PARAMETER ESTIMATION WITH NEURAL NETWORKS SUBJECT TO PDE CONSTRAINTS

Consider

$$\nabla \cdot (k(x)\nabla u(x)) = 0 \tag{1}$$

subject to the appropriate BCs.

Here, $k(x)$ is the unknown coefficient. We assume that N_k measurements of k and N_u measurements of u are available: $k^*(x_i)$ $(i = 1, ..., N_k)$ and $u^*(x_i)$ $(i = 1, ..., N_u)$.

Define NNs for $k(x)$ and $u(x)$, $\hat{k}(x;\gamma) = \mathcal{NN}_k(x;\gamma)$ and $\hat{u}(x;\theta) = \mathcal{NN}_u(x;\theta)$. Substituting this in the government equation yields:

$$f(x;\gamma,\theta) = \nabla \cdot (\hat{k}(x)\nabla \hat{u}(x)) = \mathcal{NN}(x;\gamma,\theta). \tag{2}$$

Then, we define the loss function as:

(3)

$$\mathcal{L}(\theta,\gamma) = \frac{1}{N_k}\sum_{i=1}^{N_k}(\hat{k}(x_i;\gamma)-k^*(x_i))^2+\frac{1}{N_u}\sum_{i=1}^{N_u}(\hat{u}(x_i;\theta)-u^*(x_i))^2+\frac{1}{N_c}\sum_{i=1}^{N_c}f(x_i;\gamma,\theta)^2$$

In the last term, the N_c collocation points could be chosen uniformly or non-uniformly depending on the problem.