Runtime Analysis

Big O, Θ, some simple sums.

(see section 1.2 for motivation)

Notes, examples and code adapted from Data Structures and Other Objects Using C++ by Main & Savitch

Sums - review

- Some quick identities (just like integrals):

$$\sum_i cf(i) = c\sum_i f(i), \text{ where c is constant}$$

$$\sum \left(f(i) + g(i)\right) = \sum f(i) + \sum g(i)$$

- Not like the integral:

$$\sum_{i=b}^{a} f(i) = \sum_{i=a}^{b} f(i)$$

Closed form for simple sums

- A couple easy ones you really should stick in your head:

$$\sum_{i=a}^{b} c = c + c + \ldots + c = (|b - a| + 1)c, \text{ where c is constant}$$

Remember, +1 for the mule you're sitting on

$$\sum_{i=1}^{m} i = 1 + 2 + 3 + \ldots + m = \frac{m(m+1)}{2}$$

(Thank Prof. Gauss for that one.)

Example (motivation), sect 1.2

- **Problem:** to count the # of steps in the Eiffel Tower
- **Setup:** Jack and Jill are at the top of the tower w/paper and a pencil
- Let $n \equiv$ # of stairs to climb the Eiffel Tower

Eiffel Tower, alg. 1

- 1st attempt:
  - Jack takes the paper and pencil, walks stairs
  - Makes mark for each step
  - Returns, shows Jill
- Runtime:
  - # steps Jack took: 2n
  - # marks: n
  - So,

    $T_1(n) = 2n + n = 3n$

Eiffel Tower, alg. 2

- 2nd plan:
  - Jill doesn't trust Jack, keeps paper and pencil
  - Jack descends first step
  - Marks step w/hat
  - Returns to Jill, tells her to make a mark
  - Finds hat, moves it down a step
  - etc.

Eiffel Tower, alg. 2 (cont.)

- Analysis:
  - # marks = n
  - # steps = 2 (1+2+3+…+n)

    = 2 * n(n+1)/2

    = $n^2 + n$
  - So,

    $T_2(n) = n^2 + 2n$

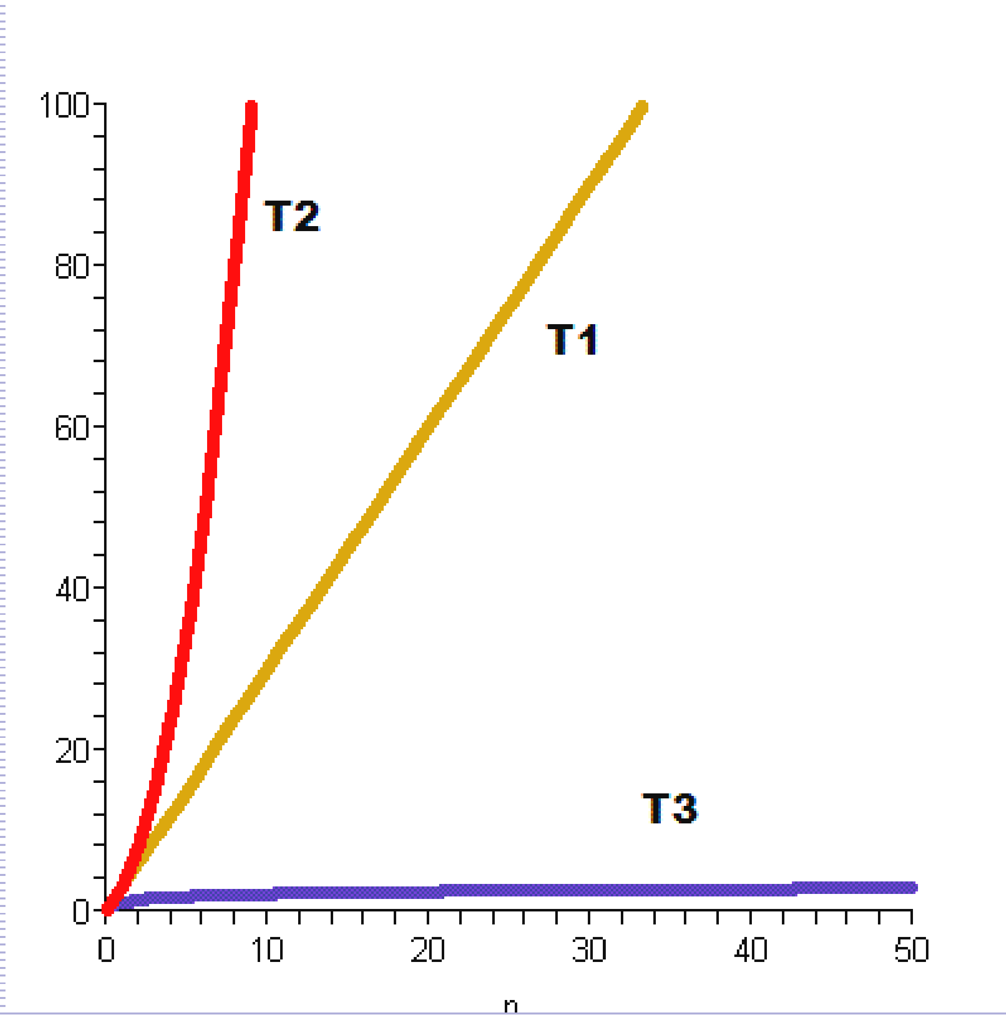
Eiffel Tower, alg. 3

- 3^{rd} try:
  - Jack & Jill see their friend Steve on the ground
  - Steve points to a sign
  - Jill copies the number down
- Runtime:
  - # steps Jack took: 0
  - # marks: $\log_{10}(n)$
  - So,

    $T_3(n) = \log_{10}(n)$

Comparison of algorithms

- Cost of algorithm (T(n)) vs. number of inputs, n, (the number of stairs).



Wrap-up

- $T_1(n)$ is a line
  - So, if we double the # of stairs, the runtime doubles
- $T_2(n)$ is a parabola
  - So, if we double the # of stairs, the runtime quadruples (roughly)
- $T_3(n)$ is a logarithm
  - We'd have to multiply the number of stairs by a factor of 10 to increase T by 1 (roughly)
  - Very nice function

Some “what ifs”

- Suppose 3 stairs (or 3 marks) can be made per 1 second
- (There are really 2689 steps)

	n	2n	3n
$T_1(n)$	45 min	90 min	135 min
$T_2(n)$	28 days	112 days	252 days
$T_3(n)$	2 sec	2 sec	2 sec

More observations

- While the relative times for a given n are a little sobering, what is of larger importance (when evaluating algorithms) is how each function grows
  - T_3 apparently doesn't grow, or grows very slowly
  - T_1 grows linearly
  - T_2 grows quadratically

Asymptotic behavior

- When evaluating algorithms (from a design point of view) we don’t want to concern ourselves with lower-level details:
  - processor speed
  - the presence of a floating-point coprocessor
  - the phase of the moon
- We are concerned simply with how a function grows as n grows *arbitrarily large*
- I.e., we are interested in its *asymptotic behavior*

Asymptotic behavior (cont.)

- As n gets large, the function is dominated more and more by its highest-order term (so we don't really need to consider lower-order terms)
- The coefficient of the leading term is not really of interest either. A line w/a steep slope will eventually be overtaken by even the laziest of parabolas (concave up). That coefficient is just a matter of scale. Irrelevant as $n \to \infty$

Big-O, Θ

- A formal way to present the ideas of the previous slide:

T(n) = O(f(n)) iff there exist constants k, n_0 such that:

k*f(n) > T(n)

for all $n>n_0$

- In other words, T(n) is bound above by f(n). I.e., f(n) gets on top of T(n) at some point, and stays there as $n \to \infty$
- So, T(n) grows no faster than f(n)

Θ

- Further, if f(n) = O(T(n)), then T(n) grows no slower than f(n)
- We can then say that T(n) = Θ(n)
- I.e., T(n) can be bound both above and below with f(n)

Setup

- First we have to decide what it is we're counting, what might vary over the algorithm, and what actions are constant
- Consider:

```
for( i=0; i<5; ++i )
  ++cnt;
```

  - i=0 happens exactly once
  - ++i happens 5 times
  - i<5 happens 6 times
  - ++cnt happens 5 times

- If **`i`** and **`cnt`** are **`int`**s, then assignment, addition, and comparison is constant (exactly 32 bits are examined)
- So, i=0, i<5, and ++i each take some constant time (though probably different)
- We may, for purposes of asymptotic analysis, ignore the overhead:
  - the single i=0
  - the extra i<5

  and consider the cost of executing the loop a *single* time

Setup (cont.)

- We decide that ++cnt is a constant-time operation (integer addition, then integer assignment)
- So, a single execution of the loop is done in constant time
- Let this cost be c :

Setup (cont.)

- So, the total cost of executing that loop can be given by:

$$T(n) = \sum_{i=0}^{4} c = 5c$$

- Constant time
- Makes sense. Loop runs a set number of times, and each loop has a constant cost

Setup (cont.)

- $T(n) = 5c$, where c is constant
- We say $T(n) = O(1)$
- From the definition of Big-O, let $k = 6c$:

  $6c(1) > 5c$
- This is true everywhere, for all n, so, for all $n>0$, certainly. Easy

Eg 2

- Consider this loop:

```
for( i=0; i<n; ++i )
  ++cnt;
```

- Almost just like last one:

$$T(n) = \sum_{i=0}^{n-1} c = cn$$

Eg 2 (cont.)

- Now, T(n) is linear in n
- T(n) = O(n)
- This means we can multiply n by something to get bigger than T(n):

  $\_\_$ n > cn ,   let k = 2c

  2cn > cn
- This isn't true everywhere. We want to know that it becomes true somewhere and stays true as $n \to \infty$

Eg. 2 (cont.)

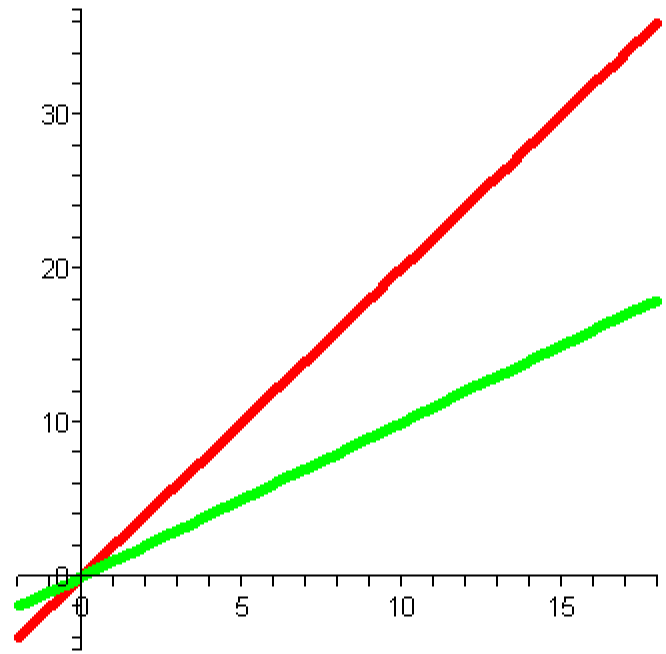
- Solve the inequality:

  2cn > cn

  cn > 0

  n > 0 (c is strictly positive, so, not 0)

- cn gets above T(n) at n=0 and stays there as n gets large
- So, T(n) grows no faster than a line



Eg. 3

- Let’s make the previous example a little more interesting:
- Say $T(n) = 2cn + 13c$
- $T(n) = O(n)$
- So, find some k such that

  $kn > 2cn + 13c$ (past some n_0)
- Let $k = 3c$. =>

  $3cn > 2cn + 13c$

Eg. 3 (cont.)

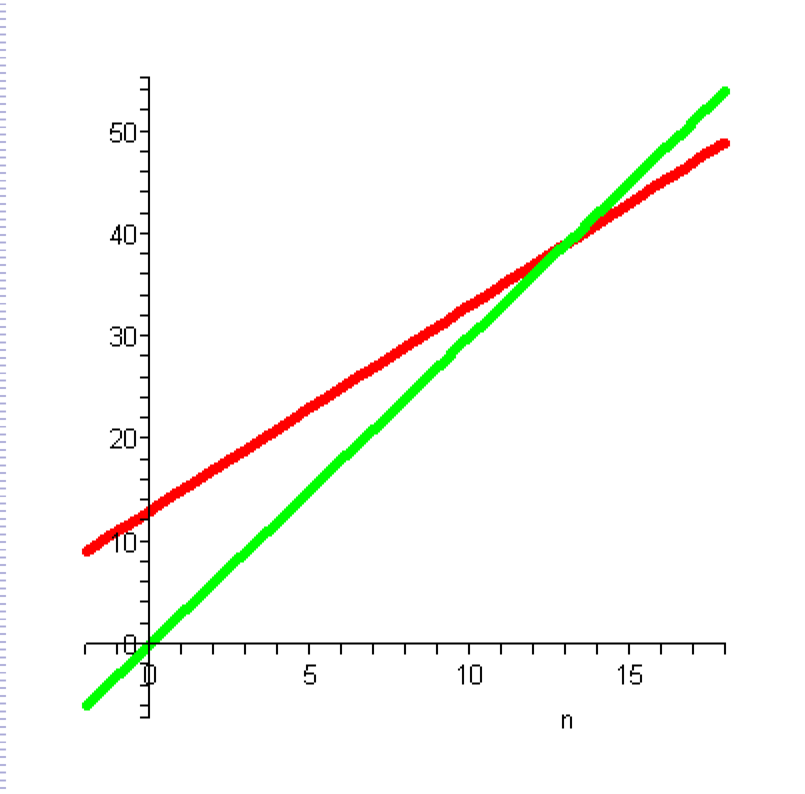
- Find values of n for which the inequality is true:

  $3cn > 2cn + 13c$

  $cn > 13c$

  $n > 13$

- 3cn gets above T(n) at n=13, and stays there.
- T(n) grows no faster than a line



Eg 4

- Nested loops:

```
for( i=0; i<n; ++i )
  for( j=1; j<=n; ++j )
    ++cnt;
```

- Runtime given by:

$$T(n) = \sum_{i=0}^{n-1} \left(\sum_{j=1}^{n} c \right) = \sum_{i=0}^{n-1} cn = cn^2$$

Eg. 4 (cont.)

- Claim: $T(n) = O(n^2)$

$\Rightarrow$there exists a constant k such that

$kn^2 > cn^2$, let $k = 2c$:

$2cn^2 > cn^2$

- Where is this true?

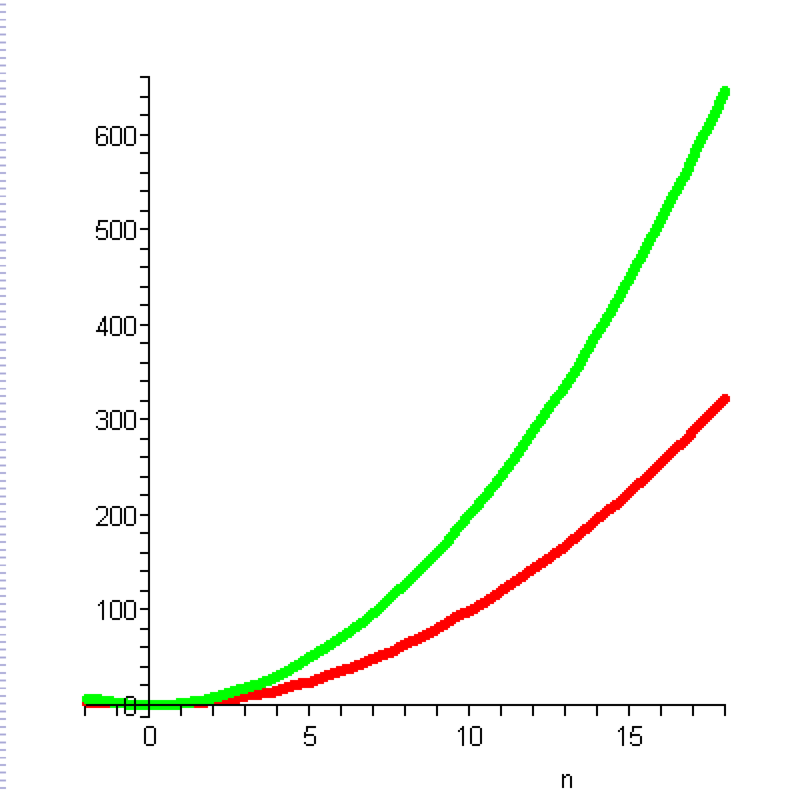
$cn^2 > 0$

$n^2 > 0$

$n > 0$

Eg. 4 (cont.)

- $2cn^2$ gets above cn^2 at n=0 and stays there
- T(n) is bound above by a parabola
- T(n) grows no faster than a parabola



Eg. 5

- Let's say

  $T(n) = 2cn^2 + 2cn + 3c$
- Claim: $T(n) = 0(n^2)$
- We just need to choose a k larger than the coefficient of the leading term.
- Let k = 3c

$\Rightarrow 3cn^2 > 2cn^2 + 2cn + 3c$

- Where? (Gather like terms, move to one side)

  $cn^2 - 2cn - 3c > 0$

Eg. 5 (cont.)

- This one could be solved directly, but it is usually easier to find the roots of the parabola on the left (which would be where our original 2 parabolas intersected)
- This happens at n=-1 and n=3
- So, plug something like n=4 into our original inequality. Is it true?
- Then it's true for all n>3 (since we know they don't intersect again)

Eg. 5 (cont.)

- $3cn^2$ gets above T(n) at n=3 and stays there
- T(n) grows no faster than a parabola
- T(n) can be bound above by a parabola