Notes on Hash Tables

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April 26, 2018

Dictionary Operations

- Insert
- Search
- Delete

Hash table implementation of Dictionary

- Expected search time: $O(1)$
- Worst case search: $O(n)$

Hash table is generalization of an ordinary array

- With array, the key k is the position k in the array.
- Given a key k, we find the element with key k by **direct addressing**.
- Direct addressing only applicable when we can afford to allocate an array with one position for every key.

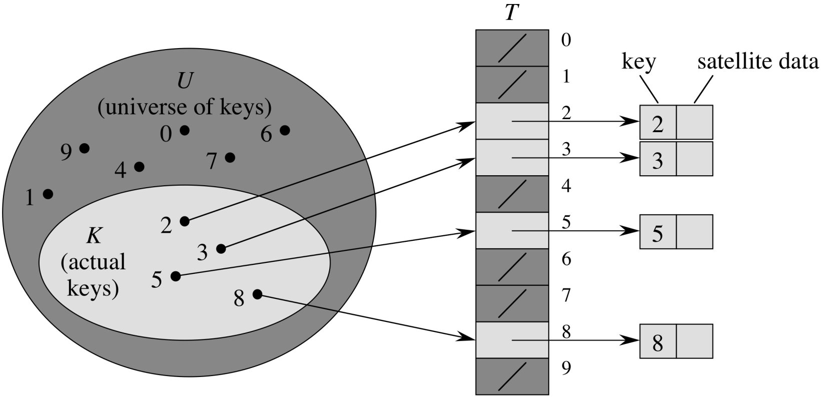
Use hash table when we don't have one position for each key

- Number of keys stored is small relative to the number of possible keys.
- Hash table is an array with size proportional to the number of keys stored, not the number of possible keys.
- Given a key k, don't use k to index the array.
- Instead, compute a function of k and use that to index the array.
- This function is called a **hash function**.
- Have to solve issue of what to do when hash function maps multiple keys to same table entry.
    - chaining
    - open addressing

Direct-address tables

- Scenario:
  - Maintain a dynamic set
  - Each element has a key drawn from a universe $U = \{0, 1, \ldots, m-1\}$ where m isn't too large.
  - No two elements have the same key.
- Represent by a **direct-address table**, or array, $T[0 \ldots m-1]$:
  - Each *slot*, or position, corresponds to a key in U.
  - If there's an element x with key k, then $T[k]$ contains a pointer to x.
  - Otherwise, $T[k]$ is empty, represented by NIL.

Direct-address table



Direct-Address-Search(T,k)

```
1  **return** $T[k]$
```

Direct-Address-Insert(T,k)

```
1  $T[key[x]] = x$
```

Direct-Address-Delete(T,k)

```
1  $T[key[x]] = \text{NIL}$
```

All operations $O(1)$.

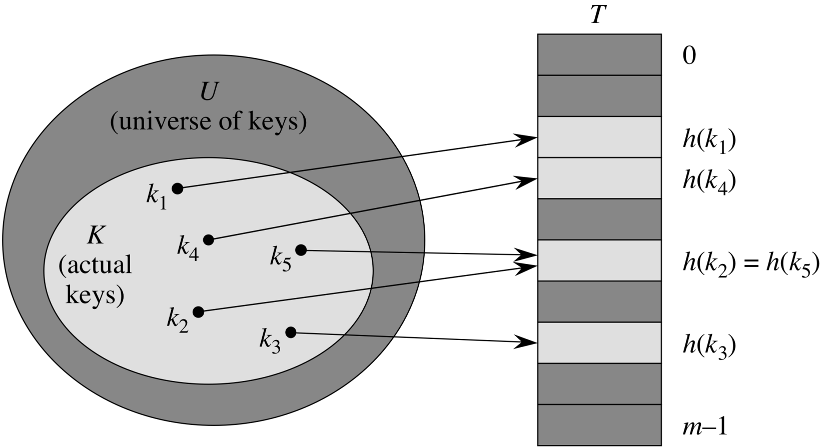
Hash tables

- If U is large, storing a table of size $|U|$ is impractical.
- Often the set K of keys actually used is small compared to U.
  - Most of the space in a direct-access table is wasted.
- When K is much smaller than U, a hash table requires much less space than a direct-address table.
- Can reduce storage requirements to $\Theta(|K|)$
- Can still get $O(1)$ search time on *average*, but not *worst* case.

Hash table idea

- Instead of storing an element with key k in slot k, use a function h and store the element in slot $h(k)$.
- h is called a **hash function**
- $h : U \rightarrow \{0, 1, \ldots, m-1\}$
- $m \ll |U|$
- $h(k)$ is a legal slot number in T
- We say k *hashes* to $h(k)$

Collisions

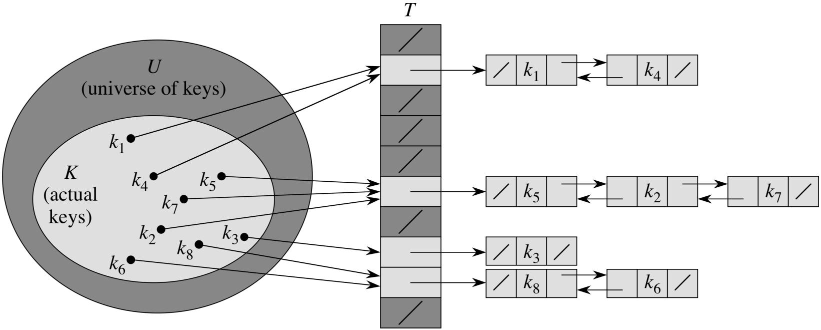

T
0
U
(universe of keys)
$h(k_1)$
$h(k_4)$
k_1
K
(actual
keys)
k_4
k_5
$h(k_2) = h(k_5)$
k_2
k_3
$h(k_3)$
$m–1$

Collisions

- When two or more keys hash to the same slot.
- Can happen when there are more possible keys than slots ($|U| > m$).
- For a given set K of keys with $|K| \leq m$, may or may not happen.
- Definitely happens when $|K| > m$.
- Must be prepared to handle collisions in all cases.
- Two methods:
  - chaining
  - open addressing
- Chaining is usually better.

Collision resolution by chaining

- Put all elements that hash to the same slot into a linked list.



- Doubly linked list allows easy deletion.

Implementation of heap with chaining

```
Chained-Hash-Insert($T, x$)
insert $x$ at the head of list $T[h(key[x])]$
```

- Worst case $O(1)$
- Assumes element inserted not already in list.
- Would take an additional search to see if it was already inserted.

Implementation of heap with chaining

```
Chained-Hash-Search($T$, $k$)
    search for element with key $k$ in list $T[h(k)]$
```

- Running time proportional to length of list in slot $h(k)$

Implementation of heap with chaining

```
Chained-Hash-Delete($T, x$)
    delete $x$ from the list $T[h(key[x])]$
```

- Given pointer x to the element to delete, so no search is needed to find this element.
- Worst case $O(1)$ if lists are doubly linked.
- If lists are singly linked, deletion takes as long as search, because we must find x's predecessor.

Analysis of hashing with chaining

- Given a key, how long does it take to find an element with that key, or determine that there is no element with that key?
- Analysis is in terms of the **load factor** $\alpha = n/m$
- $n =\#$ elements in the table
- $m =\#$ slots in the table
- Load factor is average number of elements per linked list.
- Can have $\alpha < 1$, $\alpha = 1$, or $\alpha > 1$
- Worst case is when all n keys hash to the same slot:
  - a single list of length n
  - worst case is $\Theta(n)$ plus time to compute h
- Average case depends on how well the hash function distributes keys among slots.

Average-case analysis of hashing with chaining

- Assume **simple uniform hashing**: any given element is equally likely to hash to any of the m slots.
- For $j = 0, 1, \ldots, m-1$, denote the length of the list $T[j]$ by n_j.
- $n = n_0 + n_1 + \cdots + n_{m-1}$
- Average value of n_j is $E[n_j] = \alpha = n/m$
- Assume we can compute h in $O(1)$ time, so that the time required to search for k depends on the length $n_{h(k)}$ of the list $T[h(k)]$.
- Two cases:
    - Unsuccessful search: hash table has no element with key k
    - Successful search: hash table contains an element with key k

Unsuccessful search

Theorem

An unsuccessful search takes expected time $\Theta(1+\alpha)$.

Proof

- Simple uniform hashing means any key not already in the table is equally likely to hash to any of the m slots.
- To search unsuccessfully for any key k, need to search to the end of the list $T[h(k)]$.
- This list has expected length α.
- Adding the time to compute the hash function gives $\Theta(1+\alpha)$.

Successful search

- The expected time for a successful search is also $\Theta(1 + \alpha)$.
- The probability that each list is searched is proportional to the length of the list.

Successful search

Theorem

A successful search takes expected time $\Theta(1+\alpha)$.

Proof

- Assume the element x is equally likely to be any of the n elements stored in the table.
- The number examined during the search for x is 1 more than the number of elements that appear before x in x's list.
- These are the elements inserted *after* x was inserted.
- We need to find the average, over n elements, of how many elements were inserted into x's list after x was inserted.
- Let x_i be the ith element inserted, and let $k_i = key[x_i]$.
- For all i and j, let $X_{ij} = I\{h(k_i) = h(k_j)\}$
- Simple uniform hashing means

$$Pr\{h(k_k) = h(k_j)\} = 1/m = E[X_{ij}]$$

Expected number of elements examined, successful search

$$\begin{aligned}
E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right] &= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}E[X_{ij}]\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right)\\
&= 1+\frac{1}{nm}\sum_{i=1}^{n}(n-i)\\
&= 1+\frac{1}{nm}\sum_{i=0}^{n-1}i\\
&= 1+\frac{1}{nm}\frac{n(n-1)}{2}\\
&= 1+\frac{n-1}{2m} = 1+\frac{\alpha}{2}-\frac{\alpha}{2n} = \Theta(1+\alpha)
\end{aligned}$$

Alternative analysis

- $X_{ij\ell} = I\{\text{the search is for } x_i,\ h(k_i) = h(k_j) = \ell\}$
- Simple uniform hashing means $Pr\{h(k_i) = \ell\} = Pr\{h(k_j) = \ell\} = 1/m$
- $Pr\{\text{the search is for } x_i\} = 1/n$
- All these are independent: $Pr\{X_{ij\ell} = 1\} = E[X_{ij\ell}] = 1/nm^2$

$$
\begin{aligned}
Y_j &= I\{x_j \text{ appears in a list prior to the } x_i\} \\
&= \sum_{i=1}^{j-1} \sum_{\ell=0}^{m-1} X_{ij\ell}
\end{aligned}
$$

Alternative analysis, continued

$$
\begin{aligned}
E\left[1+\sum_{j=1}^{n} Y_j\right] &= 1 + E\left[\sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} X_{ij\ell}\right] \\
&= 1 + \sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} E[X_{ij\ell}] \\
&= 1 + \sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} \frac{1}{nm^2} \\
&= 1 + \binom{n}{2} \cdot m \cdot \frac{1}{nm^2} \\
&= 1 + \frac{n-1}{2m} \\
&= 1 + \frac{\alpha}{2} - \frac{\alpha}{2n} = \Theta(1+\alpha)
\end{aligned}
$$

Interpretation

- If $n = O(m)$ then $\alpha = n/m = O(1)$, which means searching takes constant time on average.
- Since insertion and deletion take $O(1)$ worst case time, all dictionary operations take average time $O(1)$.

Hash functions

- Ideally, satisfies the assumption of simple uniform hashing.
- In practice, impossible since we don't know the distribution of input keys.
- In practice, many functions work OK.

Keys as natural numbers

- Can interpret any computer data as natural number.
- Strings, for example: CLRS
    - ASCII: 67, 76, 82, 83
    - There are 128 ASCII values
    - $h(\mathrm{CLRS}) = 67(128^3) + 76(128^2) + 82(128^1) + 83(128^0) = 141,764,947$

Division method for hash functions

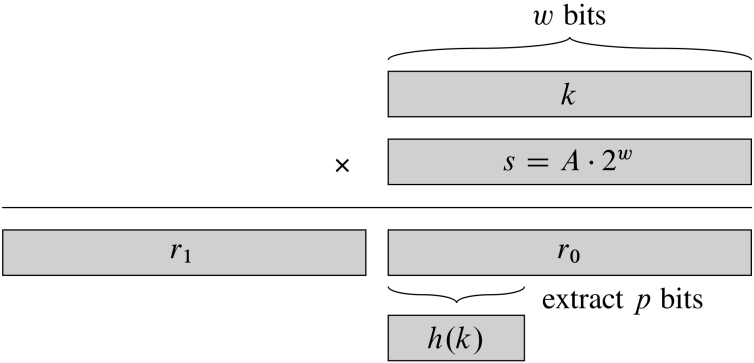
$$h(k) = k \bmod m$$

- Fast
- Powers of 2 are bad values for m:
  just uses least significant bits.
- Good choice for m:
  prime number not too close to a power of 2.
- Example:
  - Store $n \approx 2000$ character strings.
  - Don't mind searching 3 strings per unsuccessful search.
  - Choose $m = 701$.
  - This is a prime near $2000/3$ but not near any power of 2.

Multiplication method for hash functions

$$h(k) = \lfloor m((kA) \bmod 1) \rfloor \qquad 0 < A < 1$$

$$m = 2^p$$



Multiplication method for hash functions

$$h(k) = \lfloor m((kA) \bmod 1) \rfloor$$

- Slower than division method.
- Value of m is not critical.
  - Can even use $m = 2^p$
- Knuth suggests a good value for A:

$$\frac{\sqrt{5}-1}{2} = 0.6180339887\ldots$$

Open addressing

- Instead of chaining, store all keys in hash table.
- Must have $\alpha \leq 1$.
- Use $h(key) + i \mod m, i = 0, 1, 2, \ldots m - 1$
- Example: $h(n) = n \mod 10$

Index:	0	1	2	3	4	5	6	7	8	9
Insert 12:	x	x	12	x	x	x	x	x	x	x
Insert 14:	x	x	12	x	14	x	x	x	x	x
Insert 32:	x	x	12	32	14	x	x	x	x	x
Insert 92:	x	x	12	32	14	92	x	x	x	x
Insert 53:	x	x	12	32	14	92	53	x	x	x

- Linear probing like this leads to **clustering**.

Open addressing

- More generally, use a hash function that takes both a key and a position and returns a position:

$$h : U \times \{0, 1, ..., m-1\} \to \{0, 1, ..., m-1\}$$

- Then use probe sequence

$$h(k, 0), h(k, 1), ..., h(k, m-1)$$

- We require that for every key, k, the probe sequence be a permutation of

$$(0, 1, ..., m-1)$$

  so that all possible probes are examined in m probes.
- Linear probing satisfies this requirement, but has bad clustering.

Hash insertion and search, open addressing

```
HASH-INSERT$(T, k)$
  $i = 0$
  repeat
      $j = h(k, i)$
      if $T[j]$ == NIL
          $T[j] = k$
          return $j$
      else $i = i + 1$
  until $i$ == $m$
  error "hash table overflow"
```

```
HASH-SEARCH$(T, k)$
  $i = 0$
  repeat
      $j = h(k, i)$
      if $T[j]$ == $k$
          return $j$
      $i = i + 1$
  until $T[j]$ == NIL or $i = m$
  return NIL
```

Hash deletion, open addressing

- We cannot simply replace a deleted element with NIL.
- This might make search halt prematurely if clustering has occurred.
- Instead we insert a special DELETED value.
- Insert will treat DELETED as available.
- Search will treat DELETED as full.
- Search time no longer depends on α alone.

Uniform hashing with open addressing

- In our analysis, we assume **uniform hashing**:
  - The probe sequence for a key k is equally likely to be any of the $m!$ possible sequences.
- Open addressing has to generalize the notion of uniform hashing to a function that generates an entire sequence of probes.
- True open uniform hashing is difficult.
- In practice approximations are used.

Linear probing

- Given an ordinary hash function $h' : U \to \{0, ..., m-1\}$, called an **auxiliary hash function**, use

$$h(k, i) = (h'(k) + i) \mod m$$

- Only generates m of the $m!$ possible permutations of $\{0, ..., m-1\}$
- Extremely susceptible to **primary clustering**.
- Any slot preceded by i full slots gets filled with probability

$$\frac{i+1}{m}$$

and hence long chains get longer.

Linear probing

- Given an ordinary hash function $h' : U \to \{0, ..., m-1\}$, called an **auxiliary hash function**, use

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- Only generates m of the $m!$ possible permutations of $\{0, ..., m-1\}$
- Extremely susceptible to **primary clustering**.
- Any slot preceded by i full slots gets filled with probability

$$\frac{i+1}{m}$$

  and hence long chains get longer.
- Note: it does not matter if we use

$$h(k, i) = h'(k) + ai + b$$

  . Why not?

Quadratic probing

$$h(k,i) = (h'(k) + c_1 i + c_2 i^2) \mod m$$

- c_1, c_2, m must be carefully selected.
- Primary clustering is eliminated.
- If two keys have the same initial probe, their entire sequence is the same.
- This is called **secondary clustering** and is not as serious.
- Only m of the $m!$ possible permutations of $\{0, \ldots, m-1\}$ are used.

Double hashing

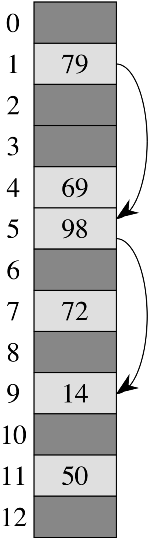
$$h(k,i) = (h_1(k) + ih_2(k)) \mod m$$

- Needs two auxiliary hash functions, h_1, h_2.
- $h_2(k)$ must be relatively prime to m:
  - Let $m = 2^p$ and make $h_2(k)$ odd.
  - Let m be prime and make $h_2(k) < m$:

$$\begin{aligned} h_1(k) &= k \mod m \\ h_2(k) &= 1 + (k \mod (m-1)) \end{aligned}$$

- $\Theta(m^2)$ probe sequences used, instead of $\Theta(m)$.
- Performance is very close to ideal uniform hashing.

Double hashing



0	
1	79
2	
3	
4	69
5	98
6	
7	72
8	
9	14
10	
11	50
12	

$$h_1(k) = k \mod 13$$
$$h_2(k) = 1 + (k \mod 11)$$

$$h_1(14) = 1$$
$$h_2(14) = 4$$

Analysis of open-address hashing, unsuccessful search

Theorem 11.6
Assuming uniform hashing and an open-address hash table with load factor $\alpha = n/m < 1$ the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$.
Proof

$$
\begin{aligned}
X &= \text{number of probes in unsuccessful search} \\
A_k &= \{k\text{th probe is to an occupied slot}\} \\
\{X \geq i\} &= \bigcap_{k=1}^{i-1} A_k \\
\Pr\{X \geq i\} &= \Pr\{A_1\} \cdot \Pr\{A_2 | A_1\} \cdot \Pr\{A_3 | A_1 \cap A_2\} \\
& \cdots \\
& \cdot \Pr\{A_{i-1} | A_1 \cap A_2 \cap ... \cap A_{i-2}\}
\end{aligned}
$$

Probability that i probes find occupied slot

$$
\begin{aligned}
X &= \text{number of probes in unsuccessful search} \\
A_k &= \{k\text{th probe is to an occupied slot}\} \\
\{X \geq i\} &= \bigcap_{k=1}^{i-1} A_k \\
\Pr\{X \geq i\} &= \Pr\{A_1\} \cdot \Pr\{A_2 | A_1\} \cdot \Pr\{A_3 | A_1 \cap A_2\} \\
&\cdots \\
&\cdot \Pr\{A_{i-1} | A_1 \cap A_2 \cap \ldots \cap A_{i-2}\} \\
&= \frac{n}{m} \cdot \frac{n-1}{m-1} \cdot \frac{n-2}{m-2} \cdots \frac{n-i+2}{m-i+2} \\
&\leq \left(\frac{n}{m}\right)^{i-1} \\
&= \alpha^{i-1}
\end{aligned}
$$

Expected number of probes in unsuccessful search

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} \Pr\{X \geq i\} \\
&\leq \sum_{i=1}^{\infty} \alpha^{i-1} \\
&= \sum_{i=0}^{\infty} \alpha^{i} \\
&= \frac{1}{1-\alpha} \\
&= 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \cdots
\end{aligned}
$$

- The last expression gives us an intuitive picture.

Analysis of open-address hashing, unsuccessful search

Theorem 11.6
Assuming uniform hashing and an open-address hash table with load factor $\alpha = n/m < 1$ the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$.

- If $\alpha = 0.5$, then we expect less than 2 probes on average.
- If $\alpha = 0.9$, then we expect less than 10 probes on average.

Expected number of probes for Hash-Insert

- An element is inserted after finding an open slot.
- This is the same procedure followed by an unsuccessful search.
- Therefore the expected number of probes is at most

$$\frac{1}{1-\alpha}$$

Analysis of open address hashing, successful search

Theorem 11.8

Assuming uniform hashing and every key is equally likely, in an open-address hash table with load factor $\alpha < 1$ the expected number of probes in a successful search is at most

$$\frac{1}{\alpha} \ln \frac{1}{1-\alpha}$$

Proof

- A successful search does the same as when the k was inserted.
- If k was the ith key inserted, then $\alpha = i/m$ when inserted.
- Therefore there were at most $1/(1-i/m) = m/(m-i)$ probes
- The average over all n keys is then

$$\frac{1}{n} \sum_{i=0}^{n-1} \frac{m}{m-i}$$

Averaging over the n keys in the table:

$$
\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1}\frac{m}{m-i} &= \frac{m}{n}\sum_{i=0}^{n-1}\frac{1}{m-i} \\
&= \frac{1}{\alpha}\sum_{k=m-n+1}^{m}\frac{1}{k} \\
&\leq \frac{1}{\alpha}\int_{m-n}^{m}(1/x)dx \\
&= \frac{1}{\alpha}\ln\frac{m}{m-n} \\
&= \frac{1}{\alpha}\ln\frac{1}{1-\alpha}
\end{aligned}
$$

Analysis of open address hashing, successful search

Theorem 11.8

Assuming uniform hashing and every key is equally likely, in an open-address hash table with load factor $\alpha < 1$ the expected number of probes in a successful search is at most

$$\frac{1}{\alpha} \ln \frac{1}{1 - \alpha}$$

- If $\alpha = 0.5$, then we expect less than 1.387 probes on average.
- If $\alpha = 0.9$, then we expect less than 2.559 probes on average.