Book of Proof: Part III, More on Proof

October 15, 2018

If-and-Only-If Proof

Outline for If-and-Only-If Proof

Proposition P if and only if Q.

Proof.

"Only if"

[Prove $P \Rightarrow Q$ by whatever means you can.]

"If"

[Prove $Q \Rightarrow P$ by whatever means you can.]

Equivalent Statements

Theorem Suppose A is an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.

b. $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

c. $Ax = 0$ has only the trivial solution.

d. The reduced row echelong form of A is I_n.

e. $\det(A) \neq 0$.

f. The matrix A does not have 0 as an eigenvector.

Equivalent Statements

$$
\begin{array}{ccccc}
a & \Rightarrow & b & \Rightarrow & c \\
\Uparrow & & & & \Downarrow \\
f & \Leftarrow & e & \Leftarrow & d
\end{array}
$$

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Existence Proofs

Proposition There exists an even prime number.

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Proof. Two is an even prime number. ■

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Proposition There exists an integer that can be expressed as the sum of two perfect cubes in two different ways.
Proof.

$$1^3 + 12^3 = 1729$$
$$9^3 + 10^3 = 1729$$

∎

Example

Proposition 7.1 If $a, b \in \mathbb{N}$ then
there exist $k, \ell \in \mathbb{Z}$ for which $\gcd(a, b) = ak + b\ell$.

For example:

$$\gcd(12, 18) = 6 \text{ and } 6 = (-1)12 + (1)18$$
$$\gcd(9, 21) = 3 \text{ and } 3 = (-2)9 + (1)21$$

Example

Proposition 7.1 If $a, b \in \mathbb{N}$ then
there exist $k, \ell \in \mathbb{Z}$ for which $\gcd(a, b) = ak + b\ell$.

Proof. Suppose $a, b \in \mathbb{N}$.
Consider the set $A = \{ax + by : x, y \in \mathbb{Z}\}$.
A contains positive integers and 0.
Let $d \in A$ be the smallest positive integer.
$d = ak + b\ell$ for some $k, \ell \in \mathbb{Z}$.
We will show that $d = \gcd(a, b)$.
First, prove that $d \mid a$ and $d \mid b$.
Then show that it is the largest such number.

Example

Proposition 7.1 If $a, b \in \mathbb{N}$ then
there exist $k, \ell \in \mathbb{Z}$ for which $\gcd(a, b) = ak + b\ell$.

Proof (continued).
$d = ak + b\ell$ is the smallest positive element of
$A = \{ax + by : x, y \in \mathbb{Z}\}$.
Show that $d \mid a$.
Use division algorithm: $a = qd + r$, where $0 \leq r < d$.

$$\begin{aligned} r &= a - qd \\ &= a - q(ak + b\ell) \\ &= a(1 - qk) + b(-q\ell) \in A \end{aligned}$$

But d is the smallest element of A, and $0 \leq r < d$, so $r = 0$.
So $a = qd + r = qd$ and so $d \mid a$.

Example

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But d is the smallest element of A, and $0 \leq r < d$, so $r = 0$.
So $a = qd + r = qd$ and so $d \mid a$.
A similar argument shows $d \mid b$.
So $d \leq \gcd(a, b)$.

Example

Proposition 7.1 If $a, b \in \mathbb{N}$ then
there exist $k, \ell \in \mathbb{Z}$ for which $\gcd(a, b) = ak + b\ell$.
Proof (continued). $d = ak + b\ell$ is the smallest positive element of $A = \{ax + by : x, y \in \mathbb{Z}\}$, and $d \mid a$ and $d \mid b$ so $d \leq \gcd(a, b)$.

$$
\begin{aligned}
a &= \gcd(a,b) \cdot m && m > 0 \\
b &= \gcd(a,b) \cdot n && n > 0 \\
d &= ak + b\ell \\
&= \gcd(a,b) \cdot mk + \gcd(a,b) \cdot n\ell \\
&= \gcd(a,b)(mk + n\ell) > 0 \\
d &\geq \gcd(a,b) \\
d &\leq \gcd(a,b) && d \mid a \text{ and } d \mid b \\
d &= \gcd(a,b)
\end{aligned}
$$

Proofs involving sets

How to show $a \in \{x : P(x)\}$

Show that $P(a)$ is true. $\blacksquare$

How to show $a \in \{x \in S : P(x)\}$

1. Verify that $a \in S$.
2. Show that $P(a)$ is true. $\blacksquare$

Proofs involving sets

How to Prove $A \subseteq B$
(Direct approach)

Proof. Suppose $a \in A$.
$\vdots$
Therefore $a \in B$. ■

How to Prove $A \subseteq B$
(Contrapositive approach)

Proof. Suppose $a \notin B$.
$\vdots$
Therefore $a \notin A$. ■

Proofs involving sets

How to Prove $A = B$

Proof.
[Prove that $A \subseteq B$.]
[Prove that $B \subseteq A$.]
$\blacksquare$

Disproof

How to disprove P:
Prove $\sim P$. $\blacksquare$

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How to disprove $\forall x \in S, P(x)$:
Produce an example of $x \in S$ where $P(x)$ is false. $\blacksquare$

Disproof

How to disprove P:
Prove $\sim P$. $\blacksquare$

How to disprove $\forall x \in S, P(x)$:
Produce an example of $x \in S$ where $P(x)$ is false. $\blacksquare$

How to disprove $P(x) \Rightarrow Q(x)$:
Produce an example of x where $P(x)$ is true but $Q(x)$ is false. $\blacksquare$

Proving facts about $\mathbb{N}$

n	sum of the first n odd natural numbers	n^2
1	$1 =$	1
2	$1+3 =$	4
3	$1+3+5 =$	9
4	$1+3+5+7 =$	16
5	$1+3+5+7+9 =$	25
$\vdots$	$\vdots$	$\vdots$
n	$1+3+5+7+9+11+\cdots+(2n-1) =$	n^2
$\vdots$	$\vdots$	$\vdots$

Proving facts about $\mathbb{N}$

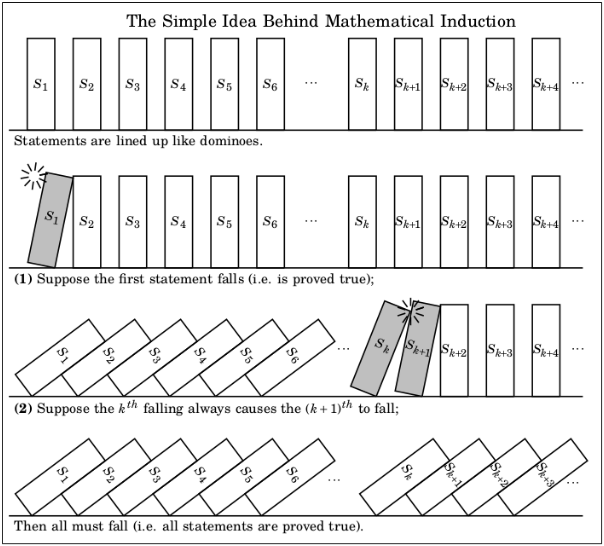
For all $n \in \mathbb{N}$,

$$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$$

$$\sum_{i=1}^{n} (2i - 1) = n^2$$

- Does not appear to be a conditional we can work from.
- Negating it does not lead to an easy contradiction.

Mathematical Induction



Mathematical Induction

Outline for Proof by Induction

Proposition The statements $S_1, S_2, S_3, \ldots$ are all true.

Proof.

(1) Prove that S_1 is true.

(2) Prove that for $k \in \mathbb{N}$, $S_k \Rightarrow S_{k+1}$ is true. ∎

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Proposition For all $n \in \mathbb{N}$, S_n.

Proof.

(1) Prove that S_1 is true.

(2) Prove that for $k \in \mathbb{N}$, $S_k \Rightarrow S_{k+1}$ is true. ∎

Example Proof by Induction

Proposition If $n \in \mathbb{N}$, then $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$.
(1) If $n = 1$, then we need to prove $1 = 1^2$, which is obviously true.
(2) Assume

$$1 + 3 + 5 + 7 + ... + (2k - 1) = k^2 \qquad \text{for some } k \in \mathbb{N}.$$

$$\vdots$$

Therefore,

$$1 + 3 + 5 + 7 + ... + (2(k + 1) - 1) = (k + 1)^2$$

$\blacksquare$

Example Proof by Induction

Proposition If $n \in \mathbb{N}$, then $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$.
(1) If $n = 1$, then $1 = 1^2$, which is true.
(2) Assume $1 + 3 + 5 + 7 + ... + (2k - 1) = k^2$ for some $k \in \mathbb{N}$.
Then

$$\begin{aligned} 1 + 3 + 5 + 7 + ... + 2(k+1) - 1 &= \\ 1 + 3 + 5 + 7 + ... + (2k-1) + (2(k+1) - 1) &= k^2 + (2(k+1) - 1) \\ &= k^2 + 2d + 1 \\ &= (k+1)^2 \end{aligned}$$

Therefore,

$$1 + 3 + 5 + 7 + ... + (2(k+1) - 1) = (k+1)^2$$

$\blacksquare$

Example Proof by Induction

Proposition For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(2i-1) = n^2$$

(1) If $n = 1$, then we need to prove $1 = 1^2$, which is obviously true.
(2) Assume, for some $k \in \mathbb{N}$ (the induction hypothesis):

$$\sum_{i=1}^{k}(2i-1) = k^2$$

$$\vdots$$

Therefore,

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$

Example Proof by Induction

Proposition If $n \in \mathbb{N}$, then

$$\sum_{i=1}^{n}(2i-1) = n^2$$

(1) If $n = 1$, then we need to prove $1 = 1^2$, which is obviously true.
(2)Assume

$$\sum_{i=1}^{k}(2i-1) = k^2 \qquad \text{for some } k \in \mathbb{N}.$$

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + (2(k+1)-1) \\
&= k^2 + 2k + 1 && \text{by induction hypothesis} \\
&= (k+1)^2
\end{aligned}$$

Example Proof by Induction

Proposition If $n \in \mathbb{N}^0$, then $5 \mid (n^5 - n)$.

Proof.

(1) If $n = 0$, then we need to prove $5 \mid (0^5 - 0)$, which is true.

(2) Assume $5 \mid (k^5 - k)$ for some $k \in \mathbb{N}^0$.

$\vdots$

Therefore $5 \mid ((k+1)^5 - (k+1))$. ■

What can we get from definitions?

Example Proof by Induction

Proposition If $n \in \mathbb{N}^0$, then $5 \mid (n^5 - n)$.
Proof.
(1) If $n = 0$, then we need to prove $5 \mid (0^5 - 0)$, which is true.
(2) Assume $5 \mid (k^5 - k)$ for some $k \in \mathbb{N}^0$.
Then $(k^5 - k) = 5a$ for some $a \in \mathbb{N}$.

$\vdots$

Then $((k+1)^5 - (k+1)) = 5b$ for some $b \in \mathbb{N}$.
Therefore $5 \mid ((k+1)^5 - (k+1))$. $\blacksquare$

Example Proof by Induction

Proposition If $n \in \mathbb{N}^0$, then $5 \mid (n^5 - n)$.

Proof.

(1) If $n = 0$, then we need to prove $5 \mid (0^5 - 0)$, which is true.

(2) Assume $5 \mid (k^5 - k)$ for some $k \in \mathbb{N}^0$.

Then $(k^5 - k) = 5a$ for some $a \in \mathbb{N}$.

$$\begin{aligned}
(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\
&= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k \\
&= 5a + 5k^4 + 10k^3 + 10k^2 + 5k \\
&= 5(a + k^4 + 2k^3 + 2k^2 + k)
\end{aligned}$$

Then $((k+1)^5 - (k+1)) = 5b$ for some $b \in \mathbb{N}$.

Therefore $5 \mid ((k+1)^5 - (k+1))$. ∎

SymPy

For some help with large algebraic expressions:
http://www.sympy.org/

```
>>> from sympy import *
>>> k = symbols('k')
>>> expr = (k+1)**5 - (k + 1)
>>> expand(expr)
k**5 + 5*k**4 + 10*k**3 + 10*k**2 + 4*k
>>> expand(expr - (k**5 - k))
5*k**4 + 10*k**3 + 10*k**2 + 5*k
```

Strong Induction

Outline for Proof by Induction

Proposition The statements $S_1, S_2, S_3, \ldots$ are all true.

Proof.

(1) Prove that S_1 is true.

(2) Prove that for $k \in \mathbb{N}$, $S_k \Rightarrow S_{k+1}$ is true. ■

Outline for Proof by Strong Induction

Proposition The statements $S_1, S_2, S_3, \ldots$ are all true.

Proof.

(1) Prove that S_1 is true. (Or the first several S_n.)

(2) Prove that for $k \in \mathbb{N}$, $(S_1 \wedge S_2 \wedge S_3 \wedge \ldots \wedge S_k) \Rightarrow S_{k+1}$ is true.

■

Smallest Counterexample

Outline for Proof by Induction

Proposition The statements $S_1, S_2, S_3, \ldots$ are all true.

Proof.

(1) Prove that S_1 is true.

(2) Prove that for $k \in \mathbb{N}$, $S_k \Rightarrow S_{k+1}$ is true. ∎

Outline for Proof by Smallest Counterexample

Proposition The statements $S_1, S_2, S_3, \ldots$ are all true.

Proof.

(1) Prove that S_1 is true.

(2) Suppose that not every S_n is true.

(3) Let S_{k+1} be the smallest false one.

(4) Then S_k is true and S_{k+1} is false.

(5) Use this to get a contradiction. ∎

Really just normal induction using proof by contradiciton.