## Big O definition

$$O(g(n)) = \{f(n) : \text{ there exists positive constants } c \text{ and } n_0 \\ \text{such that } f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

When we say

$$f(n) = O(g(n))$$

we really mean

$$f(n) \in O(g(n))$$

For example

$$n^2 + 3n + 7 = O(n^2)$$

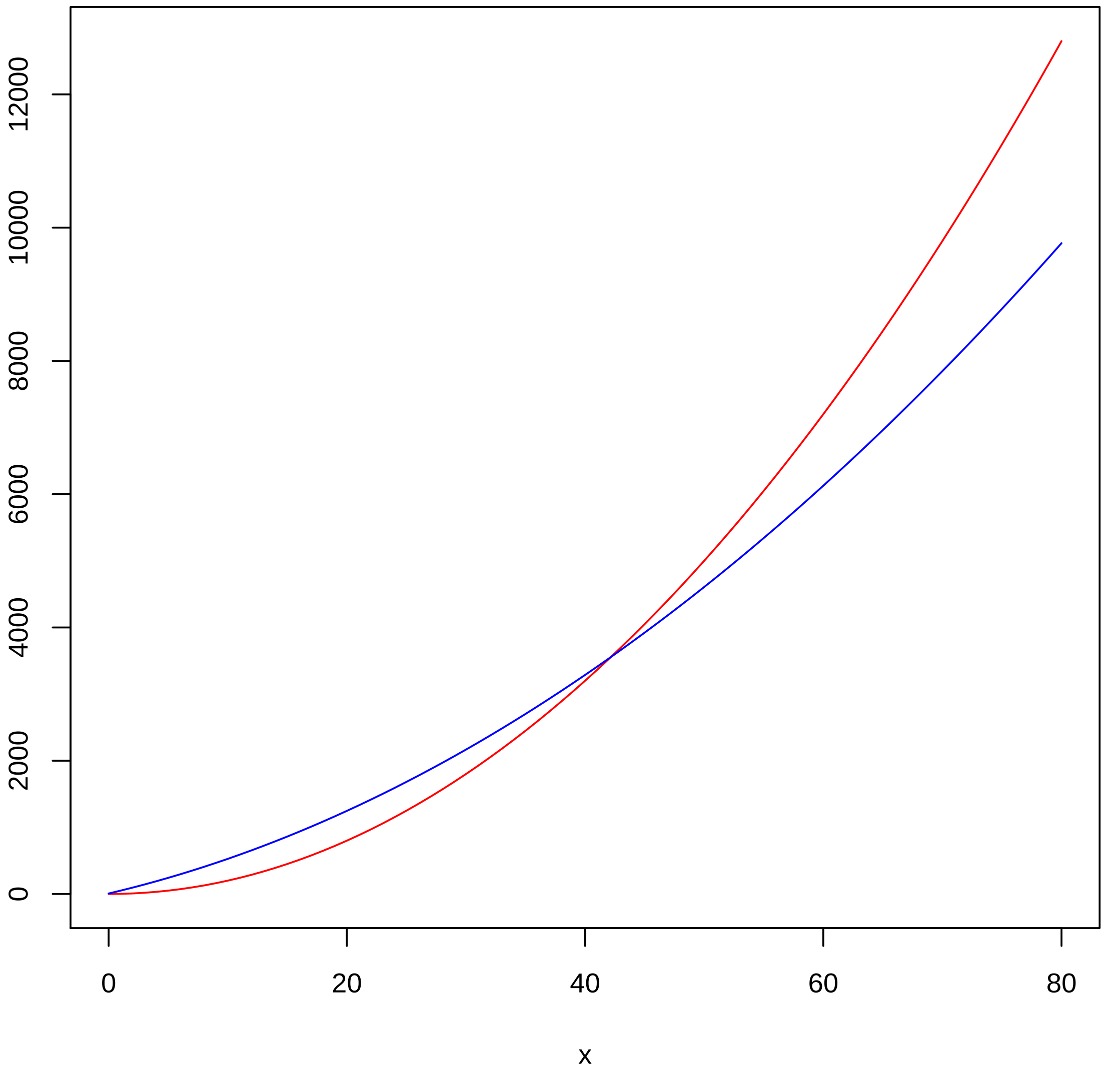
means

$$f(n) = n^2 + 3n + 7$$

is in the set

$$O(n^2)$$

$$n^2 + 42n + 7 = O(n^2)$$
$$n^2 + 42n + 7 \le 2n^2 \text{ for all } n \ge 50$$



**Proof of** $n^2 + 42n + 7 = O(n^2)$

$$\begin{aligned} n^2 + 42n + 7 &\leq n^2 + 42n^2 + 7n^2 \qquad \text{for } n \geq 1 \\ &= 50n^2 \end{aligned}$$

So $n^2 + 42n + 7 = O(n^2)$, with $c = 50$ and $n_0 = 1$

**Proof of** $4n^2 + 5n + 3 = O(n^2)$

$$\begin{aligned} 4n^2 + 5n + 3 &\le 4n^2 + 5n^2 + 3n^2 \qquad\qquad n \ge 1 \\ &= 12n^2 \end{aligned}$$

so $4n^2 + 5n + 3 = O(n^2)$ with $c = 12$ and $n_0 = 1$

**Proof of** $5n \lg n + 8n - 200 = O(n \lg n)$
Note: if $n \geq 2$ then $\lg n \geq 1$.

$$\begin{aligned} 5n \lg n + 8n - 200 &\leq 5n \lg n + 8n \\ &\leq 5n \lg n + 8n \lg n \qquad \text{for } n \geq 2 \\ &\leq 13n \lg n \end{aligned}$$

So

$$5n \lg n + 8n - 200 = O(n \lg n)$$

with $c = 13$ and $n_0 = 2$

**Proof of** $(n+5)\lg(3n^2+7) = O(n\lg n)$

$$\begin{aligned}
(n+5)\lg(3n^2+7) &\le (n+5n)\lg(3n^2+7n^2) && n \ge 1 \\
&= 6n\lg(10n^2) \\
&\le 6n\lg(n^3) && n \ge 10 \\
&= 6n(3\lg(n)) \\
&= 18n\lg(n)
\end{aligned}$$

So $(n+5)\lg(3n^2+7) = O(n\lg n)$ for $c = 18$ and $n_0 = 10$

**Proof of** $(n^2 + 5\lg n)/(2n+1) = O(n)$

$$\begin{aligned}
\frac{n^2 + 5\lg n}{2n+1} &\le \frac{n^2 + 5n^2}{2n+1} && n \ge 1 \\
&\le \frac{n^2 + 5n^2}{2n} \\
&= 3n
\end{aligned}$$

So $(n^2 + 5\lg n)/(2n+1) = O(n)$ for $c = 3$ and $n_0 = 1$

## Useful facts

- For any $a < b$:
$$O(n^a) \subset O(n^b)$$
- For any $a, b > 0, c > 1$:
$$O(a) \subset O(\lg n) \subset O(n^b) \subset O(c^n)$$
You can multiply to find
$$O(an) = O(n) \subset O(n \lg n) \subset O(n^{b+1}) \subset O(nc^n)$$

## Asymptotic notation in equations

**Right hand side:**

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

means

$$2n^2 + 3n + 1 = 2n^2 + f(n)$$

for *some* $f(n) \in \Theta(n)$.

**Left hand side:**

$$2n^2 + \Theta(n) = \Theta(n^2)$$

means for *all* $f(n) \in \Theta(n)$,

$$2n^2 + f(n) = \Theta(n^2)$$

**We can chain them:**

$$\begin{aligned} 2n^2 + 3n + 1 &= 2n^2 + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

## Relational properties

**Reflexive:**

$$f(n) = \Theta(f(n))$$

**Symmetric:**

$$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$$

**Transitive:**

$$(f(n) = \Theta(g(n)) \wedge g(n) = \Theta(h(n))) \Rightarrow f(n) = \Theta(h(n))$$

**Transpose symmetry:**

$$f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$$

$$f(n) = o(g(n)) \;\Leftrightarrow\; g(n) = \omega(f(n))$$

Note: We may have two functions $f$ and $g$ such that

$$\begin{gathered} f(n) \neq O(g(n)) \\ f(n) \neq \Theta(g(n)) \\ f(n) \neq \Omega(g(n)) \end{gathered}$$