THE VIRTUAL STRUCTURE PROBLEM FOR HIGHER MODULAR GROUPS

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Dedicated to the 70^{th} birthday of Eric Jespers

Abstract. We classify the finite groups G such that the finitely presented group $\mathcal{U}(\mathbb{Z}G)$ has the good property. Furthermore we obtain several characterisations in terms of properties of the simple factors of $\mathcal{U}(\mathbb{Q}G)$. Ring theoretically it is shown that it coincides with having only low-dimensional $\mathbb{Q}G$-components (i.e. at most 1×1 and exceptional 2×2 components). In particular, we solve a new instance of the virtual structure problem, generalising the free-by-free work. Cohomologically this happens if and only if all simple factors have virtual cohomological dimension a divisor of 4. Geometrically it is proven to be equivalent to the components acting discontinuously on $\mathbb{H}^5$. The latter properties are investigated for general lattices in semisimple algebraic $\mathbb{Q}$-groups of (inner) type A where in general the properties are no longer equivalent.

Contents

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1. Introduction

Finally, we consider the the Virtual Structure Problem, which askes for a unit theorem. A very concrete idea of a unit theorem was given by Kleinert [22] in the context of orders:

> *A unit theorem for a finite dimensional semisimple rational algebra A consists of the definition, in purely group theoretical terms, of a class of groups $C(A)$ such that almost all generic unit groups of A are members of $C(A)$.*

Recall that a generic unit group of A is a subgroup of finite index in the group of reduced norm 1 elements of an order in A.

The Virtual structure problem for exceptional components. reformulate

Question 1.1 (Virtual Structure Problem)**.** Let $\mathcal{G}$ be a class of groups. Classify the finite groups G such that $\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index lying in $\mathcal{G}$.

Till recently, the finite groups G for which a unit theorem, in the sense of Kleinert, was known for $\mathcal{U}(\mathbb{Z}G)$ are those for which the class of groups considered are either finite groups (Higman), abelian groups (Higman), or direct products of free-by-free groups [17, 14, 27, 19]. Remarkably, the latter class can also be described in terms of the rational group algebra: every simple quotient of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or $\mathrm{M}_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$. Such type of unit theorem was also obtained recently in [4, 3] for several geometric properties such as property (T) and (HFA). To our knowledge this result covers all the known unit theorems on $\mathcal{U}(\mathbb{Z}G)$.

The aim of this article is to solve the above for the following class of groups

$$\mathcal{G}_{am} = \{\mathbb{Z}^n \times \prod_{i\in I} A_i \star_{C_i} B_i \mid [A_i : C_i], [B_i : C_i] \text{ are finite but not } 1\}.$$

The main bulk of the paper will be about classifying the finite groups G such that the only non-division algebra components of $\mathbb{Q}G$ are exceptional 2×2 components. Following result completes a line of research started more than 20 years ago with the papers ... .

Theorem 1.2. *Let G be a finite group. The following are equivalent*

(1) All simple components $\mathrm{M}_n(D)$, with $n \geq 2$, of $\mathbb{Q}G$ are exceptional matrix components, i.e. $n = 2$ and D isomorphic to $\{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right)\}$,

(2) $\mathrm{vcd}(\mathrm{SL}_1(\mathbb{Z}Ge)) \in \{0, 1, 2, 4\}$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$

(3) G is a quotient of one of the following families of groups : **blabla**

Moreover, in that case $\mathrm{SL}_1(\mathbb{Z}Ge)$ is discrete subgroup of $\mathrm{SL}_4(\mathbb{C})$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$. The converse holds iff **to complete**

Remark 1.3.
- The fourth condition in Theorem 1.2 can also be equivalently stated that $\mathrm{SL}_1(\mathbb{Z}Ge)$ acts discontinuously on $\mathbb{H}_5$ or $\mathbb{H}_3$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Or PSL?

- one can be more precise in the actual components that appear. Namely **put final list**

Denote by $\mathcal{C}(\mathbb{Q}G)$ the isomorphism types of the simple components of $\mathbb{Q}G$. Next, we record a list of geometric group theoretical properties of $\mathcal{U}(\mathbb{Z}G)$ that is equivalent to $\mathcal{C}(\mathbb{Q}G)$ to be as in Theorem 1.2.

Is het eerder $V(\mathbb{Z}G)$?

Theorem 1.4. *Let G be a finite group. The following are equivalent*

(1) $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_{am}$

(2) $\mathcal{U}(\mathbb{Z}G)$ is good

(3) Stuff Angel-Zalesski paper.

As a corollary we get a statement which appeared without proof in ??. For this recall the following two classes of groups

$$\mathcal{G}_{pab} = \{\prod_i A_{i,1} \star \cdots \star A_{i,t_i} \mid A_{i,j} \text{ are finitely generated abelian } \}$$

and

$$\mathcal{G}_{\neq 1} := \{\prod_i \Gamma_i \mid e(\Gamma_i) \neq 1\}.$$

The following was announced without proof in [??].

Corollary 1.5. *The following classes are equal*

$$\{G \textit{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \textit{ is virtually-}\mathcal{G}_{\neq 1}\} = \{G \textit{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \textit{ is virtually-}\mathcal{G}_{pab}\}.$$

Theorem 1.2 and its predecessors suggest to investigate the following ring theoretical variant of the virtual structure problem for $\mathcal{U}(\mathbb{Z}G)$. **put the statement that is now later in the paper**

How to call the problem?

Problem. Let $\mathcal{P}$ be a set of isomorphism classes of finite dimensional simple algebras over $\mathbb{Q}$. Classify all finite groups G such that $\mathcal{C}(\mathbb{Q}G) \subseteq \mathcal{P}$.

The blockwise Zassenhaus property. In the 70's Zassenhaus formulated a set of conjectures which had to clarify the origin of the conjectural isomorphism between two group bases. The strongest of these, called the Third Zassenhaus conjecture, asserted that every finite subgroup H of $V(\mathbb{Z}G)$ is conjugated over $\mathbb{Q}G$ to a subgroup of G. This has been disproven in [??]. It nevertheless an important problem to determine which classes of groups satisfy the property asserted by the conjecture.

We define the Zassenhaus property, with respect to a type of subgroups, for any semisimple algebra. Subsequently we consider what we call the blockwise Zassenhaus property. For this no counterexample is yet known and in this paper we prove the following.

Theorem 1.6. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge$ is a field, totally definite quaternion algebra or an exceptional simple algebra. Then $(\mathbb{Q}G)e$ satisfy the Third Zassenhaus property.*

Corollary 1.7. *If G is a finite group such that $\mathbb{Q}G$ XX. Then it satisfies the blockwise Zassenhaus property.*

To finish, we would like to advertise the study of the block-wise version of the Isomorphism problem and the Zassenhaus conjectures. It can namely be verified that the known counterexamples to those conjectures are not counterexamples to the block-wise version!

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Conventions and notations. Throughout the full article all groups denotes by a latin letter will be a finite group. All orders will be understood to be $\mathbb{Z}$-orders. We also use the following notations:

- $\mathrm{PCI}(FG)$ for the set of primitive central idempotents of FG
- $\pi_e : \mathcal{U}(FG) \twoheadrightarrow FGe$ projection to a simple component
- $\mathcal{C}(FG) = \{FGe \mid e \in \mathrm{PCI}(FG)\}$ for the set of isomorphism types of the simple components of FG.
- Degree and index of a central simple algebra is **blabla**
- By $\phi(n)$ we denote Euler's phi function.

2. Finite groups with only exceptional higher simple components

As explained in the introduction, one old approach to representation theory of finite groups G over the ring of integers R of a number field F is to understand torsion-free normal subgroups in $\mathcal{U}(RG)$ and their quotients. The properties (such as the index) of these normal subgroups depend on the form of the simple quotients of FG. For instance the presence of the following type of simple algebras often breaks the methods used (see [4, Section 6.1] for more details):

Definition 2.1 (Exceptional components)**.** A finite dimensional simple algebra B is called *exceptional* if it is isomorphic to one of the following:

(1) a non-commutative division algebra which is not a totally definite quaternion algebra over a number field,
(2) a matrix algebra $\mathrm{M}_2(D)$ with $D \in \{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right) \mid a, b, d \in \mathbb{N}_0\}$.

If B is of the latter form, then we speak of an *exceptional matrix* algebra and in the former case of a *exceptional division* algebra. If $B \cong FGe$ for some $e \in \mathrm{PCI}(FG)$, then B is called an exceptional component of FG.

The name was coined in [18] and the reason that in practice they are exceptional (i.e. require 'other methods') is different. The exceptional matrix algebras are exactly those $\mathrm{M}_n(D)$ for which the S-rank of $\mathrm{SL}_n(D)$ is 1, where S is the set of Archimedean places of $\mathcal{Z}(D)$ (see [4, Remark 6.7] for details and eq. (3.1) for the definition of SL over non-commutative rings). Consequently, they are exactly those having a maximal order[1] $\mathrm{M}_n(\mathcal{O})$ such that $\mathrm{SL}_n(\mathcal{O})$ does not satisfy the subgroup congruence problem. In particular there exist non-central normal subgroups which are not of finite index.

The non-exceptional division algebras D are exactly those having an order $\mathcal{O}$ such that $\mathrm{SL}_1(\mathcal{O})$ is finite. The others are problematic because $\mathrm{SL}_1(D)$ has no unipotent elements which is an ingredient for most generic constructions of units in $\mathcal{U}(RG)$.

The aim of this section is to characterise the finite groups G such that FG has the following property:

$$(\mathrm{M}_{\mathrm{exc}}) \quad \text{all } FGe \cong \mathrm{M}_n(D), \text{ with } n \geq 2, \text{ are of the form } \mathrm{M}_2(\mathbb{Q}(\sqrt{-d})) \text{ or } \mathrm{M}_2(\left(\tfrac{-a,-b}{\mathbb{Q}}\right)) \text{ with } a, b, d \in \mathbb{N}.$$

So with respect to the general theory, they represent the 'most degenerate groups'. The groups G such that $\mathbb{Q}G$ has the stronger property that (i) all matrix component are of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{N}$ *and* (ii) $\mathbb{Q}G$ has no exceptional division components were classified in [19]. So considering property $(\mathrm{M}_{\mathrm{exc}})$ generalises their considerations in two ways.

Considering the role of the ground field we directly note the following:

Lemma 2.2. *Let G be a finite group and $F \subseteq L$ fields of characteristic* 0*. If LG has* $(\mathrm{M}_{\mathrm{exc}})$*, then FG also has* $(\mathrm{M}_{\mathrm{exc}})$*.*

Proof. Recall that there is a bijection between the simple component of LG and the absolutely irreducible characters $\mathrm{Irr}(G)$ of G. For $\chi \in \mathrm{Irr}(G)$ denote the associated component of LG by $A_L(\chi)$. Now, see [15, Theorem 3.3.1], $A_L(\chi)$ and $L(\chi) \otimes_{F(\chi)} A_F(\chi)$ are isomorphic as $F(\chi)$-algebras.

If $L = F$ nothing needs to be proved, so let $F \subsetneq L$. Note that the aforementioned results imply that the reduced degree of $A_F(\chi)$ is smaller or equal to the one of $A_L(\chi)$. Hence if LG has only division algebra components, then so does FG. Furthermore if LG has $(\mathrm{M}_{\mathrm{exc}})$, then also all matrix components of FG have reduced degree two. Next suppose that $A_F(\chi)$ is a matrix component and hence also $A_L(\chi)$. Then by the above see that $L(\chi) = \mathcal{Z}(A_L(\chi))$ contains $F(\chi)$ in its center. As LG has $(\mathrm{M}_{\mathrm{exc}})$, this means that $[F(\chi) : \mathbb{Q}] \mid [L(\chi) : \mathbb{Q}] \mid 2$. Consequently, either $F(\chi) = \mathbb{Q}$ or $F(\chi) = L(\chi)$ is an imaginary quadratic extension by $(\mathrm{M}_{\mathrm{exc}})$. ☐

Remark 2.3. Finite groups G for which $\mathbb{Q}G$ has only division algebra components have been classified in [29, Theorem 3.5]. The latter result implies that LG has only division components exactly when $\mathbb{Q}G$ does and L is not a splitting field of $\left(\frac{-1,-1}{\mathbb{Q}}\right)$. Hence we may suppose that LG has a matrix component. First note that if LG has a matrix component,

[1] It is well-known that a maximal order in $\mathrm{M}_n(D)$ is of the form $\mathrm{M}_n(\mathcal{O})$ with $\mathcal{O}$ a maximal order in D. Furthermore in case of an exceptional matrix algebra, the division algebras have up to isomorphism a unique maximal order. Thus the property doesn't depend on the order chosen.

then, as $L \subseteq \mathcal{Z}(A_L(\chi))$ for all $\chi \in \mathrm{Irr}(G)$, the $(\mathrm{M_{exc}})$ assumption implies that L is either the rationals or an imaginary quadratic extension of $\mathbb{Q}$. Consider $L = \mathbb{Q}(\sqrt{-d})$ for some square-free positive integer d.

Then the proof of Lemma 2.2 yields that LG can only have $(\mathrm{M_{exc}})$ if $\mathbb{Q}G$ does and moreover $F(\chi, \sqrt{-d}) = F(\sqrt{-d})$ for all $\chi \in \mathrm{Irr}(G)$. In other words $F(\chi) \subseteq \mathbb{Q}(\sqrt{-d})$ for all χ. Note that the converse also holds. In summary:

$$\mathbb{Q}(\sqrt{-d})[G] \text{ has } (\mathrm{M_{exc}}) \Leftrightarrow \mathbb{Q}G \text{ has } (\mathrm{M_{exc}}) \text{ and } \mathcal{Z}(\mathbb{Q}Ge) \subseteq \mathbb{Q}(\sqrt{-d}) \text{ for all } e \in \mathrm{PCI}(\mathbb{Q}G).$$

This shows the importance of first understanding the case of rational group algebras. Most of the work will be dedicated to that. We start by recalling methods to describe the simple components of $\mathbb{Q}G$.

2.1. **Preliminaries on describing simple components and role coefficients.** We need to recall some methods to construct primitive central idempotents of $\mathbb{Q}G$. These methods were introduced by Olivieri-del Río-Simón [26], see [15, Chapter 3] for a good introduction. To start, recall that if $H \trianglelefteq G$, then

$$\widehat{H} := \frac{1}{|H|}\sum_{h\in H} h$$

is a central idempotent in $\mathbb{Q}G$. Now, set $\epsilon(H,H) = \widehat{H}$ and for a strict normal subgroup K of H define

$$\epsilon(H,K) = \prod_{M/K \in \mathcal{M}(H/K)} (\widehat{K} - \widehat{M}), \tag{2.1}$$

where $\mathcal{M}(H/K)$ denotes the set of the non-trivial minimal normal subgroups of H/K. The construction results in a central idempotent in $\mathbb{Q}H$. To obtain a central idempotent in $\mathbb{Q}G$ one associates to $K \trianglelefteq H$ the element

$$e(G,H,K) = \sum_{t\in\mathcal{T}} \epsilon(H,K)^t \tag{2.2}$$

where $\mathcal{T}$ is a right transversal of $\mathrm{Cen}_G(\epsilon(H,K))$ in G. The element $e(G,H,K)$ is central in $\mathbb{Q}G$ and is a primitive idempotent when (H,K) is a *Strong Shoda pair* of G, *SSP in short.* A tuple (H,K) is called a strong Shoda pair when $K \leq H \trianglelefteq N_G(K)$, H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$, and the G-conjugates of $\epsilon(H,K)$ are orthogonal.

To a central idempotent e there is also the associated epimorphism

$$\varphi_e : G \to Ge, \quad g \mapsto ge. \tag{2.3}$$

Note that Ge is a finite subgroup of $\mathcal{U}(\mathbb{Q}G)e$.

The following is a combination of [15, Proposition 3.4.1, Theorems 3.4.2 & 3.5.5 and Problem 3.5.1] and [11, Lemma 3.4].

Theorem 2.4 ([26])**.** *With notations as above, $e := e(G,H,K)$ is a primitive central idempotent of $\mathbb{Q}G$ if (H,K) is a strong Shoda pair. Moreover, in that case $\mathrm{Cen}_G(\epsilon(H,K)) \cong N_G(K)$, $\ker(\varphi_e) = core_G(K) = \bigcap_{g\in G} K^g$ and*

$$\dim_{\mathbb{Q}} \mathbb{Q}Ge = [G:H][G:N_G(K)]\phi([H:K]).$$

Furthermore, $\mathbb{Q}Ge \cong \mathrm{M}_{[G:N_G(K)]}\big(\mathbb{Q}(\zeta_{[H:K]}) * N_G(K)/H\big)$ for some explicit crossing. In particular, $\deg(\mathbb{Q}Ge) = [G:H]$.

The notation deg denotes *the degree* of the central simple algebra $\mathbb{Q}Ge(G,H,K))$, i.e. $\deg(A) = \sqrt{n}$ if $A \otimes_{\mathcal{Z}(A)} \mathbb{C} \cong \mathrm{M}_n(\mathbb{C})$.

Remark. For a SSP (H,K) it might happen that $H = G$. This happens exactly when $G' \leq K$ which in turn is equivalent to $\mathbb{Q}Ge(G,H,K)$ being commutative, see [13, Lemma 2.4].

Next we recall, following [15, Remark 3.5.6], the structure of the crossed product of $\mathbb{Q}(\zeta_{[H:K]}) * N_G(K)/H)$ mentioned in Theorem 2.4. For ease denote $N := N_G(K), m := [H : K]$ and $H/K := \langle y \rangle$. The twisting is given by:

$$\alpha : N/H \to \mathrm{Aut}(\zeta_m) : \overline{x} \mapsto \alpha_{\overline{x}}(\zeta_m) = \zeta_m^j \tag{2.4}$$

where j is determined by $(yK)^x = (yK)^j$ with $xH = \overline{x}$.

Next, the crossing is given by:

$$f : N/H \times N/H \to \mathbb{Q}(\zeta_m)^* : (\overline{x_1}, \overline{x_2}) \mapsto \zeta_m^\ell \tag{2.5}$$

with ℓ being determined by $t_{\overline{n_1}}.t_{\overline{n_2}} = y^\ell.k.t_{\overline{n_1 n_2}}$ where $\overline{n_i} = t_{\overline{n_i}}H$ for a left transversal $\mathcal{T}$ of H in N and $k \in K$.

Finally, we will mainly use the theory of SSP in case that G is metabelian in which case they are easily describable.

Theorem 2.5 (Theorem 3.5.12 in [15]). *Let G be a finite metabelian group and let B be a maximal abelian subgroup of G containing G'. Let $K \leq G$ be such that $C' \leq K \leq C$ for some $B \leq C \leq G$. Then (H, K) is a strong Shoda pair if and only if the following hold:*

(1) H/K is cyclic,

(2) H is maximal in the set $\{C \leq G \mid A \leq C$ and $C' \leq K \leq C\}$.

2.2. **Restrictions on Division algebras and the Group.** In this section we first give several group theoretical properties on the finite groups G such that FG has $(\mathrm{M}_{\mathrm{exc}})$. Under that condition we also obtain which simple algebras can occur as simple component of FG. For instance the possible division algebra that can arise as $\mathbb{Q}Ge$ are still restricted. Thereafter we will show that there exists subgroups of index two such that the matrix components are as in [19].

Description components and first properties. To start we describe the possible simple algebras that can appear as a component of $\mathbb{Q}G$ with $(\mathrm{M}_{\mathrm{exc}})$. To do so, we will describe which finite groups can be isomorphic to Ge with $e \in \mathrm{PCI}(\mathbb{Q}G)$. This has the advantage to also yield a first set of interesting properties.

In [29, Theorem 3.5] it was proven that $\mathbb{Q}G$ has only division algebra components if and only if G is of the form $A \times C_2^n \times Q_8$ with A an odd abelian group such that $|A|$ and $o_{|A|}(2)$ are odd. Hence from now we may assume that $\mathbb{Q}G$ has at least one matrix component.

Theorem 2.6. *Let G be a finite group and suppose that $\mathbb{Q}G$ has a matrix component. If $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$, then*

(1) the 1×1 components of $\mathbb{Q}G$ are either fields or quaternion algebras. More precisely, the non-commutative possibilities are[2]:

$$\{\left(\frac{\zeta_{2^t}, -3}{\mathbb{Q}(\zeta_{2^t})}\right), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right) \mid t \in \mathbb{N}_{\geq 3}\}$$

(2) The only exceptional matrix components are

$$\{\mathrm{M}_2(\mathbb{Q}(\sqrt{-d})), \mathrm{M}_2(\left(\frac{-1,-1}{\mathbb{Q}}\right)), \mathrm{M}_2(\left(\frac{-1,-3}{\mathbb{Q}}\right)) \mid d = 0,1,2,3\}.$$

(3) G has[3] an abelian normal subgroup B with $\exp(G/B) \mid 2$.

(4) $\pi(G) \subseteq \{2,3\}$.

(5) $\exp(\mathcal{Z}(G) \cap G') \mid 2$ and $exp(G/\mathcal{Z}(G)) \mid 4$.

(6) if $\mathbb{Q}G$ has no exceptional division components, then

$$\exp(\mathcal{Z}(G)) \mid 4 \text{ or } 6 \text{ and } \exp(G) \mid 24.$$

In particular, G is metabelian with $\mathrm{cd}(G) \subseteq \{1,2,4\}$. Furthermore, $\deg(\mathbb{Q}Ge) \mid 4$ for every $e \in \mathrm{PCI}(\mathbb{Q}G)$.

[2]Note that by [5, Theorem 3.5] the only exceptional matrix component not appearing is $\mathrm{M}_2(\mathbb{H}_5)$.

[3]The proof will furthermore prove that $[Ge : Ae] \mid 4$ for every $e \in \mathrm{PCI}(\mathbb{Q}G)$.

In Section 2.4 we will see that several properties hold in a stronger form, e.g. G/B is an elementary abelian 2-group of rank at most 2.

Remark 2.7. In the proof we will obtain that for $C_3 \rtimes C_{2^n}$, where the action is by inversion, the rational group algebra has all non-division simple components of the form $\mathrm{M}_2(\mathbb{Q})$ and $\mathrm{M}_2(\mathbb{Q}(i))$. Furthermore it has a division component of the form $\left(\frac{\zeta_{2^{n-1}},-3}{\mathbb{Q}(\zeta_{2^{n-1}})}\right)$. Hence it gives an example of a group satisfying that all matrix components are of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{N}$ but not the stronger property considered in [19] where they need that $\mathbb{Q}G$ has no exceptional division components.

One interesting class of groups for which $\mathbb{Q}G$ has no exceptional division components is the class of *cut groups*, see [4, Proposition 6.12]. Recall that a group is called cut if $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite.

The proof of Theorem 2.6 will go through understanding the quotient groups Ge for $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Lemma 2.8. *For any finite group G, normal subgroup $N \leqslant G$ and $e \in \mathrm{PCI}(\mathbb{Q}G)$ holds:*

$$\mathcal{C}(\mathbb{Q}[G/N]) \subseteq \mathcal{C}(\mathbb{Q}G) \text{ and } \mathcal{C}(\mathbb{Q}[Ge]) \subseteq \mathcal{C}(\mathbb{Q}G).$$

Consequently,

$$\mathcal{C}(\mathbb{Q}G) = \bigcup_{e\in \mathrm{PCI}(\mathbb{Q}G)} \mathcal{C}(\mathbb{Q}[Ge]).$$

In particular, property $(\mathrm{M}_{\mathrm{exc}})$ *is inherited by quotients.*

Proof. The first claim follows from the fact that $\mathbb{Q}[G/N]$ is a semisimple subalgebra of $\mathbb{Q}G$. Indeed, it is immediately semisimple (since it is a group algebra), and a straightforward calculation shows that $\mathbb{Q}[G/N] \cong \mathbb{Q}G\widetilde{N} \leq \mathbb{Q}G$ with $\widetilde{N}$ the central idempotent $\frac{1}{|N|}\sum_{n\in N} n$. The second inclusion follows from the first since the group Ge is an epimorphic image of G. The rest is now also a direct consequence as every simple component of $\mathbb{Q}G$ corresponds to a primitive central idempotent $e \in \mathrm{PCI}(\mathbb{Q}G)$. ☐

Using Lemma 2.8, the proof of Theorem 2.6 reduces to a study of the fixed-point free groups classified by Amitsur [1] and the finite subgroups of exceptional components classified in [5]. In fact the conclusion of Theorem 2.6 already holds under the weaker condition that each Ge is embedded in a division algebra or an exceptional matrix algebra.

Proof of Theorem 2.6. For a group G having $(\mathrm{M}_{\mathrm{exc}})$, the set $\mathrm{PCI}(\mathbb{Q}G)$ naturally decomposes into $\mathrm{PCI}_1 := \{e \mid \mathbb{Q}Ge \cong D\}$ and $\mathrm{PCI}_2 := \{e \mid \mathbb{Q}Ge \cong \mathrm{M}_2(D)\}$ where D always signifies a rational division algebra. Hence, with Lemma 2.8 in mind, for the first statement it suffices to analyse the components possibly appearing in $\mathbb{Q}[Ge]$ for $e \in \mathrm{PCI}_1$ or PCI_2.

Let's start with PCI_2. The finite subgroups $\mathcal{G}$ of $\mathrm{GL}_2(\mathbb{Q}(\sqrt{-d}))$ or $\mathrm{GL}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ with $a, b, d \in \mathbb{N}$ with the property that $\mathrm{span}_{\mathbb{Q}}(\mathcal{G})$ is the respective $\mathrm{M}_2(\cdot)$ have been classified[4] in [5, Theorem 3.7]. This classification consists of 55 groups and in particular the groups Ge for $e \in \mathrm{PCI}_2$ must be among these. One can compute the simple components for example in GAP using the Wedderga package, see **table ?? in Appendix ??** for the result. This would moreover show that for groups Ge with $e \in \mathrm{PCI}_2$ the property $(\mathrm{M}_{\mathrm{exc}})$ is equivalent to the weaker property that each non-division component has reduced degree 2. A case-by-case verification also shows that each of these groups Ge contain an abelian normal subgroup of index a divisor of 4. Furthermore, as written in the table the only 1×1 components appearing are Still to add the table!

$$\mathbb{Q}, \mathbb{Q}(\zeta_3), \mathbb{Q}(i), \mathbb{Q}(\zeta_8), \mathbb{Q}(\zeta_{12}), \left(\frac{-1,-1}{\mathbb{Q}}\right), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right).$$

[4]In [5, Table 2] a group was missing, see [4, Appendix A] for a complete list.

Inspection of the table also shows that all other statements hold for such groups.

Next we consider the case that $e \in \mathrm{PCI}_1$ for which a similar reasoning applies. Indeed, the finite subgroups (such as Ge for $e \in \mathrm{PCI}_1$) of rational division algebras have been classified by Amitsur in [1]. We will use the rephrasing from [30, Theorem 2.1.4] which asserts that they are:

a) a **Z**-group, i.e. a subgroup of a rational division algebra with cyclic Sylow-subgroups.

b) i) the binary octahedral group O^* of order 48:
$$\{\pm 1, \pm i, \pm j, \pm ij, \frac{\pm 1 \pm i \pm j \pm ij}{2}\} \cup \{\frac{\pm a \pm b}{\sqrt{2}} \mid a, b \in \{1, i, j, ij\}\}.$$

ii) $C_m \rtimes Q$, where m is odd, Q is quaternion of order 2^t for some $t \geq 3$, an element of order 2^{t-1} centralizes C_m and an element of order 4 inverts C_m.

iii) $M \times Q_8$, with M a **Z**-group of odd order m and the (multiplicative) order of 2 mod m is odd.

iv) $M \times \mathrm{SL}_2(\mathbb{F}_3)$, where M is a **Z**-group of order m coprime to 6 and the (multiplicative) order of 2 mod m is odd.

c) $\mathrm{SL}_2(\mathbb{F}_5)$.

Neither $O^*, \mathrm{SL}_2(\mathbb{F}_3)$ nor $\mathrm{SL}_2(\mathbb{F}_5)$ have ($\mathrm{M}_{\mathrm{exc}}$). Indeed using[5] the Wedderga package in Gap one learns that $\mathrm{M}_3(\mathbb{Q})$ is a simple component over $\mathbb{Q}$ of $O^* \cong \mathrm{SU}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_3)$ and $\mathrm{M}_5(\mathbb{Q})$ for $\mathrm{SL}_2(\mathbb{F}_5)$. Consequently, they can not be epimorphic images of the group G. Hence:

the cases b.i), b.iv) and c) *do not* appear as groups Ge for $e \in \mathrm{PCI}_1$ with $\mathbb{Q}[G]$ ($\mathrm{M}_{\mathrm{exc}}$).

The groups in b) ii) are actually dicyclic groups of order $2^t m$, i.e. Dic_{4n} with $n = 2^{t-2}m$ and $t \geq 3$. The case of odd n will be case (b) in the family of **Z**-groups. Therefore consider a general dicyclic group:
$$Dic_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1}\rangle.$$

Being a metabelian group its Strong Shoda pairs are described by [15, Theorem 3.5.12] which we apply now. Note that the commutator subgroup $Dic_{4n}' = \langle a^2 \rangle$ and $\langle a \rangle$ is the maximal abelian containing it. For any (H, K) SSP holds that $\langle a \rangle \subseteq H$. In other words $H = \langle a \rangle$ or Dic_{4n}. If $H = Dic_{4n}$ then the simple component associated to (H, K) is a field, [13, Lemma 2.4]. Via [15, Theorem 3.5.12] it is a direct verification that for $d \mid 2n$ the tuple $(\langle a \rangle, \langle a^d \rangle)$ is a SSP if and only if $d \neq 1, 2$. Note that K is normal in Dic_{4n}, hence the associated primitive central idempotent is $\epsilon(\langle a \rangle, \langle a^d \rangle)$. Now, in [15, Example 3.5.7], it is noted that

$$\mathbb{Q}Dic_{4n}\epsilon(\langle a \rangle, \langle a^d \rangle) \cong \mathrm{M}_2(\mathbb{Q}(\zeta_d + \zeta_d^{-1})) \text{ if } d \mid n \text{ and } d \nmid 2 \tag{2.6}$$

where ζ_d denotes a complex primitive d-th root of unity. Since $\mathbb{Q}(\zeta_d + \zeta_d^{-1}) = \mathbb{Q}(\Re(\zeta_d))$ and $\Re(\zeta_d) = \cos \frac{2\pi}{d} \in \mathbb{R} \setminus \mathbb{Q}$, Niven's theorem tells that $\mathrm{M}_2(\mathbb{Q}(\zeta_d + \zeta_d^{-1}))$ is exceptional[6] if and only if $\varphi(d) \leq 2$. The latter is equivalent to $d \in \{1, 2, 3, 4, 6\}$. In conclusion if Dic_{4n} is non-abelian and has ($\mathrm{M}_{\mathrm{exc}}$), then it must be isomorphic to $Dic_{4.2} = Q_8, Dic_{4.4} = Q_{16}$, $Dic_{4.6} = C_3 \rtimes Q_8$ or $Dic_{4.3} = C_3 \rtimes C_4$. The first three groups are in the family b) ii). Moreover these groups indeed have ($\mathrm{M}_{\mathrm{exc}}$) with Wedderburn-Artin decomposition:

$$\mathcal{C}(\mathbb{Q}Q_8) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right)\}, \qquad \mathcal{C}(\mathbb{Q}Q_{16}) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right), \mathrm{M}_2(\mathbb{Q})\} \text{ and}$$
$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes Q_8]) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right), \mathrm{M}_2(\mathbb{Q})\}$$

[5]The SmallGroup ID of the three groups are respectively [48,28], [24,3] and [120,5].

[6]Recall that $[\mathbb{Q}(\Re(\zeta_d)) : \mathbb{Q}] = \varphi(d)/2$. In particular, in contrast to the case $e \in \mathrm{PCI}_2$, it can happen that all non-division components are exceptional without the group having ($\mathrm{M}_{\mathrm{exc}}$). Even more it can happen that all non-division components are of the form $\mathrm{M}_2(F)$ with F a quadratic extension of $\mathbb{Q}$. As shown by (2.6) this namely holds for Dic_{4n} with $n = 5, 10, 8, 12$.

Before we consider case b) iii), we will discuss a), the **Z**-groups.

The **Z**-groups themselves have also been classified, see [30, Theorem 2.1.5]. They are the following:

a) cyclic.
b) $C_m \rtimes C_4$, where m is odd and C_4 acts by inversion.
c) $G_0 \times G_1 \times \ldots \times G_s$, with $s \geq 1$, $gcd(|G_i|, |G_j|) = 1$ for all $0 \leq i \neq j \leq s$ and G_0 is the only cyclic subgroup amongst them. Furthermore each of the G_i, for $i \neq 0$, is of the form
$$C_{p^a} \rtimes \left(C_{q_1^{b_1}} \times \ldots \times C_{q_r^{b_r}} \right),$$
for $p, q_1, \ldots, q_r$ distinct primes. Moreover, each of the groups $C_{p^a} \rtimes C_{q_j^{b_j}}$ is non-cyclic (i.e. if $C_{q^{\alpha_j}}$ denotes the kernel of the action of $C_{q_j^{b_j}}$ on C_{p^a}, then $\alpha_j \neq b_j$) and satisfies the following properties:
 (i) $q_j o_{q_j^{\alpha_j}}(p) \nmid o_{\frac{|G|}{|G_i|}}(p)$.
 (ii) one of the following is true:
  - $q_j = 2, p \equiv -1 \mod 4,$ and $\alpha_j = 1$,
  - $q_j = 2, p \equiv -1 \mod 4,$ and $2^{\alpha_j+1} \nmid p^2 - 1$,
  - $q_j = 2, p \equiv 1 \mod 4,$ and $2^{\alpha_j+1} \nmid p - 1$,
  - $q_j > 2,$ and $q_j^{\alpha_j+1} \nmid p - 1$.

It is clear the cyclic groups have $(\mathrm{M}_{\mathrm{exc}})$ since $\mathbb{Q}C_n$ is abelian. Moreover, by the well known theorem of Perlis-Walker, $\mathcal{C}(\mathbb{Q}C_n) = \{\mathbb{Q}(\zeta_d) \mid d \text{ divides } n\}$.

Case b), i.e Dic_{4n} with n odd, was already handled via (2.6). The conclusion was that the only possible (non-abelian) such group having $(\mathrm{M}_{\mathrm{exc}})$ is $C_3 \rtimes C_4$. In this case
$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_4])) = \{\mathbb{Q}, \mathbb{Q}(i), \left(\frac{-1,-3}{\mathbb{Q}}\right), \mathrm{M}_2(\mathbb{Q})\}.$$

Next consider case c). We first show that $(\mathrm{M}_{\mathrm{exc}})$ enforces $2 \mid |G_i|$, for $1 \leq i \leq s$ and hence $s = 1$ by the coprime condition. For this consider $A_i = \prod_{j=1}^{s} C_{q^{\alpha_j}}$, the kernel of the action. Then $B := G_i/A_i \cong C_{p^a} \rtimes C_{q_1^{k_1} \cdot \ldots \cdot q_s^{k_s}}$ where $k_j = b_j - \alpha_j > 0$ and the action is non-trivial and faithful. Denote $B = \langle x \rangle \rtimes \langle y \rangle$. By Lemma 2.8 the group B also has $(\mathrm{M}_{\mathrm{exc}})$. Note that $C_{p^a} = \langle x \rangle$ is a maximal abelian subgroup of B containing B'. Now using [15, Theorem 3.5.12] it is a direct verification that $(H, K) = (\langle x \rangle, 1)$ is a SSP of B. Moreover $\mathbb{Q}Be(G, \langle x \rangle, 1) \cong \mathbb{Q}(\zeta_{o(x)}) * \langle y \rangle$ for some explicit crossing (see [15, Remark 3.5.6]) which imply that the component is non-division. Now using [11, Lemma 3.4], we compute that
$$\dim_{\mathbb{Q}} \mathbb{Q}Be(G, \langle x \rangle, 1) = [G : \langle x \rangle]\phi(o(x)) = q_1^{k_1} \cdot \cdots \cdot q_s^{k_s} p^{a-1}(p-1).$$
On the other $\dim_{\mathbb{Q}} \mathbb{Q}Be(G, \langle x \rangle, 1) \mid 16$ as B has $(\mathrm{M}_{\mathrm{exc}})$. Combining both with the fact that p and the q_i are different primes, we obtain that $s = 1, q_1 = 2$ and $p^a = 3$. Thus $B \cong C_3 \rtimes C_{2^{k_1}}$. Furthermore, as the action is faithful, we obtain that $k_1 = 1$, i.e. $B \cong C_3 \rtimes C_2$ where the action is by inversion. Consequently $G_1 \cong C_3 \rtimes C_{2^{b_1}}$ with the action being inversion (as $\alpha_1 = b_1 - 1$). For such groups, using [15, Theorem 3.5.12], one can verify that *for* $n \geq 4$
$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_{2^n}]) = \{\mathbb{Q}(\zeta_{2^\ell}), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right), \mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(\mathbb{Q}(i)) \mid 1 \leq \ell \leq n,\ 3 \leq t \leq n-1\}.$$
And if $n \leq 3$:
$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_2]) = \{\mathbb{Q}, \mathrm{M}_2(\mathbb{Q})\}, \qquad \mathcal{C}(\mathbb{Q}[C_3 \rtimes C_4]) = \{\mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \left(\frac{-1,-3}{\mathbb{Q}}\right), \mathrm{M}_2(\mathbb{Q})\}$$
$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_8]) = \{\mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\zeta_8), \left(\frac{-1,-3}{\mathbb{Q}}\right), \mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(\mathbb{Q}(\sqrt{-1}))\}$$
Hence we see that such G_1 has $(\mathrm{M}_{\mathrm{exc}})$.

It remains to consider $G_0 \times G_1$ with $G_0 \cong C_m$ cyclic. From the Wedderburn-Artin decomposition of $\mathbb{Q}[G_1]$ above we see that $\mathbb{Q}[G_0 \times G_1]$ contains as a simple component $\mathbb{Q}(\zeta_m) \otimes_{\mathbb{Q}} \mathrm{M}_2(\mathbb{Q}) \cong \mathrm{M}_2(\mathbb{Q}(\zeta_m)) \oplus \mathrm{M}_2(\mathbb{Q}(\zeta_m))$ with $d \in \mathbb{N}$ (potentially zero). As $G_0 \times G_1$ is assumed to have ($\mathrm{M}_{\mathrm{exc}}$) this implies that $[\mathbb{Q}(\zeta_m) : \mathbb{Q}] \leq 2$. The latter happens exactly when $m \in \{1, 2, 3, 4, 6\}$. Now recall that $|G_0| = m$ and $|G_1| = 3.2^{b_1}$ are relatively prime, yielding that $m = 1$. Thus in conclusion[7]:

(2.7) $\quad$ **Z**-groups of type c) with ($\mathrm{M}_{\mathrm{exc}}$) are the $C_3 \rtimes C_{2^n}$ with action by inversion

The last case to handle is b) iii), i.e. $M \times Q_8$ with M a **Z**-group of odd order m and also the multiplicative order of 2 modulo m is odd. By Lemma 2.8 also M has ($\mathrm{M}_{\mathrm{exc}}$), but looking at the possible such **Z**-groups of odd order we see that M must be cyclic. Now recall that $\mathcal{C}(\mathbb{Q}Q_8) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right)\}$. If M is cyclic, then $\mathcal{C}(\mathbb{Q}[M \times Q_8]) = \{\mathbb{Q}(\zeta_d), \left(\frac{-1,-1}{\mathbb{Q}(\zeta_d)}\right) \mid d \text{ divides } m\}$. As m and $o_m(2)$ are odd all the components are division algebras, a conclusion that also directly would have followed from [29]. In conclusion[8]:

Groups of type *b.iii)* with ($\mathrm{M}_{\mathrm{exc}}$) are the $C_m \times Q_8$ with m and $o_m(2)$ odd.

To summarize, with the analysis above we have shown part (1) and (2) from the statement by describing $\prod_{e \in \mathrm{PCI}(\mathbb{Q}G)} Ge$. Note that all allowed groups Ge have been highlighted in the proof. We see that they all have an abelian normal subgroup B_e with Ge/B_e isomorphic to C_2 or $C_2 \times C_2$. Hence $B := G \cap \prod_{e \in \mathrm{PCI}(\mathbb{Q}G)} B_e$ is an abelian normal subgroup of G with $\exp(G/B) \mid 2$. Consequently, G is metabelian. Also, for the simple algebras $\mathrm{M}_n(D)$ allowed by ($\mathrm{M}_{\mathrm{exc}}$), we see that $\mathrm{M}_2(D) \otimes_{\mathcal{Z}(D)} \mathbb{C}$ is isomorphic to $\mathrm{M}_2(\mathbb{C})$ or $\mathrm{M}_4(\mathbb{C})$. By the first part if $\mathbb{Q}Ge \cong D$, then $D \otimes_{\mathbb{Z}(D)} \mathbb{C}$ is either $\mathbb{C}$ or $\mathrm{M}_2(\mathbb{C})$. So indeed $\mathrm{cd}(G) \subseteq \{1, 2, 4\}$ and $\deg(\mathbb{Q}Ge) \mid 4$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Since $(\chi(1), q) = 1$ for any odd prime q and any $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ it follows from the Ito-Michler theorem [10, Corollary 12.34] that $G \cong A \rtimes Q$ with A an abelian subgroup of odd order and Q a Sylow 2-subgroup of G.

This detailed analysis also shows that $\pi(Ge) \subseteq \{2, 3\}$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that Ge is non-abelian and not isomorphic to $C_m \times Q_8$ in which case $\mathbb{Q}Ge \cong \left(\frac{-1,-1}{\mathbb{Q}(\zeta_m)}\right)$. Therefore if we take $g \in G$ with $o(g)$ prime different of 2 or 3, then $ge \in \ker(\varphi_e)$ except possibly for primitive central idempotents e such that $\mathbb{Q}Ge$ is a field or a quaternion algebra with $ge \in \mathcal{Z}(\mathbb{Q}Ge)$. Therefore $g \in \mathcal{Z}(G)$. Now, as $G \cong A \rtimes Q$ with Q a 2-group, we can decompose G into $\langle g \rangle \times H$ for some $H \leq G$. However taken together with the assumption that $\mathbb{Q}G$ has a matrix component, which by necessarily is exceptional, we obtain that $o(g) \mid 4$ or 6, a contradiction.

Notice that the preceding argument proved that $C_m \times Q_8$ could only occur as quotient of G if $m = 3$. However we also had the assumption that $o_m(2)$ needs to odd, which also discards the option $m = 3$.

to complete the exponent stuff

Finally we consider the statements considering $\mathcal{Z}(G)$. First consider the set

$$\mathcal{E} := \{e \in \mathrm{PCI}(\mathbb{Q}G) \mid \mathbb{Q}Ge \text{ is not a division component }\}.$$

As shown in **Table ??** $\exp(\mathcal{Z}(Gf)) \mid 4$ or 6 where $f := \sum_{e \in \mathcal{E}} e$. The same holds for $\exp(\mathcal{Z}Ge)$ when $\mathbb{Q}Ge$ is a non-commutative division component different from $\left(\frac{\zeta_{2^t}, -3}{\mathbb{Q}(\zeta_{2^t})}\right)$. Indeed, verifying all Ge which are ($\mathrm{M}_{\mathrm{exc}}$) one notices that such components either occur from a Ge in the table or when $Ge \cong Q_{16}$ or $C_3 \rtimes Q_8$. In all cases $\exp(\mathcal{Z}(Ge)) \mid 4$ or 6. Hence when $\mathbb{Q}G$ has no exceptional division components it remains to understand $\mathcal{Z}(Ge)$ the field components.

[7]To reach this conclusion we only used that every non-division component is of the form $\mathrm{M}_2(D)$ with $[\mathcal{Z}(D) : \mathbb{Q}] \leq 2$.

[8]Also this conclusion only requires that every non-division component is of the form $\mathrm{M}_2(D)$ with $[\mathcal{Z}(D) = \mathbb{Q}] \leq 2$ and not the full strength of ($\mathrm{M}_{\mathrm{exc}}$).

Consider $1 \neq g \in \mathcal{Z}(G)$. By [12, Lemma 5.20].

□

On nice index 2 *subgroups.* **At the end still see if this subsection is required !**

To prove Corollary 2.22, we first need the following result of independent interest which will also be instrumental in Section 2.4.

Lemma 2.9. *If $\mathbb{Q}G$ has* $(\mathrm{M}_{\mathrm{exc}})$*, then so does $\mathbb{Q}H$, for any subnormal subgroup H of G.*

Proof. It suffices to prove that $(\mathrm{M}_{\mathrm{exc}})$ is inherited by any normal subgroup H of G, since then a recursive argument finishes the proof.

Consider a simple $\mathbb{Q}H$-module N, and decompose the induced $\mathbb{Q}G$-module $\mathrm{Ind}_H^G(N) = M_1 \oplus \ldots \oplus M_t$ into simple $\mathbb{Q}G$-modules. Then N is a direct summand of the restriction $\mathrm{Res}_H^G \mathrm{Ind}_H^G(N)$, and hence of one of the $\mathbb{Q}H$-modules $\mathrm{Res}_H^G(M_i)$. Therefore, in order to describe the simple $\mathbb{Q}H$-components of a $\mathbb{Q}G$-module, it suffices to investigate the modules $\mathrm{Res}_H^G(M)$ with M a simple $\mathbb{Q}G$-module. In the case at hand, we are interested in describing the simple $\mathbb{Q}H$-components of the $\mathbb{Q}H$-module $\mathrm{Res}_H^G(\mathbb{Q}G)$ obtained from the regular $\mathbb{Q}G$-module. The simple $\mathbb{Q}G$-components M of $\mathbb{Q}G$ are columns of matrix algebras $\mathrm{M}_n(D)$, and since $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ by assumption, the form of $\mathrm{M}_n(D)$ is described by Theorem 2.6.

We recall Clifford's theorem, **[cite]**, which, since H is normal in G, implies that M decomposes as a $\mathbb{Q}H$-module into a direct sum

$$\mathrm{Res}_H^G(M) \simeq N_1 \oplus \cdots \oplus N_r$$

of simple $\mathbb{Q}H$-modules N_i which are all in the same G-orbit. Thus for all $1 \leq i, j \leq r$, $N_i = g \cdot N_j$ for some $g \in G$. In particular all N_i have the same $\mathbb{Q}$-dimension. Each N_i is a column of a matrix algebra $\mathrm{M}_{n_i}(D_i)$, with $\mathrm{M}_{n_i}(D_i)$ a component of $\mathbb{Q}H$. We investigate the N_i appearing in the above decomposition of M, for each possible simple $\mathbb{Q}G$-component M.

To be continued ☺ □

Proof of Corollary 2.22. By Theorem 2.6, G has character degrees $\mathrm{cd}(G) \subseteq \{1,2,4\}$. If G is non-nilpotent, then by [9, Theorem 1.1], either G has a subgroup H of index 2 such that $\mathrm{cd}(H) \subseteq \{1,2\}$, or $G/\mathcal{Z}(G) \cong (C_3 \rtimes C_2) \wr C_2$. But a calculation using GAP, shows immediately that $\mathbb{Q}((C_3 \rtimes C_2) \wr C_2)$ does not have the $(\mathrm{M}_{\mathrm{exc}})$ property, and in particular this would imply by Lemma 2.8 that $\mathbb{Q}G$ does not have $(\mathrm{M}_{\mathrm{exc}})$, which is a contradiction with the assumption. Hence G has a subgroup H of index two with $\mathrm{cd}(H) \subseteq \{1,2\}$. Now since H is normal, Lemma 2.9 implies that $\mathbb{Q}H$ has $(\mathrm{M}_{\mathrm{exc}})$. If $\mathbb{Q}H$ had a component of the form $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$, this would imply[9] that $4 \in \mathrm{cd}(H)$, a contradiction. Hence $\mathbb{Q}H$ has the desired components.

Suppose now that G is nilpotent. □

2.3. **Groups with low character degrees.** This section and the next aim to give a precise description of the groups satisfying $(\mathrm{M}_{\mathrm{exc}})$. By Theorem 2.6 we should first consider the more general class of groups with the character degrees of the irreducible complex characters all divisors of 4. We denote by $\mathrm{cd}(G)$ the set of character degrees of irreducible complex representations of G. In this section we focus on the groups G with $\mathrm{cd}(G) = \{1,4\}$.

More generally, whenever $\mathrm{cd}(G) \subseteq \{1, p^j, p^k\}$ then G is solvable of derived length at most 3 [10, Theorem 12.15]. Furthermore, $(\chi(1), q) = 1$ for any prime q different from p and any $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. This allows to apply the Ito-Michler theorem [10, Corollary 12.34] which in this case yields that

$$G \cong A \rtimes Q, \tag{2.8}$$

[9] Add justification...

where A is an abelian p'-subgroup and Q a Sylow p-subgroup of G. More precisely, A is the direct product of all Sylow q-subgroups of G, for all $q \neq p$, which are normal and abelian under the above assumption on $\mathrm{cd}(G)$.

Using character theory and [9], the following results will give further restrictions on the decomposition (eq. (2.8)) in the case that $|\mathrm{cd}(G)| = 2$. **merge following with 2.13 in order to have a clean theorem for the case Q is abelian.**

Proposition 2.10. *Let G be a non-nilpotent group such that $\mathrm{cd}(G) = \{1, p^k\}$, with $k > 1$. Then the Fitting subgroup $F(G)$ of G is the unique maximal abelian subgroup of index p^k. Furthermore, all Sylow subgroups of G are abelian, and G decomposes as a semidirect product*

$$G \cong N \rtimes C,$$

with N an abelian subgroup of $F(G)$ and C a 2-generated p-group $\langle x, y\rangle$, with x acting with order p^k, and y central in G. Conversely, a group G as above with $|\mathrm{cd}(G)| = 2$ must have $\mathrm{cd}(G) = \{1, p^k\}$.

Remark 2.11. The proof of Proposition 2.10 will also yield extra restrictions on the structure of G. For example $F(G) = C_G(A) \cong Z(G) \times [G, G]$ by (2.9) and (2.10). Additionally, denoting $C = \langle x, y\rangle$ and using [9, Theorem 3.1.(iii)], one can prove that the action of x^i, for $1 \leq i \leq p^k$, on $[G, G]$ is free.

Proof of Proposition 2.10. As noticed earlier G has the form (2.8) with A and Q as described there. If Q is non-abelian, then [10, Exercise 12.6] implies that G is nilpotent, which is in contradiction with the hypothesis. Hence Q is abelian.

Now we show that the Fitting subgroup is the centraliser of A, i.e. $F(G) = C_G(A)$. By[10] [9, Theorem 2.2 (ii)], $C_G(A)$ is a normal abelian subgroup of G. As $F(G)$ is the unique maximal normal nilpotent subgroup, the inclusion $C_G(A) \subseteq F(G)$ follows. But by [25, Lemma 1.2 (a)], there is some character χ of G such that $[G : F(G)] = \chi(1)$. Since G is non-nilpotent by assumption, we obtain that

$$p^k = [G : F(G)] \leq [G : C_G(A)] = p^k,$$

where the last equality follows from [9, Theorem 2.2 (ii)]. Thus in particular

$$C_G(A) = F(G). \tag{2.9}$$

Next, since Q is abelian one can apply [9, Theorem 3.1 (ii)], implying that $G/F(G)$ is cyclic of order p^k. As G is of the form (2.8), we can choose an $x \in Q$ such that $\bar{x} := xF(G)$ generates $G/F(G)$. As $\bar{x}$ has order p^k, it follows that $F(G) \cap \langle x\rangle = \langle x^{p^k}\rangle$.

Let $\langle y\rangle$ be the cyclic subgroup of $F(G)$ containing x^{p^k} and which is maximal amongst the cyclic subgroups of $Q \cap F(G)$ for this property. As Q is abelian, there exists a $M \leqslant Q$ such that $Q \cap F(G) \cong M \times \langle y\rangle$. Note that since $F(G)$ is abelian, eq. (2.8) implies that

$$F(G) = A \times (Q \cap F(G)),$$

and it follows that

$$F(G) = N \times \langle y\rangle, \text{ with } N = A \times M.$$

We conclude the proof by showing that

$$G \cong N \rtimes \langle x, y\rangle.$$

Firstly, it is clear that $N \cap \langle x, y\rangle = \{1\}$, since if $t \in N \cap \langle x, y\rangle$, then $t = x^r y^s$ for some $r, s \in \mathbb{Z}$. But then $ty^{-s} \in F(G)$, which since $F(G) \cap \langle x\rangle = \langle x^{p^k}\rangle$, implies that $ty^{-s} = (x^{p^k})^\ell$ for some $\ell \in \mathbb{Z}$. But $\langle x^{p^k}\rangle \leqslant \langle y\rangle$, and in particular there exists an $s' \in Z$ with $t = y^{s'}$. But $N \cap \langle y\rangle = \{1\}$ by construction, and hence $t = 1$.

[10]The fact that the Sylow p-subgroup Q is abelian yields that G is a $\{p\}$-character group as in [9, Definition 2.1.]

Additionally, $N \,.\, \langle x, y\rangle = G$, because

$$|N||\langle x,y\rangle| = |N||\langle y\rangle|\frac{|\langle x\rangle|}{|\langle x^{p^k}\rangle|} = |F(G)|[G:F(G)].$$

Finally, we show that N is normal in G. Combining [9, Theorem 3.1 (iii)] with eq. (2.9), [25, Lemma 1.6 (d)] now implies that

$$F(G) = C_G(A) \cong \mathcal{Z}(G) \times G'. \tag{2.10}$$

Since $N = A \times M$, and A is normal in G by construction, it suffices to show that M is normal. We show that in fact $\langle y\rangle \times M$ is even central in G. Indeed, if $z \in \langle y\rangle \times M \leqslant F(G)$, it may be written as $z = z_1 z_2$ for unique $z_1 \in \mathcal{Z}(G)$ and $z_2 \in G'$ by the above. Furthermore, since Q is an abelian Sylow p-subgroup and so $G' \leqslant A$, one has that $(o(z_2), p) = 1$. But by definition z has order a power of p, and it follows that

$$\langle z\rangle = \langle z^{o(z_2)}\rangle \subseteq \mathcal{Z}(G),$$

as claimed. Finally, the action of x on N is of order p^k, since x^{p^k} is the smallest non-trivial power of x which commutes with N, because $F(G) = \langle N, x^{p^k}\rangle$ is the maximal normal abelian subgroup.

For the converse, as $[G : F(G)] = p^k$ and $F(G)$ is a normal abelian subgroup, Ito's theorem [10, Theorem 6.15] yields that $\chi(1)$ divides p^k for all irreducible characters χ. Note also that $G' \leq N \leq F(G)$, hence by [8, Lemma 1] there exists some $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ with $\chi(1) = [G : F(G)] = p^k$. The assumption $|\operatorname{cd}(G)| = 2$ concludes the proof. □

Example 2.12. Consider the group $G := C_5 \rtimes C_8 = \langle a, b \mid a^5, b^8, a^b = a^3\rangle$. It is easily shown that $\operatorname{cd}(G) = \{1, 4\}$ and $F(G) = C_G(\langle a\rangle) = \langle a, b^4\rangle$. Thus, in the notation of Proposition 2.10, $N = \langle a\rangle$ and $C = \langle b\rangle$. We see that in this example one can indeed not find a complement for $F(G)$ itself. Thus the splitting in Proposition 2.10 is the finest possible in general.

Proposition 2.13. *Let G be a non-nilpotent finite group such that $\operatorname{cd}(G) \subseteq \{1, p^i, p^k\}$ with $i < k$. Using the notations from (2.8), if Q is abelian then the following hold:*

(1) $C_G(A) = F(G) = \mathcal{Z}(G) \times G'$,
(2) $F(G)$ is the unique maximal abelian subgroup, of index p^k,
(3) $G/C_G(A)$ is either cyclic of order p^k or isomorphic to $C_{p^i} \times C_{p^{k-i}}$.

Proof. As explained at the beginning of the section, the group G has the form (2.8) with A and Q as described there, and G is solvable. Since Q is assumed abelian, in particular G is solvable of derived length at most 2. It follows from the work of Taunt, [7, VI, Satz 14.7 b)] that $F(G) = \mathcal{Z}(G) \times G'$. Additionally, $C_G(A)$ and $G/C_G(A)$ are abelian and in particular it follows from [7, VI, Satz 14.7 a)] that $C_G(A) = \mathcal{Z}(G) \times (C_G(A) \cap G') = \mathcal{Z}(G) \times G'$. It now also immediately follows that $F(G)$ is the unique maximal *abelian* subgroup. By the fact that G is non-nilpotent by assumption, we obtain as in the proof of Proposition 2.13 that $[G : F(G)] \in \{p^i, p^k\}$, and $[G : C_G(A)] = p^k$. Hence $F(G)$ is of index p^k. Now, by [9, Theorem 2.2], $C_G(A)$ is either isomorphic to C_{p^k} or to $C_{p^i} \times C_{p^{k-i}}$. □

Next consider the nilpotent case. Then the decomposition in (2.8) is a direct product $G \cong A \times Q$. A precise classification in the nilpotent case seems hard. Nevertheless note that $\{1, 4\} = \operatorname{cd}(G) = \operatorname{cd}(Q)$. Now applying [9, Theorem 3.10 & Lemma 5.4] yields the following.

Lemma 2.14. *Let G be a nilpotent group with $\operatorname{cd}(G) = \{1, 4\}$, then $G \cong A \times Q$ with A an odd abelian group and Q a 2-group satisfying the following:*

(1) Q has nilpotency class 2,
(2) $[Q, Q]$ and $Q/\mathcal{Z}(Q)$ are elementary abelian 2-groups.

Remark 2.15. If $\operatorname{cd}(G) = \{1, 2, 4\}$, then the statement of Lemma 2.14 does not hold in general. For example the group

$$C_8 \rtimes (C_2 \times C_2) := \langle a, b, c \mid a^8 = b^2 = c^2 = 1, bab = a^3, cac = a^5, bc = cb\rangle$$

has[11] $\mathrm{cd}(G) = \{1, 2, 4\}$ but is nilpotent of class 3 and $G' \cong C_4$.

2.4. **Characterisation of the groups.** In this section we will give a complete characterisation of groups satisfying property $(\mathrm{M}_{\mathrm{exc}})$. Recall from (2.8) that $G \cong A \rtimes Q$ with A an odd abelian group and Q a 2-group. Denote by $\varphi : Q \to \mathrm{Aut}(A)$ the action of Q on A. In Theorem 2.6 it was proven that $\mathrm{cd}(G) \subseteq \{1, 2, 4\}$. In [19] the groups H such that the components $\mathbb{Q}H$ are of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{Z}_{\geq 0}$ and there is no exceptional division components have been addressed. We will not require the latter condition and note that the former condition yields the restriction $\mathrm{cd}(G) \subseteq \{1, 2\}$. Now whether we are in the more general setting where $4 \in \mathrm{cd}(G)$ can be read off from both the Wedderburn-Artin decomposition of $\mathbb{Q}G$ and from G. If G *is not-nilpotent:*

$$\begin{aligned} 4 \in \mathrm{cd}(G) \quad &\Leftrightarrow \quad \mathrm{M}_2\left(\left(\frac{-a,-b}{\mathbb{Q}}\right)\right) \in \mathcal{C}(\mathbb{Q}G) \text{ for some a,b} > 0 \\ &\Leftrightarrow \quad \text{maximal abelian normal subgroup has index } 4 \end{aligned}$$

The last equivalence will follow from Lemma 2.20 and the first is also true for nilpotent groups. If G *is nilpotent*, then $4 \in \mathrm{cd}(G)$ exactly when G is a 2-group with $G/\mathcal{Z}(G) \cong C_2^n$ with $n \geq 4$ or $cl(G) = 3$ and the maximal abelian normal subgroup has index 4.

2.4.1. *A characterization via Strong Shoda Pairs.* **put this section later? I.e. so long necessary conditions don't use it, then wait with it. Maybe also before the description of the SSP.**

To start we study $\mathrm{PCI}(\mathbb{Q}G)$. Recall that G is called *strongly monomial* if each primitive central idempotent e of $\mathbb{Q}G$ comes from a SSP, i.e. $e = e(G, H, K)$ for some SSP (H, K). For example all abelian-by-supersolvable groups are strongly monomial [15, Theorem 3.5.10]. In particular, by Theorem 2.6, $(\mathrm{M}_{\mathrm{exc}})$ implies strongly monomial. Therefore we now describe in terms of (H, K) when a simple component associated to a SSP yields an exceptional component. This consequently also yields a characterization of $(\mathrm{M}_{\mathrm{exc}})$ in terms of the SSP.

We will use the terminology that "$\mathbb{Q}Ge$ *is* $(\mathrm{M}_{\mathrm{exc}})$*-exceptional* " to mean that $\mathbb{Q}Ge$ is an exceptional algebra that can occur as a simple component of a $\mathbb{Q}G$ with $(\mathrm{M}_{\mathrm{exc}})$. In other words if $\mathbb{Q}Ge$ is isomorphic to one of the simple algebras listed in Theorem 2.6. In particular if $\mathbb{Q}Ge$ is an exceptional division algebra, then it is isomorphic to $\left(\frac{\zeta_{2^t}, -3}{\mathbb{Q}(\zeta_{2^t})}\right)$.

Lemma 2.16. *Let (H, K) be a SSP of G with $H \neq G$ and denote $e := e(G, H, K)$ the associated primitive central idempotent.*

(1) If $N_G(K) = H$, then:

$$\mathbb{Q}Ge \textit{ is exceptional } \Leftrightarrow \phi([H : K]) = \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \leq 2 \textit{ and } [G : H] = 2,$$

(2) If $H \lneq N_G(K) \lneq G$, then $\mathbb{Q}Ge$ is exceptional if and only if $[H : K] = 4$ or 6, $[G : H] = 4$, and

$$\textit{for all } h \in H \setminus K,\, y \in N_G(K) \setminus H \;:\; h.h^y,\, y^2.h^{o(h)/2} \in K.$$

(3) For $N_G(K) = G$ we suppose $G/H \not\cong C_4$. Then $\mathbb{Q}Ge$ is $(\mathrm{M}_{\mathrm{exc}})$-exceptional matrix algebra if and only if one of the following cases holds[12]*:*

(3.i) $[G : H] = 2$, $[H : K] = 4$ or 6 and $h.h^y, y^2 \in K$ for all $h \in H \setminus K, y \in G \setminus H$.

(3.ii) $[G : H] = 2$, $[H : K] = 8$ or 12 and for some $h \in H$ such that $H/K = \langle h \rangle$ it holds

- *either $h^{-5}.h^y, y^2.h^{i.o(h)/4} \in K$ for some $0 \leq i \leq 3$,*
- *or $h^{-7}.h^y, y^2.h^{2j} \in K$ for some $0 \leq j \leq 4$ (only if $[H : K] = 12$)*

for all $y \in G \setminus H$.

for a choice of y! Tell the grp they yield rather !

[11]This group has Small Group Id [32,43] and the stated information can for example be retrieved via gap or the GroupNames database.

[12]$\phi([H : K]) \leq 2$ means that it divides 4 or 6. Furthermore, when $5 \nmid [H : K]$, $\phi([H : K]) \mid 4$ means that it divides 8 or 12.

(3.iii) $[G:H]=4$, $[H:K]=8$ *or* 12. *Furthermore, for all* $h \in H \setminus K$ *such that* $H/K = \langle h \rangle$ *and* $y \neq t \in G \setminus H$ *one has that*

$$y^4 \in K,\, h^y.(h^{-1})^t \notin K$$

and $y^2 h^{o(h)/2} \in K$ *if* $h.h^y \in K$.

The value of $\phi([H:K])$ in case (3.i)-(3.ii) can be summarized in a similar way as in case (1). Namely if $N_G(K) = G$ and $[G:H]=2$, then the proof yields that

$$\phi([H:K]) = 2 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \in \{2,4\}.$$

Also note that in case (3.iii) the condition that $h^y K \neq h^t K$ implies that for every $1 \neq d$ relatively prime to $o(h) = [H:K]$ there is a $t \in G \setminus H$ such that $h^t = h^d$.

Remark 2.17. The condition that $G/H \ncong C_4$ when K is normal in G is satisfied when $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$, hence sufficient for our purposes. However it could be removed by adding extra cases (whose description could be found via similar methods). To see that it is satisfied consider the abelian normal subgroup B from Theorem 2.6. As $\exp(G/B) = 2$ it contains the commutator subgroup G'. Consequently, Theorem 2.5 tells that $B \leq H$ which entails $\exp(G/H) = 2$ and hence $G/H \ncong C_4$.

Note also that verifying when $\mathbb{Q}Ge$ is $(\mathrm{M}_{\mathrm{exc}})$-exceptional hides the assumption that $5 \nmid [H:K]$. Indeed, as $\ker(\varphi_e \colon G \to Ge) = core_G(K) \leq K$ one has that $[H:K] = [\varphi_e(H) : \varphi_e(K)]$. Now Theorem 2.6 tells us that $\pi(G) \subseteq \{2,3\}$ and thus $5 \nmid [\varphi_e(H) : \varphi_e(K)]$. Also this case could be handled by adding extra cases.

Remark 2.18. The proof of Lemma 2.16 will furthermore tell the isomorphism type of $\mathbb{Q}Ge(G,H,K)$ for each case. More precisely, the following holds:

- $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q})$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(i))$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-2})$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-3})$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\left(\frac{-1,-1}{\mathbb{Q}}\right))$ if and only if ...
- $\mathbb{Q}Ge \cong \mathrm{M}_2(\left(\frac{-1,-3}{\mathbb{Q}}\right))$ if and only if ...

Proof of Lemma 2.16. For the remainder of the proof denote $N := N_G(K)$ and $e := e(G,H,K)$. Recall that by Theorem 2.4

$$\mathbb{Q}Ge \cong \mathrm{M}_{[G:N_G(K)]}(\mathbb{Q}(\zeta_{[H:K]}) * N_G(K)/H)$$

with the crossed product defined in (2.4) and (2.5). Furthermore, $\deg(\mathbb{Q}Ge) = [G:H]$ and hence if $\mathbb{Q}Ge$ is an exceptional matrix algebra one needs that $[G:H] \mid 4$.

To start suppose that $N = H$. Then $\mathbb{Q}Ge \cong \mathrm{M}_{[G:H]}(\mathbb{Q}(\zeta_{[H:K]}))$. As $G \neq H$, this is exceptional if and only if $[G:H]=2$ and $[\mathbb{Q}(\zeta_{[H:K]}) : \mathbb{Q}] \leq 2$. Note that $\dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) = [\mathbb{Q}(\zeta_{[H:K]}) : \mathbb{Q}] = \phi([H:K])$, implying statement (1).

Next consider $H \lneq N \lneq G$. Since $N \lneq G$ the component $\mathbb{Q}Ge$ is not a division algebra. Hence, as noticed earlier, if exceptional the index $[G:H]$ divides 4. Since also $H \lneq N$, we obtain that $[G:N] = 2 = [N:H]$. Now note that these indices imply that $\mathbb{Q}Ge$ is exceptional if and only if it is isomorphic to $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some $a,b \in \mathbb{N}_0$. Using the dimension formula in Theorem 2.4 we obtain that

$$16 = \dim_{\mathbb{Q}} \mathbb{Q}Ge = 4.2.\phi([H:K]).$$

Thus $\phi([H:K]) = 2$ and hence $[H:K] = 3, 4$ or 6. For the other conditions we need to be more precise. First denote $m := [H:K]$, $N/H = \langle \overline{y} \rangle$ and $H/K = \langle hK \rangle$. Now remark that

$$\mathbb{Q}(\zeta_m) * N/H = \mathrm{span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\}.$$

By (2.4) one has that $\zeta_m^{\overline{y}} = \zeta_m^i$ with i such that $h^i.h^y \in K$. Furthermore, using that $[N : H] = 2$, the crossing (2.5) tells us that $\overline{y}^2 = \zeta_m^j$ with j such that $y^2h^{-j} \in K$. In summary, as an abstract group we obtained that

$$\langle \zeta_m, \overline{y}\rangle \cong \langle a, b \mid a^m = 1, b^2 = a^j, a^b = a^i\rangle.$$

Recall that, when $[G : N] = 2 = [N : H]$, we already knew that $\mathbb{Q}Ge$ will be exceptional exactly when $\mathbb{Q}(\zeta_m) * N/H \cong \left(\frac{-a,-b}{\mathbb{Q}}\right)$ for some $a, b \in \mathbb{N}_0$. This will yield the remaining conditions on (m, j, i). Indeed, looking at the centre $\mathbb{Q} = \mathcal{Z}(\mathbb{Q}(\zeta_m) * N/H)$, combined with $m \mid 4$ or 6, we see that necessarily $i = -1$. For the other restrictions the required observation is that one needs that $\mathcal{C}(\mathbb{Q}[\langle\zeta_m, \overline{y}\rangle])$ contains such a quaternion algebra. Now if $a^j = 1$ then $\langle\zeta_m, \overline{y}\rangle \cong D_{2m}$. However $\mathcal{C}(\mathbb{Q}D_{2m}) = \{\mathbb{Q}, \mathrm{M}_2(\mathbb{Q})\}$ for those m, so $a^j \neq 1$. If $a^j = a^{\pm 1}$, then $\langle\zeta_m, \overline{y}\rangle$ is cyclic and so its group algebra has only commutative components. In particular, for $m = 3$ there is no admissible j, yielding $[H : K] = 4$ or 6, as desired. And for $m = 4$ we necessarily have that $b^2 = a^2 = a^{o(a)/2}$. Finally, for $m = 6$ if $i = 2, 4$ then $\langle\zeta_m, \overline{y}\rangle \cong C_2 \times C_2$ abelian which yield $i = 3$ as desired. Altogether we obtained that $\langle\zeta_m, \overline{y}\rangle \cong Q_8$ if $m = 4$ and to $Dic_{4.3} := C_3 \rtimes C_4$ if $m = 6$. Inspecting their simple component, we conclude that the former case ($m = 4$) yields $\mathrm{span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\} = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ and the latter ($m = 6$) that $\mathrm{span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\} = \left(\frac{-1,-3}{\mathbb{Q}}\right)$ (to conclude that $\mathrm{span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\} \neq \mathrm{M}_2(\mathbb{Q})$, which is another simple component of $\mathbb{Q}[Dic_{12}]$ we use that $\langle\zeta_m, \overline{y}\rangle \cong Dic_{12}$ must be embedded in the component). Hence the conditions written are also enough to conclude that $\mathbb{Q}Ge$ is exceptional.

Finally suppose that K is normal in G. Then $\mathbb{Q}Ge \cong \mathbb{Q}(\zeta_{[H:K]}) * G/H$. We will obtain the desired statement via the same methods as in the case $H \lneq N \lneq G$. Recall $\deg(\mathbb{Q}Ge) = [G : H]$ which need to divide 4 in order for $\mathbb{Q}Ge$ to be an exceptional matrix algebra.

First suppose that $[G : H] = 2$. Hence $\mathbb{Q}Ge$ is exceptional matrix component if and only if $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{N}$. Therefore, $4 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) = \dim_{\mathbb{Q}} \mathbb{Q}Ge = 2.1.\phi([H : K])$ and consequently

$$\phi([H : K]) = 2 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \in \{2, 4\}. \tag{2.11}$$

The value $\phi([H : K]) = 4$ is equivalent to $[H : K] \in \{5, 8, 10, 12\}$. As we assumed that $5 \nmid [H : K]$ we conclude that $[H : K] = 8$ or 12 when $\phi([H : K]) = 4$. We consider both values in (2.11) separately.

Firstly *suppose that* $\phi([H : K]) = 2$, i.e. $[H : K] = 3, 4$ or 6 and $\mathcal{Z}(\mathbb{Q}Ge) = \mathbb{Q}$. In this case $\mathbb{Q}Ge$ is an exceptional matrix component if and only if $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q})$. For this we need to describe $\mathbb{Q}Ge \cong \mathbb{Q}(\zeta_{[H:K]}) * N/H$. Note that $\phi([H : K]) = 2 = [N : H]$ as in case (2). Therefore the computations are literally the same as there. The only difference is that now we need to understand which of the groups $\langle\zeta, \overline{y}\rangle$ yield that $\mathrm{span}_{\mathbb{Q}}\{\zeta_m, \overline{y}\} = \mathrm{M}_2(\mathbb{Q})$. Looking back to the cases we see that this exactly happens when $\langle\zeta, \overline{y}\rangle \cong D_{2m}$, whose presentation represents the relations mentioned in case (3.i).

Next *suppose that* $\phi([H : K]) = 4$, i.e. $[H : K] = 8$ or 12 and $[\mathcal{Z}(\mathbb{Q}Ge) : \mathbb{Q}] = 2$. **to write**

It remains to consider *the case that* $[G : H] = 4$. Then $\mathbb{Q}Ge$ is an exceptional matrix component if and only if $\mathbb{Q}Ge \cong \mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some $a, b \in \mathbb{N}_0$. The dimension formula now gives $16 = 4.1.\phi([H : K])$, i.e. $\phi([H : K]) = 4$. As noticed earlier, as $5 \nmid [H : K]$, this means that $[H : K] = 8$ or 12. For the stated relation we again follow the same strategy. **part proof still need to write**

□

2.4.2. *Description of Strong Shoda Pairs and main necessary conditions.*

Lemma 2.19. *Let G be a finite group such that $\mathbb{Q}G$ has* $(\mathrm{M}_{\mathrm{exc}})$*. Then the following holds:*

(1) $G \cong A \rtimes Q$ with A an elementary abelian 3-group and Q a 2-group,

(2) $[G : F(G)] = [Q : \mathrm{Ker}(\varphi)]$ *divides* 2,
(3) G *contains an abelian normal subgroup* B *such that* $G/B \cong C_2^m$ *with* $m = 1$ *or* 2,
(4) $|Q' \cap \mathcal{Z}(Q)| \leq 2$ *if* G *not nilpotent.*

The condition that $[G : F(G)] \mid 2$, together with the form of G in (1), can be interpreted very concretely. If $[G : F(G)] = 1$, then clearly G is nilpotent and $A \subseteq \mathcal{Z}(G)$ as A is abelian and $G \cong A \rtimes Q$. More generally one can decompose A as

$$A = A_{-1} \times (A \cap \mathcal{Z}(G)) \tag{2.12}$$

with $A_{-1} := \{a \in A \mid a^y = a^{-1} \text{ for all } y \in G \setminus F(G)\}$. As $F(G) = \mathrm{Cen}_G(A)$ and $[G : F(G)] \mid 2$ it suffices to verify the condition $a^y = a^{-1}$ for a single element $y \in G \setminus F(G)$.

To see why (2.12) holds, notice that for a given $a \in A$ and $y \notin \mathrm{Cen}_G(\langle a \rangle)$ one has that $a^y = a^{-1}$ or $a^y \notin \langle a \rangle$ since $o(a) = 3$ and $y^2 \in F(G)$. Now the elements in A for which $\langle a \rangle$ is not normalized can be obtained from an element in A_{-1} and $A \cap \mathcal{Z}(G)$. Indeed, if $a^y \notin \langle a \rangle$ then $a^{-1}a^y \in A_{-1}$ and $aa^y \in A \cap \mathcal{Z}(G)$. Furthermore, $a = (a^{-1}a^y).(aa^y)^{-1}$ which proves (2.12).

The decomposition (2.12) has the advantage that we can describe more precisely the commutator subgroup G' of G:

$$G' = A_{-1} \times Q'. \tag{2.13}$$

Indeed, as $G \cong A \rtimes Q$ one has that $G' = \langle [A, Q], Q' \rangle$. Now if **TO COMPLETe**

Proof Lemma 2.19. **Add first a proof of (1) for nilpotent** Next suppose that G is non-nilpotent. Then we show statement (1) by induction on the cardinality of Q. Indeed, suppose that when $|Q| = 2$, A is an elementary abelian 3-group. Now suppose $G = A \rtimes Q$ has ($\mathrm{M}_{\mathrm{exc}}$) for some 2-group Q. Let $P \leqslant Q$ of index 2, and define the subgroup $H = A \rtimes P$. Then H is a normal subgroup of G, since it is of index 2. In particular, H has ($\mathrm{M}_{\mathrm{exc}}$) by Lemma 2.9. **we don't have it yet !** If H is non-nilpotent, it follows by the induction hypothesis that A is an elementary abelian 3-group. If H is nilpotent, then $H = A \times P$, and in particular the kernel $\mathrm{Ker}(\varphi)$ of the action of Q on A contains P. Since G is non-nilpotent by assumption and P has index 2 in Q, $\mathrm{Ker}(\varphi) = P$. Then $G/\mathrm{Ker}(\varphi) \cong A \rtimes (Q/\mathrm{Ker}(\varphi)) \cong A \rtimes C_2$. By Lemma 2.8, $G/\mathrm{Ker}(\varphi)$ has ($\mathrm{M}_{\mathrm{exc}}$), and it follows by assumption that A is an elementary abelian 3-group.

We proceed to show that when $Q \cong C_2$, A is indeed an elementary abelian 3-group. Since A is abelian, in particular it is given by $A = \prod_{q \in \pi(A)} A_q$, with A_q the Sylow q-subgroup of G, which is a characteristic subgroup of G for every $q \in \pi(A)$. We can choose $p \in \pi(A)$ such that G/R_p, with $R_p := \prod_{q \in \pi(A) \setminus \{p\}} A_q$, is non-nilpotent. Indeed, if $G/R_p \cong A_p \rtimes C_2$ were nilpotent for every $p \in \pi(A)$, it would follow that C_2 acts trivially on every Sylow q-subgroup of G, and in particular $G \cong A \times C_2$, a contradiction with the non-nilpotency of G. Now from [9, Lemma 1.2] it follows that $A_p \cong (A_p \cap \mathcal{Z}(A_p \rtimes C_2)) \times B_p$ for some characteristic subgroup $B_p \leqslant A_p \rtimes C_2$. Additionally, B_p is non-trivial, since otherwise $A_p \leqslant \mathcal{Z}(A_p \rtimes C_2)$, a contradiction with the assumption on p. Now,

$$B_p \rtimes C_2 \cong G/\left(R_p \cdot (\mathcal{Z}(A_p \rtimes C_2) \cap A_p)\right),$$

and in particular $B_p \rtimes C_2$ has ($\mathrm{M}_{\mathrm{exc}}$) by Lemma 2.8. Let $x \in B_p$ have maximal order, say $o(x) = p^m$. Then there is a subgroup $K \leqslant B_p$ such that $B_p \cong \langle x \rangle \times K$ (since $B_p \leqslant A_p$ is abelian). We claim that (B_p, K) is a strong Shoda pair for $B_p \rtimes C_2$. Indeed, to find an $H \leqslant B_p \rtimes C_2$ such that (H, K) is a SSP for $B_p \rtimes C_2$, by [16, Theorem 3.5.12] it suffices to show that B_p is a maximal element in the set

$$S := \{D \leqslant B_p \rtimes C_2 \mid B_p \leqslant D \text{ and } D' \leqslant K \leq D\}.$$

Since $[B_p \rtimes C_2 : B_p] = 2$, it follows that $D \in S$ can only occur if $D = B_p$ or $D = B_p \rtimes C_2$. We claim that $(B_p \rtimes C_2)' \not\leqslant K$, and hence $D \in S$ if and only if $D = B_p$. Indeed, for any $\bar{a} \in B_p$, let $a \in A_p$ such that $a \mapsto \bar{a}$ under the quotient map $\varpi \colon G \to B_p \rtimes C_2$. Let $C_2 = \langle y \rangle$. Then aa^y commutes with y: $(aa^y)^y = a^y a^{y^2} = a^y a = aa^y$, where the last equality follows since $a^y, a \in A_p$ by definition, and A_p is abelian. Hence $aa^y \in \mathcal{Z}(A_p \rtimes C_2)$. It follows by definition of B_p that $\bar{a}\bar{a}^y = \varpi(aa^y) = 1$. In particular, $\bar{a}^y = \bar{a}^{-1}$ in $B_p \rtimes C_2$. We conclude that $[\bar{a}, y] = \bar{a}^{-1}\bar{a}^y = \bar{a}^{-2} \in \langle \bar{a} \rangle$. In particular, for $\bar{a} = x$, it follows that $[x, y] \notin K$, and more generally we obtain that $y \in N_{B_p \rtimes C_2}(B_p)$.

Combining the latter fact with [11, Lemma 3.4], it follows that

$$\begin{aligned}\dim_{\mathbb{Q}}(\mathbb{Q}[B_p \rtimes C_2]e(B_p \rtimes C_2, B_p, K)) &= [B_p \rtimes C_2 : B_p] \cdot [B_p \rtimes C_2 : N_{B_p \rtimes C_2}(K)] \cdot \phi([B_p : K]) \\ &= 2 \cdot 1 \cdot p^{m-1}(p-1).\end{aligned}$$

However, since $B_p \rtimes C_2$ has ($\mathrm{M_{exc}}$) by construction, the above $\mathbb{Q}$-dimension is an element of $\{4, 8, 16\}$. It follows that $p \in \{3, 5\}$ and $m = 1$. In particular, by definition of m, A_p is an elementary abelian p-group. Suppose $p = 5$. Then since y acts on B_p by inversion, it follows from [16, Theorem 3.5.5 (4) and Remark 3.5.6] that

$$\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) \subseteq \mathcal{Z}(\mathbb{Q}[B_p \rtimes C_2]e(B_p \rtimes C_2, B_p, K)).$$

However, $|\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) : \mathbb{Q}| = 2$, and $\mathbb{Q}(\zeta_5 + \zeta_5^{-1})$ is a totally real field. But by Theorem 2.6, if $B_p \rtimes C_2$ has ($\mathrm{M_{exc}}$), the centres of its 1×1 components intersected with $\mathbb{R}$ should equal $\mathbb{Q}$. Since $B_p \rtimes C_2$ has ($\mathrm{M_{exc}}$) by construction, we obtain a contradiction. Hence A_p is an elementary abelian 3-group.

In summary, we have proven that G decomposes as $G \cong A_{3'} \times (A_3 \rtimes Q)$ with A_3 an elementary abelian subgroup. **add why there can't be $p \geq 5$ in the center (or certainly not as direct factor), maybe by refering to previous result. Hence $A = A_3$.**

Next we prove statement (2). By Corollary 2.22 G has a subgroup H of index 2 such that all its non-division components are of the form $\mathrm{M}_n(\mathbb{Q}(\sqrt{-d}))$. In particular, $\mathrm{cd}(H) \subseteq \{1, 2\}$ by Theorem 2.6. Hence, from [2, Theorem 3] it follows that either H has an abelian subgroup $B \leqslant H$ of index 2, or $H/\mathcal{Z}(H) \cong C_2 \times C_2 \times C_2$. In the latter case, $H' \subseteq \mathcal{Z}(H)$, and in particular H is nilpotent of class 2. Thus independent of the case, H contains a nilpotent subgroup C of index at most 2 in H. Hence there exists a nilpotent subgroup $C \leqslant G$ such that $[G : C] \mid 4$. Now it follows that $A \leqslant C$, since $|A|$ is of odd order. In particular, $C = A \times (C \cap Q) \leqslant A \times \mathrm{Ker}(\varphi)$ since C is nilpotent. But then $[Q : \mathrm{Ker}(\varphi)] \mid 4$, since

$$[G : C] = [G/A : C/A] = [Q : (C \cap Q)] = [Q : \mathrm{Ker}(\varphi)][\mathrm{Ker}(\varphi) : (C \cap Q)],$$

and $[G : C] \mid 4$. The fact that $[Q : \mathrm{Ker}(\varphi)] = [G : F(G)]$ follows immediately since $A \leq F(G)$. ☐

In order to obtain more essential properties *we describe the tuples (H, K) that yield a strong Shoda pair for $\mathbb{Q}G$ such that $\mathbb{Q}Ge(G, H, K)$ is not commutative.* Recall that the latter, see [13, Lemma 2.4], means that $H \neq G$ or equivalently that $G' \nsubseteq K$.

By Lemma 2.19 and Theorem 2.5 the group H contains a maximal abelian normal subgroup B such that $G/B \cong C_2$ or $C_2 \times C_2$. Consequently, we need to describe for $H = B$ and $H = \langle B, t \rangle$, with $t \in Q \setminus (B \cap Q)$, what the possible subgroups K are such that (H, K) is a SSP. For this we need that H/K is cyclic. As $G \cong A \rtimes Q$ with A an elementary abelian 3-group and Q a 2-group, one is able to describe the possible groups K by saying which $a \in A$ and $c \in Q$ (and until which power) 'survive' in H/K. We now formalize this, which requires to define the following sets:

$$\mathcal{M}_{B_2} := \{x \in Q \cap B \mid \langle x \rangle \text{ is maximal cyclic in } Q \cap B\},$$

$$\mathcal{S}_B := \{(x, m) \in (Q \cap B) \cup \{1\} \times \mathbb{N} \mid 1 \neq x \in \mathcal{M}_{B_2} \text{ and } 0 \leq m \leq v_2(o(x))\}$$

and

$$\mathcal{S}_{tB} := \{(x, m) \in t(Q \cap B) \times \mathbb{N} \mid m = \begin{cases} 1, & \text{if } x^2 \notin \mathcal{M}_{B_2}, \\ \in [0, v_2(o(x))], & \text{else.} \end{cases} \}$$

To a triple $(a,x,m) \in A \times (\mathcal{S}_B \cup \mathcal{S}_{tB})$ we associate a subgroup of H:

$$K_{(a,x,m)} := \langle A \setminus \{a^{\pm 1}\}\rangle \rtimes \left\langle ((H \cap Q) \setminus \langle x\rangle) \cup \{x^{2^m}\}\right\rangle . \tag{2.14}$$

It is readily verified that any subgroup K in a SSP (H,K) with $H \neq G$ is of the form $K_{(a,c,m)}$. We now say which triples yield which overgroup H.

(I) SSP with $H = B$:

a tuple $(B, K_{(a,x,m)})$ is a SSP if and only if

- $(a,x,m) \in A \times \mathcal{S}_B$,
- $G' \nsubseteq K_{(a,x,m)}$,
- and there is no $t \in \mathrm{Cen}_Q(\langle a\rangle) \setminus (Q \cap B)$ such that $x = t^2$.

(II) SSP with $H = \langle B, t\rangle$:

Consider a fixed $t \in Q \setminus (Q \cap B)$. Then a tuple $(\langle B,t\rangle, K_{(a,x,m)})$ is a SSP if and only if either

- $(a,x,m) \in A \times \mathcal{S}_t$,
- $G' \nsubseteq K_{(a,c,m)}$,
- and $x \in \langle t^2\rangle$.

or

- $(a,x,m) \in A \times \mathcal{S}_{tB}$,
- and $G' \nsubseteq K_{(a,x,m)}$.

2.4.3. *On the Sylow* 2*-subgroup of a* $(\mathrm{M}_{\mathrm{exc}})$ *group.* With the description of the SSP just obtained we now have the necessary tools to obtain the remaining essential necessary conditions for $\mathbb{Q}G$ to have $(\mathrm{M}_{\mathrm{exc}})$. For one of the properties the following two groups will appear:

$$(C_2 \times C_2) \rtimes C_4 := \langle a,b,c \mid a^2 = b^2 = c^4 = 1, a^c = ab, b^c = b, ab = ba\rangle, \tag{2.15}$$

$$(C_2 \times C_2) \wr C_2 := \langle x_1,\dots,x_4,y \mid \forall i,j : y^2 = x_i^2 = 1,\ [x_i,x_j] = 1,\ x_1^y = x_3, x_2^y = x_4\rangle. \tag{2.16}$$

These groups have respectively SmallGroupId $[16,3]$ and $[32,27]$ **for nilpotent there is more! For example [32,34]...**

Lemma 2.20. *Let G be a finite group such that $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ and has a matrix component. Then the following holds:*

1. *The nilpotency class of Q is at most 3,*
2. *if Q is non-abelian, then $\mathbb{Q}G$ has no exceptional division components,*
3. *if Q is class 2, then $\exp(Q/\mathcal{Z}(Q)) = 2$. If G is not-nilpotent, then Q has an abelian subgroup of index 2,*
4. *if Q is class 3 then $Q/\mathcal{Z}(Q)$ is isomorphic to $C_2^n \times K$ with K one of the following groups*

$$D_8, \quad (C_2 \times C_2) \rtimes C_4, \quad \text{or } (C_2 \times C_2) \wr C_2.$$

Furthermore, the latter two groups only occur for G nilpotent with $4 \in \mathrm{cd}(G)$. Also, if G is non-nilpotent then $F(G)$ is abelian if and only if $\mathrm{cd}(G) = \{1,2\}$.

In the next section we will give necessary and sufficient conditions for $\mathbb{Q}G$ to have $(\mathrm{M}_{\mathrm{exc}})$ depending on nilpotency class of a Sylow 2-subgroup.

Remark 2.21. **keep this remark here?**

(i) Note that $cl(Q) \leq 3$ combined with property (1) yields that also the nilpotency class of the Fitting subgroup $F(G)$ is at most 3. It will turn out that in case that G is not-nilpotent, then even $cl(F(G)) \leq 2$.

(ii) Suppose that the abelian subgroup B yielded by **??** never has index 2 and hence $G/B \cong C_2 \times C_2$. When $cl(Q) = 2$ or $Q/\mathcal{Z}(Q) \cong D_8$, then there exists a $t \in G \setminus B$ such that $t \in C_Q(B \cap Q)$. Indeed, if $Q/\mathcal{Z}(Q) \cong D_8$, then there is a $t \in Q$ such that $\langle \mathcal{Z}(Q), t\rangle$ is an

abelian subgroup of index 2 in Q. As $[Q : B \cap Q] = 4$ this means that G is not nilpotent and $t \notin F(G) \supset B \cap Q$. But $exp(G/B) = 2$, so $t^2 \in B$ and hence $B = \langle \mathcal{Z}(Q), t^2 \rangle$. Thus indeed $t \in C_Q(B \cap Q)$. Now, if $cl(Q) = 2$, then **to complete**

Proof of Lemma 2.20. □

All finite groups such that $\mathrm{SL}_1(\mathbb{Q}Ge)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ have been classified in [19]. In loc.cit. presentations for such groups were even given. Furthermore, they showed that the aforementioned property is equivalent to saying that all simple components of $\mathbb{Q}G$ are either fields, totally definite quaternion algebras or of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \geq 0$. The following result shows that groups with $(\mathrm{M}_{\mathrm{exc}})$ are index 2 overgroups of the groups classified in [19].

Corollary 2.22. *If $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ then G has an index two subgroup H whose non-division components are all of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \geq 0$.*

Is het waar dat H ook effectief minstens 1 niet-division component heeft? I.e. hebben we $\mathrm{cd}(H) = \{1, 2\}$?

Proof. □

Example 2.23. The converse of Corollary 2.22 is not true in general. For example consider the following extraspecial group of order 2^5 (whose SmallGroup ID is [32,49]):

$$\begin{aligned} D_8 \circ D_8 := \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, c^2 = a^2, a^b = a^{-1}, c^d = a^2 c, \\ [a, c] = [a, d] = [b, c] = [b, d] = 1 \rangle. \end{aligned}$$

It can be verified that $\mathcal{C}(\mathbb{Q}[D_8 \circ D_8]) = \{\mathbb{Q}, \mathrm{M}_4(\mathbb{Q})\}$ (e.g. via the well-known description of its complex irreducible representations). On the other, e.g. to be proven via a manual check (via GAP), all the 2-groups until order 16, except D_{16}, have only matrix components of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \geq 0$.

2.4.4. *Groups with* $(\mathrm{M}_{\mathrm{exc}})$ *and* Q *of class at most* 2.

Theorem 2.24. *Let G be a finite group with abelian Sylow 2-subgroup. Then $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ if and only if*

$$G \cong D \times (A \rtimes \langle y : y^{2^m} = 1 \rangle)$$

with $a^y = a^{-1}$ for all $a \in A$, D an abelian group with $\exp(D) \mid 4, 6$ and $m = 1, 2$ if $3 \mid |D|$.

Note that the condition on D written means that either $D \cong C_3^{n_3} \times C_2^{n_2}$ or $D \cong C_2^{n_2} \times C_4^{n_4}$.

Proof. **TO write** □

Next,

Theorem 2.25. *Let G be a non-nilpotent group with Sylow 2-subgroup of nilpotency class 2. Then $\mathbb{Q}G$ is $(\mathrm{M}_{\mathrm{exc}})$ if and only if the following holds:*

(1) $G \cong A \rtimes Q$ *with A an elementary abelian 3-group and Q a 2-group,*
(2) $[G : F(G)] = 2$,
(3) $\exp(Q/\mathcal{Z}(Q)) = 2$,
(4) $\exp(\mathcal{Z}(Q)) \mid 4$,
(5) and $x^y = x^{-1}$ *for all $x \in F(G)$ and $y \in G \setminus F(G)$.* **this is wrong, to correct**

Proof. TO WRITE (and think if not possible to do nitpotent at the same time !!) □

2.4.5. *Groups with* $(\mathrm{M}_{\mathrm{exc}})$ *and* Q *of class* 3.

Theorem 2.26. *Let G be a non-nilpotent group with Sylow 2-subgroup of nilpotency class 3. Then $\mathbb{Q}G$ is $(\mathrm{M}_{\mathrm{exc}})$ if and only if the following holds:*

(1) $G \cong A \rtimes Q$ *with A an elementary abelian 3-group and Q a 2-group,*
(2) $[G : F(G)] = 2$,
(3) $Q/\mathcal{Z}(Q) \cong D_8$,
(4) $\exp(\mathcal{Z}(Q)) = 2$,
(5) and ???.

Next,

Theorem 2.27. *Let G be a nilpotent group with $4 \in \mathrm{cd}(G)$. Then $\mathbb{Q}G$ has* $(\mathrm{M_{exc}})$ *if and only if the following conditions hold:*

(1) *G is a 2-group,*
(2) *$G/\mathcal{Z}(G)$ is isomorphic to $C_2 \times C_2, D_8, C_2 \times D_8$ or $(C_2 \times C_2) \rtimes C_4$,*
(3) $\exp(\mathcal{Z}(Q)) \mid 4$.

We can now also filter when $\mathbb{Q}G$ has $(\mathrm{M_{exc}})$ with $\mathrm{cd}(G) = \{1, 4\}$. It turns out that most $(\mathrm{M_{exc}})$ groups have three different character degrees.

Corollary 2.28. *Let G be a finite group satisfying* $(\mathrm{M_{exc}})$*. Then $\mathrm{cd}(G) = \{1, 4\}$ if and only if the following conditions holds:*

(1) *G is a 2-group of class 2,*
(2) $\exp(G/\mathcal{Z}(G)) = \exp(\mathcal{Z}(G)) = 2$,
(3) **and ??**

Proof. TO DO ☐

3. The block Virtual Structure Problem

We now recall the concept of reduced norm. First, let A be a finite dimensional central simple algebra over a field K of characteristic 0 and E be a splitting field of A (i.e. $A \otimes_K E \cong \mathrm{M}_n(E)$ for some n). Then the *reduced norm* of $a \in A$ is defined as

$$\mathrm{RNr}_{A/K}(a) = \det(1_E \otimes_K a).$$

Note that $\mathrm{RNr}_{A/K}(\cdot)$ is a multiplicative map, $\mathrm{RNr}_{A/K}(A) \subseteq K$ and $\mathrm{RNr}_{A/K}(a)$ does only depend on K and $a \in A$ (and not on the chosen splitting field E and isomorphism $E \otimes_K A \cong \mathrm{M}_n(E)$), see [15, page 51]. For a subring R of A, put

$$\mathrm{SL}_1(R) = \{\ a \in \mathcal{U}(R) \mid \mathrm{RNr}_{A/K}(a) = 1\ \}, \tag{3.1}$$

which is a (multiplicative) group. If $A = \mathrm{M}_n(A')$ and $R = \mathrm{M}_n(R')$ with A' a finite dimensional central simple algebra over K and R' a subring of A', then we also write $\mathrm{SL}_1(A) = \mathrm{SL}_n(A')$ and $\mathrm{SL}_1(R) = \mathrm{SL}_n(R')$. Next, if $A = \prod \mathrm{M}_{n_i}(D_i)$ is semisimple and h_i is the projection onto the i-th component, then

$$\mathrm{SL}_1(R) := \{\ a \in R \mid \forall\ i\colon\ \mathrm{RNr}_{\mathrm{M}_{n_i}(D_i)/\mathcal{Z}(D_i)}(h_i(a)) = 1\ \}.$$

The overarching spirit of this paper is to determine properties of a finite group that are determined by its irreducible representations over a number field. For example, one could wonder which groups are fully determined by certain interesting predescribed condition on them (such as property $((\mathrm{M_{exc}}))$):

Question 3.1 (block Virtual Structure problem)**.** Let $\mathcal{P}$ be a property. Classify the group algebras FG such that for every $e \in \mathrm{PCI}(FG)$

- either FGe has property $\mathcal{P}$
- or $\mathrm{SL}_1(\mathcal{O}_e)$ has property $\mathcal{P}$ for any order $\mathcal{O}_e$ in FGe.

Note that, in the case that $\mathcal{P}$ is a property that behaves well with direct products and is constant on commensurability classes, Question 3.1 is equivalent to the classical Virtual Structure Problem. This thanks to the general fact that, for any order $\mathcal{O}$ in FG, $\mathrm{SL}_1(\mathcal{O})$ is a finite index subgroup of $\prod_{e \in \mathrm{PCI}(FG)} \mathrm{SL}_1(\mathcal{O}e)$.

3.1. **The case of properties defined on commensurability classes.** Let $\mathcal{P}$ be a group theoretical property such that

- $\mathcal{P}$ implies *not* FA or not SCP
- SL_1 of all exceptional 2×2 components satisfy it
- it is a property of commensurability classes.

Think which props exactly to put

Example 3.2. **Is $\mathcal{G}_{am}$ er zo eentje? Als niet, welk wel?** Al die uit de paper van Zalesski-Del rio

By $\prod \mathcal{P}$ we mean that the group is a direct product of groups satisfying $\mathcal{P}$ *and* an abelian group. Combining the methods in the proof of [19, Theorem 2.1.] with the results of [21] we obtain the following.

Proposition 3.3. *Let A be a finite dimensional semisimple F-algebra with F a number field and $\mathcal{O}$ an order in A. If $\mathcal{U}(\mathcal{O})$ is virtually-$\prod \mathcal{P}$, then for every $e \in \mathrm{PCI}(A)$:*

(1) $\mathrm{SL}_1(\mathcal{O}e)$ is either virtually-$\mathbb{Z}$ or virtually-$\mathcal{P}$ (Is general!!!!)
(2) The degree of Ae, as CSA, is at most 4

Proof. **to adapt** Denote $S_j := \mathrm{SL}_{n_j}(\mathcal{O}_j) \cap H$ which is of finite index in $\mathrm{SL}_{n_j}(\mathcal{O}_j)$, hence it is enough to proof that $e(S_j) = \infty$. Let p_k be the projection of H on H_k. Fix some j as in the claim. The condition is equivalent with saying that $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is infinite [20]. In particular there exists some k such that $p_k(S_j)$ is infinite[13]. For such k we will now prove that $|p_k(\prod_{i \neq j} S_i)| < \infty$. For this consider $S := S_j \times \prod_{i \neq j} S_i$ which by the first claim is of finite index in H. Therefore $p_k(S)$ is of finite index in H_k and hence $e(p_k(S)) = \infty$. However, $p_k(S_j)$ and $p_k(\prod_{i \neq j} S_i)$ are subgroups as in the second claim[14], yielding the desired. Indeed, the two subgroups clearly commute, are normal in $\pi_k(S)$ and $p_k(S_j) \cap p_k(\prod_{i \neq j} S_i) \subseteq \mathcal{Z}(p_k(S))$ which is finite since $p_k(S)$ has infinitely many ends.

Now consider the set $\mathcal{I}_j := \{k \mid |p_k(S_j)| < \infty\}$. From the previous it follows that if $k \in \{1, \ldots, q\} \setminus \mathcal{I}_j$, then $p_k(S_j)$ is of finite index in H_k. Hence $S_j / \big(S_j \cap \prod\limits_{i \in \mathcal{I}_j} H_i\big)$ is a subgroup of finite index in $\prod_{k \notin \mathcal{I}_j} H_k$. As the quotient was with a finite subgroup, we obtain that S_j is virtually-$\mathcal{G}_\infty$ and hence $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ also. However under the conditions above SL_1 is virtual indecomposable [21, Theorem 1]. Therefore $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ in fact is even virtually a group with infinitely many ends and so in fact $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$, as claimed.

Part 2: This is more generally the case for a property which imply not FA, cf. work of Kleinert-Del Rio where they deduce this by considering the associated semisimple Lie group and do rank computations. ☐

3.2. **Groups of virtual cohomological dimension 4.** Let Γ be a discrete group. Then the *cohomological dimension* of Γ over the ring R is

$$\mathrm{hdim}_R\, \Gamma := \min\{n \mid H^k(G, M) = 0 \text{ for all } k > n \text{ and } M \in \mathrm{mod}(RG)\}.$$

If no such n exists one says that $\mathrm{hdim}_R\, \Gamma = \infty$. A usual obstruction to have a finite cohomological dimension is torsion in Γ. However, if Γ has a torsion-free subgroup of finite index (e.g. when Γ is linear), then each of such finite index subgroups has the same cohomological dimension. Hence

$$\mathrm{vcd}(\Gamma) := \{\mathrm{hdim}_{\mathbb{Z}}\, \Gamma' \mid [\Gamma : \Gamma'] < \infty \text{ and } \Gamma' \text{ torsion-free}\}.$$

In particular, as the unit group of two orders $\mathcal{O}_1$ and $\mathcal{O}_2$ in A are commensurable, one has that $\mathrm{vcd}(\mathcal{O}_1) = \mathrm{vcd}(\mathcal{O}_2)$. Furthermore, as shown by eq. (3.2) their virtual cohomological dimension can be expressed purely in terms of number theoretical properties of A. In [19, Proposition 3.3] the finite dimensional simple F-algebras A, over a number field F, such that $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) \leq 2$ for some order $\mathcal{O}$ in A (and hence for all orders) have been classified. We extend this result to virtual cohomological dimension 4. See Remark 3.5 below for $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 3$.

Proposition 3.4. *Let A be a finite dimensional simple F-algebra with F a number field. If $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 4$ for an order $\mathcal{O}$ in A, then A is isomorphic as F-algebra to:*

[13] Otherwise S_j would be finite and hence also the overgroup of finite index $\mathrm{SL}_{n_j}(\mathcal{O}_j)$.

[14] Instead of claim 2 one could have used the well known [**?**, 4.A.6.3.] saying that infinite finitely generated normal subgroups of a group with infinitely many ends need to have finite index.

(1) $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ *with* $a, b \in \mathbb{N}_0$,
(2) $\mathrm{M}_2(F)$*, with* F *a cubic field with precisely one real embedding and one pair of complex embeddings,*
(3) or to $\left(\frac{-a,-b}{F}\right)$ *such that it is non-ramified at exactly two real places and* F *is totally real.*

Proof. Let $A = M_n(D)$, $F = \mathcal{Z}(D)$ for an integer $n \geq 1$ and D a division ring of degree d. We make use of the following formula, as stated in [19, Eq. (1)].

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_1\frac{(nd-2)(nd+1)}{2} + r_2\frac{(nd+2)(nd-1)}{2} + s(n^2d^2-1) - n + 1, \tag{3.2}$$

where s is the number of pairs of non-real complex embeddings of F, r_1 is the number of real embeddings of F at which A is ramified, and r_2 the number of real embeddings of F at which A is not ramified. We may assume that $nd > 1$, since when $nd = 1$, A is a field, which implies that $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 0$ by [19, Proposition 3.3]. Note that for any choice of $nd \geq 2$, the first two terms of eq. (3.2) are non-negative. Additionally, when d is odd, it is well-known that $r_1 = 0$.

Suppose $s \geq 2$. Then for any choice of n or d such that $nd = 2$,

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) \geq s(n^2d^2-1) - n + 1 > 4,$$

and this expression is strictly increasing in both n and d. Hence $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) > 4$ for any $s \geq 2$, and in particular F has at most one pair of complex embeddings.

Suppose $s = 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) \geq s(n^2d^2-1) - n + 1 = n^2d^2 - n > 4,$$

when $n \geq 3$. Hence n is at most 2 in this case. Suppose first $n = 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) \geq d^2 - 1 > 4 \text{ when } d > 2.$$

Thus $d = 2$ because $nd \geq 2$ by assumption. Then we find

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_2\frac{4.1}{2} + 4 - 1 = 4 \iff r_2 = -\frac{1}{2},$$

which is a contradiction since $r_2 \in \mathbb{N}$. Hence we cannot have $n = 1$. The only other option is $n = 2$. Then $n^2d^2 - n = 4d^2 - 2 \leq 4$ if and only if $d = 1$, in which case

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_2\frac{4.1}{2} + 2 = 4 \iff r_2 = 1.$$

In this case we find that $A = M_2(F)$ with F a cubic number field with precisely one real embedding and one pair of complex embeddings.

Suppose now $s = 0$. We examine the case $r_2 = 0$ first. Suppose $r_2 = 0$. Then $r_1 \geq 1$ and

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_1\frac{(nd-2)(nd+1)}{2} - n + 1,$$

and hence $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 4$ necessarily implies that $nd \geq 3$. Suppose first that $n = 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = r_1\frac{(d-2)(d+1)}{2} = 4 \iff r_1 \leq 2, \text{ since } d \geq 3.$$

If $r_1 = 2$, then $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = (d-2)(d+1) = 4$ implies that $d = 3$, but we have already remarked that $r_1 = 0$ when d is odd, a contradiction. If $r_1 = 1$, then $(d-2)(d+1) = 8$, which implies that $d = \frac{2\pm\sqrt{32}}{2}$, which is not an integer, a contradiction. We conclude that if $s = 0$ and $r_2 = 0$, then $n > 1$. Suppose $n = 2$. Then necessarily $d \geq 2$ since $nd \geq 3$. It follows that

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = \frac{r_1}{2}(2d-2)(2d+1) - 1 \geq 4,$$

with equality if and only if $r_1 = 1$ and $d = 2$. We find that in this case, $A = M_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some positive integers a, b. Now the expression $\frac{r_1}{2}(nd-2)(nd+1) - n + 1$ is strictly increasing in n if and only if $n > \frac{d-1}{r_1 d^2}$, which in the case at hand is satisfied since $d \geq 1$ and $r_1 \geq 1$ by assumption. It follows that $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) > 4$ whenever $n \geq 3$.

Still under the assumption that $s = 0$, we now turn our attention to the case $r_2 \neq 0$, meaning $r_2 \geq 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = \frac{r_2}{2}(nd+2)(nd-1) + \frac{r_1}{2}(nd-2)(nd+1) - n + 1.$$

If $r_1 \geq 1$, then d is even, meaning that $nd \geq 4$ when $n \geq 2$. But when $n = 2$, then

$$\frac{r_2}{2}(2d+2)(2d-1) + \frac{r_1}{2}(2d-2)(2d+1) - 1 \geq 9r_2 + 5r_1 - 1 > 4,$$

and again the expression above for $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O}))$ is strictly increasing in n. If $r_1 = 0$, then $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 4$ if and only if $\frac{r_2}{2}(nd+2)(nd-1) - n = 3$. This expression is strictly increasing in n if and only if $n > \frac{1-d}{r_2 d^2}$, which is always satisfied by assumption on r_2 and d. But when $n \geq 2$ and $d \geq 2$, meaning in particular that $nd \geq 4$, one finds that

$$\frac{r_2}{2}(nd+2)(nd-1) - n \geq 7,$$

implying that only the cases $n = 1$, and $(n, d) = (2, 1)$ need to be investigated. If $n = 2$ and $d = 1$, then

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = \frac{r_2}{2}(nd+2)(nd-1) - n + 1 = 2r_2 - 1,$$

which equals 4 if and only if $r_2 = \frac{5}{2}$, a contradiction.

In particular, for any value of r_1 only the case $n = 1$ remains. Assuming $n = 1$, we have in particular that $d \geq 2$, and

$$\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = \frac{r_2}{2}(d+2)(d-1) + \frac{r_1}{2}(d-2)(d+1) \geq \frac{r_2}{2}(d+2)(d-1) \geq 4,$$

where the last inequality becomes an equality if and only if $d = 2 = r_2$. In that case one also finds that $\frac{r_1}{2}(d-2)(d+1) = 0$, and hence $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 4$ if $s = 0$, $r_2 = d = 2$ and r_1 takes any arbitrary integer value. We obtain that $A = \left(\frac{-a,-b}{F}\right)$ with F a totally real number field and A is non-ramified at precisely two places. If $d \geq 3$, it now immediately follows that $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) \geq 9r_2 + 4r_1 > 4$, and this concludes our analysis. $\square$

Remark 3.5. With a similar proof one can verify that $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 3$ if and only if A is isomorphic to one of the following simple algebras:

- $\mathrm{M}_3(\mathbb{Q})$,
- $\mathrm{M}_2(\mathbb{Q}(\sqrt{d}))$ with $d \in \mathbb{N}$ square-free,
- $\left(\frac{-a,-b}{F}\right)$ such that F has one pair of non-real complex embeddings and is ramified at all real places.

Concerning $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) \leq 2$, [19, Proposition 3.3] tells that

- $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 0$ if and only if A is a field or a totally definite quaternion algebra,
- $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 1$ if and only if $A \cong \mathrm{M}_2(\mathbb{Q})$
- $\mathrm{vcd}(\mathrm{SL}_1(\mathcal{O})) = 2$ if and only if $A \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ or a quaternion algebra with a totally real centre and which is non-ramified at exactly one infinite place.

We now have all ingredients for our following main theorem.

Theorem 3.6. *Let G be a finite group, F a number field with $[F : \mathbb{Q}] \leq 2$ and R its ring of integers. Then the following are equivalent:*

(1) FG has $(\mathrm{M}_{\mathrm{exc}})$,
(2) $\mathrm{vcd}(\mathrm{SL}_1(RGe)) \mid 4$ for every $e \in \mathrm{PCI}(FG)$ such that FGe is a non-division simple c component.

Furthermore, if FG has $(\mathrm{M}_{\mathrm{exc}})$ and FGe is an exceptional division component, then $\mathrm{vcd}(RGe) > 4$.

Remark 3.7. That (1) implies (2) in Theorem 3.6 is true for any field F. However for the converse the condition that F is a quadratic number field is required. For example for any field $F \supseteq \mathbb{Q}$ one has that $FD_8 \cong F \times F \times \mathrm{M}_2(F)$. In particular if F is a cubic number field with one real embedding and one pair of complex embeddings (e.g. $F = \mathbb{Q}(\sqrt[3]{d})$), then $\mathrm{vcd}(\mathrm{SL}_2(R))) = 4$, but FD_8 has not $(\mathrm{M}_{\mathrm{exc}})$.

The proof of Theorem 3.6 will follow quickly out of the results obtained in earlier sections together with following fact which is of independent interest.

Lemma 3.8. *Let G be a finite group. Let F be a cubic number field. Suppose that the simple algebra $\mathrm{M}_2(F)$ is a quotient of $\mathbb{Q}G$, say $\mathrm{M}_2(F) \cong \mathbb{Q}Ge$ with $e \in \mathrm{PCI}(\mathbb{Q}G)$. Then $\pi(Ge) \subseteq \{2, 3, 7\}$, and $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ or $F = \mathbb{Q}(\zeta_9 + \zeta_9^{-1})$.*

Example 3.9. Recall that from eq. (2.6) it follows that when $G = Dic_{4n}$ with $7 \mid n$ (or $9 \mid n$), then $\mathbb{Q}Dic_{4n}$ has a component which is isomorphic to $\mathrm{M}_2(F)$, where $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ (respectively $F = \mathbb{Q}(\zeta_9 + \zeta_9^{-1})$), is a cubic extension of $\mathbb{Q}$.

Proof. Let $\lambda \in F$ be a torsion element, say of order n. Then it is a primitive n^{th} root of unity, denoted ζ_n, and

$$3 = |F : \mathbb{Q}| = |F : \mathbb{Q}(\zeta_n)||\mathbb{Q}(\zeta_n) : \mathbb{Q}| \geq \phi(n). \tag{3.3}$$

It follows that $n \in \{1, 2, 3, 4, 6\}$.

Let $g \in G$. By fixing a $\mathbb{Q}$-basis of F as a 3-dimensional $\mathbb{Q}$-space, one can realise g as a 6-by-6 matrix over $\mathbb{Q}$. We denote by $\chi_{F,g}$ and $\chi_{\mathbb{Q},g}$ respectively the characteristic polynomials of g over F and over $\mathbb{Q}$. Similarly, we write $\mu_{F,g}$ and $\mu_{\mathbb{Q},g}$ for the minimal polynomials of g over respectively F and $\mathbb{Q}$. By definition, any minimal polynomial $\mu_{\mathbb{Q},g}$ has degree at most 6, and any $\mu_{F,g}$ has degree at most 2. Remark that $\mu_{F,g}$ has degree 1 if and only if g is a scalar matrix over F.

From [23, Page 147], it follows that for any $g \in G$ of prime power order p^k, the $p^{k\text{th}}$ cyclotomic polynomial Φ_{p^k} equals $\mu_{\mathbb{Q},g}$. In particular, $\mathbb{Q}(\zeta_{p^k}) \cong \frac{\mathbb{Q}[X]}{(\mu_{\mathbb{Q},g})}$. The latter also holds over F, when $p^k > 4$. Indeed, $\mu_{F,g}$ is given as the unique monic polynomial generating the ideal $I_g := \{P \in F[X] \mid P(g) = 0\}$, and the minimal polynomial of ζ_{p^k} over F is given by the unique monic polynomial generating the ideal $I_{\zeta_{p^k}} := \{Q \in F[X] \mid Q(\zeta_{p^k}) = 0\}$. We claim that $I_g = I_{\zeta_{p^k}}$. Indeed, by [23, Page 146], there is some matrix $B \in \mathrm{GL}_2(\mathbb{C})$ such that $BgB^{-1} = \mathrm{diag}(\lambda_1, \lambda_2)$, with each λ_i a $p^{k\text{th}}$ root of unity, amongst which at least one primitive (otherwise BgB^{-1} and hence g would have order strictly smaller than p^k). Without loss of generality, assume $\lambda_1 = \zeta_{p^k}$. Remark that since conjugation by an invertible matrix induces an algebra automorphism of $\mathrm{M}_2(\mathbb{C})$, it follows that $P \in I_g$ if and only if $P(BgB^{-1}) = 0$. In particular, $P \in I_g$ if and only if $P(\zeta_{p^k}) = 0 = P(\lambda_2)$. We conclude that $I_g \subseteq I_{\zeta_{p^k}}$. But since $\deg(\mu_{F,g}) = 2$, and $p^k > 4$ (meaning that $\zeta_{p^k} \notin F$), it follows that I_g is a maximal ideal of $F[X]$, and hence $I_g = I_{\zeta_{p^k}}$. In particular, $F(\zeta_{p^k}) \cong \frac{F[X]}{(\mu_{F,g})}$.

Since Φ_p divides $\mu_{\mathbb{Q},g}$ when $g \in G$ is an element of prime order p, and the degree of $\mu_{\mathbb{Q},g}$ is at most 6 as remarked earlier, it follows that $p \in \{2, 3, 5, 7\}$, and in particular $\pi(G) \subseteq \{2, 3, 5, 7\}$. Suppose $g \in G$ has order $p \in \{5, 7\}$. Remark that $\deg(\chi_{F,g}) = 2$, since otherwise g would be a scalar matrix with a p^{th} root of unity on the diagonal, which is a contradiction with the description of torsion elements in F as given in eq. (3.3). Over $F(\zeta_p)$, $\chi_{F,g}$ splits as $(X - \zeta_p^i)(X - \zeta_p^k)$ for some $0 \leq i \neq k \leq p - 1$. Now $\zeta_p^{i+k} \neq 1$ would imply that F has a p^{th} root of unity, a contradiction by eq. (3.3). Thus, $k = -i$, and the degree 1 coefficient in $\chi_{F,g}$ is equal to $-(\zeta_p^i + \zeta_p^{-i})$. In particular $\mathbb{Q}(\zeta_p^i + \zeta_p^{-i}) \subseteq F$. If $p = 5$, then since $|\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) : \mathbb{Q}| = 2$ does not divide $|F : \mathbb{Q}| = 3$, we obtain a contradiction. When $p = 7$, $|\mathbb{Q}(\zeta_7 + \zeta_7^{-1}) : \mathbb{Q}| = 3$. It follows that $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.

Let now $\pi(G) \subseteq \{2, 3\}$. We bound the exponent of G. Let $g \in G$, say of order 2^i3^j for some non-negative integers i, j. If $i = 0$ or $j = 0$, then $o(g) = p^n$ since Φ_{p^n} divides $\mu_{\mathbb{Q},g}$ for each $p \in \{2, 3\}$ and $\deg(\mu_{\mathbb{Q},g}) \leq 6$, it follows that $o(g) \in \{p, p^2\}$. If $i \neq 0 \neq j$, then considering a realisation of g as an element in $\mathrm{GL}_6(\mathbb{Q})$, [23, Page 147] implies that there exist $m_1, \dots, m_r$ such that $o(g) = \mathrm{lcm}\{m_1, \dots, m_r\}$, Φ_{m_i} divides $\mu_{\mathbb{Q},g}$, and $6 = \sum_{i=1}^{r} d_i \phi(m_i)$, for some $d_i \geq 1$. In particular each $\phi(m_i) \leq 6$. Since $\pi(G) \subseteq \{2,3\}$, from a case-by-case analysis it follows that $m_i \in \{1,2,3,4,6,9,18\}$. In particular, $\exp(G) \mid 36$.

Suppose that G contains an element g of order 9. Then since $F(\zeta_9) \cong \frac{F[X]}{(\mu_{F,g})}$, and $\deg(\mu_{F,g}) = 2$,

$$|F(\zeta_9) : \mathbb{Q}| = |F(\zeta_9) : F||F : \mathbb{Q}| = 6,$$

and $|\mathbb{Q}(\zeta_9) : \mathbb{Q}| = 6$, it follows that $F(\zeta_9) = \mathbb{Q}(\zeta_9)$, and in particular $F \subseteq \mathbb{Q}(\zeta_9)$. The only subfields contained in $\mathbb{Q}(\zeta_9)$ are $\mathbb{Q}$, $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$, a contradiction, and of these only $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$ is of degree 3 over $\mathbb{Q}$. In particular, we conclude that if $\pi(G) \subseteq \{2,3\}$ and elements of order 3^n necessarily have order 3, then $\exp(G) \mid 12$. Suppose we are in this case, and let $e \in \mathrm{PCI}(G)$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2(F)$. Then by Brauer's splitting field theorem, $\mathbb{Q}(\zeta_{12})$ is a splitting field for G, since G has exponent a divisor of 12. In particular, $F \subseteq \mathbb{Q}(\zeta_{12})$. However, $\mathrm{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$, which has order coprime to 3, which together with the fundamental theorem of Galois theory implies a contradiction. ☐

We are now able to prove Theorem 3.6.

Proof of Theorem 3.6. First suppose that FG has $(\mathrm{M}_{\mathrm{exc}})$ and let $e \in \mathrm{PCI}(FG)$. If FGe is an exceptional matrix component then $\mathrm{vcd}(RGe) = 2$ or 4 by Proposition 3.4 and Remark 3.5. Next, if FGe is a division algebra, then by Theorem 2.6 either it is a field or one of the following quaternion algebras:

$$\{\left(\frac{-1,-1}{\mathbb{Q}(\zeta_m)}\right), \left(\frac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right) \mid m \in 2\mathbb{N}+1,\, t \in \mathbb{N}_{\geq 3}\}.$$

Of these only those with a non totally-real center (i.e. the first two) are exceptional. Inspecting the possible quaternions algebras with virtual cohomological dimension smaller than 4, we see that they are not part of the list.

It remains to prove the converse, so suppose that $\mathrm{vcd}(\mathrm{SL}_1(RGe)) \mid 4$ for every $e \in \mathrm{PCI}(FG)$ such that FGe is a non-division simple component. By the results referred to above, the only simple algebras not allowed by the property $(\mathrm{M}_{\mathrm{exc}})$ are those of the form $\mathrm{M}_2(K)$, with K a cubic number field with one real embedding and one pair of complex embeddings. Now as $F \subseteq \mathcal{Z}(FGe) = K$ and $[F : \mathbb{Q}] \mid [K : \mathbb{Q}] = 3$, one has that $F = \mathbb{Q}$. However by Lemma 3.8 the algebra $\mathrm{M}_2(K)$ can't be the simple component of FG, finishing the proof. ☐

3.3. **Higher Kleinian groups: discrete subgroups of $\mathrm{SL}_4(\mathbb{C})$.**

Another interesting property is a kind of higher Kleinian property:

Definition 3.10. A group Γ is said to *have property* Di_n if it is a discrete subgroup of $\mathrm{SL}_n(\mathbb{C})$, but not of $\mathrm{SL}_{n-1}(\mathbb{C})$.

We will be interested in the case that Γ has Di_n with n a divisor of 4. An alternative way to look at this is via the 5-dimensional hyperbolic space, as one has the following isomorphism:

$$\mathrm{Iso}^+(\mathbb{H}_5) \cong \mathrm{PGL}_2(\left(\frac{-1,-1}{\mathbb{Q}}\right))$$

In particular a group Γ acts discontinuously on[15] $\mathbb{H}_5$ if and only if Γ has Di_4 .

The finite dimensional simple algebras A such that $\mathrm{SL}_1(\mathcal{O})$ is Kleinian, i.e. is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$, for an order $\mathcal{O}$ in A were classified in [19, Proposition 3.2]. Note that if $\mathrm{SL}_1(\mathcal{O})$ has property Di_n, then so does $\mathrm{SL}_1(\mathcal{O}')$ for any other order $\mathcal{O}'$ in A (as both groups are commensurable).

Proposition 3.11. *Let A be a finite dimensional simple F-algebra with F a number field and $\mathcal{O}$ an order in A. If $\mathrm{SL}_1(\mathcal{O})$ has property Di_4, then A has one of the following forms:*

[15]Hereby one assumes that the action on $\mathbb{H}_5$ doesn't come from the inbedding of an action on $\mathbb{H}_4$.

(1) $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ *with* $a, b \in \mathbb{N}_0$,
(2) $\mathrm{M}_4(\mathbb{Q})$,
(3) $\left(\frac{-a,-b}{\mathbb{Q}(\sqrt{-d})}\right)$ *with* $a, b \in \mathbb{N}_0$ *and* $d \in \mathbb{N}_{>1}$ *square-free,*
(4) *a division algebra of degree* 4 *which is non-ramified at at most one infinite place.*

Proof. Write $A = \mathrm{M}_n(D)$ with D a finite dimensional division algebra and denote $K = \mathcal{Z}(D)$. As mentioned earlier, property Di_n doesn't depend on the order chosen. For ease we choose one of the form $\mathrm{M}_n(\mathcal{O})$ with $\mathcal{O}$ an order in D. We will consider the set of infinite non-compact places:

$$V^{nc} := V_\infty(K) \setminus \{v \in V_\infty(K) \mid SL_1(\mathrm{M}_n(D \otimes_K K_v) \text{ is compact }\}.$$

The places in V^{nc} are exactly those at which we can use strong approximation. Note that if $n \geq 2$, then $V^{nc} = V_\infty(K)$.

Now suppose that $\mathrm{SL}_1(\mathcal{O})$ is discrete in $\mathrm{SL}_4(\mathbb{C})$ and take $v_0 \in V_\infty(K)$ such that $\mathrm{SL}_n(\mathcal{O}) \leq \mathrm{SL}_n(D \otimes_K K_{v_0})$ embeds discretely in $\mathrm{SL}_4(\mathbb{C})$. **explain why the place can ve taken non-compact**

Next consider another place $v_0 \neq v_1 \in V^{nc}$ and consider the diagonal embedding

$$\Delta : \mathrm{SL}_1(\mathrm{M}_n(D) \hookrightarrow \prod_{v \in V^{nc} \setminus \{v_1\}} \mathrm{SL}_1(\mathrm{M}_n(D \otimes_K K_v))$$

By strong approximation, see [28, Theorem 7.12, pg 418], $\mathrm{Im}(\Delta)$ is dense. However $v_0 \in V^{nc} \setminus \{v_1\}$ yielding a combination of dense and discrete and hence a contradiction (as $\mathrm{SL}_n(\mathcal{O})$ being discrete in $\mathrm{SL}_4(\mathbb{C})$ can't be finite). Consequently,

$$|V^{nc}| \leq 1 \tag{3.4}$$

Now, notice that being discrete in $\mathrm{SL}_4(\mathbb{C})$ implies that $\dim_{\mathbb{C}} \mathrm{M}_n(D \otimes_K \mathbb{C}) \leq 16$. Furthermore if it is not discrete in a lower $\mathrm{SL}_m(\mathbb{C})$, then $\dim_{\mathbb{C}} \mathrm{M}_n(D \otimes_K \mathbb{C}) = 16$. The latter implies that $A = \mathrm{M}_4(F), \mathrm{M}_2(D)$ or D' with D a quaternion algebra and D' a division algebra of degree 4. This combined with eq. (3.4), yields the stated possibilities. Indeed, simply recall that $\mathrm{SL}_1(\mathrm{M}_n(D \otimes_K K_v)$ is compact if and only if $n = 1$ an A is ramified at v. **currentely the statement is for $F = \mathbb{Q}$ I think. To correct !! And ideally into an iff statement.** □

Remark 3.12. The proof of Proposition 3.11 also yields that $\mathrm{SL}_1(\mathcal{O})$ has property Di_3 if and only if A is isomorphic to $\mathrm{M}_3(\mathbb{Q}), \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ or a division algebra of degree 3 which is non-ramified at at most one infinite place.

Concerning Di_2, [19, Remark 3.5] says that $\mathrm{SL}_1(\mathcal{O})$ has Di_2 (i.e. is Kleinian) if and only if $\mathrm{vcd}(A) \leq 2$ or A is quaternion division algebra which is ramified at all its infinite places and having exactly one pair of complex embeddings.

Explain, with Free-by-free in mind it is natural to put the condition that there is no exceptional type I

Theorem 3.13. *Let G be a finite group. Then the following are equivalent:*

(1) $\mathbb{Q}G$ *has* $(\mathrm{M}_{\mathrm{exc}})$ *(resp. and also no exceptional division algebra components)*
(2) $[G : F(G)] = 2$ *and* $\mathrm{SL}_1(\mathbb{Z}Ge)$ *has* Di_n *for* $n \mid 4$ *and* $e \in \mathrm{PCI}(\mathbb{Q}G)$ *such that* $\mathbb{Q}Ge$ *a non-division algebra (resp. for each* $e \in \mathrm{PCI}(\mathbb{Q}G)$*)*

Proof. Next consider property Di_n with n a divisor of 4. Combining Remark 3.12, Proposition 3.11 and Theorem 2.6 we see that $(\mathrm{M}_{\mathrm{exc}})$ implies the desired property. **To complete the converse once the statement is chosen** □

Remark 3.14. The condition that there are no exceptional division algebra components is an important one. Indeed, **The groups $C_3 \rtimes C_{2^n}$ and $C_m \times Q_8$ have** $(\mathrm{M}_{\mathrm{exc}})$ **but not** Di_n **with $n \mid 4$ due to bad division components!!!** In particular we see that in Free-by-free, i.e. when all components are even nicer, then also the condition is required. Consider the group XX. This has Di_n for $n \mid 4$, but no $(\mathrm{M}_{\mathrm{exc}})$. For that group however the fitting subgroup has index 4.

3.4. **The good property. The aim of this section will be to give several equivalent geometric group theoretical properties for** $(\mathrm{M}_{\mathrm{exc}})$

For this section we fix for each $e \in \mathrm{PCI}(G)$ a maximal order $\mathrm{M}_{n_e}(\mathcal{O}_e)$ in $\mathbb{Q}Ge$. The arguments will however be independent of this choice. Further denote by $\widehat{\Gamma}$ the profinite completion of a group Γ. Recall that Γ is called *good* if the map

$$H^j(\widehat{\Gamma}, M) \to H^j(\Gamma, M),$$

induced by the inclusion of Γ in its completion, is an isomorphism for any j and finite Γ-module M. By [6] the property to be good is one of commensurability classes and is closed under direct product. In particular $\mathrm{SL}_1(\mathbb{Z}G)$ is good exactly when $\mathrm{SL}_n(\mathcal{O}_e)$ is good for all $e \in \mathrm{PCI}(G)$. **Still to clarify role of the center!!** Combining results in the literature one obtains:

Theorem 3.15. *Let G be a finite group with* $(\mathrm{M}_{\mathrm{exc}})$. *Then* $\mathrm{SL}_1(\mathbb{Z}Ge)$ *satisfies the good property for all* $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Lemma 3.16.

Proof. From the literature In ?? it was proven that $\mathrm{SL}_n(\mathcal{O})$ is not good if it enjoys the subgroup congruence property. In particular if $n \geq 3$ or $n = 2$ and $\mathcal{U}(\mathcal{O})$ is infinite, then it is not good, e.g. see ??.

Put Zaleskii stuff together for the matrix components. But for quaternion algebras really need to think!! ☐

Using this we know can solve the virtual structure problem for $\mathcal{U}(\mathbb{Z}G)$ and the good property.

Theorem 3.17. *Let G be a finite group. Then $\mathcal{U}(\mathbb{Z}G)$ is good if and only if een van de karakterisaties van de vorige secties.*

3.5. **Further questions.** blabla

4. The rational isomorphism problem for $(\mathrm{M}_{\mathrm{exc}})$ groups

A classical problem is how much of the group structure of G is determined by the R-algebra structure the group ring RG. In this section we consider the rational isomorphism problem:

Problem 1. Let G and H be finite groups. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, is then $G \cong H$?

In general the answer to Problem 1 is negative as shown for the first time by the counterexample of Dade REF. A positive answer is known for abelian groups REF and among metayclic groups REF. The authors are only aware of these classes. Our aim is to prove that if $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ then Problem 1 has a positive answer.

On the multiplicities of a component when $(\mathrm{M}_{\mathrm{exc}})$**.** Suppose that $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$. Then Theorem 2.6 tells us what are the possible simple components of $\mathbb{Q}G$. We will now count how many of each component arises in terms of various structural properties of G obtained in Section 2.4.

For this recall that, when $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$, every simple component of $\mathbb{Q}G$ is of the form $\mathbb{Q}Ge(G, H, K)$ for (H, K) a strong Shoda pair. Recall that the latter were described: $G \cong A \rtimes Q$ contains a maximal abelian subgroup B such that $G/B \cong C_2$ or $C_2 \times C_2$. The group H is either B or $\langle B, t\rangle$ for some $t \in Q \setminus (Q \cap B)$. Furthermore K is of the form $K_{(a,x,m)}$ defined in (2.14).

Now two strong Shoda pairs (H_1, K_1) and (H_2, K_2) yield the same component if and only if the associated primitive central idempotents are equal. By [15, Exercise 3.4.3], *if H_1 is normal*, this can be group theoretically determined as following:

$$e(G, H_1, K_1) = e(G, H_2, K_2) \Leftrightarrow \exists\, g \in G : H_1 \cap K_2 = H_2 \cap K_1^g. \tag{4.1}$$

Furthermore if $H_1 = H_2$ is normal in G, then (4.1) holds if and only if K_1 and K_2 are conjugated inside G.

Lemma 4.1. *Suppose $\mathbb{Q}G$ has* $(\mathrm{M}_{\mathrm{exc}})$ *and with notations as above, we have that:*

(1) the number of SSP (B, K) with K normal in G is ...
(2) the number of SSP $(\langle B, t\rangle, K)$ with K normal in G and $t \in Q \setminus (Q \cap B)$ is ...

5. The blockwise Zassenhaus property

An interesting corollary of **??** is that the finite groups such that $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$ satisfy a kind of component-wise third Zassenhaus conjecture.**to update to now notations**

Corollary 5.1. *Let G be a finite group such that $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$. If H is a finite subgroup of $V(\mathbb{Z}G)$, then for all $e \in PCI(\mathbb{Q}G)$ we have that He is conjugated, inside $\mathcal{U}(\mathbb{Z}Ge)$, to a finite subgroup of Ge.*

Recall that $\mathcal{U}(\mathbb{Z}G) = \pm 1.V(\mathbb{Z}G)$ where $V(\mathbb{Z}G)$ are the units of augmentation one and that $\mathcal{U}(\mathbb{Z}G))$ satisfies the third Zassenhaus conjecture if any $H \leq V(\mathbb{Z}G)$ is conjugated inside $\mathbb{Q}G$ to a subgroup of G. Using the recent survey [**?**] one can check that the 12 families of groups satisfying virtually-$\mathcal{G}_\infty$ (see [24, Theorem 1]) are not all among a known case of the third Zassenhaus conjecture. In case of the Zassenhaus conjectures the conjugation is expected to be in $\mathcal{U}(\mathbb{Q}G)$, hence it is remarkable that for this class of groups one can perform the conjugation inside an order of $\mathcal{U}(\mathbb{Q}Ge)$.

The proof of Corollary 5.1 will in fact be a corollary of Lemma 5.2 and the investigation of independant interest of the "Strong Zassenhaus property", which we introduce in Section 5.1, for exceptional components.

5.1. Zassenhaus property for semisimple algebras. write my notes here

To proof Corollary 5.1 we first record the following lemma which is of independent interest. **Make following more general !!**

Lemma 5.2. *Let G be a finite group and $H \leq V(\mathbb{Z}G)$ a finite subgroup. Then $|He| \mid |Ge|$ for every primitive central idempotent e of $\mathbb{Q}G$.***also prove exponente and set of order elements**

Proof. Fix a primitive central idempotent e of $\mathbb{Q}G$ and consider the associated epimorphism $\varphi : G \to Ge$. We $\mathbb{Z}$-linearly extend the latter to the ring epimorphism $\Phi : \mathbb{Z}[G] \to \mathbb{Z}[Ge]$. Denote $N := \ker(\varphi) = \{g \in G \mid ge = e\}$. Note that $\ker(\Phi) = \omega(G, N)$, the relative augmentation ideal. Also, by definition of the map, $\Phi(V(\mathbb{Z}[G])) \subseteq V(\mathbb{Z}[Ge])$. Hence $\Phi(H)$ is a finite subgroup of $V(\mathbb{Z}[Ge])$ hence $|\Phi(H)| \mid |Ge|$ (e.g. see [**?**, Corollary 2.7]).

It remains to prove that $|He| \mid |\Phi(H)|$. For this define the ring epimorphism $\pi : \mathbb{Z}[G] \to \mathbb{Z}[G]e : x \mapsto xe$. Since $\pi(n-1) = ne - e = 0$ for all $n \in N$, one has that $\omega(G, N) \subseteq \ker(\pi)$. Therefore we have a unique morphism $\sigma : \mathbb{Z}[Ge] \to \mathbb{Z}[G]e$ such that $\pi = \sigma\,\Phi$. In particular $He = \pi(H) = \sigma(\Phi(H))$ is an epimorphic image of $\Phi(H)$ and hence $|He| \mid |\Phi(H)|$, as needed. ☐

5.2. Strong Zassenhaus property for some simple algebras.

5.2.1. *Fields and quaternion algebras.*

Proposition 5.3. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge$ is some quaternion algebra or a field. Then for any $H \leq \mathrm{V}(\mathbb{Z}G)$ the groups He and Ge are conjugated in $\mathbb{Q}Ge$.*

Proof. **TO COMPLETE to all quaternions** Suppose $\mathbb{Q}G_e$ is isomorphic to a field F. The unit group of its unique maximal order (i.e. its rings of integers) is a finitely generated abelian group. Thus He and Ge are subgroups of the torsion group which is cyclic. Hence the dividing orders yield that $He \leq Ge$, as desired. Suppose $\mathbb{Q}Ge \cong \left(\frac{-a,-b}{\mathbb{Q}}\right)$, then by [31, Theorem 11.5.14] $\mathcal{U}(\mathbb{Z}Ge)$ is cyclic except if $(a,b) = (-1,-1)$ or $(a,b) = (-1,-3)$ and $\mathbb{Z}Ge$ is the Lipschitz order, the Hurwitz order or the maximal order[16] of $\left(\frac{-1,-3}{\mathbb{Q}}\right)$. Recall that the last two cases are the unique maximal order, thus we already have that $He \leq Ge$ if $\mathbb{Z}Ge$ is not the Lipschitz order in $\left(\frac{-1,-1}{\mathbb{Q}}\right)$. In the remaining case $Ge \cong Q_8$ and He is some subgroup of the unit group of the Hurwitz quaternions (i.e. a subgroup of $\mathrm{SL}(2,3) \cong Q_8 \rtimes C_3$). Since $|He| \mid |Ge|$ we see that in fact $He \leq Ge$, finishing all possible cases. □

5.2.2. *The exceptional* $\mathrm{GL}_2(\mathcal{O})$ *case.* Do here like we did for $\mathrm{GL}_2(\mathbb{Z})$ in our previous paper.

Theorem 5.4. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2(D)$ with*[17] *$D \in \{\mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right) \mid a,b,d \in \mathbb{N}\}$. Then for any $H \leq \mathrm{V}(\mathbb{Z}G)$ the groups He and Ge are conjugated in $\mathbb{Q}Ge$.*

plan: component per component werken via expliciete amalgam van 'grote' finite index deelgroepen. Probleem is dat niet steeds de hele component zo'n decompositie heeft, maar mischien $\mathbb{Z}Ge$ desondanks toch steeds in eentje?

Proof. If it is $\mathrm{M}_2(\mathbb{Q})$, then $\mathcal{U}(\mathbb{Z}Ge) = \mathrm{GL}_2(\mathbb{Z}) \cong D_8 *_{C_2 \times C_2} D_{12}$. Therefore there exists some $\alpha_e \in \mathcal{U}(\mathbb{Z}Ge)$ such that $\alpha_e^{-1} Ge \alpha_e$ is a subgroup of D_8 or D_{12}. Since the $\mathbb{Q}$-span of Ge is $\mathrm{M}_2(\mathbb{Q})$ we get that $\alpha_e^{-1} Ge \alpha_e = D_6, D_{12}$ or D_8. In particular $|Ge|$ determines uniquely its isomorphism type. The same holds for He if He is non-abelian. Therefore, as $|He| \mid |Ge|$ by Lemma 5.2, we have the desired statement in that case. Suppose now that $|He| = 4$. If $He \cong C_4$, then He is up to conjugation a subgroup of D_8 and due to the dividing of the orders, again $He \leq Ge$ after conjugation. If it is an elementary abelian 2-group, then it is uniquely defined in both D_{12} and D_8. As this subgroup is amalgamated we are done. □

Proof of Corollary 5.1. By Lemma 5.2 $|He| \mid |Ge|$ for every primitive central idempotent e of $\mathbb{Q}G$. By Theorem 2.6 the simple algebra $\mathbb{Q}Ge$ is either a field, a specific type of quaternion algebra or some exceptional simple algebras. The strong Zassenhaus property of the former is proven in ?? and for the latter in Theorem 5.4. **TO UPDATE**. □

[16] A short concrete summary of the above orders can also be found before and after Theorem 3.14. of [4].
[17] In other words $\mathbb{Q}Ge$ is an exceptional component of $\mathbb{Q}G$.

TO DO list
Het volgende is een lijst van precieze te typen zaken die we al hebben geïdentificeerd (maar niet de heel kleine to do's die vaak in vorm van marginpar in het document staan):

SEctie 2:

(1) SSP for $cl(Q) = 2$ (geof)
(2) update theorem 2.5 to final form en dus nodige condities (geof)
(3) SSP for $cl(Q) = 1, 3$ (Robynn)
(4) Make final form of the table for proof theorem 2.1 (Doryan maakt, Geof bepaalt wat erin afhaneklijk van final theorem 2.5)
(5) Bepalen wat nog nodig is voor ($\mathrm{M}_{\mathrm{exc}}$) overgaan naar subnormale deelgroepen
(6) Karakter theorie sectie afwerken, i.e. schrijf een equivalent van proposition 2.9 voor het geval dat Q abels is en $\mathrm{cd}(G) = \{1, 2, 4\}$ (**Wie?**)
—> bepalen als we echt willen.

Sectie 3:

(1) uitleg rond Block VSP en context (Geof)
(2) Bewijs Proposition 3.3. afwerken (Geof)
(3) Bewijs van 3.11 afwerken (Geof)
(4) Bepalen van de exacte statement + bewijs van Theorem 3.13 (Robynn)
–> eenmaal sectie 2 klaar.
(5) Zet statement in sectie 3.4 (Geof)
(6) Bewijzen en zo Sectie 3.4 (Robynn)

Rest:

(i) Tonen dat ($\mathrm{M}_{\mathrm{exc}}$) aan de rationel ISO.
(ii) Wat the Zassenhaus property for semisimple algebra is, i.e. sectie 5.2 (Geof)
(iii) Lemma 4.2 vervoledigen met $exp(He)$ en order van de elementen.
(iv) Aantonen dat sommige exceptionele 2×2 de strong zassenhaus hebben.

References

[1] S. A. Amitsur. Finite subgroups of division rings. *Trans. Amer. Math. Soc.*, 80:361–386, 1955. 7, 8
[2] S. A. Amitsur. Groups with representations of bounded degree. II. *Illinois J. Math.*, 5:198–205, 1961. 18
[3] Andreas Bächle, Geoffrey Janssens, Eric Jespers, Ann Kiefer, and Doryan Temmerman. A dichotomy for integral group rings via higher modular groups as amalgamated products. *J. Algebra*, 604:185–223, 2022. 2
[4] Andreas Bächle, Geoffrey Janssens, Eric Jespers, Ann Kiefer, and Doryan Temmerman. Abelianization and fixed point properties of units in integral group rings. *Math. Nachr.*, 296(1):8–56, 2023. 2, 3, 4, 7, 30
[5] Florian Eisele, Ann Kiefer, and Inneke Van Gelder. Describing units of integral group rings up to commensurability. *J. Pure Appl. Algebra*, 219(7):2901–2916, 2015. 6, 7
[6] F. Grunewald, A. Jaikin-Zapirain, and P. A. Zalesskii. Cohomological goodness and the profinite completion of Bianchi groups. *Duke Math. J.*, 144(1):53–72, 2008. 28
[7] B. Huppert. *Endliche Gruppen. I*, volume 134 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1967. 13
[8] Bertram Huppert and Olaf Manz. Degree-problems. I. Squarefree character degrees. *Arch. Math. (Basel)*, 45(2):125–132, 1985. 13
[9] I. M. Isaacs and D. S. Passman. A characterization of groups in terms of the degrees of their characters. II. *Pacific J. Math.*, 24:467–510, 1968. 11, 12, 13, 17
[10] I. Martin Isaacs. *Character theory of finite groups*. Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)]. 10, 11, 12, 13
[11] Geoffrey Janssens and Leo Margolis. On integral decomposition of unipotent elements in integral group rings, 2023. 5, 9, 18
[12] Geoffrey Janssens, Doryan Temmerman, and François Thilmany. Simultaneous ping-pong for finite subgroups of reductive groups, 2025. 11
[13] E. Jespers and W. Sun. Nilpotent decomposition in integral group rings. *J. Algebra*, 575:127–158, 2021. 5, 8, 18
[14] Eric Jespers and Angel del Río. A structure theorem for the unit group of the integral group ring of some finite groups. *J. Reine Angew. Math.*, 521:99–117, 2000. 2

[15] Eric Jespers and Ángel del Río. *Group ring groups. Vol. 1. Orders and generic constructions of units*. De Gruyter Graduate. De Gruyter, Berlin, 2016. 4, 5, 6, 8, 9, 14, 21, 29
[16] Eric Jespers and Ángel del Río. *Group ring groups. Vol. 1. Orders and generic constructions of units*. De Gruyter Graduate. De Gruyter, Berlin, 2016. 17, 18
[17] Eric Jespers, Guilherme Leal, and Angel del Río. Products of free groups in the unit group of integral group rings. *J. Algebra*, 180(1):22–40, 1996. 2
[18] Eric Jespers, Gabriela Olteanu, Ángel del Río, and Inneke Van Gelder. Central units of integral group rings. *Proc. Amer. Math. Soc.*, 142(7):2193–2209, 2014. 4
[19] Eric Jespers, Antonio Pita, Ángel del Río, Manuel Ruiz, and Pavel Zalesskii. Groups of units of integral group rings commensurable with direct products of free-by-free groups. *Adv. Math.*, 212(2):692–722, 2007. 2, 4, 6, 7, 14, 20, 22, 23, 24, 26, 27
[20] E. Kleinert. Two theorems on units of orders. *Abh. Math. Sem. Univ. Hamburg*, 70:355–358, 2000. 22
[21] E. Kleinert and Á. del Río. On the indecomposability of unit groups. *Abh. Math. Sem. Univ. Hamburg*, 71:291–295, 2001. 22
[22] Ernst Kleinert. Units of classical orders: a survey. *Enseign. Math. (2)*, 40(3-4):205–248, 1994. 2
[23] Reginald Koo. A Classification of Matrices of Finite Order over $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$. *Math. Mag.*, 76(2):143–148, 2003. 25, 26
[24] Guilherme Leal and Angel del Río. Products of free groups in the unit group of integral group rings. II. *J. Algebra*, 191(1):240–251, 1997. 29
[25] Thomas Noritzsch. Groups having three complex irreducible character degrees. *J. Algebra*, 175(3):767–798, 1995. 12, 13
[26] A. Olivieri, Á. del Río, and J. J. Simón. On monomial characters and central idempotents of rational group algebras. *Commun. Algebra*, 32(4):1531–1550, 2004. 5
[27] Antonio Pita, Ángel Del Río, and Manuel Ruiz. Groups of units of integral group rings of Kleinian type. *Trans. Amer. Math. Soc.*, 357(8):3215–3237, 2005. 2
[28] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. 27
[29] Sudarshan K. Sehgal. Nilpotent elements in group rings. *Manuscripta Math.*, 15:65–80, 1975. 4, 6, 10
[30] M. Shirvani and B. A. F. Wehrfritz. *Skew linear groups*, volume 118 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1986. 8, 9
[31] John Voight. *Quaternion algebras, v.0.9.13*. June 10, 2018. https://math.dartmouth.edu/~jvoight/quat/quat-book-v0.9.13.pdf. 30

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