

Summations 2

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Source

These notes are based on chapter 6 of

`http:`

`//www.cs.yale.edu/homes/aspnes/classes/202/notes.pdf`

and

`http://www.cs.yale.edu/homes/aspnes/pinewiki/`

`attachments/SummationNotation/summation-notation.pdf`

Summations

$$\sum_{i=0}^n (2i + 1) = 1 + 3 + 5 + 7 + \dots + (2n + 1)$$

Definition using recurrences

$$\sum_{i=a}^b f(i) = \begin{cases} 0 & \text{if } b < a \\ f(a) + \sum_{i=a+1}^b f(i) & \text{otherwise.} \end{cases}$$

$$\sum_{i=a}^b f(i) = \begin{cases} 0 & \text{if } b < a \\ \sum_{i=a}^{b-1} f(i) + f(b) & \text{otherwise.} \end{cases}$$

Scope

- ▶ Scope extends to the first addition or subtraction symbol.
- ▶ Best to include parentheses to avoid confusion,
- ▶ or move trailing terms to the beginning.

$$\begin{aligned}\sum_{i=1}^n i^2 + 1 &= \left(\sum_{i=1}^n i^2 \right) + 1 \\ &= 1 + \left(\sum_{i=1}^n i^2 \right) \\ &= 1 + \sum_{i=1}^n i^2 \\ &\neq \sum_{i=1}^n (i^2 + 1)\end{aligned}$$

Summation is linear

$$\sum_{i=n}^m ax_i = a \sum_{i=n}^m x_i$$

$$\sum_{i=n}^m (x_i + y_i) = \sum_{i=n}^m x_i + \sum_{i=n}^m y_i$$

Multiple sums

- ▶ The order is not important, provided the bounds of the inner sum don't depend on the index of the outer sum:

$$\sum_{i=a}^b \sum_{j=c}^d x_{ij} = \sum_{j=c}^d \sum_{i=a}^b x_{ij}$$

Products of sums

$$(a + b)(x + y + z) = ax + ay + az + bx + by + bz$$

$$\left(\sum_{i=a}^b x_i \right) \left(\sum_{j=c}^d y_j \right) = \sum_{i=a}^b \sum_{j=c}^d x_i y_j$$

Change of variables

Let

$$j = i - 1$$

$$i = j + 1$$

then,

$$\begin{aligned}\sum_{i=1}^n (i-1) &= \sum_{(j+1)=1}^n j \\ &= \sum_{j=0}^{n-1} j \\ &= \sum_{i=0}^{n-1} i\end{aligned}$$

Change of variables

Let

$$j = i + 1$$

$$i = j - 1$$

then,

$$\begin{aligned}\sum_{i=a}^b (i+1)^2 &= \sum_{(j-1)=a}^b j^2 \\ &= \sum_{j=a+1}^{b+1} j^2 \\ &= \sum_{i=a+1}^{b+1} i^2\end{aligned}$$

Sums over index sets

$$\sum_{i \in \{3,5,7\}} i^2 = 3^2 + 5^2 + 7^2$$

$$\sum_{A \subseteq S} |A|$$

$$\sum_{p < 1000, p \text{ is prime}} p^2$$

Confusing index sets

$$\sum_{1 \leq i < j \leq n} \frac{i}{j}$$

- ▶ The sum over all pairs of values (i,j) such that
 - ▶ $1 \leq i$
 - ▶ $i < j$
 - ▶ $j \leq n$

with each pair appearing exactly once.

Confusing index sets

$$\sum_{x \in A \subseteq S} |A|$$

- ▶ The sum over all sets A such that

- ▶ $x \in A$
- ▶ $A \subseteq S$

assuming that x and S are defined outside the summation.

Sums without explicit bounds

When the index set is understood from context some books use:

$$\sum_i i^2$$

Very sloppy, and discouraged.

Double sums

$$\sum_{i=0}^n \sum_{j=0}^i (i+1)(j+1)$$

When $n = 1$ we get

$$(0+1)(0+1) + [(1+1)(0+1) + (1+1)(1+1)] = 7$$

Closed forms

$$\sum_{i=1}^n 1 = n$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\begin{aligned}\sum_{i=0}^n r^i &= \frac{1 - r^{n+1}}{1 - r} \\ &= \frac{r^{n+1} - 1}{r - 1}\end{aligned}$$

Quick “proof”

Start with

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

Then

$$\begin{aligned}\sum_{i=0}^n r^i &= \sum_{i=0}^{\infty} r^i - r^{n+1} \sum_{i=0}^{\infty} r^i \\ &= \frac{1}{1-r} - \frac{r^{n+1}}{1-r} \\ &= \frac{1-r^{n+1}}{1-r}\end{aligned}$$

Solving summations

$$\begin{aligned}\sum_{i=0}^n (3(2^n) + 5) &= 3 \sum_{i=0}^n 2^n + 5 \sum_{i=0}^n 1 \\ &= 3(2^{n+1} - 1) + 5(n + 1) \\ &= 3(2^{n+1}) + 5n + 2\end{aligned}$$

Guess but verify

$$S(n) = \sum_{k=1}^n (2k - 1)$$

n	$S(n)$
0	0
1	1
2	$1 + 3 = 4$
3	$4 + 5 = 9$
4	$9 + 7 = 16$
5	$16 + 9 = 25$

$$S(n) = n^2$$

Guess!

Prove guess by induction

$$S(n) = \sum_{k=1}^n (2k - 1) \stackrel{?}{\iff} S(n) = n^2$$

$$S(0) = \sum_{k=1}^0 (2k - 1) = 0$$

Base case

$$S(n+1) = \sum_{k=1}^{n+1} (2k - 1)$$

Step

$$= \sum_{k=1}^n (2k - 1) + (2(n+1) - 1)$$

$$= n^2 + 2(n+1) - 1$$

Inductive hypothesis

$$= (n+1)^2$$

Strategies for asymptotic estimates

Pull out constant factors

$$\begin{aligned}\sum_{i=1}^n \frac{n}{i} &= n \sum_{i=1}^n \frac{1}{i} \\ &= nH_n \\ &= \Theta(n \log n)\end{aligned}$$

Strategies for asymptotic estimates

Bound using geometric series

$$\sum_{i=1}^n x^i = \frac{x^{n+1} - 1}{x - 1} = \Theta(x^n)$$

$$\sum_{i=1}^n 2^i = \Theta(2^n)$$

$$\sum_{i=1}^n 2^{-i} = \Theta(1)$$

Strategies for asymptotic estimates

Harmonic series

$$\sum_{i=1}^n \frac{1}{i} = H_n = \Theta(\lg n)$$

Strategies for asymptotic estimates

Bound part of the sum.

$$\begin{aligned}\sum_{i=1}^n i^3 &\leq \sum_{i=1}^n n^3 \\ &= O(n^4)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^n i^3 &\geq \sum_{i=n/2}^n i^3 \\ &\geq \sum_{i=n/2}^n (n/2)^3 \\ &= \Omega(n^4)\end{aligned}$$

Strategies for asymptotic estimates

Integrate!

For increasing functions:

$$\int_{a-1}^b f(x) dx \leq \sum_{i=a}^b f(i) \leq \int_a^{b+1} f(x) dx$$

Most of the functions we see in algorithms yield to this method!

Strategies for asymptotic estimates

Grouping terms.

The standard trick for showing that the harmonic series is unbounded.

$$\begin{aligned} &1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \dots \\ &\geq \\ &1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \dots \\ &= \\ &1 + 1/2 + 1/2 + 1/2 + \dots \end{aligned}$$

Final Notes

In practice, almost any sum you come across will be of the form:

$$\sum_{i=1}^n f(i)$$

where either:

- ▶ $f(n)$ is exponential.

Then it's bounded by a geometric series and the largest term dominates.

- ▶ $f(n)$ is polynomial.

Then $f(n/2) = \Theta(f(n))$ and the sum is $\Theta(nf(n))$ using the lower bound:

$$\sum_{i=n/2}^n f(n) = \Omega(nf(n))$$