SIMULTANEOUS PING-PONG FOR FINITE SUBGROUPS OF REDUCTIVE GROUPS

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Dedicated to the memory of Jacques Tits with great admiration for his legacy.

Abstract. Let Γ be a Zariski-dense subgroup of a reductive group $\mathbf{G}$ defined over a field F. Given a finite collection of finite subgroups H_i $(i \in I)$ of $\mathbf{G}(F)$ avoiding the center, we establish a criterion to ensure that the set of elements of Γ that form a free product with every H_i (so-called ping-pong partners for H_i) is both Zariski- and profinitely dense in Γ. This criterion applies to direct products of inner $\mathbb{R}$-forms of GL_n, and implies a particular case (the case of torsion elements in such products) of a 1994 question of Bekka-Cowling-de la Harpe. We also point out an obstruction to the question. Subsequently, for such $\mathbf{G}$ we give constructive methods to obtain free products between two given finite subgroups.

Next, we investigate the case where $\mathbf{G}(F) = \mathcal{U}(FG)$ for G a finite group and $\Gamma = \mathcal{U}(RG)$ for R an order in F. Hereby our main theorem is that the set of bicylic unit ping-pong partners of a given shifted bicyclic unit is profinitely dense, answering a long standing belief in the field. This is deduced as an application of the above with some new existence results of especially nice irreducible representations of G that are faithful on a given subgroup. Finally, we answer Kleinert's virtual structure problem for the property to have an amalgam or HNN spliting over a finite group.

Contents

2020 *Mathematics Subject Classification.* 20G25,20E06, 20C05.

Key words and phrases. linear algebraic groups, ping-pong, free amalgamated products, group rings, virtual structure problem, bicyclic units, almost-embeddings.

The first and third authors are grateful to the Fonds Wetenschappelijk Onderzoek vlaanderen – FWO (grants 2V0722N & 1221722N) and to the Fonds de la Recherche Scientifique – FNRS (grants 1.B.239.22 & FC 4057) for their financial support. The second author is grateful to the VUB Department of Math and Data Science – VUB for its hospitality.

1. Introduction

1.1. **Background.** The construction and study of free products in linear groups is a classical topic going back to the early days of group theory. A groundbreaking step was Tits' celebrated alternative [82], establishing existence of free subgroups in linear groups which a re not virtually solvable. In fact he proved the stronger statement that if Γ is a finitely generated Zariski-connected linear group over a field F, then either Γ is solvable, or it contains a Zariski-dense free subgroup of rank 2. A version of this statement for non-Zariski-connected groups was given by Breuillard and Gelander in [13], where they prove an analogous (but stronger) theorem for the topology induced by a local field F. The speed at which a given finite set produces two elements generating a free subgroup, has mostly been determined, see [1, 14, 16, 15, 17, 3] for some recent results.

In the present article, we are interested in constructing free subgroups of linear groups, one of whose generators is prescribed. More generally, given a finite subset S of a linear group Γ, the question of interest is: does there exists $\gamma \in \Gamma$ such that for all $h \in S$, the subgroup $\langle h, \gamma \rangle$ is *freely* generated by h and γ? Such an element γ is called a *simultaneous ping-pong partner (in* Γ*) for the set* S.

Following [7], a discrete group Γ is said to have *property (P$_{naï}$)* if any finite subset S of Γ admits a simultaneous ping-pong partner. In 1995, Bekka, Cowling and de la Harpe proved [7, Theorem 3] that Zariski-dense subgroups of connected simple real Lie groups of real rank 1 with trivial center have property (P$_{\mathrm{naï}}$), and asked in [7, Remark 3] whether the same holds for semisimple groups of arbitrary rank. This question was again highlighted in 2007 by de la Harpe [23, Question 17], and we record it here under the following form.

Question 1.1 (Bekka–Cowling–de la Harpe)**.** Let G be a connected adjoint semisimple real Lie group without compact factors, and let Γ be a Zariski-dense subgroup of G. Let S be a finite subset of Γ. Does there exist an element $\gamma \in \Gamma$ such that for every $h \in S$, the subgroup $\langle h, \gamma \rangle$ is canonically isomorphic to the free product $\langle h \rangle * \langle \gamma \rangle$?

It is well known (see [7, Lemmas 2.1 & 2.2]) that property (P$_{\mathrm{naï}}$) for Γ implies that the reduced C*-algebra $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ of Γ is simple and has a unique tracial state; this is, in fact, one of the historical reasons for the interest in property (P$_{\mathrm{naï}}$). Over the years, the simplicity and unique trace property of $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ was established for large classes of groups (see namely [66, 23, 67, 18]). The stronger property (P$_{\mathrm{naï}}$) however remains poorly understood.

Besides [7, Theorem 3] just mentioned, we are aware of the work of Soifer and Vishkautsan [77, Theorem 1.3], which gives a positive answer for $\Gamma = \mathrm{PSL}_n(\mathbb{Z})$ and S only containing elements whose semisimple part is either biproximal[1] or torsion. Contemporary to this work, a positive answer was claimed by Poznansky in his thesis [67, Theorem 6.5], for arbitrary finite subsets S of a semisimple algebraic group $\mathbf{G}$ containing no factor of type $\mathrm{A}_n, \mathrm{D}_{2n+1}$ or E_6. Unfortunately the proof of this theorem contains an error, as it relies on [67, Proposition 2.11] whose statement is not true. Nonetheless, if one assumes that S consists of elements satisfying the conclusion of [67, Proposition 2.11] (that is, of elements whose conjugacy class intersect the big Bruhat cell, see Remark 3.15 for more details) and the elements have an almost-embedding (see later for definition), then the proof of [67, Theorem 6.5] given by Poznansky goes through to the best of our knowledge, and was an instructive source for our work.

1.2. **Outline.** This article consists of essentially two parts.

In the first part, we consider the variant of Bekka, Cowling and de la Harpe's question in which S is actually a finite set of *finite subgroups* in a reductive algebraic group $\mathbf{G}$. In this setting, the statement of Theorem 3.2 gives two conditions jointly implying the

[1][77] uses for *biproximal* the term 'hyperbolic', whereas [67] uses 'very proximal'; see Definition 3.7 for the terminology used here.

existence of simultaneous ping-pong partners for S inside a fixed Zariski-dense subgroup Γ of $\mathbf{G}$. When these conditions are satisfied, we show that the set of simultaneous ping-pong partners for S is both Zariski- and profinitely-dense in Γ.

Subsequently, in Theorem 3.23, we establish these conditions for $\mathbf{G}$ a product of inner $\mathbb{R}$-forms of GL_n or SL_n. As a corollary, we answer in the affirmative (a slightly stronger version of) Bekka, Cowling and de la Harpe's question for subsets S consisting of torsion elements.

In fact one needs to be careful, because unexpectedly Question 1.1 is ill-posed when $\mathbf{G}$ is semisimple, but not simple. Indeed the elements in the set S must all (almost) embeds in a simple quotient of $\mathbf{G}$ (see Proposition 2.7). Theorem 3.23 shows that this is in fact sufficient.

A particular instance of this setting is when $\mathbf{G}$ is the unit group of a finite-dimensional semisimple algebra A over a number field. In that instance, we show in Theorem 4.1 that ping-pong partners exists for a finite subgroup H if and only if H has an almost-embedding in a "good" factor of A, that is, in a factor that is neither a field nor a totally definite quaternion algebra. When such almost-embedding exists, Section 4 aims at providing a constructive method to construct a simultaneous partner. We introduce there the convenient formalism of first-order deformations of a subgroup (see Definition 4.4), which will be used to deform finite subgroups H_1 and H_2 of the unit group $\mathcal{U}(A)$ of A to put them in ping-pong position (Theorem 4.10). In passing, we record that such deformations are in fact obtained by specific conjugation inside A (Theorem 4.7), a consequence of Hochschild's cohomology theory.

In the second part of the paper, we will delve into the case of the unit group $\mathcal{U}(FG)$ of the group algebra of a finite group G. For these groups, interesting interplay with the representation theory of G arises. As an illustration, it follows from Theorem 5.7 (see Corollary 5.10) that if a subgroup H of G embeds in a simple factor of FG, then there is also an almost embedding in a "good" simple factor.

The goal of the second part is to obtain answers to old open problems in the field of group rings. There has been for long an active interest in this field for a particular kind of unipotent elements, the so-called *bicyclic units*, which arise naturally in the study of the group ring RG where R is the rings of integers of F. For several decades it has been a long standing belief that two bicyclic units should generically generate a free group, a claim that we substantiate in Theorem 5.17.

This is done by first proving that the group of bicyclic units is always Zariski-dense in the group of unimodular elements of an "appropriate part" of the group ring. At that stage, using the main results of the first part of the paper, the aforementioned belief is reduced to show that any cyclic subgroup $\langle h \rangle$ of G has an almost-embedding into that "appropriate part". Concretely, our next main result is Theorem 5.7 where we obtain that any finite subgroup H having a faithful irreducible F-representation (e.g. nilpotent with cyclic center) has an almost embedding in a simple component of FG where the projection $p(G)$ of G is not fixed point free, a result of independent interest. That $p(G)$ is not fixed point free for instance imply that $\mathrm{span}_F\{p(G)\}$ is not a division algebra.

As a final application of the methods, we study the virtual structure problem, which asks to classify all finite groups G for which the unit group of every order in FG satisfies some prescribed structural property (see below for a precise question). In Theorem 6.2, we establish an explicit description of these groups, for the property of admitting an amalgamated or HNN splitting over a finite group.

We now give a more detailed account of the main results.

1.3. **Simultaneous ping-pong with finite subgroups.** After a short recollection on the structure of free amalgamated products in Section 2, we will consider in Section 3 a slightly more general version of Question 1.1, as we allow the finite set S to consist of subgroups (not just elements). To this end, we study the dynamics of linear transformations on projective spaces over division algebra; the details are contained in Section 3.2, and are mostly a rework of classical results of Tits, themselves already revisited by several authors. The main result of that section is Proposition 3.11, stating the abundance of simultaneously biproximal elements (when they exist).

These developments are necessary to prove the main result of Section 3:

Theorem 3.2. *Let $\mathbf{G}$ be a connected algebraic F-group with center $\mathbf{Z}$. Let Γ be a Zariski-connected subgroup of $\mathbf{G}(F)$. Let $(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(F)$, and set $C_i = H_i \cap \mathbf{Z}(F)$. Assume that for each $i \in I$ there exists a local field K_i containing F and a projective K_i-representation $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$, where V_i is a finite-dimensional module over a finite division K_i-algebra D_i, with the following properties:*

(Proximality) *$\rho_i(\Gamma)$ contains a proximal element;*

(Transversality) *For every $h \in H_i\backslash C_i$ and every $p \in \mathbb{P}(V_i)$, the span of the set $\{\rho_i(xhx^{-1})p \mid x \in \Gamma\}$ is the whole of $\mathbb{P}(V_i)$.*

Let S be the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order, such that for all $i \in I$, the canonical map

$$(\langle\gamma\rangle \times C_i) *_{C_i} H_i \to \langle\gamma, H_i\rangle \leq \mathbf{G}(F)$$

is an isomorphism. Then S is dense in Γ for the join of the profinite topology and the Zariski topology.

The transversality condition gets its name for its role in Lemma 3.14. As hinted in Lemma 3.14 and Remark 3.4, transversality is a kind of "higher irreducibility condition".

Next, in Section 3.4 we verify the proximality and transversality conditions for finite subgroups in products of inner forms of SL_n and GL_n. As a consequence of Theorem 3.2, this proves the abundance of simultaneous ping-pong partners in Zariski-dense subgroups in this setting:

Theorem 3.23. *Let $\mathbf{G}$ be a reductive $\mathbb{R}$-group, and let Γ be a subgroup of $\mathbf{G}(\mathbb{R})$ whose image in $\mathrm{Ad}\,\mathbf{G}$ is Zariski-dense. Let $(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(\mathbb{R})$.*

Suppose that for each $i \in I$, the subgroup H_i almost embeds in a simple quotient $\mathbf{Q}_i$ of $\mathbf{G}$ isogenous to $\mathrm{PGL}_{D_i^{n_i}}$, for D_i some finite dimensional division $\mathbb{R}$-algebra and $n_i > 1$. Then the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order such that for all $i \in I$, the canonical map

$$(\langle\gamma\rangle \times C_i) *_{C_i} H_i \to \langle\gamma, H_i\rangle \leq \mathbf{G}(\mathbb{R})$$

is an isomorphism, is dense in Γ for the join of the profinite topology and the Zariski topology.

Given a reductive F-group $\mathbf{G}$ with center $\mathbf{Z}$ and a subgroup $H \leq \mathbf{G}(F)$, we say that *H almost embeds in a (simple) quotient $\mathbf{Q}$ of $\mathbf{G}$* if there exists a (simple) quotient $\mathbf{Q}$ of $\mathbf{G}$ for which the kernel of the restriction $H \to \mathbf{Q}(F)$ is contained in $\mathbf{Z}(F)$. As mentioned in Remark 3.24, this is a necessary condition for the subgroup H to admit a ping-pong partner in $\mathbf{G}(F)$.

Despite the abundance of simultaneous ping-pong partners, the above theorem does not immediately give an explicit construction of such partners. In Section 4, we will provide an explicit construction for certain Zariski-dense subgroups of algebraic groups $\mathbf{G}$ which are a direct product of inner forms of SL_n for $n \geq 2$. We also investigate whether two given finite subgroups H and K of $\mathbf{G}(F)$ can appear as the factors of a free product $H * K$ inside $\mathbf{G}(F)$. To this end, we introduce *first-order deformations* of H in Section 4.2, which are linear deformations of H suitable to obtain appropriate dynamics. The main result of that section is Theorem 4.10.

1.4. **The case of semisimple algebras and the unit group of a group ring.** Let A be a finite dimensional semisimple algebra over F. By the Artin–Wedderburn theorem,

$$A \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_m}(D_m).$$

as F-algebra, for D_i some finite-dimensional division algebras over F. In particular, the F-group $\mathcal{U}(A)$ of units of A is the reductive group

$$\mathrm{GL}_{D_1^{n_1}} \times \cdots \times \mathrm{GL}_{D_m^{n_m}} .$$

Furthermore, if $\mathcal{O}$ is an order in A, then by a classical result of Borel and Harish-Chandra $\Gamma = \mathcal{U}(\mathcal{O})$ is an arithmetic subgroup of $\mathcal{U}(A)$, placing us in the setting of Theorem 3.23.

Existence of almost-embeddings and ping-pong partners. We deduce from the aforementioned the following criterion for the existence of simultaneous ping-pong partners for a finite subgroup of $\mathcal{U}(A)$.

Theorem 4.1. *Let F be a number field, A be a finite semisimple F-algebra, and $\mathcal{O}$ be an order in A. Let Γ be a Zariski-dense subgroup of $\mathcal{U}(\mathcal{O})$. Let H be a finite subgroup of $\mathcal{U}(A)$, and C be its intersection with the center of A.*

There exists $\gamma \in \Gamma$ of infinite order with the property that the canonical map

$$(\langle\gamma\rangle \times C) *_C H \to \langle\gamma, H\rangle$$

is an isomorphism, if and only if H almost embeds in a simple factor of A which is neither a field nor a totally definite quaternion algebra. Moreover, in the affirmative, the set of such elements γ is dense in the join of the Zariski and the profinite topology.

In the specific case where $A = FG$ is a group algebra, the simple components $\mathrm{M}_{n_i}(D_i)$ of A are not arbitrary. Indeed, each $\mathrm{M}_{n_i}(D_i)$ is precisely the F-span of the projection of G in the ith factor of this decomposition of FG. Thanks to this the simple factors are related to each other via the representation theory of one common finite group G. For instance, as shown in Corollary 5.11, the condition in Theorem 4.1 to have an almost embedding in an appropriate simple factor simplifies to having an almost embedding in any type of simple factor. This is a consequence of our main almost-embedding theorem:

Theorem 5.7. *Let F be a field of characteristic 0, G a finite group and $H \leq G$ such that it admit a faithful irreducible F-representation $\psi \in \mathrm{Irr}(H)$. Then there exists an irreducible F-representation $\rho \in \mathrm{Irr}(G)$ such that*

(1) $H \cap \ker(\rho) \subseteq \mathcal{Z}(G)$,

(2) $\rho(G)$ is not *a Frobenius complement.*

if and only if G is not a Dedekind group. In particular, $\rho(FG)$ is not a division algebra.

The work above also allows us to give for most subgroups H necessary and sufficient conditions on when an amalgamated product of the form $H *_{H\cap\mathcal{Z}(G)} H$ exists in $\mathcal{U}(RG)$.

Corollary 5.10. *Let F be a number field and R its ring of integers. Further let G be a finite group and $H \leq G$ such that H embeds in a simple factor of FG. Then, there exists some $t \in \mathcal{U}(RG)$ such that*

$$\langle H, t\rangle \cong H *_C \langle t, C\rangle \cong H *_C (\mathbb{Z} \times C).$$

where $C = \langle h\rangle \cap \mathcal{Z}(G)$ if and only if $G \not\cong Q_8 \times C_2^n$ for some n. In particular, $H *_C H$ exists in $\mathcal{U}(RG)$ in that case.

When $G \cong Q_8 \times C_2^n$, then $\mathcal{U}(RG)$ consists only of trivial units and hence contain no non-trivial amalgamated products.

Bicyclic units yield generically free products. Next, we focus on the construction of free products with certain specific units in $\mathcal{U}(RG)$. More precisely, consider in RG the unipotent elements of the form

$$b_{\tilde{h},x} = 1 + (1-h)x\widetilde{h} \quad \text{and} \quad b_{x,\tilde{h}} = 1 + \widetilde{h}x(1-h), \quad \text{for } x \in RG, h \in G, \text{ and } \widetilde{h} := \textstyle\sum_{i=1}^{o(h)} h^i,$$

called *bicyclic units.* The group they generate is denoted

$$\mathrm{Bic}_R(G) = \langle b_{\tilde{h},x}, b_{x,\tilde{h}} \mid h \in G, x \in RG \rangle.$$

These bicyclic elements constitute one of the few known generic constructions of units in RG. We record in passing the new construction given in [42].

For almost 30 years it has been conjectured in the field of group rings that two *generically chosen* bicyclic units generate a free group. That being said, the meaning to give to 'generic' has, to our knowledge, never been made precise. See [38] for a quit complete survey until 2013 and also see [35, 37, 36, 39, 49, 70] (and the references therein). This conjecture, for which we obtain an affirmative answer, was the original motivation for this work.

To start we propose a new point of view on bicyclic units, as being first-order deformations of torsion units. These deformations are maps of the form

$$\Delta : H \to \mathcal{U}(FG) : h \mapsto \Delta(h) = h + \delta_h$$

which furthermore are group morphisms. The latter is encoded in two properties on $\delta = \Delta - 1$, see Definition 4.4. In the case of bicyclic units $\delta_h = (1-h)x\widetilde{h}$ (or $\widetilde{h}x(1-h)$). In particular, the image of Δ lies inside $\mathcal{U}(RG)$, despite Theorem 4.7 saying that first order deformations are given by conjugation in FG.

In the literature it has often been an obstacle that $\mathrm{Bic}_R(G)$ might be of infinite index in $\mathrm{SL}_1(RG)$ if FG has certain low rank simple factors. Lemma 5.19 forms a next important observation: $\mathrm{Bic}_R(G)$ is nevertheless Zariski dense in $SL_1(FGf)$ with f the sum of all primitive central idempotents e such that Ge is not fixed point free. The subalgebra FGf is the 'appropriate part" referred to earlier. This restriction arises due to the fact that Ge is fixed point free if and only if $\widetilde{h}e$ is central for all $1 \neq h \in G$. Hence bicyclic units project to the identity in such simple components. Due to this, Proposition 2.7 tells that a necessary condition for $\langle \alpha h \rangle$ to have a bicyclic ping-pong partner is that $\langle h \rangle$ embeds in a component where Ge is not fixed point free.

The main result states that if one replaces the given bicyclic unit $b_{\tilde{h},x}$ by its closely related variant $b_{\tilde{h},x}h = h + (1-h)x\widetilde{h}$ (called *shifted bicyclic units* in the literature), this long-standing conjecture holds true for a profinitely-dense and Zariski-dense subset of units.

Theorem 5.17. *Let F be a number field and R be its rings of integers. Further let G be finite group and $\alpha = 1 + (1-h)x\widetilde{h}$ be a non-trivial bicyclic unit for some $h \in G$ and $x \in RG$. Denote $C = \langle h \rangle \cap \mathcal{Z}(G)$. Then*

$$S_\alpha = \{\beta \in \mathrm{Bic}_R(G) \mid \langle \alpha h, \beta \rangle \cong \langle \alpha h \rangle *_C (\langle \beta \rangle \times C)\}$$

is dense in $\mathrm{Bic}_R(G)$ *for the join of the profinite and Zariski topologies.*

Theorem 5.17 follows from a combination of two interesting results on its own. The first is a general statement, Theorem 5.15, about the existence of bicyclic ping-pong partners for arbitrary first-order deformations of a finite group H in RG. Namely, there exists one (and subsequently densely many) if and only if H has an almost-embedding in a simple component where the projection of G is not fixed point free. This theorem is an application of the Zariski density result and of Theorem 3.23.

The second ingredient for Theorem 5.17 is the almost-embedding result, i.e. Theorem 5.7 mentioned earlier. It namely shows the existence of such nice almost-embedding for every subgroup H of G that embeds in a simple factor of FG. This situation is shown to occur when H has a faithful irreducible F-representation (a condition well understood in the literature).

Finally, note that such embedding result does not hold for any (cyclic) subgroup of FG and hence Theorem 5.7 is truly a representation theoretical result. As such Theorem 5.17 turned out to be a combination of a result in algebraic groups and one in representation theory.

1.5. **The virtual structure problem for amalgams or HNN extensions over finite groups.** Lastly, we consider the the Virtual Structure Problem, which loosly asks for a structure theorem for unit groups of orders. The latter is also called a 'unit theorem' and a concrete meaning was formulated in Kleinert's 1994 survey [55]:

> *"A unit theorem for a finite-dimensional semisimple rational algebra A consists of the definition, in purely group-theoretical terms, of a class of groups $C(A)$ such that almost all generic unit groups of A are members of $C(A)$."*

Recall that a generic unit group of A is a subgroup of finite index in the group of elements of reduced norm 1 of an order[2] in A.

One instance of a unit theorem in the sense of Kleinert is when the class $C(A)$ consists of all groups satisfying some prescribed group theoretical property $\mathcal{P}$. For instance when A is chosen to be a group algebra FG, then such interpretation reformulates to:

Classify all finite groups G such that (almost) all generic unit groups of FG satisfy property $\mathcal{P}$.

Up to our knowledge, the only properties $\mathcal{P}$ for which such a unit theorem is known are the following:

- $\mathcal{P} = \{$ finite groups$\}$, [46, Corollary 5.5.8]
- $\mathcal{P} = \{$ abelian groups$\}$, [53]
- $\mathcal{P} = \{$ solvable groups$\}$, [53, Theorem 2]
- $\mathcal{P} = \{$ direct product of free-by-free groups $\}$ ([48, 45, 64, 50]),
- $\mathcal{P}$ are groups satisfying some fixed point property such as property (T) or HFA ([5, 4]).

Remarkably, in all these case the property can also be described in terms of the rational group algebra. For example in case that $\mathcal{P}_{free} := \{$ direct product of free-by-free groups $\}$, one has that all generic unit groups of $\mathbb{Q}G$ have $\mathcal{P}_{free}$ if and only if every simple factor of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or $\mathrm{M}_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$.

In this article, we address the problem above for the class of groups

$$\mathcal{P}_\infty := \{\prod_i \Gamma_i \mid \Gamma_i \text{ has infinitely many ends}\}.$$

By Stallings' theorem [79, 78], a group has infinitely many ends if and only if it can be decomposed as a non-trivial amalgamated product or HNN extension over a finite group. (In fact, we will mostly work with this characterization.)

Theorem 6.2. *Let G be a finite group and F a number field. The following are equivalent:*

(i) *$H \leq \mathcal{U}(\mathcal{O})$ has $\mathcal{P}_\infty$ for all orders $\mathcal{O}$ in FG and finite index subgroup $H \leq \mathcal{U}(\mathcal{O})$;*

(ii) *$F = \mathbb{Q}$ and $\mathcal{U}(\mathbb{Z}G)$ has $\mathcal{P}_\infty$;*

(iii) *$F = \mathbb{Q}$ and $\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of non-abelian free groups;*

[2]Here, and in the remaining of this article, an order refers to a $\mathbb{Z}$-order, but the question also makes sense for more general orders.

(iv) *all the simple components of FG are of the form $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{N}$, $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with non-zero $a, b \in \mathbb{N}$ or $\mathrm{M}_2(\mathbb{Q})$, and the latter needs to occur.*

Moreover, only the parameters $(-1,-1)$ and $(-1,-3)$ can occur for $(-a,-b)$. Also, $\mathcal{U}(\mathbb{Z}G)$ itself has infinitely many ends if and only if it is virtually free, if and only if G is isomorphic to D_6, D_8, Dic_3, or $C_4 \rtimes C_4$.

The finite groups satisfying assertion (iii) in Theorem 6.2 have been classified in [45], so the theorem does indeed answer the Virtual Structure problem for the class $\mathcal{P}_\infty$.

Acknowledgments. Andreas Bächle was involved in earlier stages of this project, and we thank him heartily for many helpful discussions and enjoyable moments. We thank Emmanuel Breuillard for interesting conversations regarding Section 3. We also thank Leo Margolis for his is interest and some clarifications around the Zassenhaus conjecture, and Špela Špenko and Eric Jespers for helpful conversations on Section 3, resp. Section 6. Furthermore we are grateful to Jean-Pierre Tignol for providing us with a better proof of Theorem 4.7, and to Justin Vast and Miquel Martínez for useful discussions on Theorem 5.7. Lastly, the second author wish to thank Jairo Zacarias Gonçalves for his hospitality during a visit at the Univeristy of Sao Paulo, where Section 4 was started.

2. Amalgams in almost-direct products

In this section, we recall a variant for amalgamated products of the classical ping-pong lemma. Thereafter we exhibit a necessary condition for a subgroup of an almost-direct product to be an amalgamated product.

Given a subgroup C of a group G, we will denote by T_C^G a set of representatives of the left cosets of C in G, containing the identity element.

The ping-pong lemma for amalgams and its variant for HNN extensions can be found in [57, Propositions 12.4 & 12.5]. For the convenience of the reader, we provide a proof as it will be instrumental in the rest of this paper.

Lemma 2.1 (Ping-pong for amalgams)**.** *Let A, B be subgroups of a group G and suppose $C = A \cap B$ satisfies $|A : C| > 2$. Let G act on a set X. If $P, Q \subset X$ are two subsets with $P \not\subset Q$, such that for all elements $a \in T_C^A \setminus \{e\}$, $b \in T_C^B \setminus \{e\}$ and $c \in C$, we have*

$$aP \subset Q, \quad bQ \subset P, \quad cP \subset P, \quad \text{and} \quad cQ \subset Q,$$

then the canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism.

As in the case of free products, the proof of Lemma 2.1 is straightforward once one knows the normal form for elements in an amalgamated product. The normal form also allows us to unambiguously speak of *words starting with A* and *words starting with B*. In the next lemma, these are the elements for which $\dot{a}_1 \notin C$, resp. for which $\dot{a}_1 \in C$.

Lemma 2.2 (Normal form in amalgams)**.** *Let $A, B \le G$ be groups and $C \le A \cap B$. The following are equivalent.*

(i) *The canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism.*

(ii) *Every element in $\langle A, B\rangle$ has a unique decomposition of the form $\dot{a}_1 b_1 \cdots a_n \dot{b}_n c$, where $a_i \in T_C^A \setminus \{e\}$, $b_i \in T_C^B \setminus \{e\}$, $\dot{a}_1 \in T_C^A$, $\dot{b}_n \in T_C^B$, and $c \in C$.*

(iii) *Given $a_i \in A \setminus C$, $b_i \in B \setminus C$, $\dot{a}_1 \in A$, and $\dot{b}_n \in B$, the product $\dot{a}_1 b_1 \cdots a_n \dot{b}_n$ belongs to C only if $n = 1$ and $\dot{a}_1, \dot{b}_n \in C$.*

In consequence of the affirmative, $C = A \cap B$.

Sketch of proof. The implication (i) $\Longrightarrow$ (ii) is the existence and uniqueness of a normal form (see for instance [74, Theorem 1]), and its converse amounts to checking the injectivity of the canonical map, which follows from the uniqueness of the decomposition in $\langle A, B\rangle$.

After replacing $\dot{b}_n, a_n, \ldots, b_1, \dot{a}_1$ by the appropriate coset representatives, (ii) $\Longrightarrow$ (iii) becomes obvious. For the contrapositive of its converse, note that two different decompositions of an element in $\langle A, B\rangle$ result in a non-trivial expression of the form $\dot{a}_1 b_1 \cdots a_n \dot{b}_n$ in C. □

Proof of Lemma 2.1. Note that the assumptions imply that $aP_1 \subset P_2$ for all $a \in A \setminus C$, $bP_2 \subset P_1$ for all $b \in B \setminus C$, and $cP_1 = P_1$, $cP_2 = P_2$ for every $c \in C$.

Suppose that given $a_i \in A \setminus C$, $b_i \in B \setminus C$, $\dot{a}_1 \in A$ and $\dot{b}_n \in B$, the non-empty word $c = \dot{a}_1 b_1 \cdots a_n \dot{b}_n$ lies in C. The possible cases for $\dot{a}_1$ and $\dot{b}_n$ to belong to C are:

- $\dot{a}_1 \notin C$, $\dot{b}_n \in C$. We have $\dot{b}_n P_1 = P_1$, $a_n \dot{b}_n P_1 \subset P_2$, $b_{n-1} a_n \dot{b}_n P_1 \subset P_1$, etc., so that eventually, $cP_1 = \dot{a}_1 b_1 \cdots a_n P_1 \subset P_2$. Since $cP_1 = P_1$ and $P_1 \not\subset P_2$, this case cannot occur.
- $\dot{a}_1 \in C$, $\dot{b}_n \notin C$. Pick $a \in A \setminus C$, and let $a' \in A$ and $c' \in C$ be such that $a^{-1} c a = a' c'$. We have $aa' \notin C$, hence the word $c' = (aa')^{-1} b_1 \cdots a_n \dot{b}_n a$ starts and ends with an element of $A \setminus C$. This case thus reduces to the first one.

- $\dot{a}_1 \notin C$, $\dot{b}_n \notin C$. As $|A : C| > 2$, we may pick $a \in A \setminus (C \cup \dot{a}_1 C)$, so that $a^{-1}\dot{a}_1 P_1 \subset P_2$ hence $\dot{a}_1 P_1 \subset aP_2$. As in the first case, we have $cP_2 \subset \dot{a}_1 P_1$. Since $cP_2 = P_2$, this would imply $aP_1 \subset P_2 \subset aP_2$, hence this case does not occur either.
- $\dot{a}_1 \in C$, $\dot{b}_n \in C$. If $n > 1$, replacing c by c^{-1} reduces to the third case. The only remaining possibility is thus $n = 1$ and $\dot{a}_1, \dot{b}_n \in C$, as expected.

We conclude from Lemma 2.2 that the canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism. □

Lemma 2.3. *Let $A*_C B$ be a free amalgamated product. If f is a surjective morphism from a group Γ to $A *_C B$, then Γ is the free product with amalgamation $f^{-1}(A) *_{f^{-1}(C)} f^{-1}(B)$.*

If moreover Γ is generated by two subgroups Γ_1, Γ_2 with the properties $f(\Gamma_1) \subseteq A$, $f(\Gamma_2) \subseteq B$, the induced map $\Gamma_1 \to A/C$ is injective, and $\Gamma_1(\Gamma_2 \cap f^{-1}(C))$ is a subgroup, then $f^{-1}(B) = \Gamma_2$ and $\Gamma \cong (\Gamma_1 f^{-1}(C)) *_{f^{-1}(C)} \Gamma_2$.

Proof. The first part of the lemma is standard (see for instance [86, Lemma 3.2]). For the second part, let $g = b_0 a_1 b_1 \cdots a_n b_n$ with $a_i \in \Gamma_1 \setminus \{e\}$ and $b_i \in \Gamma_2$ be an element of $f^{-1}(B)$. Since $(\Gamma_2 \cap f^{-1}(C))\Gamma_1 = \Gamma_1(\Gamma_2 \cap f^{-1}(C))$, after perhaps reducing the expression for g, we may assume that $b_i \notin f^{-1}(C)$ for $0 < i < n$. Because $f(\Gamma) = A *_C B$ and $f(b_i) \in B \setminus C$, Lemma 2.2 implies that $n = 0$, hence $g = b_0 \in \Gamma_2$. Thus $f^{-1}(B) = \Gamma_2$, and in consequence $f^{-1}(C) \le \Gamma_2$. On the other hand, if $g = b_0 a_1 b_1 \cdots a_n b_n \in f^{-1}(A)$, we may assume as before that $a_i \neq e$ for $1 \le i \le n$ and $b_i \notin f^{-1}(C)$ for $0 < i < n$. Applying f again then shows that $n \le 1$ and $b_i \in f^{-1}(C)$ for $i \le n$, so that $g \in \Gamma_1 f^{-1}(C)$. □

The following folkloric terminology is inspired by Lemma 2.1.

Definition 2.4. Let A and B be subgroups of a group G. We say that A is *a ping-pong partner for B*, or that *A and B play ping-pong*, if the subgroup $\langle A, B\rangle$ is freely generated by A and B, or in other words if the canonical map $A * B \to \langle A, B\rangle$ is an isomorphism. Similarly, we say that $a \in A$ is *a ping-pong partner for B in A*, or that *a and B play ping-pong*, if the subgroup $\langle a, B\rangle$ is freely generated by $\langle a\rangle$ and B. When B is generated by a single element b, we also say that *a is a ping-pong partner for b*.

Sets P_1 and P_2 to which one can apply Lemma 2.1 are sometimes called a *ping-pong table for A and B*.

In the subsequent sections, we will look to play ping-pong inside a group $G = \prod_{i=1}^n G_i$ which decomposes into a direct product of subgroups G_i. Using some simple facts about free (amalgamated) products, the next proposition will show that this requires an embedding of the ping-pong partners in one of the factors G_i.

Given subgroups $H_1, \ldots, H_n$ of a group G, let $[H_1, \ldots, H_n] = [H_1, [H_2, \ldots, H_n]]$ denote the *left-iterated (or right-normed) commutator subgroup* of the H_i.

Lemma 2.5. *Let N, N_1, $\ldots$, N_n be normal subgroups of $A *_C B$, where $|A : C| > 2$.*

(i) *Either $N \subset C$, or N contains a non-abelian free group.*

(ii) *If $[N_1, N_2] \subset C$, then either $N_1 \subset C$ or $N_2 \subset C$.*

In consequence, if $[N_1, \ldots, N_n]$ admits no non-abelian free subgroups, there exists $i \in \{1, \ldots, n\}$ for which $N_i \subset C$.

Proof. First, suppose that N is a normal subgroup of $A *_C B$ not contained in C. Pick $x \in N \setminus C$; by Lemma 2.2, we may assume after conjugation that x either belongs to $B \setminus C$, belongs to $A \setminus C$, or is cyclically reduced starting with $a_1 \in A \setminus C$.

- If $x \in B \setminus C$, pick $a, a' \in A \setminus C$ such that $a \notin a'C$. Using Lemma 2.2, one readily checks that the cyclically reduced words $w = [a, x]$ and $w' = [a', x]$ generate a free group, as every non-empty word in w and w' remains a non-empty word alternating in elements of $A \setminus C$ and $B \setminus C$. (Only simplifications of the form $[a, x][a', x]^{-1} = ax(a^{-1}a')x^{-1}a'^{-1}$ occur, and the condition on a and a' ensures no further cancellations arise.)

- If $x \in A \setminus C$, pick $b \in B \setminus C$ and $a, a' \in A \setminus C$ such that $a \notin a'C$, and consider $w = [x, bab^{-1}]$ and $w' = [x, ba'b^{-1}]$ instead.
- In the last case, write $x = a_1b_1 \cdots, a_nb_n$ with $n \geq 1$ and $a_i \in A \setminus C$, $b_i \in B \setminus C$. Pick $b \in B \setminus C$ and $a \in A \setminus C$ such that $a \notin a_1C$. Then the words $w = x$ and $w' = aba^{-1}xab^{-1}a^{-1}$ generate a free group.

This proves part (i).

Second, suppose that there exist elements $x \in N_1 \setminus C$ and $x' \in N_2 \setminus C$. By Lemma 2.2, we may assume after conjugation that x, x' either belong to $A \setminus C$, belongs to $B \setminus C$, or is cyclically reduced starting with A. We exhibit in each case a commutator in $[N_1, N_2] \setminus C$.

- If $x = a_1$ and $x' = b_1'$, then $[x, x'] \notin C$.
- If x is cyclically reduced starting with a_1 and $x' = a_1'$, then $[x, bx'b^{-1}] \notin C$ for any $b \in B \setminus (C \cup b_n^{-1}C)$.
- If x is cyclically reduced starting with a_1 and $x' = b_1'$, then $[a^{-1}xa, x'] \notin C$ for any $a \in A \setminus (C \cup a_1C)$.
- If $x = a_1$ and $x' = a_1'$, then $[x, bx'b^{-1}] \notin C$ for any $b \in B \setminus C$.
- If $x = b_1$ and $x' = b_1'$, then $[axa^{-1}, x'] \notin C$ for any $a \in A \setminus C$.
- If x, x' are both cyclically reduced starting with a_1, and ending with $b_{n'}'$ respectively, then $[a^{-1}xa, b^{-1}x'b] \notin C$ for any $a \in A \setminus (C \cup a_1C)$ and $b \in B \setminus (C \cup b_{n'}'^{-1}C)$.

This proves part (ii).

Lastly, if $[N_1, \dots, N_n]$ admits no non-abelian free subgroups, we deduce from part (i) that $[N_1, \dots, N_n] \subset C$. Part (ii) then implies that either $N_1 \subset C$, or $[N_2, \dots, N_n] \subset C$, and recursively, that eventually $N_i \subset C$ for some $i \in \{1, \dots, n\}$. □

Definition 2.6. Let $\mathcal{S}$ be a class of groups closed under subquotients and extensions. For the purposes of the following proposition, we will say that G is an *$\mathcal{S}$-almost direct product of* $G_1, \dots, G_n$ if G has a normal subgroup $K \in \mathcal{S}$ such that G/K is the direct product $G_1 \times \cdots \times G_n$.

Equivalently, if there are normal subgroups $M_1, \dots, M_n$ of G such that $\bigcap_{i=1}^n M_i \in \mathcal{S}$ and $M_i(M_{i+1} \cap \cdots \cap M_n) = G$ for $i = 1, \dots, n-1$, then G is the $\mathcal{S}$-almost direct product of $G/M_1, \dots, G/M_n$. Indeed, the second condition ensures that the canonical map $G/\bigcap_{i=1}^n M_i \to G/M_1 \times \cdots \times G/M_n$ is surjective; conversely, writing M_i for the kernel of $G \to G_i$, it is obvious that $K = \bigcap_{i=1}^n M_i$ and $M_j(\bigcap_{i \neq j} M_i) = G$.

Almost direct products with respect to the class containing only the trivial group are just direct products. In the literature, almost direct products appear most often for $\mathcal{S}$ the class of finite groups. Here are a few straightforward observations:

- Any group in $\mathcal{S}$ is an $\mathcal{S}$-almost empty direct product; so of course the notion is meaningful only for groups outside of $\mathcal{S}$.
- An $\mathcal{S}$-almost direct product of groups $G_1, \dots, G_n$ themselves $\mathcal{S}$-almost direct products of respectively $H_{i1}, \dots, H_{in_j}$ $(i = 1, \dots, n)$, is an $\mathcal{S}$-almost direct product of the H_{ij}, $i = 1, \dots, n$, $j = 1, \dots, n_j$.
- Any quotient or extension of an $\mathcal{S}$-almost direct product by a group in $\mathcal{S}$ is again an $\mathcal{S}$-almost direct product.

Sometimes, almost direct products are defined by the following variant: G is the quotient of a direct product $G_1 \times \cdots \times G_n$ by a normal subgroup $H \in \mathcal{S}$. An almost direct product in this second sense is also an $\mathcal{S}$-almost direct product in the sense of Definition 2.6. Indeed, if $G = (G_1 \times \cdots \times G_n)/H$, denoting π_i the projection onto G_i and $K = \pi_1(H) \times \cdots \times \pi_n(H)$, we see that $G/(K/H) \cong (G_1 \times \cdots \times G_n)/K = G_1/\pi_1(H) \times \cdots \times G_n/\pi_n(H)$. The converse however does not always hold, as the images of the factors G_i in $(G_1 \times \cdots \times G_n)/H$ are commuting normal subgroups, and this may not happen in G even if G/K is a direct product.

Proposition 2.7 (Free subgroups in almost direct products)**.** *Let $\mathcal{S}$ be the class of groups not containing a non-abelian free group. Let G be the $\mathcal{S}$-almost direct product of groups $G_1, \dots, G_m$, and suppose that $G_{n+1}, \dots, G_m$ belong to $\mathcal{S}$. If A and B are subgroups of G whose intersection C satisfies $|A : C| > 2$, and are such that the canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism, then there exists $i \in \{1, \dots, n\}$ for which the kernel of the projection $\langle A, B\rangle \to G_i$ is contained in C.*

Proof. Since $G_{n+1}, \dots, G_m$ belong to $\mathcal{S}$, it is clear that G is also the $\mathcal{S}$-almost direct product of $G_1, \dots, G_n$. Let π_i denote the projection $G \to G_i$ and set $M_i = \ker \pi_i$. Identify $\langle A, B\rangle$ with $A *_C B$ and set $N_i = M_i \cap (A *_C B)$.

By assumption, $\bigcap_{i=1}^n M_i$ does not contain a non-abelian free group. The same then holds for $[N_1, \dots, N_n] \subset [M_1, \dots, M_n] \subset \bigcap_{i=1}^n M_i$, and Lemma 2.5 implies the existence of an index $i \in \{1, \dots, n\}$ for which $N_i \subset C$. □

There are versions of Lemma 2.5 and Proposition 2.7 for HNN extensions. We leave their statement and proof to the reader.

3. Simultaneous ping-pong partners for finite subgroups of reductive groups

Let F be a field. Let $\mathbf{G}$ be a reductive[3] algebraic F-group, Γ a Zariski-dense subgroup of $\mathbf{G}(F)$, and H a finite subgroup of $\mathbf{G}(F)$. This section is concerned with finding elements γ of Γ which are ping-pong partners for H.

3.1. **Existence in connected groups.** The construction and study of free products in linear groups is a classical topic, going back way beyond Tits' celebrated work [82] establishing existence of free subgroups in linear groups which are not virtually solvable. Given a subset F of a linear group G, the existence of *simultaneous* ping-pong partners for elements of F (that is, elements which are ping-pong partners for every $h \in F$) has also been studied, see namely the works of Poznansky [67, Theorem 6.5] and Soifer & Vishkautsan [77, Theorem 1.3]. We also recall from the introduction the open question going back to Bekka, Cowling and de la Harpe, cases of which are answered in the two works just cited.

Question 3.1 (see [7, Remark 3] and [23, Question 17])**.** Let G be a connected semisimple adjoint real Lie group without compact factors, and let Γ be a Zariski-dense subgroup of G. Let F be a finite set of non-trivial elements of Γ. Does there exists an element $\gamma \in \Gamma$ of infinite order such that $\langle h, \gamma\rangle \cong \langle h\rangle * \langle\gamma\rangle$ for every $h \in F$?

Of course, if F is a subgroup, the condition that $\langle h, \gamma\rangle$ be freely generated for every element $h \in F$ does not imply that the subgroup $\langle F, \gamma\rangle$ is freely generated by F and γ. For instance, even when the subgroup $\langle (h_1, h_2), (\gamma_1, \gamma_2)\rangle$ of $G_1 \times G_2$ is freely generated for every pair $(h_1, h_2) \in F_1 \times F_2$, the group $\langle F_1 \times F_2, (\gamma_1, \gamma_2)\rangle$ is never freely generated, as $(\gamma_1 h_1 \gamma_1^{-1}, 1)$ commutes with $(1, h_2)$.

Similarly, even if the projections $\langle h_i, \gamma_i\rangle$ $(i = 1, 2)$ of a subgroup $\langle (h_1, h_2), (\gamma_1, \gamma_2)\rangle$ of $G_1 \times G_2$ are freely generated subgroups, it may be that $\langle (h_1, h_2), (\gamma_1, \gamma_2)\rangle$ is itself not freely generated: if h_1, h_2 have distinct finite orders n_1 and n_2, then again $(\gamma_1, \gamma_2)(h_1, h_2)^{n_2}(\gamma_1, \gamma_2)^{-1}$ and $(h_1, h_2)^{n_1}$ commute. Proposition 2.7 shows in fact that for Question 1.1 to possibly have a positive answer, one must at the very least require that each $h \in F$ generates a subgroup $\langle h\rangle$ that embeds into one of the factors of G (cf. Remark 3.24 in general).

For these reasons and others, we cannot directly use the works mentioned above; but we will use similar techniques to prove the following.

Theorem 3.2. *Let $\mathbf{G}$ be a connected algebraic F-group with center $\mathbf{Z}$. Let Γ be a Zariski-connected subgroup of $\mathbf{G}(F)$. Let $(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(F)$, and set $C_i = H_i \cap \mathbf{Z}(F)$. Assume that for each $i \in I$ there exists a local field K_i containing F and a projective K_i-representation $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$, where V_i is a finite-dimensional module over a finite division K_i-algebra D_i, with the following properties:*

(Proximality) *$\rho_i(\Gamma)$ contains a proximal element;*

(Transversality) *For every $h \in H_i \backslash C_i$ and every $p \in \mathbf{P}(V_i)$, the span of the set $\{\rho_i(xhx^{-1})p \mid x \in \Gamma\}$ is the whole of $\mathbf{P}(V_i)$.*

Let S be the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order, such that for all $i \in I$, the canonical map

$$(\langle\gamma\rangle \times C_i) *_{C_i} H_i \to \langle\gamma, H_i\rangle \leq \mathbf{G}(F)$$

is an isomorphism. Then S is dense in Γ for the join of the profinite topology and the Zariski topology.

[3]In this paper, all reductive (in particular, all semisimple) algebraic groups are connected by definition. This convention sometimes differs in the literature. We also call *simple* a non-commutative algebraic group whose proper normal subgroups are finite (sometimes called 'almost simple' in the literature).

Remark 3.3. The conclusion of the theorem amounts to the kernel of the canonical map

$$\langle\gamma\rangle * H_i \to \langle\gamma, H_i\rangle \leq \mathbf{G}(F)$$

being $\langle\!\langle[\gamma, C_i]\rangle\!\rangle$. Note that when $\mathbf{Z}(F)$ is trivial, the theorem states that for any $\gamma \in S$ and for all $i \in I$, the subgroup $\langle\gamma, H_i\rangle$ is freely generated by γ and H_i.

Remark 3.4. Note that the transversality condition implies that every ρ_i is irreducible. Moreover, the transversality condition holds equivalently for Γ or for its Zariski closure (it is a *Zariski-closed* condition). Thus, if Γ happens to be Zariski-dense (as is most common), this condition can be replaced by the analogue for $\mathbf{G}(K_i)$:

(Transversality′) *For every $h \in H_i \backslash C_i$ and every $p \in \mathbf{P}(V_i)$, the span of the set $\{\rho_i(xhx^{-1})p \mid x \in \mathbf{G}(K_i)\}$ is the whole of $\mathbf{P}(V_i)$.*

Remark 3.5. Theorem 3.2 is only meaningful for pseudo-reductive groups. Indeed, the F-unipotent radical $\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ must acts trivially under ρ_i, as the fixed-point set of $\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ is non-empty by the Lie–Kolchin theorem, hence is the whole of V_i. Thus each ρ_i factors through the pseudo-reductive quotient $\mathbf{G}/\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ of $\mathbf{G}$. We remind the reader that if $\operatorname{char} F = 0$, the full unipotent radical $\mathrm{R}_{\mathrm{u}}(\mathbf{G})$ of $\mathbf{G}$ is defined over F, hence pseudo-reductive groups are reductive (the converse always holding).

In subsequent sections, we will mostly be concerned with number fields and their archimedean completions, leaving aside the usual complications arising in positive characteristic.

Remark 3.6. There is no obvious analogue of Theorem 3.2 for HNN extensions. Indeed, $\mathbf{G}(F)$ may admit finite subgroups H containing a proper subgroup H_1 whose centralizer in $\mathbf{G}(F)$ is trivial. For instance, $\mathrm{PGL}_2(\mathbb{C})$ contains a copy of the symmetric group on 4 letters, whose alternating subgroup has trivial centralizer (see for instance [6, Proposition 1.1]). In such a situation, there is no HNN extension in $\mathbf{G}(F)$ of H with respect to the identity $H_1 \to H_1$, as any $g \in \mathbf{G}(F)$ centralizing H_1 is trivial, but $H*_{H_1}$ is not.

3.2. **Proximal dynamics in projective spaces.** Before proving Theorem 3.2, we need to extend a few known facts about the dynamics of the action of $\mathrm{GL}(V)$ on $\mathbf{P}(V)$ to projective spaces over division algebras. Foremost, we will need the contents of [82, §3] over a division algebra, but the proofs given by Tits are valid with minor adaptations to keep track of the D-structure and the fact D is not necessarily commutative. All of this is straightforward, so we will not rewrite arguments whenever they apply in the same way.

In this subsection, let K be a local field, D a division algebra of dimension d over K, and V a finite-dimensional right D-module. Recall that the absolute value $|\cdot|$ of K extends uniquely to an absolute value on D which will also denote by $|\cdot|$; we have the formula $|x| = |\mathrm{N}(x)|^{1/d}$ for $x \in D$. For each K-variety $\mathbf{V}$, the topology of K induces a locally compact topology on $\mathbf{V}(K)$; this topology is often called the *local topology*, to distinguish it from the Zariski topology when needed.

With little deviation, we will follow the notations and conventions of [81] and [82], which the reader may consult along with [10] for background material on the representation theory of algebraic groups (including over division algebras).

Recall that GL_V is the algebraic K-group of automorphisms of the D-module V, so that for any F-algebra A, the group $\mathrm{GL}_V(A)$ is the group of automorphisms of the right $(D \otimes_K A)$-module $V \otimes_K A$. Provided $\dim V \geq 2$, the K-group PGL_V is the quotient of GL_V by its center (which is the multiplicative group of the center of D). The *projective general linear group* PGL_V acts on the *projective space* $\mathbf{P}(V)$ of V, which is the space of right D-submodules of V of dimension 1. The D-submodules of V and their images in $\mathbf{P}(V)$ are both called *(D-linear) subspaces*. A projective representation $\rho : \mathbf{G} \to \mathrm{PGL}_V$ of a K-group $\mathbf{G}$ is called *irreducible* if there are no proper non-trivial linear subspaces of

$\mathbf{P}(V)$ stable under $\rho(\mathbf{G})$. A representation $\mathbf{G} \to \mathrm{GL}_V$ is then irreducible if and only if its projectivization is.

Given two subspaces X, Y of $\mathbf{P}(V)$, we denote their span by $X \vee Y$. If $X \cap Y = \emptyset$ and $X \vee Y = \mathbf{P}(V)$, we denote by $\mathrm{proj}(X, Y)$ the mapping $\pi : X \to Y$ defined by $\{\pi(p)\} = (X \vee \{p\}) \cap Y$. We will denote by $\mathring{C}$ the interior (for the local topology) of a subset C of $\mathbf{P}(V)$.

When it is needed to view V as a K-module instead of a D-module, we will add the corresponding subscript to the notation.

Definition 3.7. Let g be an element of $\mathrm{GL}_V(K)$ or $\mathrm{PGL}_V(K)$.

(1) Momentarily view V as a vector K-space, so as to identify GL_V with the subgroup of $\mathrm{GL}_{V,K}$ centralizing the right action of D on V, and likewise for PGL_V. The *attracting subspace of* g is the subspace $\mathrm{A}(g)$ of V which is the direct sum of the generalized eigenspaces (over some algebraic closure) associated to the eigenvalues of maximal absolute value of (any lift to GL_V of) g. The complementary set $\mathrm{A}'(g)$ is defined to be the direct sum of the remaining generalized eigenspaces of g. By construction, $V = \mathrm{A}(g) \oplus \mathrm{A}'(g)$.

Note that since the Galois group of any extension of K preserves the absolute value, it permutes the generalized eigenspaces of maximal absolute value, hence $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ are stable under the Galois group and are indeed defined over K. Moreover, if g commutes with the action of D, then D preserves the generalized eigenspaces of g (after perhaps extending scalars). In this case, $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ are themselves stable under D, i.e. they are D-subspaces of V.

The subspaces $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ only depend on the image of g in PGL_V. In what follows, we will often omit projectivization from the notation as long as it causes no confusion between V and $\mathbf{P}(V)$.

(2) We call g *proximal* if $\dim_D \mathrm{A}(g) = 1$, in other words if $\mathrm{A}(g)$ is a point in $\mathbf{P}(V)$. In case $D = K$, this means that g has a unique eigenvalue (counting with multiplicity) of maximal absolute value. In general, this means that g has d (possibly different) eigenvalues of maximal absolute value. If both $\mathrm{A}(g)$ and $\mathrm{A}(g^{-1})$ are one-dimensional, we call g *biproximal*[4]. We call a (projective) representation $\rho : \Gamma \to \mathrm{(P)GL}_V(K)$ *proximal* if $\rho(\Gamma)$ contains a proximal element.

Proximal elements have contractive dynamics on $\mathbf{P}(V)$: if g is proximal, then for any $p \in \mathbf{P}(V) \setminus \mathrm{A}'(g)$ the sequence $(g^n \cdot p)_{n \in \mathbb{N}}$ converges to the point $\mathrm{A}(g)$ (see Lemma 3.8).

The complement $\mathbf{P}(V) \setminus X$ of a hyperplane $X \subset \mathbf{P}(V)$ can be identified with an affine space over D by choosing for V a system of coordinate functions $\xi = (\xi_0, \ldots, \xi_{\dim \mathbf{P}(V)})$, $\xi_i \in V^*$, such that $X = \ker \xi_0$. The functions $\xi_i \xi_0^{-1}$ $(i = 1, \ldots, \dim \mathbf{P}(V))$ then define affine coordinates on $\mathbf{P}(V) \setminus X$. If $g \in \mathrm{PGL}_V(K)$ stabilizes X, its restriction to $\mathbf{P}(V) \setminus X$ need not be an affine map in these coordinates, but will be semiaffine (with respect to conjugation by the factor by which g scales ξ_0). In particular, if $\mathbf{P}(V) \setminus X$ is seen as an affine space over K, then the restriction of g is K-affine.

For the rest of this section, we fix an *admissible* distance d on $\mathbf{P}(V)$, that is, a distance function $d : \mathbf{P}(V) \times \mathbf{P}(V) \to \mathbf{P}(V)$ inducing the local topology on $\mathbf{P}(V)$ and satisfying the property that for every compact subset C contained in an affine subspace of $\mathbf{P}(V)$, there exist constants $M, M' \in \mathbb{R}$ such that

$$M \cdot d_\xi|_{C \times C} \leq d|_{C \times C} \leq M' \cdot d_\xi|_{C \times C}.$$

Here d_ξ is the supremum distance with respect to the affine coordinates $(\xi_i \xi_0^{-1})_{i=1}^{\dim \mathbf{P}(V)}$ described above. Note that two different coordinate systems on the same affine subspace A

[4]Biproximal elements are sometimes called 'very proximal' or 'hyperbolic' in the literature.

of $\mathbf{P}(V)$ define comparable distance functions on this affine subspace. Moreover, if instead of using D-coordinates one views A as an affine K-space, the supremum distance in any set of affine K-coordinates will again be comparable to d_ξ.

As indicated by Tits, when K is an archimedean local field, any elliptic metric on $\mathbf{P}(V)$ is admissible. Tits also indicates in [82, §3.3] how to construct an admissible metric in the non-archimedean case by patching together different d_ξ's; this construction works identically over a division algebra.

Having fixed an (admissible) distance d on $\mathbf{P}(V)$, the *norm* of a mapping $f : X \to \mathbf{P}(V)$ defined on some subset $X \subset \mathbf{P}(V)$ is the quantity

$$\|f\| = \sup_{\substack{p,q\in X\\ p\neq q}} \frac{d(f(p), f(q))}{d(p,q)}.$$

Note that the norm is submultiplicative: given mappings $f : X \to \mathbf{P}(V)$ and $g : Y \to X$, we have $\|f\circ g\| \leq \|f\|\cdot\|g\|$. Projective transformations always have finite norm [82, Lemma 3.5]. Indeed, given $g \in \mathrm{PGL}_V(K)$, the distance function d^g defined by $d^g(p,q) = d(gp, gq)$ is again admissible. Since $\mathbf{P}(V)$ is compact, it can be covered by finitely many compact sets contained in affine subspaces, on which the ratio between d^g and d is uniformly bounded, by admissibility.

We can now state the needed results from [82, §3] in our setting. The following two lemmas describe the dynamics of D-linear transformations.

Lemma 3.8 (Lemma 3.8 in [82])**.** *Let $g \in \mathrm{PGL}_V(K)$, let C be a compact subset of $\mathbf{P}(V)$ and let $r \in \mathbb{R}_{>0}$.*

(i) *Suppose that g is proximal and that $C \cap \mathrm{A}'(g) = \emptyset$. Then there exists an integer N such that $\|g^n|_C\| < r$ for all $n > N$; and for any neighborhood U of $\mathrm{A}(g)$, there exists an integer N' such that $g^nC \subset U$ for all $n > N'$.*

(ii) *Assume that, for some $m \in \mathbb{N}$, one has $\|g^m|_C\| < 1$ and $g^mC \subset \mathring{C}$. Then $\mathrm{A}(g)$ is a point contained in $\mathring{C}$.*

Note that in loc. cit. Tits assumes the existence of a semisimple proximal element; but as he indicates in the footnotes, this assumption is superfluous and the proof of the lemma is identical with an arbitrary proximal element.

Proof. The argument given by Tits applies, taking into account the following adaptations.

In part (i), the transformation g restricted to $\mathbf{P}(V) \setminus A'(g)$ is not necessarily D-linear, as was already mentioned. It is nevertheless K-linear, with eigenvalues of absolute value strictly smaller than 1 by assumption. So one can apply [82, Lemma 3.7 (i)] over K and use that the norms defined over D or K are comparable to conclude.

In part (ii), one cannot pick a representative of g in GL_V whose eigenvalues corresponding to the fixed point $p \in \mathbf{P}(V)$ equal one (as g may have different eigenvalues on the D-line p). Nevertheless, they are all of the same absolute value, which we can assume to be 1. If there is another eigenvalue of the same absolute value (i.e. if $\mathrm{A}(g) \neq \{p\}$), then the restriction of g to $\mathrm{A}(g)$ is a block-upper-triangular matrix in a well-chosen basis. Since the compact set C has non-empty interior, this contradicts the hypothesis of (ii). $\square$

Lemma 3.9 (Lemma 3.9 in [82])**.** *Let $g \in \mathrm{PGL}_V(K)$ be semisimple, let $\bar{g} \in \mathrm{GL}_V(K)$ be a representative of g, let Ω be the set of eigenvalues of $\bar{g}$ (over an appropriate field extension of K) whose absolute value is maximum, let C be a compact subset of $\mathbf{P}(V) \setminus \mathrm{A}'(g)$, set $\pi = \mathrm{proj}(\mathrm{A}'(g), \mathrm{A}(g))$, and let U be a neighborhood of $\pi(C)$ in $\mathbf{P}(V)$.*

(i) *There exists an infinite set $N \subset \mathbb{N}$ such that $\lim_{\substack{n\in N\\ n\to\infty}} (\lambda^{-1}\mu)^n = 1$ for all $\lambda, \mu \in \Omega$.*

(ii) *The set* $\{\|g^n|_C\| \mid n \in \mathbb{N}\}$ *is bounded.*

(iii) *If* N *is as in (i),* $g^n C \subset U$ *for almost all* $n \in N$*.*

Proof. The easiest way to obtain this lemma over the division algebra D is to take a representative of g in GL_V, see it as an K-linear transformation in $\mathrm{GL}_{V,K}$ and apply Tits' original lemma [82, Lemma 3.9]. Part (i) is then immediate.

For part (ii) and (iii), denote $\mathbf{P}_K(V)$ the projective space of V seen as a vector K-space. Since the canonical GL_V-equivariant map $q : \mathbf{P}_K(V) \to \mathbf{P}(V)$ is proper and continuous, $C' = q^{-1}(C)$ is compact, and $U' = q^{-1}(U)$ is open. Thus [82, Lemma 3.9] applies with C' and U' over K, and in turn yields the same conclusions over D, since the norms of g restricted to C and C' bound each-other. □

We will also make use of a version of part (i) of Lemma 3.9 for multiple representations, due to Margulis and Soifer. They initially stated it for multiple vector spaces over the same local field, but as already observed in [67, Lemma 3.1], the proof is identical.

Lemma 3.10 (Lemma 3 in [61])**.** *Let* $\{K_i\}_{i\in I}$*, be a finite collection of local fields and* V_i *be a finite-dimensional vector* K_i*-space. Let* g_i *be a semisimple element of* $\mathrm{GL}_{V_i}(K)$*, and let* $\Omega(g_i)$ *be the set of eigenvalues of* g_i *whose absolute value is maximum. There exists an infinite subset* $N \subset \mathbb{N}$ *such that* $\lim_{\substack{n\in N\\ n\to\infty}} (\lambda^{-1}\mu)^n = 1$ *for all* $i \in I$ *and* $\lambda, \mu \in \Omega(g_i)$*.*

We are now ready to prove the following slight generalization of [67, Corollary 3.7], which is itself a refinement of both [82, Proposition 3.11] and [61, Lemma 8]. This proposition is a crucial piece of the proof of Theorem 3.2: it will be used to find enough biproximal elements in Γ.

Proposition 3.11 (Abundance of simultaneously biproximal elements)**.** *Let* $\mathbf{G}$ *be a connected algebraic* F*-group and let* Γ *be a Zariski-dense subgroup of* $\mathbf{G}(F)$*. Let* $\{K_i\}_{i\in I}$ *be a finite collection of local fields each containing* F*. For each* $i \in I$*, let* $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$ *be an irreducible projective* K_i*-representation, where* V_i *is a finite-dimensional module over a finite division* K_i*-algebra* D_i*.*

Suppose that for each $i \in I$*,* $\rho_i(\Gamma)$ *contains a proximal element. Then the set of regular semisimple elements* $\gamma \in \Gamma$ *such that* $\rho_i(\gamma)$ *is biproximal for every* $i \in I$*, is dense in* Γ *for the join of the Zariski topology and the profinite topology.*

Proof. We follow the line of arguments given in [82, 61, 67], keeping track of the different representations, and using the extension of Tits' work to projective representations over a division algebra laid out above.

Given an arbitrary element $g \in \mathbf{G}(F)$, let us abbreviate $\rho_i(g)$ by g_i.

<u>Step 1:</u> The set of simultaneously proximal elements in Γ is Zariski-dense if it is non-empty.

Let $g \in \Gamma$ be such that g_i is proximal for all $i \in I$. Since ρ_i is irreducible, for each $i \in I$ the set of elements x of $\mathbf{G}(F)$ such that $x_i\mathrm{A}(g_i) \notin \mathrm{A}'(g_i)$ is non-empty and Zariski-open. Because $\mathbf{G}$ is Zariski-connected, the intersection of these sets remains non-empty (and Zariski-open). Let us then pick $x \in \Gamma$ satisfying $x_i\mathrm{A}(g_i) \notin \mathrm{A}'(g_i)$ for every $i \in I$.

By construction of x, we can pick a compact neighborhood C_i of $\mathrm{A}(g_i)$ in $\mathbf{P}(V_i)$ such that x_iC_i is disjoint from $A'(g)$. Since projective transformations have finite norm, we have $\max_{i\in I} \|x_i|_{C_i}\| < r$ for some $r \in \mathbb{R}$. By Lemma 3.8 (i), for each $i \in I$ there exists an integer N_i such that

$$\|g_i^n|_{x_iC_i}\| < r^{-1} \qquad \text{and} \qquad g_i^n(x_iC_i) \subset \mathring{C}_i \qquad \text{for } n > N_i.$$

Set $N_x = \max_{i \in I} N_i$. Then for any $i \in I$, we have that

$$\|g_i^n x_i|_{C_i}\| < 1 \qquad \text{and} \qquad (g_i^n x_i) C_i \subset \mathring{C}_i \qquad \text{for } n > N_x.$$

We deduce from Lemma 3.8 (ii) that $g_i^n x_i = \rho_i(gx)$ is proximal for every $n > N_x$.

Observe that the Zariski closure Z of $\{g^n \mid n > N_x\}$ in Γ has the property that $gZ \subset Z$. Since the Zariski topology is Noetherian, we deduce that $g^{m+1}Z = g^m Z$ for some $m \in \mathbb{N}$. This implies that $g^n Z = Z$ for every $n \in \mathbb{Z}$, and in particular that $g \in Z$. Let now $\overline{S}$ denote the Zariski closure in Γ of the set S of elements of Γ which are proximal under every ρ_i. We have shown that S contains $g^n x$ for each $x \in \Gamma$ chosen as above and $n > N_x$. By our last observation, $\overline{S}x^{-1}$ contains g, hence $gx \in \overline{S}$. As this holds for every x in a Zariski-dense (open) subset of Γ, we conclude that $\overline{S}$ contains $g\Gamma = \Gamma$, as claimed.

Step 2: Γ contains a semisimple element that is simultaneously proximal.

We argue by induction on $\#I$. Fix $j \in I$, and suppose that there are elements $g, h \in \Gamma$ such that $\rho_j(h)$ is proximal and $\rho_i(g)$ is proximal for $i \in I \setminus \{j\}$. By Step 1, we may in addition assume that g and h are semisimple. Write $\pi_i = \operatorname{proj}(\mathrm{A}'(h_i), \mathrm{A}(h_i))$ for $i \neq j$, and $\pi_j = \operatorname{proj}(\mathrm{A}'(g_j), \mathrm{A}(g_j))$.

Let $N \subset \mathbb{N}$ be an infinite set such as afforded by Lemma 3.10 applied to the elements h_i for $i \neq j$ and g_j for $i = j$, so that we have $\lim\limits_{\substack{n \in N \\ n \to \infty}} (\lambda^{-1}\mu)^n = 1$ for $\lambda, \mu \in \Omega(h_i)$ if $i \neq j$, and for $\lambda, \mu \in \Omega(g_j)$.

Since ρ_i is irreducible and Γ is Zariski-dense, as before we can fix $x \in \Gamma$ such that

$$x_i \mathrm{A}(g_i) \not\subset \mathrm{A}'(h_i) \quad \text{for every } i \in I.$$

Similarly, the elements $y \in \mathbf{G}(F)$ satisfying

$$\begin{aligned} y_i \cdot \pi_i(x_i \mathrm{A}(g_i)) &\notin \mathrm{A}'(g_i) \quad \text{for } i \in I \setminus \{j\}, \\ \text{and} \quad y_j \mathrm{A}(h_j) &\notin \left(x_j^{-1}\mathrm{A}'(h_j) \cap \mathrm{A}(g_j)\right) \vee \mathrm{A}'(g_j), \end{aligned}$$

form a non-empty Zariski-open subset of $\mathbf{G}(F)$. Let us then fix y such an element in Γ.

For $i \neq j$, let B_i be a compact neighborhood of $y_i \cdot \pi_i(x_i \mathrm{A}(g_i))$ disjoint from $\mathrm{A}'(g_i)$, and let B_j be a compact neighborhood of $x_j \cdot \pi_j(y_j \mathrm{A}(h_j))$ disjoint from $\mathrm{A}'(h_j)$. The latter exists because $\pi_j^{-1}(x_j^{-1}\mathrm{A}'(h_j)) \subset (x_j^{-1}\mathrm{A}'(h_j) \cap \mathrm{A}(g_j)) \vee \mathrm{A}'(g_j)$ does not contain $y_j \mathrm{A}(h_j)$. We also choose for $i \neq j$ a compact neighborhood C_i of $\mathrm{A}(g_i)$ disjoint from $x_i^{-1}\mathrm{A}'(h_i)$ and small enough to satisfy $y_i \cdot \pi_i(x_i C_i) \subset \mathring{B}_i$; and choose a compact neighborhood C_j of $\mathrm{A}(h_j)$ disjoint from $y_j^{-1}\mathrm{A}'(g_j)$ and satisfying $x_j \cdot \pi_j(y_j C_j) \subset \mathring{B}_j$.

The careful choice of B_i, C_i and N sets us up for the following applications of Lemmas 3.8 and 3.9. By Lemma 3.9, for each $i \neq j$ there exists $r_i \in \mathbb{R}$ and $N_i \in \mathbb{N}$ such that

$$\|h_i^n|_{x_i C_i}\| < r_i \text{ for } n \in \mathbb{N} \quad \text{and} \qquad y_i h_i^n x_i C_i \subset \mathring{B}_i \quad \text{for } n \in N,\, n > N_i.$$

Similarly, there exists $N_j \in \mathbb{N}$ and $r_j \in \mathbb{R}$ such that

$$\|g_j^n|_{y_j C_j}\| < r_j \text{ for } n \in \mathbb{N} \quad \text{and} \qquad x_j g_j^n y_j C_j \subset \mathring{B}_j \quad \text{for } n \in N,\, n > N_j.$$

By Lemma 3.8 (i), for each $i \neq j$ there exists $N_i' \in \mathbb{N}$ such that

$$\|g_i^n|_{B_i}\| < \left(\|y_i|_{y_i^{-1}B_i}\| \cdot r_i \cdot \|x_i|_{C_i}\|\right)^{-1} \quad \text{and} \qquad g_i^n B_i \subset \mathring{C}_i \quad \text{for } n > N_i'.$$

Similarly, there exists $N_j' \in \mathbb{N}$ such that

$$\|h_j^n|_{B_j}\| < \left(\|x_j|_{x_j^{-1}B_j}\| \cdot r_j \cdot \|y_j|_{C_j}\|\right)^{-1} \quad \text{and} \qquad h_j^n B_j \subset \mathring{C}_j \quad \text{for } n > N_j'.$$

Set $N' = \{n \in N \mid n > N_i \text{ and } n > N_i' \text{ for all } i \in I\}$. For $i \neq j$, we have by construction that

$$\|g_i^m y_i h_i^n x_i|_{C_i}\| < 1 \quad \text{and} \quad g_i^m y_i h_i^n x_i C_i \subset \mathring{C}_i \quad \text{for } m, n \in N'.$$

Similarly, we have that

$$\|h_j^n x_j g_j^m y_j|_{C_j}\| < 1 \quad \text{and} \quad h_j^n x_j g_j^m y_j C_j \subset \mathring{C}_j \quad \text{for } m, n \in N'.$$

We conclude from Lemma 3.8 (ii) that for all $m, n \in N'$, the element $g_i^m y_i h_i^n x_i$ is proximal for $i \neq j$, and so is $h_j^n x_j g_j^m y_j$. But $h_j^n x_j g_j^m y_j$ and $g_j^m y_j h_j^n x_j$ are conjugate, so $g^m y h^n x \in \Gamma$ is proximal under ρ_i for every $i \in I$.

In view of Step 1, the set of simultaneously proximal elements in Γ is Zariski-dense, so there is also a semisimple one as claimed.

Step 3: Γ contains an element which is simultaneously biproximal.

By Steps 1–2, there is a semisimple element $g \in \Gamma$ such that $\rho_i(g^{-1})$ is proximal for every $i \in I$. Let N be an infinite set such as afforded by Lemma 3.10. Replacing N by an appropriate subset, we may assume that the set $g^N = \{g^n \mid n \in N\}$ is Zariski-connected.

Since ρ_i is irreducible and Γ is Zariski-dense, the elements $x \in \mathbf{G}(F)$ such that

$$x_i \mathrm{A}(g_i) \not\subset \mathrm{A}'(g_i^{-1}) \quad \text{and} \quad x_i^{-1} \mathrm{A}(g_i) \not\subset \mathrm{A}'(g_i^{-1}) \quad \text{for every } i \in I$$

form a non-empty Zariski-open subset. Fix such an element $x \in \Gamma$. For the same reasons, the set U of elements $y \in \mathbf{G}(F)$ satisfying

$$\begin{aligned} & y_i \mathrm{A}(g_i^{-1}) \notin x_i \mathrm{A}'(g_i) \vee (x_i \mathrm{A}(g_i) \cap \mathrm{A}'(g_i^{-1})), \\ \text{and} \quad & y_i^{-1} x_i \mathrm{A}(g_i^{-1}) \notin \mathrm{A}'(g_i) \vee (\mathrm{A}(g_i) \cap x_i \mathrm{A}'(g_i^{-1})) \qquad \text{for every } i \in I \end{aligned}$$

is also non-empty and Zariski-open; fix $y \in U \cap \Gamma$.

Write $\pi_i = \mathrm{proj}(\mathrm{A}'(g_i), \mathrm{A}(g_i))$ and $\pi_i' = \mathrm{proj}(x_i \mathrm{A}'(g_i), x_i \mathrm{A}(g_i))$. For each $i \in I$, let B_i be a compact neighborhood of $\pi_i'(y_i \mathrm{A}(g_i^{-1}))$ disjoint from $\mathrm{A}'(g_i^{-1})$, and let B_i' be a compact neighborhood of $\pi_i(y_i^{-1} x_i \mathrm{A}(g_i^{-1}))$ disjoint from $x_i \mathrm{A}'(g_i^{-1})$. We also choose a compact neighborhood C_i of $\mathrm{A}(g_i^{-1})$ disjoint from $y_i^{-1} x_i \mathrm{A}'(g_i)$ satisfying $\pi_i'(y_i C_i) \subset \mathring{B}_i$, and a compact neighborhood C_i' of $y_i^{-1} x_i \mathrm{A}(g_i^{-1})$ disjoint from $\mathrm{A}'(g_i)$ satisfying $\pi_i(C_i') \subset \mathring{B}_i'$.

By Lemma 3.9 (ii), for each $i \in I$ there exist $N_i, N_i' \in \mathbb{N}$ and $r_i, r_i' \in \mathbb{R}$ such that

$$\begin{aligned} \|x_i g_i^n x_i^{-1}|_{y_i C_i}\| < r_i \text{ for } n \in \mathbb{N} \quad &\text{and} \quad x_i g_i^n x_i^{-1} y_i C_i \subset \mathring{B}_i \quad \text{for } n \in N,\ n > N_i, \\ \|g_i^n|_{C_i'}\| < r_i' \text{ for } n \in \mathbb{N} \quad &\text{and} \quad g_i^n C_i' \subset \mathring{B}_i' \quad \text{for } n \in N,\ n > N_i'. \end{aligned}$$

By Lemma 3.8 (i), for each $i \in I$ there exist $M_i, M_i' \in \mathbb{N}$ such that

$$\begin{aligned} \|g_i^{-n}|_{B_i}\| < \left(r_i \cdot \|y_i|_{C_i}\|\right)^{-1} \quad &\text{and} \quad g_i^{-n} B_i \subset \mathring{C}_i \quad \text{for } n > M_i, \\ \|x_i g_i^{-n} x_i^{-1}|_{B_i'}\| < \left(\|y_i^{-1}|_{y_i C_i'}\| \cdot r_i'\right)^{-1} \quad &\text{and} \quad x_i g_i^{-n} x_i^{-1} B_i' \subset y_i \mathring{C}_i' \quad \text{for } n > M_i'. \end{aligned}$$

Set $N_{x,y} = \{n \in N \mid n > \max \bigcup_{i \in I} \{N_i, N_i', M_i, M_i'\}\}$. We then have by construction that

$$\begin{aligned} \|g_i^{-n} x_i g_i^n x_i^{-1} y_i|_{C_i}\| < 1 \quad &\text{and} \quad g_i^{-n} x_i g_i^n x_i^{-1} y_i C_i \subset \mathring{C}_i \quad \text{for } n \in N_{x,y}, \\ \|y_i^{-1} x_i g_i^{-n} x_i^{-1} g_i^n|_{C_i'}\| < 1 \quad &\text{and} \quad y_i^{-1} x_i g_i^{-n} x_i^{-1} g_i^n C_i' \subset \mathring{C}_i' \quad \text{for } n \in N_{x,y}. \end{aligned}$$

We conclude from Lemma 3.8 (ii) that for all $n \in N_{x,y}$ and for each $i \in I$, the element $g^{-n} x g^n x^{-1} y$ is biproximal under ρ_i.

Step 4: The set of regular semisimple simultaneously biproximal elements is dense.

Let S denote the set of elements in Γ which are biproximal under every ρ_i. Let Λ be a normal subgroup of finite index in Γ, and let $\gamma \in \Gamma$. Because the set of regular semisimple elements is Zariski-open, it suffices to show that $S \cap \Lambda\gamma$ is Zariski-dense to prove the proposition.

Since Γ is Zariksi-connected and Λ has finite index in Γ, every coset of Λ is Zariski-dense. Moreover, if $h \in \Gamma$ is such that h_i is proximal, then $h^{|\Gamma:\Lambda|}$ is also proximal under ρ_i, and belongs to Λ. We can thus apply Steps 1–3 to Λ, to find an element $g \in \Lambda$ such that g_i is biproximal for every $i \in I$.

As before, the set U of elements $x \in \mathbf{G}(F)$ such that

$$x_i\gamma_i\mathrm{A}(g_i) \notin \mathrm{A}'(g_i) \quad \text{and} \quad \gamma_i^{-1}x_i^{-1}\mathrm{A}(g_i^{-1}) \notin \mathrm{A}'(g_i^{-1}) \quad \text{for every } i \in I$$

is Zariski-open and non-empty. In particular, $\Lambda \cap U$ is Zariski-dense in Γ; pick $x \in \Lambda \cap U$.

Let $C_i^{\pm}$ be a compact neighborhood of $\mathrm{A}(g_i^{\pm 1})$ such that $(x\gamma)_i^{\pm 1}C_i^{\pm}$ is disjoint from $\mathrm{A}'(g_i^{\pm 1})$. Since projective transformations have finite norm, we have that $\max_{i\in I} \|(x\gamma)_i^{\pm 1}|_{C_i^{\pm}}\| < r$ for some $r \in \mathbb{R}$. By Lemma 3.8 (i), there exist integers N_i^+ and N_i^- such that

$$\begin{aligned} &\|g_i^n|_{x_i\gamma_iC_i^+}\| < r^{-1} && \text{and} && g_i^n x_i\gamma_i C_i^+ \subset \mathring{C}_i^+ && \text{for } n > N_i^+.\\ &\|g_i^{-n}|_{(x\gamma)_i^{-1}C_i^-}\| < r^{-1} && \text{and} && g_i^{-n}(x\gamma)_i^{-1}C_i^- \subset \mathring{C}_i^- && \text{for } n > N_i^-. \end{aligned}$$

For $N_x = \max \bigcup_{i\in I}\{N_i^+, N_i^-\}$, we then have for every $i \in I$ that

$$\begin{aligned} &\|g_i^n x_i\gamma_i|_{C_i^+}\| < 1 && \text{and} && g_i^n x_i\gamma_i C_i^+ \subset \mathring{C}_i^+ && \text{for } n > N_x.\\ &\|g_i^{-n}\gamma_i^{-1}x_i^{-1}|_{C_i^-}\| < 1 && \text{and} && g_i^{-n}\gamma_i^{-1}x_i^{-1}C_i^- \subset \mathring{C}_i^- && \text{for } n > N_x. \end{aligned}$$

We deduce from Lemma 3.8 (ii) that $g_i^n x_i\gamma_i$ and $g_i^{-n}\gamma_i^{-1}x_i^{-1}$ are proximal for every $i \in I$ and for $n > N_x$. But $g_i^{-n}\gamma_i^{-1}x_i^{-1}$ and $\gamma_i^{-1}x_i^{-1}g_i^{-n}$ are conjugate, so $g_i^n x_i\gamma_i$ is in fact biproximal for every $i \in I$. Of course $g^n x\gamma \in \Lambda\gamma$, so we have shown that $S \cap \Lambda\gamma$ contains $g^n x\gamma$ for every $x \in \Lambda \cap U$ and $n > N_x$.

As was observed in Step 1, the Zariski closure of $\{g^n \mid n > N_x\}$ in Γ contains g. Thus the Zariski closure of $S \cap \Lambda\gamma$ contains $gx\gamma$ for every $x \in \Lambda \cap U$. As $\Lambda \cap U$ is Zariski-dense, so is $S \cap \Lambda\gamma$. This concludes the proof of the proposition. $\square$

3.3. **Towards the proof of Theorem 3.2.** Before starting the proof of Theorem 3.2, we record the following lemmas.

Lemma 3.12. *Let K, D and V be as in §3.2. Let $\mathbf{G}$ be a connected K-subgroup of PGL_V, acting irreducibly on $\mathbf{P}(V)$. Suppose that $\mathbf{G}(K)$ contains a proximal element g_0. Then the set*

$$X = \{\mathrm{A}(g) \mid g \in \mathbf{G}(K) \textit{ is proximal}\} \subseteq \mathbf{P}(V)$$

coincides with the orbit $\mathbf{G}(K)\cdot\mathrm{A}(g_0)$ and constitutes the unique irreducible projective subvariety of $\mathbf{P}(V)$ stable under $\mathbf{G}(K)$. In consequence, $\mathrm{Stab}_{\mathbf{G}}(\mathrm{A}(g_0))$ is a parabolic subgroup of $\mathbf{G}$.

Proof. By a theorem of Chevalley, there is a Zariski-closed $\mathbf{G}(K)$-orbit $Y \subseteq \mathbf{P}(V)$. Let $g \in \mathbf{G}(K)$ be proximal. Because $\mathbf{G}$ acts irreducibly on $\mathbf{P}(V)$, there exists $y \in Y \setminus \mathrm{A}'(g)$. We then have $g^n \cdot y \xrightarrow{n\to\infty} \mathrm{A}(g)$, thus $\mathrm{A}(g)$ lies in the closure of Y in the local hence in the Zariski topology. As Y was Zariski-closed, $\mathrm{A}(g) \in Y$. Since this happens for any proximal element g, we deduce that $X \subseteq Y$. As X is $\mathbf{G}(K)$-stable and Y is a single orbit, equality holds. It is now clear that X is the set of K-points of a projective variety $\mathbf{X}$, which is irreducible because $\mathbf{G}$ is.

Let $\mathbf{P} = \mathrm{Stab}_{\mathbf{G}}(\mathrm{A}(g))$ denote the stabilizer of $\mathrm{A}(g)$ in $\mathbf{G}$. The above shows that orbit map yields an isomorphism $\mathbf{G}/\mathbf{P} \to \mathbf{X}$, hence $\mathbf{G}/\mathbf{P}$ is a complete variety, meaning that $\mathbf{P}$ is parabolic. The same holds for every other proximal element. □

Remark 3.13. Lemma 3.12 can also be proven by arguing that if g_0 is proximal, $\mathrm{A}(g_0)$ must be a highest weight line.

Lemma 3.14 (Transversality)**.** *Let* $\mathbf{G}$ *be as in Lemma 3.12, and suppose that* $\mathbf{G}(K)$ *contains a proximal element* g*. For any* $h \in \mathbf{G}(K)$*, the set*

$$U_{h,g} = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g) \cup \mathrm{A}'(g^{-1})\}$$

is Zariski-open in $\mathbf{G}(K)$*. If* $h \in \mathbf{G}(K)$ *is such that the span of* $\{xhx^{-1}\mathrm{A}(g) \mid x \in \mathbf{G}(K)\}$ *is the whole of* $\mathbf{P}(V)$*, then* $U_{h,g}$ *is non-empty.*

Proof. The two sets

$$U_1 = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g)\}$$
$$U_2 = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g^{-1})\}$$

are Zariski-open by a standard argument: for any subspaces $W_1, W_2 \subseteq V$, the set $\{x \in \mathbf{G}(K) \mid x \cdot W_1 \subseteq W_2\}$ is Zariski-closed. We have to show they are both non-empty.

There is a minimal parabolic K-subgroup $\mathbf{B}$ of $\mathbf{G}$ that contains h. By Lemma 3.12, there is a conjugate $x\mathbf{B}x^{-1}$ of $\mathbf{B}$ which fixes $\mathrm{A}(g)$. But then for this choice of x, we surely have $xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g)$. This shows that U_1 is not empty.

Finally, U_2 is non-empty because of the assumption made on h. Indeed, U_2 being empty means $xhx^{-1}\mathrm{A}(g) \in \mathrm{A}'(g^{-1})$ for every $x \in \mathbf{G}(K)$, but the latter is a proper subspace of $\mathbf{P}(V)$. □

Remark 3.15. At first glance, Lemma 3.14 above may seem to be weaker than [67, Proposition 2.17]. Unfortunately, the proof of [67, Proposition 2.17] relies on [67, Proposition 2.11], whose statement is erroneous. The set of elements whose conjugacy class intersects a big Bruhat cell is in fact smaller than stated there (see for instance [27, 28, 20] for a description in the case of SL_n). In consequence, the results of [67] are only valid under the additional assumption that the conjugacy classes of the elements h under consideration intersect a big Bruhat cell. Note that there are non-central torsion elements whose conjugacy class does not intersect the big Bruhat cell. We will address this in the next section by arranging for the transversality assumption of Lemma 3.14 and Theorem 3.2 to hold.

We note in addition that the proof of [67, Theorem 6.5] overlooks the possibility that a given torsion element h may not embed in any simple quotient of $\mathbf{G}$. As will be emphasized in Remark 3.24, this condition is necessary for constructing a ping-pong partner for h.

Proof of Theorem 3.2. For an arbitrary element $g \in \mathbf{G}(F)$, let us abbreviate $\rho_i(g)$ by g_i. For simplicity, we also write $H_i^* = H_i \setminus C_i$.

Fix a normal subgroup Λ of finite index in Γ, and fix $\gamma_0 \in \Gamma$. First, because of the proximality hypothesis, Proposition 3.11 applied to the Zariski-closure $\mathbf{H}$ of Γ in $\mathbf{G}$ states that the set S' of regular semisimple elements $\gamma' \in \Lambda\gamma_0$ such that $\rho_i(\gamma')$ is biproximal for every $i \in I$, is Zariski-dense in Γ. Pick $\gamma' \in S'$.

Second, using the transversality hypothesis on ρ_i, we exhibit a simultaneously biproximal element in $\Lambda\gamma_0$ acting transversely to every H_i. By Lemma 3.14, for every $i \in I$ and every $h \in H_i^*$ the sets

$$U_{i,h,\gamma'^{\pm 1}} = \{x \in \mathbf{H}(F) \mid x_i h_i x_i^{-1}\mathrm{A}(\gamma_i'^{\pm 1}) \notin \mathrm{A}'(\gamma_i') \cup \mathrm{A}'(\gamma_i'^{-1})\}$$

are Zariski-open and non-empty. In consequence, we can pick an element λ in the Zariski-dense set $\Lambda \cap U_{\gamma'}$, where $U_{\gamma'} = \bigcap_{i \in I} \bigcap_{h \in H_i^*} (U_{i,h,\gamma'} \cap U_{i,h,\gamma'^{-1}})$. Setting $\gamma = \lambda^{-1}\gamma'\lambda$, we see that $\gamma \in S'$, while for any $h \in H_i^*$,

$$h_i \mathrm{A}(\gamma_i) \notin \mathrm{A}'(\gamma_i) \cup \mathrm{A}'({\gamma_i}^{-1}) \quad \text{and} \quad h_i \mathrm{A}({\gamma_i}^{-1}) \notin \mathrm{A}'(\gamma_i) \cup \mathrm{A}'({\gamma_i}^{-1}).$$

Next, we construct the sets that will allow us to apply Lemma 2.1. Given $i \in I$, let $P_i^{\pm}$ be a compact neighborhood of $\mathrm{A}({\gamma_i}^{\pm 1})$ in $\mathbf{P}(V_i)$ small enough to achieve $(H_i^* \cdot P_i^{\pm}) \cap (\mathrm{A}'(\gamma_i) \cup \mathrm{A}'({\gamma_i}^{-1})) = \emptyset$. Such a set exists by construction of γ: by local compactness, the complement of the closed set $H_i^* \cdot (\mathrm{A}'(\gamma_i) \cup \mathrm{A}'({\gamma_i}^{-1}))$ contains a compact neighborhood of $\mathrm{A}({\gamma_i}^{\pm 1})$. In the same way, we can arrange that also

$$(H_i^* \cdot P_i^{\pm}) \cap (P_i^+ \cup P_i^-) = \emptyset.$$

Note that $\mathbf{Z}(F)$ fixes $\mathrm{A}(\gamma_i)$ and $\mathrm{A}({\gamma_i}^{-1})$. The finite intersection $\bigcap_{c \in C_i} (c \cdot P_i^{\pm})$ is thus again a compact neighborhood of $\mathrm{A}({\gamma_i}^{\pm 1})$. Replacing $P_i^{\pm}$ by this intersection, we will further assume that $P_i^{\pm}$ is stable under C_i.

Set $P_i = P_i^+ \cup P_i^-$ and set

$$Q_i = H^* \cdot P_i;$$

these two subsets of $\mathbf{P}(V_i)$ are compact, disjoint, and preserved by C_i. As $Q_i \cap (\mathrm{A}'(\gamma_i) \cup \mathrm{A}'({\gamma_i}^{-1})) = \emptyset$, Lemma 3.8 (i) shows that there exists $N \in \mathbb{N}$ such that for any $n > N$,

$${\gamma_i}^n Q_i \subset P_i \quad \text{and} \quad {\gamma_i}^{-n} Q_i \subset P_i.$$

Pick $N_1 > N$ with $N_1 = 1 \mod |\Gamma : \Lambda|$, so that $\gamma^{N_1 + n|\Gamma:\Lambda|} \in \Lambda\gamma_0$ for every $n \in \mathbb{Z}$.

For each $i \in I$, Lemma 2.1 now applies to the subgroups $\langle \gamma^{N_1+n|\Gamma:\Lambda|} \rangle \times C_i$ and H_i of $\mathbf{G}(F)$, with the sets P_i and Q_i constructed above. We conclude that for every $i \in I$ and all $n \in \mathbb{N}$, the subgroup $\langle \gamma^{N_1+n|\Gamma:\Lambda|}, H_i \rangle$ is the free amalgamated product $(\langle \gamma^{N_1+n|\Gamma:\Lambda|} \rangle \times C_i) *_{C_i} H_i$.

This establishes that $S \cap \Lambda\gamma_0$ contains $\gamma^{N_1+n|\Gamma:\Lambda|}$ for every $n \in \mathbb{N}$; it remains to show that $S \cap \Lambda\gamma_0$ is Zariski-dense.

The Zariski closure Z of $\{\gamma^{N_1+n|\Gamma:\Lambda|} \mid n \in \mathbb{N}\}$ satisfies $\gamma^{|\Gamma:\Lambda|} Z \subset Z$. Since the Zariski topology is Noetherian, it follows that $\gamma^{(m+1)|\Gamma:\Lambda|} Z = \gamma^{m|\Gamma:\Lambda|} Z$ for some $m \in \mathbb{N}$, and in turn that $\gamma \in Z$.

We have seen that S' is Zariski-dense, and that for each $\gamma' \in S'$, the set $\Lambda \cap U_{\gamma'}$ is Zariski-dense. In consequence, the set $S'' = \{(\gamma', \lambda) \in \Gamma \times \Gamma \mid \gamma' \in S', \lambda \in \Lambda \cap U_{\gamma'}\}$ is Zariski-dense in $\Gamma \times \Gamma$. Indeed, its closure contains $\overline{\{\gamma'\} \times S_{\gamma'}} = \{\gamma'\} \times \Gamma$ for each $\gamma' \in S'$, therefore contains $\overline{S' \times \{\gamma\}} = \Gamma \times \{\gamma\}$ for each $\gamma \in \Gamma$.

Since the conjugation map $\mathbf{H} \times \mathbf{H} \to \mathbf{H} : (x, y) \mapsto y^{-1}xy$ is dominant, it sends S'' to a Zariski-dense subset of Γ. Following the argument above, the Zariski closure of $S \cap \Lambda\gamma_0$ contains the image of S''. This proves the theorem. $\square$

Remark 3.16. Each of the two properties assumed in Theorem 3.2 can be satisfied individually. Given a finitely generated Zariski-dense subgroup of a (connected) semisimple algebraic group, the existence of a local field and a representation satisfying the proximality property was first shown by Tits (see the proof of [82, Proposition 4.3]). A refinement to non-connected simple groups can also be found in [61, Theorem 1].

The second property, transversality, can be established for one given element $h \in H_i \setminus C_i$ using representation-theoretic techniques. However, it is not always possible to find a representation that works for all $h \in H_i$ at the same time.

Even so, it may not always be possible to find a single representation which satisfies both properties of Theorem 3.2 simultaneously. Our next task will be to construct such a representation for real inner forms of SL_n and $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_n)$. This will be sufficient for the applications appearing in §4 & §5.

3.4. **Constructing a proximal and transverse representation for inner $\mathbb{R}$-forms of SL_n and GL_n.** Let D be a finite division $\mathbb{R}$-algebra and set $d = \dim_{\mathbb{R}} D$. Let $n \geq 2$ and let $\mathbf{H}$ be any algebraic $\mathbb{R}$-group in the isogeny class of SL_{D^n} or GL_{D^n}, viewing D^n as a right D-module. For example, if $D = \mathbb{C}$ this means that $\mathbf{H}$ is a quotient of the $\mathbb{R}$-group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_n)$ or $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_n)$ by a (finite) central subgroup. The *standard projective representation* of $\mathbf{H}$ is the canonical morphism $\rho_{\mathrm{st}} : \mathbf{H} \to \mathrm{PGL}_{D^n}$. This is the projective representation which will exhibit both proximal and transverse elements.

First, we recall that an element $g \in \mathbf{G}(\mathbb{R})$, in some reductive $\mathbb{R}$-group $\mathbf{G}$, is called $\mathbb{R}$-*regular* if the number of eigenvalues (counted with multiplicity) of $\mathrm{Ad}(g)$ of absolute value 1 is minimal. Any $\mathbb{R}$-*regular* element is semisimple (see [69, Remark 1.6.1]), and when $\mathbf{G}$ is split, every $\mathbb{R}$-regular element is regular.

With $\mathbf{H}$ as specified above, an element $g \in \mathbf{H}(\mathbb{R})$ is $\mathbb{R}$-regular if and only if some (any) representative of $\rho_{\mathrm{st}}(g)$ in $\mathrm{GL}_{D^n}(\mathbb{R})$ is conjugate to a diagonal n-by-n matrix with entries in D of distinct absolute values. Indeed, if $\rho_{\mathrm{st}}(g)$ is represented by $\mathrm{diag}(a_1, \ldots, a_n)$ with $|a_i| \neq |a_j|$ for $i \neq j$, the absolute values of the eigenvalues of $\mathrm{Ad}(g)$ are $\{|a_i a_j^{-1}|\}_{1 \leq i,j \leq 1}$ (with the correct multiplicities) and are equal to 1 only for $i = j$, which are the least possible occurrences. Conversely, if g is $\mathbb{R}$-regular, the centralizer of the $\mathbb{R}$-regular element $\rho_{\mathrm{st}}(g)$ contains a unique maximal $\mathbb{R}$-split torus $\mathbf{S}$ of PGL_{D^n} (see [69, Lemma 1.5]). Thus $\rho_{\mathrm{st}}(g)$ belongs to the centralizer of $\mathbf{S}(\mathbb{R})$, which, up to conjugation, is the subgroup of (classes of) diagonal n-by-n matrices with entries in D; say $\rho_{\mathrm{st}}(g)$ is represented by $\mathrm{diag}(a_1, \ldots, a_n)$. The absolute values of the eigenvalues of $\mathrm{Ad}(g)$ are again $\{|a_i a_j^{-1}|\}_{1 \leq i,j \leq 1}$. From the $\mathbb{R}$-regularity of $\rho_{\mathrm{st}}(g)$, we deduce that each value $|a_i a_j^{-1}|$ with $i \neq j$ must differ from 1, as claimed.

It follows from this description that if $\ell_{\max}$ (resp. $\ell_{\min}$) denotes the D-line in D^n on which a $\mathbb{R}$-regular element $g \in \mathbf{H}(\mathbb{R})$ acts by multiplication by an element of $D^\times$ of largest (resp. smallest) absolute value, then $\ell_{\max} = \mathrm{A}(g)$ is the attracting subspace of g (resp. $\ell_{\min} = \mathrm{A}(g^{-1})$), so that g is biproximal.[5] We record this here.

Lemma 3.17. *Let $\mathbf{H}$ and ρ_{st} be as above. Any $\mathbb{R}$-regular element $g \in \mathbf{H}(\mathbb{R})$ is biproximal under ρ_{st}.*

So, in order to exhibit proximal elements in $\rho_{\mathrm{st}}(\Gamma)$ for $\Gamma \leq \mathbf{H}(\mathbb{R})$ a Zariski-dense subgroup, it suffices to show Γ admits a $\mathbb{R}$-regular element. This is the content of the following theorem, due to Benoist and Labourie [8, A.1 Théorème]. We also refer the reader to the direct proof given by Prasad in [68].

Theorem 3.18 (Abundance of $\mathbb{R}$-regular elements, A.1 Théorème in [8])**.** *Let $\mathbf{G}$ be a reductive $\mathbb{R}$-group. Let Γ be a Zariski-dense subgroup of $\mathbf{G}(\mathbb{R})$. The subset of $\mathbb{R}$-regular elements in Γ is Zariski-dense.*

Corollary 3.19. *Let $\mathbf{H}$ and ρ_{st} be as above. Let Γ be a Zariski-dense subgroup of $\mathbf{H}(\mathbb{R})$. The elements $g \in \Gamma$ such that $\rho_{\mathrm{st}}(g)$ is biproximal, form a Zariski-dense subset of Γ.*

Remark 3.20. The existence of elements proximal under ρ_{st} in any Zariski-dense sub(semi)group can also be established using the results of Goldsheid and Margulis [33, Theorem 6.3] (see also [1, 3.12–14]). This approach is more tedious, as the standard representation of GL_{D^n} does not admit proximal elements if D^n is seen as a vector $\mathbb{R}$-space (which is in fact one of the motivations to extend the framework of [82] to division algebras). Instead, one should embed $\mathbf{P}_D(D^n)$ inside $\mathbf{P}_{\mathbb{R}}(\bigwedge_{\mathbb{R}}^d D^n)$ via the Plücker embedding, and exhibit proximal elements in that projective representation.

[5] Conversely, there exists a representation under which any proximal element is $\mathbb{R}$-regular, see [69, Lemma 3.4].

Next, we move on to the question of transversality. It turns out that under ρ_{st}, every non-central element $h \in \mathbf{H}(\mathbb{R})$ satisfies the transversality condition of Theorem 3.2.

Proposition 3.21. *Let $\mathbf{H}$ and ρ_{st} be as above. Let $h \in \mathbf{H}(\mathbb{R})$ be non-central. For every $p \in \mathbf{P}(D^n)$, the span of $\{\rho_{\mathrm{st}}(xhx^{-1})p \mid x \in \mathbf{H}(\mathbb{R})\}$ is the whole of $\mathbf{P}(D^n)$.*

Proof. Taking preimages in GL_{D^n}, we may without loss of generality work with the action of GL_{D^n} on D^n instead of $\rho_{\mathrm{st}}(\mathbf{H}) = \mathrm{PGL}_{D^n}$ on $\mathbf{P}(D^n)$. We will show in this setting that, for every non-zero $v \in D^n$ and every non-central $h \in \mathrm{GL}_{D^n}(\mathbb{R})$, the $\mathbb{R}$-span of $\{xhx^{-1} \cdot v \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ is the whole of D^n. The statement of the proposition then follows immediately by projectivization.

Viewing $\mathrm{End}_D(D^n)$ as a vector $\mathbb{R}$-space, the conjugation action defines a linear representation of SL_{D^n} on $\mathrm{End}_D(D^n)$. This representation decomposes into two irreducible components: a copy of the trivial representation given by the action of SL_{D^n} on the center of $\mathrm{End}_D D^n$, and a copy of the adjoint representation given by the action of SL_{D^n} on the subspace $\mathfrak{sl}_n(D)$ of traceless endomorphisms.

When h is not central, it admits a distinct conjugate xhx^{-1} of the same trace, hence the $\mathbb{R}$-span W_h of $\{xhx^{-1} \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ contains for some $g \in \mathrm{SL}_{D^n}(\mathbb{R})$ the nonzero traceless element $h' = h - ghg^{-1}$. In turn, W_h contains the $\mathbb{R}$-span $W_{h'}$ of $\{xh'x^{-1} \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$, a SL_{D^n}-stable subspace of $\mathfrak{sl}_n(D)$ which must equal $\mathfrak{sl}_n(D)$, as the latter is irreducible for the adjoint action. Thus, either $W_h = \mathfrak{sl}_n(D)$ if $\mathrm{Tr}(h) = 0$, or $W_h = \mathrm{End}_D(D^n)$ if $\mathrm{Tr}(h) \neq 0$.

Finally, for any non-zero $v \in D^n$ we have that $\mathfrak{sl}_n(D) \cdot v = D^n$, from which we conclude that the $\mathbb{R}$-span of $\{xhx^{-1} \cdot v \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ contains $W_h \cdot v = D^n$. □

Definition 3.22. Given a reductive F-group $\mathbf{G}$ with center $\mathbf{Z}$ and a subgroup $H \leq \mathbf{G}(F)$, for the purposes of this paper, we will say that H *almost embeds in a (simple) quotient* $\mathbf{Q}$ *of* $\mathbf{G}$ if there exists a (simple) quotient $\mathbf{Q}$ of $\mathbf{G}$ for which the kernel of the restriction $H \to \mathbf{Q}(F)$ is contained in $\mathbf{Z}(F)$.

It is clear that if $\mathbf{Q}$ is a simple factor of $\mathbf{G}$ and H is a subgroup of $\mathbf{Q}(F)$, then H almost embeds in $\mathbf{Q}$. In particular, if $\mathbf{G}$ is itself simple, every subgroup almost embeds in a simple quotient.

With this, we are ready to prove the following application of Theorem 3.2, establishing the abundance of simultaneous ping-pong partners for finite subgroups in products of inner forms of SL_n and GL_n which almost embed in a factor.

Theorem 3.23. *Let $\mathbf{G}$ be a reductive $\mathbb{R}$-group with center $\mathbf{Z}$. Let Γ be a subgroup of $\mathbf{G}(\mathbb{R})$ whose image in $\mathrm{Ad}\,\mathbf{G}$ is Zariski-dense. Let $(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(\mathbb{R})$, and set $C_i = H_i \cap \mathbf{Z}(\mathbb{R})$.*

Suppose that for each $i \in I$, there exists a simple quotient $\mathbf{Q}_i$ of $\mathbf{G}$ which is isogenous to $\mathrm{PGL}_{D_i^{n_i}}$ for D_i some finite division $\mathbb{R}$-algebra and $n_i \geq 2$, and for which the kernel of the projection $H_i \to \mathbf{Q}_i(\mathbb{R})$ is contained in C_i. Then the set of regular semisimple elements $\gamma \in \Gamma$ of infinite order such that for all $i \in I$, the canonical map

$$(\langle\gamma\rangle \times C_i) *_{C_i} H_i \to \langle \gamma, H_i\rangle \leq \mathbf{G}(\mathbb{R})$$

is an isomorphism, is dense in Γ for the join of the profinite topology and the Zariski topology.

Proof. For $i \in I$, let ρ_i denote the composite of the quotient map $\mathbf{G} \to \mathbf{Q}_i$ with the standard projective representation $\mathbf{Q}_i \to \mathrm{PGL}_{D_i^{n_i}}$, where D_i, n_i are an appropriate division $\mathbb{R}$-algebra and integer. Note that ρ_i factorizes $\mathbf{G} \to \mathrm{Ad}\,\mathbf{G} \to \mathrm{PGL}_{D_i^{n_i}}$.

Corollary 3.19 shows that the set of elements in $\rho_i(\Gamma)$ which are biproximal is Zariski-dense in $\mathrm{PGL}_{D_i^{n_i}}$; a fortiori, $\rho_i(\Gamma)$ contains a proximal element. Moreover, since C_i is the kernel of $\rho_i : H_i \to \mathrm{PGL}_{D_i^{n_i}}(\mathbb{R})$ by construction, every $h \in H_i \setminus C_i$ maps to a non-central element under ρ_i. Proposition 3.21 then precisely states that ρ_i satisfies the transversality condition of Theorem 3.2. We are thus at liberty to apply Theorem 3.2 to $\Gamma \leq \mathbf{G}(\mathbb{R})$ and the collection $(H_i)_{i\in I}$ (see also Remark 3.4), deducing this theorem. □

Remark 3.24. Let F be any field, and let $\mathbf{G}$ be a reductive F-group with center $\mathbf{Z}$. In order for a subgroup $H \leq \mathbf{G}(F)$ to admit a ping-pong partner in $\mathbf{G}(F)$, it is necessary that H almost embeds in a simple factor. Indeed, if the subgroup $\langle \gamma, H\rangle$ is the free amalgamated product of $\langle\gamma\rangle \times C$ and H over $C = H \cap \mathbf{Z}(F)$, then in the quotient $\mathbf{G}/\mathbf{Z}$, the image of $\langle\gamma, H\rangle$ is certainly freely generated by the images of γ and H. But $\mathbf{G}/\mathbf{Z}$ is the direct product of adjoint simple quotients of $\mathbf{G}$, so by Proposition 2.7, H/C embeds in (the F-points of) one of these factors.

In other words, Theorem 3.23 states that a collection of finite subgroups $(H_i)_{i\in I}$ in a group $\mathbf{G}$ whose simple quotients $\mathbf{Q}$ are each isogenous to some $\mathrm{PGL}_n(D)$, admits simultaneous ping-pong partners in Γ *if and only if* each H_i almost embeds in a simple factor.

Remark 3.25. There are versions of Theorem 3.23 for semisimple $\mathbb{R}$-groups of other types, but proving them requires a more careful study of the representation theory of $\mathbf{G}$ to exhibit a representation playing the role of ρ_{st}. However, as indicated in Remark 3.16, there are also cases where one needs additional information on the H_i to get a representation satisfying the transversality assumption of Theorem 3.2.

There are also versions of the theorem for other local fields. However, to prove those one needs additional information on Γ. Indeed, over a local field different from $\mathbb{R}$, bounded Zariski-dense subgroups exist, and a bounded subgroup obviously never admits proximal elements.

4. Free products between finite subgroups of units in a semisimple algebra

Conventions: throughout the remainder of this article, all orders will be understood to be $\mathbb{Z}$-orders. We also use the following notations:

- Whenever we say that a given F-algebra A is a finite algebra we mean that A is finite dimensional over F.
- RG denotes the group ring of G with coefficients in a ring R and $V(RG)$ the group of units in RG whose augmentation equals 1.
- $\mathrm{PCI}(A)$ is the set of primitive central idempotents of a finite (semisimple) algebra A. For each $e \in \mathrm{PCI}(A)$, there is a projection $\pi_e : \mathcal{U}(A) \twoheadrightarrow Ae$ onto the simple factor Ae of A.

4.1. Simultaneous partners in the unit group of an order.

By Wedderburn's theorem, every semisimple F-algebra A factors as

$$A = \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_m),$$

for V_i an n_i-dimensional right module over some finite division F-algebra D_i, $i = 1, \dots m$. In consequence, the F-group of units of A is the reductive group

$$\mathbf{G} = \mathrm{GL}_{D_1^{n_1}} \times \cdots \times \mathrm{GL}_{D_m^{n_m}} . \tag{4.1}$$

The original motivation for this project was the study of free amalgamated products inside $\mathcal{U}(\mathcal{O})$, the unit group of an order $\mathcal{O}$ in A; more precisely, the aim was to answer Conjecture 5.14. In this context, Theorem 3.23 yields the following neat necessary and sufficient condition for a given finite subgroup of $\mathcal{U}(\mathcal{O})$ to admit a ping-pong partner.

Theorem 4.1. *Let F be a number field, A be a finite semisimple F-algebra, and $\mathcal{O}$ be an order in A. Let Γ be a subgroup of $\mathcal{U}(\mathcal{O})$ whose image*[6] *in $\mathrm{Ad}(\mathbf{G})$ is Zariski-dense. Let H be a finite subgroup of $\mathcal{U}(A)$, and C be its intersection with the center of A.*

There exists $\gamma \in \Gamma$ of infinite order with the property that the canonical map

$$(\langle\gamma\rangle \times C) *_C H \to \langle \gamma, H\rangle$$

is an isomorphism, if and only if H almost embeds in Ae for some $e \in \mathrm{PCI}(A)$ for which Ae is neither a field nor a totally definite quaternion algebra.

Moreover, in the affirmative, the set of such elements γ is dense in the join of the Zariski and the profinite topology.

In particular, a free product $\mathbb{Z} * H$ exists in $\mathcal{U}(\mathcal{O})$ if and only if C is trivial and H embeds in a factor Ae which is neither a field nor a totally definite quaternion algebra.

Proof. We can base-change $\mathbf{G}$ to the $\mathbb{R}$-group $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$, whose $\mathbb{R}$-points $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ are a product of groups of the form $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, or $\mathrm{GL}_n(\mathbb{H})$, for various $n \geq 1$.

Any subgroup H of $\mathcal{U}(A) = \mathbf{G}(F)$ embeds in $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$. In fact, H almost embeds in a F-simple factor of $\mathbf{G}$ if and only if it does so in a $\mathbb{R}$-simple factor of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$. More precisely, let $K_1, \dots, K_s$ denote the summands of the étale $\mathbb{R}$-algebra $F \otimes_{\mathbb{Q}} \mathbb{R}$; they are precisely the different archimedean completions of F. Given a finite division algebra D over F, let D_{ij} be the division $\mathbb{R}$-algebras such that $D \otimes_F K_i \cong \prod_{j=1}^{m_i} \mathrm{M}_{r_{ij}}(D_{ij})$ as $\mathbb{R}$-algebras. The group $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{D^n} \times_{\mathbb{Q}} \mathbb{R}$ then factors into the product $\prod_{i=1}^{s} \prod_{j=1}^{m_i} \mathrm{GL}_{D_{ij}^{nr_{ij}}}$. The image of $\mathrm{GL}_{D^n}(F)$ in this product is obtained by embedding it diagonally using the canonical maps $\mathrm{GL}_{D^n}(F) \to \mathrm{GL}_{D^n}(K_i) \to \mathrm{GL}_{D_{ij}^{nr_{ij}}}(\mathbb{R})$. Thus if H (almost) embeds in a factor $(\mathrm{P})\mathrm{GL}_{D^n}$ over F, then it does so in any of the $(\mathrm{P})\mathrm{GL}_{D_{ij}^{nr_{ij}}}$ over $\mathbb{R}$, and the converse is obvious.

[6] This condition is for example satisfied if Γ is Zariski-dense in $\mathcal{U}(\mathcal{O})$ or $\mathrm{SL}_1(\mathcal{O})$.

Now, a simple quotient $\mathrm{PGL}_{D_{ij}^{nr_{ij}}}$ over $\mathbb{R}$ of a given factor GL_{D^n} of $\mathbf{G}$ satisfies $nr_{ij} = 1$, if and only if the jth factor in $Ae \otimes_F K_i$ is a division algebra, where e is the projection onto the factor of A corresponding to GL_{D^n}. In other words, the factor GL_{D^n} has a simple quotient $\mathrm{PGL}_{D_{ij}^{nr_{ij}}}$ with $nr_{ij} \geq 2$ for some i, j, if and only if Ae is not a division algebra which remains so under every archimedean completion of its center. This amounts in turn to Ae not being a field nor a totally definite quaternion algebra.

Next, let $\mathbf{G}_{\mathrm{is}}$ denote the $\mathbb{R}$-subgroup of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$ which is the direct product of those subgroups $\mathrm{GL}_{D_{ij}^{nr_{ij}}}$ for which $nr_{ij} \geq 2$. Since $\mathcal{U}(\mathcal{O})$ is an arithmetic subgroup of $\mathcal{U}(A) = \mathbf{G}(F)$, a classical theorem of Borel and Harish-Chandra [9] attests that the connected component of $\mathcal{U}(\mathcal{O})$ in $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$ is a lattice in the derived subgroup $\mathcal{D}\mathbf{G}_{\mathrm{is}}$ of $\mathbf{G}_{\mathrm{is}}$. In consequence, the image of Γ in $\mathrm{Ad}\,\mathbf{G}_{\mathrm{is}}$ is Zariski-dense.

Let f denote the canonical map $\mathbf{G}(\mathbb{R}) \to \mathrm{Ad}\,\mathbf{G}_{\mathrm{is}}(\mathbb{R})$, whose kernel is the product of the compact factors of $\mathbf{G}(\mathbb{R})$ with the center of $\mathbf{G}(\mathbb{R})$. Note that $\ker f$ commutes with $\mathbf{G}_{\mathrm{is}}(\mathbb{R})$, and that $\ker f \cap \Gamma$ is finite.

In view of all the above, provided H satisfies the embedding condition, we deduce from Theorem 3.23 applied to $\mathrm{Ad}\,\mathbf{G}_{\mathrm{is}}$ the existence of a dense set $S \subset f(\Gamma)$ of ping-pong partners for $f(H)$. By Lemma 2.3, the preimage $f^{-1}(S) \cap \Gamma$ consists of elements $\gamma \in \Gamma$ for which the canonical map $(\langle\gamma\rangle \times C) *_C H \to \langle\gamma, H\rangle$ is an isomorphism.

As S is dense in the join of the Zariski and the profinite topology, the same holds for $f^{-1}(S) \cap \Gamma$. Indeed, if $\Lambda\gamma_0$ is a coset of finite index in Γ, and U is a Zariski-open subset of Γ intersecting it, perhaps after shrinking and translating by $\ker f \cap \Gamma$, we can arrange that $\Lambda\gamma_0$ and U are contained in the connected component Γ° of Γ, and that $(\ker f \cap \Gamma^\circ) \cdot U = U$. Then $f(\Lambda\gamma_0 \cap U)$ equals the open set $f(\Lambda\gamma_0) \cap f(U)$. We may thus pick $x \in S \cap f(\Lambda\gamma_0 \cap U)$, implying that $f^{-1}(S) \cap \Lambda\gamma_0 \cap U$ is non-empty.

It remains to verify that the embedding condition is necessary. Suppose $\gamma \in \Gamma$ is such that $(\langle\gamma\rangle \times C) *_C H \to \langle\gamma, H\rangle$ is an isomorphism. Let $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) denote the product of the factors of $\mathbf{G}$ over F for which the corresponding factor Ae of A is not (resp. is) a field or a totally definite quaternion algebra. Because this product decomposition is defined over F, the projections of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_1(F \otimes_{\mathbb{Q}} \mathbb{R})$ and $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ are discrete. Since $\mathcal{D}\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact, the image of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ is in fact finite.

As $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$, Proposition 2.7 shows that one of the kernels N_1, N_2 of the respective projections $\pi_i : \langle\gamma, H\rangle \to \mathbf{G}_i(F \otimes_{\mathbb{Q}} \mathbb{R})$, is contained in C. Of course, N_2 can not be contained in C, otherwise the image of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ would contain the infinite group $(\langle\gamma\rangle \times C/N_2) *_{C/N_2} (H/N_2)$. We deduce that $N_1 \subset C$, that is, $\langle\gamma, H\rangle$ almost embeds in $\mathbf{G}_1$. Another application of Proposition 2.7 then shows that $\langle\gamma, H\rangle$ almost embeds in some factor of $\mathbf{G}_1$ over F, hence in a factor of A which is not a field nor a totally definite quaternion algebra, as claimed. □

Example 4.2. If $A = FG$ and $\mathcal{O} = RG$ for some order R in the number field F and G a finite group, then by the theorem of Berman-Higman [51, Theorem 2.3.] the only torsion central units are the trivial ones (i.e. the elements of $\mathcal{U}(R) \cdot \mathcal{Z}(G)$). Thus if we take $H \leq V(RG)$, then $C = H \cap \mathcal{Z}(G)$. In particular, $G * \mathbb{Z}$ exists if and only if G itself embeds in a simple factor and intersects the center of $\mathcal{U}(R)$ trivially (this happens when G is simple, for instance).

Although Theorem 3.23 and Theorem 4.1 are nice existence results, they leave open the following two questions.

Questions 4.3. With the notation of Theorem 4.1:

(i) How can we construct the ping-ping partner γ concretely?

(ii) When does H embed in a simple factor?

In the remainder of this section, we present a method to approach question (i), which will reduce the problem to constructing certain *deformations* of H (see Definition 4.4); the main result is Theorem 4.10. In Section 5, we will propose a general method to construct such deformations when A is a group algebra. All of this entails the question of the existence, given two (finite) subgroups H and K of Γ, of a copy of $H * K$ inside Γ (see Question 4.11). Question (ii) will be addressed in Section 5.2.

4.2. **Deforming finite subgroups and subalgebras.** Keeping the notation of Section 4.1, the aim of this section is to introduce an explicit linear method that allows to replace a finite subgroup $H \leq \mathcal{U}(A)$ by an isomorphic copy which has the necessary ping-pong dynamics. Concretely, we want to construct a group morphism of the form

$$\Delta : H \to \mathcal{U}(A) : h \mapsto \Delta(h) = h + \delta_h.$$

This is possible when the map $\delta = \Delta - 1$ satisfies the following conditions.

Definition 4.4. Let H be a subgroup of $\mathcal{U}(A)$. We call an F-linear map $\Delta : H \to A$ *a first-order deformation of* H if the map $\delta = \Delta - 1$ satisfies the following conditions:

(Derivation) $\delta_{hk} = \delta_h k + h\delta_k$ for all $h, k \in H$;

(Order 1) $\delta_h \delta_k = 0$ for all $h, k \in H$.

A straightforward calculation shows that if Δ is a first-order deformation of H, then the maps $\Delta_t(h) = h + t\delta_h$ for $t \in F$ are group morphisms from H to $\mathcal{U}(A)$, interpolating between the identity Δ_0 and $\Delta_1 = \Delta$. In fact, if a linear map $\Delta : H \to A$ is a group morphism and $\delta = \Delta - 1$ satisfies either (Derivation) or (Order 1), then δ also satisfies the remaining property. Moreover, since Δ_t is assumed to be linear, it extends uniquely to an algebra morphism $FH \to A$. We define a *first-order deformation* of a subalgebra B of A analogously, so that first-order deformations of subalgebras are algebra morphisms, and the linear extension of a deformation of H is a deformation of FH. We say that Δ is an *inner (first-order) deformation* when the derivation δ is inner over A, that is, when $\delta_h = [n, h]$ for some $n \in A$.

Examples 4.5. Let $H \leq \mathcal{U}(A)$.

(i) If n is an element of A satisfying $nhn = 0$ for all $h \in H$, then the assignment

$$\delta_h = [n, h]$$

defines a first-order deformation of H, which is actually given by the conjugation

$$\Delta(h) = (1 + n)h(1 + n)^{-1} = h + [n, h].$$

This deformation is inner by construction.

(ii) If $m \in A$ satisfies $mh = m$ for all $h \in H$, then the assignment

$$\delta_h = (1 - h)m$$

defines a first-order deformation of H. (The assignment $\delta_h = m(1 - h) = 0$ defines the trivial deformation.)

Assume for a moment that H is finite and that $\operatorname{char} F$ does not divide $|H|$; set $e = \frac{1}{|H|}\sum_{h \in H} h$. Then this deformation is in fact of the first kind with $n = (1-e)m$, as $(1 - e)m \cdot h \cdot (1 - e)m = 0$ since $m(1 - e) = 0$, and

$$\delta_h = [(1 - e)m, h] = (1 - e)m - (h - e)m = (1 - h)m.$$

Note that under this additional assumption on H, the condition $mh = m$ for all $h \in H$ is equivalent to $me = m$. This deformation might be trivial, for instance when $m = e$, but if H is F-linearly independent[7] and not central, one can find some m for which it is not.

(iii) Any example of the second kind obviously satisfies $\delta_g h = \delta_g$ for all $g, h \in H$. The converse holds under the assumption that H is finite and that $\operatorname{char} F$ does not divide $|H|$. Set $e = \frac{1}{|H|}\sum_{h\in H} h$ as above. If $\Delta = 1 + \delta$ is a first-order deformation which happens to satisfy $\delta_g h = \delta_g$ for all $g, h \in H$, then the equation $\delta_e = \delta_{he} = \delta_h e + h\delta_e = \delta_h + h\delta_e$ implies that

$$\delta_h = (1-h)\delta_e.$$

Thus, this deformation is of the second kind with $m = \delta_e$. Since $\delta_e = e\delta_e + \delta_e e$ implies $n = (1-e)\delta_e = \delta_e$, we deduce as in the first two examples that

$$\Delta(h) = (1+\delta_e)h(1+\delta_e)^{-1}.$$

Note that when Δ is an inner first-order deformation, $\Delta(h) = h$ for every $h \in H \cap \mathcal{Z}(A)$. Also, examples (ii) and (iii) above can only occur if $H \cap F = \{1\}$.

Lemma 4.6. *The kernel of a first-order deformation $\Delta : H \to A$ consists of unipotent elements. In consequence, if H is finite and* $\operatorname{char} F$ *does not divide $|H|$, every first-order deformation $H \to A$ is injective.*

Proof. An element h lies in the kernel of Δ if and only if $\delta_h = 1 - h$. By assumption, $(\delta_h)^2 = (1-h)^2 = 0$, showing that h is a unipotent element of $A^\times$.

Over a field of characteristic p (resp. 0), any non-trivial unipotent element has order p (resp. infinite order); so if H has no elements of order p nor ∞, the map Δ is injective. $\square$

If H is infinite, or if $\operatorname{char} F$ divides $|H|$, there are first-order deformations which are not injective. For instance, assuming $\operatorname{char} F \neq 2$, the trivial map

$$H = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \middle| \, x \in F \right\} \to \mathrm{M}_2(F) : h \to 1$$

is an inner first-order deformation, associated with $\delta_h = [n, h] = 1 - h$ for $n = \left[\begin{smallmatrix} 1/2 & 0 \\ 0 & -1/2 \end{smallmatrix}\right]$.

In more generality, we have the following description of first-order deformations of semisimple subalgebras, which may be of independent interest.

Theorem 4.7. *Let A be an F-algebra, and let B be a separable subalgebra of A. Then every first-order deformation $\Delta : B \to A$ is given by conjugation, that is, there exists $a \in A$ such that $\Delta(b) = aba^{-1}$ for every $b \in B$. In particular, any such deformation extends to an automorphism of A and fixes $B \cap \mathcal{Z}(A)$.*

Remark 4.8. In Section 5, the F-algebra A will be taken finite-dimensional and semisimple, and the first-order deformations $\Delta : B \to A$ under consideration will restrict to ring morphisms between an order $\mathcal{O}_B$ of B and an order $\mathcal{O}_A$ of A. In that setting, it will be convenient to use the formalism of first-order deformations, despite the fact that in the end, they are given by conjugation in the ambient algebra A.

Proof of Theorem 4.7. Recall that the separability of B means that B admits a *separating idempotent*, that is, there is an element $e \in B \otimes_F B$ whose image under the multiplication map $\mu : B \otimes_F B \to B$ is 1 and which satisfies $(b \otimes 1)e = e(1 \otimes b)$ for every $b \in B$.

Let I denote the kernel of the (B, B)-bimodule map μ. Recall that the canonical map

$$d : B \to I : b \mapsto b \otimes 1 - 1 \otimes b$$

[7]This condition is natural for the applications later on. Indeed, it is well-known that if H is a finite subgroup of $\mathcal{V}(RG)$, with R a $|G|$-adapted ring (i.e. $|G|$ is not invertible in R), then H is F-linearly independent in FG for $F = \mathrm{Frac}(R)$.

identifies I with the bimodule of (non-commutative) differentials of the algebra B. That is, for any derivation $\delta : B \to M$ to a (B,B)-bimodule M, there exists a unique bimodule map $f : I \to M$ such that $\delta = f \circ d$ (see [12, A III §10 N°10 Proposition 17]). Applying this to the derivation $\delta = \Delta - 1$, for Δ a first-order deformation $B \to A$, we obtain existence of a unique bimodule map $f : I \to A$ satisfying $\delta(b) = f(b \otimes 1 - 1 \otimes b)$ for every $b \in B$.

Note that $e - 1 \in I$ by construction. Inside A, we now compute

$$\begin{aligned}[f(e-1), b] &= f(e-1)\cdot b - b \cdot f(e-1) = f\big((e-1)(1\otimes b) - (b \otimes 1)(e-1)\big) \\ &= f(e(1\otimes b) - 1 \otimes b - e(1 \otimes b) + b \otimes 1) \\ &= f(b \otimes 1 - 1 \otimes b) = f(d(b)) = \delta(b),\end{aligned}$$

showing that the derivation $\delta : B \to A$ is inner, given by the adjoint of $f(e-1) \in A$.

The bimodule I of differentials of B is generated as a left (respectively right) module by the image of d (see [12, A III §10 N°10 Lemme 1]), hence the same holds for the image of f; in other words, $f(e-1) \in B \cdot \delta(B) = \delta(B) \cdot B$. As by assumption $\delta(B)\cdot\delta(B) = 0$, we conclude that $f(e-1)bf(e-1) = 0$ for every $b \in B$. This shows that

$$\Delta(b) = b + f(e-1)b - bf(e-1) = (1 + f(e-1))b(1 - f(e-1))$$

is given by conjugating b by $1 + f(e-1)$, proving the theorem. □

Remark 4.9. Recall that an F-algebra B is separable if and only if B is absolutely semisimple, which is in turn equivalent to B being semisimple with étale center (see respectively [11, A VIII §13 N°5 Théorème 2 & N°3 Théorème 1]). The center of a finite-dimensional semisimple algebra is always a product of fields, and is automatically étale if the base field F is perfect. The example above the theorem illustrates why it is essential to assume that B is semisimple.

It follows from the general theory that when B is a separable algebra, every derivation of B with values in a (B,B)-bimodule is inner (see [11, A VIII §13 N°7, Corollaire]). However, without additional information on n, this does not formally imply that a first-order deformation $\Delta : B \to A : b \mapsto b + [n, b]$ is given by conjugation. The proof of Theorem 4.7 exhibits a suitable such element n.

4.3. **Ping-pong between two given finite subgroups of $\mathcal{U}(A)$.** Given two finite subgroups H, A in $\mathrm{GL}_n(D)$, the aim of this subsection is to provide a constructive method to obtain a subgroup of $\mathrm{GL}_n(D)$ isomorphic to $A * H$ by deforming H using the first-order deformations introduced in Section 4.2.

Theorem 4.10. *Let K be a local field, D be a finite-dimensional division K-algebra, and A, H be finite subgroups of $\mathrm{GL}_n(D)$. Set $C = A \cap H$, and assume that $|A : C| > 2$ or $|H : C| > 2$. Suppose that $\Delta_t : H \to \mathrm{GL}_n(D) : h \mapsto h + t\delta_h$ (with $t \in F^\times$) is a family of first-order deformations satisfying*

(i) $\delta_g = 0 \iff g \in C$, *and*

(ii) $a\,\mathrm{im}(\delta_h) \cap \ker(\delta_{h'}) = \{0\}$ *for every $a \in A \setminus C$, $h, h' \in H \setminus C$.*

Then there exists $N \in \mathbb{R}$ such that

$$\langle A, \Delta_t(H)\rangle \cong A *_C \Delta_t(H) \cong A *_C H$$

when $|t| \geq N$.

In practice, checking the conditions $a\,\mathrm{im}(\delta_h) \cap \ker(\delta_{h'}) = \{0\}$ can be difficult, but luckily many are superfluous. For example, one can prove that $\ker(\delta_h) = \ker(\delta_{h^t})$ when $\gcd(o(h), t) = 1$. Building on Examples 4.5, we will lay out in Section 5 a way to construct a first-order deformation Δ_t to which Theorem 4.10 applies, in the case where A and H are finite subgroups of the unit group of a group ring.

Note that there is certainly a restriction on the existence of free products $A * H$ in $\mathrm{GL}_n(D)$. For example, $H * H$ exists if and only if $H * \langle t \rangle$ exists for some $t \in \mathrm{GL}_n(D)$. According to Theorem 4.1, inside of the unit group of an order the latter happens precisely when H contains no scalar matrices and $\mathrm{GL}_n(D)$ is not the multiplicative group of a field or of a totally definite quaternion algebra. Consequently, if for some $x \in \mathrm{GL}_n(D)$ the group $\langle A, H^x \rangle$ is finite and not intersecting the center, then there is some $y \in \mathrm{GL}_n(D)$ for which the subgroup $\langle A, H^y \rangle$ is freely generated by A and H^y. In fact the results in [62, 35] can be reformulated to say that H^y can be taken to be the image of H under some first order deformation. Note that the condition that H does not intersect $\mathcal{Z}(D)$ is not essential, and can be resolved by amalgamating along $H \cap \mathcal{Z}(D)$. On the other hand, the case where $\langle A, H^x \rangle$ is infinite for any conjugate H^x of H seems much more difficult to understand. This can happen, as for instance $GL_2(\mathbb{Z}) \cong D_8 *_{C_2 \times C_2} D_{12}$. In fact, $\langle A, H^x \rangle$ will generically be infinite for every x.

Altogether, the following question arises naturally.

Question 4.11. Let D be a finite division algebra over a number field F. Let A and H be finite subgroups of $\mathrm{GL}_n(D)$. Suppose that $\mathrm{GL}_n(D)$ with $n \geq 2$. Is there a copy of $A *_{A \cap H} H$ in $\mathrm{GL}_n(D)$?

Remark 4.12. Note that the conditions from Theorem 4.10 imply that

$$A \cap F^\times = H \cap F^\times \subset C.$$

Indeed, if $a \in A \cap F^\times \setminus C$ and $h \in H \setminus C$, then the second condition implies that $\mathrm{im}(\delta_h) \cap \ker(\delta_h) = \{0\}$. Since $\mathrm{im}(\delta_h) \subset \ker(\delta_h)$, it would follow that $\delta_h = 0$, a contradiction. Similarly, if $h \in H \cap F^\times$, then it is a general property of first-order deformations that $\delta_h = 0$, hence $h \in C$ by the first condition.

The next lemma will serve to replace Lemma 3.9 for the proof of Theorem 4.10.

Lemma 4.13. *Let $h \in \mathrm{GL}_V(K)$, let n be a non-zero nilpotent transformation in $\mathrm{End}_D(V)$, and set $h_t = h + tn$ for $t \in K$. Let C be a compact subset of $\mathbf{P}(V) \setminus \ker(n)$, and let U be a neighborhood of $\mathrm{im}\, n$ in $\mathbf{P}(V)$. There exists $N > 0$ such that if $|t| \geq N$, then $h_t \in \mathrm{GL}_V(K)$ and $h_t C \subset U$.*

Proof. Since $\mathrm{Nrd}(h) \neq 0$, the polynomial $\mathrm{Nrd}(h + Xn) \in K[X]$ is not zero, hence has only finitely many roots. For $|t|$ strictly larger than the maximum absolute value N_0 of these roots, $h + tn \in \mathrm{GL}_V(K)$.

Let now $p \in \mathbf{P}(V) \setminus \ker n$, and let v represent p in V. Since $h_t(v) = h(v) + tn(v)$, it follows that $h_t p$ converges to $np \in \mathbf{P}(V)$ as $|t| \to \infty$. So for each $p \in C$, there exists $N_p > 0$ and a neighborhood U_p of p in $\mathbf{P}(V)$ such that $h_t(U_p) \subset U$ if $|t| \geq N_p$. By the compactness of C, there is a finite collection $U_{p_1}, \ldots, U_{p_r}$ covering C. Setting $N = \max\{N_0, N_{p_1}, \ldots, N_{p_r}\}$, we see that $h_t C \subset U$ when $|t| \geq N$, as claimed. $\square$

Remark 4.14. When K is a non-archimedean local field, care has to be taken that the condition $|t| \geq N$ is not preserved by addition. In other words, if the conclusion of the lemma holds for h_t, one cannot deduce that $gC \subset U$ for every $g \in \langle h_t \rangle$. This fails already for $h = 1$, in which case the subgroup $\langle 1 + tn \rangle$ accumulates at 1.

This mistake was made in [35]. Some of the results in this paper, namely [35, Theorems 2.3, 2.6, and 2.7], therefore only hold over archimedean local fields.

Proof of Theorem 4.10. As in the proof of Theorem 3.2, it will be convenient to write $H^* = H \setminus C$ and $A^* = A \setminus C$.

Let W denote the union of the proper subspaces $\mathrm{im}(\delta_h)$ for $h \in H^*$, and let W' denote the union of the proper subspaces $\ker(\delta_h)$ for $h \in H^*$, all viewed in $\mathbf{P}(V)$. Note right away that $W \subset W'$. By the first assumption, if $c \in C$ then $c\, \mathrm{im}(\delta_h) = \mathrm{im}(c\delta_h) = \mathrm{im}(\delta_{ch})$, so W

is stable under C. Moreover, $aW \cap W \subset aW \cap W' = \emptyset$ for every $a \in A^*$. Indeed, if this last intersection were non-empty, then so would be $a\,\mathrm{im}(\delta_h) \cap \ker(\delta_{h'})$ for some $h, h' \in H^*$, contradicting the second assumption.

In consequence, we can construct a compact neighborhood P of W with the properties that P is stable under C, and $aP \cap (P \cup W') = \emptyset$ for every $a \in A^*$. For instance, start with a neighborhood of W whose translates under A^* are disjoint from W', remove from it a sufficiently small open neighborhood of the union of its translates under A^*, then intersect the result with its translates under C. Set $Q = A^* \cdot P = \bigcup_{a \in A^*} aP$; by construction, Q is a compact set disjoint from P, and from $\ker(\delta_h)$ for every $h \in H^*$.

In order to conclude the proof, it remains to verify the conditions of Lemma 2.1. We already arranged for $A^* \cdot P \subset Q$ and $C \cdot P = P$. The fact that $C \cdot Q = Q$ is an obvious consequence of $C \cdot A^* = A^*$. Lastly, since Q is disjoint from $\ker(\delta_h)$, Lemma 4.13 yields for each choice of $h \in H^*$ a positive number N_h such that $\Delta_t(h) = h + t\delta_h$ sends Q into P when $|t| \geq N_h$. Set $N = \max_{h \in H^*} N_h$ and pick $|t| \geq N$, so that $\Delta_t(h) \cdot Q \subset P$ for every $h \in H^*$. An application of Lemma 2.1 (to A and $\Delta_t(H)$ with the sets P and Q) finally shows that when $|t| \geq N$, $\langle A, \mathrm{im}(\Delta_t) \rangle$ is the free product of A and $\Delta_t(H)$ amalgamated along their intersection C. $\square$

5. Generic constructions of amalgams and the embedding property for group rings

In Section 4, via the first-order deformation introduced in Definition 4.4, we proposed a constructive method to obtain free products of finite groups inside the unit group of an order in a finite-dimensional semisimple algebra A. The present and final section will focus on the case where $A = FG$ is the group ring of a finite group G and $\Gamma = \mathcal{U}(RG)$ is the group of units of A over some order R in F. To this choice of A and Γ the results of Sections 3 and 4 are more readily applicable, in part thanks to a theorem of Cohn and Levingstone attesting that finite subgroups of $\mathcal{U}(RG)$ are R-linearly independent. Section 2

More precisely, in Section 5.1 we develop further the first-order deformations from Examples 4.5.(iii), which are inspired from the construction of shifted bicyclic units (see Definition 5.1). Conjecture 5.6 formulates our presumption that the images of two opposite shifted bicyclic maps always form a free product amalgamated along their intersection. In particular we address the first of Questions 4.3 for group rings.

We discuss the second of Questions 4.3 in Section 5.2. Theorem 5.7 answers that question in the affirmative for any subgroup H of G admitting a faithful irreducible representation (e.g. any subgroup whose Sylow subgroups have cyclic center). Consequently, for such subgroups H we obtain in Corollary 5.10 the existence of a free product $H *_C H$ amalgamated along $C = H \cap \mathcal{Z}(G)$, inside $\mathcal{U}(RG)$.

Finally, and most importantly, we prove in Section 5.3 that profinitely-generically, pairs of shifted bicyclic units generate a free group. As a consequence of all this work, we can precisely determine when a given finite subgroup admits a bicyclic unit as ping-pong partner.

Throughout this section, R will denote a commutative Noetherian domain of characteristic 0, and F its field of fractions. Recall that $\mathcal{U}(RG)$ stands for the group of units of the algebra RG; we will denote $\mathcal{V}(RG)$ the kernel of the augmentation map $\epsilon : FG \to F : \sum_i a_i g_i \mapsto \sum a_i$ restricted to $\mathcal{U}(RG)$.

5.1. Shifted bicyclic units and a conjecture on amalgams.

We will now apply the construction from Examples 4.5.(iii) to the case of the group ring $A = FG$ and finite subgroups of $\mathcal{V}(RG)$. Note that $\mathcal{U}(RG) = \mathcal{U}(R) \cdot \mathcal{V}(RG)$. The advantage of working with $\mathcal{V}(RG)$ is that its finite subgroups are R-linearly independent, by a theorem of Cohn and Livingstone [21][8].

Definition 5.1. Let G be a finite group, H be a finite subgroup of $\mathcal{U}(RG)$, and pick $x \in RG$. Set $\widetilde{H} = \sum_{h\in H} h$. The maps

$$b_{x,H} : H \to \mathcal{U}(RG) : h \mapsto h + (1-h)x\widetilde{H}$$

and

$$b_{H,x} : H \to \mathcal{U}(RG) : h \mapsto h + \widetilde{H}x(1-h)$$

will be called the *(left, resp. right) shifted bicyclic maps associated with H and x*. An element in $\mathcal{U}(RG)$ of the form $b_{x,H}(h)$ or $b_{H,x}(h)$ will be called a *(left, resp. right) shifted bicyclic unit.*

Remark 5.2. When $H = \langle h \rangle$ is a cyclic group and $x \in G$, the elements $b_{x,\langle h\rangle}(h)$ and $b_{\langle h\rangle,x}(h)$ have been called Bovdi units in [43] in honor of Victor Bovdi, who initiated the

[8] In [21] this result is shown only for F a number field and R its ring of integers. However the proof of [24, Corollary 2.4] combined with the general version of Berman's theorem stated in [71, Theorem III.1], implies it in the generality claimed here.

study of such units. In [58] these units were renamed shifted bicyclic units. Recall that *bicyclic units* are elements of the form

$$b_{x,h} = 1 + (1-h)x\widetilde{\langle h\rangle} \text{ and } b_{h,x} = 1 + \widetilde{\langle h\rangle}x(1-h) \tag{5.1}$$

for $h, x \in G$. Note that one can rewrite $b_{x,H}(h) = h(1+(1-h)h^{-1}x\widetilde{H})$, hence $b_{x,\langle h\rangle}(h) = hb_{h^{-1}x,h}$. In this sense, shifted bicyclic units are slight (torsion) adaptations of bicyclic units. As the terminology used in [58] better reflects the nature of these units, we adopt it here.

Note that the shifted bicyclic maps from Definition 5.1 are instances of first-order deformations (with $\delta_h := (1-h)x\widetilde{H}$, cf. Examples 4.5.(ii)). As such, the first two properties below follow from the considerations of Section 4.2.

Proposition 5.3. *Let G be a finite group, H be a finite subgroup of $\mathcal{V}(RG)$, and pick $x \in RG$.*

(i) *The shifted bicyclic maps $b_{x,H}$ and $b_{H,x}$ are injective group morphisms.*
(ii) *The subgroups H, $\mathrm{im}(b_{x,H})$ and $\mathrm{im}(b_{H,x})$ are conjugate in $\mathcal{U}(FG)$.*
(iii) *If in addition $H \leq G$ and $x \in G$, then $\mathrm{Im}(b_{x^{-1},H}) \cap \mathrm{Im}(b_{H,x}) = H \cap H^x$.*

Proof. As we just noted, $b_{x,H}$ and $b_{H,x}$ are first-order deformations of H in FG; it follows from Definition 4.4 that they are group morphisms. Theorem 4.7 states that the identity map and the maps $b_{x,H}$ and $b_{H,x}$ are all conjugate by $\mathcal{U}(FG)$. In particular, $b_{x,H}$ and $b_{H,x}$ are injective, and points (i) and (ii) are proved.

For point (iii), note that $H \cap H^g = \{h \in H \mid [h, g^{-1}] \in H\}$. Therefore if $h \in H \cap H^g$, then $B_{g^{-1},H}(h) = h + g^{-1}(1-h)[h, g^{-1}]\widetilde{H} = h$. Similarly $h = B_{H,g}(h)$ and so $h \in \mathrm{Im}(b_{x^{-1},H}) \cap \mathrm{Im}(b_{H,x})$. Conversely, suppose that

$$h + (1-h)g^{-1}\widetilde{H} = k + \widetilde{H}g(1-k)$$

for some $h, k \in H$. In other words,

$$h - k + g^{-1}\widetilde{H} - hg^{-1}\widetilde{H} - \widetilde{H}g + \widetilde{H}gk = 0. \tag{5.2}$$

If $g \in H$, then the converse inclusion trivially holds, so suppose $g \notin H$. By Cohn-Livingstone's result finite subgroups of $\mathcal{U}(RG)$ are $R-$linear independent, thus we will look at the support of the elements. Note that $h \notin \mathrm{Supp}\{hg^{-1}\widetilde{H}\} \cup \mathrm{Supp}\{\widetilde{H}g\} \cup \mathrm{Supp}\{g^{-1}\widetilde{H}\} \cup \mathrm{Supp}\{\widetilde{H}gk\}$ as otherwise $g \in H$. Thus $h = k$. We will proof that $h \in H \cap H^g$. For this take $g^{-1}l \in g^{-1}\widetilde{H}$ which by (5.2) must cancel with either an element of the form $hg^{-1}t$ or tg for $t \in H$. In the former case $h = (lt^{-1})^g$, as desired. Thus we may suppose that $\mathrm{Supp}\{g^{-1}\widetilde{H}\} = \mathrm{Supp}\{\widetilde{H}g\}$. In particular $g \in g^{-1}H$, i.e. $g^2 \in H$. On this turn this entails that $g^{-1}h \in Hg$, hence also $gh = g^2g^{-1}h \in Hg$. This finishes the proof. □

The shifted bicyclic maps can be used to construct generically several types of subgroups of $\mathcal{U}(RG)$. For example, using other terminology, in [43, Proposition 3.2.] they were used to produce solvable subgroups and free subsemigroups. Another construction is the one below. Recall that by $I(RG)$ we denote the *kernel of the augmentation map* ϵ as a ring morphism. Moreover,

$$I(RG) = \sum_{g\in G}(1-g)RG = \sum_{g\in G} R(1-g).$$

Proposition 5.4. *Let G be a finite group, $H \leq G$, $g \in G$ and set $C = H \cap H^g$. Then*

$$\langle H, \mathrm{im}(b_{g,H})\rangle \simeq I(R[H/C]) \rtimes H,$$

where H acts on $I(R[H/C])$ cia left multiplication by inverses. In particular it is abelian-by-finite.

When C is not normal in H, the group $I(R[H/C])$ is meant to mean the kernel of the R-module morphism ϵ, which element wise is the same as the ring morphism ϵ, between the R-modules $R[H/C]$ and R.

Proof. For notation's sake, put $b = b_{g,H}$. Set $U = \langle H, b(H)\rangle \leq \mathcal{U}(RG)$. Remember that a shifted bicyclic unit is the product of a (generalized) bicyclic unit and an element of H:

$$b(h) = h + (1-h)g\widetilde{H} = (1 + (1-h)g\widetilde{H})h = b_h h;$$

where $b_h := 1 + (1-h)g\widetilde{H}$. So,

$$U = \langle h, b_k \mid h, k \in H\rangle.$$

Define $N = \langle b_k \mid k \in H\rangle$. We will first show that N is a normal complement of H in U and thus $U \simeq N \rtimes H$. Recall that $b_h^n = 1 + n(1-h)g\widetilde{H}$ and hence b_h is a torsion unit if and only if it is equal to 1 which happens exactly when $h^g \in H$. In particular N and H have trivial intersection. Also from the previous follows that N consists exactly of the elements of the form $b_a := 1 + ag\widetilde{H}$ with $a \in I(RH)$. Using this remark we see that N is normal:

$$b_a^x = x^{-1}(1 + ag\widetilde{H})x = 1 + x^{-1}ag\widetilde{H} = b_{x^{-1}a} \in N. \tag{5.3}$$

for all $x \in H$ and $a \in I(RH)$.

It remains to prove that N is isomorphic to $I(R[H/C])$. Clearly $b_{a_1}b_{a_2} = b_{a_1+a_2}$ for all $a_1, a_2 \in I(RH)$ so that we have a group epimorphism $\varphi\colon I(RH) \to N\colon a \mapsto b_a = 1 + ag\widetilde{H}$. Note that for $x, y \in H$ we have $\mathrm{Supp}(xg\widetilde{H}) \cap \mathrm{Supp}(yg\widetilde{H}) \neq \emptyset$ if and only if $xg\widetilde{H} = yg\widetilde{H}$ if and only if $xC = yC$. Thus

$$\varphi\left(\sum_{x\in H} a_x x\right) = 1 + \sum_{hC\in H/C}\left(\sum_{x\in hC} a_x\right) hg\widetilde{H},$$

and hence

$$\mathrm{Ker}(\varphi) = \bigoplus_{t\in T} tI(RC),$$

for some T a left-transversal of C in H and $N \simeq I(R[H/C])$.

Finally note that if we identify N with $I(R[H/C])$ then H acts on $I(R[H/C])$ via $\varphi\colon H \to \mathrm{Aut}(I(R[H/C]))\colon h \mapsto (a \mapsto h^{-1}a)$ by (5.3). ☐

The proof of Proposition 5.4 shows that the group $\langle 1 + (h-1)g\widetilde{H} \mid h \in H\rangle$ is a free-abelian group of rank $|H : H \cap H^g| - 1$. In particular, if $H \cap H^g = 1$, $\langle H, B_{-g,H}(H)\rangle \simeq I(RH) \rtimes H$ yields a free-abelian subgroup of rank $|H| - 1$.

Corollary 5.5. *Let G be a finite group and H a cyclic subgroup of G of prime order. If $g \in G$ does not normalise H then $\mathcal{U}(RG)$ contains a subgroup isomorphic to $\mathbb{Z}^{p-1} \rtimes C_p$. In particular, if $p = 2$, then $\mathcal{U}(RG)$ contains*

$$\langle C_2, B_{g,C_2}(C_2)\rangle \cong \mathbb{Z} \rtimes C_2 \cong C_2 * C_2,$$

the infinite dihedral group.

Remark. In general the existence of an abelian subgroup $H \leq G$ yields a free-abelian subgroup $F \leq \mathcal{U}(\mathbb{Z}H) \leq \mathcal{U}(\mathbb{Z}G)$ of rank $e = \frac{1}{2}(|H| + 1 + n_2 - 2\ell)$, where n_2 is the number of involutions in H and ℓ the number of cyclic subgroups of H, cf. [65, Exercise 8.3.1] or [46, Theorem 7.1.6.]. Corollary 5.5 therefore yields a larger than usually expected free-abelian subgroup.

The part of Corollary 5.5 for the prime 2 also suggests that it might be possible to make free products of finite groups using appropriate shifted bicyclic maps. This is further supported with reformulating via Theorem 4.10 some results in the literature in terms of first order deformations as in Examples 4.5. All this gives evidence for the following which is a precise version of Question 4.11 in case of FG.

Conjecture 5.6. *Let $H \leq G$ be finite groups such that H has an almost embedding in a simple factor of $\mathcal{U}(FG)$. Further let $g \in G$ and denote $C = H \cap H^g$, then*

$$\langle \mathrm{im}(b_{g,H}), \mathrm{im}(b_{H,g^{-1}}) \rangle \cong H *_C H \cong \langle \mathrm{im}(b_{g,H}), \mathrm{im}(b_{g,H})^* \rangle$$

where $(\cdot)^*$ is the canonical involution on FG.

If $F = \mathbb{Q}$, G is nilpotent of class 2, $H \cong C_n$ and $g \in G$ such that $H \cap H^g = 1$, then [43, Theorem 4.1] shows that the conditions of Theorem 4.10 are satisfied and so $H * H$ can be constructed in the conjectured way. If n is prime, this was also obtained for arbitrary (finite) nilpotent groups. In all these cases an explicit embedding of H in a simple component of $\mathbb{Q}G$ was constructed. Recently Marciniak - Sehgal [58] were able to drop the condition on n without the use of such an embedding. The literature on constructing copies of F_2 using bicyclic units is much richer as will be recalled in the next section.

Remark. Recall that the condition that H must have an embedding in a simple component is necessary by Proposition 2.7. Also, the reason why the amalgamated subgroup needs to contain C is the third part of Proposition 5.3. Note that this issue exactly corresponds to the first extra condition for a first order deformation in Theorem 4.10.

Remark. One could hope to generalize Proposition 5.4 to a result where H is replaced by a conjugate. However, known instances of Conjecture 5.6 combined with the second part of Proposition 5.3 seem to say that such generality does not hold.

5.2. **On the embedding condition for group rings.** In this section we consider the group algebra FG and wish to understand when a finite subgroup H of $\mathcal{U}(RG)$ has the embedding condition from Theorem 4.1 to find a ping-pong partner for H.

Recall that a group G is called *Dedekind* if every subgroup is normal. The Baer-Dedekind classification theorem tells that such groups are either abelian or isomorphic to $Q_8 \times C_2^n \times A$ with $n \in \mathbb{N}$ and A an odd abelian group. Our main result is the following:

Theorem 5.7. *Let F be a field of characteristic 0, G a finite group and $H \leq G$ such that it admit a faithful irreducible F-representation $\psi \in \mathrm{Irr}(H)$. Then there exists an irreducible F-representation $\rho \in \mathrm{Irr}(G)$ such that*

(1) $H \cap \ker(\rho) \subseteq \mathcal{Z}(G)$,

(2) $\rho(G)$ is not *a Frobenius complement.*

if and only if G is not a Dedekind group. In particular, $\rho(FG)$ is not a division algebra.

It is well-known, e.g. see [46, Section 11.3], that finite subgroups H of a finite dimensional division F-algebra with $\mathrm{char}(F) = 0$ are *fixed point free*. The latter meaning that H has an irreducible $\mathbb{C}$-representation σ such that 1 is not an eigenvalue of $\sigma(g)$ for all $1 \neq g \in G$. Such groups coincide with the group-theoreticaly defined class of Frobenius complements, see [46, Proposition 11.4.6]. We refer the reader to [63, section 18] for background on Frobenius groups and the main structural results on Frobenius complements. The importance of constructing an irreducible representation such that $\rho(FG)$ is not a division algebra follows from Theorem 4.1, i.e. in order to have a ping-pong partner. The importance of having the stronger property that $\rho(G)$ is not a Frobenius complement, is to obtain a bicyclic ping-pong partner. This will be explained further in Section 5.3.

Remark 5.8. If G is not a Dedekind group and H is a finite subgroup of $\mathcal{U}(FG)$ which is FG-conjugated to a subgroup of G, then Theorem 5.7 also yields an irreducible F-representation $\rho : \mathcal{U}(FG) \to \mathrm{GL}_n(D)$ such that $H \cap \ker(\rho) \subseteq \mathcal{Z}(G)$ and $\rho(G)$ not a

Frobenius complement. This conjugation condition is reminiscent of the Zassenhaus conjectures. The latter states that, for finite G, any $H \leq \mathcal{U}(\mathbb{Z}G)$ is conjugated in $\mathbb{Q}G$ to a subgroup of $\pm G$. If H is non-cyclic, counterexamples were obtained by Roggenkamp-Scott [72] and for cyclic subgroup by Eisele-Margolis [25]. However for large classes of groups the conjectures holds. For instance for all H and G *nilpotent group* [84, 85] or for G *cyclic-by-abelian groups* and H cyclic [19]. See [59] for a survey. Thus for $F = \mathbb{Q}$ and G nilpotent the conclusion of Theorem 5.7 holds for all finite subgroups and not only those contained in a group basis.

Remark 5.9. There also exists a variant of Theorem 5.7 where condition (2) is replaced by "$\rho(G)$ is neither a field or totally definite quaternion algebra". Such representation ρ namely exists if and only if G is not $Q_8 \times C_2^n$. To prove this, it remains to investigate Dedekind groups and can be done with similar methods as in step 6.3 of the proof of Theorem 5.7.

Remark. Step 1 in the proof of Theorem 5.7 will consist in proving that if H has a faithful irreducible F-representation, then there exists a $\rho \in \mathrm{Irr}(G)$ satisfying $H \cap \ker(\rho) \subseteq \mathcal{Z}(G)$. The converse of this is however not true. For instance, every finite group H can be embedded[9] in an alternating group A_n with $n \geq 5$. The latter being simple has a faithful irreducible representation, but if $\mathcal{Z}(H)$ is not cyclic then H does not.

An interesting consequence of Theorem 5.7, Remark 5.9 and Theorem 4.1 is the following.

Corollary 5.10. *Let F be a number field and R its ring of integers. Further let G be a finite group and $H \leq G$ such that it admit a faithful irreducible F-representation. Then, there exists some $t \in \mathcal{U}(RG)$ such that*

$$\langle H, t \rangle \cong H *_C \langle t, C \rangle \cong H *_C (\mathbb{Z} \times C).$$

where $C = \langle h \rangle \cap \mathcal{Z}(G)$ if and only if $G \ncong Q_8 \times C_2^n$ for some n. In particular, $H *_C H$ exists in $\mathcal{U}(RG)$ in that case.

Explicitly the copy $H *_C H$ is given by $\langle H, H^t \rangle$. If H is a cyclic group of prime order and $R = \mathbb{Z}$, the preceding was obtained by Goncalves-Passman [34]. They furthermore proved that $C_p * \mathbb{Z}$ exists in $\mathcal{U}(\mathbb{Z}G)$ if and only if G has a non-central element of order p. The latter also follows from our results, using the positive solution on the Kimmerle problem for prime order elements [52, Corollary 5.2.].

Recall that for every $e \in \mathrm{PCI}(FG)$ we denote by $FGe \cong \mathrm{M}_{n_e}(D_e)$ the associated simple quotient and by

$$\pi_e : \mathcal{U}(FG) \twoheadrightarrow FGe \cong \mathrm{GL}_{n_e}(D_e)$$

the map induced by the projection onto FGe. Define the following set:

$$\begin{aligned} \mathrm{Emb}_{G,F}(H) &= \{e \in \mathrm{PCI}(FG) \mid H \cap \ker(\pi_e) = 1\}. \\ \mathrm{AEmb}_{G,F}(H) &= \{e \in \mathrm{PCI}(FG) \mid H \cap \ker(\pi_e) \text{ is central }\}. \end{aligned} \tag{5.4}$$

When $H = G$ we denote $\mathrm{AEmb}_F(G)$. Also, when F is clear from the context, then we will simply write $\mathrm{AEmb}_G(H)$. With this notation Theorem 5.7 says that if $\mathrm{AEmb}_F(H) \neq \emptyset$, then $\mathrm{AEmb}_{G,F}(H) \neq \emptyset$ and Ge is not a Frobenius complement for some $e \in \mathrm{AEmb}_{G,F}(H)$ if and only if G is not Dedekind.

Another important corollary is the following showing that in general the condition that H must have an almost embedding in a component which is neither a field or totally definite quaternion algebra is not an extra condition in case of group rings.

[9]Cayley's theorem yields an embedding in some S_m which on turn can be embedded in A_{m+2}.

Corollary 5.11. *Let F be a field of characteristic 0 and $H \leq G$ be a finite groups. Suppose that G is not a Dedekind group and $\mathrm{Emb}_{G,F}(H)$ is non-empty. Then there exists an idempotent $e \in \mathrm{AEmb}_{G,F}(H)$ such that Ge is not a Frobenius complement.*

Proof. Take $e \in \mathrm{Emb}_{G,F}(H)$ and suppose that Ge is a Frobenius complement. Then $H \cong He \leq Ge$ for some central subgroup Z of G. As Frobenius complements coincide with the class of fixed point free groups, we see that the property is inherited by subgroups. Therefore, $H \cong He$ has a faithful irreducible F-representation [75, Theorem 6.13]. Now Theorem 5.7 yields the desired statement. ☐

5.2.1. *Faithful irreducible embedding over different fields.* The existence of irreducible faithful complex representations for finite groups has already been intensively studied, see [80, Section 2] for a survey. We however also need to understand the existence of such representations for smaller fields. So let F be a field with $\mathrm{char}(F) = 0$ and let $FG \cong \prod_{i=1}^{q} \mathrm{M}_{n_i}(D_i)$ be its Wedderburn-Artin decomposition.

Classical results over $\mathbb{C}$ imply the following on the existence of an almost faithful embedding.

Lemma 5.12. *Let G be a finite group, $F \subseteq L$ be fields of characteristic 0 and $H \leq \mathcal{U}(FG)$ a finite subgroup. Then*

(i) *If $e \in \mathrm{AEmb}_F(G)$, then $\mathcal{Z}(G)/\ker(\pi_e)$ is cyclic.*
(ii) *If $\mathrm{AEmb}_{G,L}(H) \neq \emptyset$, then also $\mathrm{AEmb}_{G,F}(H) \neq \emptyset$.*
(iii) *If G is nilpotent, then $\mathrm{Emb}_F(G) \neq \emptyset$ if and only if $\mathcal{Z}(G)$ is cyclic.*
(iv) *$\mathrm{AEmb}_F(G) \neq \emptyset$ if and only if there exists a central subgroup, such that the quotient has socle normally generated by a single element.*

Another handy sufficient condition for $\mathrm{Emb}_{\mathbb{C}}(G)$ to be non-empty is that all Sylow subgroups have a cyclic center [41, Exercise 5.25].

Proof. Since finite subgroups of a field are cyclic and $\mathcal{Z}(G)e \subseteq \mathcal{Z}(FGe)$ for any $e \in \mathrm{PCI}(FG)$ (as Ge generates the simple component as F-vector space), it follows that $\mathcal{Z}(Ge)$ is cyclic. Therefore, as $\ker(\pi_e) \leq \mathcal{Z}(G)$ by assumption, one has that $\mathcal{Z}(G)/\ker(\pi_e) \cong \mathcal{Z}(Ge)$ is cyclic.

Next, note that $\mathbb{C}G \cong \mathbb{C} \otimes_F FG \cong \oplus_{f \in \mathrm{PCI}(FG)} (\mathbb{C} \otimes_F FGf)$ and $\mathbb{C} \otimes FGf$ might be semisimple but not simple over $\mathbb{C}$. Clearly the kernel of the projection to any $\mathbb{C}$-simple component of $\mathbb{C}Gf$ contains $\ker(\pi_f)$. Therefore if there exists an $e \in \mathrm{PCI}(LG)$ such that $H \cap \ker(\pi_e) = 1$, then $H \cap \ker(\pi_f) = 1$ for some $f \in \mathrm{PCI}(FG)$. Similarly for when the intersection is central. In conclusion, the second assertion holds.

For part (iii) it now suffices to recall the classical result that if G is nilpotent and $\mathcal{Z}(G)$ is cylic, then $\mathrm{Emb}_{\mathbb{C}}(G)$ is non-empty.

Part (iv) follows from part (ii) and the classical result of Gaschütz [32] stating that $\mathrm{Emb}_{\mathbb{C}}(G) \neq \emptyset$ if and only if $\mathrm{soc}(G) = \langle h^{-1}gh \mid h \in G \rangle$ for some $g \in G$. ☐

When G is nilpotent Lemma 5.12 gives the existence of an embedding in a simple factor, however even in that case it does not say how to construct the representation, or alternatively the necessary primitive central idempotent. This we do in the next example.

Example 5.13. Let G be a finite nilpotent group with cyclic center. Then $\mathrm{Emb}_F(G)$ is non-empty by Lemma 5.12. An $e \in \mathrm{Emb}_F(G)$ can be constructed as following:

Write $\mathcal{Z}(G) = \langle z_1 \rangle \times \ldots \times \langle z_n \rangle$ with each z_i of order $p_i^{n_i}$ where $p_1, \ldots, p_n$ are distinct prime numbers and n_i is a positive integer. Let $c_i = z_i^{p_i^{n_i - 1}}$, an element of order p_i. Then $f = \prod_{i=1}^{n} (1 - \widehat{c_i})$ is a central idempotent of FG and thus $f = \sum_{i=1}^{t} e_i$ is a sum of primitive central idempotents of FG.

Claim: the natural epimorphism $G \to Ge_i \leq \mathcal{U}((RG)e_i)$ is an embedding[10] for each i. Moreover, if there is some $H \leq G$ with $H \cap \mathcal{Z}(G) = 1$, then Ge_i is not a fixed point free group. In particular, FGe_i is not a division F-algebra.

Proof. To see the first part suppose the contrary. Then let $G_{e_i} = \{y \in G \mid ye_i = e_i\}$ be a non-trivial normal subgroup of G for some i. Hence $G_{e_i} \cap \mathcal{Z}(G) \neq \{1\}$ and thus G_{e_i} contains a c_j for some $1 \leq j \leq n$. Hence, $e_i = fe_i = \left(\prod_{k=1,k\neq j}^{n} (1-\widehat{c_k})\right)(1-\widehat{c_j})e_i = 0$, a contradiction.

For the second part, suppose G is non-abelian. The last part will follow from the second as finite subgroups of D^* are fixed point free (e.g. see [46, page 347]). Recall that fixed point free groups are exactly the Frobenius complements, [46, Proposition 11.4.6.]. As G is nilpotent, if this would be the case, then by [46, Corollary 11.4.7.] $G \cong Ge_i$ would be either cyclic or isomorphic to $Q_{2^t} \times C_p$ for some prime $p \neq 2$.

Consider the second case. Recall that $\mathcal{Z}(Q_{2^t}) \cong C_2$, whose generator we denote by -1. Then, if H contains some $(x,c) \in Q_{2^t} \times C_p$ with $x \neq 1$, we can take a 2-power q such that $(x,c)^q = (-1, c^q) \in H \cap \mathcal{Z}(G)$, a contradiction. Thus $H \leq C_q \subset \mathcal{Z}(G)$, also a contradiction. ☐

Note that, in view of the proof, we could also have supposed that G is a non-abelian p-group instead of the existence of such subgroup H. In fact if $G \cong Q_{2^t} \times C_p$, then it might happen that e.g. $\mathbb{Q}Ge_i$ is a division algebra. Nevertheless in that case $G/\langle -1 \rangle \cong D_{2^{t-1}} \times C_p$ which embeds in a non-division simple factor over $\mathbb{Q}$. In other words, without that assumption G has still an almost embedding in a non-division component if $t \geq 4$.

5.2.2. *Proof of the main embedding theorem.* We now proceed with the proof of the main result.

Proof of Theorem 5.7. Since cyclic groups are Frobenius complements, such ρ can not exist if G is abelian. Thus for the remained of the proof we assume that G is non-abelian.

First we construct a non-linear irreducible representation σ of G that restrictions to a faithful representation of H. Afterwards σ will be adapted to yield the required ρ.

Step 1: $\mathrm{Emb}_{G,F}(H) \neq \emptyset$.

Let ψ be an irreducible faithful F-representation[11] of H, and pick any irreducible quotient σ of $\mathrm{Ind}_H^G(\psi)$, the induction of ψ from H to G. By construction, the space $\mathrm{Hom}_{FG}(\mathrm{Ind}_H^G(\psi), \sigma)$ contains the projection p from $\mathrm{Ind}_H^G(\psi)$ to σ. By Frobenius' reciprocity,

$$\mathrm{Hom}_{FH}(\psi, \mathrm{Res}_H^G(\sigma)) \cong \mathrm{Hom}_{FG}(\mathrm{Ind}_H^G(\psi), \sigma) \ni p.$$

Thus, the image of p in $\mathrm{Hom}_{FH}(\psi, \mathrm{Res}_H^G(\sigma))$ is a non-zero morphism from the irreducible representation ψ to $\mathrm{Res}_H^G(\sigma)$, hence an embedding. Since ψ was faithful by assumption, the same holds for $\mathrm{Res}_H^G(\sigma)$. The primitive central idempotent associated to σ now gives an element in $\mathrm{Emb}_{G,F}(H)$.

Step 2: Existence of $\sigma \in \mathrm{Irr}(G)$ such that $\sigma(G)$ is not abelian and restricting faithfully to H.

From now on we decompose

$$\mathrm{Ind}_H^G(\psi) = \sigma_1 \oplus \cdots \oplus \sigma_\ell \tag{5.5}$$

into irreducible sub-representations. Recall that $\ker(\mathrm{Ind}_H^G(\psi)) = \mathrm{core}_G(\ker(\psi)) = 1$, i.e. the induction is faithful.

[10]Note that Ge_i is indeed a group since e_i is central

[11]This step in fact works for F replaced by any coefficient ring R such that $\mathrm{Ind}_H^G(\psi)$ has an irreducible quotient. For instance it works also in dividing characteristic.

Now if $\sigma_i(G)$ is abelian, then the commutator subgroup G' is contained in $\ker(\sigma_i)$. Hence if all σ_i are linear, then $G' \leq \ker(\mathrm{Ind}_H^G(\psi))$. Due to the faithfulness this implies that $G' = 1$, i.e. G is abelian, which contradicts the non-abelian assumption on G. Now, any non-linear σ restricts faithfully to H by the proof of step 1, finishing this step.

Step 3: If B is a non-abelian Frobenius complement, then there exists $N \leq \mathcal{Z}(B)$ such that B/N is *not* a Frobenius complement, $\mathrm{Emb}_F(B/N) \neq \emptyset$ and N is a cyclic p-group (with $N \cong C_2$ if Sylow 2-subgroup of B is non-abelian). Furthermore, B/N is abelian if and only if B is isomorphic to $Q_8 \times A$ with A an odd abelian group.

By the Thompson–Frobenius–Zassenhaus theorem [46, Theorem 11.4.5] a Frobenius complement has the following restrictions on its Sylow subgroups:

- Every odd-prime Sylow subgroup of B is cyclic.
- The Sylow 2-subgroups of B are either cyclic or a quaternion group
$$Q_{4m} = \langle a, b \mid a^{2m} = 1, b^2 = a^m, b^{-1}ab = a^{-1} \rangle$$
with $m = 2^{n-2}$ for some $n \geq 2$.

We need to consider both cases separately:

Case A: The Sylow 2-subgroups are B is quaternion.

Recall that $\mathcal{Z}(Q_{4m}) = \langle a^m \rangle \cong C_2$ and $Q_{4m}/\langle a^m \rangle \cong D_{2m}$ is a dihedral group. Furthermore, if 2 divides $|B|$, a^m is the unique involution in B, see [46, Theorem 11.4.5], and therefore it is central. Thus we may consider the quotient $B/\langle a^m \rangle$ which does not depend on the chosen Sylow 2-subgroup and also is not a Frobenius complement as the Sylow 2-subgroups of $B/\langle a^m \rangle$ are dihedral. Next, as all Sylow subgroups of $B/\langle a^m \rangle$ have cyclic center, it follows from [41, Exercise (5.25)] that $\mathrm{Emb}_{\mathbb{C}}(B/\langle a^m \rangle)$ is non-empty (hence also over F by Lemma 5.12).

Suppose that $B/\langle a^m \rangle$ is abelian. Then B would be nilpotent of class 2. However a nilpotent Frobenius complement is either cyclic or isomorphic to $Q_{2^n} \times A$ for some odd abelian group A and $n \geq 3$, see [46, Corollary 11.4.7]. Since the nilpotency class of $Q_{2^n} \times A$ is $n-1$, we obtain that $n = 3$, as claimed.

Case B: All Sylow subgroups of B are cyclic.

Such group is also called a Z-group and it was proven by Zassenhaus [63, Theorem 18.2] that they are of the form:

$$B \cong C_m \rtimes C_n := \langle a, b \mid a^m = 1,\ b^n = 1,\ b^{-1}ab = a^r \rangle, \tag{5.6}$$

with $r^n \equiv 1 \mod m$ and $\gcd(m,n) = \gcd(m, r-1) = 1$. This presentation entails that a Sylow p-subgroups is normal if and only if $p \mid n$. Furthermore,

$$C_m \rtimes C_n \text{ as in (5.6) is a Frobenius complement } \Leftrightarrow r^{n/n_0} \equiv 1 \mod m \tag{5.7}$$

with n_0 the product of all prime divisors of n. In other words, $C_m \rtimes C_n$ is a Frobenius complement if and only if all elements of prime order commute. Or yet reformulated, $r^{n/p} \equiv 1 \mod m$ for all primes p dividing n.

We need the following claim:

$$\mathcal{Z}(B) = \langle b^d \rangle \text{ with } d = \mathrm{ord}_{(\mathbb{Z}/m)^\times}(r). \tag{5.8}$$

To see this consider an arbitrary element $a^i b^j$ of B with $0 \leq i < m$ and $0 \leq j < n$. Since $b^{-1}(a^i b^j)\, b = a^{ir}\, b^j$, in order $a^i b^j$ to be central we need that $ir \equiv i \mod m$. However, $\gcd(m, r-1) = 1$ which entails that $i \equiv 0 \mod m$, i.e. $a^i b^j$ is a power of b. Next note that $b^{-j}ab^j = a^{r^j}$. Hence they would commute if and only if $r^j \equiv 1 \mod m$. In other words, b^j is central if $d \mid j$, finishing the proof of the claim.

Next write $o(b^d) = \frac{n}{d} = p^\ell.s$ with $(p,s) = 1$ and consider the associated central element $(b^d)^{n/(d.p^\ell)} = b^{n/p^\ell}$ of order p^ℓ. From (5.8) it follows that

$$B/\langle b^{n/p^\ell}\rangle \cong C_m \rtimes C_{n/p^\ell},$$

where p does not divide the order of the kernel of the action of C_{n/p^ℓ} on C_m. However p does divide $\frac{n}{p^\ell}$ as otherwise the p-part of C_n would have been central, which is not the case in the Zassenhaus presentation considered. Now note that the 'p-faithfulness' of the action makes it impossible that $r^{(n/p^\ell)/p} \equiv 1 \mod m$. Therefore the criterion in (5.7) shows that $B/\langle b^{n/p^\ell}\rangle$ is not a Frobenius complement if $n \neq d$, which holds as B was assumed non-abelian. Hence in this case we consider $N := \langle b^{n/p^\ell}\rangle$. It remains to verify the other properties.

As all Sylow subgroups of B/N are cyclic, it again follows from [41, Exercise (5.25)] and Lemma 5.12 that $\mathrm{Emb}_F(B/N)$ is non-empty. Finally, since $B/N \cong C_m \rtimes C_{n/p^\ell}$ where the action is p-faithful it can not be abelian, except if B was abelian which we assumed to not be the case.

Step 4: A simple summand in (5.5) induces the desired representation via step 3 under some assumptions on G and H.

Take some σ_i as in step 2. If $\sigma_i(G)$ is not a Frobenius complement, then it yields the desired representation with the even stronger property that $H \cap \ker(\sigma_i) = 1$. Thus we need to consider the setting where $\sigma_i(G)$ is a Frobenius complement for all $1 \leq i \leq \ell$.

The induced representation from (5.5) was proven to be faithful. Therefore G both surjects onto $\sigma_i(G)$ and

$$G \hookrightarrow \prod_{i=1}^{\ell} \sigma_i(G). \tag{5.9}$$

Both properties imply that $G \cap \prod_{i=1}^{\ell} \mathcal{Z}(\sigma_i(G)) = \mathcal{Z}(G)$. Furthermore, all elements of order 2 in G are central in G. The latter follows from the observation on $\mathcal{Z}(G)$ and the fact that when $|\sigma_i(G)|$ is even it has a unique involution.

Now, for every i consider a quotient $\sigma_i(G)/N_i$ as in step 3 and an associated faithful representation ρ_i of $\sigma_i(G)/N_i$. Denote by $\overline{\rho_i}$ its inflation to $\sigma_i(G)$. Then $\ker(\overline{\rho_i}) = N_i$ and $N := \prod_{i=1}^{\ell} N_i$ is a central subgroup of the direct product. Thus the image of G into $\prod_{i=1}^{\ell} \sigma_i(G)/N_i$ is isomorphic to $G/(G \cap N)$ with $G \cap N \leq \mathcal{Z}(G)$.

First *suppose that* $\sigma_i(H) \cap N_i = 1$ *for some* i. Then if $\sigma_i(G)/N_i$ is non-abelian then we are finished. Hence suppose also that all $\sigma_i(G)/N_i$ are abelian. Then by the description given in step 3, this would mean that G is a subgroup of $\prod_{i=1}^{\ell} \sigma_i(G)$ with each $\sigma_i(G)$ either cyclic or isomorphic to $Q_8 \times A_i$ with A_i some odd abelian group. In other words G is a subgroup of $G \leq Q_8^t \times A$ with $t \in \mathbb{N}$ and A an abelian group.

Besides, note that if $\sigma_i(G)/N_i$ is abelian, then $(\overline{\rho_i} \circ \sigma_i)(H) \cong H/H \cap N_i$ is a subgroup of a field and hence cyclic. As $\sigma_i(H) \cap N_i$ is central in $\sigma_i(G)$, this implies that $H \cong \sigma_i(H)$ is abelian. Recall that furthermore H was assumed to have a faithful irreducible F-representation, implying that in that case H is already cyclic. Thus it remains to consider the case that H is cyclic. *In step* 6 we will consider separately the case that H is cyclic and $G \leq Q_8^t \times A$.

Next *suppose that* $\sigma_i(H) \cap N_i \neq 1$ *for all* i. In this case we are finished whenever

$$H \cap \sigma_i(N_i)^{-1} \leq H \cap (\sigma_1(G) \times \cdots \times N_i \times \cdots \times \sigma_\ell(G)) \leq \mathcal{Z}(G). \tag{5.10}$$

Suppose that $\sigma_j(G)$ has a non-abelian Sylow 2-subgroup for some $1 \leq j \leq \ell$, then step 3 tells that $N_j \cong C_2$. Hence if $x \in H \cap \sigma_i(N_i)^{-1}$, then $x^2 \in H \cap \ker(\sigma_i) = 1$. As involutions in G are central, we obtain that $H \cap \sigma_i(N_i)^{-1} \leq \mathcal{Z}(G)$, as desired.

In conclusion: When G is not a Dedekind group, we have obtained the desired representation except for the following two families:

(1) G is such that all $\sigma_i(G)$ have abelian Sylow 2-subgroups,
(2) G is a subgroup of $Q_8^t \times A$ with A abelian and H is cyclic a subgroup.

The family in (1) we will handled in step 6 and the one in (2) in the next step.

Step 5: The case that all $\sigma_i(G)$ have abelian Sylow 2-subgroups.

If $\sigma_i(G)$ has abelian Sylow 2-subgroups, then $\sigma_i(G)$ is even a Z-group. In that case, by (5.9), G is metabelian, all its Sylow subgroups are abelian and all elements of prime order commute. Furthermore, by Ito's theorem [41, Corollary 12.34] a Sylow p-subgroup is normal if and only if $p \in \pi(\mathrm{cd}(G))$ where $\pi(\mathrm{cd}(G)) := \{p \text{ prime } \mid \exists \chi \in \mathrm{Irr}_{\mathbb{C}}(G) : p \mid \chi(1)\}$. Combined with the theorem of Schur-Zassenhaus we obtain that

$$G \cong \Big(\prod_{p \notin \pi(\mathrm{cd}(G))} D_p\Big) \rtimes E \tag{5.11}$$

with D_p abelian normal Sylow p-subgroups and E an abelian group with $\pi(E) = \pi(\mathrm{cd}(G))$. This same explanation also yields that each $\sigma_i(G)$ is of the form $C_{m_i} \rtimes C_{n_i}$ with $\pi(\mathrm{cd}(\sigma_i))$ equal to the set of prime divisors of n_i. Furthermore $\pi(\mathrm{cd}(\sigma_i)) \subseteq \pi(\mathrm{cd}(G))$.

Consider $x \in H \cap \sigma_i(N_i)^{-1}$. If $g \in G$ is such that $[x,g] \in H$, then $[x,g] \in H \cap \ker(\sigma_i) = 1$. In particular, $x \in \mathcal{Z}(H)$ and if H is normal then $x \in \mathcal{Z}(G)$. Hence we may assume that H is non-normal. Next recall that H is supposed to have an irreducible faithful F-representation and hence $\mathcal{Z}(H)$ is cyclic, say $\langle y \rangle$. Therefore $H \cap \sigma_i(N_i)^{-1} \leq \langle y \rangle$ for all i. Thus in order to obtain (5.10) it is enough to do so for the case that H is cyclic. This case will follows from several observations.

Observation 1: $x \in H \cap \sigma_i(N_i)^{-1}$ is a p_i-power element where N_i is a p_i-group and this for every i. Furthermore, $p_i \in \pi(\mathrm{cd}(\sigma_i))$.

Indeed, fix i and suppose that $\sigma_j(x)$ is not a p_i-power for some $j \neq i$. Then $x^{|N_i|}$ would be a non-trivial element in $H \cap \ker(\sigma_i) = 1$, a contradiction.

Observation 2: the Sylow p-subgroups of $\sigma_i(G)$ for $p \in \pi(\mathrm{cd}(G)) \setminus \pi(\mathrm{cd}(\sigma_i))$ are central.

Suppose that $\mathrm{cd}(\sigma_i(G))$ is strictly smaller as $\mathrm{cd}(G)$, say the prime p misses. Then either the Sylow p-subgroups of $\sigma_i(G)$ are trivial or $p \mid m_i$. However in the latter case a Sylow p-subgroup P_i is central. Indeed, P_i also commutes with the C_{n_i}-part due to the combination of (i) it is the image of a Sylow p-subgroup of G and (ii) the Sylow p-subgroups of G are subgroups of E which maps onto C_{n_i}.

Now fix some $p \in \pi(\mathrm{cd}(G))$ which also divides $|H|$. By (5.9) and the fact that G maps onto $\sigma_i(G)$, there is some i for which $p \in \mathrm{cd}(\sigma_i(G))$. For all such images take N_i to be of p-power (which we can by the proof of step 3). Let N_j be the smallest p-power subgroup constructed in this way. Then both observations together yield that all coordinates of x are either 1 or central in the component, hence $x \in \mathcal{Z}(G)$, as desired. If no $p \in \pi(\mathrm{cd}(G))$ divides $|H|$, then $x \in H \cap \sigma_i(N_i)^{-1}$ is trivial by the first observation.

Step 6: The case that H is a cyclic subgroup and $G \leq Q_8^t \times A$ with $t \in \mathbb{N}$ and A any abelian group.

As G is nilpotent we can decompose it into $G_2 \times A_{2'}$ where G_2 is the Sylow 2-subgroup of G and $G_{2'}$ its Hall $2'$-subgroup which is abelian. On $G_{2'} \leq A$ there is no restriction, thus we will now focus on classifying the possibilities for $G_{2'}$.

Step 6.1: Classifying subgroups of $Q_8^t \times A_2$ with A_2 an abelian 2-group.

Denote $Z_0 := \mathcal{Z}(Q_8^t)$. Recall that $Z_0 \cong C_2^t$ and $V := Q_8^t / Z_0 \cong C_2^{2t}$. We will view V as a $\mathbb{F}_2$-vector space. Since Q_8^t is of nilpotency class 2, the group-commutator induces a non-degenerate alternating bilinear form

$$[\cdot,\cdot] : V \times V \to Z_0.$$

Every subgroup M of Q_8^t is uniquely described by a triple (U, K_0, λ) where

- $U \subseteq V := Q_8^t/Z_0$ is an $\mathbb{F}_2$-subspace,
- $K_0 \leq Z_0$ is a subgroup containing the commutator $[U, U] \subseteq Z_0$ (i.e. U is K_0-isotropic for $[\cdot, \cdot]$).
- $\lambda \in Z^2(U, K_0)$ or equivalently a section $\sigma : U \to Q_8^t$ such that

$$\sigma(u).\sigma(v) = [u, v]\sigma(u + v).$$

and $o(\sigma(u)) = 4$ for all non-trivial $u \in U$.

Concretely, to such triple one can associate the following set

$$M_{(U,K_0)} := \{\sigma(u)\, k \mid u \in U,\ k \in K_0\}$$

which is a subgroup thanks to the 2-cocycle condition. Note that up to isomorphism the section σ involved in the definition is unique up to a 1-cocycle, so in the sequel we will omit to describe it. Note that the condition $o(\sigma(u)) = 4$ comes from the fact that $\exp(Q_8^t) = 4$ and all elements of order 2 are central. Thus U in fact corresponds to choosing order 4 elements.

Next, for a given $G \leq Q_8^t \times A_2$ we consider the projection $\pi : G \to Q_8^t : (g, b) \mapsto g$ and $B := \ker(\pi) = G \cap A_2$. Then the subgroup $G \leq Q_8^t \times A_2$ fits into a central extension

$$1 \to B \to G \to G_t \to 1, \tag{5.12}$$

where $G_t := \operatorname{im}(\pi) \leq Q_8^t$. In other words, G corresponds to a 2-cohomology class $H^2(G_t, B)$ where G_t can be described via a triple as above.

Step 6.2: Determining possible images under irreducible representations.

If ρ is an irreducible F-representation of G, then the image $\rho(G)$ must have a faithful irreducible representation. Furthermore, in step 5 the group G is nilpotent and thus by Lemma 5.12 a quotient G/N has a faithful irreducible F-representation if and only if $\mathcal{Z}(G/N)$ is cyclic.

Claim: if $G \leq Q_8^t \times A$ and $N \triangleleft G$ is such that $\mathcal{Z}(G/N)$ is cyclic, then G/N is either cyclic or isomorphic to $E \times C_m$ with E an extraspecial 2-group and m odd.

Now consider the description of G given in step 5.1. Thus $G \cong G_2 \times G_{2'}$ with $G_{2'}$ abelian and the group G_2 can be written as in (5.12). Furthermore, $\pi(G_2)$ corresponds to a tuple (U, K_0).

Due to the relative prime orders and $G_{2'}$ being abelian, a quotient $(G_2 \times G_{2'})/N$ has cyclic center if and only if $N = N_2 \times N_{2'}$ with $\mathcal{Z}(G_2)/N_2)$ and $G_{2'}/N_{2'}$ cyclic. Thus we are reduced to understanding cyclic center quotients of G_2.

We start by investigating the quotients G_2/N with $B \leq N$. In other words, quotients of $\pi(G_2)$. For this, consider a cyclic quotient K_0/N_0 of K_0. Then the group $\pi(G_2)/N_0$ corresponds to the tuple $(U, K_0/N_0)$ and has cyclic center if and only if $[\cdot, \cdot]|_{U \times U}$ is still non-degenerate. Otherwise, $\mathcal{Z}(\pi(G_2)/N_0) = \pi^{-1}(U_c).K_0/N_0$ with $U_c := \operatorname{span}_{\mathbb{F}_2}\{u \in U \mid [u, U] \in N_0\}$. Thus if $N_0 = K_0$, then $\mathcal{Z}(\pi(G_2)/N_0) = \pi^{-1}(U) = \pi(G_2)/N_0$ is abelian. So suppose that $K_0 \gneq N_0$. Then we need that K_0/N_0 is cyclic and $\pi^{-1}(U_c)$ is trivial. Therefore in that case $\pi(G_2)/N_0$ has center of order 2 such that the quotient is an elementary abelian 2-group, i.e. it is extraspecial.

Now consider a quotient G_2/N with $B \not\leq N$. In this case, in order to have a cyclic center one needs that $[B : N \cap B] = 2$. Let $\mathcal{T}^B_{N \cap B} = \{1, x\}$ be left coset representatives and let $\langle y \rangle \leq B$ be a maximal cyclic subgroup above $\langle x \rangle$.

Next note that $\sigma(\pi(G_2)') = 1$ since $G_2' \cap B = 1$ (because G is class 2 and B abelian). Thus $\mathcal{Z}(G_2) = \pi^{-1}(\mathcal{Z}(\pi(G_2)))$. Therefore we need that $\mathcal{Z}(G_2/N) = \langle z \rangle$ with z such that $\langle y \rangle \leq \langle z \rangle$ and $\mathcal{Z}(\pi(G_2)) = \langle \pi(z) \rangle$. The latter is only possible for particular choices of (U, K_0) for which the quotient will be an extraspecial group of order larger than 8.

Step 6.3: Characterizing when all images are Frobenius complements and the almost-embedding for H.

Claim: a subgroup $G \leq Q_8^t \times A$ has the property that all images $\rho(G)$ for $\rho \in \mathrm{Irr}_F(G)$ are Frobenius complements if and only if G is a Dedekind group. Furthermore, if G is not Dedekind and H cyclic, then H almost-embeds in an extraspecial quotient different of Q_8.

For the characterization, it follows from (5.14) that all image are Frobenius complements if and only $\tilde{g}e$ is central in FG for all $e \in \mathrm{PCI}(FG)$ and all $g \in G$ (if e corresponds to a non-linear representation, then even $\tilde{g}e = 0$). In other words, if and only if $\frac{1}{o(g)}\tilde{g}$ is central in FG for all $g \in G$. Recall that the idempotent $\frac{1}{o(g)}\tilde{g}$ is central exactly when $\langle g \rangle$ is normal in G. The latter is the defining definition of being Dedekind, finishing the equivalence.

Now we will prove the 'furthermore part' and hence assume G is not Dedekind. As mentioned earlier, the set of images $\rho(G)$ coincide with the set of quotients G/N such that $\mathcal{Z}(G/N)$ is cyclic. By step 5.2, such a quotient G/N is either cyclic or $E \times C_m$ with E an extraspecial 2-group and m odd. By the description of Sylow subgroups of Frobenius complements, E is a Frobenius complement if and only if $E \cong Q_8$.

Now write $H = \langle h \rangle = \langle h_2 \rangle \times \langle h_{2'} \rangle$ the decomposition into the odd and even order part. One can always decompose $G_{2'} = \langle y \rangle \times R$ with $\langle y \rangle$ a maximal cyclic above $\langle h_{2'} \rangle$. Thus the normal subgroup N to mod out will contain R and the problem is reduced to finding an almost-embedding for h_2. For ease of notation we now assume that G is a 2-group and hence $h = h_2$.

Consider the quotient $G/B \cong \pi(G) \leq Q_8^t$ and denote by (U, K_0) the associated tuple. Recall that $\sigma(u)$ has order 4 for each non-zero $u \in U$ and $\sigma(u)^2 \in K_0$. Denote by $\{u_1, \ldots, u_\ell\}$ a $\mathbb{F}_2$-basis of u and consider the subgroup $N_0 := \langle \sigma(u_i)^2\sigma(u_j)^{-2} \mid 1 \leq i \neq j \leq \ell \rangle$ of K_0. Then K_0/N_0 is cyclic and $\pi(G)/N_0$ is extraspecial. By construction $\langle \pi(h) \rangle$ almost-embeds in $\pi(G)/N_0$. Thus if $\ell \geq 3$, then we are finished. If $\dim_{\mathbb{F}_2} U \leq 1$, then G/BK_0 is cyclic with $BK_0 \leq \mathcal{Z}(G)$ and hence G abelian, a contradiction with the assumption.

Thus it remains to consider the case that $\dim_{\mathbb{F}_2} U = 2$. Then a quotient with cyclic center of $\pi(G_2)$ is either Q_8 or D_8. As $\langle h \rangle$ will almost-embed, the latter type of quotient would be good. Thus consider the former case. Note that all restrictions obtained yields that $\pi(G_2) \cong Q_8 \times C_2^\ell$ for some ℓ. As we assumed that G is not Dedekind, it implies that $B \neq 1$. Now considering a quotient G/N with $B \not\leq N$ yields the desired type of quotient, as explained at the end of step 6.2.

To conclude for $G \leq Q_8^t \times A$: This case included Dedekind groups and for this we showed that the desired representation can not exist. Besides, in step 1 till 4 we obtained that if H is not a cyclic subgroup, then the desired statement holds. When H is cyclic and G not Dedekind we proved in step 5.3 that there exists a $\rho \in \mathrm{Irr}_F(G)$ such that $\ker(\rho) \cap H \leq \mathcal{Z}(G)$ and $\rho(G)$ is an extraspecial group E different of Q_8. As E is nilpotent with cyclic center, it has a faithful irreducible F-representation which combined yields the desired ρ. $\square$

5.3. **Bicyclic units generically play ping-pong.** In RG one can consider following elements which slightly generalize those in (5.1) to those of the form

$$b_{\tilde{h},x} = 1 + (1-h)x\widetilde{h} \text{ and } b_{x,\tilde{h}} = 1 + \widetilde{h}x(1-h) \tag{5.13}$$

with $x \in RG$ and $\widetilde{h} := \sum_{i=1}^{o(h)} h^i$. As $(1-h)\widetilde{h} = 0 = \widetilde{h}(1-h)$, all elements in (5.13) are unipotent units and called *bicyclic units*. The group generated by them we denote

$$\mathrm{Bic}_R(G) := \langle b_{\tilde{h},x}, b_{x,\tilde{h}} \mid x \in RG, h \in G \rangle.$$

It is a fact that

$$\mathrm{Bic}_R(G) \neq 1 \text{ if and only if } G \text{ is not a Dedekind group.}$$

For many years an overarching belief in the field of group rings has been that two bicyclic units should generically generate a free group:

Conjecture 5.14. *Let G be a finite (non Dedekind) group and α bicyclic. Then the set $\{\beta \in \mathrm{Bic}_R(G) \mid \langle \alpha, \beta \rangle \cong \langle \alpha \rangle * \langle \beta \rangle\}$ is 'large' in $\mathrm{Bic}_R(G)$.*

The above conjecture has been intensively investigated for $R = \mathbb{Z}$. See [38] for a quit complete survey untill 2013 and also see [35, 37, 36, 39, 49, 70] and the references therein.

In this section we obtain, as a main application of Theorem 3.23 and Theorem 5.7, a concrete version of Conjecture 5.14, modulo a deformation to a shifted bicyclic unit. We also obtain a variant for a given image under a first order deformation. For the latter we need to consider the following set

$$\mathrm{PCI}_{fpf}(FG) = \{e \in \mathrm{PCI}(FG) \mid Ge \text{ is not fixed point free}\}.$$

Take $e \in \mathrm{PCI}(FG)$ such that FGe is non-commutative. It is well known to experts, [46, Section 11.4], that for such e the following hold:

$$Ge \text{ is fixed point free} \quad \Leftrightarrow \quad \forall g \in G : \widetilde{g}e = 0 \tag{5.14}$$

In particular if FGe is a division algebra, then Ge is fixed point free. Using the notations from (5.4), we have the following necessary and sufficient condition for the existence of a bicyclic ping-pong partner.

Theorem 5.15. *Let F be a number field, R its ring of integers, $H \leq G$ be finite groups and $D : H \to \mathcal{U}(FG)$ a first-order deformation of H. Denote $C := H \cap \mathcal{Z}(G)$. Then the set*

$$S_\alpha := \{\beta \in \mathrm{Bic}_R(G) \setminus \{1\} \mid \langle D(H), \beta \rangle \cong \langle D(H) \rangle *_C (\langle \beta \rangle \times C)\}$$

is non-empty if and only if $\mathrm{AEmb}_{G,F}(H) \cap \mathrm{PCI}_{fpf}(FG) \neq \emptyset$. Furthermore,

(1) when $S_\alpha \neq \emptyset$ then it is dense in $\mathrm{Bic}(G)$ for the join of the profinite and Zariski topologies.

(2) if $\mathrm{Emb}_{G,F}(H) \neq \emptyset$ (e.g. H has a faithful irreducible F-representation) and $\mathrm{Bic}_R(G) \neq \{1\}$, then $S_\alpha \neq \emptyset$.

Remark 5.16. The proof of the first part of Theorem 5.15 is ultimately an application of Theorem 3.23. Hence the statement of Theorem 5.15 also holds for a finite family of finite subgroups $H_i \leq G$.

Theorem 5.15 now directly yields an answer to Conjecture 5.14:

Theorem 5.17. *Let F be a number field and R be its rings of integers. Further let G be finite group and $\alpha = 1 + (1-h)x\widetilde{h}$ be a non-trivial bicyclic unit for some $h \in G$ and $x \in RG$. Denote $C = \langle h \rangle \cap \mathcal{Z}(G)$. Then*

$$S_\alpha = \{\beta \in \mathrm{Bic}_R(G) \mid \langle \alpha h, \beta \rangle \cong \langle \alpha h \rangle *_C (\langle \beta \rangle \times C)\}$$

is dense in $\mathrm{Bic}_R(G)$ for the join of the profinite and Zariski topologies.

Note that theorem 5.17 gives a concrete interpretation of 'large' in Conjecture 5.14 for two of the natural topologies. Also remark that Conjecture 5.14 is not a direct instance of Question 1.1 as we still need the non-trivial statement that cyclic subgroups have an appropriate almost-embedding.

Remark 5.18. By enlarging $\mathrm{Bic}(G)$, the condition that $e \in \mathrm{PCI}_{fpf}(FG)$ can be weakened to supposing that FGe is not a division algebra, i.e. to

$$e \in \mathrm{PCI}_{div}(FG) := \{e \in \mathrm{PCI}(FG) \mid FGe \text{ is not a division algebra}\}.$$

More precisely, consider

$$\mathcal{U}(RG)_{un} = \{\alpha \in \mathcal{U}(RG) \mid \alpha \text{ is unipotent }\}.$$

Then exactly the same proof as Theorem 5.17 yields the following statement:

Consider F, G, H, C and D as in Theorem 5.15. Suppose that there exists $e \in \mathrm{AEmb}_{G,F}(H)$ such that FGe is not a division algebra, then $\{\beta \in \langle \mathcal{U}(RG)_{un}\rangle \mid \langle D(H), \beta\rangle \cong H *_C (\langle \beta \rangle \times C)\}$ is dense in $\langle \mathcal{U}(RG)_{un}\rangle$ for the join of the profinite and Zariski topologies.

The statements in Theorem 5.15 and Remark 5.18 require to relate it to the setting of Question 1.1. More precisely, the next lemma essentially enables us to use Theorem 3.23. First recall that :

$$\mathrm{SL}_1(FG) := \{x \in \mathcal{U}(FG) \mid \forall e \in \mathrm{PCI}(FG) : \mathrm{RNr}_{FGe/\mathcal{Z}(FGe)}(\pi_e(x)) = 1\}$$

where Rnr denotes the *reduced norm*, i.e. $\mathrm{Rnr}_{A/F}(a) = \det(1_E \otimes_F a)$ for a central simple F-algebra A and a splitting field E of A.

Lemma 5.19. *Let F be a number field and R its ring of integers. Then the following hold:*

- $\mathrm{Bic}_R(G)$ *is Zariski-dense in* $\mathrm{SL}_1(FG)f$ *with* $f = \sum_{e \in \mathrm{PCI}_{fpf}(FG)} e$.
- $\langle \mathcal{U}(RG)_{un}\rangle$ *is Zariski-dense in* $\mathrm{SL}_1(FG)f$ *with* $f = \sum_{e \in \mathrm{PCI}_{div}(FG)} e$.

Proof. Notice that a unipotent unit α in RG projects in any simple component to a unipotent element and in particular has reduced norm 1 there. Thus $\langle \mathcal{U}(RG)_{un}\rangle$ can be viewed as a subgroup of $\mathrm{SL}_1(RG)f$ with $f = \sum_{e \in \mathrm{PCI}_{div}(FG)} e$. As $\mathrm{Bic}_R(G)$ is generated by unipotent elements it is a subgroup of $\langle \mathcal{U}(RG)_{un}\rangle$. By (5.14) the elements in $\mathrm{Bic}_R(G)$ have a trivial projection for $e \in \mathrm{PCI}_{div}(FG) \setminus \mathrm{PCI}_{fpf}(FG)$. In short, the groups in the statement are indeed subgroups of the SL_1's mentioned.

We now proceed to prove the Zariski-density. For this, consider the Wedderburn-Artin decomposition $FG \cong \prod_{e \in \mathrm{PCI}(FG)} \mathrm{M}_{n_e}(D_e)$. Further let $\mathcal{O}_e$ be a maximal order in D_e such that $RGe \subseteq \mathrm{M}_{n_e}(\mathcal{O}_e)$. For every $e \in \mathrm{PCI}(FG)_{div}$ one can clearly choose an idempotent f_e in FG such that ef_e is non-central in FGe. Associated to f_e one has the group of generalized bicyclic units $\mathrm{GBic}^{\{f_e\}}(RG)$, see [46, Section 11.2] for the definition, which is generated by unipotent elements in RG.

If $e \in \mathrm{PCI}_{fpf}(FG)$, then by (5.14) there exists a $g \in G$ with $\widetilde{g}e \neq 0$. As $FGe \cong \mathrm{M}_{n_e}(D_e)$ with $n_e > 1$ and Ge is a F-spanning set, there must even exist a $g \in G$ such that $\widetilde{g}e \notin \mathcal{Z}(FGe)$. Therefore, for such components the choice $f_e = \hat{g} := \frac{1}{o(g)}\widetilde{g}$ can be made for some $g \in G$. By definition, for $f_e = \hat{g}$ one has that $\mathrm{GBic}^{\{f_e\}}(RG) = \mathrm{Bic}_R(G)$. For $e \in \mathrm{PCI}_{fpf}(FG)$ we assume such a choice in the remaining of the proof.

Now, thanks to [42, Theorem 6.3] the group $\mathrm{GBic}^{\{f_e\}}(RG)$ contains a subgroup U_e of the form $1 - e + E_{n_e}(I_e)$ for some non-zero ideal I_e of $\mathcal{O}_e$. Recall that

$$E_n(I_e) := \langle e_{ij}(r) \mid 1 \leq i \neq j \leq n, \, r \in I_e\rangle,$$

where $e_{ij}(r)$ is the elementary matrix in $\mathrm{GL}_{n_e}(\mathcal{O}_e)$ which has 1 on the diagonal and r in the (i,j)-entry.

Consequently the desired statement reduces to prove that $E_n(I)$ is Zariski-dense in $\mathrm{SL}_n(D)$ for I a non-zero ideal in an order $\mathcal{O}$ in a finite dimensional division algebra D and $n \geq 2$. This follows from the classical fact that the Zariski-closure of $\{e_{ij}(x) \mid x \in I\}$ contains the set $\{e_{ij}(y) \mid y \in D\}$. Therefore the Zariski-closure of $E_n(I) = \langle e_{ij}(x) \mid 1 \leq i, j \leq n,\ x \in I\rangle$ equals $E_n(D)$. As for any finite dimensional division algebra, $E_n(D) = \mathrm{SL}_n(D)$, finishing the proof. □

Theorem 5.15 now follows readily.

Proof of Theorem 5.17. By Theorem 4.7 D is given by conjugation in FG. Hence $D(H) \cap \mathcal{Z}(G) = H \cap \mathcal{Z}(G)$. Also recall that $H \cap \mathcal{Z}(G) = H \cap \mathcal{Z}(\mathcal{U}(FG))$, as explained in Example 4.2. Thus the choice of C in Theorem 5.17 corresponds to the one in Theorem 3.23.

Next, as $D(H)$ is a finite subgroup of $\mathcal{U}(FG)$ conjugate to H, one has that $\mathrm{AEmb}_{G,F}(H) = \mathrm{AEmb}_{G,F}(D(H))$. Thus Theorem 3.23 and Remark 3.24 yields that $D(H)$ has a ping-pong partner in $\mathcal{U}(RG)$ if and only if H almost-embeds in a non-division component. However $\mathrm{Bic}_R(G)$ is a subgroup of $\mathrm{SL}_1(FG)f$ with $f = \sum_{e \in \mathrm{PCI}_{fpf}(FG)} e$. Thus clearly a necessary condition is that H almost-embeds in a component where Ge is fixed point free. Also the converse holds by Lemma 5.19 and since that not being fixed point free avoids that $\mathrm{span}_F\{Ge\}$ is a division algebra.

The densely many partners is also provided by Theorem 3.23. Finally, that $\mathrm{AEmb}_{G,F}(H) \cap \mathrm{PCI}_{fpf}(FG) \neq \emptyset$ if $\mathrm{Emb}_{G,F}(H)$ is non-empty is Corollary 5.11. That subgroups H having a faithful irreducible representation satisfies the latter condition is a part of Theorem 5.7. □

Finally we record the following neat corollary of independent interest.

Corollary 5.20. *Let F be a number field and R its ring of integers. Furhter let G be a non-abelian finite simple group and $H_1, \ldots, H_\ell$ non-trivial finite subgroups of G. Then there exists a unit $b \in \mathrm{Bic}_R(G)$ such that*

$$\langle H_i, b\rangle \cong H_i * \langle b\rangle$$

for all $1 \leq i \leq \ell$

Proof. In case G is simple, the morphism $G \to Ge$ is clearly an embedding for every primitive central idempotent. Also, from Amitsur's classification [2] (or see the more compact formulation in [76, Theorem 2.1.4]) it follows that the multiplicative group of a division algebra does not contain a non-abelian simple group. In particular, if FGe is a division algebra, then it must be a field. As G is assumed non-abelian and simple the only 1-dimensional representation must be the trivial one.

Next, recall that a group is fixed point free if and only if it is a Frobenius complement (e.g. see [46, Proposition 11.4.6]). Furthermore, by [46, Theorem 11.4.6], a Frobenius complement K contains a normal subgroup N such that K/N is a either a normal subgroup of S_4 or S_5 or it is a Z-group (i.e. all Sylow subgroup are cyclic). Thus in all cases the only simple subgroups of K/N are abelian. By the above this implies that Ge is only fixed point free when FGe is the trivial representation. The desired conclusion now follows from Theorem 5.15 and Remark 5.16. □

6. Virtual structure problem for product of amalgam and HNN over finite groups

In this final section we consider the virtual structure problem which was for the first time explicitly formulated in [45] but in fact goes back to the question on 'unit theorems' by Kleinert [55], as explained in Section 1.5.

Question 6.1 (Virtual Structure Problem)**.** Let $\mathcal{G}$ be a class of groups. Classify the finite groups G and ring of integers R such that $\mathcal{U}(RG)$ has a subgroup of finite index in $\mathcal{G}$.

Note that Question 6.1 differs from Kleinert's version as it only asks for the existence of a single finite index subgroup satisfying property $\mathcal{G}$ instead of (almost) all finite index subgroups in any order in FG with $F = \mathrm{Frac}(R)$. In the case that the property $\mathcal{G}$ is defined on commensurability classes, as unit groups of orders are commensurable, then both versions are equivalent and boil down to asking for which G and R the unit group $\mathcal{U}(RG)$ has $\mathcal{G}$. Also note that in the literature only the case $R = \mathbb{Z}$ is considered.

In [45], building on [44, 48, 56, 47], Jespers-Del Rìo answered Question 6.1 for

$$\mathcal{G}_{pab} = \{\prod_i A_{i,1} * \cdots * A_{i,t_i} \mid A_{i,j} \text{ are finitely generated abelian } \}$$

where $t_i = 1$ is allowed (i.e. an abelian factor). It turns out the classification coincide with the case of products of free groups (where again $\mathbb{Z}$ is also allowed). Moreover the problem for the classes $\{A * B \mid A, B \text{ f.g. abelian } \}$ and $\{ \text{ free groups } \}$ coincide and there is only four finite groups satisfying this (in all these cases $\pm G$ has a free normal complement in $\mathcal{U}(\mathbb{Z}G)$ [44]).

We will now consider the case

$$\mathcal{G}_\infty := \{\prod_i \Gamma_i \mid \Gamma_i \text{ has infinitely many ends } \}.$$

By Stallings theorem [79, 78] a group has infinitely many ends if and only if it can be decomposed as an amalgamated product or HNN extension over a finite group. In fact we will mainly work with this characterisation. Recall that given a finitely generated group Γ, then the number of ends $e(\Gamma)$ is the defined in terms of its Cayley graph $\mathrm{Cay}(\Gamma, S)$ with S a finite generating set[12]. More precisely, $e(\Gamma)$ is the smallest number m such that for any finite set F the graph $\mathrm{Cay}(\Gamma, S) \setminus F$ has at most m infinite connected components. If no finite m exists one defines $e(\Gamma) = \infty$.

6.1. Contributions to Kleinert's Virtual Structure Problem.

Despite that the class $\mathcal{G}_\infty$ is much larger than the aforementioned classes, we will now prove that the virtual structure problem for it coincide. We will use the terminology *virtually-$\mathcal{P}$* to say that a given group has a subgroup of finite index with property $\mathcal{P}$.

Theorem 6.2. *Let G be a finite group and F a number field. The following are equivalent:*

(i) *$H \leq \mathcal{U}(\mathcal{O})$ is virtually-$\mathcal{G}_\infty$ for all orders $\mathcal{O}$ in FG and finite index subgroup $H \leq \mathcal{U}(\mathcal{O})$;*

(ii) *$F = \mathbb{Q}$ and $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$;*

(iii) *all the simple components of FG are of the form $\mathbb{Q}(\sqrt{-d})$, with $d \in \mathbb{N}$, $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with non-zero $a, b \in \mathbb{N}$ or $\mathrm{M}_2(\mathbb{Q})$ and the latter needs to occur.*

Moreover, only the parameters $(-1,-1)$ and $(-1,-3)$ can occur for $(-a,-b)$. Also, $e(\mathcal{U}(\mathbb{Z}G)) = \infty$ if and only if $\mathcal{U}(\mathbb{Z}G)$ is virtually free, if and only if G is isomorphic to D_6, D_8, Dic_3, or $C_4 \rtimes C_4$.

[12] The number of ends is known to be independent of the chosen generating set.

In the statement above we used the notation $D_{2n} = \langle a, b \mid a^n = 1 = b^2, a^b = a^{-1}\rangle$, $Dic_3 = \langle a, b \mid a^6, a^3 = b^2, a^b = a^{-1}\rangle$ and $C_4 \rtimes C_4 = \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1}\rangle$. That these groups are exactly those for which $\mathcal{U}(\mathbb{Z}G)$ is virtually free is known since [44, 45], but we will give a short new proof using amalgamated product methods.

Using the description obtained in [56, Theorem 1] in terms of simple components we see that the classes indeed correspond:

Corollary 6.3. *Let G be a finite group. The following are equivalent:*

(i) *$\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$;*

(ii) *$\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of non-abelian free groups.*

In particular, in those cases G is a cut group, i.e. $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite.

It is well known that $e(\Gamma) \in \{0, 1, 2, \infty\}$ for a finitely generated group. By definition, $e(\Gamma) = 0$ if and only if Γ is finite. Moreover, $e(\Gamma) = 2$ if and only if Γ has a subgroup of finite index isomorphic to $\mathbb{Z}$. In case $\Gamma = \mathcal{U}(\mathbb{Z}G)$ the former happens exactly when G is abelian with $\exp(G) \mid 4, 6$ or $G \cong Q_8 \times C_2^n$ for some n (see [46, Theorem 1.5.6.], as proven by Higman). The case that $\mathcal{U}(\mathbb{Z}G)$ is $\mathbb{Z}$-by-finite has not yet been recorded in the literature, but can be obtained using classical methods:

Proposition 6.4. *Let G be a finite group, F a number field and R its ring of integers. Then, the following are equivalent:*

(1) $e(H) = 2$ for some finite index subgroup H in the unit group of an order $\mathcal{O}$ in FG;
(2) $e(\mathcal{U}(RG)) = 2$ and $F = \mathbb{Q}(\sqrt{-d})$ for $d \in \mathbb{N}$;
(3) $\mathcal{U}(RG)$ is $\mathbb{Z}$-by-finite and $F = \mathbb{Q}(\sqrt{-d})$ for $d \in \mathbb{N}$;
(4) G is isomorphic to C_n with $n = 5, 8$ or 12 and $F = \mathbb{Q}(\zeta_d)$ with $d \nmid n$.

Note that the equivalence (1) $\Leftrightarrow$ (2) imply that if $e(H) = 2$ for some H, then also all other finite index subgroups in an order will have two ends.

Remark 6.5. The proof of Proposition 6.4 also indicates that $\mathcal{U}(RG)$ is virtually a direct product of groups with two ends if and only if all the simple components of FG are either fields or totally definite quaternion algebras. Such finite groups can be described using [73, Theorem 2.6].

In light of Theorem 6.2 and Proposition 6.4, it would be natural to consider the class

$$\mathcal{G}_{\neq 1} := \{\prod_i \Gamma_i \mid e(\Gamma_i) \neq 1\}.$$

In fact, with a bit more of work one can prove that

$$\{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{\neq 1}\} = \{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{pab}\}. \tag{6.1}$$

In the upcoming work [22] applications of Theorem 6.2 to the "blockwise Zassenhaus conjecture" will be investigated. In other words applications to the question whether He is conjugated inside $\mathcal{U}(RGe)$ to a subgroup of Ge for any finite subgroup H of $V(RG)$ for R the ring of integers of some number field.

6.2. **Proofs of the statements.** We start with:

Proof of Proposition 6.4. For the equivalence (1) $\Leftrightarrow$ (2) recall that, see [78, pg38], if Γ_1 and Γ_2 are commensurable then $e(\Gamma_1) = e(\Gamma_2)$. Moreover, the unit group of two orders are commensurable [46, lemma 4.6.9]. Therefore, $e(H) = 2$ if and only if $e(\mathcal{U}(RG)) = 2$ for R the ring of integers of F. Furthermore, for general finitely generated groups Γ one has that $e(\Gamma) = 2$ if and only if Γ is $\mathbb{Z}$-by-finite, see [40, 30] or [78, pg 38]. All this settles (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1).

We will now prove simultaneously that (1) implies (4) and (2). To do so, recall a result by Kleinert [53] (or [46, Corollary 5.5.7]) saying that $\mathcal{U}(RG)$ is abelian-by-finite if and only if all the simple components of FG are either fields or totally definite quaternion algebras. In particular FG has no non-trivial nilpotent elements in which case [73, Theorem 2.6] tells that G is either abelian or $G \cong Q_8 \times C_2^m \times A$ with $m \geq 0$ and A an abelian group of odd order. We first consider the case that $G \cong Q_8 \times C_2^m \times A$. Then

$$FG \cong F[Q_8 \times C_2^m] \otimes_F FA \cong (4m\, F \oplus m \left(\frac{-1,-1}{F}\right)) \otimes_F FA.$$

We now see that in order to obtain a *single* copy of $\mathbb{Z}$ in $\mathcal{U}(RG)$ that this will have to come from a component of FA. However this component will appear at least 4 times and hence such groups are never $\mathbb{Z}$-by-finite.

Now suppose that G is abelian. By the theorem of Perlis-Walker [65, Theorem 3.5.4]

$$FG \cong \bigoplus_{d \mid |G|} a_d \frac{[\mathbb{Q}(\zeta_d) : \mathbb{Q}]}{[F(\zeta_d) : F]} F(\zeta_d) \tag{6.2}$$

with a_d the number of different cyclic subgroups of order d. Now denote by $R_{F,d}$ the ring of integers of $F(\zeta_d)$ and recall that by Dirichlet Unit theorem [46, Theorem 5.2.4] the rank of the finitely generated abelian group $\mathcal{U}(R_{F,d})$ is $n_1 + n_2 - 1$ with n_1 the number of real embeddings of $F(\zeta_d)$ and r_2 the number of pairs of complex embeddings. Note that the rank of $\mathcal{U}(R_{F,d})$ is at least the one of the unit group of the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_d)$. The latter rank is well-known to be $\frac{\varphi(d)}{2} - 1$. A direct computation yields that $\varphi(d) \leq 4$ if and only if $d \in \{2, 3, 4, 5, 68, 10, 12\}$ with equality only for $\{5, 8, 10, 12\}$. This combined with (6.2) we see that we have *exactly one* copy of $\mathbb{Z}$ if and only if G is isomorphic to C_n with $n = 5, 8$ or 12, $\operatorname{rank} \mathcal{U}(R_{F,n}) = \operatorname{rank} \mathcal{U}(R_{\mathbb{Q},n}) = 1$ and $\operatorname{rank} \mathcal{U}(R) = 0$. The latter means that F is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$ and hence yields the implication (1) $\Rightarrow$ (2). For the restriction on F written in (4) note that due to the values of n the field $\mathbb{Q}(\zeta_d)$ with $d \nmid n$ is either $\mathbb{Q}$ or an imaginary quadratic extension and hence $\operatorname{rank} \mathcal{U}(R) = 0$. Furthermore, $F(\zeta_n) = \mathbb{Q}(\zeta_n)$ and hence $\operatorname{rank} \mathcal{U}(R_{F,n}) = \operatorname{rank} \mathcal{U}(R_{\mathbb{Q},n})$ as desired. $\square$

We will now start with the proof of Theorem 6.2. This requires the following lemma that is a generalisation of [45, prop. 4.5.].

Lemma 6.6. *Let G be a finite group, D be a finite dimensional division algebra over F with $\operatorname{char}(F) = 0$, different*[13] *of $\left(\frac{-2,-5}{\mathbb{Q}}\right)$, and suppose $\mathrm{M}_n(D)$ with $n \geq 2$ is a simple component of FG. If $\mathcal{O}$ is an order in $\mathrm{M}_n(D)$, then $e(\mathcal{U}(\mathcal{O})) = \infty$ if and only if $n = 2$ and $D = F = \mathbb{Q}$.*

Proof. Suppose $e(\mathcal{U}(\mathcal{O})) = \infty$. As pointed out in the proof of Proposition 6.4, $e(\mathcal{U}(\mathcal{O}_1)) = e(\mathcal{U}(\mathcal{O}_2))$ for two orders $\mathcal{O}_1, \mathcal{O}_2$ in FG, as the number of ends is constant on commensurability classes [78, pg38] and unit groups of orders are commensurable [46, lemma 4.6.9]. Thus without lose of generality we will assume that $\mathcal{O}$ is a maximal order in $\mathrm{M}_n(D)$. It is well known that in that case $\mathcal{O} \cong \mathrm{M}_n(\mathcal{O}_{max})$ with $\mathcal{O}_{max}$ a maximal order in D.

Next recall that any group with infinitely many ends has finite center (as central elements need to be in the subgroup over which the amalgam and HNN are constructed, which is now finite). Therefore, $\mathrm{SL}_n(\mathcal{O}_{max})$ has finite index in $\mathrm{GL}_n(\mathcal{O}_{max})$ and hence $\mathrm{SL}_n(\mathcal{O}_{max}))$ also has infinitely many ends. This implies that $\mathrm{SL}_n(\mathcal{O}_{max})$ has S-rank 1, with S the set of infinite places, as otherwise it has hereditarily Serre's property FA (even property T [60, 29]) and hence can not have a non-trivial amalgam or HNN splitting.

The S-rank being one means that $n = 2$ and D is either $\mathbb{Q}(\sqrt{-d})$, with $d \geq 0$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with a, b strictly positive integers (e.g. see [5, Theorem 2.10.]). Furthermore it was proven

[13] This condition is not necessary, i.e the number of ends of $GL_2(\mathcal{O})$ for $\mathcal{O}$ an order in $\left(\frac{-2,-5}{\mathbb{Q}}\right)$ is not infinite. However including this case would make the proof unnecessarily lengthy.

in [26] that the condition that $\mathrm{M}_2(D)$ is a component of a group algebra yields that $d \in \{0, -1, -2, -3\}$ and $(a, b) \in \{(1, 1), (1, 3), (2, 5)\}$. All these division algebras are (right norm) Euclidean and due to this have a unique maximal order (see [5, remark 3.13]), which we denote $\mathcal{O}_D$. By assumption $(a, b) = (2, 5)$ doesn't occur. Now, following [5, Theorem 5.1] $\mathrm{GL}_2(\mathcal{O}_D)$ has property FA except if $D = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-2})$. In case of $D = \mathbb{Q}(\sqrt{-2})$ one can use the amalgam decomposition of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ given in [31, Theorem 2.1] to see that the group doesn't admit a splitting over a finite group. Finally, $\mathrm{GL}_2(\mathbb{Z}) = D_8 *_{C_2 \times C_2} D_{12}$ and hence $e(\mathrm{GL}_2(\mathbb{Z})) = \infty$, finishing the proof. ☐

We now proceed to the main proof.

Proof of Theorem 6.2. It is well known (e.g. see [78, pg38]) that if Γ_1 and Γ_2 are two groups such that $\Gamma_1 \cap \Gamma_2$ has finite index in the both (i.e. the Γ_i are commensurable), then $e(\Gamma_1) = e(\Gamma_2)$. Also if N is a finite normal subgroup, then $e(\Gamma_1) = e(\Gamma_1/N)$. Using this it is readily observed that the property to be virtually-$\mathcal{G}_\infty$ also enjoy these two properties.

This observation entails that if $H \leq \mathcal{U}(\mathcal{O})$ is virtually-$\mathcal{G}_\infty$ for some order $\mathcal{O}$ in FG, then also all other finite index subgroups of the unit group of any other order is by [46, lemma 4.6.9]. In particular we obtained:

Claim 0: Statement (i) holds if and only if $\mathcal{U}(RG)$, with R the ring of integers in F, is virtually-$\mathcal{G}_\infty$.

The above claim *yields the implication* $(ii) \Rightarrow (i)$ and the converse follows once we prove that virtually-$\mathcal{G}_\infty$ forces F to be the field of rational numbers.

Next, for the remaining of the proof fix a Wedderburn-Artin decomposition $FG = \bigoplus_{i=1}^q \mathrm{M}_{n_i}(D_i)$. Furthermore, fix a maximal order $\mathcal{O}_i$ in D_i for each i such that $\mathcal{U}(RG)$ is a subgroup of $\prod_{i=1}^q \mathrm{GL}_{n_i}(\mathcal{O}_i)$.

By our starting observation and [46, Lemma 4.6.9.] one has that $\mathcal{U}(RG)$ is virtually-$\mathcal{G}_\infty$ if and only if $\prod_{i=1}^q \mathrm{GL}_{n_i}(\mathcal{O}_i)$ is. Now if $\mathrm{M}_{n_i}(D_i)$ is either a field or a totally definite quaternion algebra, then $\mathrm{SL}_1(\mathcal{O}_i)$ is finite by [46, Proposition 5.5.6]. Also recall that F is in the center of every simple component of FG. Hence, if $\mathrm{M}_2(\mathbb{Q})$ is a simple component, then $F = \mathbb{Q}$. All this combined with Lemma 6.6 now *yields the implication* $(iii) \Rightarrow (ii)$.

Now suppose condition (i). Equivalently, by claim 0, suppose that $\mathcal{U}(RG)$ has virtually-$\mathcal{G}_\infty$. Let $H = \prod_{i=1}^m H_i \in \mathcal{G}_\infty$ be such subgroup of finite index in $\mathcal{U}(RG)$ (so $e(H_i) = \infty$ for all i). We need several observations:

Claim 1: $\mathcal{Z}(\mathcal{U}(RG))$ is finite and hence also $\mathcal{Z}(\mathcal{O}_i)$ is finite for all i.

For this remark that if $e(\Gamma) = \infty$ for some finitely generated group Γ, then $\mathcal{Z}(\Gamma)$ is finite. Therefore also $\mathcal{Z}(H)$ is finite and hence[14] $\mathcal{Z}(\mathcal{U}(RG))$ too. Due to the choice of the order $\mathcal{O}_i$ the latter implies that also $\mathcal{Z}(\mathcal{O}_i)$ is finite for all i.

Next,

Claim 2: Let T be a finitely generated group with $e(T) = \infty$. If $P, Q \lneq T$ are normal finitely generated subgroups such that $|P \cap Q| < \infty$ and PQ of finite index, then P or Q is finite.

Suppose such would exists. Then $e(PQ) = \infty$. Since by assumption $P \times Q \cong PQ/(P \cap Q)$ is commensurable with PQ also $e(P \times Q) = \infty$. However, the Cayley graph of a direct product is the cartesian product of the Cayley graphs. Using this one can see that the number of ends of a direct product of finitely generated groups is always one if P and Q are infinite, a contradiction.

Claim 3: $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$ for all j such that $M_{n_j}(D_j)$ is different of a field or totally definite quaternion algebra (e.g. all j for which $n_j \geq 2$).

Denote $S_j := \mathrm{SL}_{n_j}(\mathcal{O}_j) \cap H$ which is of finite index in $\mathrm{SL}_{n_j}(\mathcal{O}_j)$, hence it is enough to proof that $e(S_j) = \infty$. Let p_k be the projection of H on H_k. Fix some j as in the

[14]The subgroup $H \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \leq \mathcal{Z}(H)$ is of finite index in $\mathcal{Z}(\mathcal{U}(RG))$.

statement of claim 3. The condition is equivalent with saying that $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is infinite [53]. In particular there exists some k such that $p_k(S_j)$ is infinite[15]. For such k we will now prove that $|p_k(\prod_{i\neq j} S_i)| < \infty$. For this consider $S := S_j \times \prod_{i\neq j} S_i$ which by the first claim is of finite index in H. Therefore $p_k(S)$ is of finite index in H_k and hence $e(p_k(S)) = \infty$. However, $p_k(S_j)$ and $p_k(\prod_{i\neq j} S_i)$ are subgroups as in the second claim[16], yielding the desired. Indeed, the two subgroups clearly commute, are normal in $\pi_k(S)$ and $p_k(S_j) \cap p_k(\prod_{i\neq j} S_i) \subseteq \mathcal{Z}(p_k(S))$ which is finite since $p_k(S)$ has infinitely many ends.

Now consider the set $\mathcal{I}_j := \{k \mid |p_k(S_j)| < \infty\}$. From the previous it follows that if $k \in \{1, \ldots, q\} \setminus \mathcal{I}_j$, then $p_k(S_j)$ is of finite index in H_k. Hence $S_j/(S_j \cap \prod\limits_{i\in\mathcal{I}_j} H_i)$ is a subgroup of finite index in $\prod_{k\notin\mathcal{I}_j} H_k$. As the quotient was with a finite subgroup, we obtain that S_j is virtually-$\mathcal{G}_\infty$ and hence $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ also. However, under the conditions stated in claim 3, SL_1 is virtual indecomposable [54, Theorem 1]. Therefore $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ in fact is even virtually a group with infinitely many ends and so in fact $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$, as claimed.

Altogether: Claim 1 says that $\mathcal{Z}(\mathcal{O}_i)$ is finite and consequently $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is of finite index in $\mathrm{GL}_{n_j}(\mathcal{O}_j)$ for all j. In particular, $e(\mathrm{GL}_{n_j}(\mathcal{O}_j)) = \infty$ if $n_j \geq 2$. Now Lemma 6.6 implies that $n_j = 2$, i.e. no higher matrix algebras appear in the decomposition of FG. In such a case no $\left(\frac{-2,-5}{\mathbb{Q}}\right)$ component arises. Indeed, following [5, table appendix] such a component can only arise if $F = \mathbb{Q}$ and G maps onto one of the groups with SmallGroupID [40,3], [240,89] or [240,90]. But a direct verification, e.g. via the Wedderga package in GAP, shows that these groups all have higher matrix components.

Consequently, Lemma 6.6 says that all matrix components of FG must be isomorphic to $\mathrm{M}_2(\mathbb{Q})$ and in particular $F = \mathbb{Q}$ (as F is contained in the center of every simple component). As pointed out earlier, together with claim 0 this finishes the proof the equivalence $(i) \Leftrightarrow (ii)$.

Furthermore, by [5, Th. 2.10. & Prop. 6.11.], if $\mathbb{Q}Ge$ is a division algebra D for some primitive central idempotent e of $\mathbb{Q}G$ then D is $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{Z}_{\geq 0}$ or a totally definite quaternion algebra over $\mathbb{Q}$. In summary, we obtained that all components of $\mathbb{Q}G$ are of the desired form, *yielding the remaining implication* $(i) \Rightarrow (iii)$.

Next, that only the parameters $(-1,-1)$ and $(-1,-3)$ appear is due to [83, Theorem 11.5.14] saying that else $\mathcal{U}(\mathcal{O})$ is cyclic for any order in $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. In those cases $Ge \leq \mathcal{U}(\mathbb{Z}Ge)$ would have an abelian $\mathbb{Q}$-span and thus $\mathbb{Q}Ge \neq \left(\frac{-a,-b}{\mathbb{Q}}\right)$, a contradiction.

For the last part, first recall that by the commensurability of unit groups of orders $e(\mathcal{U}(\mathbb{Z}G)) = e(\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i))$. However the Cayley graph of a direct product is the cartesian product of the Cayley graphs. Using this we see that $e(Q \times P) = 1$ for any finitely generated group P, Q. Therefore $e(\mathcal{U}(\mathbb{Z}G)) = \infty$ if and only if $e(\mathrm{GL}_{n_{i_0}}(\mathcal{O}_{i_0})) = \infty$ for exactly one i_0 and the other factors are finite. In light of Lemma 6.6 and [5, Theorem 2.10.] this happens exactly when $\mathbb{Q}G$ has exactly one $\mathrm{M}_2(\mathbb{Q})$ component and all the others are $\mathbb{Q}, \mathbb{Q}(\sqrt{-d})$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. Since $\mathrm{GL}_2(\mathbb{Z})$ is virtually free we see that in those cases $\mathcal{U}(\mathbb{Z}G)$ is indeed virtually free.

It remains to prove that the only finite groups for which this happens are D_6, D_8, Dic_3 and $C_4 \rtimes C_4$. Recall that the unit group of the maximal orders of $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ and $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ are respectively $\mathrm{SL}(2,3) \cong Q_8 \rtimes C_3$ and Dic_3. Thus by the work done till now we already know that $\mathcal{U}(\mathbb{Z}G)$ is a subgroup of finite index in $(D_8 \times U) *_{C_2 \times C_2 * U} (D_{12} \times U)$ where $U = A \times$

[15]Otherwise S_j would be finite and hence also the overgroup of finite index $\mathrm{SL}_{n_j}(\mathcal{O}_j)$.

[16]Instead of claim 2 one could have used the well known [78, 4.A.6.3.] saying that infinite finitely generated normal subgroups of a group with infinitely many ends need to have finite index.

$\mathrm{SL}(2,3)^s \times Dic_3^t$ for some s,t and with A abelian with $\exp(A) \mid 4,6$. Using the description of torsion subgroups in amalgamated products we know that, up to conjugation, G is a subgroup of $C_2 \times C_2 * U$ or its contains transversal elements in one of the factors (i.e. D_8 or D_{12}). First suppose G is conjugated to a subgroup of U. Recall that all subgroups of Dic_3 are cyclic and the only non-cyclic one $\mathrm{SL}(2,3)$ is Q_8. As $\mathbb{Q}[\mathrm{SL}(2,3)]$ has a component $\mathrm{M}_3(\mathbb{Q})$ one can conclude that the only way to have exactly matrix component, which moreover is $\mathrm{M}_2(\mathbb{Q})$, is for G to be Dic_3. No suppose G is not conjugated to a subgroup of the amalgamated part. Then we know from Proposition 2.7 that $G \setminus (G \cap (C_2 \times C_2 \times U))$ embeds in $\mathrm{GL}_2(Z)$. If G contains no amalgamated element, then G embeds it needs to be isomorphic to D_6 or D_8 (as D_{12} has two matrix components). In general since $G \cap (C_2 \times C_2 \times U)$ will be a strict subgroup it will be central. Moreover, in order to have not more matrix components, the intersection clearly has to be a central subgroup of order 2. Thus G is a central extension of D_6 or D_8 with a C_2. A look at the groups of order 12 and 16 tells us that G is isomorphic to either Dic_3 or $C_4 \times C_4$, finishing the proof. □

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