deaps

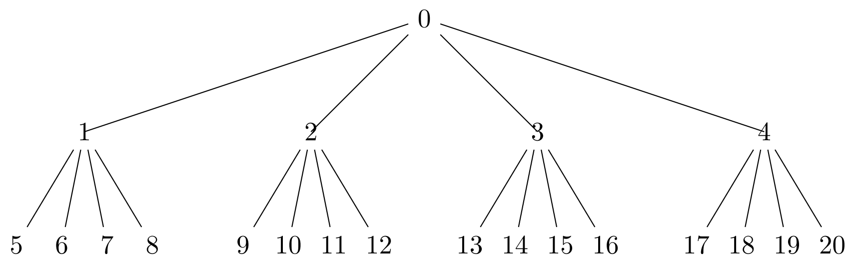
CSCI 112, Labs 9

d-ary heaps: A **d-ary heap** is like a binary heap, but (with one possible exception) non-leaf nodes have d children instead of 2 children.

Indexing in the array:

- The root of the tree is stored at position 0.
- The d children of node at position n are stored at positions $dn + i$ for $i = 1, \ldots, d$.

  For example, in a 4-ary heap, the root is at 0, the children of 0 are at $0+1 = 1$ through $0+4 = 4$, the children of 1 are at $4+1 = 5$ through $4+4 = 8$, and so on, as in the figure:



- To find the parent of a node at i, take $\lfloor((i-1)/d)\rfloor$. Some examples from the tree above:

$$\begin{aligned}
\text{parent}(8) &= \lfloor((8-1)/4)\rfloor = \lfloor 7/4 \rfloor = \lfloor 1.75 \rfloor = 1 \\
\text{parent}(9) &= \lfloor((9-1)/4)\rfloor = \lfloor 2 \rfloor = 2 \\
\text{parent}(10) &= \lfloor((10-1)/4)\rfloor = \lfloor 9/4 \rfloor = 2
\end{aligned}$$

What is the height of a d-ary heap of n elements in terms of n and d?

As can be seen by examining the above figure, the maximum number of elements that can be stored in a d-ary heap of height h is

$$\max(n) = \sum_{i=1}^{h} d^i = \frac{d^{h+1}-1}{d-1}$$

In the above example, $d = 4$, $h = 2$, giving

$$\frac{d^{h+1}-1}{d-1} = \frac{4^{2+1}-1}{4-1} = \frac{64-1}{3} = 21$$

So it checks out.

To find h, given d and n, we need to find the smallest h such that the above fraction is greater than or equal to n, in other words,

$$h = \min\left\{ h : \frac{d^{h+1}-1}{d-1} \geq n \right\}$$

Working with that inequality, we want the smallest h such that

$$\begin{aligned}
\frac{d^{h+1}-1}{d-1} &\geq n \\
d^{h+1}-1 &\geq n(d-1) \\
d^{h+1} &\geq n(d-1)+1 \\
\log_d(d^{h+1}) &\geq \log_d(n(d-1)+1) \\
h+1 &\geq \log_d(n(d-1)+1) \\
h &\geq \log_d(n(d-1)+1)-1
\end{aligned}$$

and hence

$$h = \lceil \log_d(n(d-1)+1)-1 \rceil$$

Let's try that on our example. With $d=4$ and $n=21$ we get

$$\begin{aligned}
h &= \lceil \log_d(n(d-1)+1)-1 \rceil \\
&= \lceil \log_4((21)(3)+1)-1 \rceil \\
&= \lceil \log_4(63+1)-1 \rceil \\
&= \lceil \log_4(64)-1 \rceil \\
&= \lceil 3-1 \rceil \\
&= 2
\end{aligned}$$

So that works. With one more item, though, $d=4$ and $n=22$, and we get

$$\begin{aligned}
h &= \lceil \log_d(n(d-1)+1)-1 \rceil \\
&= \lceil \log_4((22)(3)+1)-1 \rceil \\
&= \lceil \log_4(66+1)-1 \rceil \\
&= \lceil \log_4(67)-1 \rceil \\
&= \lceil 3.033-1 \rceil \\
&= 3
\end{aligned}$$

So, yes, with one more item we will need another level in the height of our tree, which is obvious from the example and so our formula checks out here, too.

- We follow one path from root to leaf, looping over up to d children at each level, so running time is $O(d \log_d n)$. We loop over d, but for a d-ary heap d is a constant and does not depend on the input, so the loop is constant for all inputs, so we could say $O(\log_d n)$

Give an efficient implementation of INSERT in a d-ary max-heap. Analyze its running time in terms of d and n.

- Nothing has to be changed as long as we use the new definition of PARENT. Time is $O(\log_d n)$.

Give an efficient implementation of INCREASE-KEY(A, i, k), which flags an error if $k < A[i]$, but otherwise sets $A[i] = k$ and then updates the d-ary max-heap structure appropriately. Analyze its running time in terms of d and n.

- Nothing has to be changed as long as we use the new definition of PARENT. Time is $O(\log_d n)$.