# Notes on Hash Tables

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# Dictionary Operations

- Insert
- Search
- Delete

# Hash table implementation of Dictionary

- Expected search time: $O(1)$
- Worst case search: $O(n)$

# Hash table is generalization of an ordinary array

- With array, the key $k$ is the position $k$ in the array.
- Given a key $k$, we find the element with key $k$ by **direct addressing**.
- Direct addressing only applicable when we can afford to allocate an array with one position for every key.

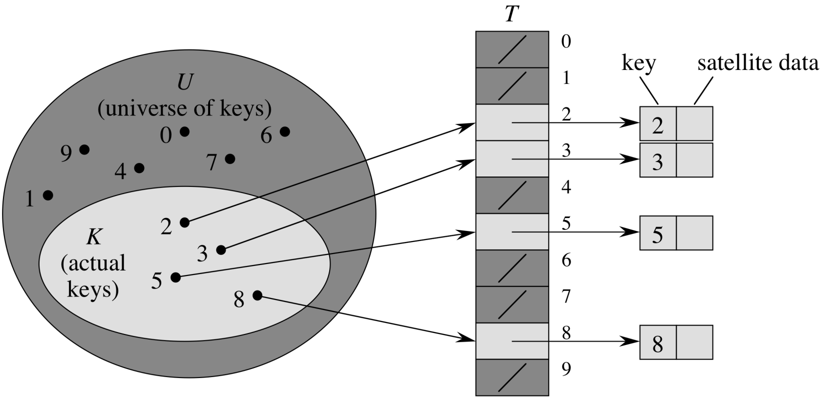
# Use hash table when we don't have one position for each key

- Number of keys stored is small relative to the number of possible keys.
- Hash table is an array with size proportional to the number of keys stored, not the number of possible keys.
- Given a key $k$, don't use $k$ to index the array.
- Instead, compute a function of $k$ and use that to index the array.
- This function is called a **hash function**.
- Have to solve issue of what to do when hash function maps multiple keys to same table entry.
  - chaining
  - open addressing

# Direct-address tables

- Scenario:
  - Maintain a dynamic set
  - Each element has a key drawn from a universe $U = \{0, 1, \ldots, m-1\}$ where $m$ isn't too large.
  - No two elements have the same key.
- Represent by a **direct-address table**, or array, $T[0 \ldots m-1]$:
  - Each *slot*, or position, corresponds to a key in $U$.
  - If there's an element $x$ with key $k$, then $T[k]$ contains a pointer to $x$.
  - Otherwise, $T[k]$ is empty, represented by $\mathrm{NIL}$.

# Direct-address table




Direct-Address-Search(T,k)

1 **return** $T[k]$

Direct-Address-Insert(T,k)

1 $T[key[x]] = x$

Direct-Address-Delete(T,k)

1 $T[key[x]] = \text{NIL}$

All operations $O(1)$.

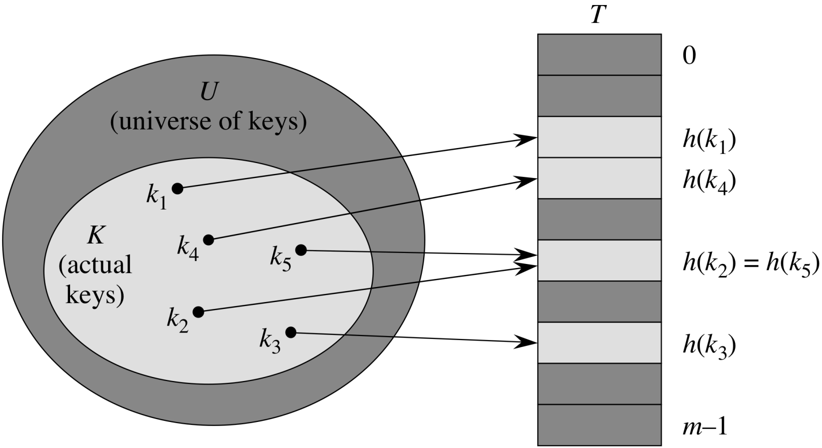
# Hash tables

- If $U$ is large, storing a table of size $|U|$ is impractical.
- Often the set $K$ of keys actually used is small compared to $U$.
  - Most of the space in a direct-access table is wasted.
- When $K$ is much smaller than $U$, a hash table requires much less space than a direct-address table.
- Can reduce storage requirements to $\Theta(|K|)$
- Can still get $O(1)$ search time on *average*, but not *worst* case.

# Hash table idea

- Instead of storing an element with key $k$ in slot $k$, use a function $h$ and store the element in slot $h(k)$.
- $h$ is called a **hash function**
- $h : U \to \{0, 1, \ldots, m-1\}$
- $m \ll |U|$
- $h(k)$ is a legal slot number in $T$
- We say $k$ *hashes* to $h(k)$

# Collisions



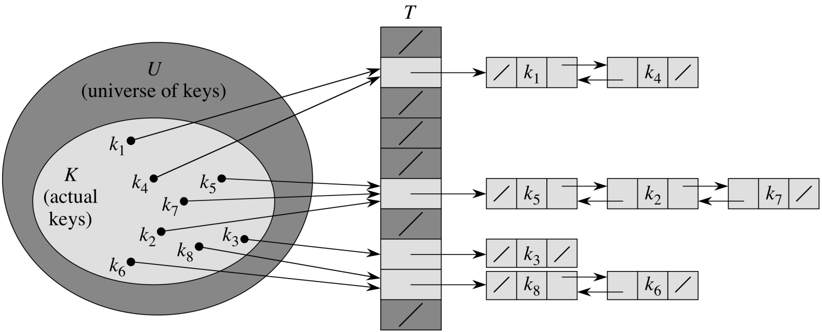
$T$
$U$
(universe of keys)
$K$
(actual keys)
$k_1$
$k_4$
$k_5$
$k_2$
$k_3$
0
$h(k_1)$
$h(k_4)$
$h(k_2) = h(k_5)$
$h(k_3)$
$m-1$

# Collisions

- When two or more keys hash to the same slot.
- Can happen when there are more possible keys than slots ($|U| > m$).
- For a given set $K$ of keys with $|K| \leq m$, may or may not happen.
- Definitely happens when $|K| > m$.
- Must be prepared to handle collisions in all cases.
- Two methods:
  - chaining
  - open addressing
- Chaining is usually better.

# Collision resolution by chaining

- Put all elements that hash to the same slot into a linked list.




- Doubly linked list allows easy deletion.

# Implementation of hash table with chaining

Chained-Hash-Insert($T, x$)

1 insert $x$ at the head of list $T[h(key[x])]$

- Worst case $O(1)$
- Assumes element inserted not already in list.
- Would take an additional search to see if it was already inserted.

# Implementation of hash table with chaining

Chained-Hash-Search($T$, $k$)

1 search for element with key $k$ in list $T[h(k)]$

- Running time proportional to length of list in slot $h(k)$

# Implementation of hash table with chaining

Chained-Hash-Delete($T, x$)

delete $x$ from the list $T[h(key[x])]$

- Given pointer $x$ to the element to delete, so no search is needed to find this element.
- Worst case $O(1)$ if lists are doubly linked.
- If lists are singly linked, deletion takes as long as search, because we must find $x$'s predecessor.

# Analysis of hashing with chaining

- Given a key, how long does it take to find an element with that key, or determine that there is no element with that key?
- Analysis is in terms of the **load factor** $\alpha = n/m$
- $n = \#$ elements in the table
- $m = \#$ slots in the table
- Load factor is average number of elements per linked list.
- Can have $\alpha < 1$, $\alpha = 1$, or $\alpha > 1$
- Worst case is when all $n$ keys hash to the same slot:
  - a single list of length $n$
  - worst case is $\Theta(n)$ plus time to compute $h$
- Average case depends on how well the hash function distributes keys among slots.

# Average-case analysis of hashing with chaining

- Assume **simple uniform hashing**: any given element is equally likely to hash to any of the $m$ slots.
- For $j = 0, 1, \ldots, m-1$, denote the length of the list $T[j]$ by $n_j$.
- $n = n_0 + n_1 + \cdots + n_{m-1}$
- Average value of $n_j$ is $E[n_j] = \alpha = n/m$
- Assume we can compute $h$ in $O(1)$ time, so that the time required to search for $k$ depends on the length $n_{h(k)}$ of the list $T[h(k)]$.
- Two cases:
  - Unsuccessful search: hash table has no element with key $k$
  - Successful search: hash table contains an element with key $k$

# Unsuccessful search

**Theorem**

An unsuccessful search takes expected time $\Theta(1 + \alpha)$.

**Proof**

- Simple uniform hashing means any key not already in the table is equally likely to hash to any of the $m$ slots.
- To search unsuccessfully for any key $k$, need to search to the end of the list $T[h(k)]$.
- This list has expected length $\alpha$.
- Adding the time to compute the hash function gives $\Theta(1 + \alpha)$.

# Successful search

- The expected time for a successful search is also $\Theta(1 + \alpha)$.
- The probability that each list is searched is proportional to the length of the list.

# Successful search

**Theorem**
A successful search takes expected time $\Theta(1 + \alpha)$.
**Proof**

- Assume the element $x$ is equally likely to be any of the $n$ elements stored in the table.
- The number examined during the search for $x$ is 1 more than the number of elements that appear before $x$ in $x$'s list.
- These are the elements inserted *after* $x$ was inserted.
- We need to find the average, over $n$ elements, of how many elements were inserted into $x$'s list after $x$ was inserted.
- Let $x_i$ be the $i$th element inserted, and let $k_i = key[x_i]$.
- For all $i$ and $j$, let $X_{ij} = I\{h(k_i) = h(k_j)\}$
- Simple uniform hashing means

$$Pr\{h(k_k) = h(k_j)\} = 1/m = E[X_{ij}]$$

# Expected number of elements examined, successful search

$$
\begin{aligned}
E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right] &= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}E[X_{ij}]\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right) \\
&= 1+\frac{1}{nm}\sum_{i=1}^{n}(n-i) \\
&= 1+\frac{1}{nm}\sum_{i=0}^{n-1} i \\
&= 1+\frac{1}{nm}\frac{n(n-1)}{2} \\
&= 1+\frac{n-1}{2m} = 1+\frac{\alpha}{2}-\frac{\alpha}{2n} = \Theta(1+\alpha)
\end{aligned}
$$

# Alternative analysis

- $X_{ij\ell} = I\{\text{the search is for } x_i,\ h(k_i) = h(k_j) = \ell\}$
- Simple uniform hashing means $Pr\{h(k_i) = \ell\} = Pr\{h(k_j) = \ell\} = 1/m$
- $Pr\{\text{the search is for } x_i\} = 1/n$
- All these are independent: $Pr\{X_{ij\ell} = 1\} = E[X_{ij\ell}] = 1/nm^2$

$$
\begin{aligned}
Y_j &= I\{x_j \text{ appears in a list prior to the } x_i\} \\
&= \sum_{i=1}^{j-1} \sum_{\ell=0}^{m-1} X_{ij\ell}
\end{aligned}
$$

# Alternative analysis, continued

$$
\begin{aligned}
E\left[1+\sum_{j=1}^{n} Y_j\right] &= 1 + E\left[\sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} X_{ij\ell}\right] \\
&= 1 + \sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} E[X_{ij\ell}] \\
&= 1 + \sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} \frac{1}{nm^2} \\
&= 1 + \binom{n}{2} \cdot m \cdot \frac{1}{nm^2} \\
&= 1 + \frac{n-1}{2m} \\
&= 1 + \frac{\alpha}{2} - \frac{\alpha}{2n} = \Theta(1+\alpha)
\end{aligned}
$$

# Interpretation

- If $n = O(m)$ then $\alpha = n/m = O(1)$, which means searching takes constant time on average.
- Since insertion and deletion take $O(1)$ worst case time, all dictionary operations take average time $O(1)$.

# Hash functions

- Ideally, satisfies the assumption of simple uniform hashing.
- In practice, impossible since we don't know the distribution of input keys.
- Often use heuristics, based on the domain of the keys, to create hash functions that work well.

# Keys as natural numbers

- Hash functions usually assume keys are natural numbers.
- Can interpret any computer data as natural number.
- Interpret as radix $2^p$ number.
- Strings, for example: $\mathrm{CLRS}$
  - ASCII: 67, 76, 82, 83
  - There are 128 ASCII values, use radix $2^7$:
  - $h(\mathrm{CLRS}) = 67(128^3) + 76(128^2) + 82(128^1) + 83(128^0) = 141,764,947$

# Division method for hash functions

$$h(k) = k \bmod m$$

- Example: $m = 20$ and $k = 91 \Rightarrow h(k) = 11$
- Fast: requires only one division.
- Bad ideas:
  - Powers of 2 are bad values for $m$:
    just uses least significant bits.
  - If $k$ is a character string interpreted as radix $2^p$ number, then $m = 2^p - 1$ is bad: permuting characters does not change hash value.
- Good choice for $m$:
  Prime number not too close to a power of 2.

# Division method example

- Store $n \approx 2000$ character strings.
- Don't mind searching 3 strings per unsuccessful search.
- Choose $m = 701$.
- This is a prime near $2000/3$ but not near any power of 2.

# Multiplication method for hash functions

1. Choose $0 < A < 1$
2. Multiply $k$ by $A$
3. Extract the fractional part
4. Multiply by $m$
5. Take the floor.

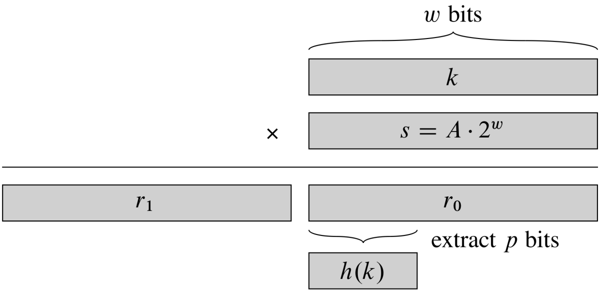
$$h(k) = \lfloor m(kA \bmod 1) \rfloor$$

where

$$kA \mod 1 = kA - \lfloor kA \rfloor$$

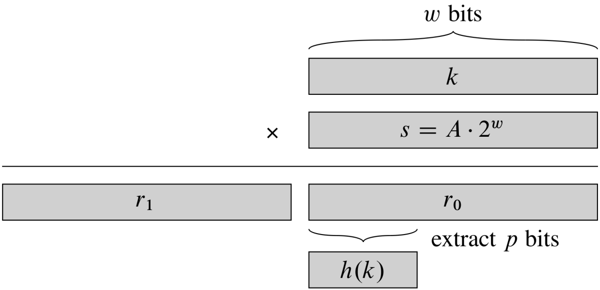
gives the fractional part.

# Easy implementation of: $h(k) = \lfloor m(kA \bmod 1) \rfloor$




- Choose $m = 2^p$.
- Let word size be $w$ bits.
- Assume $k$ fits in a single word.
- Let $s$ be an integer in the range $0 < s < 2^w$.
- Let $A$ be $s/2^w$.

# Easy implementation of: $h(k) = \lfloor m(kA \bmod 1) \rfloor$




- Multiply $k$ by $s$.
- Result is $2w$ bits.
- We can ignore $r_1$, since $r_0$ is the fractional part.
- Multiplying by $m = 2^p$ just shifts $r_0$ left $p$ places.
- Instead, just take the $p$ most significant bits of $r_0$.

# Example computation of: $h(k) = \lfloor m(kA \bmod 1) \rfloor$

- Choose $m = 2^3 = 8$, $w = 5$, $k = 21$.
- Choose $0 < s < 2^5$, s $= 13$, therefore $A = 13/32$.

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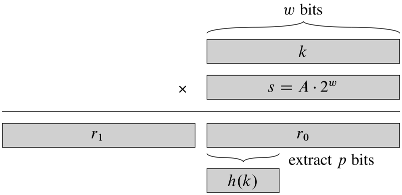
- $kA = 21(13/32) = 273/32 = 8\frac{17}{32}$
- $kA \mod 1 = 17/32$
- $m(kA \mod 1) = 8(17/32) = 17/4 = 4\frac{1}{4}$
- $\lfloor m(kA \mod 1) \rfloor = 4 = h(k)$

---

- $ks = 21(13) = 273 = 8(2^5) + 17$
- $r_1 = 8$, $r_0 = 17 = 10001_b$.
- $p = 3$ most significant bits is $100_b = 4$.



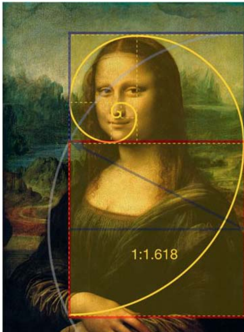
# Multiplication method for hash functions

$$h(k) = \lfloor m((kA) \bmod 1) \rfloor$$

- Disadvantage: slower than division method.
- Advantage: value of $m$ is not critical, $2^p$ a good choice.
- Knuth suggests a value for $A$:

$$\frac{\sqrt{5}-1}{2} = 0.6180339887\ldots$$

- Look up the Golden Ratio.



# Universal hashing—randomized hashing

- For any hash function, the world *could* give us keys that all hash to the same spot. If the world was very, very mean.
- To randomize this, choose a hash function randomly from a collection of hash functions, each time the program starts.
- A collection of hash functions, $\mathcal{H}$ that map a universe $U$ into keys $0 \leq k < m$ is called **universal** if for each pair, $k, \ell \in U$, $k \neq \ell$,

$$\Pr_{h \in \mathcal{H}}\{h(k) = h(\ell)\} \leq \frac{1}{m}$$

- In other words, the chance of a collision between $k$ and $\ell$ is no more than $1/m$, when $h$ is chosen at random from $\mathcal{H}$.
- Such collections of hash functions are easy to design.

# Universal hashing expected chain lengths

Using chaining and universal hashing on key $k$:

- If $k$ is not in the table,

$$E[n_{h(k)}] \leq \alpha$$

- If $k$ is in the table,

$$E[n_{h(k)}] \leq 1 + \alpha$$

- So the expected time for $\textsc{Search}$ is $O(1)$.

# Open addressing

- Instead of chaining, store all keys in hash table.
- Must have $\alpha \leq 1$.
- Use $h(key) + i \mod m, i = 0, 1, 2, \ldots m - 1$
- Example: $h(n) = n \mod 10$

| Index: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Insert 12: | x | x | 12 | x | x | x | x | x | x | x |
| Insert 14: | x | x | 12 | x | 14 | x | x | x | x | x |
| Insert 32: | x | x | 12 | 32 | 14 | x | x | x | x | x |
| Insert 92: | x | x | 12 | 32 | 14 | 92 | x | x | x | x |
| Insert 53: | x | x | 12 | 32 | 14 | 92 | 53 | x | x | x |

- Linear probing like this leads to **clustering**.

# Open addressing

- More generally, use a hash function that takes both a key and a position and returns a position:

$$h : U \times \{0, 1, ..., m-1\} \to \{0, 1, ..., m-1\}$$

- Then use probe sequence

$$h(k, 0), h(k, 1), ..., h(k, m-1)$$

- We require that for every key, $k$, the probe sequence be a permutation of

$$(0, 1, ..., m-1)$$

  so that all possible probes are examined in $m$ probes.
- Linear probing satisfies this requirement, but has bad clustering.

# Hash insertion and search, open addressing

```
HASH-INSERT(T, k)
  i = 0
  repeat
      j = h(k, i)
      if T[j] == NIL
          T[j] = k
          return j
      else i = i + 1
  until i == m
  error "hash table overflow"
```

```
HASH-SEARCH(T, k)
  i = 0
  repeat
      j = h(k, i)
      if T[j] == k
          return j
      i = i + 1
  until T[j] == NIL or i = m
  return NIL
```

# Hash deletion, open addressing

- We cannot simply replace a deleted element with NIL.
- This might make search halt prematurely if clustering has occurred.
- Instead we insert a special DELETED value.
- Insert will treat DELETED as available.
- Search will treat DELETED as full.
- Search time no longer depends on $\alpha$ alone.

# Uniform hashing with open addressing

- In our analysis, we assume **uniform hashing**:
  - The probe sequence for a key $k$ is equally likely to be any of the $m!$ possible sequences.
- Open addressing has to generalize the notion of uniform hashing to a function that generates an entire sequence of probes.
- True open uniform hashing is difficult.
- In practice approximations are used.

# Linear probing

- Given an ordinary hash function $h' : U \to \{0, \ldots, m-1\}$, called an **auxiliary hash function**, use

$$h(k, i) = (h'(k) + i) \mod m$$

- Only generates $m$ of the $m!$ possible permutations of $\{0, \ldots, m-1\}$
- Extremely susceptible to **primary clustering**.
- Any slot preceded by $i$ full slots gets filled with probability

$$\frac{i+1}{m}$$

and hence long chains get longer.

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- Extremely susceptible to **primary clustering**.
- Any slot preceded by $i$ full slots gets filled with probability

$$\frac{i+1}{m}$$

and hence long chains get longer.
- Note: it does not matter if we use

$$h(k, i) = h'(k) + ai + b$$

. Why not?

# Quadratic probing

$$h(k,i) = (h'(k) + c_1 i + c_2 i^2) \mod m$$

- $c_1, c_2, m$ must be carefully selected.
- Primary clustering is eliminated.
- If two keys have the same initial probe, their entire sequence is the same.
- This is called **secondary clustering** and is not as serious.
- Only $m$ of the $m!$ possible permutations of $\{0, \ldots, m-1\}$ are used.

# Double hashing

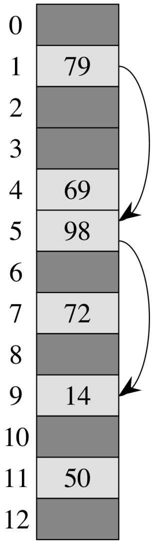
$$h(k, i) = (h_1(k) + ih_2(k)) \mod m$$

- Needs two auxiliary hash functions, $h_1, h_2$.
- $h_2(k)$ must be relatively prime to $m$:
  - Let $m = 2^p$ and make $h_2(k)$ odd.
  - Let $m$ be prime and make $h_2(k) < m$:

$$\begin{aligned} h_1(k) &= k \mod m \\ h_2(k) &= 1 + (k \mod (m-1)) \end{aligned}$$

- $\Theta(m^2)$ probe sequences used, instead of $\Theta(m)$.
- Performance is very close to ideal uniform hashing.

# Double hashing




$$h_1(k) = k \mod 13$$
$$h_2(k) = 1 + (k \mod 11)$$

$$h_1(14) = 1$$
$$h_2(14) = 4$$

# Analysis of open-address hashing, unsuccessful search

**Theorem 11.6**
Assuming uniform hashing and an open-address hash table with load factor $\alpha = n/m < 1$ the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$.
**Proof**

$$
\begin{aligned}
X &= \text{number of probes in unsuccessful search} \\
A_k &= \{k\text{th probe is to an occupied slot}\} \\
\{X \geq i\} &= \bigcap_{k=1}^{i-1} A_k \\
\Pr\{X \geq i\} &= \Pr\{A_1\} \cdot \Pr\{A_2 | A_1\} \cdot \Pr\{A_3 | A_1 \cap A_2\} \\
&\cdots \\
&\cdot \Pr\{A_{i-1} | A_1 \cap A_2 \cap \ldots \cap A_{i-2}\}
\end{aligned}
$$

# Probability that $i$ probes find occupied slot

$$
\begin{aligned}
X &= \text{number of probes in unsuccessful search} \\
A_k &= \{k\text{th probe is to an occupied slot}\} \\
\{X \geq i\} &= \bigcap_{k=1}^{i-1} A_k \\
\Pr\{X \geq i\} &= \Pr\{A_1\} \cdot \Pr\{A_2 | A_1\} \cdot \Pr\{A_3 | A_1 \cap A_2\} \\
& \cdots \\
& \cdot \Pr\{A_{i-1} | A_1 \cap A_2 \cap \ldots \cap A_{i-2}\} \\
&= \frac{n}{m} \cdot \frac{n-1}{m-1} \cdot \frac{n-2}{m-2} \cdots \frac{n-i+2}{m-i+2} \\
&\leq \left(\frac{n}{m}\right)^{i-1} \\
&= \alpha^{i-1}
\end{aligned}
$$

# Expected number of probes in unsuccessful search

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} \Pr\{X \geq i\} \\
&\leq \sum_{i=1}^{\infty} \alpha^{i-1} \\
&= \sum_{i=0}^{\infty} \alpha^{i} \\
&= \frac{1}{1-\alpha} \\
&= 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \cdots
\end{aligned}
$$

- The last expression gives us an intuitive picture.

# Analysis of open-address hashing, unsuccessful search

**Theorem 11.6**
Assuming uniform hashing and an open-address hash table with load factor $\alpha = n/m < 1$ the expected number of probes in an unsuccessful search is at most $1/(1 - \alpha)$.

- If $\alpha = 0.5$, then we expect less than 2 probes on average.
- If $\alpha = 0.9$, then we expect less than 10 probes on average.

# Expected number of probes for Hash-Insert

- An element is inserted after finding an open slot.
- This is the same procedure followed by an unsuccessful search.
- Therefore the expected number of probes is at most

$$\frac{1}{1-\alpha}$$

# Analysis of open address hashing, successful search

**Theorem 11.8**

Assuming uniform hashing and every key is equally likely, in an open-address hash table with load factor $\alpha < 1$ the expected number of probes in a successful search is at most

$$\frac{1}{\alpha} \ln \frac{1}{1-\alpha}$$

**Proof**

- A successful search does the same as when the $k$ was inserted.
- If $k$ was the $i$th key inserted, then $\alpha = i/m$ when inserted.
- Therefore there were at most $1/(1 - i/m) = m/(m - i)$ probes
- The average over all $n$ keys is then

$$\frac{1}{n} \sum_{i=0}^{n-1} \frac{m}{m-i}$$

# Averaging over the $n$ keys in the table:

$$
\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1}\frac{m}{m-i} &= \frac{m}{n}\sum_{i=0}^{n-1}\frac{1}{m-i} \\
&= \frac{1}{\alpha}\sum_{k=m-n+1}^{m}\frac{1}{k} \\
&\leq \frac{1}{\alpha}\int_{m-n}^{m}(1/x)dx \\
&= \frac{1}{\alpha}\ln\frac{m}{m-n} \\
&= \frac{1}{\alpha}\ln\frac{1}{1-\alpha}
\end{aligned}
$$

# Analysis of open address hashing, successful search

**Theorem 11.8**
Assuming uniform hashing and every key is equally likely, in an open-address hash table with load factor $\alpha < 1$ the expected number of probes in a successful search is at most

$$\frac{1}{\alpha} \ln \frac{1}{1-\alpha}$$

- If $\alpha = 0.5$, then we expect less than 1.387 probes on average.
- If $\alpha = 0.9$, then we expect less than 2.559 probes on average.

# Perfect hashing

- In some applications the keys are static:
  - Reserved words in a programming language.
  - File names on a write-only CD-rom.
- In this case we can guarantee **worst-case** $O(1)$.
- Use a double hashing scheme.
- Choose a good primary $h$ from a universal hash, $\mathcal{H}$.
- For each slot $j$, choose a $h_j$ into a secondary hash table, $S_j$.
- If size of $S_j$ is proportional to $n_j^2$, we can find $h_j$ with no collisions.
- If we choose $h$ carefully, expected size of all secondary hash tables is still $O(n)$.

# Perfect hashing