

Mergesort Recurrence

$$T(1) = c_1$$

$$T(n) = 2T(n/2) + cn \quad (\text{if } n > 1)$$

Get lots of equations:

$$T(n) = 2T(n/2) + cn$$

$$\begin{aligned} 2T(n/2) &= 2[2T(n/2^2) + cn/2] \\ &= 2^2T(n/2^2) + cn \end{aligned}$$

$$\begin{aligned} 2^2T(n/2^2) &= 2^2[2T(n/2^3) + cn/2^2] \\ &= 2^3T(n/2^3) + cn \end{aligned}$$

...

$$2^{k-1}T(n/2^{k-1}) = 2^kT(n/2^k) + cn$$

...

$$2^{\lg n - 1}T(n/2^{\lg n - 1}) = 2^{\lg n}T(n/2^{\lg n}) + cn$$

Mergesort Recurrence

$$T(1) = c_1$$

$$T(n) = 2T(n/2) + cn \quad (\text{if } n > 1)$$

Rewrite equations, sum and cancel:

$$\begin{aligned} T(n) &= \cancel{2T(n/2)} + cn \\ \cancel{2T(n/2)} &= \cancel{2^2T(n/2^2)} + cn \\ \cancel{2^2T(n/2^2)} &= \cancel{2^3T(n/2^3)} + cn \\ &\dots \\ \cancel{2^{k-1}T(n/2^{k-1})} &= \cancel{2^kT(n/2^k)} + cn \\ &\dots \\ \cancel{2^{\lg n - 1}T(n/2^{\lg n - 1})} &= 2^{\lg n}T(n/2^{\lg n}) + cn \end{aligned}$$

Which results in

$$\begin{aligned} T(n) &= 2^{\lg n}T(n/2^{\lg n}) + \sum_{k=1}^{\lg n} cn \\ &= c_1n + cn \lg n \\ &= \Theta(n \lg n) \end{aligned}$$

Mergesort Recurrence, changing function

$$T(1) = c$$

$$T(n) = 2T(n/2) + cn \quad (\text{if } n > 1)$$

Let's make this easier by assuming n is a power of two:

$$n = 2^j$$

$$j = \log_2(n)$$

Now we can rewrite our equations:

$$T(2^0) = c$$

$$T(2^j) = 2T(2^{j-1}) + cn \quad (\text{if } j > 0)$$

Now we can introduce a new function, $f(j) = T(2^j)$, and we have

$$f(0) = c$$

$$f(j) = 2f(j-1) + c2^j$$

Let's try to solve this recurrence.

Solve the recurrence

$$f(0) = c$$

$$f(j) = 2f(j-1) + c2^j$$

Get lots of equations

$$f(j) = 2f(j-1) + c2^j$$

$$2f(j-1) = 2^2f(j-2) + c2(2^{j-1})$$

$$= 2^2f(j-2) + c2^j$$

$$2^2f(j-2) = 2^3f(j-3) + c2^2(2^{j-2})$$

$$= 2^3f(j-3) + c2^j$$

...

$$2^{k-1}f(j-(k-1)) = 2^kf(j-k) + c2^j$$

...

$$2^{j-1}f(j-(j-1)) = 2^jf(j-j) + c2^j$$

$$2^{j-1}f(j-1) = c2^j + c2^j$$

Simplify and cancel

$$f(0) = c$$

$$f(j) = 2f(j-1) + c2^j$$

Cancelling:

$$f(j) = \cancel{2f(j-1)} + c2^j$$

$$\cancel{2f(j-1)} = \cancel{2^2 f(j-2)} + c2^j$$

$$\cancel{2^2 f(j-2)} = \cancel{2^3 f(j-3)} + c2^j$$

...

$$\cancel{2^{k-1} f(j-(k-1))} = \cancel{2^k f(j-k)} + c2^j$$

...

$$\cancel{2^{j-1} f(1)} = c2^j + c2^j$$

Gives

$$\begin{aligned} f(j) &= c2^j + \sum_{k=1}^j c2^j \\ &= (j+1)c2^j \end{aligned}$$

Remembering $f(j) = T(2^j)$

$$T(1) = c$$

$$T(n) = 2T(n/1) + cn \quad (\text{if } n > 1)$$

$$n = 2^j$$

$$j = \log_2(n)$$

$$f(0) = c$$

$$f(j) = 2f(j-1) + c2^j$$

$$f(j) = (j+1)c2^j$$

$$T(n) = T(2^j)$$

$$= f(j)$$

$$= (j+1)c2^j$$

$$= (\log_2(n) + 1)c2^{\log_2(n)}$$

$$= cn \log_2(n) + cn$$

$$= \Theta(n \lg(n))$$