Notes on Binary Search Trees

Geoffrey Matthews

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Search Trees

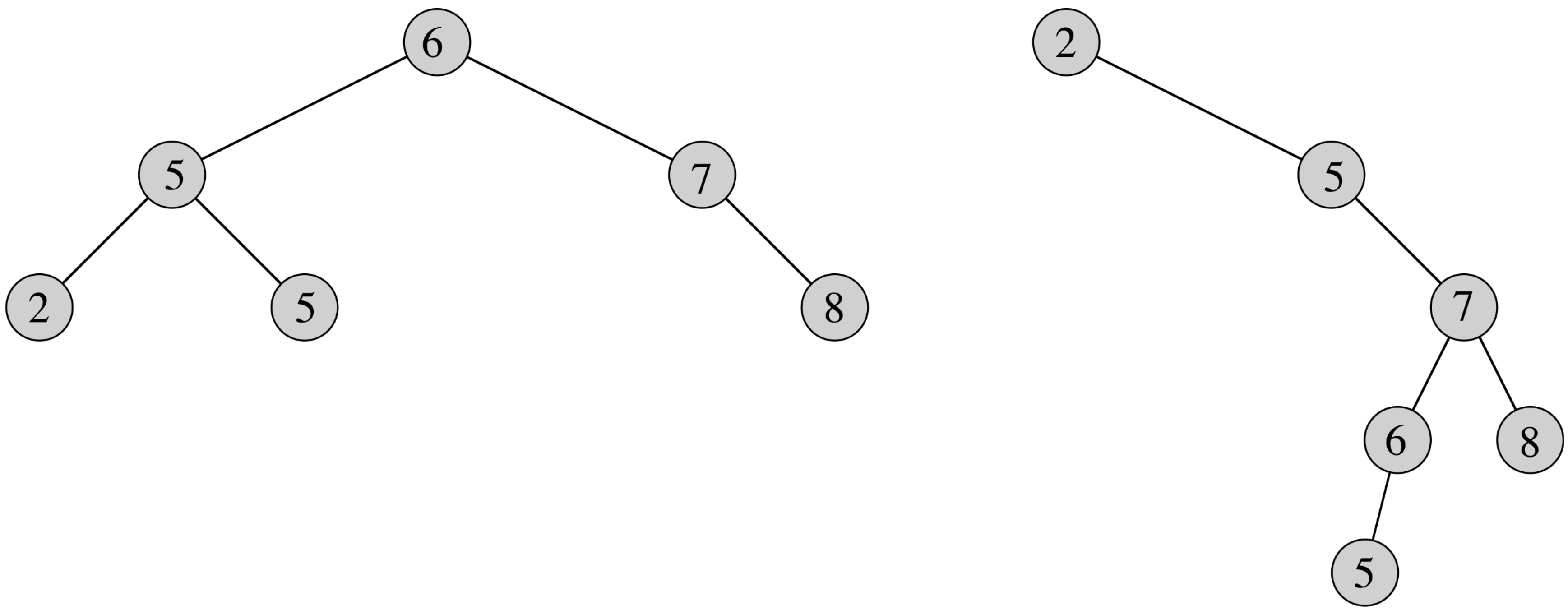
- Data structures that support many dynamic-set operations.
- Dictionaries and priority queues.
- Basic operations take time proportional to height of the tree.
  - Best case: $\Theta(\lg n)$
  - Worst case: $\Theta(n)$
- Different types of search trees:
  - binary search trees
  - red-black trees
  - B-trees

Binary search trees

- Many dynamic-set operations in $O(h)$ time, where $h =$ height of tree.
- We represent a binary tree by a linked data structure where each node is an object.
- $T.root$ points to the root of the tree T.
- Each node contains the attributes:
  - *key* (and possibly other satellite data).
  - *left*: points to left child.
  - *right*: points to right child.
  - *p:* points to parent. $T.root.p =$ NIL

Binary search tree property

- If y is in the left subtree of x, then $y.key \leq x.key$
- If y is in the right subtree of x, then $y.key \geq x.key$

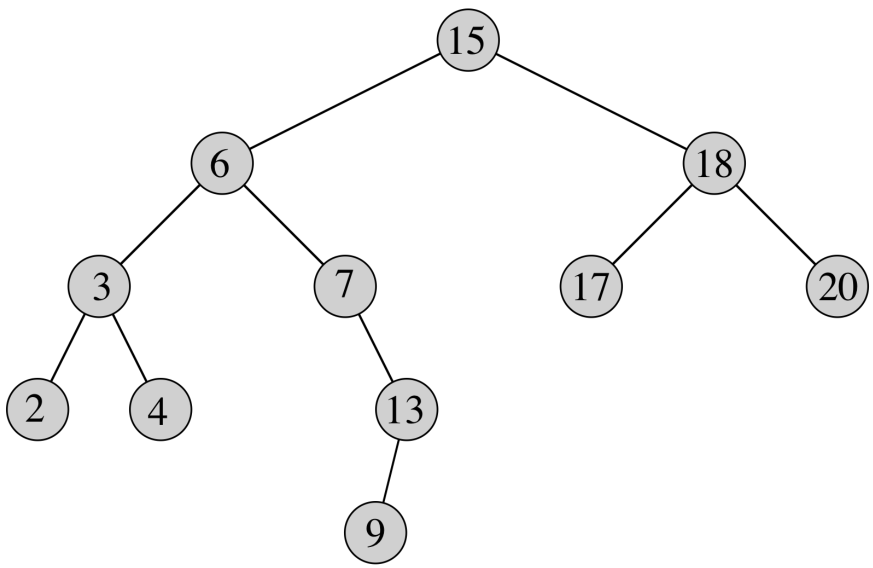


(a) (b)

- Frequently we assume keys are unique.

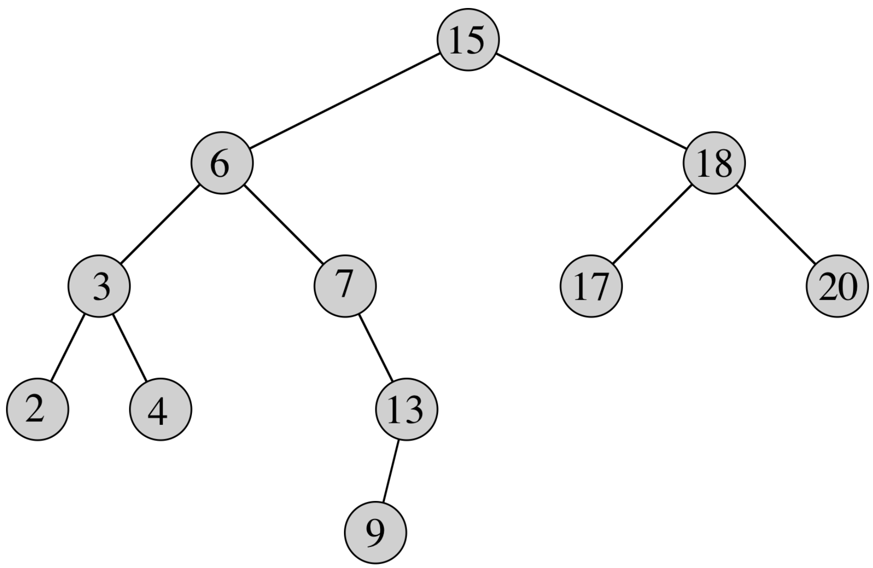
INORDER-TREE-WALK(x)

if $x \neq=$ NIL
    INORDER-TREE-WALK($x.left$)
    print $x.key$
    INORDER-TREE-WALK($x.right$)



- Correctness follows from binary search tree property.
- Time: $\Theta(n)$, because we visit and print each node once.
  - Formal proof in book.

```
TREE-SEARCH(x, k)
if x == NIL or k == x.key
    return x
if x < x.key
    return TREE-SEARCH(x.left, k)
else return TREE-SEARCH(x.right, k)
```



- The algorithm has a single recursion on a downward path from the root.
- Time: $O(h)$ where h is the height of the tree.

Iterative version

```
TREE-SEARCH(x, k)
if x == NIL or k == x.key
    return x
if x < x.key
    return TREE-SEARCH(x.left, k)
else return TREE-SEARCH(x.right, k)
```

```
ITERATIVE-TREE-SEARCH(x, k)
while x ≠= NIL and k ≠= x.key
    if x < x.key
        x = x.left
    else x = x.right
return x
```

- Tail recursion is easy to eliminate.

Minimum and maximum

```
TREE-MINIMUM(x)
while x.left ≠= NIL
        x = x.left
return x
```

```
TREE-MINIMUM-REC(x)
if x.left == NIL
        return x
return TREE-MINIMUM-REC(x.left)
```

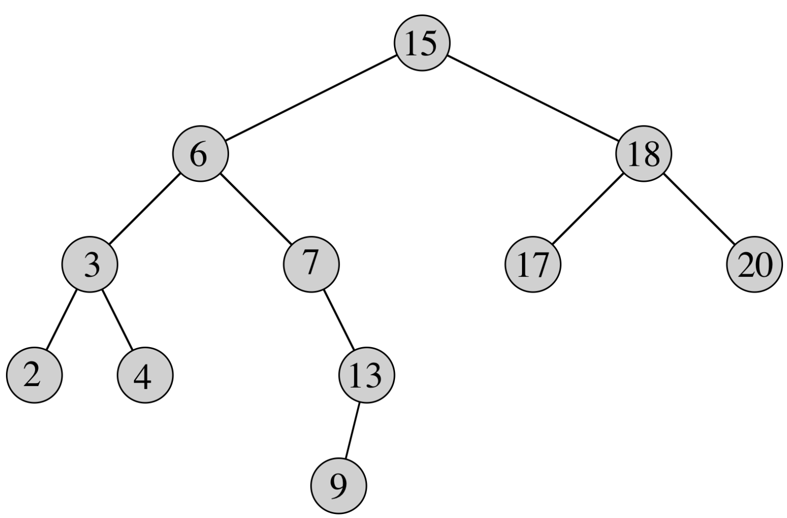
```
TREE-MAXIMUM(x)
while x.right ≠= NIL
        x = x.right
return x
```

```
TREE-MAXIMUM-REC(x)
if x.right == NIL
        return x
return TREE-MINIMUM-REC(x.right)
```

- Both procedures trace a path from root to leaf.
- $O(h)$

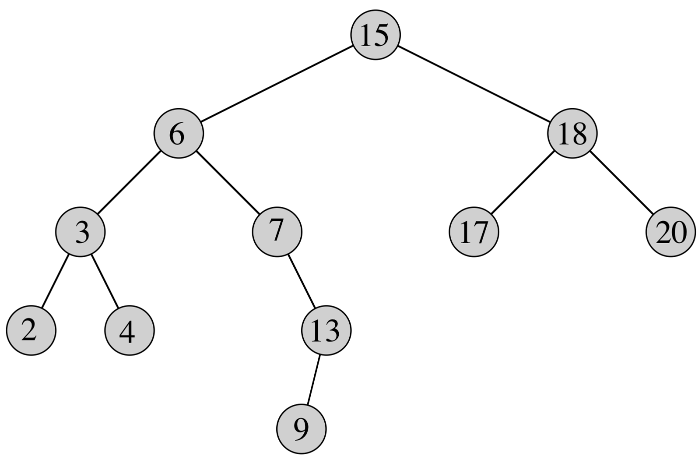
Successor and predecessor

- Assume all keys are distinct.
- The successor of a node x is the node y such that
  - $y.key$ is the smallest key $> x.key$.
- We can find successor without looking at keys.
- If x has the largest key, its successor is NIL.
- Two cases:
  1. If node x has a non-empty right subtree, return its minimum.
  2. Otherwise, move up the tree until the first right turn.



TREE-SUCCESSOR(x)

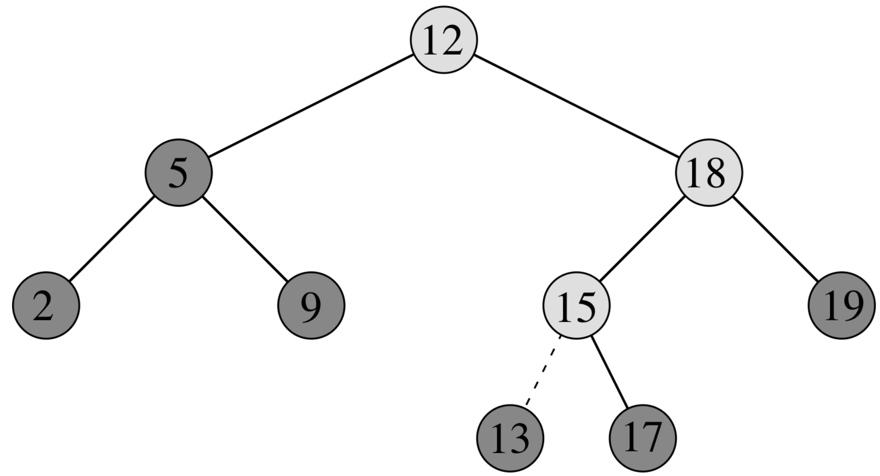
```
if x.right ≠= NIL
    return TREE-MINIMUM(x.right)
y = x.p
while y ≠= NIL and x == y.right
    x = y
    y = y.p
return y
```



- Can also move up until parent key $\geq$ child key, but that uses keys.
- TREE-PREDECESSOR similar. Both are $O(h)$.

TREE-INSERT(T, z)

```
y = NIL
x = T.root
while x ≠= NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == NIL
    T.root = z
elseif z.key < y.key
    y.left = z
else y.right = z
```



- $O(h)$
- TREE-INSERT can be used with INORDER-TREE-WALK to sort.

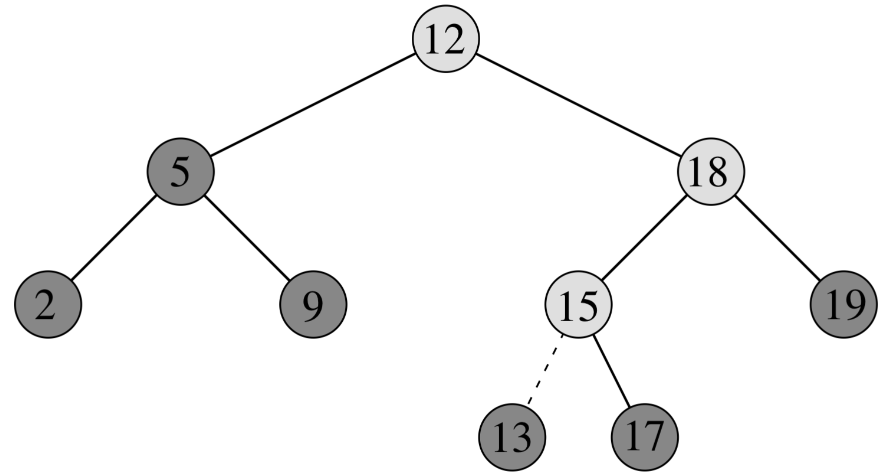
Recursive tree insert

TREE-INSERT-REC(T, z)

```
T.root = NODE-INSERT(T.root, z)
```

NODE-INSERT(x, z)

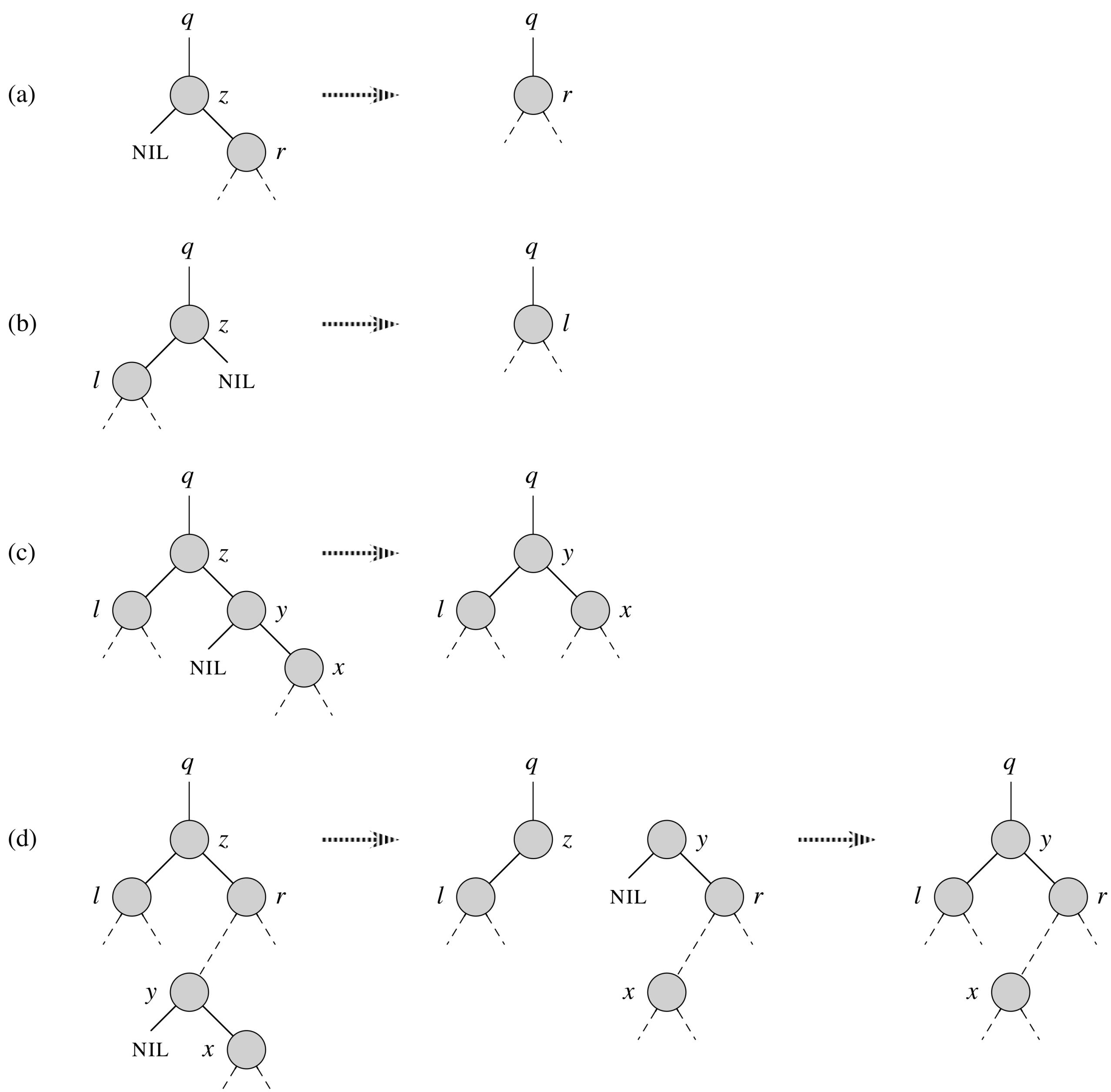
```
if x == NIL
        return z
z.p = x
if z.key < x.key
        x.left = NODE-INSERT(x.left, z)
else
        x.right = NODE-INSERT(x.right, z)
return x
```



Deletion

To delete node z from tree T:

1. If z has no children, just remove it.
2. If z has just one child, then make that child take z's position in the tree, dragging the child's subtrees along.
3. If z has two children, then
   - Find z's successor y.
   - y must be in z's right subtree and have no left child.
   - $y.key$ must be the smallest key in z's right subtree.
   - y can therefore replace z at z's position in the tree.
   - Deleting y's node from the tree is easy because it has only one child.
   - z's right subtree (now without y) becomes y's right subtree.
   - z's left child becomes y's left child.
     - This case is tricky when y is z's right child.


(a)
q
z
NIL
r
q
r
(b)
q
z
l
NIL
q
l
(c)
q
z
l
y
NIL
x
q
y
l
x
(d)
q
z
l
r
y
NIL
x
q
z
l
y
NIL
r
x
q
y
l
r
x

Transplant

- $\textsc{Transplant}(T, u, v)$ replaces the subtree rooted at u with the subtree rooted at v.

$\textsc{Transplant}(T, u, v)$

if $u.p == \text{NIL}$
    $T.root = v$
elseif $u == u.p.left$
    $u.p.left = v$
else $u.p.right = v$
if $v \neq= \text{NIL}$
    $v.p = u.p$

```
TREE-DELETE(T, z)
if z.left == NIL
      TRANSPLANT(T, z, z.right)
elseif z.right == NIL
      TRANSPLANT(T, z, z.left)
else
      y = TREE-MINIMUM(z.right)
      if y.p ≠ z
            TRANSPLANT(T, y, y.right)
            y.right = z.right
            y.right.p = y
      TRANSPLANT(T, z, y)
      y.left = z.left
      y.left.p = y
```

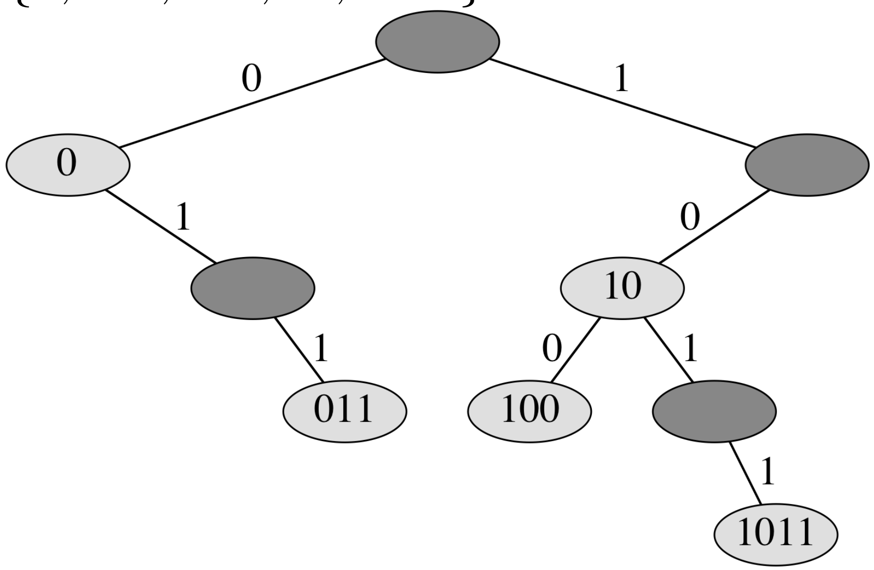
- $O(h)$

Theorem 12.4
The expected height of a randomly built binary search tree on n distinct keys is $O(\lg n)$.

- Red-black trees and B-trees actively maintain a $O(\lg n)$ height in worst case.

Problem 12-2, Radix trees

$\{0, 011, 100, 10, 1011\}$



- $a = a_0 a_1 \ldots a_p$ is **lexicolgraphically less than** $b = b_0 b_1 \ldots b_q$:
  1. there exists and integer j, where $0 \leq j \leq \min(p, q)$, such that $a_i = b_i$ for all $i = 0, 1, \ldots, j-1$ and $a_j < b_j$, or
  2. $p < q$ and $a_i = b_i$ for all $i = 0, 1, \ldots, p$.
- A set S of bit strings can be sorted lexicographically in $\Theta(n)$ time, where n is the sum of the lengths of the strings in S.