Equations and recurrences

- Let's start backwards. Suppose we know what the function is, for example:

$$f(n) = 3n^2 + 5$$

- We want to get a recurrence for this function, so let's try some of our favorite recurrences.
- First, $f(n-1)$:

$$\begin{aligned} f(n-1) &= 3(n-1)^2 + 5 \\ &= 3n^2 - 6n + 3 + 5 \\ &= (3n^2 + 5) - 6n + 3 \\ &= f(n) - 6n + 3 \end{aligned}$$

- And now we have it! Our original function is the solution to the recurrence:

$$f(n) = f(n-1) + 6n - 3$$

- We can check this by substituting the original function in both sides of this:

$$\begin{aligned} 3n^2 + 5 &= 3(n-1)^2 + 5 + 6n - 3 \\ &= 3n^2 - 6n + 3 + 5 + 6n - 3 \\ &= 3n^2 + 5 \end{aligned}$$

Equations and recurrences

- Let's try a different recurrence for the same function.

$$f(n) = 3n^2 + 5$$

- Let's try , $f(n/2)$:

$$\begin{aligned} f(n/2) &= 3(n/2)^2 + 5 \\ &= (3/4)n^2 + 5 \\ &= 3n^2 + 5 - (9/4)n^2 \\ &= f(n) - (9/4)n^2 \end{aligned}$$

- And now we have it! Our original function is the solution to the recurrence:

$$f(n) = f(n/2) + (9/4)n^2$$

- Now let's check this with our original equation plugged into both sides:

$$\begin{aligned} 3n^2 + 5 &= 3(n/2)^2 + 5 + (9/4)n^2 \\ &= (3/4)n^2 + 5 + (9/4)n^2 \\ &= 3n^2 + 5 \end{aligned}$$

- So, again, we have verified that our original function is a solution to this recurrence.

Substitution method, big-O

- Now let's go the other way. Suppose we have an innocent recurrence, like this one:

$$f(n) = f(n-1) + 6n - 3$$

- And we want to solve it. (Don't tell anybody we already know the solution.)
- And suppose that we suspect the solution is $O(n^2)$. How can we *prove* this?
- Put in other words, can we prove by induction that there exists some c such that $f(n) \leq cn^2$ for sufficiently large n? Can we prove that this is a solution by induction?
- The base case is trivial, since we just have to show there exists some c such that $f(1) \leq c$, which will always be true.

Substitution method, big-O, inductive step

- Now for the inductive step.

$$f(n) = f(n-1) + 6n - 3$$

- Since our recurrence is on $f(n-1)$, we will assume, by inductive hypothesis, that $f(n-1) \leq c(n-1)^2$ and on the basis of this try to prove that $f(n) \leq cn^2$.

$$\begin{aligned} f(n) &= f(n-1) + 6n - 3 \\ &\leq c(n-1)^2 + 6n - 3 \\ &= cn^2 - 2cn + c + 6n - 3 \\ &= cn^2 + (6-2c)n + (c-3) \\ &\leq cn^2 \end{aligned}$$

- So long as

$$\begin{aligned} (6-2c)n + (c-3) &\leq 0 \\ (6-2c)n &\leq 3 - 2 \\ n &\leq \frac{3-2}{6-2c} \\ n &\geq \frac{1}{2c-6} \end{aligned}$$

which we can use in our definition of "sufficiently large" n, so this checks out, too.

Substitution method, big-Ω, inductive step

- Now let's try that method for Ω.

$$f(n) = f(n-1) + 6n - 3$$

- Suppose $f(n-1) \geq c(n-1)^2$ and prove $f(n) \geq cn^2$

$$\begin{aligned} f(n) &= f(n-1) + 6n - 3 \\ &\geq c(n-1)^2 + 6n - 3 \\ &= cn^2 - 2cn + c + 6n - 3 \\ &= cn^2 + (6-2c)n + (c-3) \\ &\geq cn^2 \end{aligned}$$

So long as $2c < 6$, and sufficiently large n.

Substitution method

$$T(1) = c$$
$$T(n) = 8T(n/2) + \Theta(n^2)$$

- We guess that $T(n) = O(n^3)$ and prove it true by induction.
- Base case is some constant, which can always be chosen large enough.
- Induction, we need to show there is some constant c_0 such that if we assume $T(m) \le c_0 m^3$ for all $m < n$ (strong induction), we can then prove $T(n) \le c_0 n^3$ (same constant).
- First try:

$$\begin{aligned}
T(n) &= 8T(n/2) + \Theta(n^2) && \text{use definition of } \Theta \\
&\le 8T(n/2) + cn^2 && n > n_0 \\
&\le 8c_0 n^3/2^3 + cn^2 && \text{since } n/2 < n \\
&= c_0 n^3 + cn^2 \\
&\overset{?}{\le} c_0 n^3
\end{aligned}$$

- We can't really prove what we want.
- We can't guarantee it is smaller than our target, because we don't know what c is.
- Be careful! Don't be tempted to say cn^2 doesn't matter in a big-0 proof.
- It *does* here because this is just one step in an inductive proof.

Substitution method, second try

$$T(1) = c$$
$$T(n) = 8T(n/2) + \Theta(n^2)$$

- We guess that $T(n) = O(n^3)$ and prove it true by induction.
- Base case is some constant, which can always be chosen large enough.
- We use a different member of $O(n^3)$ for induction.
- We need to show that there exist some constantes c_0, c_1 such that if we assume $T(m) \le c_0 m^3 - c_1 m^2$ for all $m < n$ (strong induction), we can then prove $T(n) \le c_0 n^3 - c_1 n^2$ (same constants).

$$\begin{aligned}
T(n) &= 8T(n/2) + \Theta(n^2) && \text{use definition of } \Theta \\
&\le 8T(n/2) + cn^2 && n > n_0 \\
&\le 8(c_0 n^3/2^3 - c_1 n^2/2^2) + cn^2 && \text{since } n/2 < n \\
&= c_0 n^3 - 2c_1 n^2 + cn^2 \\
&= c_0 n^3 - c_1 n^2 && \text{choose } c_1 = c
\end{aligned}$$