

Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on nonnegative integers by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Then $T(n)$ has the following asymptotic bounds:

$f(n)$	$T(n)$
$O(n^{\log_b a - \epsilon})$	$\Theta(n^{\log_b a})$
$\Theta(n^{\log_b a})$	$\Theta(f(n) \lg n)$
$\Omega(n^{\log_b a + \epsilon})$	$\Theta(f(n))$

The last one only if $af(\frac{n}{b}) \leq cf(n)$ for some $c < 1$ and large enough n .

Master theorem examples

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n))$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$\log_b a = 1$$

$$f(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n \lg n)$$

Master theorem does not apply

$$T(n) = 2T(n/2) + n \lg n$$

$$n^{\log_b a} = n$$

$$f(n) = n \lg n$$

$$f(n) = \Omega(n^{\log_b a}) = \Omega(n)$$

$$f(n) \neq \Omega(n^{\log_b a + \epsilon}) = \Omega(n^{1+\epsilon})$$

$$n \lg n \neq \Omega(n^{1.0000000000000001})$$

$$\lg n \neq \Omega(n^{0.0000000000000001})$$

Master theorem does apply

$$T(n) = 4T(n/3) + n \lg n$$

$$n^{\log_b a} = n^{\log_3 4} = n^{1.26\dots}$$

$$f(n) = n \lg n$$

$$f(n) = O(n^{\log_b a}) = O(n^{1.26\dots})$$

$$f(n) = O(n^{\log_b a - \epsilon}) = O(n^{1.2}) \quad \epsilon = 0.06\dots$$

$$n \lg n = O(n^{1.2})$$

$$\lg n = O(n^{0.2})$$