebra M.

Proof of Theorem 4.9. Recall that the separability of N means that N admits a *separating idempotent*, that is, there is an element $e \in N \otimes_F N$ whose image under the multiplication map $\mu : N \otimes_F N \to N$ is 1 and which satisfies $(b \otimes 1)e = e(1 \otimes b)$ for every $b \in N$.

Let I denote the kernel of the (N, N)-bimodule map μ. Recall that the canonical map

$$d : N \to I : b \mapsto b \otimes 1 - 1 \otimes b$$

identifies I with the bimodule of (non-commutative) differentials of the algebra N. That is, for any derivation $\delta : N \to V$ to a (N, N)-bimodule V, there exists a unique bimodule map $f : I \to V$ such that $\delta = f \circ d$ (see [12, A III §10 N°10 Proposition 17]). Applying this to the derivation $\delta = \Delta - 1$, for Δ a first-order deformation $N \to M$, we obtain existence of a unique bimodule map $f : I \to M$ satisfying $\delta(b) = f(b \otimes 1 - 1 \otimes b)$ for every $b \in N$.

Note that $e - 1 \in I$ by construction. Inside M, we now compute

$$\begin{aligned} [f(e-1), b] &= f(e-1) \cdot b - b \cdot f(e-1) = f\big((e-1)(1 \otimes b) - (b \otimes 1)(e-1)\big) \\ &= f(e(1 \otimes b) - 1 \otimes b - e(1 \otimes b) + b \otimes 1) \\ &= f(b \otimes 1 - 1 \otimes b) = f(d(b)) = \delta(b), \end{aligned}$$

showing that the derivation $\delta : N \to M$ is inner, given by the adjoint of $f(e-1) \in M$.

The bimodule I of differentials of N is generated as a left (respectively right) module by the image of d (see [12, A III §10 N°10 Lemme 1]), hence the same holds for the image of f; in other words, $f(e-1) \in N \cdot \delta(B) = \delta(N) \cdot N$. As by assumption $\delta(N) \cdot \delta(N) = 0$, we conclude that $f(e-1)bf(e-1) = 0$ for every $b \in N$. This shows that

$$\Delta(b) = b + f(e-1)b - bf(e-1) = (1 + f(e-1))b(1 - f(e-1))$$

is given by conjugating b by $1 + f(e-1)$, proving the theorem. □

Remark 4.11. Recall that an F-algebra N is separable if and only if N is absolutely semisimple, which is in turn equivalent to N being semisimple with étale center (see respectively [11, A VIII §13 N°5 Théorème 2 & N°3 Théorème 1]). The center of a finite-dimensional semisimple algebra is always a product of fields, and is automatically étale if the base field F is perfect. The example above the theorem illustrates why it is essential to assume that N is semisimple.

It follows from the general theory that when N is a separable algebra, every derivation of N with values in a (N, N)-bimodule is inner (see [11, A VIII §13 N°7, Corollaire]). However, without additional information on n, this does not formally imply that a first-order deformation $\Delta : N \to M : b \mapsto b + [n, b]$ is given by conjugation. The proof of Theorem 4.9 exhibits a suitable such element n.

4.3. **Ping-pong between two given finite subgroups of $\mathcal{U}(M)$.** Given two finite subgroups A, B in $\mathrm{GL}_n(D)$, the aim of this subsection is to provide a constructive method to obtain a subgroup of $\mathrm{GL}_n(D)$ isomorphic to $A * B$ by deforming B using the first-order deformations introduced in Section 4.2.

Theorem 4.12. *Let K be a local field of characteristic 0, D be a finite-dimensional division K-algebra, and A, B be finite subgroups of $\mathrm{GL}_n(D)$. Set $C = A \cap B$, and assume that $|A : C| > 2$ or $|B : C| > 2$. Suppose that $\Delta_t : B \to \mathrm{GL}_n(D) : h \mapsto h + t\delta_h$ (with $t \in F^\times$) is a family of first-order deformations satisfying*

(i) $\delta_g = 0 \iff g \in C$, *and*

(ii) $a\,\mathrm{im}(\delta_h) \cap \ker(\delta_{h'}) = \{0\}$ *for every $a \in A \setminus C$, $h, h' \in B \setminus C$.*

Then there exists $N \in \mathbb{R}$ such that

$$\langle A, \Delta_t(B)\rangle \cong A *_C \Delta_t(B) \cong A *_C B$$

when $|t| \geq N$.

In practice, checking the conditions $a\,\mathrm{im}(\delta_h) \cap \ker(\delta_{h'}) = \{0\}$ can be difficult, but luckily many are superfluous. For example, one can prove that $\ker(\delta_h) = \ker(\delta_{h^t})$ when $\gcd(o(h), t) = 1$. Building on Examples 4.7, we will lay out in Section 5 a way to construct a first-order deformation Δ_t to which Theorem 4.12 applies, in the case where A and B are finite subgroups of the unit group of a group ring.

Note that the conditions from Theorem 4.12 imply that $A \cap F^\times = B \cap F^\times \subset C$. Indeed, if $a \in A \cap F^\times \setminus C$ and $h \in B \setminus C$, then the second condition implies that $\mathrm{im}(\delta_h) \cap \ker(\delta_h) = \{0\}$. Since $\mathrm{im}(\delta_h) \subset \ker(\delta_h)$, it would follow that $\delta_h = 0$, a contradiction. Similarly, if $h \in B \cap F^\times$, then it is a general property of first-order deformations that $\delta_h = 0$, hence $h \in C$ by the first condition.

When D is a division algebra over a local field K of characteristic 0, and A, B are subgroups of $\mathrm{GL}_{D^n}(K)$, Theorem 3.26 implies on the nose that there is a free amalgamated product of the form $A *_C \gamma B \gamma^{-1}$ in $\mathrm{GL}_{D^n}(K)$. The point of Theorem 4.12 is to express such conjugate of B more explicitly.

As bibliographical context, note that the related results in [61, 34] can be reformulated as follows: in the special case where B admits a conjugate B^x for which $A \cap B^x$ is trivial and $\langle A, B^x\rangle$ is finite, then there is a first-order deformation B' of B which plays ping-pong with A. Under these restrictions [61] in fact constructs a free product of the form $\langle A, B^x\rangle * \mathbb{Z}$.

The next lemma will serve to replace Lemma 3.9 in the proof of Theorem 4.12.

Lemma 4.13. *Let $h \in \mathrm{GL}_V(K)$, let n be a non-zero nilpotent transformation in $\mathrm{End}_D(V)$, and set $h_t = h + tn$ for $t \in K$. Let C be a compact subset of $\mathbf{P}(V) \setminus \ker(n)$, and let U be a neighborhood of $\mathrm{im}\, n$ in $\mathbf{P}(V)$. There exists $N > 0$ such that if $|t| \geq N$, then $h_t \in \mathrm{GL}_V(K)$ and $h_t C \subset U$.*

Proof. Since $\mathrm{Nrd}(h) \neq 0$, the polynomial $\mathrm{Nrd}(h + Xn) \in K[X]$ is not zero, hence has only finitely many roots. For $|t|$ strictly larger than the maximum absolute value N_0 of these roots, $h + tn \in \mathrm{GL}_V(K)$.

Let now $p \in \mathbf{P}(V) \backslash \ker n$, and let v represent p in V. Since $h_t(v) = h(v) + tn(v)$, it follows that $h_t p$ converges to $np \in \mathbf{P}(V)$ as $|t| \to \infty$. So for each $p \in C$, there exists $N_p > 0$ and a neighborhood U_p of p in $\mathbf{P}(V)$ such that $h_t(U_p) \subset U$ if $|t| \geq N_p$. By the compactness of C, there is a finite collection $U_{p_1}, \ldots, U_{p_r}$ covering C. Setting $N = \max\{N_0, N_{p_1}, \ldots, N_{p_r}\}$, we see that $h_t C \subset U$ when $|t| \geq N$, as claimed. □

Remark 4.14. When K is a non-archimedean local field, care has to be taken that the condition $|t| \geq N$ is not preserved by addition. In other words, if the conclusion of the lemma holds for h_t, one cannot deduce that $gC \subset U$ for every $g \in \langle h_t \rangle$. This fails already for $h = 1$, in which case the subgroup $\langle 1 + tn \rangle$ accumulates at 1.

This mistake was made in [34]. Some of the results in this paper, namely [34, Theorems 2.3, 2.6, and 2.7], therefore only hold over archimedean local fields.

Proof of Theorem 4.12. As in the proof of Theorem 3.2, it will be convenient to write $H^* = H \setminus C$ and $A^* = A \setminus C$.

Let W denote the union of the proper subspaces $\mathrm{im}(\delta_h)$ for $h \in H^*$, and let W' denote the union of the proper subspaces $\ker(\delta_h)$ for $h \in H^*$, all viewed in $\mathbf{P}(V)$. Note right away that $W \subset W'$. By the first assumption, if $c \in C$ then $c\,\mathrm{im}(\delta_h) = \mathrm{im}(c\delta_h) = \mathrm{im}(\delta_{ch})$, so W is stable under C. Moreover, $aW \cap W \subset aW \cap W' = \emptyset$ for every $a \in A^*$. Indeed, if this last intersection were non-empty, then so would be $a\,\mathrm{im}(\delta_h) \cap \ker(\delta_{h'})$ for some $h, h' \in H^*$, contradicting the second assumption.

In consequence, we can construct a compact neighborhood P of W with the properties that P is stable under C, and $aP \cap (P \cup W') = \emptyset$ for every $a \in A^*$. For instance, start with a neighborhood of W whose translates under A^* are disjoint from W', remove from it a sufficiently small open neighborhood of the union of its translates under A^*, then intersect the result with its translates under C. Set $Q = A^* \cdot P = \bigcup_{a \in A^*} aP$; by construction, Q is a compact set disjoint from P, and from $\ker(\delta_h)$ for every $h \in H^*$.

In order to conclude the proof, it remains to verify the conditions of Lemma 2.1. We already arranged for $A^* \cdot P \subset Q$ and $C \cdot P = P$. The fact that $C \cdot Q = Q$ is an obvious consequence of $C \cdot A^* = A^*$. Lastly, since Q is disjoint from $\ker(\delta_h)$, Lemma 4.13 yields for each choice of $h \in H^*$ a positive number N_h such that $\Delta_t(h) = h + t\delta_h$ sends Q into P when $|t| \geq N_h$. Set $N = \max_{h \in H^*} N_h$ and pick $|t| \geq N$, so that $\Delta_t(h) \cdot Q \subset P$ for every $h \in H^*$. An application of Lemma 2.1 (to A and $\Delta_t(H)$ with the sets P and Q) finally shows that when $|t| \geq N$, $\langle A, \mathrm{im}(\Delta_t) \rangle$ is the free product of A and $\Delta_t(H)$ amalgamated along their intersection C. □

5. The embedding condition and free products in group rings

In Section 4 we established, subject to the appropriate embedding condition, the existence of ping-pong partners for finite subgroups of the unit group of an order in a finite semisimple algebra M. We also proposed, via the first-order deformations introduced in Definition 4.6, a more constructive method to obtain free products between such finite subgroups.

The present section focuses on the case where $M = FG$ is the group algebra of a finite group G and $\Gamma = \mathcal{U}(RG)$ is the group of units of M over some order R in F. To this choice of M and Γ the results of Sections 3 and 4 are more readily applicable, thanks to the fact that the simple factors of FG are the images $\rho(FG)$ for ρ ranging over the irreducible representations of G.

Our first task will be to address Questions 4.5.(i), that is, to determine when a subgroup $H \leq G$ embeds in an appropriate simple quotient of FG. This will be the matter of

Section 5.1; Theorem 5.1 shows that this happens if G is not a Dedekind group and if H admits a faithful irreducible representation (e.g. any subgroup H whose Sylow subgroups have cyclic center). Consequently, for such a pair $H \leq G$ we obtain in Corollary 5.4 the existence inside $\mathcal{U}(RG)$ of a non-trivial free amalgamated product $H *_C H$.

In Section 5.2 we develop further the first-order deformations from Examples 4.7.(ii), in connection with the construction of shifted bicyclic units (see Definition 5.16). We formulate in Conjecture 5.22 our presumption that the images of two opposite shifted bicyclic maps always form a free product amalgamated along their intersection. This conjecture echoes Questions 4.5.(ii), putting forward two explicit conjugates of a subgroup of $\mathcal{U}(FG)$ that should play ping-pong with each other.

Finally, we prove in Section 5.3 that profinitely-generically, pairs of a shifted bicyclic unit and a bicyclic unit generate a free group. Moreover, as a consequence of all this work, we can precisely determine when a given finite subgroup admits a bicyclic unit as ping-pong partner.

5.1. **On the embedding condition for group rings.** In this subsection, we wish to determine when a finite subgroup H of $\mathcal{U}(RG)$ satisfies the embedding condition from Theorem 4.1, aiming to find a ping-pong partner for H.

We start by stating a three-headed theorem, establishing the existence of a certain center-preserving irreducible representation of G under increasingly weaker assumptions. The three parts will be proved together, after some technical preliminaries concerning faithful irreducible representations and Frobenius complements.

5.1.1. *An embedding theorem for subgroups, and its consequences.* Recall that a group is called *Dedekind* if all of its subgroups are normal. The Baer–Dedekind classification theorem tells us that such a group is either abelian, or isomorphic to $Q_8 \times C_2^n \times A$ with $n \in \mathbb{N}$ and A an abelian group of odd order.

Theorem 5.1. *Let F be a field of characteristic 0, assumed to be a number field for the purpose of part (iii). Let G be a finite group and $H \leq G$ be a subgroup admitting a faithful irreducible F-representation. If respectively*

(i) *G is not a Dedekind group,*

(ii) *FG is not a product of division algebras,*

(iii) *F is not totally real, or G is not isomorphic to $Q_8 \times C_2^n$ for any $n \in \mathbb{N}$,*

then there exists an irreducible F-representation ρ of G such that $|H \cap \mathcal{Z}(\rho) : H \cap \mathcal{Z}(G)| \leq 2$ and satisfying respectively

(i) *$\rho(G)$ is not a Frobenius complement,*

(ii) *$\rho(FG)$ is not a division algebra,*

(iii) *$\rho(FG)$ is neither a field nor a totally definite quaternion algebra.*

If moreover G does not have a 2-Sylow subgroup mapping onto Q_{2^n}, or 4 does not divide $|H|$, or 2 does not divide $|H \cap \mathcal{Z}(G)|$, then ρ can be chosen so that in fact $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$.

Remark 5.2. If G is a Dedekind group (resp. FG is a product of division algebras, F is totally real and $G \cong Q_8 \times C_2^n$), then for every $\rho \in \mathrm{Irr}_F(G)$, the image $\rho(G)$ is a Frobenius complement (resp. $\rho(FG)$ is a division algebra, $\rho(FG)$ is either a field or a totally definite quaternion algebra), hence no irreducible representation of G can possibly satisfy (NF) (resp. (ND), (NQ)). In this sense, the assumptions on G made in Theorem 5.1 are sharp.

Remark 5.3. Note that if ρ is a representation such that $\rho(G)$ is not a Frobenius complement (for instance, if ρ is afforded by Theorem 5.1), then $\rho(FG)$ is certainly not a division algebra. Indeed, it is easy to see that a finite subgroup of a division F-algebra is fixed-point-free.[8] Zassenhaus [87] showed that a group is fixed-point-free if and only if it is a Frobenius complement (in some Frobenius group). We refer the reader to [62, Section 18] for background on Frobenius groups and the main structural results concerning Frobenius complements.

The condition that FG be not a product of division algebras can also be reformulated in terms of G. Indeed, by [73, Theorem 3.5], $\mathbb{Q}G$ is a product of division algebras if and only if G is isomorphic to $Q_8 \times C_2^n \times A$ with $n \in \mathbb{N}$ and A an abelian group such that $|A|$ and the order of 2 in $(\mathbb{Z}/|A|\mathbb{Z})^\times$ are both odd. It follows that FG is a product of division algebras if and only if G is of this form and F is not a splitting field of Hamilton's quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$.

In the frame of this paper, the importance of having an irreducible representation ρ for which $\rho(FG)$ is neither a field nor a totally definite quaternion algebra stems from Theorem 4.1, which will be used to construct a ping-pong partner. The relevance of the stronger property that $\rho(G)$ be not a division algebra lies in its use to obtain a *bicyclic* ping-pong partner. The details can be found in Section 5.3. For now, combining **??** and Theorem 4.1 we record the following interesting consequence.

Corollary 5.4. *Let F be a number field and R be its ring of integers. Let G be a finite group, and H be a non-central subgroup of G which admits a faithful irreducible F-representation. Suppose that either F is not totally real, or $G \not\cong Q_8 \times C_2^n$ for any $n \in \mathbb{N}$. Then there exists a subgroup $C \leq H$ with $|C : H \cap \mathcal{Z}(G)| \leq 2$ and an element $\gamma \in \mathcal{U}(RG)$ of infinite order such that*

$$\langle \gamma, H \rangle \cong \langle \gamma, C \rangle *_C H,$$

If moreover G does not have a 2-Sylow subgroup mapping onto Q_{2^n}, or 4 does not divide $|H|$, or 2 does not divide $|H \cap \mathcal{Z}(G)|$, then C can be taken to be $H \cap \mathcal{Z}(G)$.

Note again that if F is totally real and $G \cong Q_8 \times C_2^n$, then every simple factor of FG is either a field or a totally definite quaternion algebra, implying that $\mathcal{U}(RG)/\mathcal{Z}(\mathcal{U}(RG))$ is a finite group. The attentive reader will notice that the converse must also hold: if $\mathcal{U}(RG)/\mathcal{Z}(\mathcal{U}(RG))$ is finite, then either G is abelian or F is totally real and $G \cong Q_8 \times C_2^n$ for some $n \in \mathbb{N}$. Indeed, $\mathrm{SL}_1(RG)$ is commensurable with $\mathcal{U}(RG)/\mathcal{Z}(\mathcal{U}(RG))$ by [45, Proposition 5.5.1] and finiteness of $\mathrm{SL}_1(RG)$ was described as stated by Kleinert [52] (or [45, Corollary 5.5.7]). Thus, in that case, there is no hope to observe any non-trivial free amalgamated product inside $\mathcal{U}(RG)$.

For H a cyclic group of prime order p, the existence of a free amalgamated product $H *_{H \cap \mathcal{Z}(G)} H$ inside $\mathcal{U}(\mathbb{Z}G)$ was first obtained by Gonçalves and Passman [33]. They proved furthermore that $C_p * C_\infty$ is a subgroup of $\mathcal{U}(\mathbb{Z}G)$ if and only if G has a non-central element of order p. This last fact also follows from our results thanks to the positive solution to the Kimmerle problem for prime order elements [51, Corollary 5.2.], which states that all elements of order p in $\mathcal{U}(\mathbb{Z}G)$ can be conjugated (inside some larger algebra) to one in G.

Proof of Corollary 5.4. Let ρ be a representation afforded by Theorem 5.1.(iii) (with $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$ if possible), and set $C = H \cap \mathcal{Z}(\rho)$. By construction, the simple F-algebra $\rho(FG)$ is not a field nor a totally definite quaternion algebra.

Because the Wedderburn decomposition $FG = \bigoplus_{\sigma \in \mathrm{Irr}_F(G)} \sigma(FG)$ is defined over F, the intersection $RG \cap \rho(FG)$ is an order in $\rho(FG)$, and $\Gamma = \mathcal{U}(RG \cap \rho(FG))$ is an arithmetic subgroup of $\mathrm{GL}_{\rho(FG)}(F) = \mathcal{U}(\rho(FG))$. In particular, the image of Γ in the adjoint group $\mathrm{PGL}_{\rho(FG)}$ is Zariski-dense, and we can apply Theorem 4.1 (see also Remark 4.2) with

[8] Recall that a group G is called *fixed-point-free* if it admits an irreducible representation under which no non-trivial element of G fixes a non-zero vector (over some field, hence over every field of characteristic 0).

$M = \rho(FG)$ and $A = B = \rho(H)$ to deduce the existence of an element $\gamma \in \Gamma$ of infinite order for which the canonical map

$$\langle \gamma, \rho(C) \rangle *_{\rho(C)} \rho(H) \to \langle \gamma, \rho(H) \rangle$$

is an isomorphism. Note that by construction, C is the preimage in H of $\rho(C)$ under ρ, and $\gamma = \rho(\gamma)$ commutes with C. An application of Lemma 2.3 to the surjective map

$$\rho : \langle \gamma, H \rangle \to \langle \gamma, \rho(C) \rangle *_{\rho(C)} \rho(H)$$

thus shows that $\langle \gamma, H \rangle$ and $\langle \gamma, C \rangle *_C H$ are canonically isomorphic. $\square$

Remark 5.5. The proof of Theorem 5.1 will detail when one can arrange that $H \cap \mathcal{Z}(\rho)$ and $H \cap \mathcal{Z}(G)$ coincide. Concretely, the case $|H \cap \mathcal{Z}(\rho) : H \cap \mathcal{Z}(G)| = 2$ need only be considered if the group G has an irreducible representation σ which is both faithful and centrally faithful on H, and σ is such that $\sigma(G)$ has a Sylow 2-subgroup isomorphic to the generalized quaternion group Q_{2^n} and $a^{2^{n-3}} \in \sigma(H)$ (in the notation of (5.2)). This characterization is however not very practical as it depends on $\mathrm{Irr}_F(G)$. Nevertheless, it shows that if no Sylow 2-subgroup of G maps onto a generalized quaternion group, then in fact the representation ρ in Theorem 5.1 can be taken to satisfy $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$.

Remark 5.6. Theorem 5.1 can also be applied when H is a subgroup of a conjugate of G in $\mathcal{U}(FG)$, to obtain an irreducible F-representation of $\mathcal{U}(FG)$ such that $H \cap \mathcal{Z}(\rho) \subseteq \mathcal{Z}(FG)$, and satisfying (NF), (ND) or (NQ). This conjugacy condition is reminiscent of the Zassenhaus conjectures. The strongest Zassenhaus conjecture asserts that for a finite group G, any subgroup $H \leq \mathcal{U}(\mathbb{Z}G)$ is conjugated under $\mathcal{U}(\mathbb{Q}G)$ to a subgroup of $\pm G$. Counterexamples to this conjecture for H not cyclic were obtained by Roggenkamp and Scott [72], and for H cyclic by Eisele and Margolis [25]. However, the strongest Zassenhaus conjecture does hold for some classes of groups: for instance, if G is nilpotent and H is any subgroup [85, 86], or for G cyclic-by-abelian and H cyclic [19]. We refer to [58] for an overview of this topic.

As a consequence of these facts, when $F = \mathbb{Q}$ and G is nilpotent, the statements of Theorem 5.1 hold for all finite subgroups H of $\mathcal{U}(FG)$, not just for those contained in a group basis.

Recall that for every primitive central idempotent $e \in \mathrm{PCI}(FG)$, we denote by

$$\pi_e : FG \twoheadrightarrow FGe \cong \mathrm{M}_{n_e}(D_e)$$

the projection onto the the associated simple quotient FGe of FG. Somewhat abusively, we will also use π_e to denote its restriction $\mathcal{U}(FG) \to \mathrm{GL}_{n_e}(D_e)$ between unit groups, as well as the corresponding irreducible F-representation of G on $D_e^{n_e}$. It will be convenient to switch back and forth between the language of primitive central idempotents of FG, and of irreducible F-representations of G. On the representation side, $\mathrm{Irr}_F(G)$ denotes the set of irreducible F-representations of G; so the two sets $\mathrm{Irr}_F(G)$ and $\mathrm{PCI}(FG)$ can naturally be identified.

As usual, a representation ρ of G is called *faithful on H* if $H \cap \ker(\rho) = 1$, and we will say that ρ is *center-preserving on H* if $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$. If a representation ρ of G is faithful on G then it is of course center-preserving on G, but on proper subgroups the two notions are distinct. We will make use of the following two sets, gathering the irreducible F-representations of G which are faithful, resp. center-preserving, on H:

$$\begin{aligned} \mathrm{Fir}_F^G(H) &= \{e \in \mathrm{PCI}(FG) \mid H \cap \ker(\pi_e) = 1\}, \\ \mathrm{Cir}_F^G(H) &= \{e \in \mathrm{PCI}(FG) \mid H \cap \mathcal{Z}(\pi_e) \leq \mathcal{Z}(G)\}. \end{aligned} \tag{5.1}$$

When $G = H$, we will simply write $\mathrm{Fir}_F(H)$ and $\mathrm{Cir}_F(H)$; when the field F is clear from context, we will write $\mathrm{Fir}^G(H)$ and $\mathrm{Cir}^G(H)$.

With this notation, Theorem 5.1.(i) implies for example that if $\mathrm{Fir}_F(H) \neq \emptyset$ and G has no Sylow 2-subgroup mapping onto Q_{2^n}, there exists $e \in \mathrm{Cir}_F^G(H)$ such that $\pi_e(G)$ is not a Frobenius complement.

The next corollary gives evidence that in general, whenever G has an irreducible representation which is centrally faithful on H, then there is also some representation whose associated simple quotient of FG is neither a field nor a totally definite quaternion algebra.

Corollary 5.7. *Let F be a field of characteristic 0, G a non-Dedekind finite group and $H \leq G$ a non-central subgroup. If $\mathrm{Fir}_F^G(H) \cap \mathrm{Cir}_F^G(H)$ is non-empty, then there is an irreducible F-representation ρ of G such that $\rho(G)$ is not a Frobenius complement and $|H \cap \mathcal{Z}(\rho) : H \cap \mathcal{Z}(G)| \leq 2$. If moreover $4 \nmid |H|$ or $2 \nmid |H \cap \mathcal{Z}(G)|$, then ρ can be taken in $\mathrm{Cir}_F^G(H)$.*

Proof. Pick $e \in \mathrm{Fir}_F^G(H) \cap \mathrm{Cir}_F^G(H)$. If $\pi_e(G)$ is not a Frobenius complement to begin with, there is nothing to prove. If $\pi_e(G)$ is a Frobenius complement, then $H \cong \pi_e(H)$ is a subgroup of the fixed-point-free group $\pi_e(G)$. In particular, H is itself fixed-point-free, and thus admits a faithful irreducible representation ψ (see [76, Theorem 6.13]). Theorem 5.1.(i) then implies the existence of an irreducible representation ρ of G as in the statement. □

Remark 5.8. It would be interesting to obtain a version of Theorem 5.1 for subgroups $H \leq G$ having a (perhaps not faithful) irreducible F-representation ψ such that $\ker(\psi) \leq H \cap \mathcal{Z}(G)$. If such a generality holds true, then the statement of Corollary 5.7 with $\mathrm{Fir}_F^G(H) \cap \mathrm{Cir}_F^G(H)$ replaced by $\mathrm{Cir}_F^G(H)$ would also hold. In particular, if say $4 \nmid |H|$, then one would obtain that H has a ping-pong partner if and only if $\mathrm{Cir}_F^G(H) \neq \emptyset$. Simply put, the condition appearing in Theorem 4.1, of embedding into a factor that is neither a field nor a totally definite quaternion algebra, can potentially always be satisfied for $M = FG$ with G not Dedekind, and $H \leq G$. Check for counterexamples on the computer?

Before starting the proof of Theorem 5.1, we gather several needed results about faithful irreducible representations and Frobenius complements.

5.1.2. *Faithful irreducible representations and their center.* The existence of faithful irreducible (complex) representations of finite groups has been intensively studied, see [81, Section 2] for a brief survey of its history. For the purpose of proving Theorem 5.1, we will need to understand the existence of such representations over an arbitrary field F of characteristic 0, and determine when they preserve the center. The following lemma summarizes some basic facts concerning the existence of faithful representations over F.

Lemma 5.9. *Let G be a finite group, $F \subseteq L$ be fields of characteristic 0, and H be a finite subgroup of $\mathcal{U}(FG)$.*

(i) *If $e \in \mathrm{PCI}(FG)$, then $\mathcal{Z}(G)/\ker(\pi_e)$ is cyclic.*
(ii) *If $\mathrm{Fir}_L^G(H) \neq \emptyset$ then $\mathrm{Fir}_F^G(H) \neq \emptyset$; if $\mathrm{Cir}_L^G(H) \neq \emptyset$ then $\mathrm{Cir}_F^G(H) \neq \emptyset$.*
(iii) *If all Sylow subgroups of G have a cyclic center, then $\mathrm{Fir}_F(G) \neq \emptyset$.*
(iv) *If G is nilpotent, then $\mathrm{Fir}_F(G) \neq \emptyset$ if and only if $\mathcal{Z}(G)$ is cyclic.*

Proof. Since FGe is simple, $\mathcal{Z}(FGe)$ is a field. Thus, $\pi_e(\mathcal{Z}(G)) \subseteq \mathcal{Z}(FGe)^\times$ is a finite subgroup of the multiplicative group of a field, hence is cyclic. This proves (i).

Next, note that

$$\bigoplus_{e\in \mathrm{PCI}(LG)} LGe \cong LG \cong L \otimes_F FG \cong \bigoplus_{f\in \mathrm{PCI}(FG)} (L \otimes_F FGf)\,.$$

Pick $e \in \mathrm{PCI}(LG)$, and let $f \in \mathrm{PCI}(FG)$ be the idempotent for which the quotient map $\pi_e : LG \to LGe$ factors through the summand $L \otimes_F FGf$. The kernel of the restriction to

G of π_e then obviously contains that of π_f. Similarly, the preimage of the center of LGe under π_e contains that of FGf under π_f. Hence if $e \in \mathrm{Fir}_L^G(H)$, resp. $\mathrm{Cir}_L^G(H)$, we deduce that $f \in \mathrm{Fir}_F^G(H)$, resp. $\mathrm{Cir}_F^G(H)$, proving (ii).

Part (iii) follows from part (ii) and [40, Exercise (5.25)] stating that $\mathrm{Fir}_{\mathbb{C}}(G)$ is non-empty if all Sylow subgroups have a cyclic center.

Finally, for part (iv) the necessity of $\mathcal{Z}(G)$ being cyclic follows from (i). The converse follows from (iii) as a nilpotent G is the direct product of its Sylow subgroups. ☐

The existence of irreducible representations faithful of a group G can be characterized in several ways. The most common criterion is due to Gaschütz, and states that G has a faithful irreducible representation over some (any) field of characteristic not dividing $|G|$, if and only if the socle of G is generated by a single conjugacy class. The literature contains multiple generalizations of this fact, but somewhat surprisingly, we could not find any results providing sufficient control on the center of the representation in order to apply Theorem 4.1.

With the aim to fill this gap, it is shown in [**?**] update bibliography that if H admits a faithful irreducible F-representation, then in fact $\mathrm{Fir}_F^G(H) \cap \mathrm{Cir}_F^G(H)$ is non-empty for any finite group G containing H. This will form the first step in proving Theorem 5.1, and the remainder of the proof essentially amounts to upgrading this representation to fit our needs.

Theorem 5.10 ([**?**, Theorem ?])**.** *Let G be a finite group and let F be a field whose characteristic does not divide $|G|$. Let $H \leq G$ be a subgroup possessing a faithful irreducible F-representation. Then G has an irreducible F-representation σ whose restriction to H is faithful, and such that $H \cap \mathcal{Z}(\sigma) \leq \mathcal{Z}(G)$.*

We refer the reader to [**?**]update reference for the proof of this theorem, as well as additional context and a reminder of the proof of Gaschütz' criterion.

Let us immediately record a consequence of Theorem 5.10 concerning the existence of free products in $\mathcal{U}(FG)$. This corollary is less precise than Theorem 5.1, as it exhibits ping-pong partners that need not lie in a proper subring of FG, but in return presents the advantage of avoiding the intricacies of the representation theory of FG.

Corollary 5.11. *Let F be a field of characteristic 0. Let G be a finite group, and let $H \leq G$ be a subgroup possessing a faithful irreducible F-representation. Set $C = H \cap \mathcal{Z}(G)$. The set of regular semisimple elements $\gamma \in \mathcal{U}(FG)$ of infinite order with the property that the canonical map*

$$\langle \gamma, C \rangle *_C H \to \langle \gamma, H \rangle$$

is an isomorphism, is Zariski-dense in $\mathcal{U}(FG)$.

Proof. Let $\mathbf{G}$ denote the multiplicative group scheme of $\mathbb{Q}G$. Recall that $\mathbf{G}$ is a reductive group, and that the simple factors of $\mathrm{Ad}\,\mathbf{G}$ are in natural correspondence with the irreducible $\mathbb{Q}$-representations of G, each one being of the form PGL_{D^n} for D some division algebra over $\mathbb{Q}$.

As the field F contains $\mathbb{Q}$, it suffices to prove the corollary inside $\mathcal{U}(\mathbb{Q}G)$. Indeed, $\mathcal{U}(\mathbb{Q}G)$ is Zariski-dense in $\mathbf{G}$, hence also in the extension of scalars of $\mathbf{G}$ to F (whose F-points form the group $\mathcal{U}(FG)$).

Since H possesses a faithful irreducible F-representation, it also has a faithful $\mathbb{Q}$-representation. By Theorem 5.10, we can thus find a representation $\rho \in \mathrm{Cir}_{\mathbb{Q}}^G(H)$.[9] In other words, the kernel C of the projection from H to the simple adjoint factor $\mathrm{Ad}\,\mathbf{G}_\rho = \mathrm{PGL}_{D_\rho^{n_\rho}}$ corresponding to the representation ρ, is contained in $\mathcal{Z}(G)$, hence equals $H \cap \mathcal{Z}(G)$. The

[9]Theorem 5.10 allows us to also arrange that ρ be faithful on H, although this will not be used here.

corollary is now an immediate consequence of Theorem 3.26 applied to $\mathbf{G}$, the collection of subgroups $\{(H, H)\}$, and the simple quotient $\operatorname{Ad} \mathbf{G}_\rho$. □

The proof of Theorem 5.10 is not constructive, and one can wonder how to concretely construct an element of $\mathrm{Fir}_F^G(H) \cap \mathrm{Cir}_F^G(H)$. Under some additional assumptions, we show in the next proposition that such an irreducible representation of G can in fact be found among the irreducible constituents of $\mathrm{Ind}_H^G(\psi)$.

Proposition 5.12. *Let $H \leq G$ be finite groups and F be a field whose characteristic is prime to $|G|$. Suppose that H admits a faithful irreducible F-representation ψ. Then every irreducible constituent σ of $\mathrm{Ind}_H^G(\psi)$ satisfies the following:*

(i) $H \cap \ker(\sigma) = 1$,

(ii) $\mathrm{core}_G(H) \cap \mathcal{Z}(\sigma) = H \cap \mathcal{Z}(G)$,

(iii) $H \cap \mathcal{Z}(\sigma) \leq \mathcal{Z}(\mathrm{N}_G(H))$.

Furthermore, if $\mathcal{Z}(H)$ is a p-group, then there is a constituent σ_0 of $\mathrm{Ind}_H^G(\psi)$ such that $H \cap \mathcal{Z}(\sigma_0) \leq \mathcal{Z}(G)$.

Proof. Pick any irreducible quotient σ of $\mathrm{Ind}_H^G(\psi)$, the induction of ψ from H to G. By construction, the space $\mathrm{Hom}_{FG}(\mathrm{Ind}_H^G(\psi), \sigma)$ contains the projection p from $\mathrm{Ind}_H^G(\psi)$ to σ. By Frobenius' reciprocity,

$$\mathrm{Hom}_{FH}(\psi, \mathrm{Res}_H^G(\sigma)) \cong \mathrm{Hom}_{FG}(\mathrm{Ind}_H^G(\psi), \sigma) \ni p.$$

Thus, the image of p in $\mathrm{Hom}_{FH}(\psi, \mathrm{Res}_H^G(\sigma))$ is a non-zero morphism from the irreducible representation ψ to $\mathrm{Res}_H^G(\sigma)$, hence an embedding. Since ψ was faithful by assumption, the same holds for $\mathrm{Res}_H^G(\sigma)$, meaning that $H \cap \ker(\sigma) = 1$.

Now pick $h \in H \cap \mathcal{Z}(\sigma)$, so that $[g, h] \in \ker(\sigma)$ for any $g \in G$. Then, using that $H \cap \ker(\sigma) = 1$, we have

$$\{g \in G \mid h^g \in H\} = \{g \in G \mid [g, h] \in H\} = \{g \in G \mid [g, h] = 1\} = C_G(h).$$

In particular, for $h \in \mathrm{core}_G(H) \cap \mathcal{Z}(\sigma)$ we have that $C_G(h) = G$. This proves $\mathrm{core}_G(H) \cap \mathcal{Z}(\sigma) \leq H \cap \mathcal{Z}(G)$, and it is readily seen that the reverse inclusion holds in general.

Next, let $h \in H \cap \mathcal{Z}(\sigma)$ and pick $g \in \mathrm{N}_G(H)$. Then $[g, h] \in H \cap \ker(\sigma)$ is trivial by the first part, showing that $h \in \mathcal{Z}(\mathrm{N}_G(H))$.

Finally, suppose that $\mathcal{Z}(H)$ is a p-group for some prime p. Let

$$\mathrm{Ind}_H^G(\psi) = \sigma_1 \oplus \cdots \oplus \sigma_\ell$$

be the decomposition of $\mathrm{Ind}_H^G(\psi)$ into irreducible subrepresentations of G. Recall that $\ker(\mathrm{Ind}_H^G(\psi)) = \mathrm{core}_G(\ker(\psi)) = 1$, therefore $\mathrm{Ind}_H^G(\psi)$ is faithful. In consequence, $\mathcal{Z}(G) = \bigcap_{i=1}^{\ell} \mathcal{Z}(\sigma_i)$, hence also $\mathcal{Z}(H) \cap \mathcal{Z}(G) = \mathcal{Z}(H) \cap \bigcap_{i=1}^{\ell} \mathcal{Z}(\sigma_i)$. Since $\mathcal{Z}(H)$ is an abelian p-group, its subgroups are totally ordered and there is an index i for which $\mathcal{Z}(H) \cap \mathcal{Z}(G) = \mathcal{Z}(H) \cap \mathcal{Z}(\sigma_i)$. Since σ_i is faithful on H by the first part, we have for such an index that $H \cap \mathcal{Z}(\sigma_i) = \mathcal{Z}(H) \cap \mathcal{Z}(\sigma_i) = \mathcal{Z}(H) \cap \mathcal{Z}(G)$ as desired. □

Remark 5.13. Remove? Proposition 5.12 shows that a representation in $\mathrm{Fir}_F(H)$ can be induced (so to speak) to obtain a representation in $\mathrm{Fir}_F^G(H)$. The reverse process is obviously not possible in general. For instance, every finite group H can be embedded in an alternating group A_n for some $n \geq 5$. The latter, being simple, has a faithful irreducible representation, while H need not have any.

5.1.3. *Representations whose images are (not) Frobenius complements.* In order to prove Theorem 5.1, we need to understand how to avoid representations whose images are Frobenius complements. This is of particular concern when G is itself a Frobenius complement, which is the subject of the next lemma. Recall that by the Frobenius–Thompson–Zassenhaus Theorem [45, Theorem 11.4.5], any Frobenius complement B has the following restrictions on its Sylow subgroups:

- Every odd-prime Sylow subgroup of B is cyclic.
- The Sylow 2-subgroups of B are either cyclic or a generalized quaternion group

$$Q_{4m} = \langle a, b \mid a^{2m} = 1, b^2 = a^m, b^{-1}ab = a^{-1} \rangle \tag{5.2}$$

  with $m = 2^{n-2}$ for some $n \geq 2$.

The shape of the Sylow 2-subgroup of B turns out to play an important role, as illustrated by the following technical result.

Lemma 5.14. *Let B be a non-abelian Frobenius complement. There exists a cyclic p-group $N \leq \mathcal{Z}(B)$, with p prime, such that the following properties hold:*

- *$N \cong C_2$ if some Sylow 2-subgroup of B is non-abelian,*
- *B/N is not a Frobenius complement,*
- *B/N is abelian if and only if B is isomorphic to $Q_8 \times C_m$ with m odd.*

Furthermore, if $B \not\cong Q_8 \times C_m$ with m odd, then

- *B/N has a faithful irreducible F-representation,*
- *$|\pi^{-1}(\mathcal{Z}(B/N)) : \mathcal{Z}(B)| \leq 2$, where π is the quotient map $B \twoheadrightarrow B/N$.*

Moreover, when $B \not\cong Q_8 \times C_m$, the following are equivalent:

(i) $\pi^{-1}(\mathcal{Z}(B/N)) = \mathcal{Z}(B)$,

(ii) *B does not have a Sylow 2-subgroup isomorphic to Q_{2^n} such that $\langle a^{2^{n-3}} \rangle$ is normal in B,*

(iii) *B is a Z-group, or* $\mathrm{Fit}(B)_2$, *the Sylow 2-subgroup of the Fitting subgroup of B, is isomorphic to Q_8 and $3 \mid |B : C_B(\mathrm{Fit}(B)_2)|$.*

If the above equivalent conditions do not hold, then $\pi^{-1}(\mathcal{Z}(B/N)) = \langle \mathcal{Z}(B), a^{2^{n-3}} \rangle$, with a as above.

Note that in the last case, the subgroup $\langle \mathcal{Z}(B), a^{2^{n-3}} \rangle$ does not depend on the choice of the Sylow 2-subgroup, as $\langle a^{2^{n-3}} \rangle$ lies in the intersection of all of them when it is normal in B.

Proof. We will use the notation x_p (resp. $x_{p'}$) to denote the p-part (resp p'-part) of an element $x \in B$, with p some prime. We distinguish two cases straight away, based on the isomorphism type of the Sylow 2-subgroups of B.

Case 1: All Sylow subgroups of B are cyclic.

Such groups are called Z-groups, and it was proven by Zassenhaus [62, Theorem 18.2] that they are of the form

$$B \cong C_m \rtimes_r C_n := \langle a, b \mid a^m = 1,\ b^n = 1,\ b^{-1}ab = a^r \rangle, \tag{5.3}$$

with $r^n \equiv 1 \mod m$ and $\gcd(m, n) = \gcd(m, r-1) = 1$. The presentation shows that Sylow p-subgroups are normal if and only if $p \mid m$. Furthermore,

$$C_m \rtimes_r C_n \text{ is a Frobenius complement} \iff r^{n/\operatorname{rad}(n)} \equiv 1 \mod m, \tag{5.4}$$

where $\operatorname{rad}(n)$ is the product of all prime divisors of n. In other words, $C_m \rtimes_r C_n$ is a Frobenius complement if and only if all elements of prime order commute [45, Theorme 11.4.9] (or [62, Theorem 18.2]). On turn this is equivalent to say that all prime order elements dividing n are central. Or yet reformulated, if and only if $r^{n/p} \equiv 1 \mod m$ for all primes p dividing n.

First note that the center of B is

$$\mathcal{Z}(B) = \langle b^d \rangle, \text{ for } d \text{ the order of } r \text{ in } (\mathbb{Z}/m\mathbb{Z})^\times. \tag{5.5}$$

Indeed, let $a^i b^j$ with $0 \leq i < m$ and $0 \leq j < n$, be an arbitrary element in B. Since $b^{-1}(a^i b^j)b = a^{ir}b^j$, in order for $a^i b^j$ to be central it must be $ir \equiv i \mod m$. Now, $\gcd(m, r-1) = 1$ implies that $i \equiv 0 \mod m$, hence $a^i b^j$ is a power of b. Since $b^{-j}ab^j = a^{r^j}$, the elements a and b^j commute if and only if $r^j \equiv 1 \mod m$. In other words, b^j is central if and only if $d \mid j$.

Next, write $o(b^d) = n/d = sp^\ell$ with $\gcd(p, s) = 1$, and consider the associated central element b^{n/p^ℓ} of order p^ℓ. From (5.5) it follows that $B/\langle b^{n/p^\ell} \rangle \cong C_m \rtimes_r C_{n/p^\ell}$, and p does not divide the order of the kernel of the action of C_{n/p^ℓ} on C_m. However, p does divide $n/p^\ell = sd$, as otherwise the p-Sylow of C_n would be central, which is not the case in the Zassenhaus presentation considered. I don't see why p must divide d. However, I guess that if B is not abelian, then d has a prime divisor and one can choose that prime? Now note that the "p-faithfulness" of the action makes it impossible that $r^{(n/p^\ell)/p} \equiv 1 \mod m$. Therefore the criterion in (5.4) shows that $B/\langle b^{n/p^\ell} \rangle$ is not a Frobenius complement when $n \neq d$ $d \neq 1$?, that is, when B is non-abelian. Set $N = \langle b^{n/p^\ell} \rangle$; it remains to verify the other properties.

As all Sylow subgroups of B/N are cyclic, it again follows from Lemma 5.9 that B/N has a faithfull irreducible F-representation. Finally, since $B/N \cong C_m \rtimes C_{n/p^\ell}$ where the action is p-faithful it can not be abelian, except if B was abelian which we assumed to not be the case. Finally, (5.5) implies that $\pi^{-1}(\mathcal{Z}(B/N)) = \mathcal{Z}(B)$. Thus it remains to prove that $2 \nmid m$ (from which it would follow that any non-central 2-subgroup of B is not normal in B). For this consider the quotient $C_{q^t} \rtimes C_{n/p^\ell}$ where $m = q^t.m'$, q is prime and $q \nmid m'$. As the action of C_{n/p^ℓ} is p-faithful, we have that C_{n_p/p^ℓ} (with n_p the p-part of n) acts faithfully on C_{q^t}. Therefore, n_p/p^ℓ divides $|\operatorname{Aut}(C_{q^t})| = (q-1)q^{t-1}$. Since q and p are different primes the latter implies that $q \equiv 1 \mod n_p/p^\ell$. In particular q is odd and hence $2 \nmid m$, as desired.

Case 2: The Sylow 2-subgroups of B are quaternion.

Recall that $\mathcal{Z}(Q_{4m}) = \langle a^m \rangle \cong C_2$ and $Q_{4m}/\langle a^m \rangle \cong D_{2m}$ is a dihedral group. Furthermore, if 2 divides $|B|$, a^m is the unique involution in B, see [45, Theorem 11.4.5], and therefore it is central in B. Thus we may consider the quotient $B/\langle a^m \rangle$ which does not depend on the chosen Sylow 2-subgroup and also is not a Frobenius complement as the Sylow 2-subgroups of $B/\langle a^m \rangle$ are dihedral. Next, if $B \not\cong Q_8 \times C$ with C some odd order cyclic group, then all Sylow subgroups of $B/\langle a^m \rangle$ have cyclic center and hence from Lemma 5.9 it follows that $B/\langle a^m \rangle$ has a faithful irreducible F-representation.

Suppose that $B/\langle a^m \rangle$ is abelian. Then B would be nilpotent of class 2. However a nilpotent Frobenius complement is either cyclic or isomorphic to $Q_{2^n} \times C_m$ for some odd $m$and $n \geq 3$, see [45, Corollary 11.4.7]. Since the nilpotency class of $Q_{2^n} \times C_m$ is $n-1$, we obtain that $n = 3$, as claimed.

From now on suppose that $B \not\cong Q_8 \times C_m$ and consider $\pi : B \twoheadrightarrow B/N$. We need to investigate elements g in $\pi^{-1}(\mathcal{Z}(B/N)) \setminus \mathcal{Z}(B)$.

Firstly, we claim that if $x \in \pi^{-1}(\mathcal{Z}(B/N))$ is a 2-power element, then $\langle x \rangle \lhd B$. Indeed, by definition $\pi(x^g) = \pi(x)$ for $g \in B$. Hence, $x^g = x(a^m)^\delta$ with $\delta = 0$ or 1 as $N = \langle a^m \rangle$. Now let Q_x be a Sylow 2-subgroup containing x, which is in case II is assumed quaternion Q_{4m}. Therefore the unique involution a^m is contained in $\langle x \rangle$ and hence $x^g \in \langle x \rangle$, as desired.

Now consider $g \in \pi^{-1}(\mathcal{Z}(B/N)) \setminus \mathcal{Z}(B)$ of odd order and let $y \in B$ such that $[g, y] \neq 1$. By definition $\pi([g, y]) = 1$, hence $g^y = ga^m$. However $o(g) = o(ga^m) = 2o(g)$ as a^m is a central involution and $o(g)$ is odd, a contradiction. Therefore, all odd order elements in $\pi^{-1}(\mathcal{Z}(B/N))$ must be central in B.

Next let $g \in \pi^{-1}(\mathcal{Z}(B/N)) \setminus \mathcal{Z}(B)$ be arbitrary. Let $y \in B$ be such that $\pi([g, y]) = 1$. The reasoning for $o(g)$ odd, but applied to y, yields that $[g, y] = 1$ if y is odd order. Hence, writing $y = y_{2'}.y_2$ we have that $[g, y] = [g, y_2].[g, y_{2'}]^{y_2} = [g, y_2]$. In other words, an element $g \in \pi^{-1}(\mathcal{Z}(B/N))$ must commute in B with all odd order elements and it remains to understand what conditions on g commuting with the Sylow 2-subgroup of B/N yields. Moreover, $g_{2'} \in \pi^{-1}(\mathcal{Z}(B/N))$ and hence by the above $g_{2'} \in \mathcal{Z}(B)$. Thus it remains to understand g_2, i.e. the case that g is a 2-power element, which we now assume.

Consider a Sylow 2-subgroup Q of B containing g. As observed earlier on, $\langle g \rangle$ is normal in B for a 2-power element $g \in \pi^{-1}(\mathcal{Z}(B/N))$. Hence, the upcoming reasoning will be independent of the chosen Q. Also, as we are in case II, $Q \cong Q_{4m}$ with presentation given in (5.2), and $\pi(g)$ is contained in a Sylow 2-subgroup of B/N which is isomorphic to D_{2m}. If $m > 2$, the center of the latter is $\langle \pi(a^{m/2}) \rangle$ and $\langle a^{m/2} \rangle = \pi^{-1}(\langle \pi(a^{m/2}) \rangle)$. Thus if $m > 2$ we obtained that

$$\pi^{-1}(\mathcal{Z}(B/N)) \leq \langle \mathcal{Z}(B), a^{m/2} \rangle. \tag{5.6}$$

and the right hand side is independent of the chosen Sylow 2-subgroup. It remains to understand the case that $m = 2$ (i.e. whether an element $a^j b$ can be in $\pi^{-1}(\mathcal{Z}(B/N))$) and when $\pi^{-1}(\mathcal{Z}(B/N)) = \langle \mathcal{Z}(B), a^{m/2} \rangle$. This will require more structure theory of Frobenius complements.

First we claim that $\pi^{-1}(\mathcal{Z}(B/N))$ is a subgroup of the Fitting subgroup $\mathrm{Fit}(B)$ of B. Indeed, for $x \in \pi^{-1}(\mathcal{Z}(B/N))$ all the conjugates $\pi(x)^g$ for $g \in B$ commute. Consequently, $[x, x^g] \in N = \langle a^m \rangle$ all the commutators are in a central subgroup of order 2. Therefore, the normal closure $\langle x^g \mid g \in B \rangle$ is a normal subgroup of nilpotency class 2 and hence in $\mathrm{Fit}(B)$.

Now we consider several cases. To start, suppose that B is non-solvable. Then by [45, Theorem 11.4.10] the group B is isomorphic to $M \times Y$ with Y a Z-group of order prime to 60 and M either $\mathrm{SL}_2(\mathbb{F}_5)$ or its index 2 overgroup[10] $\langle v, d \mid d^8 = v^3 = 1, (d^{-2}v)^5 = d^5, v^d = vd^2(v, d^{-2}) \rangle$. In both cases $N = \mathcal{Z}(M) = \mathrm{Fit}(M)$. Hence if $x \in Y$, then clearly $x \in \pi^{-1}(\mathcal{Z}(B/N))$ exactly when it is was in $\mathcal{Z}(B)$. Moreover if $x \in M \cap \in \pi^{-1}(\mathcal{Z}(B/N))$, then by the claim above $x \in \mathrm{Fit}(M) = N \leq \mathcal{Z}(B)$. In summary, for B non-solvable $\pi^{-1}(\mathcal{Z}(B/N)) = \mathcal{Z}(B)$.

Next suppose that B is nilpotent. As B is assumed non-abelian and not $Q_8 \times C_m$, we must have by [45, Corollary 11.4.7] that $B \cong Q_{2^n} \times C_m$ with m odd and $n \geq 4$. Hence in this case $\pi(a^{m/2}) = \mathcal{Z}(Q_{2^n}/\mathcal{Z}(Q_{2^n})) \leq \mathcal{Z}(B/N)$ and thus $\pi^{-1}(\mathcal{Z}(B/N)) = \langle \mathcal{Z}(B), a^{m/2} \rangle$.

Finally, suppose that B is solvable, but not nilpotent. In that case in the proof of [62, Theorem 18.2.] it is shown that exactly one of the following cases occur, denoting $\mathrm{Fit}(B)_2$ the Sylow 2-subgroup of $\mathrm{Fit}(B)$:

(1) If $\mathrm{Fit}(B)_2$ is cyclic, then $\mathrm{Fit}(B) \leq G_0$ with G_0 a Z-group such that G/G_0 is C_2 or $C_2 \times C_2$.
(2) If $\mathrm{Fit}(B)_2 \cong Q_8$, then $G/C_G(\mathrm{Fit}(B)_2)$ is a subgroup of $\mathrm{Aut}(Q_8) \cong S_4$ and $C_G(\mathrm{Fit}(B)_2)$ is a Z-group.
(3) If $\mathrm{Fit}(B)_2 \cong Q_{2^n} = \langle \overline{a}, \overline{b} \rangle$ with $n \geq 4$, then $C_G(\langle \overline{a} \rangle)$ is a Z-group and $G/C_G(\langle \overline{a} \rangle) \cong C_2$.

Take a 2-power element $x \in \pi^{-1}(\mathcal{Z}(B/N)) \setminus \mathcal{Z}(B)$. As noticed earlier $x \in \mathrm{Fit}(B)_2$. Consider a Sylow 2-subgroup $Q_{2^n} = \langle a, b \rangle$ containing $\mathrm{Fit}(B)_2$. If case (1) holds, then $\mathrm{Fit}(B)_2 \leq \langle a \rangle$ and each element in $\mathrm{Fit}(B)_2$ is normal in B. Consequently, the only

[10] The IdSmallGroup is [240,90].

possibility is $x = a^{m/2}$. Now note that $\text{Fit}(B)$ is contained in the Z-group G_0. In particular, $x \in \pi^{-1}(\mathcal{Z}(G_0/N))$ and thus case I yields that $x \in \mathcal{Z}(G_0)$. Since $\exp(G/G_0) = 2$ this implies that x commutes with the elements of odd order. As $a^{m/2}$ is central in the Sylow 2-subgroups of B/N, we obtained that $a^{m/2} \in \pi^{-1}(\mathcal{Z}(B/N)) \setminus \mathcal{Z}(B)$.

For the remaining cases denote $\text{Fit}(B)_2 = \langle \overline{a}, \overline{b} \rangle$ and by $S \cong Q_{4m} = \langle a, b\rangle$ a Sylow 2-subgroup of B containing $\text{Fit}(B)_2$.

In case (3) we can apply (5.6). Thus it remains to verify whether $a^{m/2} \in \pi^{-1}(\mathcal{Z}(B/N))\setminus \mathcal{Z}(B)$. As noticed earlier, this will be the case if $a^{m/2}$ commutes with all odd order elements. The latter holds because $[G : C_G(\langle \overline{a} \rangle)] = 2$..

Finally, consider case (2). Note that $\text{Fit}(B)_2 = \langle a^{m/2}, y\rangle$ with y either b or some $a^{2^j}b$. If 3 does not divide $[B : \text{Fit}(B)_2]$, then all odd order elements of B are in $C_G(\text{Fit}(B)_2)$. Hence an element in $\text{Fit}(B)_2$ is in $\pi^{-1}(\mathcal{Z}(B/N))$ whenever it is also in $\mathcal{Z}(S)$. If $S = \text{Fit}(B)_2 \cong Q_8$, then the proof of case I shows that one must have that B is nilpotent, which we assumed not. Hence S is strictly larger than $\text{Fit}(B)_2$. Therefore the only possibility for $x \in \pi^{-1}(\mathcal{Z}(B/N))$ is $a^{m/2}$. The only remaining case is when 3 divides $[B : \text{Fit}(B)_2]$. Then such element of order 3 transitively permutes the elements of order 4 [62, Proposition 9.9.] and in particular the non-central elements of $\text{Fit}(B)_2$ are even not normal in B. So $\pi^{-1}(\mathcal{Z}(B/N)) = \mathcal{Z}(B)$.

In summary, for B solvable non-nilpotent we have shown that

$$a^{m/2} \notin \pi^{-1}(\mathcal{Z}(B/N)) \Leftrightarrow \langle a^{m/2}\rangle \ntrianglelefteq B \Leftrightarrow \text{Fit}(B)_2 \cong Q_8 \text{ and } 3 \mid [G : C_G(\text{Fit}(B)_2)].$$

Moreover, note that this equivalence also holds for both the non-solvable and the nilpotent cases, finishing the proof. ☐

Lastly, we need to understand the specificity of Dedekind groups.

Lemma 5.15. *Let G be a subgroup of $Q_8^t \times A$ with $t \in \mathbb{N}$ and A a finite abelian group. Then the following hold:*

(i) *All images $\rho(G)$ for $\rho \in \text{Irr}_F(G)$ are Frobenius complements if and only if G is a Dedekind group.*

(ii) *The images $\rho(G)$ are isomorphic to $E \times C_m$ or C_{2m} with m odd and E an extraspecial group. If G is Dedekind, then $E \cong Q_8$.*

Furthermore, let H is a non-central normal subgroup having a faithful irreducible representation.

(iii) *If G is not Dedekind, then there exists a $\rho \in \text{Irr}_F(G)$ such that $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$ and the Sylow 2-subgroup of $\rho(G)$ is an extraspecial group different of Q_8.*

(iv) *if G is Dedekind group, then there is a $\rho \in \text{Irr}_F(G)$ such that $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$ and*
- *$\rho(G)$ not a division algebra if and only if FG is not a product of division algebras.*
- *$\rho(G)$ neither a field nor a totally definite quaternion algebra (when F is a number field) if and only if F is not totally real, or $G \ncong Q_8 \times C_2^n$ for any $n \in \mathbb{N}$.*

Proof. We start with proving *statement (1)*. For this note that it follows from (5.13) that all images are Frobenius complements if and only $\tilde{g}e$ is central in FG for all $e \in \text{PCI}(FG)$ and all $g \in G$ (if e corresponds to a non-linear representation, then even $\tilde{g}e = 0$). In other words, if and only if $\frac{1}{o(g)}\tilde{g}$ is central in FG for all $g \in G$. Recall that the idempotent $\frac{1}{o(g)}\tilde{g}$ is central exactly when $\langle g\rangle$ is normal in G. The latter is the defining definition of being Dedekind, finishing part (1).

For statement (2) and (3) we first need to classify the subgroups of $Q_8^t \times A$. To start, as G is nilpotent we can decompose it into $G_2 \times G_{2'}$ where G_2 is the Sylow 2-subgroup of G and $G_{2'}$ its Hall $2'$-subgroup which is abelian. On $G_{2'} \leq A$ there is no restriction, thus we will now focus on classifying the possibilities for G_2.

Preliminary work: classifying subgroups of $Q_8^t \times A_2$ with A_2 an abelian 2-group:

Denote $Z_0 := \mathcal{Z}(Q_8^t)$. Recall that $Z_0 \cong C_2^t$ and $V := Q_8^t/Z_0 \cong C_2^{2t}$. We will view V as a $\mathbb{F}_2$-vector space. Since Q_8^t is of nilpotency class 2, the group-commutator induces a non-degenerate alternating bilinear form

$$[\cdot,\cdot] : V \times V \to Z_0.$$

Every subgroup M of Q_8^t is uniquely described by a triple (U, K_0, λ) where

- $U \subseteq V := Q_8^t/Z_0$ is an $\mathbb{F}_2$-subspace,
- $K_0 \leq Z_0$ is a subgroup containing the commutator $[U,U] \subseteq Z_0$ (i.e. U is K_0-isotropic for $[\cdot,\cdot]$).
- $\lambda \in Z^2(U, K_0)$ or equivalently a section $\tau : U \to Q_8^t$ such that

$$\tau(u).\tau(v) = [u,v]\sigma(u+v).$$

and $o(\tau(u)) = 4$ for all non-trivial $u \in U$.

Concretely, to such triple one can associate the following set

$$M_{(U,K_0)} := \{\tau(u)\, k \mid u \in U,\ k \in K_0\}$$

which is a subgroup thanks to the 2-cocycle condition. Note that up to isomorphism the section τ involved in the definition is unique up to a 1-cocycle, so in the sequel we will omit to describe it. Note that the condition $o(\tau(u)) = 4$ comes from the fact that $\exp(Q_8^t) = 4$ and all elements of order 2 are central. Thus U in fact corresponds to choosing order 4 elements.

Next, for a given $G \leq Q_8^t \times A_2$ we consider the projection $\pi : G \to Q_8^t : (g,b) \mapsto g$ and $B := \ker(\pi) = G \cap A_2$. Then the subgroup $G \leq Q_8^t \times A_2$ fits into a central extension

$$1 \to B \to G \to G_t \to 1, \tag{5.7}$$

where $G_t := \mathrm{im}(\pi) \leq Q_8^t$. In other words, G corresponds to a 2-cohomology class $H^2(G_t, B)$ where G_t can be described via a triple as above.

Statement (2): Determining possible images under irreducible representations:

If ρ is an irreducible F-representation of G, then the image $\rho(G)$ must have a faithful irreducible representation. Furthermore, G is nilpotent and thus by Lemma 5.9 a quotient G/N has a faithful irreducible F-representation if and only if $\mathcal{Z}(G/N)$ is cyclic.

Claim: For $N \lhd G$ such that $\mathcal{Z}(G/N)$ is cyclic the quotient G/N is either cyclic or isomorphic to $E \times C_m$ with E an extraspecial 2-group and m odd.

From the above claim follows also the second part of statement (2). Indeed, if G is Dedekind it is isomorphic to $Q_8 \times C_2^k \times A_{odd}$ with $k \in \mathbb{N}$ and A_{odd} some odd abelian group. Therefore E is a quotient of $Q_8 \times C_2^k$ and hence is isomorphic to Q_8. As $\mathcal{Z}(G/N)$ is cyclic, we also see that A_{odd} is send to a cyclic group C_m with m odd.

We will now focus on proving the above claim. For this consider the description of G obtained earlier, i.e. $G \cong G_2 \times G_{2'}$ with $G_{2'}$ abelian and the group G_2 can be written as in (5.7). Furthermore, $\pi(G_2)$ corresponds to a tuple (U, K_0).

Due to the relative prime orders and $G_{2'}$ being abelian, a quotient $(G_2 \times G_{2'})/N$ has cyclic center if and only if $N = N_2 \times N_{2'}$ with $\mathcal{Z}(G_2)/N_2$ and $G_{2'}/N_{2'}$ cyclic. Thus we are reduced to understanding cyclic center quotients of G_2.

We start by investigating the quotients G_2/N with $B \leq N$. In other words, quotients of $\pi(G_2)$. For this, consider a cyclic quotient K_0/N_0 of K_0. Then the group $\pi(G_2)/N_0$

corresponds to the tuple $(U, K_0/N_0)$ and has cyclic center if and only if $[\cdot,\cdot]_{|_{U\times U}}$ is still non-degenerate. Otherwise, $\mathcal{Z}(\pi(G_2)/N_0) = \pi^{-1}(U_c).K_0/N_0$ with $U_c := \operatorname{span}_{\mathbb{F}_2}\{u \in U \mid [u,U] \in N_0\}$. Thus if $N_0 = K_0$, then $\mathcal{Z}(\pi(G_2)/N_0) = \pi^{-1}(U) = \pi(G_2)/N_0$ is abelian. So suppose that $K_0 \gneq N_0$. Then we need that K_0/N_0 is cyclic and $\pi^{-1}(U_c)$ is trivial. Therefore in that case $\pi(G_2)/N_0$ has center of order 2 such that the quotient is an elementary abelian 2-group, i.e. it is extraspecial.

Now consider a quotient G_2/N with $B \not\leq N$. In this case, in order to have a cyclic center one needs that $[B : N \cap B] = 2$. Let $\mathcal{T}^B_{N\cap B} = \{1, x\}$ be left coset representatives and let $\langle y\rangle \leq B$ be a maximal cyclic subgroup of B above $\langle x\rangle$.

Next note that $\tau(\pi(G_2)') = 1$ since $G_2' \cap B = 1$ (because G is class 2 and B abelian). Thus $\mathcal{Z}(G_2) = \pi^{-1}(\mathcal{Z}(\pi(G_2)))$. Therefore we need that $\mathcal{Z}(G_2/N) = \langle z\rangle$ with z such that $\langle y\rangle \leq \langle z\rangle$ and $\mathcal{Z}(\pi(G_2)) = \langle \pi(z)\rangle$. The latter is only possible for particular choices of (U, K_0) for which the quotient will be an extraspecial group of order larger than 8, except if G is a split extension of B with G_t. In that case the quotient is cyclic.

Statement (3): Centrally faithful embeddings for H if G is non-Dedekind.

Let ψ be a faithful irreducible F-representation of H and $\operatorname{Ind}^G_H(\psi) = \sigma_1 \oplus \cdots \oplus \sigma_\ell$ a decomposition of the induced representation into irreducible F-subrepresentations. Since H is normal in G, Proposition 5.12 yields that $H \cap \mathcal{Z}(\sigma_i) = H \cap \mathcal{Z}(G)$ and $H \cap \ker(\sigma_i) = 1$ for all $1 \leq i \leq \ell$. Recall that $\ker(\operatorname{Ind}^G_H(\psi)) = \operatorname{core}_G(\ker(\psi))$. Therefore $\operatorname{Ind}^G_H(\psi)$ is faithful and consequently $\mathcal{Z}(\operatorname{Ind}^G_H(\psi)) = \mathcal{Z}(G)$. Thus G embeds into $\prod_{1\leq i\leq \ell} \sigma_i(G)$. Hence, some $\sigma_i(G)$ must be non-abelian. By part (2), this means that $\sigma_i(G) \cong E \times C_{m_i}$ with m_i odd and E extraspecial. If $E \not\cong Q_8$ we are finished. Thus it remain to consider the case that $\sigma_i(G)$ is either cyclic or of the form $Q_8 \times C_{m_i}$. In particular, as each σ_i is faithful on the non-central subgroup H, we have that H is either $Q_8 \times C_d$ or C_{4d} for some odd d. In both cases H contains a subgroup C_{2d} contained in $\mathcal{Z}(G)$.

In both cases, statement (2) implies that $\sigma_i(G)$ is non-abelian for all i (as the element of order 4 in H can not embed otherwise). Moreover, the index $[\sigma_i(G)_2 : \sigma_i(H)_2]$ is the same for all i and equals 1 or 2 (where $(\cdot)_2$ denotes the Sylow 2-subgroup). Now note that $G \cap \prod_{1\leq i\leq\ell} \sigma_i(H_2) = G \cap \prod_{1\leq i\leq \ell} \sigma_i(G)_2$ is a Sylow 2-subgroup of G. Therefore it must be a subgroup of a Q_8^ℓ and in particular $B = 1$ in the extension (5.7). Consequently, G_2 has an associated tuple (U, K_0).

As mentioned earlier, the set of images $\rho(G)$ coincide with the set of quotients G/N such that $\mathcal{Z}(G/N)$ is cyclic. Denote by $\pi_N : G \to G/N$ the associated projection. We will now make an explicit quotient such that $N \cap H = 1$, $H \cap \pi_N^{-1}(\mathcal{Z}(G/N)) = H \cap \mathcal{Z}(G)$ and $\mathcal{Z}(G/N)$ cyclic.

Decompose $G_{2'} = \langle y\rangle \times R$ with $\langle y\rangle$ a maximal cyclic above $H_{2'} \cong C_d$. The normal subgroup N to mod is chosen to contain R and hence the problem is reduced to find the appropriate quotient for H_2. For ease of notation we now assume that G is a 2-group and hence $H = H_2$ either Q_8 or a C_4 in it.

Recall that G_2 has an associated tuple (U, K_0). Hereby, K_0 is central in G, $\tau(u)$ has order 4 for each non-zero $u \in U$ and $\tau(u)^2 \in K_0$. Denote by $\{u_1, \ldots, u_\ell\}$ a $\mathbb{F}_2$-basis of u and consider the subgroup $N_0 := \langle \sigma(u_i)^2\sigma(u_j)^{-2} \mid 1 \leq i \neq j \leq \ell\rangle$ of K_0. Then K_0/N_0 is cyclic and G/N_0 is extraspecial. Denote by $\overline{\rho}$ a faithful irreducible representation of G/N_0 and with ρ its inflation to G. By construction $H \cap \mathcal{Z}(\rho) = H \cap N_0$ which is central in G. Thus if $\ell \geq 3$, then we are finished. If $\dim_{\mathbb{F}_2} U \leq 1$, then G/K_0 is cyclic with $K_0 \leq \mathcal{Z}(G)$ and hence G abelian, a contradiction with the non-Dedekind assumption. Thus it remains to consider the case that $\dim_{\mathbb{F}_2} U = 2$. Then a quotient with cyclic center of G is either Q_8 or D_8. The latter case is fine and the former would imply that G is Dedekind, showing that modulo N_0 yields the desired representation ρ.

Statement (4): Centrally faithful embeddings for H if G is Dedekind.

For the first dot it is clearly a required property that FG has no division algebra component. Also, if $G \cong Q_8 \times C_2^n$, then $FG \cong F^{\oplus 2^{n+1}} \oplus \left(\frac{-1,-1}{F}\right)^{\oplus 2^n}$. Thus also in the second dot the necessary condition follows. For the necessity recall the Baer–Dedekind classification saying that $G \cong Q_8 \times C_2^n \times A$ with $n \in \mathbb{N}$ and A an odd abelian group. Now the theorem of Perlis-Walker says that

$$FA \cong \bigoplus_{d \mid\mid A|} k_d \frac{[\mathbb{Q}(\zeta_d) : \mathbb{Q}]}{[F(\zeta_d) : F]} F(\zeta_d)$$

with k_d the number of cyclic subgroups of order d and ζ_d an d-th root of unity. Hence the simple components of FG are of the form $F(\zeta_d)$ or $\left(\frac{-1,-1}{F(\zeta_d)}\right)$. Hence if F and d are not such that $\left(\frac{-1,-1}{F(\zeta_d)}\right)$ is totally definite, then the associated representation, say ρ_d, is an allowed one. Moreover, by construction $\rho_d(G) \cong Q_8 \times C_d$. Hence $\mathcal{Z}(\rho_d(G)) = \langle a^2, x, \ker(\rho_d(G))\rangle$ for some $x \in A$ which is a subgroup of $\mathcal{Z}(G)$. In particular, $H \cap \mathcal{Z}(G) = H \cap \mathcal{Z}(\rho)$. $\square$

We are now ready to prove the embedding theorem.

Proof of Theorem 5.1. Note that we may as well assume that H is not central in G. Indeed, if $H \leq \mathcal{Z}(G)$, then of course $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G) = H$ is automatically satisfied for any representation ρ of G. The remainder of the statement, concerning only G, would then follow by replacing H by any non-central cyclic subgroup of G (by assumption, G is not abelian). For the remainder of the proof we assume H to be non-central.

First we suppose that G is not a Dedekind group and tackle the existence of a center-preserving representation ρ for which $\rho(G)$ is not a Frobenius complement. By Theorem 5.10 there exists an irreducible F-representation σ of G such that $H \cap \mathcal{Z}(\sigma) = H \cap \mathcal{Z}(G)$ and $H \cap \ker(\sigma) = 1$. Since we assumed H to be non-central, $\sigma(G)$ is certainly not abelian.

If $\sigma(G)$ is already not a Frobenius complement, then there is nothing more to prove; thus we suppose that $\sigma(G)$ is a Frobenius complement. In consequence, Lemma 5.14 applies to $\sigma(G)$ and yields a subgroup $N \leq \mathcal{Z}(\sigma(G))$ and a faithful irreducible representation τ of $\sigma(G)/N$. Denote by ρ the inflation of τ to G, so that ρ is an irreducible representation of G satisfying $|H \cap \mathcal{Z}(\rho) : H \cap \mathcal{Z}(G)| \leq 2$ as desired, unless $\sigma(G) \cong Q_8 \times C_m$ with m odd. Thanks to the properties of σ, the latter would imply that $H \cong \sigma(H) \leq Q_8 \times C_m$, and in turn that $H/H \cap \mathcal{Z}(\sigma) = H/H \cap \mathcal{Z}(G)$ is a non-trivial subgroup of $C_2 \times C_2 \cong \mathcal{Z}(\sigma(G))$. We now deal with such subgroups H separately, taking into account the other irreducible representations of G.

Let ψ be a faithful representation of H and decompose the induced representation

$$\operatorname{Ind}_H^G(\psi) = \sigma_1 \oplus \cdots \oplus \sigma_\ell \tag{5.8}$$

into irreducible subrepresentations. By Proposition 5.12 the restriction of every σ_i to H is faithful. Recall that $\ker(\operatorname{Ind}_H^G(\psi)) = \operatorname{core}_G(\ker(\psi)) = 1$, therefore $\operatorname{Ind}_H^G(\psi)$ is faithful and consequently $\mathcal{Z}(\operatorname{Ind}_H^G(\psi)) = \mathcal{Z}(G)$.

Since $H/H \cap \mathcal{Z}(G) \cong C_2$ or $C_2 \times C_2$, the index $|H \cap \mathcal{Z}(\sigma_i) : H \cap \mathcal{Z}(G)|$ equals 2 or 4, with 4 only occurring if $H/H \cap \mathcal{Z}(G) \cong C_2 \times C_2$ and $H \subseteq \mathcal{Z}(\sigma_i)$. As $\mathcal{Z}(\operatorname{Ind}_H^G(\psi)) = \mathcal{Z}(G)$ and H non-central, there must be some $1 \leq i \leq \ell$ with $H \cap \mathcal{Z}(\sigma_i) \lneq H$. In particular $\sigma_i(G)$ is non-abelian and $|H \cap \mathcal{Z}(\sigma_i) : H \cap \mathcal{Z}(G)| \leq 2$. Hence if some $\sigma_i(G)$ is not a Frobenius complement, then that σ_i is the desired representation. If $\sigma_i(G)$ is a Frobenius complement, then as before we can use Lemma 5.14 and compose σ_i with the lift modulo some $N_i \leq \mathcal{Z}(\sigma_i(G))$ to obtain a representation ρ_i. Lemma 5.14 implies that $\mathcal{Z}(\rho_i) \gneq \mathcal{Z}(\sigma_i)$ exactly when $\sigma_i(G)$ is a Frobenius complement having a quaternion Sylow 2-subgroup Q_{2^n}, with presentation (5.2), such that $a^{2^{n-3}} \in \sigma_i(H)$ with $\langle a^{2^{n-3}}\rangle$ normal in $\sigma_i(G)$. In that case $4 = o(a^{2^{n-3}})$ divides $|H| = |\sigma(H)|$ and $a^{2^{n-2}} \in \mathcal{Z}(G) \cap H$. In particular, if $H/H \cap \mathcal{Z}(G) \cong C_2$, then $\sigma_i(H) = \langle a^{2^{n-3}}, H \cap \mathcal{Z}(G)\rangle$. If $H/H \cap \mathcal{Z}(G) \cong C_2 \times C_2$, then

$\sigma_i(H) = \langle a^{2^{n-3}}, b, H \cap \mathcal{Z}(G)\rangle$. Note that in both cases $H \cap \mathcal{Z}(\sigma_i(G)) = H \cap \mathcal{Z}(G)$. If $\mathcal{Z}(\rho_i) = \mathcal{Z}(\sigma_i)$, then $[H \cap \mathcal{Z}(\rho_i) : H \cap \mathcal{Z}(G)] = [H \cap \mathcal{Z}(\sigma_i) : H \cap \mathcal{Z}(G)] \leq 2$. Hence ρ_i is the desired representation except if $\sigma_i(G) \cong Q_8 \times C_{m_i}$ for some odd m_i. Thus we are reduced to the case that all $\sigma_i(G)$ are either abelian or Frobenius complements of the form $Q_8 \times C_{m_i}$. In that case G is a subgroup of $Q_8^t \times A$ with A some abelian group. Furthermore, all $\sigma_i(H)$ are normal in $\sigma_i(G)$ (as $Q_8 \times C_{m_i}$ is a Dedekind group) and hence H would be normal in G. Moreover, $H \cong \sigma(H) \leq Q_8 \times C_m$ and hence H is either cyclic or of the form $Q_8 \times C_{m/d}$ with $d \mid m$. For such groups G and H it was shown in Lemma 5.15 that such representation exists when G is not a Dedekind group.

Second, we consider the existence of an irreducible representation ρ with the weaker property that $\rho(FG)$ is not a product of division algebras, respectively $\rho(FG)$ is neither a field, nor a totally definite quaternion algebra. By the first part it remains to consider the case where G is a Dedekind group. In particular, H is a normal subgroup. For such G and H the desired representation ρ was shown to exist in Lemma 5.15. Moreover, it has the stronger property that $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$.

To finish we discuss the stronger property $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$. If $\sigma(G)$ was not a Frobenius complement, then $\rho = \sigma$ satisfied this property. Next suppose that $\sigma(G)$ is a Frobenius complement different of $Q_8 \times C_m$ and consider the representation ρ constructed in that case. From Lemma 5.14 it follows that $H \cap \mathcal{Z}(\rho)$ is larger than $H \cap \mathcal{Z}(G)$ exactly when $\sigma(G)$ has a quaternion Sylow 2-subgroup Q_{2^n}, with presentation (5.2), such that $a^{2^{n-3}} \in \sigma(H)$. In that case $4 = o(a^{2^{n-3}})$ divides $|H| = |\sigma(H)|$ and also $a^{2^{n-2}} \in \mathcal{Z}(G) \cap H$. Furthermore, the Sylow 2-subgroup of G maps onto a generalized quaternoin groups. This shows that the conditions mentioned in the statement are indeed sufficient to ensure $H \cap \mathcal{Z}(\rho) = H \cap \mathcal{Z}(G)$.

It remains to consider the case that $\sigma(G) \cong Q_8 \times C_m$. In particular, $H/H \cap \mathcal{Z}(G)$ is a subgroup of $C_2 \times C_2$. As noticed before, since H is assumed non-central then some σ_i in (5.8) satisfies $H \cap \mathcal{Z}(\sigma_i) = H \cap \mathcal{Z}(G)$, except possibily if the Sylow 2-subgroup of $\sigma(H)$ equals the one of $\sigma(G)$, which is Q_8. Therefore, as $\ker(\sigma) \cap H = 1$, the only possible problematic case is when $H \cong Q_8 \times C_{m/d}$ with $\mathcal{Z}(H) \leq \mathcal{Z}(G)$ for some $d \mid m$. This case is discarded through the assumption $4 \nmid |H|$ or $2 \nmid |H \cap \mathcal{Z}(G)|$, finishing the proof. □

5.2. **Shifted bicyclic units and a conjecture on amalgams.** We will now apply the construction from Examples 4.7.(iii) to the case of the group ring $A = FG$ and finite subgroups of $\mathcal{V}(RG)$. Recall that $\mathcal{V}(RG)$ denotes the kernel of the augmentation map $\epsilon : FG \to F : \sum_i a_i g_i \mapsto \sum a_i$ restricted to $\mathcal{U}(RG)$. Thus $\mathcal{U}(RG) = \mathcal{U}(R) \cdot \mathcal{V}(RG)$. The advantage of working with $\mathcal{V}(RG)$ is that its finite subgroups are R-linearly independent, by a theorem of Cohn and Livingstone [21][11].

Definition 5.16. Let G be a finite group, H be a finite subgroup of $\mathcal{U}(RG)$, and pick $x \in RG$. Set $\widetilde{H} = \sum_{h \in H} h$. The maps

$$\mathrm{sb}_{x,H} : H \to \mathcal{U}(RG) : h \mapsto h + (1-h)x\widetilde{H}$$

and

$$\mathrm{sb}_{H,x} : H \to \mathcal{U}(RG) : h \mapsto h + \widetilde{H}x(1-h)$$

will be called the *(left, resp. right) shifted bicyclic maps associated with H and x.* An element in $\mathcal{U}(RG)$ of the form $\mathrm{sb}_{x,H}(h)$ or $\mathrm{sb}_{H,x}(h)$ will be called a *(left, resp. right) shifted bicyclic unit.*

[11]In [21] this result is shown only for F a number field and R its ring of integers. However the proof of [24, Corollary 2.4] combined with the general version of Berman's theorem stated in [71, Theorem III.1], implies it in the generality claimed here.

Remark 5.17. When $H = \langle h \rangle$ is a cyclic group and $x \in G$, the elements $\mathrm{sb}_{x,\langle h\rangle}(h)$ and $\mathrm{sb}_{\langle h\rangle,x}(h)$ have been called Bovdi units in [42] in honor of Victor Bovdi, who initiated the study of such units. In [57] these units were renamed shifted bicyclic units. Recall that *bicyclic units* are elements of the form

$$\mathrm{b}_{x,h} = 1 + (1-h)x\widetilde{\langle h\rangle} \text{ and } \mathrm{b}_{h,x} = 1 + \widetilde{\langle h\rangle}x(1-h) \tag{5.9}$$

for $h, x \in G$. Note that one can rewrite $\mathrm{sb}_{x,H}(h) = h(1+(1-h)h^{-1}x\widetilde{H})$, hence $\mathrm{sb}_{x,\langle h\rangle}(h) = h\mathrm{b}_{h^{-1}x,h}$. In this sense, shifted bicyclic units are slight (torsion) adaptations of bicyclic units. As the terminology used in [57] better reflects the nature of these units, we adopt it here.

Note that the shifted bicyclic maps from Definition 5.16 are instances of first-order deformations (with $\delta_h := (1-h)x\widetilde{H}$, cf. Examples 4.7.(ii)). As such, the first two properties below follow from the considerations of Section 4.2.

Proposition 5.18. *Let G be a finite group, H be a finite subgroup of $\mathcal{V}(RG)$, and pick $x \in RG$.*

(i) *The shifted bicyclic maps $\mathrm{sb}_{x,H}$ and $\mathrm{sb}_{H,x}$ are injective group morphisms.*
(ii) *The subgroups H, $\mathrm{im}(\mathrm{sb}_{x,H})$ and $\mathrm{im}(\mathrm{sb}_{H,x})$ are conjugate in $\mathcal{U}(FG)$.*
(iii) *If in addition $H \leq G$ and $g \in G$, then the maps $\mathrm{sb}_{g^{-1},H}$ and $\mathrm{sb}_{H,g}$ are the identity on $H \cap H^g$, and $\mathrm{im}(\mathrm{sb}_{g^{-1},H}) \cap \mathrm{im}(\mathrm{sb}_{H,g}) = H \cap H^g$.*

Proof. As just noted, $\mathrm{sb}_{x,H}$ and $\mathrm{sb}_{H,x}$ are first-order deformations of H in FG; it follows from Definition 4.6 that they are group morphisms. Theorem 4.9 states that the identity map and the maps $\mathrm{sb}_{x,H}$ and $\mathrm{sb}_{H,x}$ are all conjugate by $\mathcal{U}(FG)$. In particular, $\mathrm{sb}_{x,H}$ and $\mathrm{sb}_{H,x}$ are injective, and parts (i) and (ii) are proved.

For part (iii), note that $H \cap H^g = \{h \in H \mid h^{-1}ghg^{-1} \in H\}$. Therefore if $h \in H \cap H^g$, then

$$\mathrm{sb}_{g^{-1},H}(h) = h + g^{-1}(1-h)h^{-1}ghg^{-1}\widetilde{H} = h.$$

Similarly $h = \mathrm{sb}_{H,g}(h)$ for $h \in H \cap H^g$, and so $h \in \mathrm{im}(\mathrm{sb}_{g^{-1},H}) \cap \mathrm{im}(\mathrm{sb}_{H,g})$. Conversely, suppose that

$$h + (1-h)g^{-1}\widetilde{H} = b_{g^{-1},H}(h) = b_{H,g}(h) = k + \widetilde{H}g(1-k)$$

for some $h, k \in H$. In other words,

$$h - k + g^{-1}\widetilde{H} - hg^{-1}\widetilde{H} - \widetilde{H}g + \widetilde{H}gk = 0. \tag{5.10}$$

We now use the fact that H is linear independent over R to do a support argument. If $g \in H$, then both $\mathrm{sb}_{g^{-1},H}$ and $\mathrm{sb}_{H,g}$ are the identity map on H and there is nothing to prove; so assume that $g \notin H$. In this case, the support of the element $g^{-1}\widetilde{H} - hg^{-1}\widetilde{H} - \widetilde{H}g + \widetilde{H}gk$ is disjoint from H, from which it follows that $h = k$.

We will proof that $h \in H \cap H^g$. For this take $g^{-1}l \in g^{-1}\widetilde{H}$ which by (5.10) must cancel with either an element of the form $hg^{-1}t$ or tg for $t \in H$ (due to the signs and linear independence of H). In the former case $h = (lt^{-1})^g$, as desired. In the latter case we have that $\mathrm{Supp}\{g^{-1}\widetilde{H}\} = \mathrm{Supp}\{\widetilde{H}g\}$. In particular $g \in g^{-1}H$, i.e. $g^2 \in H$. On this turn this entails that $g^{-1}h \in Hg$, hence also $gh = g^2g^{-1}h \in Hg$, as desired. □

The shifted bicyclic maps can be used to generically (in the sense of independent of knowing the group basis G) construct several kinds of subgroups of $\mathcal{U}(RG)$. For example, they were used (under different terminology) in [42, Proposition 3.2.] to produce solvable subgroups and free subsemigroups of $\mathcal{U}(RG)$. The next proposition displays another construction that makes use of shifted bicyclic maps.

Recall that $I(RG)$ denotes the kernel of the augmentation map $\epsilon : RG \to R$; as an R-module, $I(RG)$ has basis $\{1 - g \mid g \in G\}$. When H is a subgroup of G, we will use the same notation to denote the kernel $I(R[G/H])$ of the RG-module map $\epsilon : R[G/H] \to R : gH \mapsto 1$.

Proposition 5.19. *Let G be a finite group, $H \leq G$, $g \in G$, and set $C = H \cap H^g$. Then*

$$\langle H, \mathrm{im}(\mathrm{sb}_{g,H})\rangle \cong I(\mathbb{Z}[H/C]) \rtimes H,$$

where $h \in H$ acts on $I(\mathbb{Z}[H/C])$ via left multiplication by h^{-1}. In particular, the subgroup of G generated by H and $\mathrm{im}(\mathrm{sb}_{g,H})$ is abelian-by-finite.

Proof. For the sake of convenience, put $\mathrm{sb} = \mathrm{sb}_{g,H}$. Set $M = \langle H, \mathrm{sb}(H)\rangle \leq \mathcal{U}(\mathbb{Z}G)$. Recall that a shifted bicyclic unit is the product of a (generalized) bicyclic unit and the corresponding element of H:

$$\mathrm{sb}(h) = h + (1-h)g\widetilde{H} = (1 + (1-h)g\widetilde{H})h = \mathrm{b}_h h,$$

where we dropped g from the notation $\mathrm{b}_h = 1 + (1-h)g\widetilde{H}$. So, $M = \langle h, \mathrm{b}_k \mid h, k \in H\rangle$. Set $N = \langle \mathrm{b}_k \mid k \in H\rangle$.

We first show that N is a normal complement of H in M, implying that $M \cong N \rtimes H$. Note that $b_{h_1}.b_{h_2} = 1 + ((1-h_1) + (1-h_2))\, g\widetilde{H}$. Thus N consists exactly of the elements of the form $b_a := 1 + ag\widetilde{H}$ with $a \in I(\mathbb{Z}H)$. Moreover, $b_a^n = 1 + nag\widetilde{H}$ and thus b_a is a torsion unit if and only if it is trivial. In particular, N and H must have trivial intersection. Using this set theoretical description of N we easily see that N is normal in M:

$$b_a^x = x^{-1}(1 + ag\widetilde{H})x = 1 + x^{-1}ag\widetilde{H} = b_{x^{-1}a} \in N. \tag{5.11}$$

for all $x \in H$ and $a \in I(\mathbb{Z}H)$.

It remains to prove that N is isomorphic to $I(\mathbb{Z}[H/C])$. Clearly $b_{a_1}b_{a_2} = b_{a_1+a_2}$ for all $a_1, a_2 \in I(\mathbb{Z}H)$ so that we have a group epimorphism $\varphi\colon I(RH) \to N\colon a \mapsto b_a = 1 + ag\widetilde{H}$. Note that for $x, y \in H$ we have $\mathrm{Supp}(xg\widetilde{H}) \cap \mathrm{Supp}(yg\widetilde{H}) \neq \emptyset$ if and only if $xg\widetilde{H} = yg\widetilde{H}$ if and only if $xC = yC$. Thus if we take $a \in I(RH)$ and write it in terms of the group ring, say $a = \sum_{h\in H} a_h h$, we obtain that

$$\varphi\left(\sum_{h\in H} a_h h\right) = 1 + \sum_{hC \in H/C}\left(\sum_{x\in hC} a_x\right) hg\widetilde{H},$$

and hence

$$\mathrm{Ker}(\varphi) = \bigoplus_{t\in T} tI(\mathbb{Z}C),$$

for some left-transversal T of C in H. In conclusion, $N \simeq I(\mathbb{Z}[H/C])$.

Finally note that if we identify N with $I(\mathbb{Z}[H/C])$ then H acts on $I(\mathbb{Z}[H/C])$ via $\varphi\colon H \to \mathrm{Aut}(I(\mathbb{Z}[H/C]))\colon h \mapsto (a \mapsto h^{-1}a)$ by (5.11). □

The proof of Proposition 5.19 shows that the group $N = \langle 1 + (1-h)g\widetilde{H} \mid h \in H\rangle$ is a free abelian group of rank $|H : H \cap H^g| - 1$. In particular, if $H \cap H^g = 1$ then $\langle H, \mathrm{sb}_{-g,H}(H)\rangle \cong I(\mathbb{Z}H) \rtimes H$ is virtually {free abelian of rank $|H| - 1$}, where the action is given by (5.11).

Corollary 5.20. *Let G be a finite group and H a cyclic subgroup of G of prime order p. If $g \in G$ does not normalize H then $\mathcal{U}(RG)$ contains a subgroup isomorphic to $\mathbb{Z}^{p-1} \rtimes C_p$. The action is given by:*

$$\begin{cases} x \cdot e_i = e_{i+1} - e_1, & \textit{for } 1 \leq i \leq p-2 \\ x \cdot e_{p-1} = -e_1 \end{cases}$$

where e_i denotes the generator of the i-th copy of $\mathbb{Z}$ and $C_p = \langle x\rangle$. In particular, if $p = 2$, then $\mathcal{U}(RG)$ contains

$$\langle C_2, B_{g,C_2}(C_2)\rangle \cong \mathbb{Z} \rtimes C_2 \cong C_2 * C_2,$$

the infinite dihedral group.

Remark 5.21. In general, the existence of an abelian subgroup $H \leq G$ yields a free abelian subgroup $F \leq \mathcal{U}(\mathbb{Z}H) \leq \mathcal{U}(\mathbb{Z}G)$ of rank $e = \frac{1}{2}(|H| + 1 + n_2 - 2\ell)$, where n_2 is the number of involutions in H and ℓ is the number of cyclic subgroups of H, cf. [65, Exercise 8.3.1] or [45, Theorem 7.1.6.]. Corollary 5.20 therefore yields a larger than expected free abelian subgroup.

Corollary 5.20 for $p = 2$ suggests that it might be possible to construct free products with finite subgroups of $\mathcal{U}(RG)$ using appropriate shifted bicyclic maps. This is further supported by certain results in the literature **to put refs!** (subject to reformulation in terms of first order deformations) via Theorem 4.12. All this gives evidence for the following conjecture, which echoes Questions 4.5 in the case of a group ring.

Conjecture 5.22. *Let $H \leq G$ be finite groups such that $\mathrm{Cir}_{G,F}(H)$ is non-empty. Let $g \in G$ and denote $C = H \cap H^g$. Then*

$$\langle \mathrm{im}(\mathrm{sb}_{g,H}), \mathrm{im}(\mathrm{sb}_{H,g^{-1}}) \rangle \cong H *_C H \cong \langle \mathrm{im}(\mathrm{sb}_{g,H}), \mathrm{im}(\mathrm{sb}_{g,H})^* \rangle$$

where $(-)^*$ denotes the canonical involution on FG.

Remark 5.23. By Proposition 2.7, the condition that H has an embedding in a simple component is necessary in order for H to appear in a free product. Note in addition that as shown in Proposition 5.18.(iii), the conjectured free product

$$\langle \mathrm{im}(\mathrm{sb}_{g,H}), \mathrm{im}(\mathrm{sb}_{H,g^{-1}}) \rangle \cong H *_C H$$

must at least amalgamated along (the image of) C. In terms of first-order deformations, this is reflected by condition (i) in Theorem 4.12.

If $F = \mathbb{Q}$, G is nilpotent of class 2, $H \cong \mathbb{Z}/n\mathbb{Z}$, and $g \in G$ is such that $H \cap H^g = 1$, then [42, Theorem 4.1] shows that the conditions of Theorem 4.12 are satisfied, and so $H * H$ can be constructed in the conjectured way. If n is prime, this was obtained in [42, Theorem 4.2] for arbitrary (finite) nilpotent groups. In all these cases an explicit embedding of H in a simple component of $\mathbb{Q}G$ was constructed. Recently, Marciniak and Sehgal [57] were able to drop the condition on n without the use of such an embedding. The literature on constructing copies of a free group using bicyclic units is much richer, as we will survey in the next section.

5.3. **Bicyclic units generically play ping-pong.** In RG one can consider following elements which slightly generalize those in (5.9) to those of the form

$$b_{\tilde{h},x} = 1 + (1-h)x\widetilde{h} \text{ and } b_{x,\tilde{h}} = 1 + \widetilde{h}x(1-h) \tag{5.12}$$

with $x \in RG$ and $\widetilde{h} := \sum_{i=1}^{o(h)} h^i$. As $(1-h)\widetilde{h} = 0 = \widetilde{h}(1-h)$, all elements in (5.12) are unipotent units and called *bicyclic units*. The group generated by them we denote

$$\mathrm{Bic}_R(G) := \langle b_{\tilde{h},x}, b_{x,\tilde{h}} \mid x \in RG, h \in G \rangle.$$

It is a fact that

$$\mathrm{Bic}_R(G) \neq 1 \text{ if and only if } G \text{ is not a Dedekind group.}$$

For many years an overarching belief in the field of group rings has been that two bicyclic units should generically generate a free group:

Conjecture 5.24. *Let G be a finite (non Dedekind) group and α bicyclic. Then the set $\{\beta \in \mathrm{Bic}_R(G) \mid \langle \alpha, \beta \rangle \cong \langle \alpha \rangle * \langle \beta \rangle\}$ is 'large' in $\mathrm{Bic}_R(G)$.*

The above conjecture has been intensively investigated for $R = \mathbb{Z}$. See [37] for a quit complete survey untill 2013 and also see [34, 36, 35, 38, 48, 70] and the references therein.

In this section we obtain, as a main application of Theorem 3.23 and Theorem 5.1, a concrete version of Conjecture 5.24, modulo a deformation to a shifted bicyclic unit. We also obtain a variant for a given image under a first order deformation. For the latter we need to consider the following set

$$\mathrm{PCI}_{fpf}(FG) = \{e \in \mathrm{PCI}(FG) \mid Ge \text{ is not fixed point free}\}.$$

Take $e \in \mathrm{PCI}(FG)$ such that FGe is non-commutative. It is well known to experts, [45, Section 11.4], that for such e the following hold:

$$Ge \text{ is fixed point free} \quad \Leftrightarrow \quad \forall g \in G : \widetilde{g}e = 0 \tag{5.13}$$

In particular if FGe is a division algebra, then Ge is fixed point free. Using the notations from (5.1), we have the following necessary and sufficient condition for the existence of a bicyclic ping-pong partner.

Theorem 5.25. *Let F be a number field, R its ring of integers, $H \leq G$ be finite groups and $D : H \to \mathcal{U}(FG)$ a first-order deformation of H. Denote $C := H \cap \mathcal{Z}(G)$. Then the set*

$$S_\alpha := \{\beta \in \mathrm{Bic}_R(G) \setminus \{1\} \mid \langle D(H), \beta\rangle \cong \langle D(H)\rangle *_C (\langle\beta\rangle \times C)\}$$

is non-empty if and only if $\mathrm{Cir}_{G,F}(H) \cap \mathrm{PCI}_{fpf}(FG) \neq \emptyset$. Furthermore,

1. *when $S_\alpha \neq \emptyset$ then it is dense in $\mathrm{Bic}_R(G)$ for the join of the profinite and Zariski topologies.*
2. *if $\mathrm{Fir}_{G,F}(H) \neq \emptyset$ (e.g. H has a faithful irreducible F-representation), $\mathrm{Bic}_R(G) \neq \{1\}$ and $2 \nmid |C|$, then $S_\alpha \neq \emptyset$.*

The condition that $2 \nmid |C|$ in statement (2) of Theorem 5.25 refers to the conditions in Theorem 5.1. In particular, we could also have written $4 \nmid |H|$ or that the Sylow 2-subgroup of G does not map on a generalized quaternion group, see Remark 5.5. Besides, if one allows to enlarge C with an index 2-overgroup in the definition of S_α then no condition on C (or H) is required.

Remark 5.26. The proof of the first part of Theorem 5.25 is ultimately an application of Theorem 3.23. Hence the statement of Theorem 5.25 also holds for a finite family of finite subgroups $H_i \leq G$.

Theorem 5.25 now directly yields an answer to Conjecture 5.24:

Theorem 5.27. *Let F be a number field and R be its rings of integers. Further let G be finite group and $\alpha = 1 + (1-h)x\widetilde{h}$ be a non-trivial bicyclic unit for some $h \in G$ and $x \in RG$. If $\langle h\rangle \cap \mathcal{Z}(G) = 1$, then*

$$S_\alpha = \{\beta \in \mathrm{Bic}_R(G) \mid \langle \alpha h, \beta\rangle \cong \langle \alpha h\rangle * \langle\beta\rangle\}$$

is dense in $\mathrm{Bic}_R(G)$ for the join of the profinite and Zariski topologies.

Note that theorem 5.27 gives a concrete interpretation of 'large' in Conjecture 5.24 for two of the natural topologies. Also remark that Conjecture 5.24 is not a direct instance of Question 1.1 as we still need the non-trivial statement that cyclic subgroups have an appropriate centrally faithful embedding, as provided by Theorem 5.1.

Remark 5.28. By enlarging $\mathrm{Bic}_R(G)$, the condition that $e \in \mathrm{PCI}_{fpf}(FG)$ can be weakened to supposing that FGe is not a division algebra, i.e. to

$$e \in \mathrm{PCI}_{div}(FG) := \{e \in \mathrm{PCI}(FG) \mid FGe \text{ is not a division algebra}\}.$$

More precisely, consider

$$\mathcal{U}(RG)_{un} = \{\alpha \in \mathcal{U}(RG) \mid \alpha \text{ is unipotent }\}.$$

Then exactly the same proof as Theorem 5.27 yields the following statement:

Consider F, G, H, C and D as in Theorem 5.25. Suppose that there exists $e \in \mathrm{Cir}_{G,F}(H)$ such that FGe is not a division algebra, then $\{\beta \in \langle \mathcal{U}(RG)_{un}\rangle \mid \langle D(H), \beta\rangle \cong H *_C (\langle \beta\rangle \times C)\}$ is dense in $\langle \mathcal{U}(RG)_{un}\rangle$ for the join of the profinite and Zariski topologies.

The statements in Theorem 5.25 and Remark 5.28 require to relate it to the setting of Question 1.1. More precisely, the next lemma essentially enables us to use Theorem 3.23. First recall that :

$$\mathrm{SL}_1(FG) := \{x \in \mathcal{U}(FG) \mid \forall e \in \mathrm{PCI}(FG) : \mathrm{RNr}_{FGe/\mathcal{Z}(FGe)}(\pi_e(x)) = 1\}$$

where Rnr denotes the *reduced norm*, i.e. $\mathrm{Rnr}_{A/F}(a) = \det(1_E \otimes_F a)$ for a central simple F-algebra A and a splitting field E of A.

Lemma 5.29. *Let F be a number field and R its ring of integers. Then the following hold:*

- $\mathrm{Bic}_R(G)$ *is Zariski-dense in* $\mathrm{SL}_1(FG)f$ *with* $f = \sum_{e\in \mathrm{PCI}_{fpf}(FG)} e$.
- $\langle \mathcal{U}(RG)_{un}\rangle$ *is Zariski-dense in* $\mathrm{SL}_1(FG)f$ *with* $f = \sum_{e\in \mathrm{PCI}_{div}(FG)} e$.

Proof. Notice that a unipotent unit α in RG projects in any simple component to a unipotent element and in particular has reduced norm 1 there. Thus $\langle \mathcal{U}(RG)_{un}\rangle$ can be viewed as a subgroup of $\mathrm{SL}_1(RG)f$ with $f = \sum_{e\in \mathrm{PCI}_{div}(FG)} e$. As $\mathrm{Bic}_R(G)$ is generated by unipotent elements it is a subgroup of $\langle \mathcal{U}(RG)_{un}\rangle$. By (5.13) the elements in $\mathrm{Bic}_R(G)$ have a trivial projection for $e \in \mathrm{PCI}_{div}(FG) \setminus \mathrm{PCI}_{fpf}(FG)$. In short, the groups in the statement are indeed subgroups of the SL_1's mentioned.

We now proceed to prove the Zariski-density. For this, consider the Wedderburn-Artin decomposition $FG \cong \prod_{e\in \mathrm{PCI}(FG)} \mathrm{M}_{n_e}(D_e)$. Further let $\mathcal{O}_e$ be a maximal order in D_e such that $RGe \subseteq \mathrm{M}_{n_e}(\mathcal{O}_e)$. For every $e \in \mathrm{PCI}(FG)_{div}$ one can clearly choose an idempotent f_e in FG such that ef_e is non-central in FGe. Associated to f_e one has the group of generalized bicyclic units $\mathrm{GBic}^{\{f_e\}}(RG)$, see [45, Section 11.2] for the definition, which is generated by unipotent elements in RG.

If $e \in \mathrm{PCI}_{fpf}(FG)$, then by (5.13) there exists a $g \in G$ with $\widetilde{g}e \neq 0$. As $FGe \cong \mathrm{M}_{n_e}(D_e)$ with $n_e > 1$ and Ge is a F-spanning set, there must even exist a $g \in G$ such that $\widetilde{g}e \notin \mathcal{Z}(FGe)$. Therefore, for such components the choice $f_e = \hat{g} := \frac{1}{o(g)}\widetilde{g}$ can be made for some $g \in G$. By definition, for $f_e = \hat{g}$ one has that $\mathrm{GBic}^{\{f_e\}}(RG) = \mathrm{Bic}_R(G)$. For $e \in \mathrm{PCI}_{fpf}(FG)$ we assume such a choice in the remaining of the proof.

Now, thanks to [41, Theorem 6.3] the group $\mathrm{GBic}^{\{f_e\}}(RG)$ contains a subgroup U_e of the form $1 - e + E_{n_e}(I_e)$ for some non-zero ideal I_e of $\mathcal{O}_e$. Recall that

$$E_n(I_e) := \langle e_{ij}(r) \mid 1 \leq i \neq j \leq n,\, r \in I_e\rangle,$$

where $e_{ij}(r)$ is the elementary matrix in $\mathrm{GL}_{n_e}(\mathcal{O}_e)$ which has 1 on the diagonal and r in the (i, j)-entry.

Consequently the desired statement reduces to prove that $E_n(I)$ is Zariski-dense in $\mathrm{SL}_n(D)$ for I a non-zero ideal in an order $\mathcal{O}$ in a finite dimensional division algebra D and $n \geq 2$. This follows from the classical fact that the Zariski-closure of $\{e_{ij}(x) \mid x \in I\}$ contains the set $\{e_{ij}(y) \mid y \in D\}$. Therefore the Zariski-closure of $E_n(I) = \langle e_{ij}(x) \mid 1 \leq i, j \leq n,\, x \in I\rangle$ equals $E_n(D)$. As for any finite dimensional division algebra, $E_n(D) = \mathrm{SL}_n(D)$, finishing the proof. ☐

Theorem 5.25 now follows readily.

Proof of Theorem 5.27. By Theorem 4.9 D is given by conjugation in FG. Hence $D(H) \cap \mathcal{Z}(G) = H \cap \mathcal{Z}(G)$. Also recall that $H \cap \mathcal{Z}(G) = H \cap \mathcal{Z}(\mathcal{U}(FG))$, as explained in Example 4.4. Thus the choice of C in Theorem 5.27 corresponds to the one in Theorem 3.23.

Next, as $D(H)$ is a finite subgroup of $\mathcal{U}(FG)$ conjugate to H, one has that $\mathrm{Cir}_{G,F}(H) = \mathrm{Cir}_{G,F}(D(H))$. Thus Theorem 3.23 and Remark 3.24 yields that $D(H)$ has a ping-pong partner in $\mathcal{U}(RG)$ if and only if H almost-embeds in a non-division component. However $\mathrm{Bic}_R(G)$ is a subgroup of $\mathrm{SL}_1(FG)f$ with $f = \sum_{e \in \mathrm{PCI}_{fpf}(FG)} e$. Thus clearly a necessary condition is that H almost-embeds in a component where Ge is fixed point free. Also the converse holds by Lemma 5.29 and since that not being fixed point free avoids that $\mathrm{span}_F\{Ge\}$ is a division algebra.

The densely many partners is also provided by Theorem 3.23. Finally, that $\mathrm{Cir}_{G,F}(H) \cap \mathrm{PCI}_{fpf}(FG) \neq \emptyset$ if $\mathrm{Fir}_{G,F}(H)$ is non-empty is Corollary 5.7. That subgroups H having a faithful irreducible representation satisfies the latter condition is a part of Theorem 5.1. □

Finally we record the following neat corollary of independent interest.

Corollary 5.30. *Let F be a number field and R its ring of integers. Furhter let G be a non-abelian finite simple group and $H_1, \ldots, H_\ell$ non-trivial finite subgroups of G. Then there exists a unit $b \in \mathrm{Bic}_R(G)$ such that*

$$\langle H_i, b\rangle \cong H_i * \langle b\rangle$$

for all $1 \leq i \leq \ell$

Proof. In case G is simple, the morphism $G \to Ge$ is clearly an embedding for every primitive central idempotent. Also, from Amitsur's classification [2] (or see the more compact formulation in [77, Theorem 2.1.4]) it follows that the multiplicative group of a division algebra does not contain a non-abelian simple group. In particular, if FGe is a division algebra, then it must be a field. As G is assumed non-abelian and simple the only 1-dimensional representation must be the trivial one.

Next, recall that a group is fixed point free if and only if it is a Frobenius complement (see for instance [45, Proposition 11.4.6]). Furthermore, by [45, Theorem 11.4.6], a Frobenius complement K contains a normal subgroup N such that K/N is a either a normal subgroup of S_4 or S_5 or it is a Z-group (i.e. all Sylow subgroup are cyclic). Thus in all cases the only simple subgroups of K/N are abelian. By the above this implies that Ge is only fixed point free when FGe is the trivial representation. The desired conclusion now follows from Theorem 5.25 and Remark 5.26. □

6. Virtual structure problem for product of amalgam and HNN over finite groups

In this final section we consider the virtual structure problem which was for the first time explicitly formulated in [44] but in fact goes back to the question on 'unit theorems' by Kleinert [54], as explained in Section 1.5.

Question 6.1 (Virtual Structure Problem)**.** Let $\mathcal{G}$ be a class of groups. Classify the finite groups G and ring of integers R such that $\mathcal{U}(RG)$ has a subgroup of finite index in $\mathcal{G}$.

Note that Question 6.1 differs from Kleinert's version as it only asks for the existence of a single finite index subgroup satisfying property $\mathcal{G}$ instead of (almost) all finite index subgroups in any order in FG with $F = \mathrm{Frac}(R)$. In the case that the property $\mathcal{G}$ is defined on commensurability classes, as unit groups of orders are commensurable, then both versions are equivalent and boil down to asking for which G and R the unit group $\mathcal{U}(RG)$ has $\mathcal{G}$. Also note that in the literature only the case $R = \mathbb{Z}$ is considered.

In [44], building on [43, 47, 55, 46], Jespers-Del Rìo answered Question 6.1 for

$$\mathcal{G}_{pab} = \{\prod_i A_{i,1} * \cdots * A_{i,t_i} \mid A_{i,j} \text{ are finitely generated abelian } \}$$

where $t_i = 1$ is allowed (i.e. an abelian factor). It turns out the classification coincide with the case of products of free groups (where again $\mathbb{Z}$ is also allowed). Moreover the problem for the classes $\{A * B \mid A, B \text{ f.g. abelian }\}$ and $\{\text{ free groups }\}$ coincide and there is only four finite groups satisfying this (in all these cases $\pm G$ has a free normal complement in $\mathcal{U}(\mathbb{Z}G)$ [43]).

We will now consider the case

$$\mathcal{G}_\infty := \{\prod_i \Gamma_i \mid \Gamma_i \text{ has infinitely many ends }\}.$$

By Stallings theorem [80, 79] a group has infinitely many ends if and only if it can be decomposed as an amalgamated product or HNN extension over a finite group. In fact we will mainly work with this characterisation. Recall that given a finitely generated group Γ, then the number of ends $e(\Gamma)$ is the defined in terms of its Cayley graph $\mathrm{Cay}(\Gamma, S)$ with S a finite generating set[12]. More precisely, $e(\Gamma)$ is the smallest number m such that for any finite set F the graph $\mathrm{Cay}(\Gamma, S) \setminus F$ has at most m infinite connected components. If no finite m exists one defines $e(\Gamma) = \infty$.

6.1. **Contributions to Kleinert's Virtual Structure Problem.** Despite that the class $\mathcal{G}_\infty$ is much larger than the aforementioned classes, we will now prove that the virtual structure problem for it coincide. We will use the terminology *virtually-$\mathcal{P}$* to say that a given group has a subgroup of finite index with property $\mathcal{P}$.

Theorem 6.2. *Let G be a finite group and F a number field. The following are equivalent:*

(i) *$H \leq \mathcal{U}(\mathcal{O})$ is virtually-$\mathcal{G}_\infty$ for all orders $\mathcal{O}$ in FG and finite index subgroup $H \leq \mathcal{U}(\mathcal{O})$;*

(ii) *$F = \mathbb{Q}$ and $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$;*

(iii) *all the simple components of FG are of the form $\mathbb{Q}(\sqrt{-d})$, with $d \in \mathbb{N}$, $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with non-zero $a, b \in \mathbb{N}$ or $\mathrm{M}_2(\mathbb{Q})$ and the latter needs to occur.*

Moreover, only the parameters $(-1,-1)$ and $(-1,-3)$ can occur for $(-a,-b)$. Also, $e(\mathcal{U}(\mathbb{Z}G)) = \infty$ if and only if $\mathcal{U}(\mathbb{Z}G)$ is virtually free, if and only if G is isomorphic to D_6, D_8, Dic_3, or $C_4 \rtimes C_4$.

In the statement above we used the notation $D_{2n} = \langle a, b \mid a^n = 1 = b^2, a^b = a^{-1}\rangle$, $Dic_3 = \langle a, b \mid a^6, a^3 = b^2, a^b = a^{-1}\rangle$ and $C_4 \rtimes C_4 = \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1}\rangle$. That these groups are exactly those for which $\mathcal{U}(\mathbb{Z}G)$ is virtually free is known since [43, 44], but we will give a short new proof using amalgamated product methods.

Using the description obtained in [55, Theorem 1] in terms of simple components we see that the classes indeed correspond:

Corollary 6.3. *Let G be a finite group. The following are equivalent:*

(i) *$\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$;*

(ii) *$\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of non-abelian free groups.*

In particular, in those cases G is a cut group, i.e. $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite.

It is well known that $e(\Gamma) \in \{0, 1, 2, \infty\}$ for a finitely generated group. By definition, $e(\Gamma) = 0$ if and only if Γ is finite. Moreover, $e(\Gamma) = 2$ if and only if Γ has a subgroup of finite index isomorphic to $\mathbb{Z}$. In case $\Gamma = \mathcal{U}(\mathbb{Z}G)$ the former happens exactly when G is abelian with $\exp(G) \mid 4, 6$ or $G \cong Q_8 \times C_2^n$ for some n (see [45, Theorem 1.5.6.], as proven by Higman). The case that $\mathcal{U}(\mathbb{Z}G)$ is $\mathbb{Z}$-by-finite has not yet been recorded in the literature, but can be obtained using classical methods:

Proposition 6.4. *Let G be a finite group, F a number field and R its ring of integers. Then, the following are equivalent:*

[12]The number of ends is known to be independent of the chosen generating set.

(1) $e(H) = 2$ *for some finite index subgroup* H *in the unit group of an order* $\mathcal{O}$ *in* FG*;*
(2) $e(\mathcal{U}(RG)) = 2$ *and* $F = \mathbb{Q}(\sqrt{-d})$ *for* $d \in \mathbb{N}$ *;*
(3) $\mathcal{U}(RG)$ *is* $\mathbb{Z}$*-by-finite and* $F = \mathbb{Q}(\sqrt{-d})$ *for* $d \in \mathbb{N}$ *;*
(4) G *is isomorphic to* C_n *with* $n = 5, 8$ *or* 12 *and* $F = \mathbb{Q}(\zeta_d)$ *with* $d \nmid n$.

Note that the equivalence (1) $\Leftrightarrow$ (2) imply that if $e(H) = 2$ for some H, then also all other finite index subgroups in an order will have two ends.

Remark 6.5. The proof of Proposition 6.4 also indicates that $\mathcal{U}(RG)$ is virtually a direct product of groups with two ends if and only if all the simple components of FG are either fields or totally definite quaternion algebras. Such finite groups can be described using [74, Theorem 2.6].

In light of Theorem 6.2 and Proposition 6.4, it would be natural to consider the class

$$\mathcal{G}_{\neq 1} := \{\prod_i \Gamma_i \mid e(\Gamma_i) \neq 1\}.$$

In fact, with a bit more of work one can prove that

$$\{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{\neq 1}\} = \{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{pab}\}. \tag{6.1}$$

In the upcoming work [22] applications of Theorem 6.2 to the "blockwise Zassenhaus conjecture" will be investigated. In other words applications to the question whether He is conjugated inside $\mathcal{U}(RGe)$ to a subgroup of Ge for any finite subgroup H of $V(RG)$ for R the ring of integers of some number field.

6.2. **Proofs of the statements.** We start with:

Proof of Proposition 6.4. For the equivalence (1) $\Leftrightarrow$ (2) recall that, see [79, pg38], if Γ_1 and Γ_2 are commensurable then $e(\Gamma_1) = e(\Gamma_2)$. Moreover, the unit group of two orders are commensurable [45, lemma 4.6.9]. Therefore, $e(H) = 2$ if and only if $e(\mathcal{U}(RG)) = 2$ for R the ring of integers of F. Furthermore, for general finitely generated groups Γ one has that $e(\Gamma) = 2$ if and only if Γ is $\mathbb{Z}$-by-finite, see [39, 30] or [79, pg 38]. All this settles (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1).

We will now prove simultaneously that (1) implies (4) and (2). To do so, recall a result by Kleinert [52] (or [45, Corollary 5.5.7]) saying that $\mathcal{U}(RG)$ is abelian-by-finite if and only if all the simple components of FG are either fields or totally definite quaternion algebras. In particular FG has no non-trivial nilpotent elements in which case [74, Theorem 2.6] tells that G is either abelian or $G \cong Q_8 \times C_2^m \times A$ with $m \geq 0$ and A an abelian group of odd order. We first consider the case that $G \cong Q_8 \times C_2^m \times A$. Then

$$FG \cong F[Q_8 \times C_2^m] \otimes_F FA \cong (4m\, F \oplus m \left(\frac{-1,-1}{F}\right)) \otimes_F FA.$$

We now see that in order to obtain a *single* copy of $\mathbb{Z}$ in $\mathcal{U}(RG)$ that this will have to come from a component of FA. However this component will appear at least 4 times and hence such groups are never $\mathbb{Z}$-by-finite.

Now suppose that G is abelian. By the theorem of Perlis-Walker [65, Theorem 3.5.4]

$$FG \cong \bigoplus_{d \| G|} a_d \frac{[\mathbb{Q}(\zeta_d) : \mathbb{Q}]}{[F(\zeta_d) : F]} F(\zeta_d) \tag{6.2}$$

with a_d the number of different cyclic subgroups of order d. Now denote by $R_{F,d}$ the ring of integers of $F(\zeta_d)$ and recall that by Dirichlet Unit theorem [45, Theorem 5.2.4] the rank of the finitely generated abelian group $\mathcal{U}(R_{F,d})$ is $n_1 + n_2 - 1$ with n_1 the number of real embeddings of $F(\zeta_d)$ and r_2 the number of pairs of complex embeddings. Note that the rank of $\mathcal{U}(R_{F,d})$ is at least the one of the unit group of the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_d)$. The latter rank is well-known to be $\frac{\varphi(d)}{2} - 1$. A direct computation yields that

$\varphi(d) \leq 4$ if and only if $d \in \{2,3,4,5,68,10,12\}$ with equality only for $\{5,8,10,12\}$. This combined with (6.2) we see that we have *exactly one* copy of $\mathbb{Z}$ if and only if G is isomorphic to C_n with $n = 5, 8$ or 12, rank $\mathcal{U}(R_{F,n}) =$ rank $\mathcal{U}(R_{\mathbb{Q},n}) = 1$ and rank $\mathcal{U}(R) = 0$. The latter means that F is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$ and hence yields the implication (1) $\Rightarrow$ (2). For the restriction on F written in (4) note that due to the values of n the field $\mathbb{Q}(\zeta_d)$ with $d \nmid n$ is either $\mathbb{Q}$ or an imaginary quadratic extension and hence rank $\mathcal{U}(R) = 0$. Furthermore, $F(\zeta_n) = \mathbb{Q}(\zeta_n)$ and hence rank $\mathcal{U}(R_{F,n}) =$ rank $\mathcal{U}(R_{\mathbb{Q},n})$ as desired. $\square$

We will now start with the proof of Theorem 6.2. This requires the following lemma that is a generalisation of [44, prop. 4.5.].

Lemma 6.6. *Let G be a finite group, D be a finite dimensional division algebra over F with $\mathrm{char}(F) = 0$, different*[13] *of $\left(\frac{-2,-5}{\mathbb{Q}}\right)$, and suppose $\mathrm{M}_n(D)$ with $n \geq 2$ is a simple component of FG. If $\mathcal{O}$ is an order in $\mathrm{M}_n(D)$, then $e(\mathcal{U}(\mathcal{O})) = \infty$ if and only if $n = 2$ and $D = F = \mathbb{Q}$.*

Proof. Suppose $e(\mathcal{U}(\mathcal{O})) = \infty$. As pointed out in the proof of Proposition 6.4, $e(\mathcal{U}(\mathcal{O}_1)) = e(\mathcal{U}(\mathcal{O}_2))$ for two orders $\mathcal{O}_1, \mathcal{O}_2$ in FG, as the number of ends is constant on commensurability classes [79, pg38] and unit groups of orders are commensurable [45, lemma 4.6.9]. Thus without lose of generality we will assume that $\mathcal{O}$ is a maximal order in $\mathrm{M}_n(D)$. It is well known that in that case $\mathcal{O} \cong \mathrm{M}_n(\mathcal{O}_{max})$ with $\mathcal{O}_{max}$ a maximal order in D.

Next recall that any group with infinitely many ends has finite center (as central elements need to be in the subgroup over which the amalgam and HNN are constructed, which is now finite). Therefore, $\mathrm{SL}_n(\mathcal{O}_{max})$ has finite index in $\mathrm{GL}_n(\mathcal{O}_{max})$ and hence $\mathrm{SL}_n(\mathcal{O}_{max}))$ also has infinitely many ends. This implies that $\mathrm{SL}_n(\mathcal{O}_{max})$ has S-rank 1, with S the set of infinite places, as otherwise it has hereditarily Serre's property FA (even property T [59, 29]) and hence can not have a non-trivial amalgam or HNN splitting.

The S-rank being one means that $n = 2$ and D is either $\mathbb{Q}(\sqrt{-d})$, with $d \geq 0$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with a, b strictly positive integers (see for instance [5, Theorem 2.10.]). Furthermore it was proven in [26] that the condition that $\mathrm{M}_2(D)$ is a component of a group algebra yields that $d \in \{0,-1,-2,-3\}$ and $(a,b) \in \{(1,1),(1,3),(2,5)\}$. All these division algebras are (right norm) Euclidean and due to this have a unique maximal order (see [5, remark 3.13]), which we denote $\mathcal{O}_D$. By assumption $(a,b) = (2,5)$ doesn't occur. Now, following [5, Theorem 5.1] $\mathrm{GL}_2(\mathcal{O}_D)$ has property FA except if $D = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-2})$. In case of $D = \mathbb{Q}(\sqrt{-2})$ one can use the amalgam decomposition of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ given in [31, Theorem 2.1] to see that the group doesn't admit a splitting over a finite group. Finally, $\mathrm{GL}_2(\mathbb{Z}) = D_8 *_{C_2 \times C_2} D_{12}$ and hence $e(\mathrm{GL}_2(\mathbb{Z})) = \infty$, finishing the proof. $\square$

We now proceed to the main proof.

Proof of Theorem 6.2. It is well known (see for instance [79, p. 38]) that if Γ_1 and Γ_2 are two groups such that $\Gamma_1 \cap \Gamma_2$ has finite index in the both (i.e. the Γ_i are commensurable), then $e(\Gamma_1) = e(\Gamma_2)$. Also if N is a finite normal subgroup, then $e(\Gamma_1) = e(\Gamma_1/N)$. Using this it is readily observed that the property to be virtually-$\mathcal{G}_\infty$ also enjoy these two properties.

This observation entails that if $H \leq \mathcal{U}(\mathcal{O})$ is virtually-$\mathcal{G}_\infty$ for some order $\mathcal{O}$ in FG, then also all other finite index subgroups of the unit group of any other order is by [45, lemma 4.6.9]. In particular we obtained:
Claim 0: Statement (i) holds if and only if $\mathcal{U}(RG)$, with R the ring of integers in F, is virtually-$\mathcal{G}_\infty$.

[13] This condition is not necessary, i.e the number of ends of $GL_2(\mathcal{O})$ for $\mathcal{O}$ an order in $\left(\frac{-2,-5}{\mathbb{Q}}\right)$ is not infinite. However including this case would make the proof unnecessarily lengthy.

The above claim *yields the implication* $(ii) \Rightarrow (i)$ and the converse follows once we prove that virtually-$\mathcal{G}_\infty$ forces F to be the field of rational numbers.

Next, for the remaining of the proof fix a Wedderburn-Artin decomposition $FG = \bigoplus_{i=1}^q \mathrm{M}_{n_i}(D_i)$. Furthermore, fix a maximal order $\mathcal{O}_i$ in D_i for each i such that $\mathcal{U}(RG)$ is a subgroup of $\prod_{i=1}^q \mathrm{GL}_{n_i}(\mathcal{O}_i)$.

By our starting observation and [45, Lemma 4.6.9.] one has that $\mathcal{U}(RG)$ is virtually-$\mathcal{G}_\infty$ if and only if $\prod_{i=1}^q \mathrm{GL}_{n_i}(\mathcal{O}_i)$ is. Now if $\mathrm{M}_{n_i}(D_i)$ is either a field or a totally definite quaternion algebra, then $\mathrm{SL}_1(\mathcal{O}_i)$ is finite by [45, Proposition 5.5.6]. Also recall that F is in the center of every simple component of FG. Hence, if $\mathrm{M}_2(\mathbb{Q})$ is a simple component, then $F = \mathbb{Q}$. All this combined with Lemma 6.6 now *yields the implication* $(iii) \Rightarrow (ii)$.

Now suppose condition (i). Equivalently, by claim 0, suppose that $\mathcal{U}(RG)$ has virtually-$\mathcal{G}_\infty$. Let $H = \prod_{i=1}^m H_i \in \mathcal{G}_\infty$ be such subgroup of finite index in $\mathcal{U}(RG)$ (so $e(H_i) = \infty$ for all i). We need several observations:

Claim 1: $\mathcal{Z}(\mathcal{U}(RG))$ is finite and hence also $\mathcal{Z}(\mathcal{O}_i)$ is finite for all i.

For this remark that if $e(\Gamma) = \infty$ for some finitely generated group Γ, then $\mathcal{Z}(\Gamma)$ is finite. Therefore also $\mathcal{Z}(H)$ is finite and hence[14] $\mathcal{Z}(\mathcal{U}(RG))$ too. Due to the choice of the order $\mathcal{O}_i$ the latter implies that also $\mathcal{Z}(\mathcal{O}_i)$ is finite for all i.

Next,

Claim 2: Let T be a finitely generated group with $e(T) = \infty$. If $P, Q \lneq T$ are normal finitely generated subgroups such that $|P \cap Q| < \infty$ and PQ of finite index, then P or Q is finite.

Suppose such would exists. Then $e(PQ) = \infty$. Since by assumption $P \times Q \cong PQ/(P \cap Q)$ is commensurable with PQ also $e(P \times Q) = \infty$. However, the Cayley graph of a direct product is the cartesian product of the Cayley graphs. Using this one can see that the number of ends of a direct product of finitely generated groups is always one if P and Q are infinite, a contradiction.

Claim 3: $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$ for all j such that $M_{n_j}(D_j)$ is different of a field or totally definite quaternion algebra (e.g. all j for which $n_j \geq 2$).

Denote $S_j := \mathrm{SL}_{n_j}(\mathcal{O}_j) \cap H$ which is of finite index in $\mathrm{SL}_{n_j}(\mathcal{O}_j)$, hence it is enough to proof that $e(S_j) = \infty$. Let p_k be the projection of H on H_k. Fix some j as in the statement of claim 3. The condition is equivalent with saying that $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is infinite [52]. In particular there exists some k such that $p_k(S_j)$ is infinite[15]. For such k we will now prove that $|p_k(\prod_{i \neq j} S_i)| < \infty$. For this consider $S := S_j \times \prod_{i \neq j} S_i$ which by the first claim is of finite index in H. Therefore $p_k(S)$ is of finite index in H_k and hence $e(p_k(S)) = \infty$. However, $p_k(S_j)$ and $p_k(\prod_{i \neq j} S_i)$ are subgroups as in the second claim[16], yielding the desired. Indeed, the two subgroups clearly commute, are normal in $\pi_k(S)$ and $p_k(S_j) \cap p_k(\prod_{i \neq j} S_i) \subseteq \mathcal{Z}(p_k(S))$ which is finite since $p_k(S)$ has infinitely many ends.

Now consider the set $\mathcal{I}_j := \{k \mid |p_k(S_j)| < \infty\}$. From the previous it follows that if $k \in \{1, \ldots, q\} \setminus \mathcal{I}_j$, then $p_k(S_j)$ is of finite index in H_k. Hence $S_j/(S_j \cap \prod\limits_{i \in \mathcal{I}_j} H_i)$ is a subgroup of finite index in $\prod_{k \notin \mathcal{I}_j} H_k$. As the quotient was with a finite subgroup, we obtain that S_j is virtually-$\mathcal{G}_\infty$ and hence $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ also. However, under the conditions stated in claim 3, SL_1 is virtual indecomposable [53, Theorem 1]. Therefore $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ in fact is even virtually a group with infinitely many ends and so in fact $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$, as claimed.

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[14]The subgroup $H \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \leq \mathcal{Z}(H)$ is of finite index in $\mathcal{Z}(\mathcal{U}(RG))$.

[15]Otherwise S_j would be finite and hence also the overgroup of finite index $\mathrm{SL}_{n_j}(\mathcal{O}_j)$.

[16]Instead of claim 2 one could have used the well known [79, 4.A.6.3.] saying that infinite finitely generated normal subgroups of a group with infinitely many ends need to have finite index.

Altogether: Claim 1 says that $\mathcal{Z}(\mathcal{O}_i)$ is finite and consequently $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is of finite index in $\mathrm{GL}_{n_j}(\mathcal{O}_j)$ for all j. In particular, $e(\mathrm{GL}_{n_j}(\mathcal{O}_j)) = \infty$ if $n_j \geq 2$. Now Lemma 6.6 implies that $n_j = 2$, i.e. no higher matrix algebras appear in the decomposition of FG. In such a case no $\left(\frac{-2,-5}{\mathbb{Q}}\right)$ component arises. Indeed, following [5, table appendix] such a component can only arise if $F = \mathbb{Q}$ and G maps onto one of the groups with SmallGroupID [40,3], [240,89] or [240,90]. But a direct verification, e.g. via the Wedderga package in GAP, shows that these groups all have higher matrix components.

Consequently, Lemma 6.6 says that all matrix components of FG must be isomorphic to $\mathrm{M}_2(\mathbb{Q})$ and in particular $F = \mathbb{Q}$ (as F is contained in the center of every simple component). As pointed out earlier, together with claim 0 this finishes the proof the equivalence $(i) \Leftrightarrow (ii)$.

Furthermore, by [5, Th. 2.10. & Prop. 6.11.], if $\mathbb{Q}Ge$ is a division algebra D for some primitive central idempotent e of $\mathbb{Q}G$ then D is $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{Z}_{\geq 0}$ or a totally definite quaternion algebra over $\mathbb{Q}$. In summary, we obtained that all components of $\mathbb{Q}G$ are of the desired form, *yielding the remaining implication* $(i) \Rightarrow (iii)$.

Next, that only the parameters $(-1,-1)$ and $(-1,-3)$ appear is due to [84, Theorem 11.5.14] saying that else $\mathcal{U}(\mathcal{O})$ is cyclic for any order in $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. In those cases $Ge \leq \mathcal{U}(\mathbb{Z}Ge)$ would have an abelian $\mathbb{Q}$-span and thus $\mathbb{Q}Ge \neq \left(\frac{-a,-b}{\mathbb{Q}}\right)$, a contradiction.

For the last part, first recall that by the commensurability of unit groups of orders $e(\mathcal{U}(\mathbb{Z}G)) = e(\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i))$. However the Cayley graph of a direct product is the cartesian product of the Cayley graphs. Using this we see that $e(Q \times P) = 1$ for any finitely generated group P, Q. Therefore $e(\mathcal{U}(\mathbb{Z}G)) = \infty$ if and only if $e(\mathrm{GL}_{n_{i_0}}(\mathcal{O}_{i_0})) = \infty$ for exactly one i_0 and the other factors are finite. In light of Lemma 6.6 and [5, Theorem 2.10.] this happens exactly when $\mathbb{Q}G$ has exactly one $\mathrm{M}_2(\mathbb{Q})$ component and all the others are $\mathbb{Q}, \mathbb{Q}(\sqrt{-d})$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. Since $\mathrm{GL}_2(\mathbb{Z})$ is virtually free we see that in those cases $\mathcal{U}(\mathbb{Z}G)$ is indeed virtually free.

It remains to prove that the only finite groups for which this happens are D_6, D_8, Dic_3 and $C_4 \rtimes C_4$. Recall that the unit group of the maximal orders of $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ and $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ are respectively $\mathrm{SL}(2,3) \cong Q_8 \rtimes C_3$ and Dic_3. Thus by the work done till now we already know that $\mathcal{U}(\mathbb{Z}G)$ is a subgroup of finite index in $(D_8 \times U) *_{C_2 \times C_2 * U} (D_{12} \times U)$ where $U = A \times \mathrm{SL}(2,3)^s \times Dic_3^t$ for some s, t and with A abelian with $\exp(A) \mid 4, 6$. Using the description of torsion subgroups in amalgamated products we know that, up to conjugation, G is a subgroup of $C_2 \times C_2 * U$ or its contains transversal elements in one of the factors (i.e. D_8 or D_{12}). First suppose G is conjugated to a subgroup of U. Recall that all subgroups of Dic_3 are cyclic and the only non-cyclic one $\mathrm{SL}(2,3)$ is Q_8. As $\mathbb{Q}[\mathrm{SL}(2,3)]$ has a component $\mathrm{M}_3(\mathbb{Q})$ one can conclude that the only way to have exactly matrix component, which moreover is $\mathrm{M}_2(\mathbb{Q})$, is for G to be Dic_3. No suppose G is not conjugated to a subgroup of the amalgamated part. Then we know from Proposition 2.7 that $G \setminus (G \cap (C_2 \times C_2 \times U))$ embeds in $\mathrm{GL}_2(Z)$. If G contains no amalgamated element, then G embeds it needs to be isomorphic to D_6 or D_8 (as D_{12} has two matrix components). In general since $G \cap (C_2 \times C_2 \times U)$ will be a strict subgroup it will be central. Moreover, in order to have not more matrix components, the intersection clearly has to be a central subgroup of order 2. Thus G is a central extension of D_6 or D_8 with a C_2. A look at the groups of order 12 and 16 tells us that G is isomorphic to either Dic_3 or $C_4 \times C_4$, finishing the proof. □

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