# Notes on Linear Sorting

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# Comparison sorts

- The only operation that may be used to gain information about a sequence is comparisons between pairs of elements.
- All sorts seen so far are comparison sorts:
  - insertion sort
  - merge sort
  - quicksort
  - heapsort

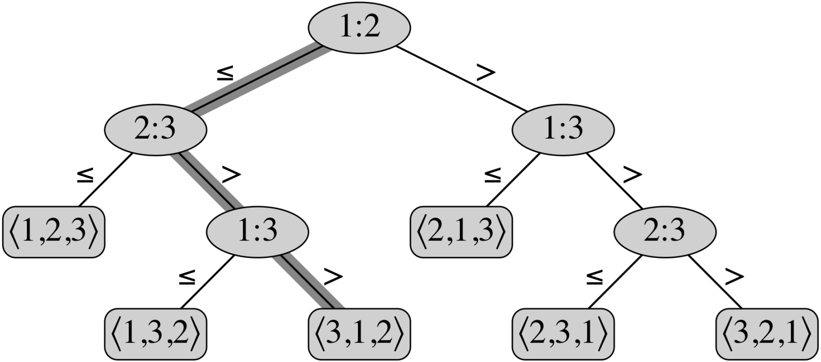
# Lower bounds for comparison sorts

- $\Omega(n)$ to examine all the input
- All sorts seen so far are $\Omega(n \lg n)$
- We will show that all comparison sorts must be $\Omega(n \lg n)$

# Decision tree

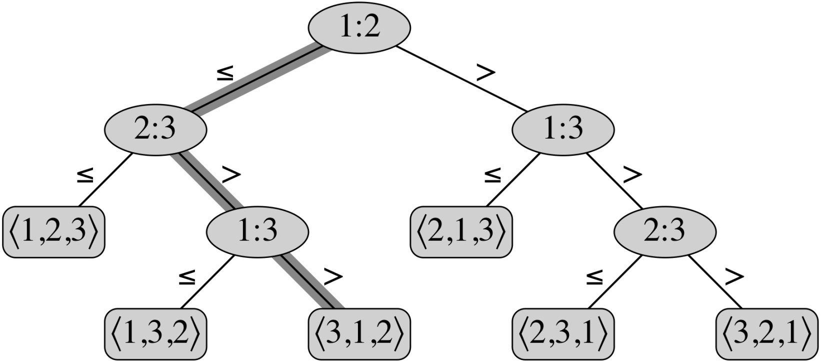
- Abstraction of any comparison sort
- Represents comparisons made by
  - a specific sorting algorithm
  - on inputs of a given size
- Abstracts away everything else: control and data movement.
- We're counting *only* comparisons.

# Insertion sort on three elements




- Internal nodes labeled by comparisons (original positions).
- Leaf nodes labeled by permutation of order from original.
- Number of leaves $\geq n!$.

# For any comparison sort




- 1 tree for each $n$
- View the tree as if the algorithm splits in two at each node.
- The tree models all possible execution traces.

# What is the longest path from root to leaf?

- Depends on the algorithm.
- Insertion sort: $\Theta(n^2)$
- Merge sort: $\Theta(n \lg n)$

## Lemma: any binary tree of height $h$ has $\leq 2^h$ leaves.

- $\ell = \#$ of leaves
- $h =$ height
- then $\ell \leq 2^h$

Proof by induction on $h$:

Base: $h = 0$. Tree is just one node, which is a leaf. $1 \leq 2^h$.

Inductive step: Assume true for $h - 1$. Extend tree with as many new leaves as possible. Each leaf becomes the parent of two new leaves.

$$\begin{aligned}
\#\text{ of leaves for } h &= 2(\#\text{ of leaves for } h-1) \\
&\leq 2(2^{h-1}) \\
&= 2^h
\end{aligned}$$

# Theorem: any decision tree that sorts $n$ elements has height $\Omega(n \lg n)$

- $\ell \geq n!$
- $n! \leq \ell \leq 2^h$
- $h \geq \lg(n!)$
- Sterling's approximation: $n! > (n/e)^n$
- Therefore:

$$
\begin{aligned}
h &\geq \lg(n!) \\
&\geq \lg(n/e)^n \\
&= n \lg(n/e) \\
&= n \lg n - n \lg e \\
&= \Omega(n \lg n)
\end{aligned}
$$

# Sorting in linear time

- Impossible with any comparison sort.
- **Counting sort**
  - Key assumption: numbers to be sorted are integers in $\{0, \ldots, k\}$.

Input: $A[1..n]$ where $A[j] \in \{0, ..., k\}$

Output: $B[1..n]$, sorted.

Auxiliary storage: $C[0..k]$

# Counting sort example

COUNTING-SORT$(A, B, n, k)$

let $C[0..k]$ be a new array
**for** $i = 0$ **to** $k$
    $C[i] = 0$
**for** $j = 1$ **to** $n$
    $C[A[j]] = C[A[j]] + 1$
**for** $i = 1$ **to** $k$
    $C[i] = C[i] + C[i-1]$
**for** $j = n$ **downto** 1
    $B[C[A[j]]] = A[j]$
    $C[A[j]] = C[A[j]] - 1$

| **A:** | $2_1$ | $5_1$ | $3_1$ | $0_1$ | $2_2$ | $3_2$ | $0_2$ | $3_3$ |
|---|---|---|---|---|---|---|---|---|

After second **for** loop:

| **C:** | 2 | 0 | 2 | 3 | 0 | 1 |
|---|---|---|---|---|---|---|

After third **for** loop:

| **C:** | 2 | 2 | 4 | 7 | 7 | 8 |
|---|---|---|---|---|---|---|

# Counting sort example

```
COUNTING-SORT(A, B, n, k)
let C[0..k] be a new array
for i = 0 to k
        C[i] = 0
for j = 1 to n
        C[A[j]] = C[A[j]] + 1
for i = 1 to k
        C[i] = C[i] + C[i − 1]
for j = n downto 1
        B[C[A[j]]] = A[j]
        C[A[j]] = C[A[j]] − 1
```

**A:**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $2_1$ | $5_1$ | $3_1$ | $0_1$ | $2_2$ | $3_2$ | $0_2$ | $3_3$ |

**C:**

| | | | | | |
|---|---|---|---|---|---|
| 2 | 2 | 4 | 7 | 7 | 8 |

**B:**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | $3_3$ | |
| | $0_2$ | | | | | $3_3$ | |
| | $0_2$ | | | | $3_2$ | $3_3$ | |
| | $0_2$ | | $2_2$ | | $3_2$ | $3_3$ | |
| $0_1$ | $0_2$ | | $2_2$ | | $3_2$ | $3_3$ | |
| $0_1$ | $0_2$ | | $2_2$ | $3_1$ | $3_2$ | $3_3$ | |
| $0_1$ | $0_2$ | | $2_2$ | $3_1$ | $3_2$ | $3_3$ | $5_1$ |
| $0_1$ | $0_2$ | $2_1$ | $2_2$ | $3_1$ | $3_2$ | $3_3$ | $5_1$ |

Counting sort is **stable**:

- Keys with the same value appear in the same order in output as in input.

# Counting sort analysis

```
COUNTING-SORT(A, B, n, k)
let C[0..k] be a new array
for i = 0 to k
        C[i] = 0
for j = 1 to n
        C[A[j]] = C[A[j]] + 1
for i = 1 to k
        C[i] = C[i] + C[i − 1]
for j = n downto 1
        B[C[A[j]]] = A[j]
        C[A[j]] = C[A[j]] − 1
```

- $\Theta(n + k)$
  - which is $\Theta(n)$ if $k = O(n)$.
- How big a $k$ is practical?
  - 64-bit values? Are you kidding?
  - 32-bit values? No.
  - 16-bit? Probably not.
  - 8-bit? Maybe, depending on $n$.
  - 4-bit? Unless $n$ is really small.

- Counting sort will be used in radix sort.

# Radix sort example

| | | | |
|---|---|---|---|
| 329 | 720 | 720 | 329 |
| 457 | 355 | 329 | 355 |
| 657 | 436 | 436 | 436 |
| 839 | 457 | 839 | 457 |
| 436 | 657 | 355 | 657 |
| 720 | 329 | 457 | 720 |
| 355 | 839 | 657 | 839 |

- Sort on each digit individually.
- Must use a stable sort subroutine.
- Subroutine only works on a small range of numbers.

# Radix sort

- IBM in early 20th century.
- Punch card sorting machines only sorted on one column.
- Humans would reload the cards and change the column.
- Human-machine cyborg algorithm!
- **Key idea:** Sort *least* significant digits first.

# Radix sort

```
RADIX-SORT(A, d)
for i = 1 to d
        use a stable sort to sort A on digit i
```

# Radix sort correctness

- Induction on number of passes.
- Assume digits $1, \ldots, i-1$ are sorted.
- Show that a stable sort on $i$ leaves $1, \ldots, i-1$ sorted:
  - If 2 digits in position $i$ are different,
    - ordering by $i$ is correct and positions $1, \ldots, i-1$ are irrelevant.
  - If 2 digits in position $i$ are equal,
    - numbers are already sorted by inductive hypothesis. Stable sort leaves them that way.

# Radix sort analysis

Assume we use counting sort on each digit.

- $\Theta(n + k)$ per digit
- $d$ digits
- $\Theta(d(n + k))$ total
- If $k = O(n)$, time $= \Theta(dn)$.

# Radix sort: How to break each key into digits?

- $n$ words
- $b$ bits/word
- Break into $r$-bit digits. $d = \lceil b/r \rceil$
- Use counting sort, $k = 2^r - 1$.
  Example: 32-bit words, 8-bit digits.

$$
\begin{aligned}
b &= 32 & r &= 8 \\
d &= \lceil 32/8 \rceil = 4 & k &= 2^8 - 1 = 255
\end{aligned}
$$

- Time $= \Theta\left(\frac{b}{r}(n + 2^r)\right)$

# How to choose $r$?

- Time $= \Theta\left(\frac{b}{r}(n + 2^r)\right)$
- Balance $b/r$ and $n + 2^r$.
- Choosing $r \approx \lg n$ gives

$$\Theta\left(\frac{b}{\lg n}(n + n)\right) = \Theta(bn/\lg n)$$

- If we choose $r < \lg n$ then $b/r > b/\lg n$ and $n + 2^r$ doesn't improve.
- If we choose $r > \lg n$ then $n + 2^r$ term gets big.
- Sort $2^{16}$ 32-bit numbers, use $r = \lg 2^{16} = 16$ bits. $\lceil b/r \rceil = 2$ passes.

# Compare radix to merge and quick

- 1 million ($2^{20}$) 32-bit integers.
- Radix sort: $\lceil 32/20 \rceil = 2$ passes.
- Merge/quick: $\lg n = 20$ passes.
- Each radix “pass” is 2 passes:
  - one to take census
  - one to move data

# How does radix sort violate the $\Omega(n \lg n)$ speed limit?

- Counting sort allows us to gain information about keys
  - other than by directly comparing 2 keys.
- Used keys as array indices,
  - thus getting far more information out of each key.
  - branching factor of the decision tree is $k$