# Notes on Quicksort

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May 1, 2018

# Quicksort

- $\Theta(n^2)$ worst case.
- $\Theta(n \lg n)$ expected running time.
- Constants are small.
- Sorts in place.

# Quicksort: three step process

- To sort $A[p..r]$:
  - **Divide:** Partition $A[p..r]$ into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$,
    such that each element in the first subarray is $\leq A[q]$ and $A[q] \leq$ each element in the second subarray.
  - **Conquer:** Sort the two subarrays by recursive calls.
  - **Combine:** Nothing needs to be done.

$\textsc{Quicksort}(A, p, r)$

**if** $p < r$
  $q = \textsc{Partition}(A, p, r)$
  $\textsc{Quicksort}(A, p, q-1)$
  $\textsc{Quicksort}(A, q+1, r)$

Initial call is $\textsc{Quicksort}(A, 1, n)$

# Compare QUICKSORT and MERGESORT

```
QUICKSORT(A, p, r)
  if p < r
      q = PARTITION(A, p, r)
      QUICKSORT(A, p, q − 1)
      QUICKSORT(A, q + 1, r)
```

```
MERGE-SORT(A, p, r)
  if p < r                          // check for base case
      q = ⌊(p + r)/2⌋               // divide
      MERGE-SORT(A, p, q)           // conquer
      MERGE-SORT(A, q + 1, r)       // conquer
      MERGE(A, p, q, r)             // combine
```

# Compare PARTITION and MERGE

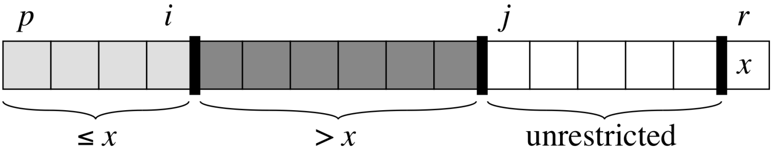
MERGE$(A, p, q, r)$

```
n1 = q - p + 1
n2 = r - q
let L[1..n1 + 1] and R[1..n2 + 1] be new arrays
for i = 1 to n1
    L[i] = A[p + i - 1]
for j = 1 to n2
    R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```

PARTITION$(A, p, r)$

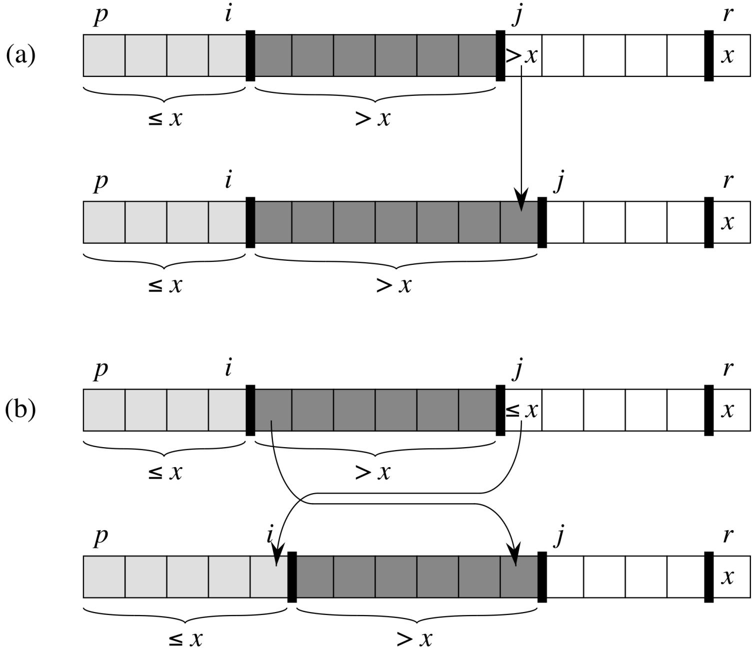
```
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```

# Partition

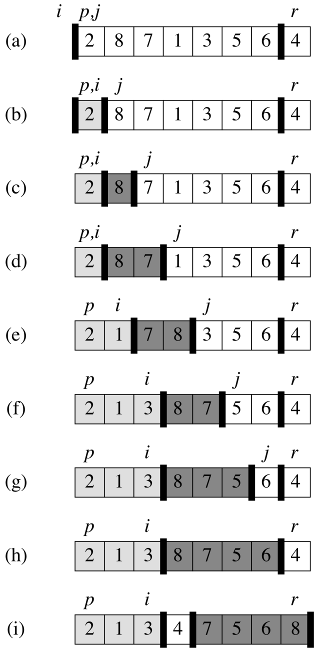



Loop invariant:

1. All entries in $A[p, \ldots, i]$ are $\leq$ pivot
2. All entries in $A[i+1, \ldots, j-1]$ are $>$ pivot
3. $A[r] =$ pivot



(a)
p
i
j
r
> x
x
≤ x
> x
p
i
j
r
x
≤ x
> x
(b)
p
i
j
r
≤ x
x
≤ x
> x
p
i
j
r
x
≤ x
> x




```
PARTITION(A, p, r)
  x = A[r]
  i = p − 1
  for j = p to r − 1
      if A[j] ≤ x
          i = i + 1
          exchange A[i] with A[j]
  exchange A[i + 1] with A[r]
  return i + 1
```

# Partition

PARTITION$(A, p, r)$

$x = A[r]$<br>
$i = p - 1$<br>
**for** $j = p$ **to** $r - 1$<br>
    **if** $A[j] \leq x$<br>
        $i = i + 1$<br>
        exchange $A[i]$ with $A[j]$<br>
exchange $A[i+1]$ with $A[r]$<br>
**return** $i + 1$

- Always selects $A[r]$ as the **pivot**
- Loop invariant:
    1. All entries in $A[p, ..., i]$ are $\leq$ pivot
    2. All entries in $A[i+1, ..., j-1]$ are $>$ pivot
    3. $A[r] =$ pivot

$O(n)$

# Can MERGESORT be done in place?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | 0 | 2 | 4 | 6 | 8 |

# Can MERGESORT be done in place?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | 0 | 2 | 4 | 6 | 8 |

- Yes, but *very* tricky!

# Running time of quicksort

- Depends on partitioning of subarrays.
- If subarrays are balanced: fast as mergesort.
- If subarrays are unbalanced: slow as insertion sort.

# Worst case for quicksort

- Arrays completely unbalanced.
- 0 elements in one and $n-1$ in the other
- Recurrence:

$$\begin{aligned} T(n) &= T(n-1) + T(0) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

- Same as insertion sort.
- Worst case for quicksort is the array is already sorted.
- This is the best case for insertion sort, which is $O(n)$.

# Best case for quicksort

- Each subarray has $n/2$ elements.
- Recurrence:

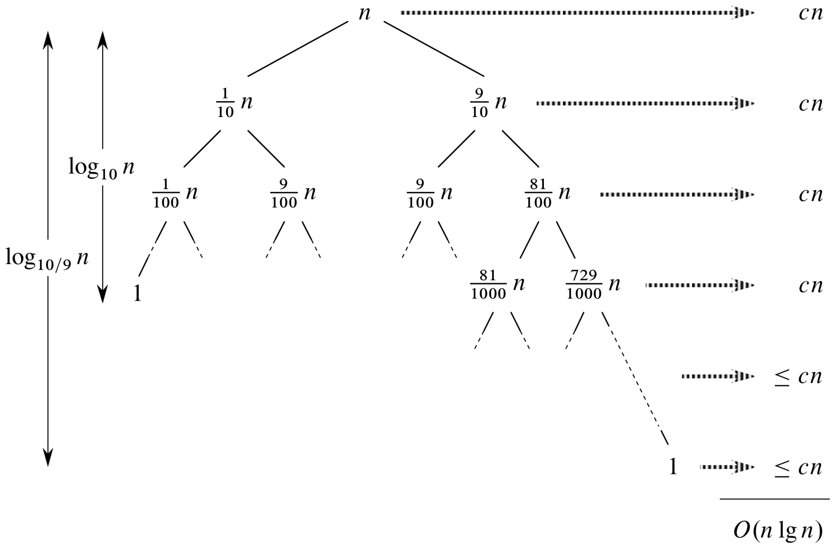
$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

- Same as mergesort.

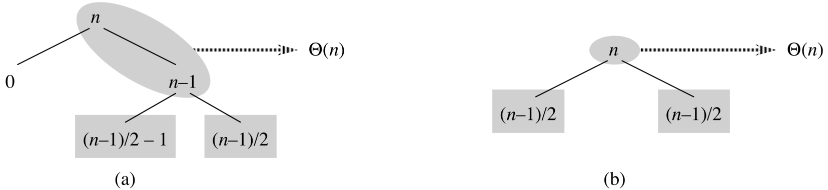
# “Average” running time for quicksort

Assume PARTITION always makes a 9-to-1 split.

$T(n) \leq T(9n/10) + T(n/10) + \Theta(n) = O(n \lg n)$



# "Average" running time for quicksort




- If levels alternate between good and bad splits, still $O(n \lg n)$
- If we randomize choice of pivot, what is the probability that all of the choices will be worst case?
- If we randomize choice of pivot, what is the probability that more than half of the choices will be worst case?

# Randomized quicksort

- Instead of randomizing the entire array, which adds a large constant factor, just randomize the choice of pivot.

  $\textsc{Randomized-Partition}(A, p, r)$

  $i = \textsc{Random}(p, r)$
  exchange $A[r]$ with $A[i]$
  **return** $\textsc{Partition}(a, p, r)$

  $\textsc{Randomized-Quicksort}(A, p, r)$

  **if** $p < r$
      $q = \textsc{Randomized-Partition}(A, p, r)$
      $\textsc{Quicksort}(A, p, q - 1)$
      $\textsc{Quicksort}(A, q + 1, r)$

- On average, the splits will be well balanced.
- $O(n \lg n)$ virtually guaranteed when $n$ is large.
- Stops any bad input from causing worst-case behavior.

# Worst-case analysis of randomized quicksort

$$T(n) = \max_{0 \le q \le n-1} (T(q) + T(n-q-1)) + \Theta(n)$$

- **Guess:** $T(n) \le cn^2$ for some $c$.

$$\begin{aligned} T(n) &\le \max_{0 \le q \le n-1} (cq^2 + c(n-q-1)^2) + \Theta(n) \\ &= c \cdot \max_{0 \le q \le n-1} (q^2 + (n-q-1)^2) + \Theta(n) \end{aligned}$$

- This is max when $q = 0$ or $q = n-1$ (parabolas).

$$\max_{0 \le q \le n-1} (q^2 + (n-q-1)^2) \le (n-1)^2 = n^2 - 2n + 1$$

$$\begin{aligned} T(n) &\le cn^2 - c(2n-1) + \Theta(n) \\ &\le cn^2 && \text{if } c(2n-1) \ge \Theta(n) \\ &= O(n^2) \end{aligned}$$

- Can also show $T(n) = \Omega(n^2)$, so $T(n) = \Theta(n^2)$.

# Average-case analysis of randomized quicksort

- The dominant cost of the algorithm is in the calls to PARTITION.
- PARTITION removes the pivot from future consideration.
- PARTITION is called at most $n$ times.
- Each call to PARTITION does a constant amount of work plus a constant times the number of comparisons done in the **for** loop.
- Let $X$ be the total number of comparisons performed in all calls to PARTITION.
- Total work done is $O(n + X)$.
- We seek a bound on the total number of comparisons.

# Average-case analysis of randomized quicksort

- Let the elements of $A$ be $z_1, z_2, \ldots, z_n$ with $z_i$ being the $i$th smallest.
- Define $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$
- Each pair of elements is compared at most once.
  - Compared only to the pivot, and then the pivot is removed.
- Let $X_{ij} = \mathbf{I}\{z_i \text{ is compared to } z_j\}$
- Since each pair is compared at most once,

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\} \end{aligned}$$

# Probability $z_i$ is compared to $z_j$

- Numbers in separate partitions will not be compared.
- If a pivot $x$ is chosen such that $z_i < x < z_j$, $z_i$ and $z_j$ will not be compared.
- If either $z_i$ or $z_j$ is chosen before any other element of $Z_{ij}$, then it will be compared to every element of $Z_{ij}$, except itself.
- The probability that $z_i$ is compared to $z_j$ is the probability that either $z_i$ or $z_j$ is chosen first.
- There are $j - i + 1$ elements of $Z_{ij}$, and pivots are chosen randomly and independently.
- The probability that any one of them is chosen first is $1/(j - i + 1)$.

$$\begin{aligned}
\Pr\{z_i \text{ is compared to } z_j\} &= \Pr\{z_i \text{ or } z_j \text{ is chosen first from } Z_{ij}\} \\
&= \Pr\{z_i \text{ is chosen first}\} + \Pr\{z_j \text{ is chosen first}\} \\
&= \frac{2}{j - i + 1}
\end{aligned}$$

# Expected number of comparisons

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\} \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \\
&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\lg n) \\
&= O(n \lg n)
\end{aligned}
$$