Notes on Hash Tables

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Dictionary Operations

- Insert
- Search
- Delete

Hash table implementation of Dictionary

- Expected search time: $O(1)$
- Worst case search: $O(n)$

Hash table is generalization of an ordinary array

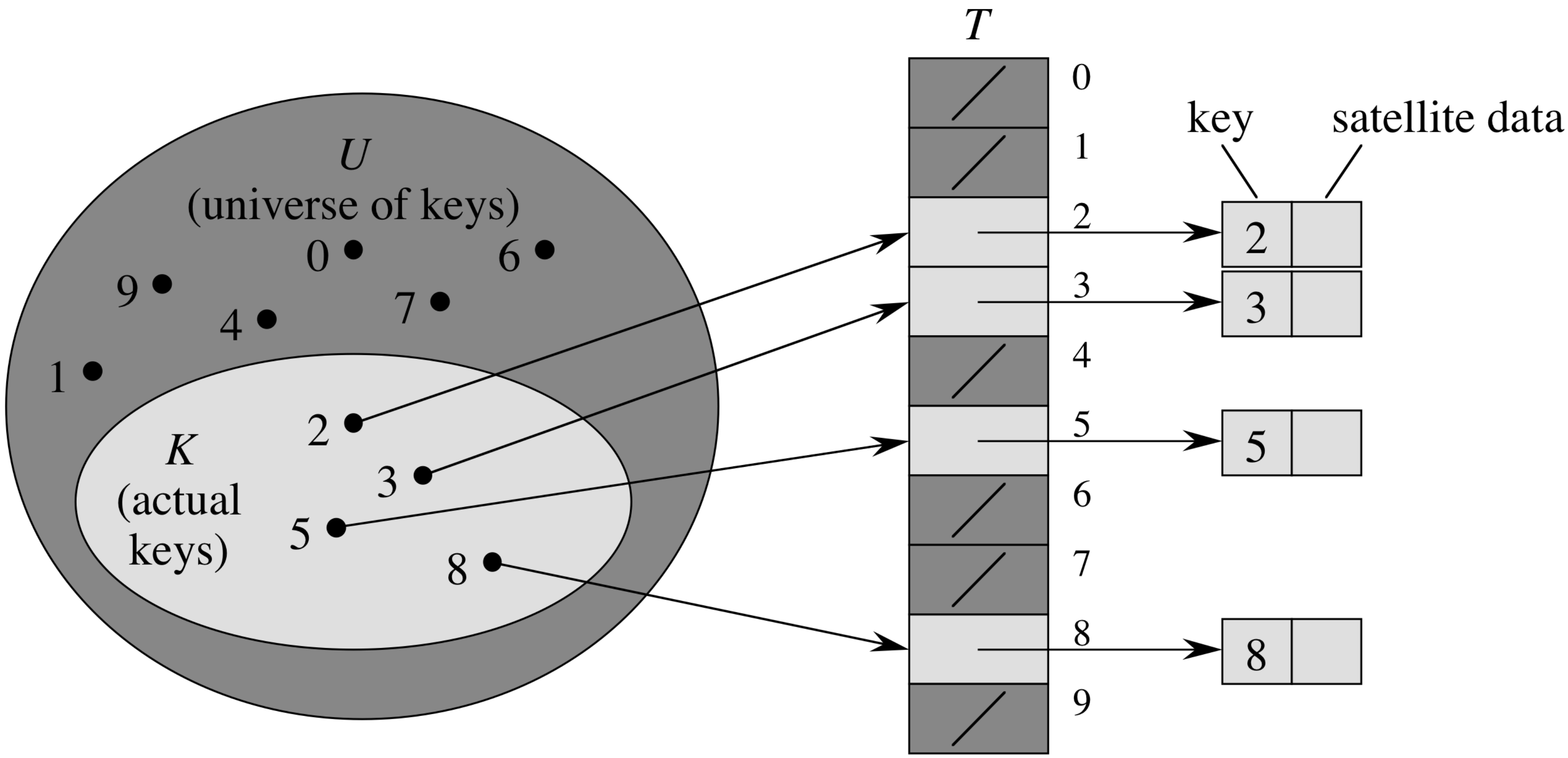
- With array, the key k is the position k in the array.
- Given a key k, we find the element with key k by **direct addressing**.
- Direct addressing only applicable when we can afford to allocate an array with one position for every key.

Use hash table when we don't have one position for each key

- Number of keys stored is small relative to the number of possible keys.
- Hash table is an array with size proportional to the number of keys stored, not the number of possible keys.
- Given a key k, don't use k to index the array.
- Instead, compute a function of k and use that to index the array.
- This function is called a **hash function**.
- Have to solve issue of what to do when hash function maps multiple keys to same table entry.
  - chaining
  - open addressing
- We will not discuss open addressing.

Direct-address tables

- Scenario:
  - Maintain a dynamic set
  - Each element has a key drawn from a universe $U = \{0, 1, \ldots, m-1\}$ where m isn't too large.
  - No two elements have the same key.
- Represent by a **direct-address table**, or array, $T[0 \ldots m-1]$:
  - Each *slot*, or position, corresponds to a key in U.
  - If there's an element x with key k, then $T[k]$ contains a pointer to x.
  - Otherwise, $T[k]$ is empty, represented by NIL.



Direct-Address-Search(T,k)

1 **return** $T[k]$

Direct-Address-Insert(T,k)

1 $T[key[x]] = x$

Direct-Address-Delete(T,k)

1 $T[key[x]] = \text{NIL}$

All operations $O(1)$.

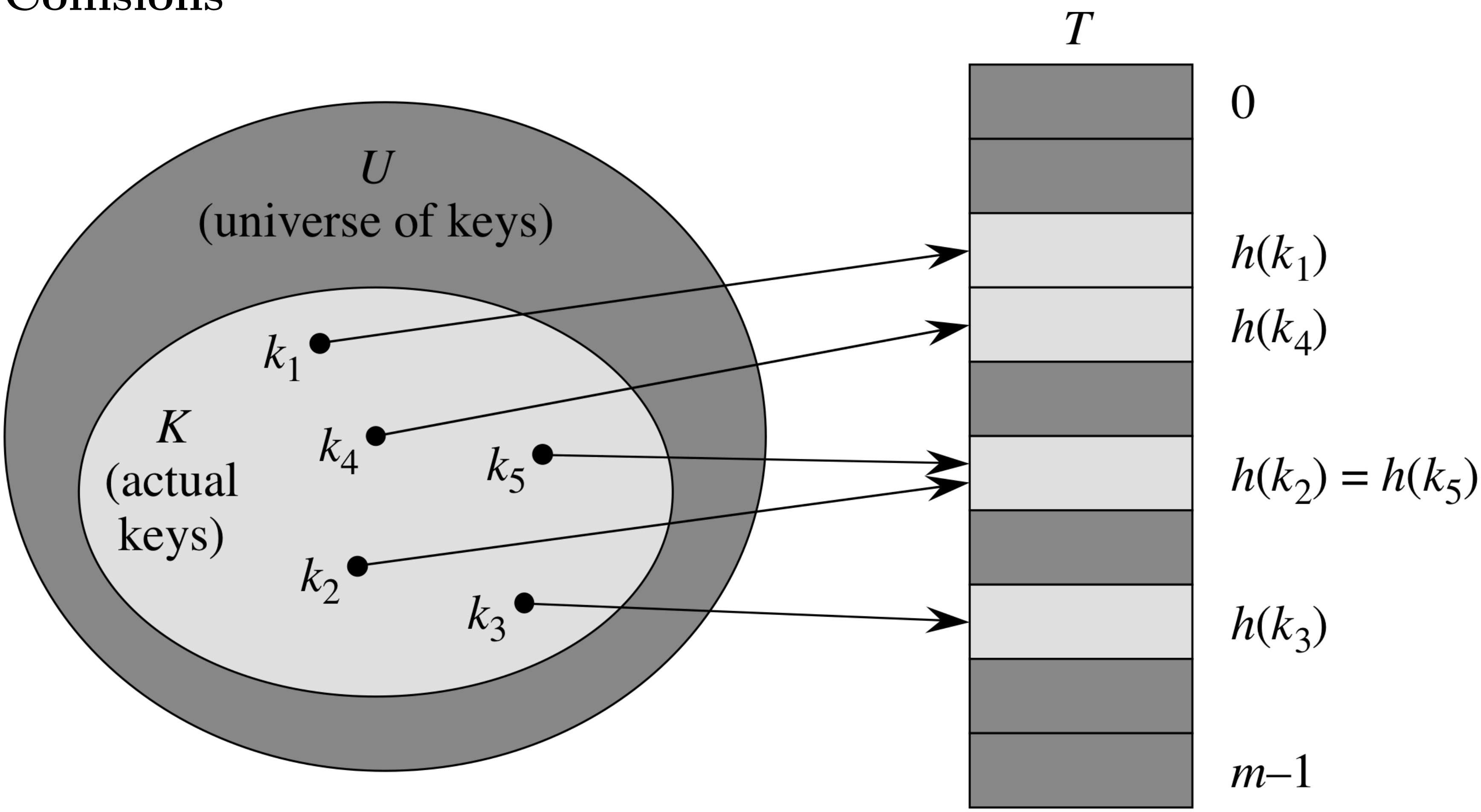
Hash tables

- If U is large, storing a table of size $|U|$ is impractical.
- Often the set K of keys actually used is small compared to U.
  - Most of the space in a direct-access table is wasted.
- When K is much smaller than U, a hash table requires much less space than a direct-address table.
- Can reduce storage requirements to $\Theta(|K|)$
- Can still get $O(1)$ search time on *average*, but not *worst* case.

Hash table idea

- Instead of storing an element with key k in slot k, use a function h and store the element in slot $h(k)$.
- h is called a **hash function**
- $h : U \to \{0, 1, \ldots, m-1\}$
- $m \ll |U|$
- $h(k)$ is a legal slot number in T
- We say k *hashes* to $h(k)$

Collisions



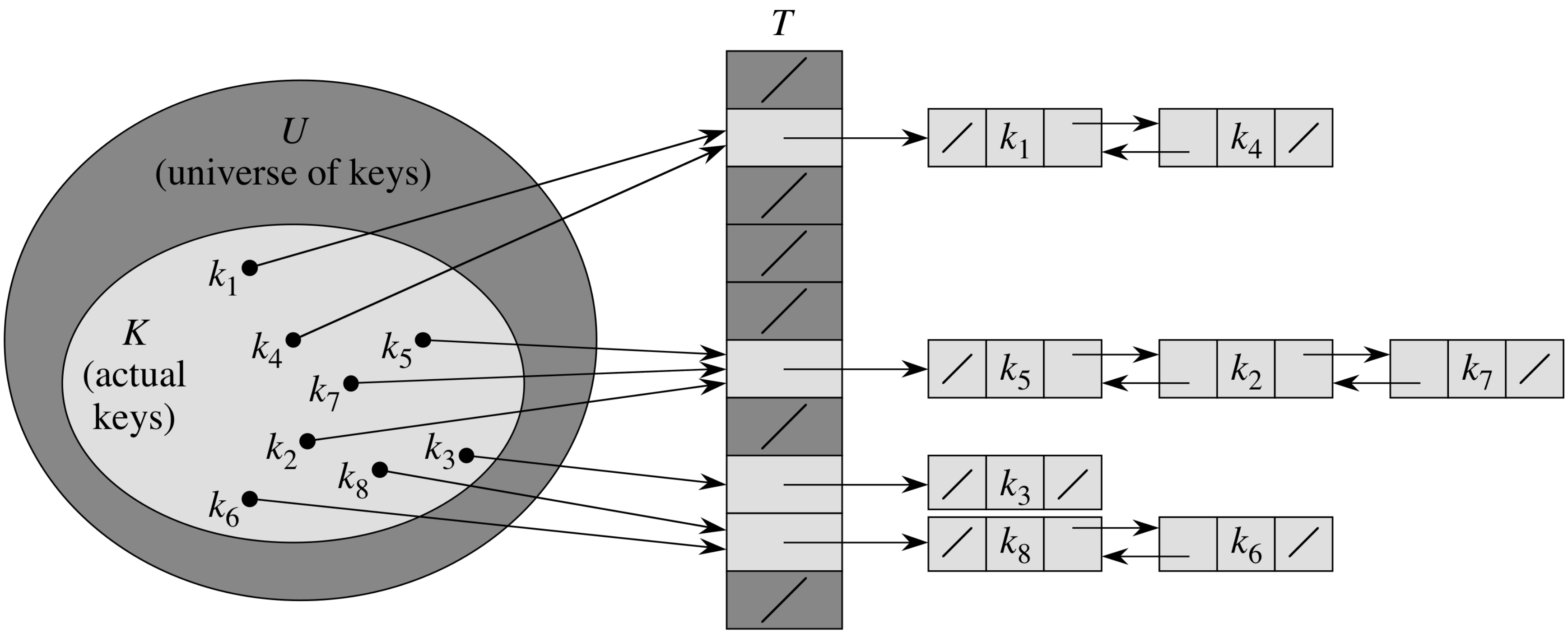
T
0
U
(universe of keys)
$h(k_1)$
$h(k_4)$
k_1
K
(actual
keys)
k_4
k_5
$h(k_2) = h(k_5)$
k_2
k_3
$h(k_3)$
m–1

Collisions

- When two or more keys hash to the same slot.
- Can happen when there are more possible keys than slots ($|U| > m$).
- For a given set K of keys with $|K| \leq m$, may or may not happen.
- Definitely happens when $|K| > m$.
- Must be prepared to handle collisions in all cases.
- Two methods:
  - chaining
  - open addressing
- Chaining is usually better.
- We will not discuss open addressing.

Collision resolution by chaining

- Put all elements that hash to the same slot into a linked list.



- Doubly linked list allows easy deletion.

Implementation of heap with chaining

Insertion:

Chained-Hash-Insert(T, x)

insert x at the head of list $T[h(key[x])]$

- Worst case $O(1)$
- Assumes element inserted not already in list.
- Would take an additional search to see if it was already inserted.

Implementation of heap with chaining

Search:

Chained-Hash-Search(T, k)

1 search for element with key k in list $T[h(k)]$

- Running time proportional to length of list in slot $h(k)$

Implementation of heap with chaining

Deletion:

```
Chained-Hash-Delete(T, x)
    delete x from the list T[h(key[x])]
```

- Given pointer x to the element to delete, so no search is needed to find this element.
- Worst case $O(1)$ if lists are doubly linked.
- If lists are singly linked, deletion takes as long as search, because we must find x's predecessor.

Analysis of hashing with chaining

- Given a key, how long does it take to find an element with that key, or determine that there is no element with that key?
- Analysis is in terms of the **load factor** $\alpha = n/m$
- $n =\#$ elements in the table
- $m =\#$ slots in the table
- Load factor is average number of elements per linked list.
- Can have $\alpha < 1$, $\alpha = 1$, or $\alpha > 1$
- Worst case is when all n keys hash to the same slot:
  - a single list of length n
  - worst case is $\Theta(n)$ plus time to compute h
- Average case depends on how well the hash function distributes keys among slots.

Average-case analysis of hashing with chaining

- Assume **simple uniform hashing**: any given element is equally likely to hash to any of the m slots.
- For $j = 0, 1, \ldots, m-1$, denote the length of the list $T[j]$ by n_n.
- $n = n_0 + n_1 + \cdots + n_{m-1}$
- Average value of n_j is $E[n_j] = \alpha = n/m$
- Assume we can compute h in $O(1)$ time, so that the time required to search for k depends on the length $n_{h(k)}$ of the list $T[h(k)]$.
- Two cases:
  - Unsuccessful search: hash table has no element with key k
  - Successful search: hash table contains an element with key k

Unsuccessful search

Theorem

An unsuccessful search takes expected time $\Theta(1 + \alpha)$.

Proof

- Simple uniform hashing means any key not already in the table is equally likely to hash to any of the m slots.
- To search unsuccessfully for any key k, need to search to the end of the list $T[h(k)]$.
- This list has expected length α.
- Adding the time to compute the hash function gives $\Theta(1 + \alpha)$.

Successful search

- The expected time for a successful search is also $\Theta(1+\alpha)$.
- The probability that each list is searched is proportional to the length of the list.

Successful search

Theorem

A successful search takes expected time $\Theta(1 + \alpha)$.

Proof

- Assume the element x is equally likely to be any of the n elements stored in the table.
- The number of elements examined during the search for x is 1 more than the number of elements that appear before x in x's list.
- These are the elements inserted *after* x was inserted.
- We need to find the average, over n elements x in the table, how many elements were inserted into x's list after x was inserted.
- For $i = 1, 2, ..., n$, let x_i be the ith element inserted, and let $k_k = key[x_i]$.
- For all i and j define

$$X_{ij} = \mathbf{I}\{h(k_i) = h(k_j)\}$$

- Simple uniform hashing means

$$Pr\{h[k_k) = h(k_j)\} = 1/m = E[X_{ij}]$$

Expected number of elements examined in successful search

$$
\begin{aligned}
E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right] &= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}E[X_{ij}]\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right) \\
&= 1+\frac{1}{nm}\sum_{i=1}^{n}(n-i) \\
&= 1+\frac{1}{nm}\sum_{i=0}^{n-1}i \\
&= 1+\frac{1}{nm}\frac{n(n-1)}{2} \\
&= 1+\frac{n-1}{2m} \\
&= 1+\frac{\alpha}{2}-\frac{\alpha}{2n} \\
&= \Theta(1+\alpha)
\end{aligned}
$$

Alternative analysis

- $X_{ij\ell} = \mathbf{I}\{\text{the search is for } x_i,\ h(k_i) = h(k_j) = \ell\}$
- Simple uniform hashing means $Pr\{h(k_i) = \ell\} = Pr\{h(k_j) = \ell\} = 1/m$
- $Pr\{\text{the search is for } x_i\} = 1/n$
- All these are independent: $Pr\{X_{ij\ell} = 1\} = \mathbf{E}[X_{ij\ell}] = 1/nm^2$

$$\begin{aligned} Y_j &= \mathbf{I}\{x_j \text{ appears in a list prior to the } x_i\} \\ &= \sum_{i=1}^{j-1} \sum_{\ell=0}^{m-1} X_{ij\ell} \end{aligned}$$

Alternative analysis, continued

$$
\begin{aligned}
E\left[1+\sum_{j=1}^{n} Y_j\right] &= 1 + E\left[\sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} X_{ij\ell}\right] \\
&= 1 + \sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} E[X_{ij\ell}] \\
&= 1 + \sum_{j=1}^{n}\sum_{i=1}^{j-1}\sum_{\ell=0}^{m-1} \frac{1}{nm^2} \\
&= 1 + \binom{n}{2} \cdot m \cdot \frac{1}{nm^2} \\
&= 1 + \frac{n-1}{2m} \\
&= 1 + \frac{\alpha}{2} - \frac{\alpha}{2n} = \Theta(1+\alpha)
\end{aligned}
$$

Interpretation

- If $n = O(m)$ then $\alpha = n/m = O(1)$, which means searching takes constant time on average.
- Since insertion and deletion take $O(1)$ worst case time, all dictionary operations take average time $O(1)$.

Hash functions

- Ideally, satisfies the assumption of simple uniform hashing.
- In practice, impossible since we don't know the distribution of input keys.
- In practice, many functions work OK.

Keys as natural numbers

- Can interpret any computer data as natural number.
- Strings, for example: CLRS
  - ASCII: 67, 76, 82, 83
  - There are 128 ASCII values
  - $h(\text{CLRS}) = 67(128^3) + 76(128^2) + 82(128^1) + 83(128^0) = 141,764,947$

Division method for hash functions

$$h(k) = k \bmod m$$

- Fast
- Powers of 2 are bad values for m, just uses least significant bits.
- Good choice for m: prime number not too close to a power of 2.

Multiplication method for hash functions

$$h(k) = \lfloor m((kA) \bmod 1) \rfloor$$

1. Choose A in range $0 < A < 1$.
2. Take fractional part of kA.
3. Multiply by m.
4. Take floor.

- Slower than division method.
- Value of m is not critical.
  - Can even use $m = 2^p$
- Knuth suggests a good value for $A \approx (\sqrt{5} - 1)/2$

Open addressing

- Store all keys in hash table, no linked lists.
- Use $i+1$ until we find an empty cell.

Example: $h(n) = n \mod 10$ (really bad hash function)

1. Insert 12:

0	1	2	3	4	5	6	7	8	9
x	x	12	x	x	x	x	x	x	x

2. Insert 14:

0	1	2	3	4	5	6	7	8	9
x	x	12	x	14	x	x	x	x	x

3. Insert 32:

0	1	2	3	4	5	6	7	8	9
x	x	12	32	14	x	x	x	x	x

4. Insert 92:

0	1	2	3	4	5	6	7	8	9
x	x	12	32	14	92	x	x	x	x

5. Insert 53:

0	1	2	3	4	5	6	7	8	9
x	x	12	32	14	92	53	x	x	x

- Linear probing leads to
  - **primary clustering**.
- Better:
  - quadratic probing
  - double hashing
- Expected number of probes

$$\frac{1}{1-\alpha}$$