Big O definition

$$O(g(n)) = \{f(n) : \quad \text{there exists positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

When we say

$$f(n) = O(g(n))$$

we really mean

$$f(n) \in O(g(n))$$

For example

$$n^2 + 3n + 7 = O(n^2)$$

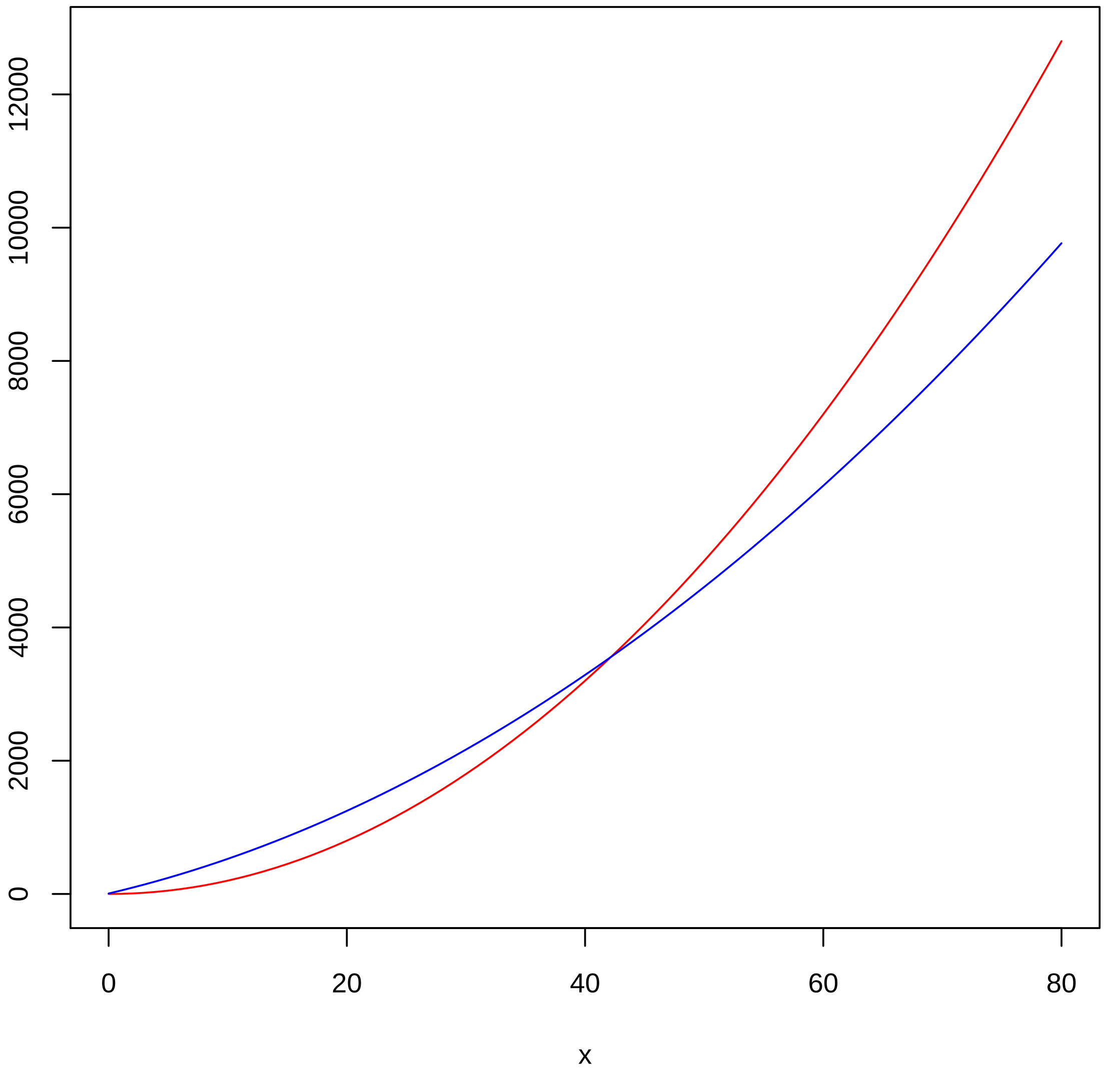
means

$$f(n) = n^2 + 3n + 7$$

is in the set

$$O(n^2)$$

$$n^2 + 42n + 7 = O(n^2)$$
$$n^2 + 42n + 7 \le 2n^2 \text{ for all } n \ge 50$$



Proof of $n^2 + 42n + 7 = O(n^2)$

$$\begin{aligned} n^2 + 42n + 7 &\leq n^2 + 42n^2 + 7n^2 \qquad \text{for } n \geq 1 \\ &= 50n^2 \end{aligned}$$

So $n^2 + 42n + 7 = O(n^2)$, with $c = 50$ and $n_0 = 1$

Proof of $4n^2 + 5n + 3 = O(n^2)$

$$\begin{aligned} 4n^2 + 5n + 3 &\le 4n^2 + 5n^2 + 3n^2 \qquad\qquad n \ge 1 \\ &= 12n^2 \end{aligned}$$

so $4n^2 + 5n + 3 = O(n^2)$ with $c = 12$ and $n_0 = 1$

Proof of $5n \lg n + 8n - 200 = O(n \lg n)$
Note: if $n \geq 2$ then $\lg n \geq 1$.

$$
\begin{aligned}
5n \lg n + 8n - 200 &\leq 5n \lg n + 8n \\
&\leq 5n \lg n + 8n \lg n \qquad \text{for } n \geq 2 \\
&\leq 13n \lg n
\end{aligned}
$$

So

$$5n \lg n + 8n - 200 = O(n \lg n)$$

with $c = 13$ and $n_0 = 2$

Proof of $(n+5)\lg(3n^2+7) = O(n \lg n)$

$$\begin{aligned}
(n+5)\lg(3n^2+7) &\le (n+5n)\lg(3n^2+7n^2) && n \ge 1 \\
&= 6n\lg(10n^2) \\
&\le 6n\lg(n^3) && n \ge 10 \\
&= 6n(3\lg(n)) \\
&= 18n\lg(n)
\end{aligned}$$

So $(n+5)\lg(3n^2+7) = O(n \lg n)$ for $c = 18$ and $n_0 = 10$

Proof of $(n^2 + 5\lg n)/(2n+1) = O(n)$

$$\begin{aligned}
\frac{n^2 + 5\lg n}{2n+1} &\le \frac{n^2 + 5n^2}{2n+1} \qquad\qquad n \ge 1 \\
&\le \frac{n^2 + 5n^2}{2n} \\
&= 3n
\end{aligned}$$

So $(n^2 + 5\lg n)/(2n+1) = O(n)$ for $c = 3$ and $n_0 = 1$

Useful facts

- For any $a < b$:

$$O(n^a) \subset O(n^b)$$

- For any $a, b > 0, c > 1$:

$$O(a) \subset O(\lg n) \subset O(n^b) \subset O(c^n)$$

  You can multiply to find

$$O(an) = O(n) \subset O(n \lg n) \subset O(n^{b+1}) \subset O(nc^n)$$

Other sets

$$
\begin{aligned}
&o(g(n)) = \\
&\{f(n) : \text{for any positive constant } c \\
&\qquad\qquad \text{there exists positive } n_0 \text{ such that} \\
&\qquad\qquad 0 \le f(n) < cg(n) \text{ for all } n \ge n_0\} \\
&O(g(n)) = \\
&\{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that} \\
&\qquad\qquad 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\} \\
&\Theta(g(n)) = \\
&\{f(n) : \text{there exist positive constants } c, d \text{ and } n_0 \text{ such that} \\
&\qquad\qquad 0 \le cg(n) \le f(n) \le dg(n) \text{ for all } n \ge n_0\} \\
&\Omega(g(n)) = \\
&\{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that} \\
&\qquad\qquad 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\} \\
&\omega(g(n)) = \\
&\{f(n) : \text{for any positive constant } c \\
&\qquad\qquad \text{there exists positive } n_0 \text{ such that} \\
&\qquad\qquad 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\}
\end{aligned}
$$

Number of anonymous functions

The number of anonymous functions is equal to the number of times the asymptotic notation appears.

$$\sum_{i=1}^{n} O(i)$$

Here we assume there is only one function, not n different functions.

Asymptotic notation in equations

Right hand side:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

means

$$2n^2 + 3n + 1 = 2n^2 + f(n)$$

for *some* $f(n) \in \Theta(n)$.

Left hand side:

$$2n^2 + \Theta(n) = \Theta(n^2)$$

means for *all* $f(n) \in \Theta(n)$,

$$2n^2 + f(n) = \Theta(n^2)$$

We can chain them:

$$\begin{aligned} 2n^2 + 3n + 1 &= 2n^2 + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

Relational properties

Reflexive:

$$f(n) = \Theta(f(n))$$

Symmetric:

$$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$$

Transitive:

$$(f(n) = \Theta(g(n)) \wedge g(n) = \Theta(h(n))) \Rightarrow f(n) = \Theta(h(n))$$

Transpose symmetry:

$$f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$$

$$f(n) = o(g(n)) \;\Leftrightarrow\; g(n) = \omega(f(n))$$

Note: We may have two functions f and g such that

$$\begin{aligned} f(n) &\neq o(g(n)) \\ f(n) &\neq O(g(n)) \\ f(n) &\neq \Theta(g(n)) \\ f(n) &\neq \Omega(g(n)) \\ f(n) &\neq \omega(g(n)) \end{aligned}$$

For example, $n^{1+\sin(n)}$ and n.

Limits

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Longrightarrow f(n) = o(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \Longrightarrow f(n) = \omega(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \Longrightarrow f(n) = O(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \Longrightarrow f(n) = \Omega(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \in (0, \infty) \Longrightarrow f(n) = \Theta(g(n))$$

L'Hospital's Rule

When

$$\lim_{x\to\infty} f(x) = \infty$$

and

$$\lim_{x\to\infty} g(x) = \infty$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$