Summation Facts

$$\sum_{i=m}^{n} c = (n - m + 1)c$$

$$\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} +b_i$$

$$\sum_{i=m}^{n} a_{i+k} = \sum_{i=m+k}^{n+k} a_i$$

$$\sum_{i=m}^{n} a_i x^{i+k} = x^k \sum_{i=m}^{n} a_i x^i$$

$$\sum_{i=m}^{n} (a_i - a_{i-1}) = a_n - a_{m-1}$$

Sum of 1

Use telescoping:

$$\sum_{i=m}^{n}(a_i - a_{i-1}) \;=\; a_n - a_{m-1}$$

Let $a_i = i$:

$$\sum_{i=1}^{n}(a_i - a_{i-1}) \;=\; \sum_{i=1}^{n}(i - (i-1))$$

$$= \sum_{i=1}^{n} 1$$

$$= a_n - a_{m-1}$$

$$= n - 0$$

$$= n$$

Sum of i

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$$
$$+ \sum_{i=1}^{n} i = n + n-1 + \ldots + 1$$

$$= n+1 + n+1 + \ldots + n+1$$
$$= n(n+1)$$

$$\Rightarrow \sum_{i=1}^{n} i = n(n+1)/2$$

Sum of i

$$
\begin{aligned}
n^2 &= n^2 - 0^2 \\
&= \sum_{i=1}^{n} (i^2 - (i-1)^2) \\
&= \sum_{i=1}^{n} (2i - 1) \\
&= 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 \\
&= 2\sum_{i=1}^{n} i - n \\
n^2 + n &= 2\sum_{i=1}^{n} i \\
\sum_{i=1}^{n} i &= \frac{n^2 + n}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}
$$

Sum of i

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1$$

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=0}^{n} (i+1)^2$$

$$= \sum_{i=0}^{n} (i^2 + 2i + 1)$$

$$= \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 2i + \sum_{i=0}^{n} 1$$

$$= \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 2i + n + 1$$

$$\sum_{i=1}^{n} i^2 + n^2 + 2n + 1 = \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 2i + n + 1$$

$$n^2 + 2n + 1 = \sum_{i=0}^{n} 2i + n + 1$$

$$\sum_{i=0}^{n} 2i = n^2 + n$$

$$\sum_{i=0}^{n} i =_5 n(n+1)/2$$

Sum of Odd Numbers

$$
\begin{aligned}
1 + 3 + \ldots + (2n + 1) &= \sum_{i=0}^{n} (2i + 1) \\
&= \sum_{i=0}^{n} 2i + \sum_{i=0}^{n} 1 \\
&= 2 \sum_{i=0}^{n} i + (n + 1) \\
&= 2n(n + 1)/2 + (n + 1) \\
&= (n + 1)^2
\end{aligned}
$$

Geometric Sum

$$
\begin{aligned}
S_n &= 1 + a + a^2 + \ldots + a^n \\
&= \sum_{i=0}^{n} a^i \\
S_{n+1} &= (1 + a + a^2 + \ldots + a^n) + a^{n+1} \\
&= S_n + a^{n+1} \\
S_{n+1} &= 1 + (a + a^2 + \ldots + a^n + a^{n+1}) \\
&= 1 + a(1 + a + a^2 + \ldots + a^n) \\
&= 1 + aS_n \\
&\Rightarrow \\
S_n + a^{n+1} &= 1 + aS_n \\
&\Rightarrow \\
\sum_{i=0}^{n} a^i &= S_n \\
&= \frac{a^{n+1} - 1}{a - 1}
\end{aligned}
$$

Geometric Sum

$$(a-1)\sum_{i=0}^{n} a^i = \sum_{i=0}^{n}(a^{i+1} - a^i)$$

$$= \sum_{j=1}^{n+1}(a^j - a^{j-1})$$

$$= a^{n+1} - a^0$$

$$= a^{n+1} - 1$$

Also do division:

$$
\require{enclose}
\begin{array}{r}
a^4 + a^3 + a^2 + a + 1 \hphantom{-1} \\
a-1 \overline{)\, a^5 \hphantom{+ a^4 + a^3 + a^2 + a} - 1} \\
\underline{-\,a^5 + a^4} \hphantom{+ a^3 + a^2 + a - 1} \\
a^4 \hphantom{+ a^3 + a^2 + a - 1} \\
\underline{-\,a^4 + a^3} \hphantom{+ a^2 + a - 1} \\
a^3 \hphantom{+ a^2 + a - 1} \\
\underline{-\,a^3 + a^2} \hphantom{+ a - 1} \\
a^2 \hphantom{+ a - 1} \\
\underline{-\,a^2 + a} \hphantom{- 1} \\
a - 1 \\
\underline{-\,a + 1} \\
0
\end{array}
$$

Sum of ia^i

$$S_n = \sum_{i=1}^{n} ia^i = 1 + a + 2a^2 + \ldots + na^n$$

$$S_{n+1} = \sum_{i=1}^{n+1} ia^i = \sum_{i=1}^{n} ia^i + (n+1)a^{n+1}$$

$$= S_n + (n+1)a^{n+1}$$

$$S_{n+1} = \sum_{i=1}^{n+1} ia^i = \sum_{i=0}^{n} (i+1)a^{i+1}$$

$$= \sum_{i=0}^{n} ia^{i+1} + \sum_{i=0}^{n} a^{i+1}$$

$$= a\sum_{i=0}^{n} ia^i + a\sum_{i=0}^{n} a^i$$

$$= aS_n + a(a^{n+1} - 1)/(a-1)$$

$$\Rightarrow$$

$$S_n + (n+1)a^{n+1} = aS_n + a(a^{n+1} - 1)/(a-1)$$

$$\Rightarrow$$

$$S_n = \frac{a - (n+1)a^{n+1} + na^{n+2}}{(a-1)^2}$$

Sum of ia^{i-1}

Don't forget your calculus!

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$
\begin{aligned}
\sum_{i=0}^{n} ia^{i-1} &= \sum_{i=0}^{n} \frac{d}{da}(a^i) \\
&= \frac{d}{da} \sum_{i=0}^{n}(a^i) \\
&= \frac{d}{da}\left(\frac{a^{n+1} - 1}{a - 1}\right)
\end{aligned}
$$

Sum of i^2

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$$

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=0}^{n} (i+1)^3$$

$$= \sum_{i=0}^{n} (i^3 + 3i^2 + 3i + 1)$$

$$= \sum_{i=0}^{n} i^3 + \sum_{i=0}^{n} 3i^2 + \sum_{i=0}^{n} 3i + \sum_{i=0}^{n} 1$$

$$= \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + (n+1)$$

$$= \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + (n+1)$$

$$\Rightarrow$$

$$(n+1)^3 = 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + (n+1)$$

$$\Rightarrow$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

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Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$\sum_{n=1}^{2^k} \frac{1}{n} \geq 1 + \frac{k}{2} \qquad\qquad k > 0$$

$$\sum_{n=1}^{k} \frac{1}{n} < \ln(k) + 1$$

Change of variables

$$\sum_{i=1}^{n} \frac{1}{n-i+1} = \sum_{i=1}^{i=n} \frac{1}{n-i+1} \qquad \text{let } i = n - j$$

$$= \sum_{n-j=1}^{n-j=n} \frac{1}{n-(n-j)+1}$$

$$= \sum_{j=n-1}^{j=0} \frac{1}{j+1}$$

$$= \sum_{j=0}^{n-1} \frac{1}{j+1}$$

$$= \sum_{j=1}^{n} \frac{1}{j}$$

$$= \sum_{i=1}^{n} \frac{1}{i}$$

Which is the Harmonic series.

Consider using Sympy

```
>>> from sympy import summation, oo, symbols, latex, log

>>> i,n,m = symbols('i n m', integer=True)

>>> summation(2*i + 1, (i, 0, n-1))
n**2

>>> summation(1/2**i, (i,0,oo))
2

>>> summation(i, (i, 0, n), (n, 0, m))
m**3/6 + m**2/2 + m/3

>>> print(latex(summation(i, (i,0,n), (n,0,m))))
\frac{m^{3}}{6} + \frac{m^{2}}{2} + \frac{m}{3}

>>> print(latex(summation(1/log(n)**n, (n, 2, oo))))
\sum_{n=2}^{\infty} \log^{- n}{\left (n \right )}
```