Notes on Quicksort

Geoffrey Matthews

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Quicksort

- $\Theta(n^2)$ worst case.
- $\Theta(n \lg n)$ expected running time.
- Constants are small.
- Sorts in place.

Quicksort: three step process

- To sort $A[p..r]$:
  - **Divide:** Partition $A[p..r]$ into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$,
    such that each element in the first subarray is $\leq A[q]$
    and $A[q] \leq$ each element in the second subarray.
  - **Conquer:** Sort the two subarrays by recursive calls.
  - **Combine:** Nothing needs to be done.

$\textsc{Quicksort}(A, p, r)$

if $p < r$
$\quad q = \textsc{Partition}(A, p, r)$
$\quad \textsc{Quicksort}(A, p, q-1)$
$\quad \textsc{Quicksort}(A, q+1, r)$

Initial call is $\textsc{Quicksort}(A, 1, n)$

Compare QUICKSORT and MERGESORT

QUICKSORT(A, p, r)

if $p < r$
  $q = $ PARTITION(A, p, r)
  QUICKSORT$(A, p, q - 1)$
  QUICKSORT$(A, q + 1, r)$

MERGE-SORT(A, p, r)

if $p < r$ // check for base case
  $q = \lfloor (p + r)/2 \rfloor$ // divide
  MERGE-SORT(A, p, q) // conquer
  MERGE-SORT$(A, q + 1, r)$ // conquer
  MERGE(A, p, q, r) // combine

Compare PARTITION and MERGE

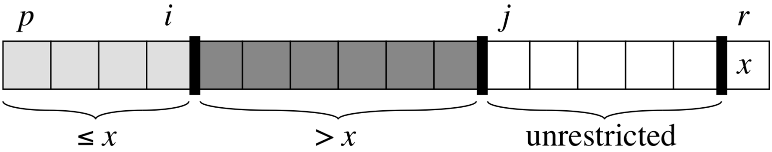
PARTITION(A, p, r)

```
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```

MERGE(A, p, q, r)

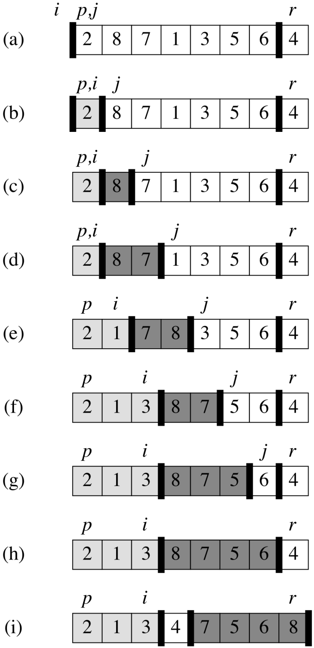
```
n1 = q - p + 1
n2 = r - q
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1 to n1
    L[i] = A[p + i - 1]
for j = 1 to n2
    R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```

Partition



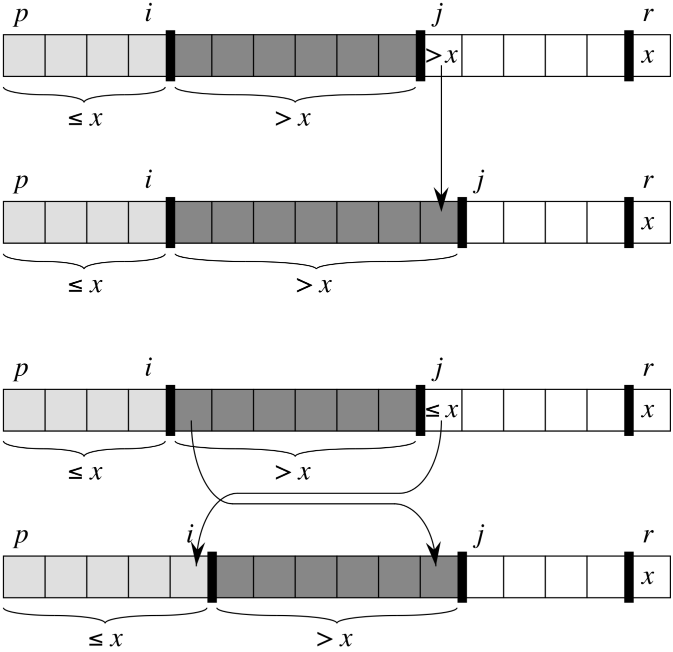
Loop invariant:

1. All entries in $A[p..i]$ are $\leq$ pivot
2. All entries in $A[i+1..j-1]$ are $>$ pivot
3. $A[r] =$ pivot


i p,j r
(a) 2 8 7 1 3 5 6 4
p,i j r
(b) 2 8 7 1 3 5 6 4
p,i j r
(c) 2 8 7 1 3 5 6 4
p,i j r
(d) 2 8 7 1 3 5 6 4
p i j r
(e) 2 1 7 8 3 5 6 4
p i j r
(f) 2 1 3 8 7 5 6 4
p i j r
(g) 2 1 3 8 7 5 6 4
p i r
(h) 2 1 3 8 7 5 6 4
p i r
(i) 2 1 3 4 7 5 6 8

(a)

(b)



Partition

```
PARTITION(A, p, r)
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```

- Always selects $A[r]$ as the **pivot**
- Loop invariant:
  1. All entries in $A[p..i]$ are $\leq$ pivot
  2. All entries in $A[i+1..j-1]$ are $>$ pivot
  3. $A[r] =$ pivot

$O(n)$

Can MERGESORT be done in place?

1	3	5	7	9	0	2	4	6	8

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1	3	5	7	9	0	2	4	6	8

- Yes, but *very* tricky!

Running time of quicksort

- Depends on partitioning of subarrays.
- If subarrays are balanced: fast as mergesort.
- If subarrays are unbalanced: slow as insertion sort.

Worst case for quicksort

- Arrays completely unbalanced.
- 0 elements in one and $n-1$ in the other
- Recurrence:

$$\begin{aligned} T(n) &= T(n-1) + T(0) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

- Same as insertion sort.
- Worst case for quicksort is the array is already sorted.
- This is the best case for insertion sort, which is $O(n)$.

Best case for quicksort

- Each subarray has $n/2$ elements.
- Recurrence:

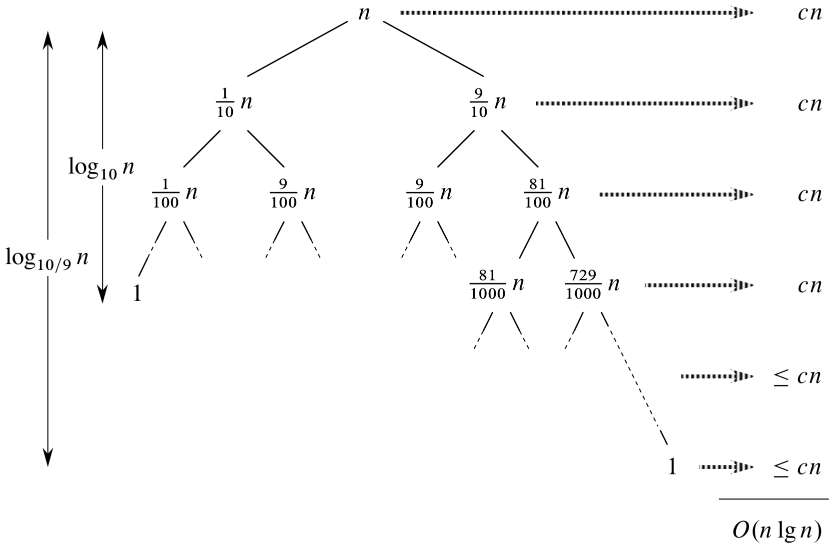
$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

- Same as mergesort.

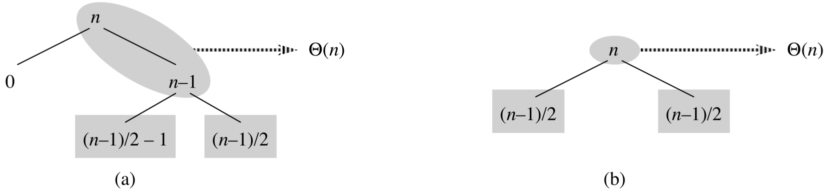
"Average" running time for quicksort

Assume PARTITION always makes a 9-to-1 split.

$T(n) \leq T(9n/10) + T(n/10) + \Theta(n) = O(n \lg n)$



"Average" running time for quicksort



- If levels alternate between good and bad splits, still $O(n \lg n)$
- If we randomize choice of pivot, what is the probability that all of the choices will be worst case?
- If we randomize choice of pivot, what is the probability that more than half of the choices will be worst case?

Randomized quicksort

- Instead of randomizing the entire array, which adds a large constant factor, just randomize the choice of pivot.

RANDOMIZED-PARTITION(A, p, r)

$i = $ RANDOM(p, r)
exchange $A[r]$ with $A[i]$
return PARTITION(a, p, r)

RANDOMIZED-QUICKSORT(A, p, r)

if $p < r$
    $q = $ RANDOMIZED-PARTITION(A, p, r)
    QUICKSORT$(A, p, q - 1)$
    QUICKSORT$(A, q + 1, r)$

- On average, the splits will be well balanced.
- $O(n \lg n)$ virtually guaranteed when n is large.
- Stops any bad input from causing worst-case behavior.

Worst-case analysis of randomized quicksort

$$T(n) = \max_{0 \le q \le n-1} (T(q) + T(n-q-1)) + \Theta(n)$$

- **Guess:** $T(n) \le cn^2$ for some c.

$$\begin{aligned} T(n) &\le \max_{0 \le q \le n-1} (cq^2 + c(n-q-1)^2) + \Theta(n) \\ &= c \cdot \max_{0 \le q \le n-1} (q^2 + (n-q-1)^2) + \Theta(n) \end{aligned}$$

- This is max when $q = 0$ or $q = n-1$ (parabolas).

$$\max_{0 \le q \le n-1} (q^2 + (n-q-1)^2) \le (n-1)^2 = n^2 - 2n + 1$$

$$\begin{aligned} T(n) &\le cn^2 - c(2n-1) + \Theta(n) \\ &\le cn^2 && \text{if } c(2n-1) \ge \Theta(n) \\ &= O(n^2) \end{aligned}$$

- Can also show $T(n) = \Omega(n^2)$, so $T(n) = \Theta(n^2)$.

Average-case analysis of randomized quicksort

- The dominant cost of the algorithm is in the calls to PARTITION.
- PARTITION removes the pivot from future consideration.
- PARTITION is called at most n times.
- Each call to PARTITION does a constant amount of work plus a constant times the number of comparisons done in the **for** loop.
- Let X be the total number of comparisons performed in all calls to PARTITION.
- Total work done is $O(n + X)$.
- We seek a bound on the total number of comparisons.

Average-case analysis of randomized quicksort

- Let the elements of A be $z_1, z_2, \ldots, z_n$ with z_i being the ith smallest.
- Define $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$
- Each pair of elements is compared at most once.
  - Compared only to the pivot, and then the pivot is removed.
- Let $X_{ij} = \mathrm{I}\{z_i \text{ is compared to } z_j\}$
- Since each pair is compared at most once,

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\} \end{aligned}$$

Probability z_i is compared to z_j

- Numbers in separate partitions will not be compared.
- If a pivot x is chosen such that $z_i < x < z_j$, z_i and z_j will not be compared.
- If either z_i or z_j is chosen before any other element of Z_{ij}, then it will be compared to every element of Z_{ij}, except itself.
- The probability that z_i is compared to z_j is the probability that either z_i or z_j is chosen first.
- There are $j - i + 1$ elements of Z_{ij}, and pivots are chosen randomly and independently.
- The probability that any one of them is chosen first is $1/(j - i + 1)$.

$$\begin{aligned}
\Pr\{z_i \text{ is compared to } z_j\} &= \Pr\{z_i \text{ or } z_j \text{ is chosen first from } Z_{ij}\} \\
&= \Pr\{z_i \text{ is chosen first}\} + \Pr\{z_j \text{ is chosen} \\
&= \frac{2}{j - i + 1}
\end{aligned}$$

Expected number of comparisons

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\} \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \\
&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\lg n) \\
&= O(n \lg n)
\end{aligned}
$$