Mergesort Recurrence

$$
\begin{aligned}
T(1) &= c \\
T(n) &= 2T(n/2) + cn \qquad\qquad \text{(if } n > 1\text{)}
\end{aligned}
$$

Get lots of equations:

$$
\begin{aligned}
T(n) &= 2T(n/2) + cn \\
2T(n/2) &= 2[2T(n/2^2) + cn/2] \\
&= 2^2T(n/2^2) + cn \\
2^2T(n/2^2) &= 2^2[2T(n/2^3) + cn/2^2] \\
&= 2^3T(n/2^3) + cn \\
&\ldots \\
2^{k-1}T(n/2^{k-1}) &= 2^kT(n/2^k) + cn \\
&\ldots \\
2^{\lg n-1}T(n/2^{\lg n-1}) &= 2^{\lg n}T(n/2^{\lg n}) + cn \\
&= nT(1) + cn \\
&= cn + cn
\end{aligned}
$$

Mergesort Recurrence

$$
\begin{aligned}
T(1) &= c \\
T(n) &= 2T(n/2) + cn \qquad\qquad (\text{if } n > 1)
\end{aligned}
$$

Rewrite equations, sum and cancel:

$$
\begin{aligned}
T(n) &= \cancel{2T(n/2)} + cn \\
\cancel{2T(n/2)} &= \cancel{2^2T(n/2^2)} + cn \\
\cancel{2^2T(n/2^2)} &= \cancel{2^3T(n/2^3)} + cn \\
&\cdots \\
\cancel{2^{k-1}T(n/2^{k-1})} &= \cancel{2^kT(n/2^k)} + cn \\
&\cdots \\
\cancel{2^{\lg n-1}T(n/2^{\lg n-1})} &= 2^{\lg n}T(n/2^{\lg n}) + cn
\end{aligned}
$$

Which results in

$$
\begin{aligned}
T(n) &= 2^{\lg n}T(N/2^{\lg n}) + \sum_{k=1}^{\lg n} cn \\
&= cn + cn \lg n \\
&= \Theta(n \lg n)
\end{aligned}
$$

Mergesort Recurrence, changing function

$$T(1) = c$$
$$T(n) = 2T(n/2) + cn \qquad \text{(if } n > 1\text{)}$$

Let's make this easier by assuming n is a power of two:

$$n = 2^j$$
$$j = \log_2(n)$$

Now we can rewrite our equations:

$$T(2^0) = c$$
$$T(2^j) = 2T(2^{j-1}) + cn \qquad \text{(if } j > 0\text{)}$$

Now we can introduce a new function, $f(j) = T(2^j)$, and we have

$$f(0) = c$$
$$f(j) = 2f(j-1) + c2^j$$

Let's try to solve this recurrence.

Solve the recurrence

$$\begin{aligned} f(0) &= c \\ f(j) &= 2f(j-1) + c2^j \end{aligned}$$

Get lots of equations

$$\begin{aligned}
f(j) &= 2f(j-1) + c2^j \\
2f(j-1) &= 2^2 f(j-2) + c2(2^{j-1}) \\
&= 2^2 f(j-2) + c2^j \\
2^2 f(j-2) &= 2^3 f(j-3) + c2^2(2^{j-2}) \\
&= 2^3 f(j-3) + c2^j \\
&\cdots \\
2^{k-1} f(j-(k-1)) &= 2^k f(j-k) + c2^j \\
&\cdots \\
2^{j-1} f(j-(j-1)) &= 2^j f(j-j) + c2^j \\
2^{j-1} f(j-1) &= c2^j + c2^j
\end{aligned}$$

Simplify and cancel

$$
\begin{aligned}
f(0) &= c \\
f(j) &= 2f(j-1) + c2^j
\end{aligned}
$$

Cancelling:

$$
\begin{aligned}
f(j) &= \cancel{2f(j-1)} + c2^j \\
\cancel{2f(j-1)} &= \cancel{2^2 f(j-2)} + c2^j \\
\cancel{2^2 f(j-2)} &= \cancel{2^3 f(j-3)} + c2^j \\
&\dots \\
\cancel{2^{k-1} f(j-(k-1))} &= \cancel{2^k f(j-k)} + c2^j \\
&\dots \\
\cancel{2^{j-1} f(1)} &= c2^j + c2^j
\end{aligned}
$$

Gives

$$
\begin{aligned}
f(j) &= c2^j + \sum_{k=1}^{j} c2^j \\
&= (j+1)c2^j
\end{aligned}
$$

Remembering $f(j) = T(2^j)$

$$\begin{aligned} T(1) &= c \\ T(n) &= 2T(n/1) + cn & \text{(if } n > 1) \end{aligned}$$

$$\begin{aligned} n &= 2^j \\ j &= \log_2(n) \end{aligned}$$

$$\begin{aligned} f(0) &= c \\ f(j) &= 2f(j-1) + c2^j \end{aligned}$$

$$f(j) = (j+1)c2^j$$

$$\begin{aligned} T(n) &= T(2^j) \\ &= f(j) \\ &= (j+1)c2^j \\ &= (\log_2(n)+1)c2^{\log_2(n)} \\ &= cn\log_2(n) + cn \\ &= \Theta(n \lg(n)) \end{aligned}$$