Summations 2

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Source

These notes are based on chapter 6 of
`http://www.cs.yale.edu/homes/aspnes/classes/202/notes.pdf`
and
`http://www.cs.yale.edu/homes/aspnes/pinewiki/attachments/SummationNotation/summation-notation.pdf`

Summations

$$\sum_{i=0}^{n}(2i+1) = 1 + 3 + 5 + 7 + ... + (2n+1)$$

Definition using recurrences

$$\sum_{i=a}^{b} f(i) = \begin{cases} 0 & \text{if } b < a \\ f(a) + \sum_{i=a+1}^{b} f(i) & \text{otherwise.} \end{cases}$$

$$\sum_{i=a}^{b} f(i) = \begin{cases} 0 & \text{if } b < a \\ \sum_{i=a}^{b-1} f(i) + f(b) & \text{otherwise.} \end{cases}$$

Scope

- Scope extends to the first addition or subtraction symbol.
- Best to include parentheses to avoid confusion,
- or move trailing terms to the beginning.

$$\begin{aligned}
\sum_{i=1}^{n} i^2 + 1 &= \left(\sum_{i=1}^{n} i^2 \right) + 1 \\
&= 1 + \left(\sum_{i=1}^{n} i^2 \right) \\
&= 1 + \sum_{i=1}^{n} i^2 \\
&\neq \sum_{i=1}^{n} (i^2 + 1)
\end{aligned}$$

Summation is linear

$$\sum_{i=n}^{m} ax_i = a \sum_{i=n}^{m} x_i$$

$$\sum_{i=n}^{m} (x_i + y_i) = \sum_{i=n}^{m} x_i + \sum_{i=n}^{m} y_i$$

Multiple sums

- The order is not important, provided the bounds of the inner sum don't depend on the index of the outer sum:

$$\sum_{i=a}^{b} \sum_{j=c}^{d} x_{ij} = \sum_{j=c}^{d} \sum_{i=a}^{b} x_{ij}$$

Products of sums

$$(a+b)(x+y+z) = ax + ay + az + bx + by + bz$$

$$\left(\sum_{i=a}^{b} x_i\right)\left(\sum_{j=c}^{d} y_j\right) = \sum_{i=a}^{b}\sum_{j=c}^{d} x_i y_j$$

Change of variables

Let

$$j = i - 1$$
$$i = j + 1$$

then,

$$\sum_{i=1}^{n}(i-1) = \sum_{(j+1)=1}^{n} j$$
$$= \sum_{j=0}^{n-1} j$$
$$= \sum_{i=0}^{n-1} i$$

Change of variables

Let

$$j = i + 1$$
$$i = j - 1$$

then,

$$\begin{aligned}
\sum_{i=a}^{b} (i+1)^2 &= \sum_{(j-1)=a}^{b} j^2 \\
&= \sum_{j=a+1}^{b+1} j^2 \\
&= \sum_{i=a+1}^{b+1} i^2
\end{aligned}$$

Sums over index sets

$$\sum_{i \in \{3,5,7\}} i^2 = 3^2 + 5^2 + 7^2$$

$$\sum_{A \subseteq S} |A|$$

$$\sum_{p < 1000,\ p \text{ is prime}} p^2$$

Confusing index sets

$$\sum_{1 \leq i < j \leq n} \frac{i}{j}$$

- The sum over all pairs of values (i, j) such that
  - $1 \leq i$
  - $i < j$
  - $j \leq n$

  with each pair appearing exactly once.

Confusing index sets

$$\sum_{x \in A \subseteq S} |A|$$

- The sum over all sets A such that
  - $x \in A$
  - $A \subseteq S$

  assuming that x and S are defined outside the summation.

Sums without explicit bounds

When the index set is understood from context some books use:

$$\sum_i i^2$$

Very sloppy, and discouraged.

Double sums

$$\sum_{i=0}^{n} \sum_{j=0}^{i} (i+1)(j+1)$$

When $n = 1$ we get

$$(0+1)(0+1) + [(1+1)(0+1) + (1+1)(1+1)] = 7$$

Closed forms

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\begin{aligned}\sum_{i=0}^{n} r^i &= \frac{1-r^{n+1}}{1-r} \\ &= \frac{r^{n+1}-1}{r-1}\end{aligned}$$

Quick "proof"

Start with

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

Then

$$\begin{aligned}
\sum_{i=0}^{n} r^i &= \sum_{i=0}^{\infty} r^i - r^{n+1} \sum_{i=0}^{\infty} r^i \\
&= \frac{1}{1-r} - \frac{r^{n+1}}{1-r} \\
&= \frac{1-r^{n+1}}{1-r}
\end{aligned}$$

Solving summations

$$\begin{aligned}
\sum_{i=0}^{n}(3(2^n)+5) &= 3\sum_{i=0}^{n}2^n + 5\sum_{i=0}^{n}1 \\
&= 3(2^{n+1}-1)+5(n+1) \\
&= 3(2^{n+1})+5n+2
\end{aligned}$$

Guess but verify

$$S(n) = \sum_{k=1}^{n} (2k-1)$$

n	$S(n)$
0	0
1	1
2	$1+3=4$
3	$4+5=9$
4	$9+7=16$
5	$16+9=25$

$S(n) = n^2$ Guess!

Prove guess by induction

$$S(n) = \sum_{k=1}^{n}(2k-1) \overset{?}{\iff} S(n) = n^2$$

$$\begin{aligned}
S(0) &= \sum_{k=1}^{0}(2k-1) = 0 && \text{Base case} \\
S(n+1) &= \sum_{k=1}^{n+1}(2k-1) && \text{Step} \\
&= \sum_{k=1}^{n}(2k-1) + (2(n+1)-1) \\
&= n^2 + 2(n+1) - 1 && \text{Inductive hypothesis} \\
&= (n+1)^2
\end{aligned}$$

Strategies for asymptotic estimates

Pull out constant factors

$$\begin{aligned}\sum_{i=1}^{n} \frac{n}{i} &= n \sum_{i=1}^{n} \frac{1}{i} \\ &= nH_n \\ &= \Theta(n \log n)\end{aligned}$$

Strategies for asymptotic estimates

Bound using geometric series

$$\sum_{i=1}^{n} x^i = \frac{x^{n+1}-1}{x-1} = \Theta(x^n)$$

$$\sum_{i=1}^{n} 2^i = \Theta(2^n)$$

$$\sum_{i=1}^{n} 2^{-i} = \Theta(1)$$

Strategies for asymptotic estimates

Harmonic series

$$\sum_{i=1}^{n} \frac{1}{i} = H_n = \Theta(n \lg n)$$

Strategies for asymptotic estimates

Bound part of the sum.

$$
\begin{aligned}
\sum_{i=1}^{n} i^3 &\le \sum_{i=1}^{n} n^3 \\
&= O(n^4) \\
\sum_{i=1}^{n} i^3 &\ge \sum_{i=n/2}^{n} i^3 \\
&\ge \sum_{i=n/2}^{n} (n/2)^3 \\
&= \Omega(n^4)
\end{aligned}
$$

Strategies for asymptotic estimates

Integrate!

$$\int_{a-1}^{b} f(x)dx \leq \sum_{i=a}^{b} f(i) \leq \int_{a}^{b+1} f(x)dx$$

Most of the functions we see in algorithms yield to this method!

Strategies for asymptotic estimates

Grouping terms.
The standard trick for showing that the harmonic series is unbounded.

$$1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots$$
$$\geq$$
$$1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \ldots$$
$$=$$
$$1 + 1/2 + 1/2 + 1/2 + \ldots$$

Final Notes

In practice, almost any sum you come across will be of the form:

$$\sum_{i=1}^{n} f(i)$$

where either:

- $f(n)$ is exponential.
  Then it's bounded by a geometric series and the largest term dominates.
- $f(n)$ is polynomial.
  Then $f(n/2) = \Theta(f(n))$ and the sum is $\Theta(nf(n))$ using the lower bound:

$$\sum_{i=n/2}^{n} f(n) = \Omega(nf(n))$$