Master Theorem

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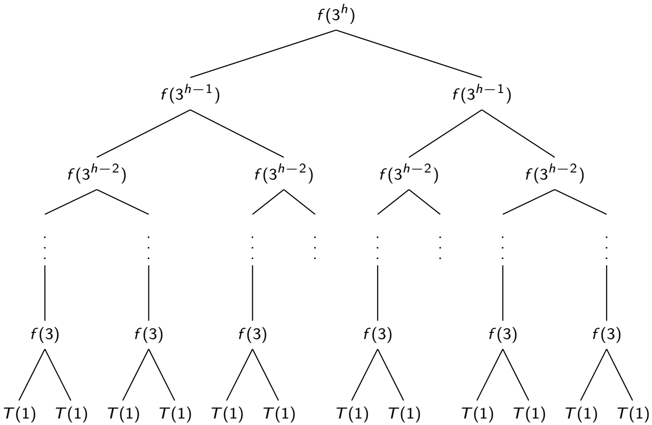
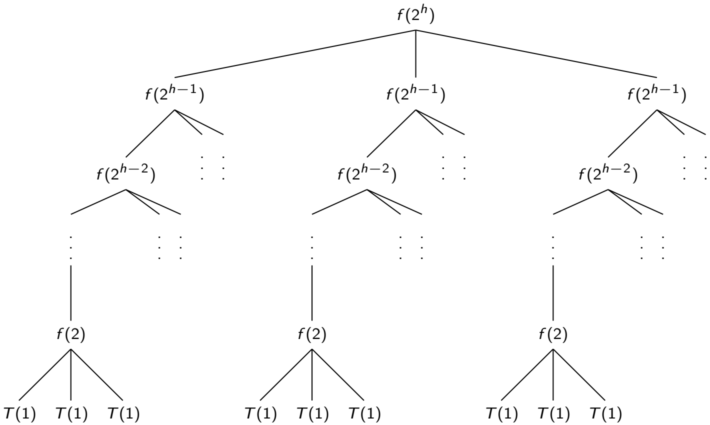
April 17, 2018

Example 1: $T(n) = aT(n/b) + f(n)$ with $a = 2, b = 3$



$h = \log_3 n$ $\qquad 2^{\log_3 n} = n^{\log_3 2} \approx n^{0.6}$ leaves $\qquad \sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$

Example 2: $T(n) = aT(n/b) + f(n)$ with $a = 3, b = 2$



$h = \log_2 n$ $\quad 3^{\log_2 n} = n^{\log_2 3} \approx n^{1.6}$ leaves $\quad \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)$

Comparing leaves to internal nodes, case 1

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- Suppose $f(n) = n$, $a = 2$, $b = 4$, $n^{\log_b a} = n^{1/2}$

$$
\begin{aligned}
\sum_{i=0}^{\log_4 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_4 n-1} 2^i (n/4^i) \\
&= n \sum_{i=0}^{\log_4 n-1} (1/2)^i \\
&= \Theta(n)
\end{aligned}
$$

$$n^{\log_b a} + \Theta(n) = \Theta(n)$$

Comparing leaves to internal nodes, case 2

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)$
- Suppose $f(n) = n$, $a = b = 4$, $n^{\log_b a} = n$

$$\begin{aligned}
\sum_{i=0}^{\log_4 n - 1} a^i f(n/b^i) &= \sum_{i=0}^{\log_4 n - 1} 4^i (n/4^i) \\
&= \sum_{i=0}^{\log_4 n - 1} n \\
&= \Theta(n \lg n)
\end{aligned}$$

$$n^{\log_b a} + \Theta(n \lg n) = \Theta(n \lg n)$$

Comparing leaves to internal nodes, case 3

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- Suppose $f(n) = n$, $a = 4$, $b = 2$, $n^{\log_b a} = n^2$

$$\begin{aligned}
\sum_{i=0}^{\log_2 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_2 n-1} 4^i (n/2^i) \\
&= n \sum_{i=0}^{\log_2 n-1} 2^i \\
&= \Theta(n^2)
\end{aligned}$$

$$n^{\log_b a} + \Theta(n^2) = \Theta(n^2)$$

Comparing leaves to internal nodes, case 4

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- Suppose $f(n) = n$, $a = 8$, $b = 2$, $n^{\log_b a} = n^3$

$$\sum_{i=0}^{\log_2 n-1} a^i f(n/b^i) = \sum_{i=0}^{\log_2 n-1} 8^i (n/2^i)$$
$$= n \sum_{i=0}^{\log_2 n-1} 4^i$$
$$= \Theta(n^3)$$

$$n^{\log_b a} + \Theta(n^3) = \Theta(n^3)$$

Comparing leaves to internal nodes, case 5

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- Suppose $f(n) = n^2$, $a = b = 4$, $n^{\log_b a} = n$

$$\begin{aligned}
\sum_{i=0}^{\log_4 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_4 n-1} 4^i (n/4^i)^2 \\
&= \sum_{i=0}^{\log_4 n-1} \left(\frac{4}{4^2}\right)^i n^2 \\
&= n^2 \sum_{i=0}^{\log_4 n-1} (1/4)^i \\
&= \Theta(n^2)
\end{aligned}$$

$$n^{\log_b a} + \Theta(n^2) = \Theta(n^2)$$

Comparing leaves to internal nodes, case 6

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- Suppose $f(n) = n^2$, $a = 4$, $b = 2$, $n^{\log_b a} = n^2$

$$
\begin{aligned}
\sum_{i=0}^{\log_b n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_2 n-1} 4^i (n/2^i)^2 \\
&= \sum_{i=0}^{\log_2 n-1} \left(\frac{4}{2^2}\right)^i n^2 \\
&= n^2 \sum_{i=0}^{\log_2 n-1} (1)^i \\
&= \Theta(n^2 \lg n)
\end{aligned}
$$

$$
n^{\log_b a} + \Theta(n^2 \lg n) = \Theta(n^2 \lg n)
$$

Comparing leaves to internal nodes, case 7

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- Suppose $f(n) = n^2$, $a = 8$, $b = 2$, $n^{\log_b a} = n^3$

$$
\begin{aligned}
\sum_{i=0}^{\log_4 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_b n-1} 8^i (n/2^i)^2 \\
&= \sum_{i=0}^{\log_b n-1} \left(\frac{8}{2^2}\right)^i n^2 \\
&= n^2 \sum_{i=0}^{\log_b n-1} 2^i \\
&= \Theta(n^3)
\end{aligned}
$$

$$n^{\log_b a} + \Theta(n^3) = \Theta(n^3)$$

Comparing leaves to internal nodes, case 8

- Number of leaves: $n^{\log_b a}$
- Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- Suppose $f(n) = n^{\log_b a}$

$$
\begin{aligned}
\sum_{i=0}^{\log_b n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_b n-1} a^i (n/b^i)^{\log_b a} \\
&= \sum_{i=0}^{\log_b n-1} \left(\frac{a}{b^{\log_b a}}\right)^i n^{\log_b a} \\
&= \sum_{i=0}^{\log_b n-1} n^{\log_b a} \\
&= n^{\log_b a} \log_b n \\
&= \Theta(f(n) \lg n)
\end{aligned}
$$

Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on nonnegative integers by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Then $T(n)$ has the following asymptotic bounds:

$f(n)$	$T(n)$
$O(n^{\log_b a - \epsilon})$	$\Theta(n^{\log_b a})$
$\Theta(n^{\log_b a})$	$\Theta(f(n) \lg n)$
$\Omega(n^{\log_b a + \epsilon})$	$\Theta(f(n))$

The last one only if $af(\frac{n}{b}) \leq cf(n)$ for some $c < 1$ and large enough n.

Master theorem example 1

$$
\begin{aligned}
T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\
O(n^{\log_b a-\epsilon}) &\Leftrightarrow \Theta(n^{\log_b a}) \\
\Theta(n^{\log_b a}) &\Leftrightarrow \Theta(f(n)\lg n) \\
\Omega(n^{\log_b a+\epsilon}) &\Leftrightarrow \Theta(f(n)) && \text{when } af(n/b) \leq cf(n)
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + n \\
\log_b a &= 1 \\
f(n) &= \Theta(n^{\log_b a}) \\
T(n) &= \Theta(n \lg n)
\end{aligned}
$$

Master theorem example 2

$$
\begin{aligned}
T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\
O(n^{\log_b a-\epsilon}) &\Leftrightarrow \Theta(n^{\log_b a}) \\
\Theta(n^{\log_b a}) &\Leftrightarrow \Theta(f(n) \lg n) \\
\Omega(n^{\log_b a+\epsilon}) &\Leftrightarrow \Theta(f(n)) && \text{when } af(n/b) \leq cf(n)
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n^2 \\
\log_b a &= 2 \\
f(n) &= \Theta(n^{\log_b a}) \\
T(n) &= \Theta(n^2 \lg n)
\end{aligned}
$$

Master theorem example 3

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a-\epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a+\epsilon}) \Leftrightarrow \Theta(f(n)) \qquad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + \lg n$$

$$\log_b a = 2$$

$$f(n) = O(n^{\log_b a-\epsilon})$$

$$T(n) = \Theta(n^2)$$

Master theorem example 4

$$
\begin{aligned}
T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\
O(n^{\log_b a - \epsilon}) &\Leftrightarrow \Theta(n^{\log_b a}) \\
\Theta(n^{\log_b a}) &\Leftrightarrow \Theta(f(n) \lg n) \\
\Omega(n^{\log_b a + \epsilon}) &\Leftrightarrow \Theta(f(n)) \qquad\qquad \text{when } af(n/b) \leq cf(n)
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n^3 \\
\log_b a &= 2 \\
f(n) &= \Omega(n^{\log_b a + \epsilon}) \\
T(n) &= \Theta(n^3)
\end{aligned}
$$

because

$$
\begin{aligned}
af(n/b) &= 4f(n/2) \\
&= 4/2^3 f(n) \leq cf(n)
\end{aligned}
$$

Master theorem does not apply

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \lg n \\
n^{\log_b a} &= n \\
f(n) &= n \lg n \\
f(n) &= \Omega(n^{\log_b a}) = \Omega(n) \\
f(n) &\neq \Omega(n^{\log_b a+\epsilon}) = \Omega(n^{1+\epsilon}) \\
n \lg n &\neq \Omega(n^{1.0000000000001}) \\
\lg n &\neq \Omega(n^{0.0000000000001})
\end{aligned}
$$

Master theorem does apply

$$
\begin{aligned}
T(n) &= 4T(n/3) + n \lg n \\
n^{\log_b a} &= n^{\log_c 4} = n^{1.26\ldots} \\
f(n) &= n \lg n \\
f(n) &= O(n^{\log_b a}) = O(n^{1.26\ldots}) \\
f(n) &= O(n^{\log_b a - \epsilon}) = O(n^{1.2}) && \epsilon = 0.06\ldots \\
n \lg n &= O(n^{1.2}) \\
\lg n &= O(n^{0.2})
\end{aligned}
$$