Probability and Counting, Appendix C

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Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$$(n + a)^b = O(n^b)$$

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$$
\begin{aligned}
(n+a)^b &= \sum_{k=0}^{b} \binom{b}{k} n^k a^{b-k} \\
&\leq \sum_{k=0}^{b} \binom{b}{k} n^b a^{b-k} \\
&= n^b \sum_{k=0}^{b} \binom{b}{k} a^{b-k} \\
&= n^b \sum_{k=0}^{b} \binom{b}{k} 1^k a^{b-k} \\
&= n^b (1+a)^b \\
&= O(n^b)
\end{aligned}
$$

Binomial Bounds

$$\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Stirling: $k! \geq (k/e)^k$

$$\binom{n}{k} \leq \frac{n^n}{k^k(n-k)^{n-k}}$$

by induction

$$\binom{n}{\lambda n} \leq 2^{nH(\lambda)}$$

$$H(\lambda) = -\lambda \lg \lambda - (1-\lambda)\lg(1-\lambda)$$

Sample Space

- Set of all possible things that can happen.
- Each thing that can happen is an **elementary event**.
- Examples:
  - Flip a coin twice and observe which side is up: $\{HH, HT, TH, TT\}$
  - Flip a coin twice and count the heads: $\{0, 1, 2\}$
  - Throw a coin down the stairs and see what step it lands on: $\{1, 2, 3, \ldots, n\}$, where n is the number of steps.
  - Deal two cards: $\{\{A\diamondsuit, 5\clubsuit\}, \{10\heartsuit, K\clubsuit\}, \{A\spadesuit, 3\heartsuit\}, \ldots\}$
  - See who wins the election: $\{Clinton, Sanders, Trump, Cruz \ldots\}$

Events

- A subset of the sample space, S.
- Examples:
  - $\{HT, TH\} \subseteq \{HH, HT, TH, TT\}$
  - $\{\{A\spadesuit, A\clubsuit\}, \{A\heartsuit, A\diamondsuit\}\} \subseteq \{\{A\diamondsuit, 5\clubsuit\}, \{10\heartsuit, K\diamondsuit\}, \{A\spadesuit, 3\heartsuit\}, \ldots\}$
  - The **certain event**: S.
  - The **null event**: $\emptyset$

A probability distribution on a sample space S

$\Pr\{\}$ is a mapping from events to real numbers such that:

1. $\Pr\{A\} \geq 0$
2. $\Pr\{S\} = 1$
3. $\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\}$ whenever $A \cap B = \emptyset$

- Theorem:

$$
\begin{aligned}
\Pr\{A \cup B\} &= \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \\
&\leq \Pr\{A\} + \Pr\{B\}
\end{aligned}
$$

Discrete probability distribution

- If S is finite or countably infinite.

$$\Pr\{A\} = \sum_{s \in A} \Pr\{s\}$$

- If S is finite and each elementary event has the same probability, we have **uniform probability distribution**.

$$\Pr\{s\} = 1/|S|$$

Continuous uniform distribution

- Each real number between a and b is equally likely.
- Not all subsets have probabilities.
- Just use intervals, and countable unions of intervals.

$$\Pr\{[c, d]\} = \frac{d - c}{b - a}$$

Conditional probability

$$\Pr\{A \mid B\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

- Probability as if B were the sample space.
- Can condition a variable on events:

$$\begin{aligned}\Pr\{B\} &= \Pr\{B \cap A\} + \Pr\{B \cap \overline{A}\} \\ &= \Pr\{A\}\Pr\{B \mid A\} + \Pr\{\overline{A}\}\Pr\{B \mid \overline{A}\}\end{aligned}$$

Conditioning example

- Suppose we flip a coin.
- If it's heads, we roll a 6-sided die; if it's tails an 8-sided die.
- What's the probability of getting a 6?

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&= \Pr\{A\}\Pr\{B \mid A\} + \Pr\{\overline{A}\}\Pr\{B \mid \overline{A}\} \\
\Pr\{6\} &= \Pr\{6 \cap H\} + \Pr\{6 \cap \overline{H}\} \\
&= \Pr\{H\}\Pr\{6 \mid H\} + \Pr\{\overline{H}\}\Pr\{6 \mid \overline{H}\} \\
&= (1/2)(1/6) + (1/2)(1/8) \\
&= 7/48
\end{aligned}$$

Independence

$$\Pr\{A \cap B\} = \Pr\{A\}\Pr\{B\}$$

This implies

$$\Pr\{A \mid B\} = \Pr\{A\}$$

Bayes's theorem

$$\Pr\{A \mid B\} = \frac{\Pr\{A\}\Pr\{B \mid A\}}{\Pr\{B\}}$$

- This follows easily from

$$\Pr\{A \cap B\} = \Pr\{A\}\Pr\{B \mid A\} = \Pr\{B\}\Pr\{A \mid B\}$$

- We can combine this with a conditioning of B getting

$$\Pr\{A \mid B\} = \frac{\Pr\{A\}\Pr\{B \mid A\}}{\Pr\{A\}\Pr\{B \mid A\} + \Pr\{\overline{A}\}\Pr\{B \mid \overline{A}\}}$$

Bayes's theorem example

- We have a fair coin and a biased coin with $\Pr\{H\} = 2/3$. We choose a coin at random and flip it twice. It comes up heads both times. What is the probability we chose the biased coin?
- Let A be the event of choosing a biased coin, and let B be the event of coming up heads twice in a row.

$$
\begin{aligned}
\Pr\{A \mid B\} &= \frac{\Pr\{A\}\Pr\{B \mid A\}}{\Pr\{A\}\Pr\{B \mid A\} + \Pr\{\overline{A}\}\Pr\{B \mid \overline{A}\}} \\
&= \frac{(1/2)(4/9)}{(1/2)(4/9) + (1/2)(1/4)} \\
&= \frac{(2/9)}{(2/9) + (1/8)} \\
&= \frac{(2/9)}{(25/72)} \\
&= 16/25
\end{aligned}
$$