Big O definition

$$O(g(n)) = \{f(n) : \quad \text{there exists positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}$$

When we say

$$f(n) = O(g(n))$$

we really mean

$$f(n) \in O(g(n))$$

For example

$$n^2 + 3n + 7 = O(n^2)$$

means

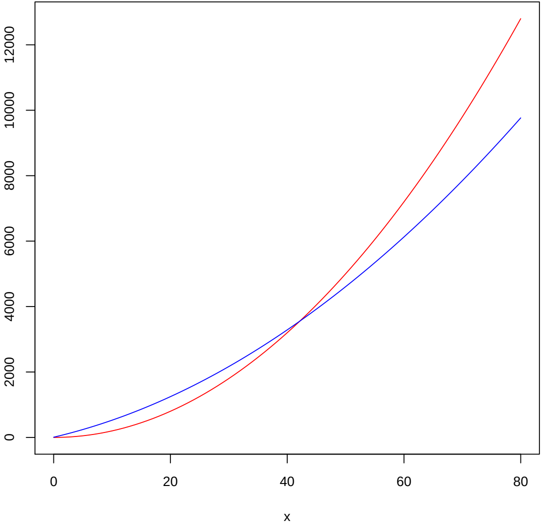
$$f(n) = n^2 + 3n + 7$$

is in the set

$$O(n^2)$$

$n^2 + 42n + 7 = O(n^2)$

$n^2 + 42n + 7 \leq 2n^2$ for all $n \geq 50$



Proof of $n^2 + 42n + 7 = O(n^2)$

$$
\begin{aligned}
n^2 + 42n + 7 &\leq n^2 + 42n^2 + 7n^2 && \text{for } n \geq 1 \\
&= 50n^2
\end{aligned}
$$

So $n^2 + 42n + 7 = O(n^2)$, with $c = 50$ and $n_0 = 1$

Proof of $4n^2 + 5n + 3 = O(n^2)$

$$
\begin{aligned}
4n^2 + 5n + 3 &\leq 4n^2 + 5n^2 + 3n^2 && n \geq 1 \\
&= 12n^2
\end{aligned}
$$

so $4n^2 + 5n + 3 = O(n^2)$ with $c = 12$ and $n_0 = 1$

Proof of $5n \lg n + 8n - 200 = O(n \lg n)$

Note: if $n \geq 2$ then $\lg n \geq 1$.

$$\begin{aligned} 5n \lg n + 8n - 200 &\leq 5n \lg n + 8n \\ &\leq 5n \lg n + 8n \lg n \qquad\qquad \text{for } n \geq 2 \\ &\leq 13n \lg n \end{aligned}$$

So

$$5n \lg n + 8n - 200 = O(n \lg n)$$

with $c = 13$ and $n_0 = 2$

Proof of $(n+5)\lg(3n^2+7) = O(n\lg n)$

$$\begin{aligned}
(n+5)\lg(3n^2+7) &\le (n+5n)\lg(3n^2+7n^2) && n \ge 1 \\
&= 6n\lg(10n^2) \\
&\le 6n\lg(n^3) && n \ge 10 \\
&= 6n(3\lg(n)) \\
&= 18n\lg(n)
\end{aligned}$$

So $(n+5)\lg(3n^2+7) = O(n\lg n)$ for $c = 18$ and $n_0 = 10$

Proof of $(n^2 + 5\lg n)/(2n+1) = O(n)$

$$\begin{aligned} \frac{n^2 + 5\lg n}{2n+1} &\le \frac{n^2 + 5n^2}{2n+1} & n \ge 1 \\ &\le \frac{n^2 + 5n^2}{2n} \\ &= 3n \end{aligned}$$

So $(n^2 + 5\lg n)/(2n+1) = O(n)$ for $c = 3$ and $n_0 = 1$

Useful facts

- For any $a < b$:
$$O(n^a) \subset O(n^b)$$
- For any $a, b > 0, c > 1$:
$$O(a) \subset O(\lg n) \subset O(n^b) \subset O(c^n)$$
You can multiply to find
$$O(an) = O(n) \subset O(n \lg n) \subset O(n^{b+1}) \subset O(nc^n)$$

Other sets

$$
\begin{aligned}
&O(g(n)) = \\
&\quad \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that} \\
&\qquad\qquad 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\} \\
&\Theta(g(n)) = \\
&\quad \{f(n) : \text{there exist positive constants } c, d \text{ and } n_0 \text{ such that} \\
&\qquad\qquad 0 \le cg(n) \le f(n) \le dg(n) \text{ for all } n \ge n_0\} \\
&\Omega(g(n)) = \\
&\quad \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that} \\
&\qquad\qquad 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}
\end{aligned}
$$

When limits exist

$$f(n) = O(g(n)) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) = \Theta(g(n)) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} \in (0, \infty)$$

$$f(n) = \Omega(g(n)) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$$

Note: $\sin(x) + 1 = \Theta(1)$ but $\lim_{x \to \infty} \sin(x)$ does not exist.

Other sets

$o(g(n)) =$

$\{f(n)$: for any positive constant c there exists positive n_0 such that
$0 \le f(n) < cg(n)$ for all $n \ge n_0\}$

$\omega(g(n)) =$

$\{f(n)$: for any positive constant c there exists positive n_0 such that
$0 \le cg(n) < f(n)$ for all $n \ge n_0\}$

For non-negative functions, these are equivalent to

$$f(n) = o(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) = \omega(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

Number of anonymous functions

The number of anonymous functions is equal to the number of times the asymptotic notation appears.

$$\sum_{i=1}^{n} O(i)$$

Here we assume there is only one function, not n different functions.

Asymptotic notation in equations

Right hand side:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

means

$$2n^2 + 3n + 1 = 2n^2 + f(n)$$

for *some* $f(n) \in \Theta(n)$.

Left hand side:

$$2n^2 + \Theta(n) = \Theta(n^2)$$

means for *all* $f(n) \in \Theta(n)$,

$$2n^2 + f(n) = \Theta(n^2)$$

We can chain them:

$$\begin{aligned} 2n^2 + 3n + 1 &= 2n^2 + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

Relational properties

Reflexive:

$$f(n) = \Theta(f(n))$$

Symmetric:

$$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$$

Transitive:

$$(f(n) = \Theta(g(n)) \wedge g(n) = \Theta(h(n))) \Rightarrow f(n) = \Theta(h(n))$$

Transpose symmetry:

$$f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$$

$$f(n) = o(g(n)) \;\Leftrightarrow\; g(n) = \omega(f(n))$$

Relations are not exhaustive

We may have two functions f and g such that

$$f(n) \neq o(g(n))$$
$$f(n) \neq O(g(n))$$
$$f(n) \neq \Theta(g(n))$$
$$f(n) \neq \Omega(g(n))$$
$$f(n) \neq \omega(g(n))$$

For example, $n^{1+\sin(n)}$ and n.

Limits

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \implies f(n) = o(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \implies f(n) = \omega(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) = O(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \implies f(n) = \Omega(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \in (0, \infty) \implies f(n) = \Theta(g(n))$$

L'Hôpital's Rule

When

$$\lim_{x \to \infty} f(x) = \infty$$

and

$$\lim_{x \to \infty} g(x) = \infty$$

then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

L'Hôpital example

$$f(n) = \lg n^2$$
$$g(n) = \lg n + 5$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{\lg n^2}{\lg n + 5} \\
&= \lim_{n\to\infty} \frac{(2 \lg e) \ln n}{(\lg e) \ln n + 5} \\
&= \lim_{n\to\infty} \frac{2 \lg e / n}{\lg e / n} \\
&= \lim_{n\to\infty} 2 = 2
\end{aligned}$$

Therefore

$$f(n) = \Theta(g(n))$$

L'Hôpital example

$$
\begin{aligned}
f(n) &= \lg n \\
g(n) &= n^c & \qquad\qquad c > 0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{\lg n}{n^c} \\
&= \lim_{n\to\infty} \frac{\lg e/n}{cn^{c-1}} \\
&= \lim_{n\to\infty} \frac{\lg e}{cn^c} \\
&= 0
\end{aligned}
$$

Therefore

$$f(n) = o(g(n))$$

L'Hôpital example

$$
\begin{aligned}
f(n) &= n^a \\
g(n) &= b^n && b > 1
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n^0}{b^n} &= 0 && a = 0 \\
\lim_{n\to\infty} \frac{n^a}{b^n} &= \lim_{n\to\infty} \frac{an^{a-1}}{(\ln b)b^n} && a > 0 \\
&= \left(\frac{a}{\ln b}\right) \lim_{n\to\infty} \frac{n^{a-1}}{b^n} \\
&= 0 && \text{by induction}
\end{aligned}
$$

Therefore

$$f(n) = o(g(n))$$