Notes on Recursion Trees

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Recursion Tree for p. 89

$$T(n) = 3T\left(\frac{n}{4}\right) + cn^2$$

Recursion Tree for p. 89

$$
\begin{aligned}
T(n) &= 3T\left(\frac{n}{4}\right) + cn^2 \\
T\left(\frac{n}{4}\right) &= 3T\left(\frac{n}{4^2}\right) + c\left(\frac{n}{4}\right)^2
\end{aligned}
$$

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T(n) &= 3\left[3T\left(\frac{n}{4^2}\right) + c\left(\frac{n}{4}\right)^2\right] + cn^2
\end{aligned}
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Recursion Tree for p. 89

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&= 3^2T\left(\frac{n}{4^2}\right) + cn^2\left(\frac{3}{4^2} + 1\right)
\end{aligned}$$

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&= 3^2T\left(\frac{n}{4^2}\right) + cn^2\left(\frac{3}{4^2} + 1\right) \\
T\left(\frac{n}{4^2}\right) &= 3T\left(\frac{n}{4^3}\right) + c\left(\frac{n}{4^2}\right)^2
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&= 3^2T\left(\frac{n}{4^2}\right) + cn^2\left(\frac{3}{4^2} + 1\right) \\
T\left(\frac{n}{4^2}\right) &= 3T\left(\frac{n}{4^3}\right) + c\left(\frac{n}{4^2}\right)^2 \\
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T\left(\frac{n}{4^2}\right) &= 3T\left(\frac{n}{4^3}\right) + c\left(\frac{n}{4^2}\right)^2 \\
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&= 3^3T\left(\frac{n}{4^3}\right) + cn^2\left(\frac{3^2}{(4^2)^2} + \frac{3}{4^2} + 1\right)
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T\left(\frac{n}{4^2}\right) &= 3T\left(\frac{n}{4^3}\right) + c\left(\frac{n}{4^2}\right)^2 \\
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&= 3^3T\left(\frac{n}{4^3}\right) + cn^2\left(\frac{3^2}{(4^2)^2} + \frac{3}{4^2} + 1\right) \quad \textbf{Aha!}
\end{aligned}
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T(n) &= 3\left[3T\left(\frac{n}{4^2}\right) + c\left(\frac{n}{4}\right)^2\right] + cn^2 \\
&= 3^2T\left(\frac{n}{4^2}\right) + cn^2\left(\frac{3}{4^2} + 1\right) \\
T\left(\frac{n}{4^2}\right) &= 3T\left(\frac{n}{4^3}\right) + c\left(\frac{n}{4^2}\right)^2 \\
T(n) &= 3^2\left[3T\left(\frac{n}{4^3}\right) + c\left(\frac{n}{4^2}\right)^2\right] + cn^2\left(\frac{3}{4^2} + 1\right) \\
&= 3^3T\left(\frac{n}{4^3}\right) + cn^2\left(\frac{3^2}{(4^2)^2} + \frac{3}{4^2} + 1\right) \quad \textbf{Aha!} \\
&= 3^3T\left(\frac{n}{4^3}\right) + cn^2\left(\frac{3^2}{(4^2)^2} + \frac{3^1}{(4^2)^1} + \frac{3^0}{(4^2)^0}\right)
\end{aligned}
$$

Recursion Tree for p. 89

$$T(n) = 3^3 T\left(\frac{n}{4^3}\right) + cn^2\left(\frac{3^2}{(4^2)^2} + \frac{3^1}{(4^2)^1} + \frac{3^0}{(4^2)^0}\right)$$

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\begin{aligned}
T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2 \left(\frac{3^2}{(4^2)^2} + \frac{3^1}{(4^2)^1} + \frac{3^0}{(4^2)^0}\right) \\
T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2 \sum_{i=0}^{2} \left(\frac{3}{16}\right)^i
\end{aligned}
$$

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T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2 \sum_{i=0}^{2}\left(\frac{3}{16}\right)^i \\
T(n) &= 3^4 T\left(\frac{n}{4^4}\right) + cn^2 \sum_{i=0}^{3}\left(\frac{3}{16}\right)^i
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T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2 \left(\frac{3^2}{(4^2)^2} + \frac{3^1}{(4^2)^1} + \frac{3^0}{(4^2)^0}\right) \\
T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2 \sum_{i=0}^{2} \left(\frac{3}{16}\right)^i \\
T(n) &= 3^4 T\left(\frac{n}{4^4}\right) + cn^2 \sum_{i=0}^{3} \left(\frac{3}{16}\right)^i \\
T(n) &= 3^k T\left(\frac{n}{4^k}\right) + cn^2 \sum_{i=0}^{k-1} \left(\frac{3}{16}\right)^i
\end{aligned}
$$

Recursion Tree for p. 89

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$$T(n) = 3^{\log_4 n} T\left(\frac{n}{4^{\log_4 n}}\right) + cn^2 \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i$$

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\begin{aligned}
T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2\left(\frac{3^2}{(4^2)^2} + \frac{3^1}{(4^2)^1} + \frac{3^0}{(4^2)^0}\right) \\
T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2 \sum_{i=0}^{2} \left(\frac{3}{16}\right)^i \\
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T(n) &= 3^k T\left(\frac{n}{4^k}\right) + cn^2 \sum_{i=0}^{k-1} \left(\frac{3}{16}\right)^i \\
T(n) &= 3^{\log_4 n} T\left(\frac{n}{4^{\log_4 n}}\right) + cn^2 \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i \\
T(n) &= n^{\log_4 3} T(1) + cn^2\left(\frac{1-(3/16)^{\log_4 n}}{1-3/16}\right)
\end{aligned}
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T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2\left(\frac{3^2}{(4^2)^2} + \frac{3^1}{(4^2)^1} + \frac{3^0}{(4^2)^0}\right) \\
T(n) &= 3^3 T\left(\frac{n}{4^3}\right) + cn^2 \sum_{i=0}^{2}\left(\frac{3}{16}\right)^i \\
T(n) &= 3^4 T\left(\frac{n}{4^4}\right) + cn^2 \sum_{i=0}^{3}\left(\frac{3}{16}\right)^i \\
T(n) &= 3^k T\left(\frac{n}{4^k}\right) + cn^2 \sum_{i=0}^{k-1}\left(\frac{3}{16}\right)^i \\
T(n) &= 3^{\log_4 n} T\left(\frac{n}{4^{\log_4 n}}\right) + cn^2 \sum_{i=0}^{\log_4 n-1}\left(\frac{3}{16}\right)^i \\
T(n) &= n^{\log_4 3} T(1) + cn^2\left(\frac{1-(3/16)^{\log_4 n}}{1-3/16}\right) = \Theta(n^2)
\end{aligned}
$$

Aha! Revisited

$$
\begin{aligned}
T(n) &= 3T\left(\frac{n}{4}\right) + cn^2 \\
3T\left(\frac{n}{4}\right) &= 3^2T\left(\frac{n}{4^2}\right) + 3c\left(\frac{n}{4}\right)^2 \\
3^2T\left(\frac{n}{4^2}\right) &= 3^3T\left(\frac{n}{4^3}\right) + 3^2c\left(\frac{n}{4^2}\right)^2 \\
3^3T\left(\frac{n}{4^3}\right) &= 3^4T\left(\frac{n}{4^4}\right) + 3^3c\left(\frac{n}{4^3}\right)^2 \\
&\cdots \\
3^{k-1}T\left(\frac{n}{4^{k-1}}\right) &= 3^kT\left(\frac{n}{4^k}\right) + 3^{k-1}c\left(\frac{n}{4^{k-1}}\right)^2 \\
&\cdots \\
3^{\log_4 n-1}T\left(\frac{n}{4^{\log_4 n-1}}\right) &= 3^{\log_4 n}T\left(\frac{n}{4^{\log_4 n}}\right) + 3^{\log_4 n-1}c\left(\frac{n}{4^{\log_4 n-1}}\right)^2
\end{aligned}
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\begin{aligned}
T(n) &= \cancel{3T\left(\frac{n}{4}\right)} + cn^2 \\
\cancel{3T\left(\frac{n}{4}\right)} &= \cancel{3^2T\left(\frac{n}{4^2}\right)} + 3c\left(\frac{n}{4}\right)^2 \\
\cancel{3^2T\left(\frac{n}{4^2}\right)} &= \cancel{3^3T\left(\frac{n}{4^3}\right)} + 3^2c\left(\frac{n}{4^2}\right)^2 \\
\cancel{3^3T\left(\frac{n}{4^3}\right)} &= \cancel{3^4T\left(\frac{n}{4^4}\right)} + 3^3c\left(\frac{n}{4^3}\right)^2 \\
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\cancel{3^{k-1}T\left(\frac{n}{4^{k-1}}\right)} &= \cancel{3^kT\left(\frac{n}{4^k}\right)} + 3^{k-1}c\left(\frac{n}{4^{k-1}}\right)^2 \\
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\cancel{3^{\log_4 n-1}T\left(\frac{n}{4^{\log_4 n-1}}\right)} &= 3^{\log_4 n}T\left(\frac{n}{4^{\log_4 n}}\right) + 3^{\log_4 n-1}c\left(\frac{n}{4^{\log_4 n-1}}\right)^2
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\cancel{3^2T\left(\frac{n}{4^2}\right)} &= \cancel{3^3T\left(\frac{n}{4^3}\right)} + 3^2c\left(\frac{n}{4^2}\right)^2 \\
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&\cdots \\
\cancel{3^{k-1}T\left(\frac{n}{4^{k-1}}\right)} &= \cancel{3^kT\left(\frac{n}{4^k}\right)} + 3^{k-1}c\left(\frac{n}{4^{k-1}}\right)^2 \\
&\cdots \\
\cancel{3^{\log_4 n-1}T\left(\frac{n}{4^{\log_4 n-1}}\right)} &= 3^{\log_4 n}T\left(\frac{n}{4^{\log_4 n}}\right) + 3^{\log_4 n-1}c\left(\frac{n}{4^{\log_4 n-1}}\right)^2 \\
T(n) &= 3^{\log_4 n}T\left(\frac{n}{4^{\log_4 n}}\right) + \sum_{i=0}^{\log_4 n-1} 3^i c\left(\frac{n}{4^i}\right)^2
\end{aligned}
$$