# Master Theorem

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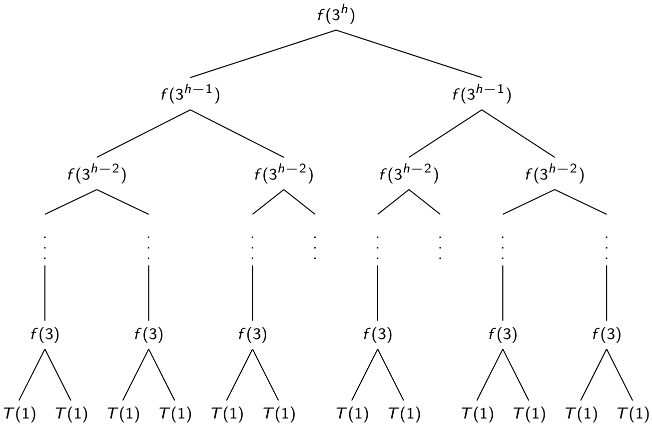
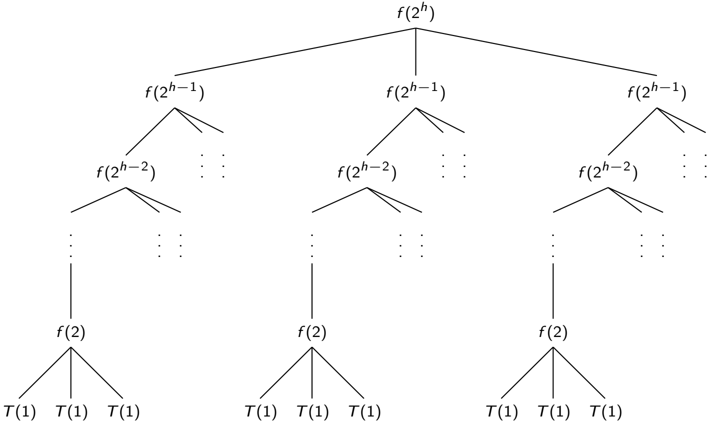
# Example 1: $T(n) = aT(n/b) + f(n)$ with $a = 2, b = 3$




$h = \log_3 n$ $\qquad$ $2^{\log_3 n} = n^{\log_3 2} \approx n^{0.6}$ leaves $\qquad$ $\sum_{i=0}^{\log_b n} a^i f(n/b^i)$

# Example 2: $T(n) = aT(n/b) + f(n)$ with $a = 3, b = 2$




$h = \log_2 n$ $\quad$ $3^{\log_2 n} = n^{\log_2 3} \approx n^{1.6}$ leaves $\quad$ $\sum_{i=0}^{\log_b n} a^i f(n/b^i)$

# Comparing leaves to internal nodes

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n} a^i f(n/b^i)$
- Suppose $f(n) = n$, $a = b = 4$, $n^{\log_b a} = n$

$$\begin{aligned}\sum_{i=0}^{\log_4 n} a^i f(n/b^i) &= \sum_{i=0}^{\log_4 n} 4^i (n/4^i) \\ &= \sum_{i=0}^{\log_4 n} n \\ &= \Theta(n \lg n)\end{aligned}$$

$$n^{\log_b a} + \Theta(n \lg n) = \Theta(n \lg n)$$

# Comparing leaves to internal nodes

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n} a^i f(n/b^i)$
- Suppose $f(n) = n$, $a = 4$, $b = 2$, $n^{\log_b a} = n^2$

$$\begin{aligned}
\sum_{i=0}^{\log_2 n} a^i f(n/b^i) &= \sum_{i=0}^{\log_2 n} 4^i (n/2^i) \\
&= n \sum_{i=0}^{\log_2 n} 2^i \\
&= \Theta(n^2)
\end{aligned}$$

$$n^{\log_b a} + \Theta(n^2) = \Theta(n^2)$$

# Comparing leaves to internal nodes

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n} a^i f(n/b^i)$
- Suppose $f(n) = n^2$, $a = b = 4$, $n^{\log_b a} = n$

$$
\begin{aligned}
\sum_{i=0}^{\log_4 n} a^i f(n/b^i) &= \sum_{i=0}^{\log_4 n} 4^i (n/4^i)^2 \\
&= \sum_{i=0}^{\log_4 n} \left(\frac{4}{4^2}\right)^i n^2 \\
&= n^2 \sum_{i=0}^{\log_4 n} (1/4)^i \\
&= \Theta(n^2)
\end{aligned}
$$

$$
n^{\log_b a} + \Theta(n^2) = \Theta(n^2)
$$

# Comparing leaves to internal nodes

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n} a^i f(n/b^i)$
- Suppose $f(n) = n^2$, $a = 4$, $b = 2$, $n^{\log_b a} = n^2$

$$\begin{aligned}
\sum_{i=0}^{\log_b n} a^i f(n/b^i) &= \sum_{i=0}^{\log_2 n} 4^i (n/2^i)^2 \\
&= \sum_{i=0}^{\log_2 n} \left(\frac{4}{2^2}\right)^i n^2 \\
&= n^2 \sum_{i=0}^{\log_2 n} (1)^i \\
&= \Theta(n^2 \lg n)
\end{aligned}$$

$$n^{\log_b a} + \Theta(n^2 \lg n) = \Theta(n^2 \lg n)$$

# Comparing leaves to internal nodes

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n} a^i f(n/b^i)$
- Suppose $f(n) = n^2$, $a = 8$, $b = 2$, $n^{\log_b a} = n^3$

$$\begin{aligned}
\sum_{i=0}^{\log_4 n} a^i f(n/b^i) &= \sum_{i=0}^{\log_b n} 8^i (n/2^i)^2 \\
&= \sum_{i=0}^{\log_b n} \left(\frac{8}{2^2}\right)^i n^2 \\
&= n^2 \sum_{i=0}^{\log_b n} 2^i \\
&= \Theta(n^2 \lg n)
\end{aligned}$$

$$n^{\log_b a} + \Theta(n^2 \lg n) = \Theta(n^2 \lg n)$$

# Comparing leaves to internal nodes

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n} a^i f(n/b^i)$
- Suppose $f(n) = n^{\log_b a}$

$$
\begin{aligned}
\sum_{i=0}^{\log_b n} a^i f(n/b^i) &= \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a} \\
&= \sum_{i=0}^{\log_b n} \left(\frac{a}{b^{\log_b a}}\right)^i n^{\log_b a} \\
&= \sum_{i=0}^{\log_b n} n^{\log_b a} \\
&= n^{\log_b a} \log_b n \\
&= \Theta(f(n) \lg n)
\end{aligned}
$$

# Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on nonnegative integers by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Then $T(n)$ has the following asymptotic bounds:

| $f(n)$ | $T(n)$ |
|---:|:---|
| $O(n^{\log_b a-\epsilon})$ | $\Theta(n^{\log_b a})$ |
| $\Theta(n^{\log_b a})$ | $\Theta(f(n)\lg n)$ |
| $\Omega(n^{\log_b a+\epsilon})$ | $\Theta(f(n))$ |

The last one only if $af(\frac{n}{b}) \leq cf(n)$ for some $c < 1$ and large enough $n$.

# Master theorem example 1

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \qquad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$\log_b a = 1$$

$$f(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n \lg n)$$

# Master theorem example 2

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a-\epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n)\lg n)$$

$$\Omega(n^{\log_b a+\epsilon}) \Leftrightarrow \Theta(f(n)) \qquad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^2$$

$$\log_b a = 2$$

$$f(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n^2 \lg n)$$

# Master theorem example 3

$$
\begin{aligned}
T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\
O(n^{\log_b a - \epsilon}) &\Leftrightarrow \Theta(n^{\log_b a}) \\
\Theta(n^{\log_b a}) &\Leftrightarrow \Theta(f(n) \lg n) \\
\Omega(n^{\log_b a + \epsilon}) &\Leftrightarrow \Theta(f(n)) \qquad\qquad \text{when } af(n/b) \leq cf(n)
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= 4T(n/2) + \lg n \\
\log_b a &= 2 \\
f(n) &= O(n^{\log_b a - \epsilon}) \\
T(n) &= \Theta(n^2)
\end{aligned}
$$

# Master theorem example 4

$$
\begin{aligned}
T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\
O(n^{\log_b a-\epsilon}) &\Leftrightarrow \Theta(n^{\log_b a}) \\
\Theta(n^{\log_b a}) &\Leftrightarrow \Theta(f(n)\lg n) \\
\Omega(n^{\log_b a+\epsilon}) &\Leftrightarrow \Theta(f(n)) \qquad\qquad \text{when } af(n/b) \le cf(n)
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n^3 \\
\log_b a &= 2 \\
f(n) &= \Omega(n^{\log_b a+\epsilon}) \\
T(n) &= \Theta(n^3)
\end{aligned}
$$

because

$$
\begin{aligned}
af(n/b) &= 4f(n/2) \\
&= 4/2^3 f(n) \le cf(n)
\end{aligned}
$$

# Master theorem does not apply

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \lg n \\
n^{\log_b a} &= n \\
f(n) &= n \lg n \\
f(n) &= \Omega(n^{\log_b a}) = \Omega(n) \\
f(n) &\neq \Omega(n^{\log_b a + \epsilon}) = \Omega(n^{1+\epsilon}) \\
n \lg n &\neq \Omega(n^{1.0000000000001}) \\
\lg n &\neq \Omega(n^{0.0000000000001})
\end{aligned}
$$

# Master theorem does apply

$$
\begin{aligned}
T(n) &= 4T(n/3) + n\lg n \\
n^{\log_b a} &= n^{\log_c 4} = n^{1.26\ldots} \\
f(n) &= n\lg n \\
f(n) &= O(n^{\log_b a}) = O(n^{1.26\ldots}) \\
f(n) &= O(n^{\log_b a-\epsilon}) = O(n^{1.2}) & \epsilon = 0.06\ldots \\
n\lg n &= O(n^{1.2}) \\
\lg n &= O(n^{0.2})
\end{aligned}
$$