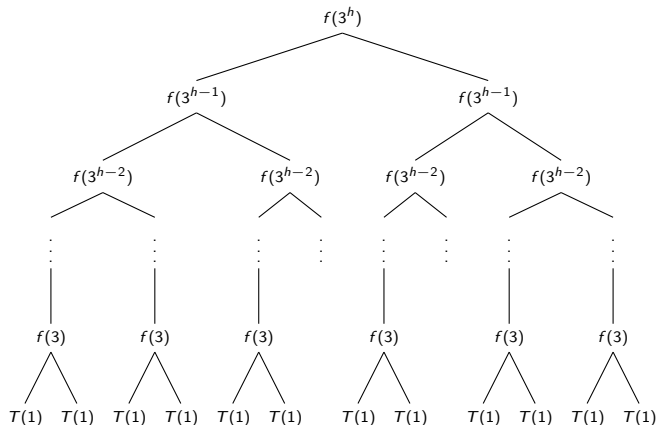


Master Theorem

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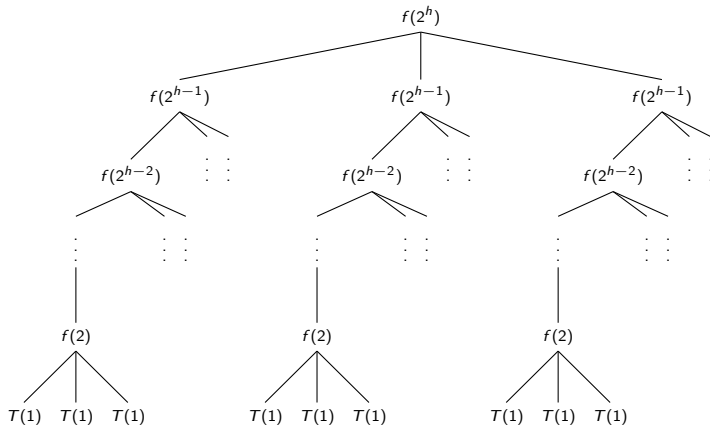
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Example 1: $T(n) = aT(n/b) + f(n)$ with $a = 2, b = 3$



$$h = \log_3 n \quad 2^{\log_3 n} = n^{\log_3 2} \approx n^{0.6} \text{ leaves} \quad \sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$$

Example 2: $T(n) = aT(n/b) + f(n)$ with $a = 3, b = 2$



$$h = \log_2 n \quad 3^{\log_2 n} = n^{\log_2 3} \approx n^{1.6} \text{ leaves} \quad \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)$$

Comparing leaves to internal nodes, case 1

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n$, $a = 2$, $b = 4$, $n^{\log_b a} = n^{1/2}$

$$\begin{aligned}\sum_{i=0}^{\log_4 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_4 n-1} 2^i (n/4^i) \\ &= n \sum_{i=0}^{\log_4 n-1} (1/2)^i \\ &= \Theta(n)\end{aligned}$$

$$n^{\log_b a} + \Theta(n) = \Theta(n)$$

Comparing leaves to internal nodes, case 2

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n$, $a = b = 4$, $n^{\log_b a} = n$

$$\begin{aligned}\sum_{i=0}^{\log_4 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_4 n-1} 4^i (n/4^i) \\ &= \sum_{i=0}^{\log_4 n-1} n \\ &= \Theta(n \lg n)\end{aligned}$$

$$n^{\log_b a} + \Theta(n \lg n) = \Theta(n \lg n)$$

Comparing leaves to internal nodes, case 3

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n$, $a = 4$, $b = 2$, $n^{\log_b a} = n^2$

$$\begin{aligned}\sum_{i=0}^{\log_2 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_2 n-1} 4^i (n/2^i) \\ &= n \sum_{i=0}^{\log_2 n-1} 2^i \\ &= \Theta(n^2)\end{aligned}$$

$$n^{\log_b a} + \Theta(n^2) = \Theta(n^2)$$

Comparing leaves to internal nodes, case 4

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n$, $a = 8$, $b = 2$, $n^{\log_b a} = n^3$

$$\begin{aligned}\sum_{i=0}^{\log_2 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_2 n-1} 8^i (n/2^i) \\ &= n \sum_{i=0}^{\log_2 n-1} 4^i \\ &= \Theta(n^3)\end{aligned}$$

$$n^{\log_b a} + \Theta(n^3) = \Theta(n^3)$$

Comparing leaves to internal nodes, case 5

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n^2$, $a = b = 4$, $n^{\log_b a} = n$

$$\begin{aligned}\sum_{i=0}^{\log_4 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_4 n-1} 4^i (n/4^i)^2 \\ &= \sum_{i=0}^{\log_4 n-1} \left(\frac{4}{4^2}\right)^i n^2 \\ &= n^2 \sum_{i=0}^{\log_4 n-1} (1/4)^i \\ &= \Theta(n^2)\end{aligned}$$

$$n^{\log_b a} + \Theta(n^2) = \Theta(n^2)$$

Comparing leaves to internal nodes, case 6

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n^2$, $a = 4$, $b = 2$, $n^{\log_b a} = n^2$

$$\begin{aligned}\sum_{i=0}^{\log_b n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_2 n-1} 4^i (n/2^i)^2 \\ &= \sum_{i=0}^{\log_2 n-1} \left(\frac{4}{2^2}\right)^i n^2 \\ &= n^2 \sum_{i=0}^{\log_2 n-1} (1)^i \\ &= \Theta(n^2 \lg n)\end{aligned}$$

$$n^{\log_b a} + \Theta(n^2 \lg n) = \Theta(n^2 \lg n)$$

Comparing leaves to internal nodes, case 7

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n^2$, $a = 8$, $b = 2$, $n^{\log_b a} = n^3$

$$\begin{aligned}\sum_{i=0}^{\log_4 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_b n-1} 8^i (n/2^i)^2 \\ &= \sum_{i=0}^{\log_b n-1} \left(\frac{8}{2^2}\right)^i n^2 \\ &= n^2 \sum_{i=0}^{\log_b n-1} 2^i \\ &= \Theta(n^3)\end{aligned}$$

$$n^{\log_b a} + \Theta(n^3) = \Theta(n^3)$$

Comparing leaves to internal nodes, case 8

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n^2$, $a = 8$, $b = 2$, $n^{\log_b a} = n^3$

$$\begin{aligned}\sum_{i=0}^{\log_4 n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_b n-1} 8^i (n/2^i)^2 \\ &= \sum_{i=0}^{\log_b n-1} \left(\frac{8}{2^2}\right)^i n^2 \\ &= n^2 \sum_{i=0}^{\log_b n-1} 2^i \\ &= \Theta(n^3)\end{aligned}$$

$$n^{\log_b a} + \Theta(n^3) = \Theta(n^3)$$

Comparing leaves to internal nodes, case 10

- ▶ Number of leaves: $n^{\log_b a}$
- ▶ Internal time: $\sum_{i=0}^{\log_b n-1} a^i f(n/b^i)$
- ▶ Suppose $f(n) = n^{\log_b a}$

$$\begin{aligned}\sum_{i=0}^{\log_b n-1} a^i f(n/b^i) &= \sum_{i=0}^{\log_b n-1} a^i (n/b^i)^{\log_b a} \\&= \sum_{i=0}^{\log_b n-1} \left(\frac{a}{b^{\log_b a}}\right)^i n^{\log_b a} \\&= \sum_{i=0}^{\log_b n-1} n^{\log_b a} \\&= n^{\log_b a} \log_b n \\&= \Theta(f(n) \lg n)\end{aligned}$$

Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on nonnegative integers by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Then $T(n)$ has the following asymptotic bounds:

$f(n)$	$T(n)$
$O(n^{\log_b a - \epsilon})$	$\Theta(n^{\log_b a})$
$\Theta(n^{\log_b a})$	$\Theta(f(n) \lg n)$
$\Omega(n^{\log_b a + \epsilon})$	$\Theta(f(n))$

The last one only if $af(\frac{n}{b}) \leq cf(n)$ for some $c < 1$ and large enough n .

Master theorem example 1

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$\log_b a =$$

Master theorem example 1

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$\log_b a = 1$$

$$f(n) =$$

Master theorem example 1

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$\log_b a = 1$$

$$f(n) = \Theta(n^{\log_b a})$$

$$T(n) =$$

Master theorem example 1

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$\log_b a = 1$$

$$f(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n \lg n)$$

Master theorem example 2

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^2$$

$$\log_b a =$$

Master theorem example 2

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^2$$

$$\log_b a = 2$$

$$f(n) =$$

Master theorem example 2

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^2$$

$$\log_b a = 2$$

$$f(n) = \Theta(n^{\log_b a})$$

$$T(n) =$$

Master theorem example 2

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^2$$

$$\log_b a = 2$$

$$f(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n^2 \lg n)$$

Master theorem example 3

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + \lg n$$

$$\log_b a =$$

Master theorem example 3

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + \lg n$$

$$\log_b a = 2$$

$$f(n) =$$

Master theorem example 3

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + \lg n$$

$$\log_b a = 2$$

$$f(n) = O(n^{\log_b a - \epsilon})$$

$$T(n) =$$

Master theorem example 3

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + \lg n$$

$$\log_b a = 2$$

$$f(n) = O(n^{\log_b a - \epsilon})$$

$$T(n) = \Theta(n^2)$$

Master theorem example 4

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^3$$

$$\log_b a =$$

Master theorem example 4

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^3$$

$$\log_b a = 2$$

$$f(n) =$$

Master theorem example 4

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^3$$

$$\log_b a = 2$$

$$f(n) = \Omega(n^{\log_b a + \epsilon})$$

$$T(n) =$$

Master theorem example 4

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^3$$

$$\log_b a = 2$$

$$f(n) = \Omega(n^{\log_b a + \epsilon})$$

$$T(n) = \Theta(n^3)$$

because

Master theorem example 4

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$O(n^{\log_b a - \epsilon}) \Leftrightarrow \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a}) \Leftrightarrow \Theta(f(n) \lg n)$$

$$\Omega(n^{\log_b a + \epsilon}) \Leftrightarrow \Theta(f(n)) \quad \text{when } af(n/b) \leq cf(n)$$

$$T(n) = 4T(n/2) + n^3$$

$$\log_b a = 2$$

$$f(n) = \Omega(n^{\log_b a + \epsilon})$$

$$T(n) = \Theta(n^3)$$

because

$$\begin{aligned} af(n/b) &= 4f(n/2) \\ &= 4/2^3 f(n) \leq cf(n) \end{aligned}$$

Master theorem does not apply

$$T(n) = 2T(n/2) + n \lg n$$

$$n^{\log_b a} = n$$

$$f(n) = n \lg n$$

$$f(n) = \Omega(n^{\log_b a}) = \Omega(n)$$

$$f(n) \neq \Omega(n^{\log_b a + \epsilon}) = \Omega(n^{1+\epsilon})$$

$$n \lg n \neq \Omega(n^{1.0000000000000001})$$

$$\lg n \neq \Omega(n^{0.0000000000000001})$$

Master theorem does apply

$$T(n) = 4T(n/3) + n \lg n$$

$$n^{\log_b a} = n^{\log_3 4} = n^{1.26\dots}$$

$$f(n) = n \lg n$$

$$f(n) = O(n^{\log_b a}) = O(n^{1.26\dots})$$

$$f(n) = O(n^{\log_b a - \epsilon}) = O(n^{1.2}) \quad \epsilon = 0.06\dots$$

$$n \lg n = O(n^{1.2})$$

$$\lg n = O(n^{0.2})$$

$$T(n) =$$

Master theorem does apply

$$T(n) = 4T(n/3) + n \lg n$$

$$n^{\log_b a} = n^{\log_3 4} = n^{1.26\dots}$$

$$f(n) = n \lg n$$

$$f(n) = O(n^{\log_b a}) = O(n^{1.26\dots})$$

$$f(n) = O(n^{\log_b a - \epsilon}) = O(n^{1.2}) \quad \epsilon = 0.06\dots$$

$$n \lg n = O(n^{1.2})$$

$$\lg n = O(n^{0.2})$$

$$T(n) = \Theta(n^{\log_3 4})$$