Mergesort Recurrence

$$
\begin{aligned}
T(1) &= c \\
f(n) &= 2T(n/2) + cn \qquad\qquad \text{(if } n > 1\text{)}
\end{aligned}
$$

Let's make this easier by assuming n is a power of two:

$$
\begin{aligned}
n &= 2^j \\
j &= \log_2(n)
\end{aligned}
$$

Now we can rewrite our equations:

$$
\begin{aligned}
T(2^0) &= c \\
f(2^j) &= 2T(2^{j-1}) + cn \qquad\qquad \text{(if } j > 0\text{)}
\end{aligned}
$$

Now we can introduce a new function, $f(j) = T(2^j)$, and we have

$$
\begin{aligned}
f(0) &= c \\
f(j) &= 2f(j-1) + c2^j
\end{aligned}
$$

Let's try to solve this recurrence.

Solve the recurrence

$$
\begin{aligned}
f(0) &= c \\
f(j) &= 2f(j-1) + c2^j
\end{aligned}
$$

Get lots of equations

$$
\begin{aligned}
f(j) &= 2f(j-1) + c2^j \\
2f(j-1) &= 2^2 f(j-2) + c2(2^{j-1}) \\
&= 2^2 f(j-2) + c2^j \\
2^2 f(j-2) &= 2^3 f(j-3) + c2^2(2^{j-2}) \\
&= 2^3 f(j-3) + c2^j \\
&\cdots \\
2^{k-1} f(j-(k-1)) &= 2^k f(j-k) + c2^j \\
&\cdots \\
2^{j-1} f(j-(j-1)) &= 2^j f(j-j) + c2^j \\
2^{j-1} f(j-1) &= c2^j + c2^j
\end{aligned}
$$

Simplify and cancel

$$
\begin{aligned}
f(0) &= c \\
f(j) &= 2f(j-1) + c2^j
\end{aligned}
$$

Cancelling:

$$
\begin{aligned}
f(j) &= \cancel{2f(j-1)} + c2^j \\
\cancel{2f(j-1)} &= \cancel{2^2 f(j-2)} + c2^j \\
\cancel{2^2 f(j-2)} &= \cancel{2^3 f(j-3)} + c2^j \\
&\cdots \\
\cancel{2^{k-1} f(j-(k-1))} &= \cancel{2^k f(j-k)} + c2^j \\
&\cdots \\
\cancel{2^{j-1} f(1)} &= c2^j + c2^j
\end{aligned}
$$

Gives

$$
\begin{aligned}
f(j) &= c2^j + \sum_{k=1}^{j} c2^j \\
&= (j+1)c2^j
\end{aligned}
$$

Remembering $f(j) = T(2^j)$

$$\begin{aligned} T(1) &= c \\ T(n) &= 2T(n/1) + cn \qquad\qquad (\text{if } n > 1) \end{aligned}$$

$$\begin{aligned} n &= 2^j \\ j &= \log_2(n) \end{aligned}$$

$$\begin{aligned} f(0) &= c \\ f(j) &= 2f(j-1) + c2^j \end{aligned}$$

$$f(j) = (j+1)c2^j$$

$$\begin{aligned} T(n) &= T(2^j) \\ &= f(j) \\ &= (j+1)c2^j \\ &= (\log_2(n)+1)c2^{\log_2(n)} \\ &= cn\log_2(n) + cn \\ &= \Theta(n\lg(n)) \end{aligned}$$