Recurrences

$$
\begin{aligned}
f(1) &= 5 \\
f(n) &= f(n-1) + 3 && \text{(if } n > 1\text{)}
\end{aligned}
$$

Find some simple cases:

$$
\begin{aligned}
f(1) &= 5 \\
f(2) &= f(1) + 3 = 8 \\
f(3) &= f(2) + 3 = 11 \\
f(4) &= f(3) + 3 = 14
\end{aligned}
$$

Guess a solution: $f(n) = 3n + 2$. This is called a *closed form* since there is no recursion in the definition.
Check that it works on the simple cases:

$$
\begin{aligned}
3(1) + 2 &= 5 \\
3(2) + 2 &= 8 \\
3(3) + 2 &= 11 \\
3(4) + 2 &= 14
\end{aligned}
$$

So it looks like we're on the right track.
How can you *prove* that it's the correct solution?

Proving recurrence solution by induction

$$
\begin{aligned}
f(1) &= 5 \\
f(n) &= f(n-1) + 3 \qquad\qquad\qquad \text{(if } n > 1\text{)}
\end{aligned}
$$

Prove $f(n) = 3n + 2$ by induction. Base:

$$f(1) = 5 = 3(1) + 2$$

So $f(n) = 3n + 2$ is true when $n = 1$.
Now assume

$$f(n) = 3n + 2$$

and prove

$$f(n+1) = 3(n+1) + 2$$

Proof:

$$
\begin{aligned}
f(n+1) &= f(n) + 3 \\
&= 3n + 2 + 3 \\
&= 3(n+1) + 2
\end{aligned}
$$

Guess checks out. We have proved it is a correct solution by induction.

Proof by induction for recurrence relations

A simpler form of proof involves just plugging the closed form in for f and checking both clauses of the recurrence.

$$\begin{aligned} f(1) &= 5 \\ f(n) &= f(n-1) + 3 \qquad \text{(if } n > 1\text{)} \end{aligned}$$

Plugging in $f(n) = 3n + 2$ gives

$$\begin{aligned} 3(1) + 2 &= 5 \\ 3(n) + 2 &= 3(n-1) + 2 + 3 \qquad \text{(if } n > 1\text{)} \end{aligned}$$

and both of these check out, so it is proved by induction.

Solving recurrences without guessing

$$
\begin{aligned}
f(1) &= 5 \\
f(n) &= f(n-1) + 3 && \text{(if } n > 1\text{)}
\end{aligned}
$$

Note that the second line represents infinitely many equations:

$$
\begin{aligned}
f(n) &= f(n-1) + 3 \\
f(n-1) &= f(n-2) + 3 \\
f(n-2) &= f(n-3) + 3 \\
f(n-3) &= f(n-4) + 3 \\
&\cdots \\
f(n-(k-1)) &= f(n-k) + 3 \\
&\cdots \\
f(n-(n-2)) &= f(n-(n-1)) + 3 \\
&= f(1) + 3 \\
&= 8
\end{aligned}
$$

As long as the argument to f never goes below 1, all these are true.

Solving recurrences without guessing

$$
\begin{aligned}
f(1) &= 5 \\
f(n) &= f(n-1) + 3 && \text{(if } n > 1\text{)}
\end{aligned}
$$

Let's **add up** the left and right hand sides of some equations:

$$
\begin{aligned}
f(n) &= f(n-1) + 3 \\
f(n-1) &= f(n-2) + 3 \\
f(n-2) &= f(n-3) + 3 \\
f(n-3) &= f(n-4) + 3
\end{aligned}
$$

Note that many of the terms appear on both sides, so we can subtract them from the final equation:

$$
\begin{aligned}
f(n) &= \cancel{f(n-1)} + 3 \\
\cancel{f(n-1)} &= \cancel{f(n-2)} + 3 \\
\cancel{f(n-2)} &= \cancel{f(n-3)} + 3 \\
\cancel{f(n-3)} &= f(n-4) + 3
\end{aligned}
$$

We are left with:

$$
\begin{aligned}
f(n) &= f(n-4) + 3 + +3 + 3 + 3 \\
&= f(n-4) + 4(3)
\end{aligned}
$$

If we added up 10 terms, we would get

$$
f(n) = f(n-10) + 10(3)
$$

Adding up recurrence equations

$$\begin{aligned} f(1) &= 5 \\ f(n) &= f(n-1) + 3 && \text{(if } n > 1\text{)} \end{aligned}$$

If we added up k terms, we would get

$$f(n) = f(n-k) + k(3)$$

We can do this as long as $n - k \geq 1$. If we let $n - k = 1$, then $k = n - 1$ and we have

$$\begin{aligned} f(n) &= f(n-(n-1)) + (n-1)3 \\ &= f(1) + (n-1)3 \\ &= 5 + (n-1)3 \\ &= 3n + 2 \end{aligned}$$

Magic!

Let's see that again

$$
\begin{aligned}
f(1) &= 5 \\
f(n) &= f(n-1) + 3 && \text{(if } n > 1\text{)}
\end{aligned}
$$

The recurrence gives us lots of equations:

$$
\begin{aligned}
f(n) &= \cancel{f(n-1)} + 3 \\
\cancel{f(n-1)} &= \cancel{f(n-2)} + 3 \\
\cancel{f(n-2)} &= \cancel{f(n-3)} + 3 \\
\cancel{f(n-3)} &= \cancel{f(n-4)} + 3 \\
&\dots \\
\cancel{f(n-(n-2))} &= f(n-(n-1)) + 3
\end{aligned}
$$

Add them all up and cancelling gives:

$$
\begin{aligned}
f(n) &= f(1) + \sum_{k=1}^{n-1} 3 \\
&= 5 + (n-1)3 \\
&= 3n + 2
\end{aligned}
$$

Recurrences

$$
\begin{aligned}
f(1) &= 1 \\
f(n) &= 2f(n-1) && \text{(if } n > 1)
\end{aligned}
$$

Find some simple cases:

$$
\begin{aligned}
f(1) &= 1 \\
f(2) &= 2f(1) = 2 \\
f(3) &= 2f(2) = 4 \\
f(4) &= 2f(3) = 8
\end{aligned}
$$

We can guess $f(n) = 2^{n-1}$. Trying some simple cases shows this is a good guess:

$$
\begin{aligned}
2^{1-1} &= 1 \\
2^{2-1} &= 2 \\
2^{3-1} &= 4 \\
2^{4-1} &= 8
\end{aligned}
$$

We can now prove this guess correct by induction. Plug the closed form solution into the recursive equations:

$$
\begin{aligned}
2^{(1-1)} &= 1 \\
2^{(n-1)} &= 2(2^{((n-1)-1)}) && \text{(if } n > 1)
\end{aligned}
$$

And it checks out.
Now let's try to find that solution without guessing.

Recurrences solved without guessing

$$
\begin{aligned}
f(1) &= 1 \\
f(n) &= 2f(n-1) & \text{(if } n > 1\text{)}
\end{aligned}
$$

Get lots of equations:

$$
\begin{aligned}
f(n) &= 2f(n-1) \\
2f(n-1) &= 2(2(f(n-2))) \\
&= 2^2 f(n-2) \\
2^2 f(n-2) &= 2^2(2f(n-3)) \\
&= 2^3 f(n-3) \\
2^3 f(n-3) &= 2^4 f(n-4) \\
&\cdots \\
2^{k-1} f(n-(k-1)) &= 2^k f(n-k) \\
&\cdots \\
2^{n-2} f(2) &= 2^{n-1} f(1)
\end{aligned}
$$

We got that last one by considering the case where

$$n - k = 1$$

or, in other words,

$$k = n - 1$$

Add and cancel

$$
\begin{aligned}
f(1) &= 1 \\
f(n) &= 2f(n-1) \qquad\qquad\qquad \text{(if } n > 1\text{)}
\end{aligned}
$$

$$
\begin{aligned}
f(n) &= \cancel{2f(n-1)} \\
\cancel{2f(n-1)} &= \cancel{2^2 f(n-2)} \\
\cancel{2^2 f(n-2)} &= \cancel{2^3 f(n-3)} \\
\cancel{2^3 f(n-3)} &= \cancel{2^4 f(n-4)} \\
&\dots \\
\cancel{2^{n-2} f(2)} &= 2^{n-1} f(1)
\end{aligned}
$$

Using the fact that $f(1) = 1$ gives

$$f(n) = 2^{n-1}$$

That's the same solution that we guessed in the beginning.

We already demonstrated that this solution works on small cases, and we proved it works on all cases by induction.

A more tricky example

$$\begin{aligned} f(1) &= 5 \\ f(n) &= 3f(n-1) + 7 \end{aligned}$$

Find some simple examples:

$$\begin{aligned} f(1) &= 5 \\ f(2) &= 3f(1) + 7 = 22 \\ f(3) &= 3f(2) + 7 = 73 \\ f(4) &= 3f(3) + 7 = 226 \end{aligned}$$

What's the next number in the sequence?

$$5, 22, 73, 226, ?$$

Guessing may not be so helpful here.

Solve without guessing

$$
\begin{aligned}
f(1) &= 5 \\
f(n) &= 3f(n-1) + 7
\end{aligned}
$$

Get lots of equations:

$$
\begin{aligned}
f(n) &= 3f(n-1) + 7 \\
3f(n-1) &= 3(3f(n-2) + 7) \\
&= 3^2 f(n-2) + 3(7) \\
3^2 f(n-2) &= 3^2(3f(n-3) + 7) \\
&= 3^3 f(n-3) + 3^2(7) \\
3^3 f(n-3) &= 3^4 f(n-4) + 3^3(7) \\
&\cdots \\
3^{k-1} f(n-(k-1)) &= 3^k f(n-k) + 3^{k-1}(7) \\
&\cdots \\
3^{n-2} f(2) &= 3^{n-1} f(1) + 3^{n-2}(7)
\end{aligned}
$$

Solve without guessing

$$
\begin{aligned}
f(1) &= 5 \\
f(n) &= 3f(n-1) + 7
\end{aligned}
$$

Add and cancel:

$$
\begin{aligned}
f(n) &= \cancel{3f(n-1)} + 7 \\
\cancel{3f(n-1)} &= \cancel{3^2 f(n-2)} + 3(7) \\
\cancel{3^2 f(n-2)} &= \cancel{3^3 f(n-3)} + 3^2(7) \\
\cancel{3^3 f(n-3)} &= \cancel{3^4 f(n-4)} + 3^3(7) \\
&\cdots \\
\cancel{3^{k-1} f(n-(k-1))} &= \cancel{3^k f(n-k)} + 3^{k-1}(7) \\
&\cdots \\
\cancel{3^{n-2} f(2)} &= 3^{n-1} f(1) + 3^{n-2} 7
\end{aligned}
$$

We are left with

$$
\begin{aligned}
f(n) &= 3^{n-1}(5) + \sum_{i=0}^{n-2} 3^i(7) \\
&= 5(3^{n-1}) + 7 \sum_{i=0}^{n-2} 3^i
\end{aligned}
$$

Solve the summation

$$
\begin{aligned}
f(1) &= 5 \\
f(n) &= 3f(n-1) + 7
\end{aligned}
$$

We have a geometric series, so

$$
\begin{aligned}
f(n) &= 5(3^{n-1}) + 7\sum_{i=0}^{n-2} 3^i \\
&= 5(3^{n-1}) + 7\frac{3^{n-1}-1}{3-1} \\
&= \frac{10(3^{n-1}) + 7(3^{n-1}) - 7}{2} \\
&= \frac{17(3^{n-1}) - 7}{2}
\end{aligned}
$$

Seriously?

Check our solution

$$f(1) = 5$$
$$f(n) = 3f(n-1) + 7$$

Closed form:

$$f(n) = \frac{17(3^{n-1}) - 7}{2}$$

Remember our sequence?

$$5, 22, 73, 226, ?$$

Let's test $f(4) = 226$:

$$\begin{aligned} f(4) &= \frac{17(3^{4-1}) - 7}{2} \\ &= \frac{17(27) - 7}{2} \\ &= \frac{459 - 7}{2} \\ &= \frac{452}{2} \\ &= 226 \end{aligned}$$

Wow.

(Next number is 685, by the way. Obvious, no?)

Prove correct by induction

$$f(1) = 5$$
$$f(n) = 3f(n-1) + 7$$

Closed form:

$$f(n) = \frac{17(3^{n-1}) - 7}{2}$$

Base:

$$\frac{17(3^{1-1}) - 7}{2} \stackrel{?}{=} 5 \qquad \text{(Correct)}$$

Step:

$$\frac{17(3^{n-1}) - 7}{2} \stackrel{?}{=} 3\left(\frac{17(3^{n-2}) - 7}{2}\right) + 7$$

$$= \frac{17(3^{n-1}) - 3(7) + 14}{2}$$

$$= \frac{17(3^{n-1}) - 7}{2} \qquad \text{(Correct)}$$

We have proved by induction that our closed form is a solution to the original recurrence.