# Summation Facts

$$\sum_{i=m}^{n} c = (n-m+1)c$$

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} +b_i$$

$$\sum_{i=m}^{n} a_{i+k} = \sum_{i=m+k}^{n+k} a_i$$

$$\sum_{i=m}^{n} a_i x^{i+k} = x^k \sum_{i=m}^{n} a_i x^i$$

$$\sum_{i=m}^{n}(a_i - a_{i-1}) = a_n - a_{m-1}$$

## Sum of 1

$$\begin{aligned}\sum_{i=1}^{n} 1 &= \sum_{i=1}^{n} (i - (i-1)) \\ &= n - 0 \\ &= n\end{aligned}$$

## Sum of $i$

$$
\begin{array}{rrcccccccc}
 & \sum_{i=1}^{n} i & = & 1 & + & 2 & + & \ldots & + & n \\
+ & \sum_{i=1}^{n} i & = & n & + & n-1 & + & \ldots & + & 1 \\
\hline
 & & = & n+1 & + & n+1 & + & \ldots & + & n+1 \\
 & & = & n(n+1) & & & & & &
\end{array}
$$

$$\Rightarrow \sum_{i=1}^{n} i = n(n+1)/2$$

## Sum of $i$

$$\begin{aligned}
n^2 &= n^2 - 0^2 \\
&= \sum_{i=1}^{n} (i^2 - (i-1)^2) \\
&= \sum_{i=1}^{n} (2i - 1) \\
&= 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 \\
&= 2\sum_{i=1}^{n} i - n
\end{aligned}$$

$$\Rightarrow \sum_{i=1}^{n} i = n(n+1)/2$$

## Sum of $i$

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\
&= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1 \\
\sum_{i=1}^{n+1} i^2 &= \sum_{i=0}^{n} (i+1)^2 \\
&= \sum_{i=0}^{n} (i^2 + 2i + 1) \\
&= \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 2i + \sum_{i=0}^{n} 1 \\
&= \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 2i + n + 1 \\
\sum_{i=1}^{n} i^2 + n^2 + 2n + 1 &= \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 2i + n + 1 \\
n^2 + 2n + 1 &= \sum_{i=0}^{n} 2i + n + 1 \\
\sum_{i=0}^{n} 2i &= n^2 + n \\
\sum_{i=0}^{n} i &= n(n+1)/2
\end{aligned}
$$

## Sum of Odd Numbers

$$
\begin{aligned}
1 + 3 + \ldots + (2n+1) &= \sum_{i=0}^{n} (2i+1) \\
&= \sum_{i=0}^{n} 2i + \sum_{i=0}^{n} 1 \\
&= 2 \sum_{i=0}^{n} i + (n+1) \\
&= 2n(n+1)/2 + (n+1) \\
&= (n+1)^2
\end{aligned}
$$

## Geometric Sum

$$\begin{aligned}
S_n &= 1 + a + a^2 + \ldots + a^n \\
&= \sum_{i=0}^{n} a^i \\
S_{n+1} &= (1 + a + a^2 + \ldots + a^n) + a^{n+1} \\
&= S_n + a^{n+1} \\
S_{n+1} &= 1 + (a + a^2 + \ldots + a^n + a^{n+1}) \\
&= 1 + a(1 + a + a^2 + \ldots + a^n) \\
&= 1 + aS_n \\
&\Rightarrow \\
S_n + a^{n+1} &= 1 + aS_n \\
&\Rightarrow \\
\sum_{i=0}^{n} a^i &= S_n \\
&= \frac{a^{n+1} - 1}{a - 1}
\end{aligned}$$

## Geometric Sum

$$
\begin{aligned}
(a-1)\sum_{i=0}^{n} a^i &= \sum_{i=0}^{n}(a^{i+1}-a^i) \\
&= \sum_{j=1}^{n+1}(a^j-a^{j-1}) \\
&= a^{n+1}-a^0 \\
&= a^{n+1}-1
\end{aligned}
$$

Also do division:

$$
\begin{array}{r|rrrrrrr}
 & & a^4 & +\,a^3 & +\,a^2 & +\,a & +\,1 \\
\hline
a-1 & a^5 & & & & & -\,1 \\
 & -\,a^5 & +\,a^4 & & & & \\
\cline{2-3}
 & & a^4 & & & & \\
 & & -\,a^4 & +\,a^3 & & & \\
\cline{3-4}
 & & & a^3 & & & \\
 & & & -\,a^3 & +\,a^2 & & \\
\cline{4-5}
 & & & & a^2 & & \\
 & & & & -\,a^2 & +\,a & \\
\cline{5-6}
 & & & & & a & -\,1 \\
 & & & & & -\,a & +\,1 \\
\cline{6-7}
 & & & & & & 0
\end{array}
$$

## Sum of $ia^i$

$$
\begin{aligned}
S_n = \sum_{i=1}^{n} ia^i &= 1 + a + 2a^2 + \ldots + na^n \\
S_{n+1} = \sum_{i=1}^{n+1} ia^i &= \sum_{i=1}^{n} ia^i + (n+1)a^{n+1} \\
&= S_n + (n+1)a^{n+1} \\
S_{n+1} = \sum_{i=1}^{n+1} ia^i &= \sum_{i=0}^{n} (i+1)a^{i+1} \\
&= \sum_{i=0}^{n} ia^{i+1} + \sum_{i=0}^{n} a^{i+1} \\
&= a\sum_{i=0}^{n} ia^i + a\sum_{i=0}^{n} a^i \\
&= aS_n + a(a^{n+1} - 1)/(a-1) \\
&\Rightarrow \\
S_n + (n+1)a^{n+1} &= aS_n + a(a^{n+1} - 1)/(a-1) \\
&\Rightarrow \\
S_n &= \frac{a - (n+1)a^{n+1} + na^{n+2}}{(a-1)^2}
\end{aligned}
$$

**Sum of $ia^{i-1}$**

Don't forget your calculus!

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\begin{aligned}\sum_{i=0}^{n} ia^{i-1} &= \sum_{i=0}^{n} \frac{d}{da}(a^i) \\ &= \frac{d}{da}\sum_{i=0}^{n}(a^i) \\ &= \frac{d}{da}\left(\frac{a^{n+1}-1}{a-1}\right)\end{aligned}$$

**Sum of $i^2$**

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
\sum_{i=1}^{n+1} i^3 &= \sum_{i=0}^{n} (i+1)^3 \\
&= \sum_{i=0}^{n} (i^3 + 3i^2 + 3i + 1) \\
&= \sum_{i=0}^{n} i^3 + \sum_{i=0}^{n} 3i^2 + \sum_{i=0}^{n} 3i + \sum_{i=0}^{n} 1 \\
&= \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + (n+1) \\
&= \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + (n+1)
\end{aligned}$$

$$\Rightarrow$$

$$(n+1)^3 = 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + (n+1)$$

$$\Rightarrow$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

## Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$\sum_{n=1}^{2^k} \frac{1}{n} \geq 1 + \frac{k}{2} \qquad k > 0$$

$$\sum_{n=1}^{k} \frac{1}{n} < \ln(k) + 1$$

## Change of variables

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{1}{n-i+1} &= \sum_{i=1}^{i=n} \frac{1}{n-i+1} \qquad\qquad \text{let } i = n-j \\
&= \sum_{n-j=1}^{n-j=n} \frac{1}{n-(n-j)+1} \\
&= \sum_{j=n-1}^{j=0} \frac{1}{j+1} \\
&= \sum_{j=0}^{n-1} \frac{1}{j+1} \\
&= \sum_{j=1}^{n} \frac{1}{j} \\
&= \sum_{i=1}^{n} \frac{1}{i}
\end{aligned}
$$

Which is the Harmonic series.

# Consider using Sympy

```
>>> from sympy import summation, oo, symbols, latex, log

>>> i,n,m = symbols('i n m', integer=True)

>>> summation(2*i + 1, (i, 0, n-1))
n**2

>>> summation(1/2**i, (i,0,oo))
2

>>> summation(i, (i, 0, n), (n, 0, m))
m**3/6 + m**2/2 + m/3

>>> print(latex(summation(i, (i,0,n), (n,0,m))))
\frac{m^{3}}{6} + \frac{m^{2}}{2} + \frac{m}{3}

>>> print(latex(summation(1/log(n)**n, (n, 2, oo))))
\sum_{n=2}^{\infty} \log^{- n}{\left (n \right )}
```